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DISCRETE s-CONVEX EXTREMAL DISTRIBUTIONS: THEORY AND APPLICATIONS

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Discrete s-convex extremal distributions: Theory and applications

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Abstract

Given a nondegenerated moment space with s fixed moments, explicit formulas for the discrete s-convex extremal distribution have been derived for $s = 1, 2, 3$ (see [1]). If $s = 4$, only the maximal distribution is known (see [2]). This paper goes beyond this limitation and proposes a method to derive explicit expressions for general nonnegative integer s. In particular, we derive explicitly the discrete 4-convex minimal distribution. For illustration, we show how this theory allows to bound the probability of extinction in a Galton-Watson branching process. The results are also applied to derive bounds for the probability of ruin in the compound binomial and Poisson insurance risk models.

Keywords : s-convex orders, moment spaces, stochastic extrema, Lundberg's bound, branching process, insurance risk model.

1 Introduction

It is well established that the theory of stochastic orderings has a considerable interest in probability for theoretical and practical purposes (see, e.g., [3] and [4]). For instance, it can be used to compare complex models with more tractable ones which are "riskier", leading thus to more conservative decisions.

In many situations, stochastic order relations are used to compare *real* random variables. Quite recently, various *discrete* stochastic orderings have been introduced to compare random variables that are *discrete* by nature as counts for instance (see, e.g., [5], [6] and [7]). A remarkable class investigated by [1] is the class of the *discrete s-convex orderings* among arithmetic random variables valued in some set $\mathcal{N}_n = \{0, 1, 2, \ldots, n\}$, $n \in \mathbb{N}$. Here s is any nonnegative integer smaller or equal to n.

Discrete s-convex orderings have been defined in [1] in the following way. Let Δ be the first order forward difference operator (with unitary increment) defined for each function $u : \mathcal{N}_n \to \mathbb{R}$ by $\Delta u(i) = u(i + 1) - u(i)$ for all $i \in \mathcal{N}_{n-1}$. Let Δ^k, $k \in \mathcal{N}_n$, be the k-th order forward difference operator defined recursively by $\Delta^k u(i) = \Delta^{k-1}u(i + 1) - \Delta^{k-1}u(i)$ for all $i \in \mathcal{N}_{n-k}$ (by convention, $\Delta^1 u \equiv \Delta u$ and $\Delta^0 u \equiv u$). If X and Y are two random variables valued in $\mathcal{N}_n$, X is said to be smaller than Y with respect to the discrete s-convex order if $\mathbb{E}[u(X)] \leq \mathbb{E}[u(Y)]$ for all $u \in \mathcal{U}^{\mathcal{N}_n}_{s-cx} = \{u : \mathcal{N}_n \to \mathbb{R} : \Delta^s u(i) \geq 0,\ \forall\ i \in \mathcal{N}_{n-s}\}$. In such a case, we write $X \preceq^{\mathcal{N}_n}_{s-cx} Y$.

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Since the power functions $x \mapsto x^k$ and $x \mapsto -x^k$ both belong to $\mathcal{U}^{\mathcal{N}_n}_{s-cx}$ for $k = 1, 2, \ldots, s-1$, we immediately get the necessary condition

$$X \preceq^{\mathcal{N}_n}_{s-cx} Y \Rightarrow \mathbb{E}X^k = \mathbb{E}Y^k \text{ for } k = 1, 2, \ldots, s-1.$$

In other words, if $X \preceq^{\mathcal{N}_n}_{s-cx} Y$ then the $s-1$ first moments of X and Y necessarily match. Consequently, the ordering relation $\preceq^{\mathcal{N}_n}_{s-cx}$ can only be used to compare the random variables with the same first $s-1$ moments. This motivates to introduce the *moment space* $\mathcal{D}_s\left(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1}\right)$ which contains all random variables valued on $\mathcal{N}_n$ such that the first $s-1$ moments are fixed to $\mathbb{E}X^k = \mu_k$, $k = 1, \ldots, s-1$, where s is a prescribed nonnegative integer. One remarkable property of s-convex orderings is the following: Provided that the moment space satisfies some reasonable conditions (in particular this space is not void), the moment space contains a minimum random variable $X^{(s)}_{\min}$ and a maximum random variable $X^{(s)}_{\max}$ with respect to $\preceq^{\mathcal{N}_n}_{s-cx}$.

However, the proof of this existence result is implicit in the sense that a formula for $X^{(s)}_{\min}$ and $X^{(s)}_{\max}$ cannot be found easily, except in the simplest cases that we recall now.

If $s = 3$, the extrema $X^{(3)}_{\min}$ and $X^{(3)}_{\max}$ have been derived in [1]. Let ξ_1 and ξ_2 be the integers in $\mathcal{N}_{n-1}$ such that $\xi_1 < \mu_2/\mu_1 \leq \xi_1 + 1$ and $\xi_2 < (n - \mu_1)^{-1}(n\mu_1 - \mu_2) \leq \xi_2 + 1$. Then the discrete 3-convex extremal distributions are given by

$$X^{(3)}_{\min} = \begin{cases} 0 & \text{with probability } p_1 = 1 - p_2 - p_3, \\ \xi_1 & \text{with probability } p_2 = \frac{(\xi_1+1)\mu_1 - \mu_2}{\xi_1}, \\ \xi_1 + 1 & \text{with probability } p_3 = \frac{\mu_2 - \xi_1\mu_1}{1+\xi_1}, \end{cases} \tag{1}$$

and

$$X^{(3)}_{\max} = \begin{cases} \xi_2 & \text{with probability } q_1 = \frac{(1+\xi_2)(n-\mu_1)+\mu_2-n\mu_1}{n-\xi_2}, \\ \xi_2 + 1 & \text{with probability } q_2 = \frac{(n+\xi_2)\mu_1-\mu_2-n\xi_2}{n-1-\xi_2}, \\ n & \text{with probability } q_3 = 1 - q_1 - q_2. \end{cases} \tag{2}$$

The proof of this result can be found in [1] and uses the theory of discrete Tchebycheff systems (see, e.g. [8]).

If $s = 4$, the same argument is used in [2] to derive the explicit formula for $X^{(4)}_{\max}$. Let ζ be the integers in $[0, n-2]$ such that $\zeta < (n\mu_1 - \mu_2)^{-1}(n\mu_2 - \mu_3) \leq \zeta + 1$. Then,

$$X^{(4)}_{\max} = \begin{cases} 0 & \text{with probability } v_1 = 1 - v_2 - v_3 - v_4, \\ \zeta & \text{with probability } v_2 = \frac{n\mu_1(\zeta+1)-\mu_2(\zeta+1+n)+\mu_3}{\zeta(n-\zeta)}, \\ \zeta+1 & \text{with probability } v_3 = \frac{\mu_2(\zeta+n)-n\mu_1\zeta-\mu_3}{(\zeta+1)(n-\zeta-1)}, \\ n & \text{with probability } v_4 = \frac{\mu_3-\mu_2(2\zeta+1)+\mu_1\zeta(\zeta+1)}{n(n-\zeta)(n-\zeta-1)}. \end{cases} \tag{3}$$

Surprisingly, no explicit formula for $X^{(4)}_{\min}$ is available in the literature. The point is that the argument based on the non-negativity of particular moment matrices is no longer valid for that case. The same phenomenon appears for the derivation of $X^{(s)}_{\min}$ or $X^{(s)}_{\max}$ with $s \geq 5$. In that sense the theory of discrete s-convex extremal distribution is limited to the case $s \leq 3$ and is partially solved for $s = 4$.

The present paper aims to go beyond this limitation and proposes new arguments, based on the so-called "majorant-minorant method" and the "cut-criterion", that allow to derive the explicit extremal distributions for all s. However these cases are far more complicated to deal with because a subtle discussion about the points of support of the extremal distribution is needed.

To illustrate that point, it is interesting to notice the close connection between the extrema (1)–(3) and the corresponding *continuous* extrema, for which a parallel theory is developed when the support of the random variable is the interval $[0, n]$. For instance, let us consider the case of $X^{(3)}_{\min}$. It can be shown (see [9]) that the continuous 3-convex minimal distribution is given by

$$X^{\text{cont.}(3)}_{\min} = \begin{cases} 0 & \text{with probability } 1 - p, \\ \mu_2/\mu_1 & \text{with probability } p = \mu_1^2/\mu_2. \end{cases} \tag{4}$$

A comparison between (1) and (4) leads to the conclusion that the discrete extremal distribution can be easily obtained from the corresponding continuous extremal distributions since the probability mass $p = \mu_1^2/\mu_2$ of the continuous distribution is spread on $\xi, \xi+1 \in \mathcal{N}_n$ such that $\xi < \mu_2/\mu_1 \leq \xi+1$. This phenomenon also arises if we compare the discrete extremal distributions (2), (3) with their corresponding continuous extremal distribution. It is then tempting to conjecture that all discrete extrema can be obtained from their continuous extrema. This would be a right strategy to solve our problem since an explicit formula for continuous extremal distributions can be written for all s.

Surprisingly, this conjecture is wrong, as we can show with a simple example. Consider for instance the moment space fixed by the moments $(\mu_0, \mu_1, \mu_2, \mu_3) = (1, 6.625, 44.8525, 313.78825)$. One can see that the corresponding continuous 4-convex minimum is given by

$$X^{\text{cont.}} = \begin{cases} 6.4 & \text{with probability } 0.95, \\ 10.9 & \text{with probability } 0.05. \end{cases}$$

Using the theory that we develop in the present article, one can show that the discrete 4-convex minimum on $\mathcal{N}_n$ is given by

$$X^{\text{disc.}} = \begin{cases} 6 & \text{with probability } 0.490875, \\ 7 & \text{with probability } 0.487025, \\ 12 & \text{with probability } 0.016725, \\ 13 & \text{with probability } 0.005375. \end{cases}$$

In other words, the support of the discrete distribution does not appear as the neighbourhood in $\mathcal{N}_n$ of the supports of the continuous distribution. Moreover, if we discretize the continuous extremal distribution on the neighbouring support $\{6, 7, 10, 11\}$ one can see that the "probability mass" at 10 would be negative (-0.0794).

This example shows that it is challenging to find the form of the support of the discrete extremal disctribution. This question is addressed in Section 2 of the article. In Subsection 2.1 we focus on the so-called "majorant/minorant method" to find the s-convex extrema. This section contains key results that characterize the discrete moment space. Then Subsection 2.2 recalls the cut-criterion [1]. Subsection 2.3 derives the support of the 4-convex minimum.

Section 3 deals with an application of this theory. We compute lower and upper bounds for the probability of extinction in a Galton-Watson branching process and for the Lundberg's coefficient in the classical insurance risk model with discrete claim amounts.

Finally, Section 4 gives some conclusions as well as the generalization of the method developed in the paper to find the s-convex extrema for $s \geq 4$.

2 Derivation of the 4-convex minimum

2.1 S-convex extrema in moment spaces

As announced, random variables are assumed to take values on the state space $\mathcal{N}_n = \{0, 1, 2, \ldots, n\}$ for some non-negative integer n. We denote by $\mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ the moment space of all the random variables valued in $\mathcal{N}_n$ and with prescribed first $s-1$ moments $\mu_k = \mathbb{E}X^k$, $k = 1, \ldots, s-1$. Henceforth, the moment sequence $(\mu_1, \mu_2, \ldots, \mu_{s-1})$ is supposed to be such that $\mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ is non void (for conditions, see [10]).

We aim to derive random variables $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ belonging to $\mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ and such that

$$X_{\min}^{(s)} \preceq_{s-cx}^{\mathcal{N}_n} X \preceq_{s-cx}^{\mathcal{N}_n} X_{\max}^{(s)} \text{ for all } X \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1}). \tag{5}$$

The determination of $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ involved in (5) has been discussed in [1]-[2]: using the cut-criterion on distribution functions (see Proposition 2.3 below), the extrema for $s = 1, 2, 3$ and the maximum for $s = 4$ were obtained explicitly. In this paper, using a method that we call the

Majorant/Minorant Method (inspired from the so-called *method of admissible measures* in [11]), we find the form of the support of the 4-convex minimum.

Instead of solving (5) directly, we first look for the random variables that achieve the bounds

$$\max_{X\in\mathcal{D}_s(\mathcal{N}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})} \mathbb{E}\left[X^s\right] \text{ and } \min_{X\in\mathcal{D}_s(\mathcal{N}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})} \mathbb{E}\left[X^s\right]. \tag{6}$$

The extrema $X_{\min}^{(s)}$ and $X_{\max}^{(s)}$ necessarily achieve the bounds in (6).

Let us consider the problem of finding the random variables that realize the bounds in (6). We have the following result.

Property 2.1. *(i) A random variable $X \in \mathcal{D}_s(\mathcal{N}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})$ achieves the maximum (6) if and only if X is sup-admissible, that is X is concentrated on the set*

$$\left\{i\in\mathcal{N}_n : i^s = c_0 + c_1\cdot i + c_2\cdot i^2 + \cdots + c_{s-1}\cdot i^{s-1}\right\}$$

where the c_i's are real constants such that

$$i^s \le c_0 + c_1\cdot i + c_2\cdot i^2 + \cdots + c_{s-1}\cdot i^{s-1}, \text{ for all } i\in\mathcal{N}_n.$$

(ii) A random variable $X \in \mathcal{D}_s(\mathcal{N}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})$ achieves the minimum (6) if and only if X is sub-admissible, that is X is concentrated on the set

$$\left\{i\in\mathcal{N}_n : i^s = c_0 + c_1\cdot i + c_2\cdot i^2 + \cdots + c_{s-1}\cdot i^{s-1}\right\}$$

where the c_i's are real constants such that

$$i^s \ge c_0 + c_1\cdot i + c_2\cdot i^2 + \cdots + c_{s-1}\cdot i^{s-1}, \text{ for all } i\in\mathcal{N}_n.$$

Proof. We only prove (i); the proof for (ii) is similar.

Sufficient condition. Henceforth, we adopt the convention that $0^0 = 1$. Let X be a random variable in $\mathcal{D}_s(\mathcal{N}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})$, i.e.

$$\sum_{i=0}^{n}\mathbb{P}\left[X=i\right]i^k = \mu_k \quad ,k=0,1,\ldots,s-1;$$

which is concentrated on the set

$$\left\{i\in\mathcal{N}_n : i^s = \sum_{k=0}^{s-1} c_k i^k\right\}$$

where the c_i's are real constants such that $i^s \le \sum_{k=0}^{s-1} c_k i^k$ for all $i\in\mathcal{N}_n$. Let also Z be some random variable in $\mathcal{D}_s(\mathcal{N}_n;\mu_1,\mu_2,\ldots,\mu_{s-1})$, i.e.

$$\sum_{i=0}^{n}\mathbb{P}\left[Z=i\right]i^k = \mu_k \quad ,k=0,1,\ldots,s-1.$$

We have

$$\begin{aligned}
\mathbb{E}\left[X^s\right] &= \sum_{i=0}^{n}\mathbb{P}\left[X=i\right]i^s = \sum_{i=0}^{n}\mathbb{P}\left[X=i\right]\sum_{k=0}^{s-1}c_k i^k = \sum_{k=0}^{s-1}c_k\sum_{i=0}^{n}\mathbb{P}\left[X=i\right]i^k \\
&= \sum_{k=0}^{s-1}c_k\mu_k = \sum_{k=0}^{s-1}c_k\sum_{i=0}^{n}\mathbb{P}\left[Z=i\right]i^k = \sum_{i=0}^{n}\mathbb{P}\left[Z=i\right]\sum_{k=0}^{s-1}c_k i^k \\
&\ge \sum_{i=0}^{n}\mathbb{P}\left[Z=i\right]i^s = \mathbb{E}\left[Z^s\right]
\end{aligned}$$

for all $Z \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$. So, $X = \arg\max_{Z \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})} \mathbb{E}[Z^s]$.

Necessary condition. Let $X = \arg\max_{Z \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})} \mathbb{E}[Z^s]$ and let us suppose that X is the s-convex maximum, i.e. $Z \preceq_{s-cx}^{\mathcal{N}_n} X$ for all $Z \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$. If X is not sup-admissible, by [11] there exists a sup-admissible random variable $Y \neq_d X$ such that $Y = \arg\max_{Z \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})} \mathbb{E}[Z^s]$, which is impossible by Proposition 3.3 of [2]. Let us now prove by absurd that X is the s-convex maximum. If not, there exists some random variable $Y \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$, $Y \neq_d X$, such that $X \preceq_{s-cx}^{\mathcal{N}_n} Y$. By Proposition 3.1 of [1], it comes particularly that $\mathbb{E}[X^s] \leq \mathbb{E}[Y^s]$, which is impossible and ends the proof. ∎

We even have the following result that enables us to identify the s-convex extrema with the random variables realizing the bounds (6). The discrete s-convex extrema are thus easily identified using Property 2.1.

Proposition 2.2. *Let X be some random variable in $\mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$. Then X is the s-convex maximum (resp. minimum) if and only if $X = \arg\max_{Z \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})} \mathbb{E}[Z^s]$ (resp. $X = \arg\min_{Z \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})} \mathbb{E}[Z^s]$).*

Proof. The necessary condition has already been proved in the the proof of the necessary part of Property 2.1 and the sufficient condition is obvious using Proposition 3.1 of [1]. ∎

2.2 Cut-criterion

We now recall the cut-criterion on the distribution functions of [1] that allows us to compare two random variables in the s-convex sense.

Let u be any real-valued function defined on a subset $\mathcal{S}$ of $\mathbb{R}$. We introduce the operator S^- which, when applied to u, counts the number of sign changes of u over its domain S. More precisely, $S^-(u) = \sup S^-[u(x_1), u(x_2), \ldots, u(x_n)]$ where the supremum is extended over all $x_1 < x_2 < \ldots < x_n \in \mathcal{S}$, n is arbitrary but finite and $S^-[y_1, y_2, \ldots, y_n]$ denotes the number of sign changes of the indicated sequence $\{y_1, y_2, \ldots, y_n\}$, zero terms being discarded. The functions u_1 and u_2 are said to cross each other k times ($k = 0, 1, 2, \ldots$) if $S^-(u_1 - u_2) = k$. Moreover, if X and Y are random variables valued in $\mathcal{N}_n$ with respective distribution functions F_X and F_Y, we say that $F_X \geq F_Y$ near n if $F_X(k) \geq F_Y(k)$ for all $k \geq k_0$, with $k_0 \leq n-1$.

Proposition 2.3 ([1]). *Let X and Y be two random variables valued in $\mathcal{N}_n$, such that $\mathbb{E}[X^k] = \mathbb{E}[Y^k]$ for $k = 1, \ldots, s-1$. Then, $S^-(F_X - F_Y) \leq s-1$ together with $F_X \geq F_Y$ near $n \Rightarrow X \preceq_{s-cx}^{\mathcal{N}_n} Y$.* ∎

2.3 Support of the 4-convex minimum

Using the cut-criterion, it can be verified that the possible structure of the supports of the 4-convex discrete extrema takes the form $\{\xi, \xi+1, \eta, \eta+1\}$ or $\{0, \zeta, \zeta+1, n\}$. It is interesting to note that those supports are identical to the ones that could be obtained calling upon the theory of the discrete Tchebycheff systems (see [8]). The Majorant/Minorant Method is then used to derive the conditions on the support points ξ, η and ζ so that the random variable corresponding to such support has moments $\mu_1, \mu_2, \ldots, \mu_{s-1}$. This is done by computing the probabilities associated to the support points as solutions to some Vandermonde system and by checking that the resulting probabilities are positive.

Property 2.4. *Consider a moment space $\mathcal{D}_4(\mathcal{N}_n; \mu_1, \mu_2, \mu_3)$ with a given sequence of moments μ_1, μ_2, μ_3. If $\xi, \eta \in \mathcal{N}_n$ are such that $0 \leq \xi < \xi+1 < \eta < \eta+1 \leq n$ and define*

$$\begin{aligned}
\alpha_1 &:= -\mu_3 + \mu_2(2\eta + \xi + 2) - \mu_1[(\xi+1)\eta + (\xi+1)(\eta+1) + \eta(\eta+1)] + (\xi+1)\eta(\eta+1) \\
\alpha_2 &:= \mu_3 - \mu_2(\xi + 2\eta + 1) + \mu_1[\xi\eta + \xi(\eta+1) + \eta(\eta+1)] - \xi\eta(\eta+1) \\
\alpha_3 &:= -\mu_3 + \mu_2(2\xi + 2 + \eta) - \mu_1[\xi(\xi+1) + \xi(\eta+1) + (\xi+1)(\eta+1)] + \xi(\xi+1)(\eta+1) \\
\alpha_4 &:= \mu_3 - \mu_2(2\xi + 1 + \eta) + \mu_1[\xi(\xi+1) + \xi\eta + \eta(\xi+1)] - \xi(\xi+1)\eta
\end{aligned} \tag{7}$$

that are positive, then the discrete 4-convex minimal distribution of $\mathcal{D}_4(\mathcal{N}_n;\mu_1,\mu_2,\mu_3)$ *is given by*

$$X_{\min}^{(4)}=\begin{cases}\xi & \textit{with probability } w_1=\alpha_1/\left(\eta-\xi\right)\left(\eta+1-\xi\right),\\ \xi+1 & \textit{with probability } w_2=\alpha_2/\left(\eta-\xi-1\right)\left(\eta-\xi\right),\\ \eta & \textit{with probability } w_3=\alpha_3/\left(\eta-\xi\right)\left(\eta-\xi-1\right),\\ \eta+1 & \textit{with probability } w_4=\alpha_4/\left(\eta+1-\xi\right)\left(\eta-\xi\right).\end{cases}\tag{8}$$

Proof. The proof gives the minimal together with the maximal distribution (3). Using the majorant/minorant method, we find out the respective supports of the 4-convex extrema $X_{\max}^{(4)}$ and $X_{\min}^{(4)}$. To that end, we just compute the polynomials $p(i)=c_0+c_1i+c_2i^2+c_3i^3$ of degree 3 (i.e. c_0, c_1, c_2 and $c_3\in\mathbb{R}$) such that $X_{\max}^{(4)}\in\mathcal{D}_4\left(\mathcal{N}_n;\mu_1,\mu_2,\mu_3\right)$ (*resp.* $X_{\min}^{(4)}$) is concentrated on the set

$$\begin{aligned}\left\{i\in\mathcal{N}_n:i^4=c_0+c_1i+c_2i^2+c_3i^3\right\} \quad &= \quad \{0,\zeta,\zeta+1,n\}\,(1\leq\zeta\leq n-2)\\ &\qquad \textit{resp.}\\ &\qquad \{\xi,\xi+1,\eta,\eta+1\}\,(0\leq\xi<\xi+1<\eta<\eta+1\leq n)\end{aligned}$$

and $i^3\leq c_0+c_1i+c_2i^2$ for all $i\in\mathcal{N}_n$ (*resp.* $\geq$).

The only polynomial of degree 3 that fulfills the conditions

$$\begin{aligned}&0=c_0\\ &\zeta^4=c_0+c_1\zeta+c_2\zeta^2+c_3\zeta^3\\ &\left(\zeta+1\right)^4=c_0+c_1\left(\zeta+1\right)+c_2\left(\zeta+1\right)^2+c_3\left(\zeta+1\right)^3\\ &n^4=c_0+c_1n+c_2n^2+c_3n^3\end{aligned}$$

is $p(i)=\zeta\left(\zeta+1\right)ni-\left[n\left(\zeta+1\right)+\zeta\left(\zeta+1\right)+n\zeta\right]i^2+\left(\zeta+\zeta+1+n\right)i^3$. The zeros of the polynomial $x^4-p(x)$ are of course 0, ζ, $\zeta+1$ and n and $x^4-p(x)$ is always negative on $\mathcal{N}_n$. So, as we have checked that $i^4\leq p(i)$ on $\mathcal{N}_n$, the random variable with support $\{0,\zeta,\zeta+1,n\}$ $(1\leq\zeta\leq n-2)$ has to be $X_{\max}^{(4)}$.

The only polynomial of degree 3 that fulfills the conditions

$$\begin{aligned}&\xi^4=c_0+c_1\xi+c_2\xi^2+c_3\xi^3\\ &\left(\xi+1\right)^4=c_0+c_1\left(\xi+1\right)+c_2\left(\xi+1\right)^2+c_3\left(\xi+1\right)^3\\ &\eta^4=c_0+c_1\eta+c_2\eta^2+c_3\eta^3\\ &\left(\eta+1\right)^4=c_0+c_1\left(\eta+1\right)+c_2\left(\eta+1\right)^2+c_3\left(\eta+1\right)^3\end{aligned}$$

is

$$\begin{aligned}p(i) \quad = \quad & -\xi\left(\xi+1\right)\eta\left(\eta+1\right)\\ &+\left[\left(\xi+\xi+1\right)\eta\left(\eta+1\right)+\xi\left(\xi+1\right)\left(\eta+\eta+1\right)\right]i\\ &-\left[\eta\left(\eta+1\right)+\left(\xi+1\right)\left(\eta+1\right)+\left(\xi+1\right)\eta+\xi\left(\eta+1\right)+\xi\eta+\xi\left(\xi+1\right)\right]i^2\\ &+\left(\xi+\xi+1+\eta+\eta+1\right)i^3\end{aligned}$$

The zeros of the polynomial $x^4-p(x)$ are of course ξ, $\xi+1$, η and $\eta+1$ and $x^4-p(x)$ is always positive on $\mathcal{N}_n$. So, as we have checked that $i^4\geq p(i)$ on $\mathcal{N}_n$, the random variable with support $\{\xi,\xi+1,\eta,\eta+1\}$ $(0\leq\xi<\xi+1<\eta<\eta+1\leq n)$ has to be $X_{\min}^{(4)}$.

Finally, we have to fix conditions on the support points to assure the non-negativity of their associated probabilities. The conditions on the support points of $X_{\max}^{(4)}$ are

$$\begin{aligned}&0<\zeta<\zeta+1<n\\ &\mu_3\leq-\zeta n\mu_1+\left(\zeta+n\right)\mu_2\\ &\mu_3\leq\zeta\left(\zeta+1\right)n-\left[\zeta\left(\zeta+1\right)+n\left(\zeta+1\right)+n\zeta\right]\mu_1+\left(\zeta+\zeta+1+n\right)\mu_2\\ &\mu_3\geq-\zeta\left(\zeta+1\right)\mu_1+\left(\zeta+\zeta+1\right)\mu_2\\ &\mu_3\geq-\left(\zeta+1\right)n\mu_1+\left(\zeta+1+n\right)\mu_2\end{aligned}$$

and because we have $\zeta(\zeta+1)n - [\zeta(\zeta+1) + n(\zeta+1) + n\zeta]i + (\zeta+\zeta+1+n)i^2 \geq i^3$ (cfr. 3-convex maximum) on $\mathcal{N}_n$ and $-\zeta(\zeta+1)i + (\zeta+\zeta+1)i^2 \leq i^3$ on $\mathcal{N}_n$ (cfr. 3-convex minimum), the second and the third condition are respectively always verified and the system of conditions reduces to

$$0 < \zeta < \zeta + 1 < n \text{ and } \zeta < \frac{n\mu_2 - \mu_3}{n\mu_1 - \mu_2} \leq \zeta + 1.$$

Henceforth, we refind the 4-convex maximum (3). The conditions on the support points of $X_{\min}^{(4)}$ are given by

$$\alpha_1 \geq 0, \alpha_2 \geq 0, \alpha_3 \geq 0 \text{ and } \alpha_4 \geq 0. \blacksquare \tag{9}$$

The solution (ξ, η) of (7) cannot be obtained explicitly. Nevertheless, it is easily obtained by testing each admissible pair (ξ, η) of $\mathcal{N}_n$.

Remark 2.5. *As it is proved in [1], the s-convex orderings with respect to $\mathcal{N}_n$ are shift invariant. In particular, this means that, for all random variable X in $\mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ and all $k = 0, 1, 2, \ldots$,*

$$X_{\min}^{(s)} \preceq_{s-cx}^{\mathcal{N}_n} X \preceq_{s-cx}^{\mathcal{N}_n} X_{\max}^{(s)} \Leftrightarrow X_{\min}^{(s)} + k \preceq_{s-cx}^{k+\mathcal{N}_n} X + k \preceq_{s-cx}^{k+\mathcal{N}_n} X_{\max}^{(s)} + k,$$

where $k+\mathcal{N}_n = \{k, k+1, \ldots, k+n\}$. Then, if the random variables are defined on $\{k, k+1, \ldots, k+n\}$, the discrete s-convex extrema can easily be obtained by shifting the discrete s-convex extrema among random variables defined on $\{0, 1, \ldots, n\}$ with appropriate moment sequence.

3 Applications

3.1 Theoretical background

Given a random variable N valued in $\mathcal{N}_n$, n being a positive integer, a classical problem consists in solving the equation

$$\varphi_N(z) = P_k(z), \tag{10}$$

in the unknown z, where $\varphi_N(z) = \mathbb{E}\left[z^N\right] = \sum_{k=0}^n z^k \mathbb{P}[N = k]$, $0 \leq z \leq 1$, is the probability generating function of N, and where $P_k(\cdot)$ is a given non-decreasing polynomial function of degree k (usually, $k \leq 2$). When all that is known about N is that it belongs to $\mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$, then (10) cannot be solved explicitely. The aim of this subsection is to show that the s-convex extrema described previously allow accurate approximations for the solution of (10). The method using the continuous s-convex extrema could of course be applied here. Nevertheless, we get better bounds if we take into account the fact that N is now valued in the arithmetic grid $\mathcal{N}_n$ rather than in the interval $[0, n]$ (see Tables 1, 3 and 4). The idea is to construct two functions $\varphi_{\min}^{(s)}(\cdot)$ and $\varphi_{\max}^{(s)}(\cdot)$ such that

$$\varphi_{\min}^{(s)}(z) \leq \varphi_N(z) \leq \varphi_{\max}^{(s)}(z) \text{ for all } 0 \leq z \leq 1. \tag{11}$$

The sequence $\left[z^k,\ k \in \mathbb{N}\right]$ being completely monotonic for $0 < z \leq 1$, we get from [1] that, when s is even, $\varphi_{\min}^{(s)}(t) = \varphi_{N_{\min}^{(s)}}(t)$ and $\varphi_{\max}^{(s)}(t) = \varphi_{N_{\max}^{(s)}}(t)$, while when s is odd, $\varphi_{\min}^{(s)}(t) = \varphi_{N_{\max}^{(s)}}(t)$ and $\varphi_{\max}^{(s)}(t) = \varphi_{N_{\min}^{(s)}}(t)$, where the $N_{\min}^{(s)}$ and $N_{\min}^{(s)}$ are the stochastic extrema in $\mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1})$ with respect to the discrete versions of the s-convex stochastic orderings.

The same problem with $\phi_N(z) = \mathbb{E}\left[e^{zN}\right]$, the moment generating function of N, can be handled similarly. Since the sequence $\left[e^{kz},\ k \in \mathbb{N}\right]$ is absolutely monotonic, we have that $\phi_{\min}^{(s)}(t) \leq \phi_N^{(s)}(t) \leq \phi_{\max}^{(s)}(t)$ with $\phi_{\min}^{(s)}(t) = \phi_{N_{\min}^{(s)}}(t)$ and $\phi_{\max}^{(s)}(t) = \phi_{N_{\max}^{(s)}}(t)$. As above, these provide bounds on the root of the equation $\phi_N(z) = P_k(z)$, where P_k is a monotone polynomial function. Solving the equation $\phi_{\min}^{(s)}(z) = P_k(z)$ yields the root $z_1^{(s)}$, say, and solving $\phi_{\max}^{(s)}(z) = P_k(z)$ yields the root $z_2^{(s)}$, say. The solution $\tilde{z}$, say, of $\phi_N^{(s)}(z) = P_k(z)$ then satisfies $z_2^{(s)} \leq \tilde{z} \leq z_1^{(s)}$.

3.2 Probability of ultimate extinction in a branching process

Let us briefly recall the definition of the Galton-Watson process. At time $t = 0$ there exists an initial population M_0. During its life span, every individual gives birth to a random number of children. During their life spans, these children give birth to a random number of children, and so on. The reproduction rules are (i) all individuals give birth according to the same probability law, independently of each other and (ii) the number of children produced by an individual is independent of the number of individuals in their generation. In the sequel, we also assume (without real loss of generality) that $M_0 = 1$. For $k \geq 1$, let M_k be the number of individuals in generation k and let N be a generic random variable valued in $\mathcal{N}_n$ representing the number of children obtained by the individuals; $\mathbb{P}[N = 1] < 1$. If you denote by α the probability of ultimate extinction of this process, i.e. $\alpha = \mathbb{P}[M_k = 0 \text{ for some } k]$, it is well-known that α is the smallest non-negative root of the equation $z = \varphi_N(z)$; $\alpha = 1$ for $\mathbb{E}[N] \leq 1$ and $\alpha < 1$ for $\mathbb{E}[N] > 1$. In order to illustrate the use of the s-convex extrema up to the order four, we consider the following example from [12] page 11.

Example 3.1. *Let us take $n = 10$ and $\mathbb{P}[N = 0] = 0.4982$, $\mathbb{P}[N = 1] = 0.2103$, $\mathbb{P}[N = 2] = 0.1270$, $\mathbb{P}[N = 3] = 0.0730$, $\mathbb{P}[N = 4] = 0.0418$, $\mathbb{P}[N = 5] = 0.0241$, $\mathbb{P}[N = 6] = 0.0132$, $\mathbb{P}[N = 7] = 0.0069$, $\mathbb{P}[N = 8] = 0.0035$, $\mathbb{P}[N = 9] = 0.0015$, $\mathbb{P}[N = 10] = 0.0005$. The exact extinction probability is $\alpha = 0.879755$. The 3- and 4-convex discrete extrema are as follows: $N_{\min}^{(3)}$ and $N_{\max}^{(3)}$ (resp. $N_{\min}^{(4)}$ and $N_{\max}^{(4)}$) have respective supports $\{0, 3, 4\}$ and $\{0, 1, 10\}$ (resp. $\{0, 1, 5, 6\}$ and $\{0, 2, 3, 10\}$) and associated probabilities $\{0.6534, 0.2415, 0.1051\}$ and $\{0.1261, 0.8438, 0.0301\}$ (resp. $\{0.3944, 0.4714, 0.1315, 0.0027\}$ and $\{0.6037, 0.1074, 0.2798, 0.0091\}$). The bounds obtained with these extrema are displayed in Table 1. The bounds obtained with $s = 4$ are remarkably accurate.*

	$\alpha_{\min}^{(s)}$	$\alpha_{\max}^{(s)}$
$s = 3$	0.8414716	0.8868653
$s = 4$	0.8791374	0.8807095

Table 1: Bounds on the probability of ultimate extinction α in Example 3.1 using the discrete s-convex extrema.

3.3 Ruin probability - Binomial risk model

In the classical discrete binomial risk model (see, e.g., [13] and [14]), the discrete claim amounts $X_1, X_2, \ldots$ recorded by an insurance company are assumed to be independent and identically distributed with common distribution function F having finite $s-1$ moments, such that $F(0) = 0$. The number of claims in the time interval $[0, t]$ is assumed to be independent of the individual claim amounts and to form a binomial process $\{N(t), t \in \mathbb{N}\}$ with parameter q, $0 < q < 1$ (i.e. in any time period there occurs 1 or 0 claim with probabilities q and $1 - q$, respectively, and occurrence of claims in different time intervals are independent events). We assume furthermore that the premium received in each period is equal to 1 and is larger than the net premium, which means that $1 > q\mathbb{E}[X_1]$.

Further, let $\psi(\kappa)$ be the ultimate ruin probability with an initial capital κ; that is, the probability that the process $Z(t) = \kappa + t - \sum_{i=1}^{N(t)} X_i$, $t \in \mathbb{N}$, describing the wealth of the insurance company, ever falls below zero. If the moment generating function of X exists, the Lundberg's inequality provides an exponential upper bound on ψ, namely $\psi(\kappa) \leq \mathrm{e}^{-z\kappa}$, where z is the Lundberg's adjustment coefficient satisfying the integral equation $\phi_{S(t)}(z) = \mathbb{E}\left[\mathrm{e}^{zS(t)}\right] = \mathrm{e}^z$ with $S(t)$ denoting the aggregate claim amount in the t-th time interval. As we are dealing with a compound binomial model, it comes easily that z is the solution of the equation $1 - q + q\mathbb{E}\left[\mathrm{e}^{zX}\right] = \mathrm{e}^z$ where $\mathbb{E}\left[\mathrm{e}^{zX}\right]$ is the moment generating funtion of the discrete claim amounts $X_1, X_2, \ldots$.

We recall that the infinite-time ruin probabilities $\psi(\kappa)$ can be computed by a recursive formula (see for example [13] and [14]). Let us also notice that, as proved in [1], $X \preceq_{2-cx}^{\mathcal{N}_n} Y \Rightarrow \psi_X(\kappa) \leq \psi_Y(\kappa)$ for all integer κ. Unfortunately, this relation is no longer true for s larger than two. Consequently, the method introduced in this paper does not allow us directly to bound the ruin probabilities. Thus, in order to make a comparison, we are going to compute the ruin probabilities using the recursive formula and the exponential Lundberg's upper bound using the 2- and 3-convex maxima.

For the application, we assume that the individual claim amount distribution is the same as in Example 3.1 except that the support is $\{1, 2, \ldots, n\}$, i.e. we take $n = 11$ and $\mathbb{P}[X = 1] = 0.4982$, $\mathbb{P}[X = 2] = 0.2103$, $\mathbb{P}[X = 3] = 0.1270$, $\mathbb{P}[X = 4] = 0.0730$, $\mathbb{P}[X = 5] = 0.0418$, $\mathbb{P}[X = 6] = 0.0241$, $\mathbb{P}[X = 7] = 0.0132$, $\mathbb{P}[X = 8] = 0.0069$, $\mathbb{P}[X = 9] = 0.0035$, $\mathbb{P}[X = 10] = 0.0015$, $\mathbb{P}[X = 11] = 0.0005$. Consequently, the first moments of the discrete claim amounts are fixed to $\mu_1 = 2.145$, $\mu_2 = 7.1454$ and $\mu_3 = 33.4896$. In addition, let $q = 0.4$. The Lundberg's adjustment coefficient is equal to $z = 0.1163$ and the ruin probabilities $\psi(\kappa)$ for some initial surplus level κ are depicted in Table 2.

The 3- and 4-convex discrete extrema are given as follows: $X_{\min}^{(3)}$ and $X_{\max}^{(3)}$ (resp. $X_{\min}^{(4)}$ and $X_{\max}^{(4)}$) have respective supports $\{1, 4, 5\}$ and $\{1, 2, 11\}$ (resp. $\{1, 2, 6, 7\}$ and $\{1, 3, 4, 11\}$) and associated probabilities $\{0.6534, 0.2415, 0.1051\}$ and $\{0.1261, 0.8438, 0.0301\}$ (resp. $\{0.3944, 0.4714, 0.1315, 0.0027\}$ and $\{0.6037, 0.1074, 0.2798, 0.0091\}$). The extremal 3- and 4- convex adjustment coefficients are respectively equal to $z_{\min}^{(3)} = 0.1053$, $z_{\max}^{(3)} = 0.1205$, $z_{\min}^{(4)} = 0.1158$ and $z_{\max}^{(4)} = 0.1166$. The exponential upper bounds obtained using these extrema are displayed in Table 1.

Initial surplus level κ	$\psi(\kappa)$	$e^{-\kappa z}$	$e^{-\kappa z_{\min}^{(3)}}$	$e^{-\kappa z_{\min}^{(4)}}$
0	0.7633	1	1	1
1	0.6842	0.8902	0.90003	0.8906
2	0.6117	0.7925	0.8101	0.7933
3	0.5461	0.7054	0.7291	0.7066
4	0.4869	0.6280	0.6562	0.6294
5	0.4338	0.5590	0.5906	0.5606
6	0.3862	0.4977	0.5315	0.4993
7	0.3438	0.4430	0.4784	0.4447
8	0.3060	0.3944	0.4306	0.3961
9	0.2724	0.3511	0.3875	0.3528
10	0.2425	0.3125	0.3488	0.3142
15	0.1355	0.1747	0.2060	0.1761
20	0.0758	0.0977	0.1217	0.0987
30	0.0237	0.0305	0.0425	0.0310
40	0.0074	0.0095	0.0148	0.0097
50	0.0023	0.0030	0.0052	0.0031

Table 2: Ruin probabilities and Lundberg's bounds when $n = 11$, $q = 0.4$.

3.4 Lundberg's coefficient - Poisson risk model

In this section, we consider the classical discrete poisson risk model. This model is the same as the one introduced in Section 3.3 except that here the number of claims is governed by a Poisson process $\{N(t), t \geq 0\}$ with constant rate λ. Let also the premium rate $c > 0$ be such that the inequality $c > \lambda\mathbb{E}[X_1]$ holds. Here, $Z(t) = \kappa + ct - \sum_{i=1}^{N(t)} X_i$ $(t \geq 0)$ and if the moment generating function of X exists, Lundberg's inequality provides again an exponential upper bound on ψ, namely $\psi(\kappa) \leq e^{-z\kappa}$, where z is the Lundberg's adjustment coefficient satisfying the integral equation $\phi_X(z) = 1 + \frac{cz}{\lambda}$.

As an illustration, let $n = 5$, $c = 12$, $\lambda = 10$ and $\mu_1 = 1$. First, consider $z_{\text{min}}^{(s)}$ and $z_{\text{max}}^{(s)}$ as functions of μ_2. Whe then get the numerical values depicted in Table 3. Second, let us fix $\mu_2 = 3$ and consider $z_{\text{min}}^{(s)}$ and $z_{\text{max}}^{(s)}$ as functions of μ_3 (see Table 4). It is seen that the bounds are quite accurate, and are particularly so when μ_3 is large.

μ_2	1.5	2	2.5	3
$z_{\text{min}}^{(3)}$, discrete	0.2144848	0.1624468	0.1324108	0.1123238
$z_{\text{max}}^{(3)}$, discrete	0.2330329	0.1771006	0.1409982	0.1180644
μ_2	3.5	4	4.5	
$z_{\text{min}}^{(3)}$, discrete	0.09778207	0.08670383	0.07794723	
$z_{\text{max}}^{(3)}$, discrete	0.1009502	0.08855031	0.07859318	

Table 3: Bounds on the Lundberg's coefficient z when $\mu_1 = 1$, $n = 5$, $c = 12$ and $\lambda = 10$.

μ_3	9.5	10	10.5	11
$z_{\text{min}}^{(4)}$, discrete	0.1172558	0.1164697	0.1157054	0.1149623
$z_{\text{max}}^{(4)}$, discrete	0.117302	0.1165591	0.1158351	0.1151295
μ_3	11.5	12	12.5	
$z_{\text{min}}^{(4)}$, discrete	0.1142785	0.1136114	0.11296	
$z_{\text{max}}^{(4)}$, discrete	0.1144	0.1136898	0.1129981	

Table 4: Bounds on the Lundberg's coefficient z when $\mu_1 = 1$, $\mu_2 = 3$, $n = 5$, $c = 12$ and $\lambda = 10$.

4 Concluding remarks and extension to $s \geq 4$

Quite surprisingly, the discrete s-convex extrema cannot be obtained by discretizing the continuous ones (contrarily to the cases treated in [1]-[2]). Using the Majorant/Minorant Method, we proved that the support of the discrete 4-convex minimum has to be of the form $\{\xi, \xi+1, \eta, \eta+1\}$ ($0 \leq \xi < \xi+1 < \eta < \eta+1 \leq n$), when ξ and η are the solutions of (9).

It is also interesting to note that the method proposed in this paper can be extended to any $s \geq 4$. It is done in the following way. Using the cut-criterion and Property 2.1, it can be seen that the most general form for the supports of the s-convex extrema, denoted by $\text{Supp}_{X_{\text{min}}^{(s)}}$ and $\text{Supp}_{X_{\text{max}}^{(s)}}$, are given as follows: for $s = 2m$, we have $\text{Supp}_{X_{\text{min}}^{(s)}} = \{\xi_1, \xi_1+1, \ldots, \xi_m, \xi_m+1\}$ ($0 \leq \xi_1 < \xi_1+1 < \ldots < \xi_m < \xi_m+1 \leq n$) and $\text{Supp}_{X_{\text{max}}^{(s)}} = \{0, \zeta_1, \zeta_1+1, \ldots, \zeta_{m-1}, \zeta_{m-1}+1, n\}$ ($0 < \zeta_1 < \zeta_1+1 < \ldots < \zeta_{m-1} < \zeta_{m-1}+1 < n$) while for $s = 2m+1$, we have $\text{Supp}_{X_{\text{min}}^{(s)}} = \{0, \xi_1, \xi_1+1, \ldots, \xi_m, \xi_m+1\}$ ($0 < \xi_1 < \xi_1+1 < \ldots < \xi_m < \xi_m+1 \leq n$) and $\text{Supp}_{X_{\text{max}}^{(s)}} = \{\zeta_1, \zeta_1+1, \ldots, \zeta_m, \zeta_m+1, n\}$ ($0 \leq \zeta_1 < \zeta_1+1 < \ldots < \zeta_{m-1} < \zeta_{m-1}+1 < n$).

Then, to express the conditions on the support points so that $X_{\text{min}}^{(s)}$ and $X_{\text{max}}^{(s)}$ have the required moments $\mu_1, \mu_2, \ldots, \mu_{s-1}$, we just have to compute the probabilities associated to the support points and to check that they are positive. We get the resulting probabilities using that

$$X \in \mathcal{D}_s(\mathcal{N}_n; \mu_1, \mu_2, \ldots, \mu_{s-1}) \text{ with } \text{Supp}_X = \{a_0, a_1, \ldots, a_k\}$$

$$\Rightarrow \mathbb{P}[X = a_i] = \frac{\mathbb{E}\left[\prod_{j \neq i}(X - a_j)\right]}{\prod_{j \neq i}(a_i - a_j)} \; (i = 0, 1, \ldots, k).$$

The solution $(\xi_1, \ldots, \xi_{s/2}, \zeta_1, \ldots, \zeta_{(s/2)-1})$ (s even) (*resp.* $(\xi_1, \ldots, \xi_{(s-1)/2}, \zeta_1, \ldots, \zeta_{(s-1)/2})$ (s odd)) cannot be obtained explicitly. Nevertheless, it is easily obtained just by testing each admissible sequence $(\xi_1, \ldots, \xi_{s/2}, \zeta_1, \ldots, \zeta_{(s/2)-1})$ (*resp.* $(\xi_1, \ldots, \xi_{(s-1)/2}, \zeta_1, \ldots, \zeta_{(s-1)/2})$) of $\mathcal{N}_n$.

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