

KLR ALGEBRAS AND THE BRANCHING RULE I: THE CATEGORICAL GELFAND-TSETLIN BASIS IN TYPE A_n

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ABSTRACT. We define a quotient of the category of finitely generated modules over the cyclotomic Khovanov-Lauda-Rouquier algebra of type A_n and show it has a module category structure over a direct sum of certain cyclotomic Khovanov-Lauda-Rouquier algebras of type A_{n-1} . This way we categorify the branching rules for $\mathfrak{sl}_{n+1} \supset \mathfrak{sl}_n$. Using this we provide an elementary proof of Khovanov-Lauda's cyclotomic conjecture. We show that continuing recursively gives the Gelfand-Tsetlin basis in type A_n . As an application we prove a conjecture of Mackaay, Stošić and Vaz concerning categorical Weyl modules.

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1. INTRODUCTION

Let A and B be associative algebras, M a (left) B -module and $f: A \rightarrow B$ a map of algebras. Then A acts on M through f by the formula $a.m = f(a)m$ for a in A and m in M . This procedure turns each (left) module over B into a (left) module over A . It is well-known that this operation defines a functor between the categories of modules over the respective algebras. Each homomorphism of algebras $f: A \rightarrow B$ gives rise to a restriction functor between their categories of representations

$$\text{res}_A^B: B\text{-mod} \rightarrow A\text{-mod}$$

defined by $M \mapsto fM = fB \otimes_B M$ for left A -modules N and B -modules M . Here fM means the structure of A -module on the M as defined above: $a.fm := f(a)m$.

In general an irreducible object M in $B\text{-mod}$ is not sent to an irreducible over A . However, if we are in categories of modules which are semisimple, then $\text{res}_A^B(M)$ decomposes as a direct sum of irreducibles over A . This procedure of writing irreducibles over B as direct sums of irreducibles over A is called the *branching rule of B with respect to A* .

The study of the branching rules has its roots in group theory. They were first obtained in a systematic way in the study of the representations of the classical groups. They were subsequently extended to categories of representations of other types of algebras like for example associative algebras, Lie algebras, Hopf algebras and quantum groups. Besides being a useful tool in the study of the representations of the objects under consideration, the branching rules have been extensively studied in theoretical physics where they have found important applications in the study of systems through reduction of its group of symmetry to one of its subgroups (see [12, 22], the review [21] and the references therein).

Let us consider the case of Kac-Moody algebras associated to finite quivers. For each embedding $\Gamma_2 \hookrightarrow \Gamma_1$ of quivers there is an embedding of the Kac-Moody algebras $\mathfrak{g}_{\Gamma_2} \hookrightarrow \mathfrak{g}_{\Gamma_1}$ associated to Γ_1 and Γ_2 . If we restrict to the categories of integrable representations, then every irreducible integral representation V_{Γ_1} of $\mathfrak{g}_{\Gamma_1}$ is isomorphic as a representation of $\mathfrak{g}_{\Gamma_2}$ to a direct sum of irreducibles [9]. In some cases a general procedure exists to obtain the branching rules for this embedding, but in the general case one has to work out the result case by case (see [6, 12, 22] for a general treatment of the branching rule for classical Lie algebras). These results extend to the quantum version of Kac-Moody algebras, which is the case we are interested in. Suppose now that there is a chain of inclusions of quivers $\Gamma_n \subset \Gamma_{n-1} \subset \cdots \Gamma_2 \subset \Gamma_1$, where Γ_i is obtained from Γ_{i+1} by adding a vertex and the corresponding edge and Γ_n is the empty quiver. We have a chain of embeddings of the corresponding Kac-Moody algebras and one may apply the branching rule recursively starting with $\mathfrak{g}_{\Gamma_1}$ until we get a direct sum of irreducibles over the one-dimensional Kac-Moody algebra. Including this collection of spaces back into V_{Γ_1} defines a distinguished basis which is an example of a canonical basis and is called the Gelfand-Tsetlin basis after [8] (see also [7]).

In a remarkable series of papers [15, 16, 24] Khovanov, Lauda and independently Rouquier introduced a family of Hecke algebras associated to a quiver (see also [25]). These *quiver Hecke* algebras, which became known as *KLR algebras*, have shown to have a rich representation theory (see for example [3, 10, 19, 20]). But more immediate to us in this paper is the fact that the KLR algebra associated to a quiver Γ categorifies the lower half of the quantum version of the Kac-Moody algebra associated to Γ , which means that the latter is isomorphic to the Grothendieck ring of the former. For each dominant integral weight λ the KLR algebra R_Γ admits a quotient, denoted by R_Γ^λ , which is called a cyclotomic quotient after [3], and whose Grothendieck group is isomorphic to the integral representation V_λ of $\mathfrak{g}_\Gamma$, *i.e.* the category of graded modules over R_Γ^λ , finite in each degree, admits a categorical action of $\mathfrak{g}_\Gamma$ which descends to the Grothendieck group yielding a representation which is isomorphic to V_λ (see [14, 29]).

In this paper we concentrate on the case where Γ_1 is the Dynkin diagram of type A_n and Γ_2 is the Dynkin diagram obtained from Γ_1 by removing the vertex labeled n (and the corresponding edge), and investigate the consequences for the corresponding KLR algebras and its cyclotomic quotients. The inclusion of quivers $A_{n-1} \hookrightarrow A_n$ determines an inclusion $R_{n-1} \hookrightarrow R_n$ between the corresponding KLR algebras. This gives rise to restriction and induction functors between their categories of representations which turn out to descend to the usual inclusion and projection maps between the corresponding (one-half) quantum Kac-Moody algebras.

This approach needs to be modified to work with cyclotomic quotients. In this case there is a projection of the cyclotomic KLR algebra $R_{A_n}^\lambda$ to a direct sum of cyclotomic KLR algebras $\bigoplus_{\mu \in \tau(\lambda)} R_{A_{n-1}}^\mu$ with the set $\tau(\lambda)$ being determined combinatorially from λ . We obtain a functor

$$\Pi: R_{A_n}^\lambda\text{-fmod} \rightarrow \bigoplus_{\mu \in \tau(\lambda)} R_{A_{n-1}}^\mu\text{-fmod}$$

between their categories of graded, finite dimensional modules which is full, essentially bijective, and commutes with the categorical action of the Kac-Moody algebra given by A_n . Continuing recursively we end up in the category of one-dimensional modules over a collection of one-dimensional algebras $R_{A_0}^{\mu'}$ which are labeled by certain sequences of partitions $(\lambda, \lambda^{(n-1)}, \dots, \lambda^{(1)})$, each $\lambda^{(i)}$ being a partition with exactly i parts.

There is a categorical action of the Kac-Moody algebra of A_n on the functors

$$R_{A_{n+1}}^\lambda\text{-pmod} \rightarrow R_{A_0}^\mu\text{-pmod} \cong \mathbb{k}\text{-pmod}$$

which descends to an action on the Grothendieck group, which means that these functors categorify the elements of the Gelfand-Tsetlin basis. These functors can be interpreted as the preimages under Π of the one-dimensional modules over $R_{A_0}^{\mu'}$, giving a realization of the Gelfand-Tsetlin basis in terms of some special objects in the category of modules over $R_{A_n}^\lambda$.

One consequence of the categorical branching rules is that we can use it to provide an easy proof of Khovanov and Lauda's cyclotomic conjecture from [15], namely that the grothendieck group $K_0(R_{A_n}^\lambda)$ is isomorphic to the irreducible representation V_λ . As another application of the categorical branching rules we prove a conjecture in [23] about categorical Weyl modules for the q -Schur algebra. Namely we prove that the cyclotomic KLR algebra is isomorphic to a certain endomorphism algebra constructed in [23] as part of the q -Schur categorification to give a conjectural categorification of the Weyl module W_λ . As a consequence we obtain that the aforementioned endomorphism algebra indeed categorifies W_λ . This way we prove a second conjecture in [23].

This paper was motivated by an attempt to lift the recursive formulas for link polynomials in [13] and [27] to statements between the corresponding link homology theories (see [27, 28] for further explanations and [26] for developments). This is the first output of the program outlined in [26]. We remark that the results in this paper are different from the branching rule obtained by Brundan, Kleshchev and Wang in [5] for graded Specht modules using the natural embedding of cyclotomic Hecke algebras together with the identification with cyclotomic KLR algebras of type A .

We have tried to make this paper reasonably self-contained with the exception of Section 6 where we assume familiarity with [23].

2. QUANTUM $\mathfrak{sl}_{n+1}$, THE BRANCHING RULE AND THE GELFAND-TSETLIN BASIS

In this section we review the basics about quantum $\mathfrak{sl}_{n+1}$, its irreducible representations, the branching rule for $\mathfrak{sl}_{n+1} \supset \mathfrak{sl}_n$, and the Gelfand-Tsetlin basis. We also fix notation and recollect some results that will be used in this paper.

2.1. Quantum $\mathfrak{sl}_{n+1}$ and its irreducible representations. We denote by Λ^{n+1} the $\mathfrak{sl}_{n+1}$ -weight lattice and by X^{n+1} the $\mathfrak{sl}_{n+1}$ -root lattice. Let $\alpha_1, \dots, \alpha_n$ be the simple roots and $\alpha_1^\vee, \dots, \alpha_n^\vee$ the simple coroots. Any weight $\bar{\lambda}$ can be written as $\bar{\lambda} = (\bar{\lambda}_1, \dots, \bar{\lambda}_n)$ where $\bar{\lambda}_i = \alpha_i^\vee(\bar{\lambda})$. Denote the set of dominant integral weights by

$$\Lambda_+^{n+1} = \{\bar{\lambda} \in \Lambda^{n+1} \mid \alpha_i^\vee(\bar{\lambda}) \in \mathbb{Z}_{\geq 0} \text{ for all } i = 1, \dots, n\}.$$

Let also

$$a_{ij} = \alpha_i^\vee(\alpha_j) = \begin{cases} 2 & \text{if } j = i, \\ -1 & \text{if } j = i \pm 1, \\ 0 & \text{else,} \end{cases}$$

be the entries of the Cartan matrix of $\mathfrak{sl}_{n+1}$.

The *quantum special linear algebra* $U_q(\mathfrak{sl}_{n+1})$ is the associative, unital $\mathbb{Q}(q)$ -algebra generated by the Chevalley generators F_i , E_i and $K_i^{\pm 1}$ for $1, \dots, n$, subject to the relations

$$\begin{aligned} K_i K_j &= K_j K_i, & K_i K_i^{-1} &= K_i^{-1} K_i = 1, \\ K_i F_j K_i^{-1} &= q^{-a_{ij}} F_j, & K_i E_j K_i^{-1} &= q^{a_{ij}} E_j, \\ E_i F_j - F_j E_i &= \delta_{i,j} \frac{K_i - K_i^{-1}}{q - q^{-1}}, \\ F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0 & \text{if } |i - j| = 1, \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 & \text{if } |i - j| = 1, \\ F_i F_j &= F_j F_i, & E_i E_j &= E_j E_i & \text{if } |i - j| > 1. \end{aligned}$$

For $\underline{i} = (i_1, \dots, i_k)$ we define $F_{\underline{i}} = F_{i_k} \cdots F_{i_1}$ and $E_{\underline{i}} = E_{i_k} \cdots E_{i_1}$. The reason for this convention will be clear later when we introduce the diagrammatics.

The *lower half* $U_q^-(\mathfrak{sl}_{n+1}) \subset U_q(\mathfrak{sl}_{n+1})$ quantum algebra is the subalgebra generated by the F 's (analogously for the *upper half* $U_q^+(\mathfrak{sl}_{n+1})$).

Recall that a subspace $V_{\bar{\mu}}$ of a finite dimensional $U_q(\mathfrak{sl}_{n+1})$ -module V is called a *weight space* if

$$K_i^{\pm 1} v = q^{\pm \bar{\mu}_i} v$$

for all $v \in V_{\bar{\mu}}$ and that V is called a *weight module* if

$$V = \bigoplus_{\bar{\mu} \in \Lambda^{n+1}} V_{\bar{\mu}}.$$

A weight module V is called a *highest weight module with highest weight $\bar{\lambda}$* if there exists a non-zero weight vector $v_{\bar{\lambda}} \in V_{\bar{\lambda}}$ such that $E_i v_{\bar{\lambda}} = 0$ for $i = 1, \dots, n$. For each $\bar{\lambda} \in \Lambda_+^{n+1}$ there exists a unique irreducible highest weight module with highest weight $\bar{\lambda}$. In the sequel we will drop the U_q and write $\mathfrak{sl}_{n+1}$ -module instead of $U_q(\mathfrak{sl}_{n+1})$ -module.

Let ϕ be the anti-involution on $U_q(\mathfrak{sl}_{n+1})$ defined by

$$\phi(K_i^{\pm 1}) = K_i^{\mp 1}, \quad \phi(F_i) = q^{-1} K_i E_i, \quad \phi(E_i) = q^{-1} K_i^{-1} F_i.$$

The *q-Shapovalov form* $\langle \cdot, \cdot \rangle$ is the unique nondegenerate symmetric bilinear form on the highest weight module $V(\bar{\lambda})$ satisfying

$$\begin{aligned} \langle v_{\bar{\lambda}}, v_{\bar{\lambda}} \rangle &= 1, \\ \langle uv, v' \rangle &= \langle v, \phi(u)v' \rangle \quad \text{for all } u \in U_q(\mathfrak{sl}_{n+1}) \text{ and } v, v' \in V(\bar{\lambda}), \\ f \langle v, v' \rangle &= \langle \bar{f}v, v' \rangle = \langle v, f v' \rangle \quad \text{for any } f \in \mathbb{Q}(q) \text{ and } v, v' \in V(\bar{\lambda}). \end{aligned}$$

2.2. The q -Schur algebra. In this subsection we give a brief review the q -Schur algebra $S_q(n, d)$ following the exposition in [23] (see [23] and the references therein for more details). The Schur algebra appears naturally in the context of (polynomial) representations of $U_q(\mathfrak{gl}_n)$, which is the starting point of this subsection. The root and weight lattices are very easy to describe for quantum $\mathfrak{gl}_n$. Let $\epsilon_i = (0, \dots, 1, \dots, 0) \in \mathbb{Z}^n$, with the 1 being on the i -th coordinate for $i = 1, \dots, n$. Let also $\tilde{\alpha}_i = \epsilon_i - \epsilon_{i+1} \in \mathbb{Z}^n$ and $(\epsilon_i, \epsilon_j) = \delta_{i,j}$ be the Euclidean inner product on $\mathbb{Z}^n$ (in this basis the $\mathfrak{sl}_n$ roots can be expressed by $\alpha_i = \tilde{\alpha}_i - \tilde{\alpha}_{i+1}$).

The quantum general linear algebra $U_q(\mathfrak{gl}_n)$ is the associative, unital $\mathbb{Q}(q)$ -algebra generated by K_i, K_i^{-1} , for $i = 1, \dots, n$, and F_i, E_i for $i = 1, \dots, n-1$, subject to the relations

$$K_i K_j = K_j K_i, \quad K_i K_i^{-1} = K_i^{-1} K_i = 1,$$

$$E_i F_j - F_j E_i = \delta_{i,j} \frac{K_i K_{i+1}^{-1} - K_i^{-1} K_{i+1}}{q - q^{-1}},$$

$$K_i F_j = q^{-(\epsilon_i, \alpha_j)} F_j K_i, \quad K_i E_j = q^{(\epsilon_i, \alpha_j)} E_j K_i,$$

and the so-called *Serre relations*

$$\left. \begin{aligned} F_i^2 F_j - (q + q^{-1}) F_i F_j F_i + F_j F_i^2 &= 0 \\ E_i^2 E_j - (q + q^{-1}) E_i E_j E_i + E_j E_i^2 &= 0 \end{aligned} \right\} \quad \text{if } |i - j| = 1,$$

$$\left. \begin{aligned} F_i F_j - F_j F_i &= 0 \\ E_i E_j - E_j E_i &= 0 \end{aligned} \right\} \quad \text{if } |i - j| > 1.$$

In the Beilinson-Lusztig-MacPherson [2] idempotent version of quantum groups the Cartan subalgebras are “replaced” by algebras generated by orthogonal idempotents corresponding to the weights. To understand their definition, recall that K_i acts as q^{λ_i} on the λ -weight space of any weight representation. The idempotent version of $U_q(\mathfrak{gl}_n)$ can be obtained from $U_q(\mathfrak{gl}_n)$ by adjoining orthogonal idempotents 1_λ , for $\lambda \in \mathbb{Z}^n$ and adding the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda, \mu} 1_\mu, \\ F_i 1_\lambda &= 1_{\lambda - \tilde{\alpha}_i} F_i, \\ E_i 1_\lambda &= 1_{\lambda + \tilde{\alpha}_i} E_i, \\ K_i 1_\lambda &= q^{\lambda_i} 1_\lambda. \end{aligned}$$

The *idempotent quantum $\mathfrak{gl}_n$* is then defined by

$$\dot{U}(\mathfrak{gl}_n) \cong \bigoplus_{\lambda, \mu \in \mathbb{Z}^n} 1_\lambda U_q(\mathfrak{gl}_n) 1_\mu.$$

Note that $\dot{U}(\mathfrak{gl}_n)$ is not unital anymore because $1 = \sum_{\lambda \in \mathbb{Z}^n} 1_\lambda$ would be an infinite sum. In this setting the q -Schur algebra occurs naturally as a quotient of idempotent $U_q(\mathfrak{gl}_n)$, which happens

to be very easy to describe. Let

$$\Lambda(n, d) = \left\{ \lambda \in \mathbb{N}^n : \sum_i \lambda_i = d \right\}$$

be a weight (sub)lattice and the highest weights be elements in

$$\Lambda^+(n, d) = \{ \lambda \in \Lambda(n, d) : \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \}.$$

The q -Schur algebra $S_q(n, d)$ can be defined as the quotient of idempotent quantum $\mathfrak{gl}_n$ by the ideal generated by all idempotents 1_λ such that $\lambda \notin \Lambda(n, d)$. Thus, we have a finite presentation of $S_q(n, d)$ as the associative, unital $\mathbb{Q}(q)$ -algebra generated by 1_λ for $\lambda \in \Lambda(n, d)$, and F_i, E_i for $i = 1, \dots, n-1$, subject to the relations

$$\begin{aligned} 1_\lambda 1_\mu &= \delta_{\lambda, \mu} 1_\lambda, \\ \sum_{\lambda \in \Lambda(n, d)} 1_\lambda &= 1, \\ F_i 1_\lambda &= 1_{\lambda - \tilde{\alpha}_i} F_i, \\ E_i 1_\lambda &= 1_{\lambda + \tilde{\alpha}_i} E_i, \\ E_i E_{-j} - E_{-j} E_i &= \delta_{ij} \sum_{\lambda \in \Lambda(n, d)} [\lambda_i - \lambda_{i+1}] 1_\lambda. \end{aligned}$$

We use the convention that $1_\mu X 1_\nu = 0$ if μ or ν is not contained in $\Lambda(n, d)$.

The irreducibles W_λ , for $\lambda \in \Lambda^+(n, d)$, can be constructed as subquotients of $S_q(n, d)$, called *Weyl modules*. Let $<$ denote the lexicographic order on $\Lambda(n, d)$. For any $\lambda \in \Lambda^+(n, d)$, we have

$$W_\lambda \cong 1_\lambda S_q(n, d) / [\mu > \lambda].$$

Here $[\mu > \lambda]$ is the ideal generated by all elements of the form $1_\mu x 1_\lambda$, for some $x \in S_q(n, d)$ and $\mu > \lambda$.

2.3. Branching rules. Recall that a partition with m parts is a sequence of nonnegative integers $(\lambda_1, \dots, \lambda_m)$ with $\lambda_1 \geq \dots \geq \lambda_m$. Partitions are in bijection with Young diagrams. We follow English notation, where Young diagrams are left justified and lines are enumerated from top to bottom. The bijection sends λ to the Young diagram with λ_i boxes in the i -th line. From now on we denote them by the same symbol.

There is a well-known relation between integral dominant weights of $\mathfrak{sl}_{n+1}$ and partitions with $n+1$ parts. For each such partition λ there is an integral dominant weight $\bar{\lambda} \in \Lambda_+^{n+1}$ defined by

$$\bar{\lambda}_i := \lambda_i - \lambda_{i+1}.$$

If we want to use partitions to describe the finite dimensional irreducibles of $\mathfrak{sl}_{n+1}$ we can write $V_\lambda^{\mathfrak{sl}_{n+1}}$ to denote the irreducible $\mathfrak{sl}_{n+1}$ -module $V_{\bar{\lambda}}$ without any ambiguity. Of course there are several partitions giving the same element of Λ_+^{n+1} , but there is only one if we fix the value of $\lambda_{n+1} = 0$.

For a partition λ with $n+1$ parts denote by $\tau(\lambda)$ the set of all partitions μ with n parts satisfying

$$(1) \quad \lambda_i \leq \mu_i \leq \lambda_{i+1}.$$

We denote by $V_\lambda^{\mathfrak{sl}_m}$ the irreducible finite dimensional representation of $\mathfrak{sl}_m$ of highest weight λ . For the embedding $\mathfrak{sl}_n \hookrightarrow \mathfrak{sl}_{n+1}$ corresponding to adding one vertex at the end of the Dynkin diagram of $\mathfrak{sl}_n$ the branching rule [12] says that

$$(2) \quad (V_\lambda^{\mathfrak{sl}_{n+1}})^{\mathfrak{sl}_n} \cong \bigoplus_{\mu \in \tau(\lambda)} V_\mu^{\mathfrak{sl}_n}$$

is an isomorphism of $\mathfrak{sl}_n$ -modules (adding the vertex in the beginning of the Dynkin diagram of $\mathfrak{sl}_n$ yields the same decomposition). This decomposition is multiplicity free that is, each of the $V_\mu^{\mathfrak{sl}_n} = V_{\bar{\mu}}^{\mathfrak{sl}_n}$ occurs at most once in the sum (2). This means that given an element on the right hand side there is a canonical way of associating it to one element in $(V_\lambda^{\mathfrak{sl}_{n+1}})^{\mathfrak{sl}_n}$.

Define $\tau_k(\lambda)$ as the set of all the Young diagrams obtained from λ by the removal of k boxes, no more than one from each column. The condition in (1) is the same as requiring that μ is in exactly one of the $\tau_k(\lambda)$ for some k . In other words,

$$\tau(\lambda) = \bigoplus_{k \geq 0} \tau_k(\lambda).$$

Let $\bar{\tau}_k(\lambda)$ be the set of all the $\bar{\mu} \in \Lambda_+^n$ for $\mu \in \tau_k(\lambda)$ and for $1 \leq i_1 \leq \dots \leq i_k \leq n+1$. Denote by $\bar{\mu}(i_1 \dots i_k) \in \bar{\tau}_k(\lambda)$ the weight obtained by removal of exactly one box from each of the $i_1, \dots, i_k$ -th lines of λ in the order given.

For practical purposes the set $\bar{\tau}_k(\lambda)$ is best described using maps from Λ^{n+1} to Λ^{n+1} and from Λ^{n+1} to Λ^n . For $p_\Lambda: \Lambda^{n+1} \rightarrow \Lambda^n$ the projection

$$(3) \quad p_\Lambda(\bar{\mu}_1, \dots, \bar{\mu}_n) = (\bar{\mu}_1, \dots, \bar{\mu}_{n-1})$$

and $(i_1, \dots, i_k)$ as above we define $\xi'_{i_1 \dots i_k}: \Lambda^{n+1} \rightarrow \Lambda^{n+1}$ and $\xi_{i_1 \dots i_k}: \Lambda^{n+1} \rightarrow \Lambda^n$ by

$$\begin{aligned} \xi'_i(\bar{\lambda}) &= (\bar{\lambda}_1, \dots, \bar{\lambda}_{i-1} + 1, \bar{\lambda}_i - 1, \dots, \bar{\lambda}_{n-1}, \bar{\lambda}_n) \\ \xi'_{i_1 \dots i_k}(\bar{\lambda}) &= \xi'_{i_k} \cdots \xi'_{i_1}(\bar{\lambda}) \end{aligned}$$

and $\xi_i(\bar{\lambda}) = p_\Lambda \xi'_i$. We say that ξ_i is λ -dominant if $\xi_i(\bar{\lambda})$ is in Λ_+^n whenever $\bar{\lambda}$ is in Λ_+^{n+1} and that $\xi_{i_1 \dots i_k}$ is λ -dominant if for each $j \leq k$ the map $\xi_{i_1 \dots i_j}$ is λ -dominant.

We have

$$\bar{\mu}(i_1 \dots i_k) = \xi_{i_k} \cdots \xi_{i_1}(\bar{\lambda}).$$

Keeping this notation in mind we denote by $\xi_i(\lambda)$ the partition obtained from λ by the removal of one box from its i -th line with $\xi_{i_1 \dots i_k}(\lambda)$ meaning the one obtained by the removal of k boxes, one for each i_r -th line. We see that λ -positivity of $\xi_{i_1 \dots i_k}$ is equivalent of the requirement that no two boxes are removed from the same column of λ . For later use we denote by $\mathcal{D}_\lambda^k$ the set of all λ -dominant $\xi_{i_1 \dots i_k}$ s and define $\mathcal{D}_\lambda = \bigcup_{k \geq 0} \mathcal{D}_\lambda^k$. This way $\tau(\lambda)$ can be also seen as the set of all the $\xi_{i_1 \dots i_j}(\lambda)$ s with $\xi_{i_1 \dots i_j}$ in $\mathcal{D}_\lambda$.

2.4. The Gelfand-Tsetlin basis. We can reapply the branching rule (2) recursively until we end up with a direct sum of one-dimensional spaces corresponding to a final decomposition of each irreducible of $\mathfrak{sl}_2$ into one-dimensional $\mathbb{Q}(q)$ -vector spaces.

We say a sequence $(\mu^{(n+1)}, \dots, \mu^{(1)})$ of partitions, where $\mu^{(j)}$ has j parts, is a *Gelfand-Tsetlin pattern* for $\mu^{(n+1)}$ if each consecutive pair $(\mu^{(j)}, \mu^{(j-1)})$ satisfy the condition in (1). Denote by $\mathcal{S}(\lambda)$ the set of all the Gelfand-Tsetlin patterns for λ .

The Gelfand-Tsetlin patterns for λ are the paths followed in the sequence of weight lattices

$$\Lambda_+^{n+1} \rightarrow \Lambda_+^n \rightarrow \dots \rightarrow \Lambda_+^1 = \mathbb{N}_0$$

by going from $V_\lambda^{\mathfrak{sl}_{n+1}}$ to each of the one-dimensional spaces occurring at the end. Since the decomposition (2) is multiplicity free there is a 1:1 correspondence between $\mathcal{S}(\lambda)$ and the set of all these one-dimensional spaces.

Let $V_{\mathcal{S}(\lambda)}$ be the $\mathbb{Q}(q)$ -linear space spanned by $\mathcal{S}(\lambda)$. We write $|s\rangle$ for a Gelfand-Tsetlin pattern s seen as an element of $V_{\mathcal{S}(\lambda)}$. It turns out that it is isomorphic to $V_\lambda^{\mathfrak{sl}_{n+1}}$ not only as a vector space but as $\mathfrak{sl}_{n+1}$ -modules.

An important fact is that the $\mathfrak{sl}_{n+1}$ -action on $V_{\mathcal{S}(\lambda)}$ can be obtained through a procedure which, in some sense, is the reverse of the direct sum decomposition (2) using the branching rule. While the generators $\{E_i, F_i\}_{i \in \{1, \dots, n-1\}}$ preserve the weight spaces $V_\mu^{\mathfrak{sl}_n}$ on the right-hand side of (2), the generators E_n and F_n move between the different $V_\mu^{\mathfrak{sl}_n}$. Let ϕ be the isomorphism between $V_\lambda^{\mathfrak{sl}_{n+1}}$ and $\bigoplus_{\mu \in \tau(\lambda)} V_\mu^{\mathfrak{sl}_n}$ in (2). Then the $\mathfrak{sl}_n$ -action on $\bigoplus_{\mu \in \tau(\lambda)} V_\mu^{\mathfrak{sl}_n}$ extends to an $\mathfrak{sl}_{n+1}$ -action if we define

$$E_n v := \phi E_n \phi^{-1} v \quad \text{and} \quad F_n v := \phi F_n \phi^{-1} v$$

for $v \in \bigoplus_{\mu \in \tau(\lambda)} V_\mu^{\mathfrak{sl}_n}$. Clearly $\phi E_i \phi^{-1} v = E_i v$ and $\phi F_i \phi^{-1} v = F_i v$ for all $i = 1, \dots, n-1$ and all $v \in \bigoplus_{\mu \in \tau(\lambda)} V_\mu^{\mathfrak{sl}_n}$. We can continue applying this procedure recursively until we get the desired one-dimensional spaces and regard the $\mathfrak{sl}_{n+1}$ -action on them as an action on $V_{\mathcal{S}(\lambda)}$. The basis of $V_{\mathcal{S}(\lambda)}$ given by the Gelfand-Tsetlin patterns is called the Gelfand-Tsetlin basis for $V_\lambda^{\mathfrak{sl}_{n+1}}$. This basis was first defined by Gelfand and Tsetlin in [8] for the Lie algebra $gl(n)$. The explicit form of action of the generators of the Lie algebra $gl(n)$ on the Gelfand-Tsetlin basis can be found for example in [22, 30].

3. KLR ALGEBRAS AND THEIR CYCLOTOMIC QUOTIENTS

In this section we describe the quiver Hecke algebras which were introduced by Khovanov and Lauda in [15] and independently by Rouquier in [24]. We concentrate on the particular case of type A_n . The KLR algebra R_{n+1} associated to the quiver A_n is the algebra generated by $\mathbb{k}$ -linear combinations of isotopy classes of braid-like planar diagrams where each strand is labeled by a simple root of $\mathfrak{sl}_{n+1}$. Strands can intersect transversely to form crossings and they can also carry dots. Multiplication is given by concatenation of diagrams and the collection of such diagrams is subject to relations (4)-(6) below (for the sake of simplicity we write i instead α_i when labeling a strand). We read diagrams left to right and from bottom to top by convention. Therefore the

diagram for the product $a.b$ is the diagram obtained by stacking the diagram for a on the top of the one for b .

$$(4) \quad \begin{array}{c} \text{diagram of two strands } i \text{ and } j \text{ crossing} \end{array} = \begin{cases} 0 & \text{if } i = j, \\ \begin{array}{c} \text{diagram of two strands } i \text{ and } j \text{ with a dot on strand } i \\ + \\ \text{diagram of two strands } i \text{ and } j \text{ with a dot on strand } j \end{array} & \text{if } j = i \pm 1, \\ \begin{array}{c} \text{diagram of two parallel strands } i \text{ and } j \end{array} & \text{else.} \end{cases}$$

$$(5) \quad \begin{array}{c} \text{diagram of three strands } i, j, k \text{ with } i \text{ and } j \text{ crossing} \\ - \\ \text{diagram of three strands } i, j, k \text{ with } j \text{ and } k \text{ crossing} \end{array} = \begin{cases} \begin{array}{c} \text{diagram of three parallel strands } i, j, k \end{array} & \text{if } i = k = j \pm 1, \\ 0 & \text{else.} \end{cases}$$

$$(6) \quad \begin{array}{c} \text{diagram of two strands } i \text{ and } j \text{ crossing with a dot on strand } i \\ - \\ \text{diagram of two strands } i \text{ and } j \text{ crossing with a dot on strand } j \end{array} = \delta_{ij} \begin{array}{c} \text{diagram of two parallel strands } i \text{ and } j \end{array} = \begin{array}{c} \text{diagram of two strands } i \text{ and } j \text{ crossing with a dot on strand } i \\ - \\ \text{diagram of two strands } i \text{ and } j \text{ crossing with a dot on strand } j \end{array}$$

The algebra R_{n+1} is graded with the degrees given by

$$(7) \quad \deg \left(\begin{array}{c} \text{diagram of two strands } i \text{ and } j \text{ crossing} \end{array} \right) = -a_{ij} \quad \deg \left(\begin{array}{c} \text{diagram of two parallel strands } i \text{ and } j \end{array} \right) = a_{ii}.$$

The following useful relation follows from (6) and will be used in the sequel.

$$(8) \quad \begin{array}{c} \text{diagram of two strands } i \text{ and } i \text{ crossing with a dot on strand } i \\ - \\ \text{diagram of two strands } i \text{ and } i \text{ crossing with a dot on strand } i \end{array} = \sum_{\ell_1 + \ell_2 = d-1} \begin{array}{c} \text{diagram of two parallel strands } i \text{ and } i \text{ with dots on strands } \ell_1 \text{ and } \ell_2 \end{array} = \begin{array}{c} \text{diagram of two strands } i \text{ and } i \text{ crossing with a dot on strand } i \\ - \\ \text{diagram of two strands } i \text{ and } i \text{ crossing with a dot on strand } i \end{array}$$

Let $\beta = \sum_{j=1}^n \beta_j \alpha_j$ and let $R_{n+1}(\beta)$ be the subalgebra generated by all diagrams of R_{n+1} containing exactly β_i strands labeled i . We have

$$R_{n+1} = \sum_{\beta \in \Lambda_+^n} R_{n+1}(\beta).$$

We also denote by

$$R_{n+1}(k\alpha_n) = \bigoplus_{\beta' \in \Lambda_+^{n-1}} R_{n+1}(\beta' + k\alpha_n)$$

the subalgebra of R_{n+1} containing exactly k strands labeled n . With this notation we have

$$(9) \quad R_{n+1} = \bigoplus_{k \geq 0} R_{n+1}(k\alpha_n).$$

For a sequence $\underline{i} = (i_1, \dots, i_k)$ with i_j corresponding to the simple root α_{i_j} we write $1_{\underline{i}}$ for the idempotent formed by k vertical strands with labels in the order given by $\underline{i}$, *i.e.*

$$1_{\underline{i}} = \begin{array}{ccccccc} & | & | & | & \dots & | & \\ & | & | & | & & | & \\ i_1 & & i_2 & & i_3 & & i_k \end{array}$$

We write $1_{*\ell*}$ for $1_{\underline{i}}$ if the sequence of labels $\underline{i} = \underline{j}\underline{\ell}\underline{j}'$ can be written as a concatenation of sequences and we are only interested in the $\underline{\ell}$ part. We also write $x_{r,\underline{i}}$ for the diagram consisting of a dot on the r -th strand of $1_{\underline{i}}$, *i.e.*

$$x_{r,\underline{i}} = \begin{array}{ccccccc} & | & \dots & | & \dots & | & \\ & | & & \bullet & & | & \\ i_1 & & i_r & & i_k \end{array}$$

For β as above we denote by $\text{Seq}(\beta)$ the set of all sequences $\underline{i}$ of simple roots in which i_j appears exactly β_j times. The identity of $R_{n+1}(\beta)$ is then given by

$$1_{R_{n+1}(\beta)} = \sum_{\underline{i} \in \text{Seq}(\beta)} \begin{array}{ccccccc} & | & | & | & \dots & | & \\ & | & | & | & & | & \\ i_1 & & i_2 & & i_3 & & i_k \end{array}$$

We have with these conventions that

$$R_{n+1}(\beta) = \bigoplus_{\underline{i}, \underline{j} \in \text{Seq}(\beta)} 1_{\underline{i}} R_{n+1} 1_{\underline{j}}.$$

If $e \in R_{n+1}$ is an idempotent, then there is a (right) projective module ${}_e P = eR_{n+1}$. For $e = 1_{\underline{i}}$ this is the projective spanned by all diagrams whose labels end up in the sequence $\underline{i}$. We can define the left projective P_e in a similar way.

We denote by $R_{n+1}\text{-mod}$ and $R_{n+1}\text{-pmod}$ the categories of graded finitely generated right R_{n+1} -modules and of graded finitely generated projective right R_{n+1} -modules respectively. For idempotents e, e' we have

$$\text{Hom}_{R_{n+1}\text{-mod}}({}_e P, {}_{e'} P) = eR_{n+1}e'.$$

For a graded algebra A we denote by $K'_0(A)$ the (split) Grothendieck group of the category of finitely generated graded projective A -modules and write $K_0(A)$ for $\mathbb{Q}(q) \otimes_{\mathbb{Z}[q, q^{-1}]} K'_0(A)$.

Theorem 3.1 (Khovanov-Lauda [15], Rouquier [24]). *The Grothendieck group $K_0(R_{n+1})$ is isomorphic to the lower half $U_q^-(\mathfrak{sl}_{n+1})$ through the map that takes $[\underline{i}P]$ to $F_{\underline{i}}$.*

3.1. Categorical inclusion and projection for KLR algebras. Let Γ_n and Γ_{n-1} be the Dynkin diagrams associated to $\mathfrak{sl}_{n+1}$ and $\mathfrak{sl}_n$ respectively and consider the inclusion $\Gamma_{n-1} \hookrightarrow \Gamma_n$ that adds a vertex at the end of Γ_{n-1} and the corresponding edge, *i.e.*

$$\begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ \\ 1 & & 2 & & & & n-2 & & n-1 \end{array} \hookrightarrow \begin{array}{ccccccc} \circ & \text{---} & \circ & \text{---} & \cdots & \text{---} & \circ & \text{---} & \circ & \text{---} & \circ \\ 1 & & 2 & & & & n-2 & & n-1 & & n \end{array}$$

This induces an inclusion of KLR algebras

$$\iota: R_n \hookrightarrow R_{n+1}, \quad x \mapsto x$$

which coincides with the obvious map coming from the decomposition

$$(10) \quad R_{n+1} = \bigoplus_{k \geq 0} R_{n+1}(k\alpha_n) \cong R_n + \bigoplus_{k \geq 1} R_{n+1}(k\alpha_n).$$

The induction and restriction functors induced by ι

$$\text{Ind}_\iota: R_n\text{-mod} \rightarrow R_{n+1}\text{-mod} \quad \text{Res}_\iota: R_{n+1}\text{-mod} \rightarrow R_n\text{-mod}$$

are biadjoint, take projectives to projectives and descend to the natural inclusion and projection maps between the Grothendieck groups.

To see this we note that the dual construction takes the projection $\rho: R_{n+1} \rightarrow R_n$ in the decomposition (10) to form the functors of restriction of scalars and its left and right adjoints the extension of scalars and coextension of scalars by ρ respectively. Recall that ρ endows R_n with a structure of (R_{n+1}, R_n) -bimodule, where the structure of left R_{n+1} -module is given by $r.b = \rho(r)b$ for $b \in R_n$, $r \in R_{n+1}$. The same procedure can be used to give R_n a structure of (R_n, R_{n+1}) -bimodule. We use the notation ${}_{n+1}(R_n)_n$ and ${}_n(R_n)_{n+1}$ for R_n seen as a (R_{n+1}, R_n) -bimodule and (R_n, R_{n+1}) -bimodule respectively. Then we have the functors

$$\text{Res}_\rho: R_n\text{-mod} \rightarrow R_{n+1}\text{-mod},$$

$$\text{Ext}_\rho, \text{CoExt}_\rho: R_{n+1}\text{-mod} \rightarrow R_n\text{-mod}$$

with

$$\text{Ext}_\rho(M) = M \bigotimes_{R_{n+1}} {}_{n+1}(R_n)_n, \quad \text{CoExt}_\rho(M) = \text{Hom}_{R_{n+1}\text{-mod}}({}_n(R_n)_{n+1}, M)$$

for a right R_{n+1} -module M . It follows that the functors Ext_ρ and CoExt_ρ coincide and we also have isomorphisms of functors $\text{Ext}_\rho \cong \text{Res}_\iota$ and $\text{Res}_\rho \cong \text{Ind}_\iota$.

3.2. Factoring idempotents. In this section we give some properties of R_{n+1} -pmod that will be used in the sequel. For $\nu = \sum_{j=1}^{n-1} \nu_j \alpha_j \in \Lambda_+^n$ and for an ordered sequence $i_1, \dots, i_k$ we define

$$(11) \quad \nu(i) = \sum_{j < i} \nu_j \alpha_j + \sum_{j \geq i} (\nu_j - 1) \alpha_j,$$

and $\nu(i_1 \cdots i_k)$ as the result of iteration of (11) from $i = i_k$ to $i = i_1$.

For each $i \in \{1, \dots, n\}$ let p_i be the idempotent $1_{i,i+1,\dots,n} \in R_{n+1}$ and for $1_{\underline{j}} \in R_{n+1}(\nu(i))$ let $e'(p_i, \underline{j}) \in R_{n+1}(\nu + \alpha_n)$ be the idempotent obtained by horizontal composition of the diagram for p_i at the left of the one for $1_{\underline{j}}$, *i.e.*

$$e'(p_i, \underline{j}) = \begin{array}{c} \begin{array}{ccccccc} | & | & \cdots & | & | & \cdots & | \\ i & i+1 & & n & j_1 & & j_m \end{array} \end{array}$$

This generalizes easily to $R_{n+1}(k\alpha_n)$. In this case we denote by $p_{i_1 \dots i_k} \in R_{n+1}(k\alpha_n)$ the idempotent obtained by horizontal concatenation $p_{i_1} p_{i_2} \cdots p_{i_k} = 1_{i_1, i_1+1, \dots, n, i_2, \dots, n, i_3, \dots, n}$. The idempotent $\tilde{e}'(p_{i_1 \dots i_k}, \underline{j}) \in R_{n+1}(\nu + k\alpha_n)$ for $1_{\underline{j}} \in R_{n+1}(\nu(i_1 \cdots i_k))$ is defined as the horizontal concatenation placing the diagram of $p_{i_1 \dots i_k}$ at the left of the one for $1_{\underline{j}}$, *i.e.*

$$\tilde{e}'(p_{i_1 \dots i_k}, \underline{j}) = \begin{array}{c} \begin{array}{cccccccc} | & \cdots & | & | & \cdots & | & | & | & \cdots & | \\ i_1 & & n & i_2 & & n & & i_k & & n & j_1 & & j_\ell \end{array} \end{array}$$

In the case $1 \leq i_1 \leq \dots \leq i_k \leq n+1$ we write $e'(p_{i_1 \dots i_k}, \underline{j})$ instead of $\tilde{e}'(p_{i_1 \dots i_k}, \underline{j})$.

We introduce now the notion of factoring a diagram through a family of idempotents e_i .

Definition 3.2. We say that $Y \in R_{n+1}(\nu + \alpha_n)$ factors through the family of idempotents $\{e_j\}_{j \in J}$ if it can be written as a sum $\sum_{j \in J} c_j X_j$ with all $c_j \in \mathbb{k}$ nonzero and where each X_j is in

$$R_{n+1}(\nu + \alpha_n) e_j R_{n+1}(\nu + \alpha_n)$$

with j minimal.

Proposition 3.3. For each $\underline{j} \in \text{Seq}(\nu + \alpha_n)$ there is an index set J such that $1_{\underline{j}} \in R_{n+1}(\nu + \alpha_n)$ factors through the family $\{e'(p_j, \underline{j})\}_{j \in J}$.

Proof. The idempotent $1_{\underline{j}}$ consists of $|\underline{j}|$ parallel vertical strands labeled $j_1, \dots, j_{|\underline{j}|}$ in that order from left to right. We give an algorithm to obtain the factorization as claimed.

Step 1: Take the single strand labeled n and start pushing it to the left by application of the move in (4) until we find a strand labeled $n-1$, *i.e.*

$$1_{\underline{j}} = \begin{array}{c} \begin{array}{ccccccc} | & | & | & \cdots & | & | & \cdots \\ j_{r-2} & n-1 & j_r & & j_{r'} & n & \end{array} \end{array} \mapsto \begin{array}{c} \begin{array}{ccccccc} | & | & | & \cdots & | & | & \cdots \\ j_{r-2} & n-1 & j_r & & j_{r'} & n & \end{array} \end{array}$$

Step 2: Pass to the strand labeled $n-1$. There are three cases to consider: We can have

- (i) $j_{r-2} = n-1$,
- (ii) $j_{r-2} = n-2$, or
- (iii) $j_{r-2} \neq n-2, n-1$.

We start with case (i).

(i) If $j_{r-2} = n - 1$, we use the identity

$$(12) \quad \begin{array}{c} | \quad | \quad | \\ n-1 \quad n-1 \quad n \end{array} = \begin{array}{c} \text{Diagram 1} \\ n-1 \quad n-1 \quad n \end{array} - \begin{array}{c} \text{Diagram 2} \\ n-1 \quad n-1 \quad n \end{array}$$

which follows easily from (4) and (6). We see that $1 \dots n-1, n-1, n \dots$ factors through $1 \dots n-1, n, n-1 \dots$. This reduces the number of strands on the left of the strand labeled n . We then apply Step 1 to the block formed by strand labeled n and the one labeled $n - 1$ immediately on its left.

(ii) If $j_{r-2} = n - 2$ we apply Step 1 to the block formed by the strands labeled $n - 2, n - 1, n$. We can proceed until we find a strand labeled $n - 3, n - 2$ or $n - 1$.

- (1) If we find a strand labeled $n - 3$, we repeat Step 1 to the block formed by the strands labeled $n - 3, n - 2, n - 1$ and n .
- (2) If we find a strand labeled $n - 2$, we are in the situation of (i) with n replaced by $n - 1$.
- (3) If we find a strand labeled $n - 1$, we use (5) to obtain that $1 \dots n-1, n-2, n-1, n \dots$ factors through $1 \dots n-1, n-1, n-2, n \dots$ and through $1 \dots n-2, n-1, n-1, n \dots$. In the first case we can apply (4) to obtain that $1 \dots n-1, n-1, n-2, n \dots$ factors through $1 \dots n-1, n-1, n \dots$ which is the case (i). In the second one we apply (i) to the strands labeled $n - 1, n - 1, n$. Either way we reduce the number of strands on the left of the one labeled n .

(iii) If $j_{r-2} \neq n - 2, n - 1$, we apply Step 1 to the block formed by the strands labeled $n - 1, n$ until we find a strand labeled $n - 2$ or $n - 1$. We then proceed as in (ii). We then proceed recursively, *i.e.* each time we get a diagram factoring through $1 \dots s, s+1, \dots, n-1, n \dots$ we apply Step 1 to the entire block formed by the strands labeled s to n until we find a strand labeled j for $s - 1 \leq j \leq n - 1$. We then apply the move in (4) to pull this strand to the left of the one labeled $j - 1$ obtaining the configuration below.

$$y = \dots \begin{array}{c} \text{Diagram} \\ j \quad s \quad j-2 \quad j-1 \quad j \quad n \end{array} \dots$$

Using (5) in the region factoring through $1 \dots j, j-1, j \dots$ one obtains that y factors through the idempotent $1 \dots j, j, j-1, j+1 \dots$ and through $1 \dots j-1, j, j, j+1 \dots$. Using (4) on the first term we can slide the strand labeled $j - 1$ to the right of the one labeled n and then apply the procedure described above in (i) to $1 \dots j, j, j+1 \dots$. To the second term we apply the procedure of (i) to $1 \dots j, j, j+1 \dots$. Again, in either case we obtain a linear combination of terms each having less strands on the left of the one labeled n . The procedure ends when we obtain a linear combination of diagrams, each one factoring through and idempotent of the form $e'(p_i, \underline{i})$ as claimed. $\square$

We now consider the case of $k > 1$.

Lemma 3.4. Suppose $k = 2$. Then $\tilde{e}'(p_{i_1 i_2}, \underline{i})$ factors through a family $\{e'(j_i j_2, \underline{j})\}_{j \in J}$ for some indexing set J .

Proof. Suppose we have

$$\tilde{e}'(p_{ji}, \underline{i}) = \begin{array}{ccccccc} | & | & \cdots & | & | & \cdots & | & | & | & \cdots & | & \cdots \\ j & j+1 & & n & i & & j-1 & j & j+1 & & n & \end{array}$$

Using (4) we slide the first strand labeled n from the left to the right until it encounters a strand labeled $n-1$ to obtain a factorization through $1_{j \cdots n-1, i \cdots n-2, n, n-1, n, *}$:

$$(13) \quad \begin{array}{ccccccc} | & | & \cdots & | & \cdots & | & | & | & | & \cdots & | & \cdots \\ j & j+1 & & n-1 & n & i & j-1 & j & j+1 & & n-2 & n-1 & n \end{array}$$

We now use (5) in the $(n) - (n-1) - (n)$ part on the right of the diagram to obtain

$$\begin{array}{ccccccc} | & \cdots & | & \cdots & | & | & | & | & | & \cdots \\ j & n-1 & n & i & j & n-2 & n-1 & n & \end{array} - \begin{array}{ccccccc} | & \cdots & | & \cdots & | & | & | & | & | & \cdots \\ j & n-1 & n & i & j & n-2 & n-1 & n & \end{array}$$

The first term factors through the idempotent $1_{j \cdots n-1, i \cdots j, n-2, n, n, n-1, *}$ and the second through $1_{j \cdots n-1, i \cdots j, n-2, n-1, n, n, *}$. For the first term it is easy to see that we can do the same as in (13) to slide the entire block formed by the strands labeled $1 \cdots j, n-2$ to the right of the two strands labeled n to obtain a factorization through $1_{j \cdots n-1, n, n, i \cdots j, n-2, n-1, *}$ which is of the form $e'(p_{jn}, \underline{j})$ for some $\underline{j}$ as wanted. For the second term we slide the first strand labeled $n-1$ from the left to the right until it hits the strand labeled $n-2$:

$$\begin{array}{ccccccc} | & | & \cdots & | & \cdots & | & | & | & | & | & | & \cdots \\ j & j+1 & & n-2 & n-1 & i & j-1 & j & j+1 & & n-3 & n-2 & n-1 & n & n \end{array}$$

Applying (5) to the part containing $(n-1) - (n-2) - (n-1)$ we see that it factors through

$$(14) \quad 1_{j \cdots n-2, i \cdots n-3, n-1, n-1, n-2, n, n, *} \quad \text{and} \quad 1_{j \cdots n-2, i \cdots n-3, n-2, n-1, n-1, n, n, *}.$$

Using (4) we can put the first term in the form

$$\begin{array}{ccccccc} | & \cdots & | & \cdots & | & | & | & | & | & | & | & \cdots \\ j & n-2 & i & n-3 & n-1 & n-1 & n-2 & n & n & \end{array}$$

We see that it factors through $1_{j \cdots n-2, n-1, n-1, n, n, *}$ which in turn factors (twice) through an idempotent of the form $e'(p_{j, n-1}, \underline{j}')$ as wanted (this uses the identity (12)).

For the second term in (14) we start by sliding the first strand labeled $n - 2$ from the left and repeat the procedure. After having slid all the strands from the left of the first one labeled i to the right we end up with a factorization through the family $\{e'(p_{j,n-r}, \underline{r})\}_{r \in \{0, \dots, n-j\}}$ and through a term of the form

$$(15) \quad 1_{i \dots j-1, j, j, j+1, j+1, \dots, n-1, n-1, n, n*}.$$

Notice that after the first strand labeled $j - 1$ all the strands are labeled in pairs each two strands with the same label appearing consecutively. Applying the identity (12) to the part labeled $(j) - (j) - (j + 1)$ we get a factorization through $1_{i \dots j-1, j, j+1, j, j+1, \dots, n-1, n-1, n, n*}$.

Doing the same to the part containing $(j + 1) - (j + 2) - (j + 2)$ we get a factorization through $1_{i \dots j-1, j, j+1, j, j+2, j+1, j+2, \dots, n-1, n-1, n, n*}$ which factors through

$$1_{i \dots j-1, j, j+1, j+2, j, j+1, j+2, \dots, n-1, n-1, n, n*}$$

(this uses (4) between the two consecutive strands labeled j and $j + 2$). It is clear that we will end up with a factorization through $e'(p_{ij}, \underline{\ell})$ for some $\underline{\ell}$.

This way we see that $\tilde{e}'(p_{i_1 i_2}, \underline{i})$ factors through the family $\{e'(p_{j,n-s}, \underline{s})\}_{s \in \{0, \dots, n-j\}} \cup \{e'(p_{ij}, \underline{\ell})\}$ for some idempotents $\{1_{\underline{r}}\}_{r=0, \dots, n-j}$ and $1_{\underline{\ell}}$. $\square$

Proposition 3.5. *For each $j \in \text{Seq}(\nu + k\alpha_n)$ there is an index set I such that $1_{\underline{j}} \in R_{n+1}(\nu + k\alpha_n)$ factors through a family $\{e'(p_{i_1 \dots i_k}, \underline{i})\}_{i_1 \dots i_k, \underline{i} \in I}$.*

Proof. We use a combination of Proposition 3.3 and Lemma 3.4. By application of the method described in the proof of Proposition 3.3 to the leftmost strand of $1_{\underline{j}}$ labeled n we factorize it through some family the idempotents $\{e'(p_{a_1}, \underline{a})\}$ where each $1_{\underline{a}}$ is in $R_n(\nu(a_1) + (k - 1)\alpha_n)$.

Repeating the procedure for the newly created $1_{\underline{r}}$ we get that $1_{\underline{j}}$ factors through $\{\tilde{e}'(p_{b_1 b_2}, \underline{b})\}$ for $1_{\underline{b}}$ in $R_n(\nu(b_1 b_2) + (k - 2)\alpha_n)$. But each of the $\tilde{e}'(p_{b_1 b_2}, \underline{b})$ factors through a family $\{e'(p_{c_1 c_2}, \underline{c})\}$ from Lemma 3.4 and therefore $1_{\underline{j}}$ factors through a family $\{e'(p_{d_1 d_2}, \underline{d})\}$.

Then pass to the third leftmost strand labeled n . From Proposition 3.3 each of the $e'(p_{d_1 d_2}, \underline{d})$ factors through a family $\{\tilde{e}'(p_{d_1 d_2 f_3}, \underline{f})\}$ with $d_1 \leq d_2$ as before and $1_{\underline{f}} \in R(\nu(d_1 d_2 f_3) + (k - 3)\alpha_n)$. For each of these terms we apply Lemma 3.4 again: if $d_2 \leq f_3$ do nothing otherwise factorize $\tilde{e}'(p_{d_1 d_2 f_3}, \underline{f})$ through $\{\tilde{e}'(p_{d_1 g_2 g_3}, \underline{g})\}$ where $g_2 \leq g_3$. If $d_1 \leq g_2$ do nothing otherwise apply Lemma 3.4 to the $p_{d_1 g_2}$ part to obtain a factorization through $\{e'(p_{h_1 h_2}, \underline{h})\}$ with $h_1 \leq h_2$.

This procedure slides strands to the space between the second and third strands labeled n and therefore we need to apply Proposition 3.3 to the third strand labeled k again and repeat the procedure described above.

Notice that applying Proposition 3.3 and Lemma 3.4 amounts of “sliding” locally some strands to the right of a strand labeled n . This means that each time we apply each of these procedures to a strand labeled n we decrease the number of strands on its left. We see that the process terminates with the factorization of $1_{\underline{j}}$ through a family $\{e'(p_{m_1 m_2 m_3}, \underline{m})\}$ with each $1_{\underline{m}} \in R_{n+1}(\nu(m_1 m_2 m_3) - (k - 3)\alpha_n)$.

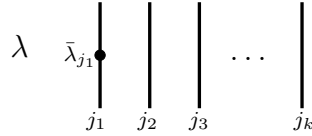
We now repeat the whole procedure to the fourth strand labeled n . Finiteness of the number of strands on its left implies that application of Proposition 3.3 and Lemma 3.4 as above allows factoring $1_{\underline{j}}$ through $\{e'(p_{n_1 n_2 n_3 n_4}, \underline{n})\}$. Proceeding recursively with the remaining strands labeled

n we get that $1_{\underline{j}}$ factors through a family $\{e'(p_{\ell_1 \ell_2 \ell_3 \dots \ell_k}, \underline{\ell})\}_{\ell \in L}$ with each 1_{ℓ} in $R_{n+2}(\nu(\ell_1 \dots \ell_k))$ as claimed. $\square$

3.3. Cyclotomic KLR-algebras. Fix a partition λ with $n + 1$ parts for the rest of the paper and let I^λ the two-sided ideal generated by $x_{1, \underline{i}}^{\bar{\lambda}_{i_1}}$ for all sequences $\underline{i}$.

Definition 3.6. . The cyclotomic KLR algebra R_{n+1}^λ is the quotient of R_{n+1} by the two-sided ideal I^λ .

Differently from the standard convention in the literature we label cyclotomic KLR algebras by partitions instead of integral dominant weights. This convention will be useful later. In terms of diagrams we are taking the quotient of R_{n+1} by the two-sided ideal generated by all the diagrams of the form



where the leftmost strand has $\bar{\lambda}_{j_1}$ dots on it. We always label the leftmost region of a diagram with a partition λ to indicate it is in R_{n+1}^λ . The following was proved in [29].

Lemma 3.7. *The cyclotomic KLR algebra R_{n+1}^λ is Frobenius.*

Projective modules over R_{n+1}^λ are defined the same way as for R_{n+1} , we write ${}_e P^\lambda$ for $e R_{n+1}^\lambda$. We denote by $R_{n+1}^\lambda\text{-mod}$ and by $R_{n+1}^\lambda\text{-pmod}$ the categories of finitely generated graded R_{n+1}^λ -modules and finitely generated graded projective R_{n+1}^λ -modules respectively. The module category structure in $R_{n+1}^\lambda\text{-mod}$ was studied in [14, 29]. Here we only describe the necessary to proceed through this paper. Let $\iota_i: R_{n+1}^\lambda(\nu) \rightarrow R_{n+1}^\lambda(\nu + \alpha_i)$ be the map obtained by adding a vertical strand labeled i on the right of a diagram from R_{n+1}^λ . The categorical $\mathfrak{sl}_{n+1}$ -action on R_{n+1}^λ is obtained by the pair of biadjoint exact functors defined by

$$(16) \quad \begin{aligned} F_i^\lambda &=: R_{n+1}^\lambda(\nu)\text{-mod} \xrightarrow{\text{Ind}_i} R_{n+1}^\lambda(\nu + \alpha_i)\text{-mod} \\ E_i^\lambda &=: R_{n+1}^\lambda(\nu + \alpha_i)\text{-mod} \xrightarrow{\text{Res}_i\{\alpha_i^\vee(\lambda - \nu) - 1\}} R_{n+1}^\lambda(\nu)\text{-mod} \end{aligned}$$

The *Khovanov-Lauda cyclotomic conjecture* [15] was proved by Brundan and Kleshchev [3] based on Ariki's categorification theorem [1], *i.e.*

Theorem 3.8. *There is an isomorphism of $\mathfrak{sl}_{n+1}$ -representations*

$$K_0(R_{n+1}^\lambda) \cong V_\lambda^{\mathfrak{sl}_{n+1}}.$$

This isomorphism sends the isomorphism class of an indecomposable $_{j_1 \dots j_r} P^\lambda$ to the weight vector $F_{j_r} \dots F_{j_1} v_\lambda$.

In Subsection 5.4 we give an alternative, elementary proof using the categorical branching rule. Theorem 3.8 was subsequently extended to affine type A by Brundan and Kleshchev [4] and to all types by Kang and Kashiwara [14] and independently by Webster [29]. Webster also proved that

$$\text{gdim Hom}_{R_{n+1}^\lambda\text{-mod}}(P, P') = \langle [P], [P'] \rangle,$$

where $\langle , \rangle$ is the q -Shapovalov form.

All the results in Subsection 3.2 descend to the cyclotomic setting. In particular they allow a presentation of the category $R_{n+1}^\lambda(\nu + k\alpha_n)$ -pmod in terms of the collection of projectives $\{e'(p_{i_1 \dots i_k, \underline{i}})P^\lambda\}$ that turns out to be useful later.

4. CATEGORICAL BRANCHING RULES

4.1. Categorical branching rules. We have a direct sum decomposition of algebras

$$R_{n+1}^\lambda \cong \bigoplus_{k \geq 0} R_{n+1}^\lambda(k\alpha_n),$$

where $R_{n+1}^\lambda(k\alpha_n) \subseteq R_{n+1}^\lambda$ is the subalgebra generated by the diagrams in R_{n+1}^λ containing exactly k strands labeled n . We also have

$$R_{n+1}^\lambda\text{-mod} \cong \bigoplus_{k \geq 0} R_{n+1}^\lambda(k\alpha_n)\text{-mod}.$$

Clearly $R_{n+1}^\lambda(0) \cong R_n^{p_\Lambda(\lambda)}$ where $p_\Lambda(\lambda): \Lambda_+^{n+1} \rightarrow \Lambda_+^n$ is the projection given in (3). We want to identify each block of $R_{n+1}^\lambda\text{-mod}$ with the categorification of the $\mathfrak{sl}_n$ -representations in (2) in the sense that $R_{n+1}^\lambda(k\alpha_n)\text{-mod}$ will give the $\mathfrak{sl}_n$ irreducibles obtained by removing exactly k boxes from the Young diagram for $V_{\mathfrak{sl}_{n+1}}^\lambda$.

We start by defining a special class of idempotents in R_{n+1}^λ .

Definition 4.1. The idempotent $e'(p_{i_1 \dots i_k, \underline{j}}) \in R_{n+1}^\lambda(\nu + k\alpha_n)$ is said to be a *special idempotent*, denoted by $e(p_{i_1 \dots i_k, \underline{j}})$, if $\xi_{i_1 \dots i_k}$ is λ -dominant.

The property of λ -dominancy of $\xi_{i_1 \dots i_k}$ implies that $\nu(i_1 \dots i_k)$ is in Λ_+^n .

In the following we give the maps between some cyclotomic KLR algebras that are necessary to obtain the categorical branching rule.

Lemma 4.2. For each $k \geq 0$ there is a surjection of algebras

$$R_{n+1}^\lambda(k\alpha_n) \xrightarrow{\pi_k} \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda^k} R_n^{\xi_{i_1 \dots i_k}(\lambda)}.$$

Proof. We first prove that for each $\xi_{i_1 \dots i_k}: \Lambda_+^{n+1} \rightarrow \Lambda_+^n$ we have a surjection of algebras

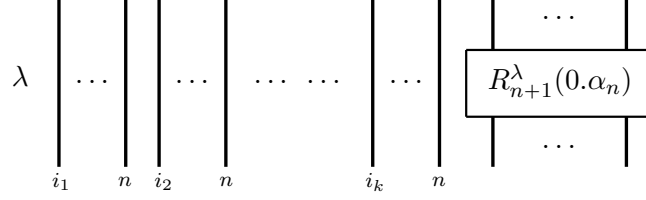
$$R_{n+1}^\lambda(k\alpha_n) \xrightarrow{\pi_{i_1 \dots i_k}} R_n^{\xi_{i_1 \dots i_k}(\lambda)}.$$

In order to prove this it is enough to show that for each $\xi_{i_1 \dots i_k}$ as above the subalgebra

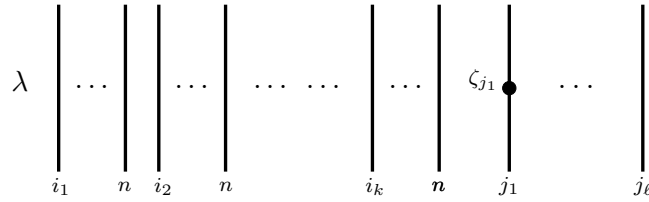
$$A_{i_1 \dots i_k} := \bigoplus_{\underline{r}, \underline{s} \in \text{Seq}\{\alpha_1, \dots, \alpha_n\}} e(p_{i_1 \dots i_k, \underline{r}})(R_{n+1}^\lambda(k\alpha_{n-1}))e(p_{i_1 \dots i_k, \underline{s}})$$

projects onto $R_n^{\xi_{i_1 \dots i_k}(\lambda)}$. Let $\tilde{A}_{i_1 \dots i_k} \subset A_{i_1 \dots i_k}$ be the subalgebra generated by all diagrams having a representative given by diagrams consisting of k blocks of vertical strands on the left, where

the $n - i_r + 1$ strands which belong to the r -th block from the left are labeled $i_r, i_r + 1, \dots, n$ in that order as illustrated below,

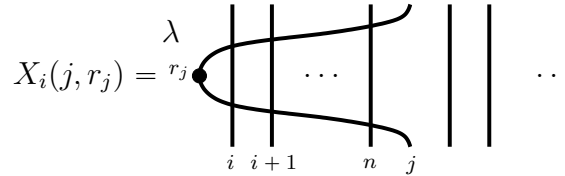


Let $\tilde{A}_{i_1 \dots i_k}^\perp$ be its complement vector space. Let also $\tilde{A}_{i_1 \dots i_k}^\lambda$ be the quotient of $\tilde{A}_{i_1 \dots i_k}$ by the two sided ideal generated by all diagrams of the form



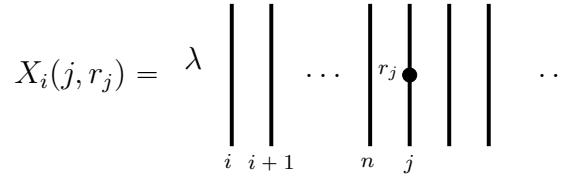
where $\zeta = \xi_{i_1 \dots i_k}(\bar{\lambda})$. The algebras $\tilde{A}_{i_1 \dots i_k}^\lambda$ and $R_n^{\xi_{i_1 \dots i_k}(\lambda)}$ are isomorphic through the map that adds $\sum_{r=1}^k (n - i_r + 1)$ vertical strands, labeled with the order determined from $p_{i_1 \dots i_k}$, on the left of a diagram from $R_n^{\xi_{i_1 \dots i_k}(\lambda)}$.

We start with the case $k = 1$ and use it to prove the general case by recursion. For this purpose we compute



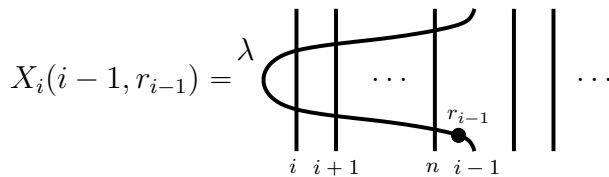
We have several cases to consider, $j < i - 1$, $j = i - 1$, $j = i$ and $i < j < n$.

For $j < i - 1$ we have



which follow easily from relations (4) and (6).

For $j = i - 1$ we have



$$= \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i-1 \text{ and a dot at } r_{i-1} \end{array} \dots + \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i-1 \text{ and a dot at } r_{i-1} \end{array} \dots$$

where we used relation (4). Sliding the newly created dot close to the original r_{i-1} dots on the first term and applying relation (4) to both terms we get

$$(17) \quad X_i(i-1, r_{i-1}) = \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i-1 \text{ and a dot at } r_{i-1} \end{array} \dots + \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i-1 \text{ and a dot at } r_{i-1} \end{array} \dots$$

which consists of a term in $\tilde{A}_i$ and a term in $\tilde{A}_i^\perp$.

For $j = i$ we compute

$$X_i(i, r_i) = \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i \text{ and a dot at } r_i \end{array} \dots = \sum_{\ell_1 + \ell_2 = r_i - 1} \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i \text{ and dots at } \ell_1, \ell_2 \end{array} \dots$$

where we used relation (8) followed by (4). Using (8) to slide the ℓ_2 dots to the upper part of the first strand from the left gives

$$X_i(i, r_i) = \sum_{\ell_1 + \ell_2 = r_i - 1} \left(\lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i \text{ and dots at } \ell_1, \ell_2 \end{array} \dots + \sum_{\kappa_1 + \kappa_2 = \ell_2 - 1} \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i \text{ and dots at } \kappa_1, \kappa_2 \end{array} \dots \right)$$

For $\bar{\lambda}_i = 1, 0$ we have $X_i(i, r_i) = 0$ for all $r_i \geq 0$ and so it is enough to consider the case $r_i > 1$ here. For $r_i > 1$ the first term is in $\tilde{A}_i^\perp$ and the second term is in $\tilde{A}_i^\perp$ unless $\ell_1 = \kappa_1 = 0$. This results in

$$\begin{aligned} X_i(i, r_i) &= \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i \text{ and a dot at } r_i \end{array} \dots + \text{terms in } \tilde{A}_i^\perp \\ &\stackrel{(17)}{=} \lambda \begin{array}{c} \text{diagram with strands } i, i+1, \dots, n, i \text{ and a dot at } r_i \end{array} \dots + \text{terms in } \tilde{A}_i^\perp \end{aligned}$$

Finally for $i < j < n$ we have

$$\begin{aligned}
 X_i(j, r_j) &= \lambda \begin{array}{c} \text{Diagram 1: A point } r_j \text{ on a line between indices } i \text{ and } i+1. \text{ A loop connects } i \text{ and } i+1. \text{ Other lines are at } n \text{ and } j. \end{array} \dots = \lambda \begin{array}{c} \text{Diagram 2: A point } r_j \text{ on a line between indices } j-1 \text{ and } j. \text{ A loop connects } j-1 \text{ and } j. \text{ Other lines are at } i \text{ and } n. \end{array} \dots \\
 &= \lambda \begin{array}{c} \text{Diagram 3: A point } r_j \text{ on a line between indices } j-1 \text{ and } j. \text{ A loop connects } j-1 \text{ and } j. \text{ Other lines are at } i \text{ and } n. \end{array} \dots + \lambda \begin{array}{c} \text{Diagram 4: A point } r_j \text{ on a line between indices } j-1 \text{ and } j. \text{ A loop connects } j-1 \text{ and } j. \text{ Other lines are at } i \text{ and } n. \end{array} \dots
 \end{aligned}$$

The first term is in $\tilde{A}_i^\perp$. For the second we use the result of the case of $i = j$ done above to obtain

$$X_i(j, r_j) = \lambda \begin{array}{c} \text{Diagram 5: A point } r_j \text{ on a line between indices } n \text{ and } j. \end{array} \dots + \text{terms in } \tilde{A}_i^\perp$$

Taking $r_i = \bar{\lambda}_i$ we see that $X_i(j, \bar{\lambda}_j)$ consists of a sum of a term in $R_n^{\xi_i(\lambda)}$ with terms in $\tilde{A}_i^\perp$ which shows that $R_{n+1}^\lambda(\alpha_n)$ projects onto $R_n^{\xi_i(\lambda)}$. We call this projection π_i . The kernel of π_i is the two-sided ideal generated by the elements in $\tilde{A}_i^\perp$ involved above. Proceeding recursively one gets that $\pi_{i_1 \dots i_k}$ is a surjection of algebras. The lemma now follows from the observation that $R_{n+1}^\lambda(k\alpha_n)$ projects canonically onto $\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda^k} A_{i_1 \dots i_k}$. $\square$

Summing over k in Lemma 4.2 we get the following.

Corollary 4.3. *We have a surjection of algebras*

$$R_{n+1}^\lambda \xrightarrow{\pi^\lambda} \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}. \quad \square$$

Fix a $k \geq 1$ and let

$$\Pi_k^\lambda := \text{ext}_k^\lambda: R_{n+1}^\lambda(k\alpha_n)\text{-mod} \rightarrow \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda^k} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod},$$

$$M \mapsto M \otimes_{R_{n+1}^\lambda(k\alpha_n)} \left(\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda^k} R_n^{\xi_{i_1 \dots i_k}(\lambda)} \right)$$

and $\text{res}_k^\lambda: \left(\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda^k} R_n^{\xi_{i_1 \dots i_k}(\lambda)} \right)\text{-mod} \rightarrow R_{n+1}^\lambda(k\alpha_n)\text{-mod}$ be respectively the functors of extension of scalars and restriction of scalars by the map π_k from Lemma 4.2.

Using the surjections $\pi_{i_1 \dots i_k}: R_{n+1}^\lambda(k\alpha_n) \rightarrow R_n^{\xi_{i_1 \dots i_k}(\lambda)}$ for each $\xi_{i_1 \dots i_k}$, that are inherited from the map π_k from Lemma 4.2, we call

$$\Pi_{i_1 \dots i_k}^\lambda: R_{n+1}^\lambda(k\alpha_n)\text{-mod} \rightarrow R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$$

and

$$\text{res}_{i_1 \dots i_k}^\lambda : R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod} \rightarrow R_{n+1}^\lambda(k\alpha_n)\text{-mod}$$

the components of Π_k^λ and res_k^λ . Since they are constructed from extension functors the functors $\Pi_{i_1 \dots i_k}^\lambda$ take projectives to projectives (contrary to res_k^λ as we will see later).

Lemma 4.4. *The functor Π_k^λ is full and essentially surjective.*

Proof. Fullness of Π_k^λ is a consequence of the surjectivity of π_k and the definition of Π_k^λ . To prove it is essentially surjective we note that this is true on projectives since every projective ${}_i P$ in $R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-pmod}$ can be obtained as $\Pi_{i_1 \dots i_k}^\lambda(e_{(p_{i_1 \dots i_k}, \underline{i})} P)$ for $e_{(p_{i_1 \dots i_k}, \underline{i})} P$ in $R_{n+1}^\lambda(k\alpha_n)\text{-pmod}$. The claim now follows from the fact that every object in these categories have a presentation by projectives. $\square$

Lemma 4.5. *Each functor $\Pi_{i_1 \dots i_k}^\lambda$ intertwines the categorical $\mathfrak{sl}_n$ -action.*

Proof. It is clear that the projection map $\pi_{i_1 \dots i_k} : R_{n+1}^\lambda(k\alpha_n) \rightarrow R_n^{\xi_{i_1 \dots i_k}(\lambda)}$ commutes with the map $\phi_j : R_{n+1}^\lambda(k\alpha_n) \rightarrow R_{n+1}^\lambda(k\alpha_n)$ that adds a vertical strand labeled $j \in \{1, \dots, n-1\}$ on the right of a diagram from $R_{n+1}^\lambda(k\alpha_n)$. This induces a natural isomorphism of functors

$$\Pi_{i_1 \dots i_k} F_j^\lambda \xrightarrow{\cong} F_j^{\xi_{i_1 \dots i_k}(\lambda)} \Pi_{i_1 \dots i_k}.$$

Now consider E_j^λ and $E_j^{\xi_{i_1 \dots i_k}(\lambda)}$. Recall that for a projective ${}_i P$ in $R_{n+1}^\lambda(k\alpha_n)$ with $\underline{i} \in \text{Seq}(\beta)$ the projective $E_j^\lambda({}_i P)$ has a basis given by all diagrams starting in a sequence $\underline{i}' j \in \text{Seq}(\beta)$ for fixed j and ending up in the sequence $\underline{i}$ and that $\Pi_{i_1 \dots i_k}({}_i P)$ has an analogous description. The isomorphism between $\Pi_{i_1 \dots i_k} E_j^\lambda$ and $E_j^{\xi_{i_1 \dots i_k}(\lambda)} \Pi_{i_1 \dots i_k}$ now follows from comparison between the vector spaces $\Pi_{i_1 \dots i_k} E_j^\lambda({}_i P)$ and $E_j^{\xi_{i_1 \dots i_k}(\lambda)} \Pi_{i_1 \dots i_k}({}_i P)$. $\square$

Using the lemmas above we have the following.

Proposition 4.6. *Each functor $\Pi_{i_1 \dots i_k}^\lambda$ descends to a surjection*

$$K_0(\Pi_{i_1 \dots i_k}^\lambda) : K_0(R_{n+1}^\lambda(k\alpha_n)) \rightarrow K_0(R_n^{\xi_{i_1 \dots i_k}(\lambda)})$$

of $\mathfrak{sl}_n$ -representations.

Finally define the functor

$$(18) \quad \Pi^\lambda := \bigoplus_{k \geq 0} \Pi_k^\lambda : R_{n+1}^\lambda\text{-mod} \rightarrow \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}.$$

Functor Π^λ is full, essentially surjective and intertwines the $\mathfrak{sl}_n$ -action by Lemmas 4.4 and 4.5.

Combining Proposition 4.6 with Theorem 3.8 we have the main result of this section, which follows easily by counting dimensions.

Theorem 4.7. *Functor Π^λ descends to an isomorphism of $\mathfrak{sl}_n$ -representations*

$$K_0(\Pi^\lambda): V_\lambda^{\mathfrak{sl}_{n+1}} \cong K_0(R_{n+1}^\lambda) \xrightarrow{\cong} K_0\left(\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\right) \cong \bigoplus_{\mu \in \tau(\lambda)} V_\mu^{\mathfrak{sl}_n}.$$

Corollary 4.8. *The functor Π^λ is injective on objects.*

Proof. It suffices to prove injectivity on projectives. Suppose Π^λ is injective on projectives and there is a non-projective object L in R_{n+1}^λ -mod with $\Pi^\lambda(L) = 0$. Since L has a presentation by a projective-injective P in R_{n+1}^λ -mod, there is another object K in R_{n+1}^λ -mod such that the (inclusion-projection) sequence $L \rightarrow P \rightarrow K$ is exact. In particular K is not projective otherwise the sequence would split and P would be the direct sum of a projective with a non-projective. Since the functor Π^λ is right-exact and takes projectives to projectives we would get an isomorphism $\Pi^\lambda(K) \cong \Pi^\lambda(P)$ forcing K to be projective by Lemma 4.4 which is a contradiction.

From the results of Subsection 3.2 translated into the cyclotomic setting we see that it is enough to prove that Π^λ is injective on the collection of objects which can be expressed as direct sums of projectives over R_{n+1}^λ of the form $e'(p_{i_1 \dots i_k})P$. We proceed by induction on the reverse order of the lexicographic order on the $p_{i_1 \dots i_k}$ s (this is induced by the lexicographic order on the k -tuples $(i_1, \dots, i_k) \in \mathbb{Z}_{\geq 0}^k$). The base case $(i_1, \dots, i_k) = (n, \dots, n)$ is clear, so we proceed to the general case. The fact that $\Pi^\lambda e'(p_{i_1 \dots i_k})P = 0$ implies that

$$\mathrm{Hom}_{R_{n+1}^\lambda\text{-mod}}(e'(p_{i_1 \dots i_k})P, e(p_{j_1 \dots j_k}, \underline{j})P) = 0$$

for all $p_{j_1 \dots j_k}$ greater than $p_{i_1 \dots i_k}$ in the lexicographic order. By the induction hypothesis this implies that

$$(19) \quad \mathrm{Hom}_{R_{n+1}^\lambda\text{-mod}}(e'(p_{i_1 \dots i_k})P, e'(p_{m_1 \dots m_k}, \underline{m})P) = 0$$

for all $p_{m_1 \dots m_k}$ greater than $p_{i_1 \dots i_k}$ in the lexicographic order, since all but the special projectives in $\{e'(p_{m_1 \dots m_k}, \underline{m})P\}_{p_{m_1 \dots m_k} > p_{i_1 \dots i_k}}$ are zero. Since every object in R_{n+1}^λ is isomorphic to a direct summand in $\bigoplus_{i_1 \dots i_k, \underline{i}} P^\lambda \{s_{i_1 \dots i_k, \underline{i}}\}^{m_{i_1 \dots i_k, \underline{i}}}$, this implies that

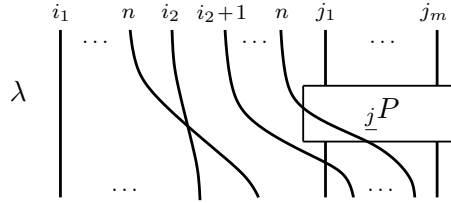
$$\mathrm{End}_{R_{n+1}^\lambda\text{-mod}}(e'(p_{i_1 \dots i_k})P) = e'(p_{i_1 \dots i_k})R_{n+1}^\lambda e'(p_{i_1 \dots i_k})$$

is one-dimensional, for any diagram other than the one consisting only of vertical strands without dots factors through elements in a sum of spaces of the form (19), as explained in Subsection 3.2. In particular this implies that $e'(p_{i_1 \dots i_k})P$ is indecomposable which contradicts Theorem 4.7 for otherwise $K_0(\Pi^\lambda)(e'(p_{i_1 \dots i_k})P)$ would be nonzero since $K_0(e'(p_{i_1 \dots i_k})P)$ is non-zero and $K_0(\Pi^\lambda)$ is an isomorphism. $\square$

Lemma 4.4 and Corollary 4.8 altogether imply that the category $\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$ contains all the objects of $R_{n+1}^\lambda\text{-mod}$.

4.2. Categorical projection. Contrary to Π_k^λ the functor res_k^λ does not have such a nice behavior on projectives. It takes a projective ${}_jP$ in $R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$ to the $R_{n+1}^\lambda(k\alpha_n)$ -module ${}_{i_1 \dots i_k; j}L$ which is a quotient of the projective ${}_{p_{i_1 \dots i_k; j}}P = {}_{i_1, \dots, n, i_2, \dots, n, \dots, i_k, \dots, n, j}P$ over $R_{n+1}^\lambda(k\alpha_n)$. To make this more precise notice that $p_{i_1 \dots i_k}$ defines $k + 1$ groups of strands in ${}_{p_{i_1 \dots i_k; j}}P$, where two strands correspond to the same group if they belong to the same p_{i_j} for some i_j or if they belong to 1_j . We say the first type of group is *special*. Let ${}_{i_1 \dots i_k; j}N \subset {}_{p_{i_1 \dots i_k; j}}P$ be the submodule spanned by all diagrams containing either a dot on any of the strands belonging to a special group or a crossing between any pair of strands belonging to the same special group. Then we have ${}_{i_1 \dots i_k; j}L = {}_{p_{i_1 \dots i_k; j}}P / {}_{i_1 \dots i_k; j}N$.

It is clear that the module ${}_{i_1 \dots i_k; j}L$ has a presentation by the span of the subset of the set of all diagrams from ${}_{p_{i_1 \dots i_k; j}}P$ whose strands which do not belong to a special group satisfy the usual KLR relations and where a strand belonging to a special group cannot intersect another strand belonging to the same special group nor carry any dots. We can also regard a diagram in ${}_{i_1 \dots i_k; j}L$ as the overlap of a diagram from ${}_jP$ and a diagram consisting of $\sum_{s=1}^k (n - i_s + 1)$ strands that do not carry dots and end up at the left of all strands from ${}_jP$ with labels determined by $p_{i_1 \dots i_k}$. Moreover, strands belonging to the same special group run parallel to each other but they can intersect strands belonging to any other group, as in the example with $p_{i_1 i_2}$ shown below.



We have that $\Pi_{\ell_1, \dots, \ell_r}^\lambda({}_{i_1 \dots i_k; j}L)$ is isomorphic to ${}_jP$ if $(\ell_1, \dots, \ell_r) = (i_1 \dots i_k)$ and is zero otherwise. We also note that the functor $\bigoplus_{\xi_{i_1, \dots, i_k} \in \mathcal{D}_\lambda} \Pi_{i_1, \dots, i_k}^\lambda \text{res}_{i_1, \dots, i_k}^\lambda$ is isomorphic to the identity functor of $\bigoplus_{\xi_{i_1, \dots, i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$. We can think of the set of the composite functors $\{\Pi_{i_1, \dots, i_k}^\lambda \text{res}_{i_1, \dots, i_k}^\lambda\}_{\xi_{i_1, \dots, i_k} \in \mathcal{D}_\lambda}$ as a categorical version of a complete system of orthogonal idempotents. This is another way of seeing that $R_{n+1}^\lambda\text{-mod}$ contains $\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$.

The collection of diagrams described above can be given the structure of associative $\mathbb{k}$ -algebra if we do not force the labels on the top to be ordered according to $e(p_{i_1 \dots i_k}, j)$ (of course the labels of the strands from the special groups have to end up in the order determined by $p_{i_1 \dots i_k}$) and impose the relations inherited from the KLR relations (4)-(6). Denote this algebra by $\tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \dots i_k}, j)$ and define

Definition 4.9. $\tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \dots i_k}) = \bigoplus_j \tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \dots i_k}, j)$.

Each element of $\tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \dots i_k})$ can be thought of as an overlap of two diagrams, one from $R_{n+1}^\lambda(\nu(i_1 \dots i_k))$ for some $\nu \in \Lambda_+^{n+1}$ (see Subsection 3.2 for the definition of $\nu(i_1 \dots i_k)$) and another one from the $\mathbb{k}$ -algebra generated by KLR-diagrams consisting of $\sum_{s=1}^k (n - i_s + 1)$

strands belonging to special groups and satisfying the conditions above. At any intersection with any horizontal line that do not pass through crossings or dots the labels of the strands from each group are ordered in the order determined by the p_{i_j} they belong to.

Definition 4.10. We define the algebras $\tilde{R}_{n+1}^\lambda(k\alpha_n)$ and $\tilde{R}_{n+1}^\lambda$ by

$$\tilde{R}_{n+1}^\lambda(k\alpha_n) = \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda^k} \tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \dots i_k})$$

and

$$\tilde{R}_{n+1}^\lambda = \bigoplus_{k \geq 0} \tilde{R}_{n+1}^\lambda(k\alpha_n).$$

We now describe $\tilde{R}_{n+1}^\lambda(k\alpha_n)$ more intrinsically. The kernel K of the action of $R_{n+1}^\lambda(\nu)$ on $_{i_1 \dots i_k; \underline{j}} L$ contains all diagrams in $\tilde{R}_{n+1}^\lambda(k\alpha_n)$ that have either a dot on a strand belonging to a special group or a crossing between strands belonging to the same special group. It is not hard to see that this collection of diagrams exhaust K . Let $J_{i_1 \dots i_k; \underline{j}} \subset R_{n+1}^\lambda(k\alpha_n)$ be the two-sided ideal generated by K and define the ideal $J_{i_1 \dots i_k} = \sum_{\underline{j}} J_{i_1 \dots i_k; \underline{j}}$ where $\underline{j}$ runs over all sequences of simple roots in Λ_+^n . All the above adds up to the following.

Lemma 4.11. *We have an isomorphism of algebras*

$$\tilde{R}_{n+1}^\lambda(k\alpha_n; p_{i_1 \dots i_k}) \cong R_{n+1}^\lambda(k\alpha_n) / J_{i_1 \dots i_k}.$$

□

Moreover, we also have the following.

Lemma 4.12. *The module $_{i_1 \dots i_k; \underline{j}} L$ is projective as a module over $\tilde{R}_{n+1}^\lambda(k\alpha_n)$.*

Proof. We have that the element $e(p_{i_1 \dots i_k}, \underline{j})$ is an idempotent in $\tilde{R}_{n+1}^\lambda(k\alpha_n)$ and the module $_{i_1 \dots i_k; \underline{j}} L = e(p_{i_1 \dots i_k}, \underline{j}) \tilde{R}_{n+1}^\lambda(k\alpha_n)$. □

Functor res_k^λ can be regarded as a functor from $\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_k^\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$ to $\tilde{R}_{n+1}^\lambda(k\alpha_n)\text{-mod}$ the latter category seen as the quotient of $R_{n+1}^\lambda(k\alpha_n)\text{-mod}$ by $\sum_{\xi_{i_1 \dots i_k} \in \mathcal{D}_k^\lambda} J_{i_1 \dots i_k}$. This functor takes projectives to projectives. With this in mind we see that the quotient functor

$$(20) \quad Q_k: R_{n+1}^\lambda(k\alpha_n)\text{-mod} \rightarrow \tilde{R}_{n+1}^\lambda(k\alpha_n)\text{-mod}$$

is isomorphic to the functor $\text{res}_k^\lambda \Pi_k^\lambda$. Moreover, the functor Π^λ descends to a functor

$$\tilde{\Pi}^\lambda: \tilde{R}_{n+1}^\lambda\text{-mod} \rightarrow \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}^\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$$

which is full and essentially surjective by Lemma 4.4. It is also faithful by the definition of the projections π_k . The categories $\tilde{R}_{n+1}^\lambda\text{-mod}$ and $\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}^\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$ are therefore equivalent. Denote by

$$\tilde{\Pi}_k^\lambda: \tilde{R}_{n+1}^\lambda(k\alpha_n)\text{-mod} \rightarrow \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_k^\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$$

and

$$\widetilde{\text{res}}_k^\lambda: \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_k^\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod} \rightarrow \widetilde{R}_{n+1}^\lambda(k\alpha_n)\text{-mod}$$

the functors induced by Π_k^λ and res_k^λ from Subsection 4.1. It is not hard to see that the composite functors $\widetilde{\text{res}}_k^\lambda \widetilde{\Pi}_k^\lambda$ and $\widetilde{\Pi}_k^\lambda \widetilde{\text{res}}_k^\lambda$ are isomorphic to the identity functors on $\widetilde{R}_{n+1}^\lambda\text{-mod}$ and $\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}^\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$.

5. CATEGORIFYING THE GELFAND-TSETLIN BASIS

5.1. Recovering the categorical $\mathfrak{sl}_{n+1}$ -action. The crucial step in constructing a categorical $\mathfrak{sl}_{n+1}$ -action on the categorified Gelfand-Tsetlin basis consists in recovering the categorical $\mathfrak{sl}_{n+1}$ -action on $\Pi^\lambda(R_{n+1}^\lambda\text{-mod})$ from its categorical action on $R_{n+1}^\lambda\text{-mod}$. This amounts to understand the interplay between the functors $E_n^\lambda, F_n^\lambda, \Pi^\lambda$ and res_k^λ . For a module M in $R_{n+1}^\lambda\text{-mod}$ we want to see how the functors E_n^λ, F_n^λ allow to move between the components $\Pi_{i_1 \dots i_k}(M)$ of the different categories $R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$.

Contrary to the case of $\Pi_{i_1 \dots i_k}^\lambda$ the functors $\text{res}_{i_1 \dots i_k}^\lambda$ do not intertwine the categorical $\mathfrak{sl}_n$ -action. Nevertheless we can define functors

$$E_j^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)}, F_j^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)}: R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod} \rightarrow R_n^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)}\text{-mod}$$

for $j = 1, \dots, n-1$ by

$$E_j^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)} = \Pi_{j_1 \dots j_r}^\lambda E_j^\lambda \text{res}_{\xi_{i_1 \dots i_k}}^\lambda$$

and

$$F_j^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)} = \Pi_{j_1 \dots j_r}^\lambda F_j^\lambda \text{res}_{\xi_{i_1 \dots i_k}}^\lambda$$

We start as on the non-categorified picture. We have the following.

Lemma 5.1. *For $j \in \{1, \dots, n-1\}$ the functors $E_j^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)}$ and $F_j^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)}$ are zero unless $(i_1, \dots, i_k) = (j_1, \dots, j_r)$. In this case they coincide with the functors $E_j^{\xi_{i_1 \dots i_k}(\lambda)}$ and $F_j^{\xi_{i_1 \dots i_k}(\lambda)}$ inherit from the structure of categorical $\mathfrak{sl}_{n+1}$ -module on $R_{n+1}^\lambda\text{-mod}$.*

Proof. From Lemma 4.5 we have for $j \in \{1, \dots, n-1\}$,

$$E_j^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)} = \Pi_{j_1 \dots j_r}^\lambda E_j^\lambda \text{res}_{\xi_{i_1 \dots i_k}}^\lambda \cong E_j^{\xi_{j_1 \dots j_r}(\lambda)} \Pi_{j_1 \dots j_r}^\lambda \text{res}_{\xi_{i_1 \dots i_k}}^\lambda.$$

The claim follows from the fact that the functor $\Pi_{j_1 \dots j_r}^\lambda \text{res}_{\xi_{i_1 \dots i_k}}^\lambda$ is the identity functor acting on $R_n^{\xi_{i_1 \dots i_k}(\lambda)}$, if the sequences $(i_1, \dots, i_k)$ and $(j_1, \dots, j_r)$ are equal, or the zero functor, if they are different. The same reasoning proves the case of $E_j^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)}$. $\square$

Definition 5.2. For $i \in \{1, \dots, n-1\}$ we define the functors

$$F_j^\xi := \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} F_j^{\xi_{i_1 \dots i_k}^{i_1 \dots i_k}(\lambda)} \quad \text{and} \quad E_j^\xi := \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} E_j^{\xi_{i_1 \dots i_k}^{i_1 \dots i_k}(\lambda)}$$

with the obvious source and target categories.

Let us now treat the case of the functors F_n^λ and E_n^λ . Each object $_{i_1 \dots i_k, j} L$ in $\tilde{R}_{n+1}^\lambda$ -pmod is also an object in R_{n+1}^λ -mod which is not projective in general. It is not hard to see that the projective cover of $_{i_1 \dots i_k, j} L$ in R_{n+1}^λ -mod is $_{e(p_{i_1 \dots i_k, j})} P$. The Lemmas 3.7 and 4.4 together with Corollary 4.8 imply that every object in R_{n+1}^λ -pmod arises in this way. For an endofunctor G acting on R_{n+1}^λ -pmod we define a functor $\tilde{G}$ on $\tilde{R}_{n+1}^\lambda$ -pmod as follows. For an object M in $\tilde{R}_{n+1}^\lambda$ -pmod we define $\tilde{G}(M)$ as $QG(P(M))$. Here $P(M)$ is the projective cover of M in R_{n+1}^λ -mod and $Q = \bigoplus_{k \geq 0} Q_k$ is the quotient functor from Equation (20). The action of $\tilde{G}$ on a morphism f in $\text{Hom}_{\tilde{R}_{n+1}^\lambda\text{-pmod}}(M, M')$ is defined in an analogous way. This operations are well defined, because the composite $QG'P(QGP(M))$ is isomorphic to $QG'GP(M) = \widetilde{G'}G(M)$ for each endofunctor G' on R_{n+1}^λ -pmod. For morphisms f, f' we observe that $P(QGP(f))$ equals $GP(f)$ yielding $\tilde{G}(f'f) = \tilde{G}(f')\tilde{G}(f)$.

Lemma 5.3. *The pair of (biadjoint) endofunctors $\{\tilde{F}_n^\lambda, \tilde{E}_n^\lambda\}$ take projectives to projectives and define a categorical $\mathfrak{sl}_2$ -action on $\tilde{R}_{n+1}^\lambda$ -pmod.*

This action extends canonically to a categorical action on $\tilde{R}_{n+1}^\lambda$ -mod. We now use this result to construct a $\mathfrak{sl}_2$ -pair of functors $\{F_n^\xi, E_n^\xi\}$ acting on the category $\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}$ -mod. We first define the functors

$$F_n^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)}, E_n^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)} : R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod} \rightarrow R_n^{\xi_{j_1 \dots j_r}(\lambda)}\text{-mod}$$

by

$$F_n^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)} = \tilde{\Pi}_{j_1 \dots j_r}^\lambda \tilde{F}_n^\lambda \widetilde{\text{res}}_{\xi_{i_1 \dots i_k}}^\lambda$$

and

$$E_n^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)} = \tilde{\Pi}_{j_1 \dots j_r}^\lambda \tilde{E}_n^\lambda \widetilde{\text{res}}_{\xi_{i_1 \dots i_k}}^\lambda.$$

Both functors are zero if the sequences $(i_1, \dots, i_k)$ and $(j_1, \dots, j_r)$ have the same length, since this would mean that the source and target categories would correspond to weights of the form $\nu + k\alpha_n$ with $\nu = \nu_1\alpha_1 + \dots + \nu_{n-1}\alpha_{n-1}$ with common k (all diagrams in $\tilde{R}_{n+1}^\lambda$ -pmod and R_{n+1}^λ -pmod would contain the same number of strands labeled n) and both functors E_n^λ and F_n^λ (and $\tilde{E}_n^\lambda, \tilde{F}_n^\lambda$) change the number of strands labeled n . Summing over all $i_1 \dots i_k$ and all k we get the functors

$$F_n^\xi, E_n^\xi : \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod} \rightarrow \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}$$

given by

$$F_n^\xi := \bigoplus_{\substack{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda \\ \xi_{j_1 \dots j_r} \in \mathcal{D}_\lambda}} F_n^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)} \quad \text{and} \quad E_n^\xi := \bigoplus_{\substack{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda \\ \xi_{j_1 \dots j_r} \in \mathcal{D}_\lambda}} E_n^{\xi_{i_1 \dots i_k}^{j_1 \dots j_r}(\lambda)}.$$

Proposition 5.4. *The functors F_n^ξ and E_n^ξ take projectives to projectives and define a categorical $\mathfrak{sl}_2$ -action on*

$$\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}.$$

Proof. The first claim is a consequence of the definition of the functors $\{\tilde{F}_n^\lambda, \tilde{E}_n^\lambda\}$. Biadjointness is a consequence of biadjointness of the pair $\{F_n, E_n\}$ and the definition of $\{F_n^\xi, E_n^\xi\}$. Since $\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda^k} \widetilde{\text{res}}_{i_1 \dots i_k}^\lambda \tilde{\Pi}_{i_1 \dots i_k}^\lambda$ is the identity functor on $\tilde{R}_{n+1}^\lambda(k\alpha_n)\text{-mod}$ we have

$$\begin{aligned} F_n^\xi E_n^\xi &= \bigoplus_{\xi_{i_1 \dots i_k}, \xi_{j_1 \dots j_r}, \xi_{\ell_1 \dots \ell_s}, \xi_{m_1 \dots m_t} \in \mathcal{D}_\lambda} \tilde{\Pi}_{j_1 \dots j_r}^\lambda \tilde{F}_n^\lambda \widetilde{\text{res}}_{\xi_{j_1 \dots j_r}}^\lambda \tilde{\Pi}_{\ell_1 \dots \ell_s}^\lambda \tilde{E}_n^\lambda \widetilde{\text{res}}_{\xi_{m_1 \dots m_t}}^\lambda \\ &\cong \bigoplus_{\xi_{i_1 \dots i_k}, \xi_{m_1 \dots m_t} \in \mathcal{D}_\lambda} \tilde{\Pi}_{j_1 \dots j_r}^\lambda \tilde{F}_n^\lambda \tilde{E}_n^\lambda \widetilde{\text{res}}_{\xi_{m_1 \dots m_t}}^\lambda \end{aligned}$$

and

$$E_n^\xi F_n^\xi \cong \bigoplus_{\xi_{i_1 \dots i_k}, \xi_{j_1 \dots j_r} \in \mathcal{D}_\lambda} \tilde{\Pi}_{i_1 \dots i_k}^\lambda \tilde{E}_n^\lambda \tilde{F}_n^\lambda \widetilde{\text{res}}_{\xi_{j_1 \dots j_r}}^\lambda$$

and the claim follows. $\square$

Corollary 5.5. *The functors $\{F_j^\xi, E_j^\xi\}_{j \in \{1, \dots, n\}}$ define a categorical $\mathfrak{sl}_{n+1}$ -action on*

$$\bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} R_n^{\xi_{i_1 \dots i_k}(\lambda)}\text{-mod}.$$

$\square$

Corollary 5.6. *With the action E_n^ξ and F_n^ξ as above the surjection $K_0(\Pi^\lambda)$ in Proposition 4.6 is a surjection of $\mathfrak{sl}_{n+1}$ -representations.*

Proof. It is enough to show that the functor Π^λ intertwines the categorical $\mathfrak{sl}_2$ -action defined by $\{F_n^\xi, E_n^\xi\}$. For an object M in $R_{n+1}^\lambda\text{-pmod}$ we have

$$\begin{aligned} F_n^\xi \Pi^\lambda(M) &= \tilde{\Pi}^\lambda \tilde{F}_n^\lambda \widetilde{\text{res}}^\lambda \Pi^\lambda(M) \\ &= \tilde{\Pi}^\lambda \mathcal{Q}_{F_n^\lambda} P(\widetilde{\text{res}}^\lambda \Pi^\lambda(M)) \\ &= \tilde{\Pi}^\lambda \widetilde{\text{res}}^\lambda \Pi^\lambda F_n^\lambda P(\widetilde{\text{res}}^\lambda \Pi^\lambda(M)) \\ &= \Pi^\lambda F_n^\lambda P(\widetilde{\text{res}}^\lambda \Pi^\lambda(M)). \end{aligned}$$

The claim now follows from a comparison between the vector spaces $P(\widetilde{\text{res}}^\lambda \Pi^\lambda(M))$ and M . $\square$

5.2. Classes of special indecomposables and the Gelfand-Tsetlin basis. Applying the procedure described in Section 4 recursively we end up with a direct sum of $d_\lambda = \dim(V_\lambda^{\text{sl}_{n+1}})$ one-dimensional $\mathbb{k}$ -vector spaces. We now reverse this procedure.

Definition 5.7. For each Gelfand-Tsetlin pattern $s \in \mathcal{S}(\lambda)$ we define a functor

$$\text{res}^s := \text{res}_{\mu^n}^\lambda \text{res}_{\mu^{n-1}}^{\mu^n} \cdots \text{res}^{\mu^1} : \mathbb{k}\text{-mod} \rightarrow R_{n+1}^\lambda\text{-mod}.$$

Here each of the functors $\text{res}_{\mu^{j-1}}^{\mu^j}$ is the restriction functor corresponding to the surjection $\pi_{\mu^j, \mu^{j-1}} : R_j^{\mu^j} \rightarrow R_{j-1}^{\mu^{j-1}}$ as in Lemma 4.2 which uniquely determines a sequence $(i_{1_j} \cdots i_{k_j})$ and therefore special idempotent $p_{i_{1_j} \cdots i_{k_j}}$ in $R_j^{\mu^j}$.

To keep the notation simple from now on we use s to denote $i_{1_1} \cdots i_{k_1}; i_{1_2} \cdots i_{k_2}; \dots; i_{1_n} \cdots i_{k_n}$, we use π_s to denote the composite $\pi_{i_{1_n} \cdots i_{k_n}} \cdots \pi_{i_{1_1} \cdots i_{k_1}}$ and we write $e(s)$ instead of $e(p_{i_{1_1} \cdots i_{k_1}}, p_{i_{1_2} \cdots i_{k_2}}, \dots, p_{i_{1_n} \cdots i_{k_n}})$.

All the above defines a special idempotent $e(s)$ in R_{n+1}^λ where each $p_{i_{1_j} \cdots i_{k_j}}$ can be seen as special idempotent in R_j . It is not hard to see that the functor res^s takes the one-dimensional $\mathbb{k}$ -module $\mathbb{k}$ to the module $e(s)L$ which is a quotient of the projective R_{n+1}^λ -module $e(s)P$ by the submodule $e(s)N$ which we now describe. It is spanned by diagrams consisting of n sets of strands where strands belonging to different groups satisfy the usual KLR relations while the i -th set consists of strands labeled within $\{1, \dots, i\}$ which are in turn grouped and are subject to the same conditions as the special groups of strands given in Subsection 4.2 for $R_j^{\mu^j}$.

Definition 5.8. The algebra $\check{R}_{n+1}^\lambda$ is defined as the quotient

$$(21) \quad \check{R}_{n+1}^\lambda = \bigoplus_{s \in \mathcal{S}(\lambda)} R_{n+1}^\lambda / \ker(\pi_s).$$

As in the case of the algebras $\check{R}_{n+1}^\lambda$ of Subsection 4.2 each algebra $R_{n+1}^\lambda / \ker(\pi_s)$ admit a presentation by diagrams consisting of groups of strands labeled by the entries $s(j)$ of the string s which are subject to the rules given above for $e(s)N$.

Lemma 5.9. We have $\text{Hom}_{\check{R}_{n+1}^\lambda\text{-mod}}(e(s)L, e(s')L) = 0$ if $s \neq s'$. □

Lemma 5.10. The module $e(s)L$ is projective indecomposable as a module over $\check{R}_{n+1}^\lambda$. □

Denote by $\check{\psi} : R_{n+1}^\lambda \rightarrow \check{R}_{n+1}^\lambda$ the projection map. The action of functors $\{F_i^\lambda, E_i^\lambda\}_{i \in \{1, \dots, n\}}$ on the collection of all the modules $e(s)L$ in $\check{R}_{n+1}^\lambda\text{-mod}$ inherited from the one on $R_{n+1}^\lambda\text{-mod}$ does not commute with the quotient functor $\check{\Psi} : R_{n+1}^\lambda\text{-mod} \rightarrow \check{R}_{n+1}^\lambda\text{-mod}$ induced by $\check{\psi}$. Nevertheless we can try change the action of the F s and of the E s on $\check{R}_{n+1}^\lambda\text{-mod}$ aiming at the obtention of a commutative diagram

$$(22) \quad \begin{array}{ccc} R_{n+1}^\lambda\text{-mod} & \xrightarrow{F_i^\lambda, E_i^\lambda} & R_{n+1}^\lambda\text{-mod} \\ \downarrow \check{\Psi} & & \downarrow \check{\Psi} \\ \check{R}_{n+1}^\lambda\text{-mod} & \xrightarrow{\check{F}_i^\lambda, \check{E}_i^\lambda} & \check{R}_{n+1}^\lambda\text{-mod} \end{array}$$

To prove this we define

$$\check{F}_j^\lambda, \check{E}_j^\lambda: \check{R}_{n+1}^\lambda\text{-mod} \rightarrow \check{R}_{n+1}^\lambda\text{-mod}$$

as the composite functors

$$\check{F}_j^\lambda = \bigoplus_{s_1, s_2, s_3 \in \mathcal{S}(\lambda)} \widetilde{\text{res}}_{s_3}^\lambda \widetilde{\Pi}_{s_2}^\lambda F_j^\lambda \widetilde{\text{res}}_{s_1}^\lambda \Pi_{s_1}^\lambda \quad \check{E}_j^\lambda = \bigoplus_{s_1, s_2, s_3 \in \mathcal{S}(\lambda)} \widetilde{\text{res}}_{s_3}^\lambda \widetilde{\Pi}_{s_2}^\lambda E_j^\lambda \widetilde{\text{res}}_{s_1}^\lambda \Pi_{s_1}^\lambda.$$

Lemma 5.11. *Functors $\check{F}_j^\lambda$ and $\check{E}_j^\lambda$ are biadjoint and take projectives to projectives. The collection of endofunctors $\{\check{F}_i^\lambda, \check{E}_i^\lambda\}_{i \in \{1, \dots, n\}}$ defines a categorical $\mathfrak{sl}_{n+1}$ -action on $\check{R}_{n+1}^\lambda\text{-mod}$. The functor $\check{\Psi}$ intertwines the categorical $\mathfrak{sl}_{n+1}$ -action.* $\square$

Theorem 5.12. *There is an isomorphism of $\mathfrak{sl}_{n+1}$ representations*

$$K_0(\check{R}_{n+1}^\lambda) \xrightarrow{\cong} V_\lambda^{\mathfrak{sl}_{n+1}}$$

taking the projective $e(s)L$ to the Gelfand-Tsetlin basis element $|s\rangle$.

Proof. The surjection $\check{\Psi}$ of algebras induces a surjective map between the Grothendieck groups

$$K_0(\check{\psi}): K_0(R_{n+1}^\lambda) \xrightarrow{\cong} K_0(\check{R}_{n+1}^\lambda)$$

intertwining the action of $\mathfrak{sl}_{n+1}$ which is an isomorphism if $K_0(\check{R}_{n+1}^\lambda)$ is not zero by Schur's lemma. To prove it is not zero we use the categorical branching rule to reduce the size of the category $\check{R}_{n+1}^\lambda\text{-pmod}$ recursively until we get something with non-zero K_0 . Choose a string $s \in \mathcal{S}(\lambda)$. Each surjection $\pi_{i_1 \dots i_k}: R_{n+1}^\lambda(k\alpha_n) \rightarrow R_n^{\xi_{i_1 \dots i_k}(\lambda)}$ from Lemma 4.2 induces a surjection $\check{\pi}_{i_1 \dots i_k}: \check{R}_{n+1}^\lambda(k\alpha_n) \rightarrow \check{R}_n^{\xi_{i_1 \dots i_k}(\lambda)}$ which in turn results in a map

$$K_0(\check{\pi}_{i_1 \dots i_k}): K_0(\check{R}_{n+1}^\lambda(k\alpha_n)) \rightarrow K_0(\check{R}_n^{\xi_{i_1 \dots i_k}(\lambda)})$$

which is surjective. Continuing recursively we end up with a chain of surjections

$$K_0(\check{R}_{n+1}^\lambda(k\alpha_n)) \rightarrow \dots \rightarrow K_0(\check{R}_1^{\mu^1}) = K_0(\mathbb{k}) \neq 0$$

which implies that $K_0(\check{R}_{n+1}^\lambda) = \bigoplus_{k \geq 0} K_0(\check{R}_{n+1}^\lambda(k\alpha_n))$ is non-zero.

The second claim follows from the fact that every indecomposable in $R_{n+1}^\lambda\text{-pmod}$ splits under $\check{\Psi}$ into a direct sum of indecomposables in $\check{R}_{n+1}^\lambda\text{-pmod}$, each one labeled by an element of $\mathcal{S}(\lambda)$, together with the fact that the number of projective indecomposables is the same in both categories and the already established result that the map $K_0(\check{\psi})$ is an isomorphism. $\square$

The results above allow us to give a presentation of the category $R_{n+1}^\lambda\text{-pmod}$ in terms of the Gelfand-Tsetlin basis using the idempotents $e(s)$.

Proposition 5.13. *Every object in $R_{n+1}^\lambda\text{-pmod}$ is isomorphic to a direct summand of some $e(s)P\{\varepsilon_s\}$, for some $s \in \mathcal{S}(\lambda)$ and some shift ε_s .*

Proof. Every object in $\check{R}_{n+1}^\lambda\text{-pmod}$ is a quotient of an object in $R_{n+1}^\lambda\text{-pmod}$. An inductive argument, starting with the modules $_{i_1 \dots i_k; j} L$ of Subsection 4.2, shows that each object $_{e(s)} L$ in $\check{R}_{n+1}^\lambda\text{-pmod}$ has a projective cover in $R_{n+1}^\lambda\text{-mod}$ which coincides with $_{e(s)} P$. The claim follows from this observation together with the Lemmas 3.7, 4.4 and Corollary 4.8. $\square$

Theorem 5.14. *The isomorphism $K_0(R_{n+1}^\lambda) \rightarrow V_\lambda^{\mathfrak{sl}_{n+1}}$ of Theorem 3.8 sends the projective $_{e(s)} P$ to the Gelfand-Tsetlin basis element $|s\rangle$.* $\square$

This basis is not orthogonal with respect to the q -Shapovalov form, but it can be used to redefining another bilinear form $(\ , \)$ on $V_\lambda^{\mathfrak{sl}_{n+1}}$ as

$$([P], [P']) := \text{gdim Hom}_{\check{R}_{n+1}^\lambda\text{-mod}}(P, P')$$

for P, P' objects in $R_{n+1}^\lambda\text{-pmod}$. Clearly we have $(_{e(s)} P, _{e(s')} P) = 0$ if $s \neq s'$.

5.3. A functorial realization of the Gelfand-Tsetlin basis. For each $s \in \mathcal{S}(\lambda)$ we also have functors

$$\Pi^s := \Pi^{\mu^1} \dots \Pi^{\mu^n} \Pi^\lambda : R_{n+1}^\lambda\text{-mod} \rightarrow \mathbb{k}\text{-mod},$$

$$\tilde{\Pi}^s := \tilde{\Pi}^{\mu^1} \dots \tilde{\Pi}^{\mu^n} \tilde{\Pi}^\lambda : \tilde{R}_{n+1}^\lambda\text{-mod} \rightarrow \mathbb{k}\text{-mod}$$

and

$$\widetilde{\text{res}}^s := \widetilde{\text{res}}^\lambda \widetilde{\text{res}}^{\mu^n} \dots \widetilde{\text{res}}^{\mu^1} : \mathbb{k}\text{-mod} \rightarrow \tilde{R}_{n+1}^\lambda\text{-mod}$$

with the obvious definition of the categories $\tilde{R}_j^{\mu^j}\text{-mod}$.

Lemma 5.15. *Functors Π^s have orthogonal hom-spaces, in the sense that for an R_{n+1}^λ -module M we have that $\text{Hom}_{\oplus_{d \in \lambda} \mathbb{k}\text{-mod}}(\Pi^s(M), \Pi^{s'}(M)) = 0$ if $s \neq s'$.* $\square$

Let $\mathcal{GT}(\lambda)$ denote the category of functors

$$\text{Fun} : R_{n+1}^\lambda\text{-mod} \rightarrow \mathbb{k}\text{-mod}$$

There are endofunctors acting on $\mathcal{GT}(\lambda)$ defined by

$$F_i^{\mathcal{GT}} \phi(M) := \bigoplus_{r, s \in \mathcal{S}(\mu^{i-1})} \tilde{\Pi}^r F_i^{\mu^{i-1}} \widetilde{\text{res}}^s \phi(M)$$

and

$$E_i^{\mathcal{GT}} \phi(M) := \bigoplus_{r, s \in \mathcal{S}(\mu^{i-1})} \tilde{\Pi}^r E_i^{\mu^{i-1}} \widetilde{\text{res}}^s \phi(M)$$

for ϕ a functor $R_{n+1}^\lambda\text{-mod} \rightarrow \mathbb{k}\text{-mod}$, M an R_{n+1}^λ -module and $i = 1, \dots, n$.

Lemma 5.16. *Each pair of functors $F_i^{\mathcal{GT}}, E_i^{\mathcal{GT}}$ is biadjoint.*

Proposition 5.17. *The collection of functors $\{F_i^{\mathcal{GT}}, E_i^{\mathcal{GT}}\}_{i=1, \dots, n}$ defines a categorical $\mathfrak{sl}_{n+1}$ -action on $\mathcal{GT}(\lambda)$.* $\square$

Conjecture 5.18. *We have an isomorphism of $\mathfrak{sl}_{n+1}$ -modules $K_0(\mathcal{GT}(\lambda)) \cong V_\lambda^{\mathfrak{sl}_{n+1}}$ that takes Π^s to the Gelfand-Tsetlin basis element $|s\rangle$.*

5.4. The cyclotomic quotient conjecture revisited. We can now give an elementary proof of the Khovanov-Lauda cyclotomic conjecture in Type A.

Theorem 5.19. *We have an isomorphism of $\mathfrak{sl}_{n+1}$ -modules*

$$K_0(R_{n+1}^\lambda) \cong V_\lambda^{\mathfrak{sl}_{n+1}}.$$

Proof. Recall that from Proposition 4.6 we have a surjection of $\mathfrak{sl}_{n+1}$ -representations

$$K_0(\Pi^\lambda): K_0(R_{n+1}^\lambda) \rightarrow \bigoplus_{\xi_{i_1 \dots i_k} \in \mathcal{D}_\lambda} K_0(R_n^{\xi_{i_1 \dots i_k}(\lambda)})$$

and so, if we know that $K_0(R_n^{\xi_{i_1 \dots i_k}(\lambda)}) \cong V_{\xi_{i_1 \dots i_k}(\lambda)}^{\mathfrak{sl}_n}$, we are done. The cyclotomic conjecture for $\mathfrak{sl}_{n+1}$ follows from the cyclotomic conjecture for $\mathfrak{sl}_2$ by recursion, which in turn is a consequence of the fact that we have $1_{\bar{\mu}+1} = 0$ in R_2^μ , where $1_{\bar{\mu}+1} = 0$ is the diagram consisting of $\bar{\mu} + 1$ vertical parallel strands. $\square$

6. CYCLOTOMIC KLR ALGEBRAS CATEGORIFY WEYL MODULES

6.1. The q -Schur categorification. In [23] a diagrammatic categorification of the q -Schur algebra was constructed using a quotient of Khovanov and Lauda's categorified quantum groups from [17, 18]. Khovanov and Lauda's categorified quantum $\mathfrak{sl}_n$ consists of a 2-category $\mathcal{U}(\mathfrak{sl}_n)$ defined from the following data. The objects are *weights* $\lambda \in \mathbb{Z}^{n-1}$. The 1-morphisms are products of symbols $\lambda' \mathcal{F}_i \lambda$ (with $\lambda'_j = \lambda_j + 1$ if $j = i \pm 1$, $\lambda'_j = \lambda_j - 2$ if $j = i$, and $\lambda'_j = \lambda_j$ otherwise) and $\lambda' \mathcal{E}_i \lambda$ (with $\lambda'_j = \lambda_j - 1$ if $j = i \pm 1$, $\lambda'_j = \lambda_j + 2$ if $j = i$, and $\lambda'_j = \lambda_j$ otherwise) with the convention that says that $\lambda' \mathcal{F}_i \mu \nu \mathcal{F}_i \lambda$ and $\lambda' \mathcal{E}_i \mu \nu \mathcal{E}_i \lambda$ are zero unless $\mu = \nu$. The 2-morphisms of $\mathcal{U}(\mathfrak{sl}_n)$ are given by planar diagrams in a strip generated by oriented arcs that can intersect transversely and can be decorated with dots (closed oriented 1 manifolds are allowed). This graphical calculus is a generalization of the KLR algebras to a calculus where the strands can travel in all directions in the sense that it gives the KLR diagrammatics when we restrict strands to travel only downwards. The boundary of each arc is decorated with a 1-morphism. These 2-morphisms are subject to a set of relations which we do not give here (see [17, 23] for details).

In [23] Khovanov and Lauda's categorified quantum $\mathfrak{sl}_n$ was upgraded to a categorification $\mathcal{U}(\mathfrak{gl}_n)$ of quantum $\mathfrak{gl}_n$ (taking Khovanov and Lauda's diagrams and relations of $\mathcal{U}(\mathfrak{sl}_n)$ with $\mathfrak{gl}_n$ -weights) and define the categorification of $S_q(n, d)$ as the quotient of $\mathcal{U}(\mathfrak{gl}_n)$ by 2-morphisms factoring through a weight not in $\Lambda(n, d)$.

Definition 6.1. The category $\mathcal{S}(n, d)$ is the quotient of $\mathcal{U}(\mathfrak{gl}_n)$ by the ideal generated by all 2-morphisms containing a region with a label not in $\Lambda(n, d)$.

The main result of [23] is that $\mathcal{S}(n, d)$ categorifies the q -Schur algebra from Subsection 2.2.

Theorem 6.2 ([23]). *There is an isomorphism of $\mathbb{Q}(q)$ -algebras*

$$\gamma: \dot{\mathcal{S}}(n, d) \xrightarrow{\cong} K_0(\text{Kar}(\mathcal{S}(n, d))).$$

6.2. Categorical Weyl modules. Recall that

$$W_\lambda \cong 1_\lambda \dot{\mathbf{S}}(n, d) / [\mu > \lambda]$$

where “ $>$ ” is the lexicographic order, is an irreducible for $\dot{\mathbf{S}}(n, d)$ and that all irreducibles can be obtained this way. It was conjectured in [23] that it is easy to categorify the irreducible representations W_λ , for $\lambda \in \Lambda^+(n, d)$, using the category $\mathcal{S}(n, d)$.

Definition 6.3. For any $\lambda \in \Lambda^+(n, d)$ let $1_\lambda \mathcal{S}(n, d)$ be the category whose objects are the 1-morphisms in $\mathcal{S}(n, d)$ of the form $1_\lambda x$ and whose morphisms are the 2-morphisms in $\mathcal{S}(n, d)$ between such 1-morphisms. Note that $1_\lambda \mathcal{S}(n, d)$ does not have a monoidal structure because two 1-morphisms $1_\lambda x$ and $1_\lambda y$ cannot be composed in general. Alternatively one can see $1_\lambda \mathcal{S}(n, d)$ as a graded ring whose elements are the morphisms.

Definition 6.4. Let $\mathcal{V}_\lambda$ be the quotient of $1_\lambda \mathcal{S}(n, d)$ by the ideal generated by all diagrams which contain a region labeled by $\mu > \lambda$.

There is a natural categorical action of $\mathcal{S}(n, d)$ (and therefore of $\mathcal{U}(\mathfrak{sl}_n)$) on $\mathcal{V}_\lambda$ defined by putting a diagram in $\mathcal{S}(n, d)$ on the right-hand side of a diagram in $\mathcal{V}_\lambda$. This action descends to an action of $\dot{\mathbf{S}}(n, d) \cong K_0(\text{Kar } \mathcal{S}(n, d))$ on $K_0(\text{Kar } (\mathcal{V}_\lambda))$. The map γ from Theorem 6.2 induces a well-defined linear map $\gamma_\lambda: W_\lambda \rightarrow K_0(\text{Kar } (\mathcal{V}_\lambda))$ which intertwines the $\dot{\mathbf{S}}(n, d)$ -actions. It was proven in [23] that γ_λ is surjective and it was conjectured that it is an isomorphism. Since W_λ is irreducible, we have $K_0(\text{Kar } (\mathcal{V}_\lambda)) \cong V_\lambda$ or $K_0(\text{Kar } (\mathcal{V}_\lambda)) = 0$. So it suffices to show that $K_0(\text{Kar } (\mathcal{V}_\lambda)) \neq 0$.

From now on we regard R_{n+1}^λ as the category whose objects are sequences of simple roots and morphisms are KLR diagrams. Let $\mathcal{N}_\lambda$ be the two-sided ideal generated by diagrams of R_{n+1}^λ containing a bubble of positive degree in its left-most region.

Definition 6.5. The category $\check{\mathcal{V}}_\lambda$ is the quotient of $\mathcal{V}_\lambda$ by $\mathcal{N}_\lambda$.

The ideal $\mathcal{N}_\lambda$ is virtually-nilpotent and therefore $\check{\mathcal{V}}_\lambda$ has the same Grothendieck group as $\mathcal{V}_\lambda$ (see [23, Sec. 7]) where it was also explained that this quotient satisfies the cyclotomic condition from Definition 3.6. In [23] the authors defined a functor from R_{n+1}^λ to $\check{\mathcal{V}}_\lambda$ which is the identity on objects and morphisms. Roughly speaking it just sends pictures to pictures where the strands in the diagrams of R_{n+1}^λ are seen as secretly oriented downwards. This functor is clearly full and essentially surjective and it was conjectured to be faithful. We denote this functor by Φ_λ . The main result of this section is the following.

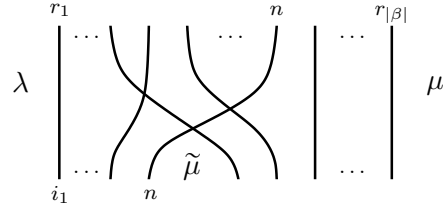
Theorem 6.6. *The functor Φ_λ is faithful and therefore an equivalence of categories.*

Proof. We can decorate the regions of the diagrams of R_{n+1}^λ with $\mathfrak{gl}_{n+1}$ -weights, starting with a λ in the leftmost region and subtracting $\varepsilon_j - \varepsilon_{i+1}$ any time we cross a strand labeled j . In other words, if the region on the left of strand labeled j is decorated with the weight λ' , then the label of its right neighboring region is $\lambda' - \varepsilon_j + \varepsilon_{j+1} = (\lambda'_1, \dots, \lambda'_j - 1, \lambda'_{j+1} + 1, \dots, \lambda'_{n+1})$.

We first prove that, if $X \in R_{n+1}^\lambda(\beta)$ contains a region labeled by $\mu \notin \Lambda^{\mathfrak{gl}_{n+1}}$, then $X = 0$. It is enough to assume that μ is the label of its rightmost region. Moreover, we can assume that

$\mu_{n+1} < 0$. For suppose $\mu_j < 0$ and $\mu_i \geq 0$ for $i > k$. Then we can use the decomposition in (18) and the fact that Π^λ is injective on objects to obtain an array of diagrams, each one in a distinct $R_n^{\xi_{i_1 \dots i_k}(\lambda)}$, but all having the weight $(\mu_1, \dots, \mu_n)$ in its rightmost region. A recursive application of this procedure yields therefore an array of diagrams in a direct sum of cyclotomic KLR algebras $\bigoplus_{\zeta} R_j^\zeta$, all of them with the rightmost region decorated with $(\mu_1, \dots, \mu_j)$. We can assume further that X is of the form $1_{\underline{r}}$ for some sequence r of simple roots.

Assume that one of the components $\Pi^{\xi_{i_1 \dots i_k}(\lambda)} 1_{\underline{r}}$ is non-zero. Then we have a non-zero diagram in R_m^λ connecting the special idempotent $e(p_{i_1 \dots i_k}, \underline{r}')$ to $1_{\underline{r}}$, as in



Recall that the strands ending in $p_{i_1 \dots i_k}$ in the bottom do not cross each other nor carry any dots. All strands labeled n must end at the bottom among the ones corresponding to $p_{i_1 \dots i_k}$. We label $\tilde{\mu}$ the region close to the bottom of $1_{\underline{r}}$ and immediately at the right of the last strand labeled n , counted from the left.

Let $|\alpha_{n-1}|_\beta$ and $|\alpha_n|_\beta$ be the number of strands labeled $n-1$ and n in $R_m^\lambda(\beta)$ respectively. Since $\mu_n < 0$ we must have $|\alpha_{n-1}|_\beta < |\alpha_n|_\beta$ which means that $\tilde{\mu}_n \leq \mu_n < 0$. This implies that $e(p_{\underline{s}}, \underline{r}')$ is not a special idempotent, which is a contradiction. This forces the component $\Pi^{\xi_{i_1 \dots i_k}(\lambda)} 1_{\underline{r}}$ to be the zero diagram. The reasoning above applies to all components $\Pi^{\xi_{i_1 \dots i_k}(\lambda)} 1_{\underline{r}}$ and altogether, it implies that $\Pi^\lambda 1_{\underline{r}} = 0$. Since Π^λ is injective, we conclude that $1_{\underline{r}} = 0$ in $R_{n+1}^\lambda(\beta)$. $\square$

Corollary 6.7. *We have an isomorphism of $\dot{S}(n, d)$ representations*

$$K_0(\text{Kar}(\mathcal{V}_\lambda)) \cong W_\lambda.$$

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REFERENCES

- [1] S. Ariki, On the decomposition numbers of the Hecke algebra of $G(m, 1, n)$. *J. Math. Kyoto Univ.* (2000), 36(4):789-808.
- [2] A. Beilinson, G. Lusztig and R. MacPherson, A geometric setting for the quantum deformation of $\mathfrak{gl}_n$. *Duke Math. J.* 61(2) (1990), 655-677.
- [3] J. Brundan and A. Kleshchev, Blocks of cyclotomic Hecke algebras and Khovanov-Lauda algebras. *Invent. Math.* (2009) 178(3), 451-484.
- [4] J. Brundan and A. Kleshchev, Graded decomposition numbers for cyclotomic Hecke algebras. *Adv. Math.* (2009) 222, 1883-1942.
- [5] J. Brundan, A. Kleshchev and W. Wang, Graded Specht modules. *J. Reine Angew. Math.* 655 (2011), 61-87.

- [6] G. Black, R. King and B. Wybourne, Kronecker products for compact semisimple Lie groups. *J. Phys. A: Math. and Gen.* (1983) 16:8, 1555-1589.
- [7] I. Cherednik, A new interpretation of Gelfand-Tsetlin bases. *Duke Math. J.* (1987) 54(2):563-577.
- [8] I. M. Gelfand and M. L. Tsetlin, Finite-dimensional representations of the group of unimodular matrices. *Dokl. Akad. Nauk SSSR* 71 (1950), 825-828 (Russian). English transl. in: I. M. Gelfand, Collected papers. Vol II, Berlin: Springer-Verlag 1988, pp. 653-656.
- [9] M. Gorelik and V. Kac, On complete reducibility for infinite-dimensional Lie algebras. *Adv. Math.* (2011) 226(2): 1911-1972.
- [10] D. Hill, G. Melvin and D. Mondragon, Representations of quiver-Hecke algebras via Lyndon bases. *J. Pure Appl. Algebra* 216 (2012) 1052-1079.
- [11] A. Hoffnung and A. Lauda, Nilpotency in type A cyclotomic quotients. *J. Algebr. Comb.* (2010) 32: 533-555.
- [12] C. Itzykson and M. Nauenberg, Unitary groups: representations and decompositions. *Rev. Mod. Phys.* (1966) 38: 95-120.
- [13] F. Jaeger, Composition products and models for the HOMFLY polynomial. *L'Enseign. Math.* 35 (1989) 323-361.
- [14] S. Kang and M. Kashiwara, Categorification of highest weight modules via Khovanov-Lauda-Rouquier algebras. *Invent. Math.* (2012) 190, 399-742.
- [15] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups I. *Repres. Theo.* 13 (2009), 309-347.
- [16] M. Khovanov and A. Lauda, A diagrammatic approach to categorification of quantum groups II. *Trans. Amer. Math. Soc.* 363 (2011), no. 5, 2685-2700.
- [17] M. Khovanov and A. Lauda, A categorification of quantum $\mathfrak{sl}(n)$. *Quantum Topol.* 1 (2010), no. 1, 1-92.
- [18] M. Khovanov and A. Lauda, Erratum to: "A categorification of quantum $\mathfrak{sl}(n)$ ". *Quantum Topol.* 2 (2011), no. 1, 97-99.
- [19] A. Kleshchev and A. Ram, Homogeneous representations of Khovanov-Lauda algebras. *J. Eur. Math. Soc.* 12 (2010), 1293-1306.
- [20] A. Kleshchev and A. Ram, Representations of Khovanov-Lauda-Rouquier algebras and combinatorics of Lyndon words. *Math. Ann.* 349 (2010), 943-975.
- [21] W. Klink and T. Ton-That, Invariant theory of tensor product decompositions of $U(N)$ and generalized Casimir products. *Notices of the AMS* 56:8 (2009), 931-940.
- [22] A. I. Molev, Gelfand-Tsetlin bases for classical Lie algebras. *Handbook of algebra*, vol 4, (M. Hazewinkel, Ed.), Elsevier, 109-170.
- [23] M. Mackaay, M. Stošić and P. Vaz, A diagrammatic categorification of the q -Schur algebra, *Quantum Topol.* 4 (2013) 1-75.
- [24] R. Rouquier, 2-Kac-Moody algebras. arXiv:0812.5023.
- [25] R. Rouquier, Quiver Hecke algebras and 2-Lie algebras. *Algebra colloquium* 19 (2012), 354-410.
- [26] P. Vaz, On Jaeger's HOMFLY-PT expansions, branching rules and link homology: a progress report. (to appear in *Boletim Soc. Portuguesa Mat.*) (2012).
- [27] P. Vaz and E. Wagner, A remark on BMW algebra, q -Schur algebras and categorification. *Canad. J. Math.*, 66 (2014), no. 2, 453-480. <http://dx.doi.org/10.4153/CJM-2013-018-1>
- [28] E. Wagner, Khovanov-Rozansky graph homology and composition product. *J. Knot Theor. Ramif.* 17-12 (2008) 1549-1559.
- [29] B. Webster, Knots and higher representation theory I: diagrammatic and geometric categorification of tensor products. arXiv:1001.2020.
- [30] D. Želobenko, Compact Lie groups and their representations. *Translations of Mathematical Monographs* 40 (1973), AMS.

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