The equation governing the evolution of the surface age distribution function of sea ice

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Introduction

The age of sea ice is usually defined as the time elapsed since accretion took place. Different approaches thereof have been suggested over past decades, focusing on either the surface or the volumetric aspects of the age distribution.

Lietaer et al. (2011) developed a method for estimating the vertical profile of sea ice age: the volumetric aspects of the age distribution were key in this study. This method leads to variables that cannot be compared straightforwardly with satellite data, since the latter provide little or no information about ice thickness. Fowler et al. (2004), on the other hand, used satellite data to track ice motion, leading to surface ice age estimates.

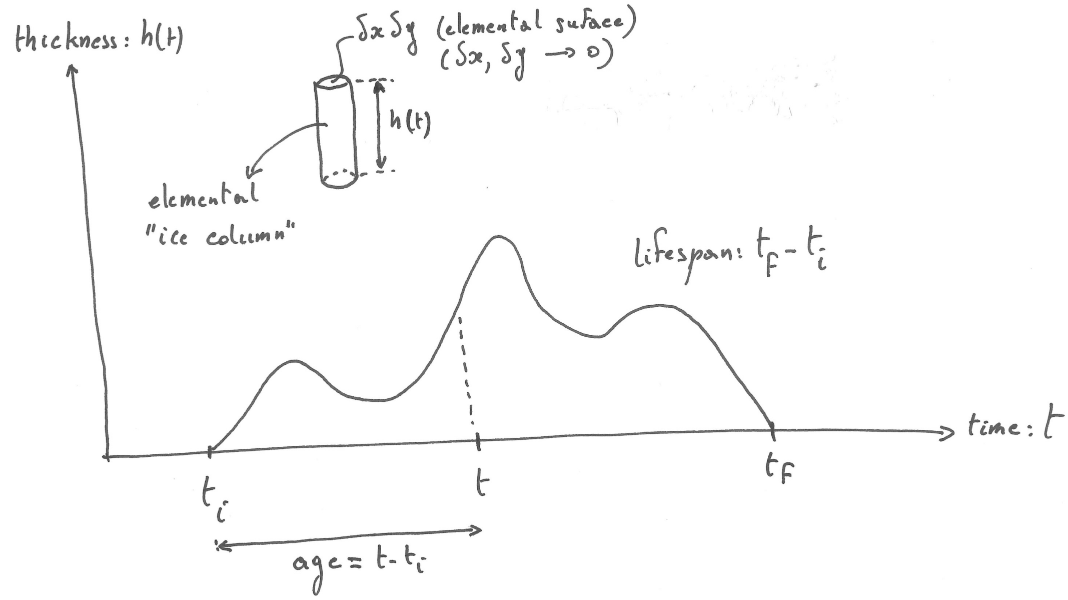

Figure 1. Illustration of the concept of ice age as is defined herein. The age of an elemental "ice column" is the time elapsed since ice was first produced (at time $t = t_i$) and is independent of subsequent ice accretion or melting. This holds valid until the aforementioned "ice column" is completely melted (at time $t = t_f$), i.e. for its whole lifespan $t_f - t_i$.

The present working note aims at deriving the equation from which the surface ice age distribution function may be computed. Accordingly, the age of an elemental "column of ice" starts increasing, at the same pace as time progresses, as soon as ice is first produced and is insensitive to subsequent accretion or melting processes as long as the aforementioned "ice column" is not entirely melted (Figure 1). Clearly, the present approach is markedly different from that of Lietaer et al. (2001). Its central purpose is to make it possible to easily compare model- and satellite-derived sea ice ages.

The surface ice age distribution function

Let t and $\mathbf{x} = (x, y)$ denote the time and the horizontal position vector, where x and y are horizontal, Cartesian coordinates. Consider a motionless elemental control surface located at $\mathbf{x}$ at time t, whose area is δS. The **sea ice concentration** $C(t,\mathbf{x})$ is a measure of the fraction of the area of the abovementioned control surface over which sea ice is present: at time t, the area covered by sea ice tends to $C(t,\mathbf{x})\delta S$ in the limit $\delta S \to 0$. Clearly, the sea ice concentration is a dimensionless variable, which, at any time and location, satisfies the inequalities $0 \le C(t,\mathbf{x}) \le 1$. When the control surface is free of ice, the concentration is zero, whilst the concentration is equal to unity if it is fully covered by sea ice.

To work out equations from which the age of sea ice may be obtained, inspiration may be sought in the developments carried out in the framework of the Constituent-oriented Age and Residence time Theory (CART, www.climate.be/cart_flyer). Accordingly, it is appropriate to introduce the **surface ice age distribution function**, $c(t,\mathbf{x},\tau)$, where τ $(0 \le \tau < \infty)$ is an independent variable representing the age. The distribution function is defined as follows: in the abovementioned elemental control surface, the area covered by sea ice whose age lies in the interval $[\tau, \tau + \delta\tau]$ tends to $c(t,\mathbf{x},\tau)\delta S \delta\tau$ in the limit $\delta S \to 0$ and $\delta\tau \to 0$. It is readily seen that the physical dimension of $c(t,\mathbf{x},\tau)$ is time^{-1}. The age distribution function may be seen as the histogram of the age.

The area covered by sea ice in the control surface under consideration is obtained by taking the sum of all the age classes elemental areas:

$$C(t,\mathbf{x})\delta S \sim \sum_{\tau=0}^{\infty} c(t,\mathbf{x},\tau)\delta S \delta\tau \quad , \quad \delta S, \delta\tau \to 0 \quad . \tag{1}$$

In the limit $\delta\tau \to 0$, this asymptotic expression transforms to

$$C(t,\mathbf{x})\,\delta S = \int_0^{\infty} c(t,\mathbf{x},\tau)\,\delta S\, d\tau \quad , \tag{2}$$

which simplifies to

$$C(t,\mathbf{x}) = \int_0^{\infty} c(t,\mathbf{x},\tau)\, d\tau \quad . \tag{3}$$

The **mean age of the ice**, $a(t,\mathbf{x})$, is to be obtained from some form of average involving the age distribution function. Here, the most natural type of mean probably is the **surface-weighted average**, i.e.

$$a(t,\mathbf{x}) \sim \frac{\sum_{\tau=0}^{\infty} \tau\, c(t,x,\tau)\, \delta S\, \delta\tau}{\sum_{\tau=0}^{\infty} c(t,x,\tau)\, \delta S\, \delta\tau} \sim \frac{\sum_{\tau=0}^{\infty} \tau\, c(t,x,\tau)\, \delta\tau}{\sum_{\tau=0}^{\infty} c(t,x,\tau)\, \delta\tau} \quad , \quad \delta S, \delta\tau \to 0 \quad . \tag{4}$$

In the limit $\delta\tau \to 0$, (4) transforms to

$$a(t,\mathbf{x}) = \frac{\int_0^{\infty} \tau\, c(t,\mathbf{x},\tau)\, d\tau}{\int_0^{\infty} c(t,\mathbf{x},\tau)\, d\tau} \tag{5}$$

It must be realised that this formulation is somewhat arbitrary, for another type of mean (e.g. a mass- or volume-weighted one) could have been resorted to. For instance, to estimate the mean age of the particles of a dissolved constituent, the usual CART practice is to have recourse to a mass-weighted average (e.g. Deleersnijder et al. 2001).

According to relation (5), the mean age is the ratio of the first-order moment of the age distribution to the zeroth-order one. The latter is the ice concentration, whilst the former may be called the **surface age concentration** in line with usual CART's denominations (e.g. Delhez et al. 1999, Deleersnijder et al. 2001), i.e.

$$\alpha(t,\mathbf{x}) = \int_0^{\infty} \tau\, c(t,\mathbf{x},\tau)\, d\tau \quad . \tag{6}$$

Obviously, the physical dimension of the age concentration $\alpha(t,\mathbf{x})$ is that of a timescale. Finally, combining relations (3), (5) and (6) yields

$$\text{mean age} = a(t,\mathbf{x}) = \frac{\alpha(t,\mathbf{x})}{C(t,\mathbf{x})} = \frac{\text{age concentration of sea ice}}{\text{concentration of sea ice}} \tag{7}$$

Ice-covered area budget

The equation governing the age distribution function of a dissolved constituent is obtained from rather straightforward mass budget considerations (Delhez et al. 1999). This approach obviously is unsuited to the needs of the present study. Instead, it is the budget of the ice-covered area that is to be made. During the time interval $[t, t+\delta t]$ ($\delta t \to 0$), this budget is to be examined over the following elemental three-dimensional control domain

$$[x-\delta x/2, x+\delta x/2] \times [y-\delta y/2, y+\delta y/2] \times [\tau-\delta\tau/2, \tau+\delta\tau/2] \quad , \tag{8}$$

with $\delta x, \delta y, \delta\tau \to 0$.

The area covered with ice in this control domain evolves under the influence of ice accretion and melting as well as transport process in the physical space and the age direction. The accretion (or formation) and melting rates are denoted below $f(t,\mathbf{x},\tau)$ and $m(t,\mathbf{x},\tau)$, respectively. Both these functions are non-negative and their physical dimension is time^{-2}. Upon denoting q_x, q_y and q_τ the fluxes along the x-, y- and τ-axes, the area covered with ice in control volume (8) evolves according to the following asymptotic expression:

$$
\begin{aligned}
& c(t+\delta t,\mathbf{x},\tau)\delta x\,\delta y\,\delta\tau \;-\; c(t,\mathbf{x},\tau)\delta x\,\delta y\,\delta\tau \\
&\qquad \sim\; f(t,\mathbf{x},\tau)\delta x\,\delta y\,\delta\tau\,\delta t \;-\; m(t,\mathbf{x},\tau)\delta x\,\delta y\,\delta\tau\,\delta t \\
&\qquad\qquad -\; \left[q_x\right]_{x-\delta x/2}^{x+\delta x/2}\delta y\,\delta\tau\,\delta t \;-\; \left[q_y\right]_{y-\delta y/2}^{y+\delta y/2}\delta x\,\delta\tau\,\delta t \;-\; \left[q_\tau\right]_{\tau-\delta\tau/2}^{\tau+\delta\tau/2}\delta x\,\delta y\,\delta t \\
&\qquad\qquad\qquad (\delta x,\delta y,\delta\tau,\delta t \to 0)
\end{aligned}
\tag{9}
$$

The terms in the above equation are to be interpreted as follows:

- $c(t+\delta t,\mathbf{x},\tau)\delta x\,\delta y\,\delta\tau$: ice-covered area whose age lies in the interval $[\tau-\delta\tau/2,\tau+\delta\tau/2]$ at time $t+\delta t$;
- $c(t,\mathbf{x},\tau)\delta x\,\delta y\,\delta\tau$: ice-covered area whose age lies in the interval $[\tau-\delta\tau/2,\tau+\delta\tau/2]$ at time t;
- $f(t,\mathbf{x},\tau)\delta x\,\delta y\,\delta\tau\,\delta t$: area of ice whose age lies in the interval $[\tau-\delta\tau/2,\tau+\delta\tau/2]$ formed during time interval $[t,t+\delta t]$;
- $m(t,\mathbf{x},\tau)\delta x\,\delta y\,\delta\tau\,\delta t$: area of ice whose whose age lies in the interval $[\tau-\delta\tau/2,\tau+\delta\tau/2]$ melted during time interval $[t,t+\delta t]$;
- $-\left[q_x\right]_{x-\delta x/2}^{x+\delta x/2}\delta y\,\delta\tau\,\delta t$: area of ice whose age lies in the interval $[\tau-\delta\tau/2,\tau+\delta\tau/2]$ that entered the control domain during time interval $[t,t+\delta t]$ due to the net action of transport processes acting in the *x*-direction (inflow - outflow);
- $-\left[q_y\right]_{y-\delta y/2}^{y+\delta y/2}\delta x\,\delta\tau\,\delta t$: area of ice whose age lies in the interval $[\tau-\delta\tau/2,\tau+\delta\tau/2]$ that entered the control domain during time interval $[t,t+\delta t]$ due to the action of transport processes acting in the *y*-direction (inflow - outflow);
- $-\left[q_\tau\right]_{y-\delta y/2}^{y+\delta y/2}\delta x\,\delta y\,\delta t$: area of ice whose age lies in the interval $[\tau-\delta\tau/2,\tau+\delta\tau/2]$ that entered the control domain during time interval $[t,t+\delta t]$ due to **ageing**, i.e. the process by which the age of ice tends to increase as time progresses.

By means of elementary algebraic manipulations, one can transform asymptotic expression (9) to

$$
\frac{c(t+\delta t,\mathbf{x},\tau)-c(t,\mathbf{x},\tau)}{\delta t} \sim f \;-\; m \;-\; \frac{\left[q_x\right]_{x-\delta x/2}^{x+\delta x/2}}{\delta x} \;-\; \frac{\left[q_y\right]_{y-\delta y/2}^{y+\delta y/2}}{\delta y} \;-\; \frac{\left[q_\tau\right]_{y-\delta y/2}^{y+\delta y/2}}{\delta\tau}
\tag{10}
$$

Taking the limit $\delta x,\delta y,\delta\tau,\delta t \to 0$ of (10) yields the generic form of the equation governing the evolution of the surface age distribution function:

$$
\frac{\partial c}{\partial t} = f \;-\; m \;-\; \frac{\partial q_x}{\partial x} \;-\; \frac{\partial q_y}{\partial y} \;-\; \frac{\partial q_\tau}{\partial \tau}
\tag{11}
$$

Formulation of the physical-space and ageing fluxes

The flux in the physical space is likely to consist of a resolved (i.e. advective) component, related to horizontal ice velocity $\mathbf{u}=(u_x,u_y)$, and a subgrid-scale or unresolved one. Assuming that diffusive processes can be parameterised *à la Fourier-Fick*, the most general expression of the diffusive flux must resort to a relevant diffusivity tensor,

$$\mathbf{K} = \begin{pmatrix} K_{xx} & K_{xy} \\ K_{yx} & K_{yy} \end{pmatrix} , \qquad (12)$$

which must be symmetric ($K_{xy}=K_{yx}$) and positive definite, requiring that inequalities $K_{xx}+K_{yy}>0$ and $K_{xx}K_{yy}>K_{xy}^2$ be satisfied. Then, the following parameterisation is to be introduced:

$$\begin{cases} q_x = cu_x - K_{xx}\dfrac{\partial c}{\partial x} - K_{xy}\dfrac{\partial c}{\partial y} \\ q_y = cu_y - K_{yx}\dfrac{\partial c}{\partial x} - K_{yy}\dfrac{\partial c}{\partial y} \end{cases} \qquad (13a)$$

It is often convenient to cast vector $\mathbf{q}=(q_x,q_y)$ into the compact form

$$\mathbf{q} = c\mathbf{u} - \mathbf{K}\bullet\nabla c \ . \qquad (13b)$$

If diffusion may be regarded as isotropic, then the diffusivity tensor simplifies to $\mathbf{K}=K\,\mathbf{I}$, where $\mathbf{I}$ is the identity tensor[1] and K (>0) is the relevant diffusivity. In this case, the diffusive flux reads $-\mathbf{K}\bullet\nabla c=-K\nabla c=-K\left(\partial c/\partial x,\partial c/\partial y\right)$. Needless to say, if diffusive processes are considered negligible, the physical-space flux is purely advective, i.e. $\mathbf{q}=c\,\mathbf{u}=\left(cu_x,cu_y\right)$.

The ageing flux q_τ takes the form of an advection term with a unit velocity:

$$q_\tau = c \ . \qquad (14)$$

This is due to ageing proceeding at the same pace as time progresses at any time and location. In other words, during the time interval $[t,t+\delta t]$, the age of the ice tends to increase by δt at any location whatever the value of t. This holds true for any ageing term, whatever the context in which it is used, as is widely acknowledged at least since the study of Bolin and Rodhe (1973).

Substituting (13)-(14) into generic equation (11) yields

$$\frac{\partial c}{\partial t} = f - m - \frac{\partial}{\partial x}\left(cu_x - K_{xx}\frac{\partial c}{\partial x} - K_{xy}\frac{\partial c}{\partial y}\right) - \frac{\partial}{\partial y}\left(cu_y - K_{yx}\frac{\partial c}{\partial x} - K_{yy}\frac{\partial c}{\partial y}\right) - \frac{\partial c}{\partial \tau} \qquad (15a)$$

or, equivalently,

$$\frac{\partial c}{\partial t} = f - m - \nabla\bullet(c\mathbf{u}-\mathbf{K}\bullet\nabla c) - \frac{\partial c}{\partial \tau} \qquad (15b)$$

Final form of the governing equations

Elementary physical reasoning suggests that, at any time and location in the domain of interest, the age distribution function must satisfy the boundary condition

[1] $\mathbf{I}=\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\left[c(t,\mathbf{x},\tau)\right]_{\tau=0} = 0 = \left[c(t,\mathbf{x},\tau)\right]_{\tau=\infty} \ . \tag{16}$$

Then, using (16) and definition (3), the following results are easily obtained:

$$\int_0^\infty \frac{\partial c}{\partial \tau} d\tau = \left[c(t,\mathbf{x},\tau)\right]_{\tau=0}^{\tau=\infty} = 0 \ , \tag{17}$$

$$\int_0^\infty \tau \frac{\partial c}{\partial \tau} d\tau = \underbrace{\int_0^\infty \frac{\partial(\tau c)}{\partial \tau} d\tau}_{=\left[\tau c\right]_{\tau=0}^{\tau=\infty}=0} - \underbrace{\int_0^\infty c \, d\tau}_{\substack{=C \\ \text{see (3)}}} = -C \ . \tag{18}$$

As will be seen, it is convenient to introduce the following functions of time and position:

$$\left(F,\mathcal{F}\right) = \int_0^\infty \left(f,\tau f\right) d\tau \ . \tag{19}$$

In the elemental control surface δS located at $\mathbf{x}$, the area of ice formed during the time interval $[t,t+\delta t]$ tends to $F(t,\mathbf{x})\delta S \delta t$ as $\delta S \to 0$ and $\delta t \to 0$. The age of the newly-formed ice is prescribed to be zero, implying that

$$f(t,\mathbf{x},\tau) = F(t,\mathbf{x})\delta(\tau - 0) \ , \tag{20}$$

where $\delta(\tau - 0)$ denotes the Dirac delta function[2]. This also leads to

$$\mathcal{F}(t,\mathbf{x}) = 0 \ . \tag{21}$$

It is readily seen that relations (19)-(21) are consistent with eachother.

The counterpart of (19) for ice melting reads

$$\left(M,\mathcal{M}\right) = \int_0^\infty \left(m,\tau m\right) d\tau \ . \tag{22}$$

In the elemental control surface δS located at $\mathbf{x}$, the area of ice, whose age lies in the interval $[\tau,\tau+\delta\tau]$, that is melted during the time interval $[t,t+\delta t]$ tends to $m(t,\mathbf{x})\delta S \delta t \delta\tau$ in the limit $\delta t, \delta S, \delta\tau \to 0$. Therefore, the mean age of the ice melted is

$$a_m(t,\mathbf{x}) = \frac{\int_0^\infty \tau \, m(t,x,\tau) \, \delta S \delta t \, d\tau}{\int_0^\infty m(t,x,\tau) \, \delta S \delta t \, d\tau} = \frac{\int_0^\infty \tau \, m(t,x,\tau) \, d\tau}{\int_0^\infty m(t,x,\tau) \, d\tau} = \frac{\mathcal{M}}{M} \tag{23}$$

If the melting processes were independent of the age of the ice, the following expression would hold valid: $m = Mc/C$. This would lead to $\mathcal{M} = Ma$ and, hence, $a_m = a$. However, melting is likely to be related to the age of the ice. Therefore, no drastic simplification of (23) can be contemplated.

Combining (16) and (20) leads to the final form of the equation obeyed by the surface ice age distribution function:

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[2] It must be borne in mind that the physical dimension of the Dirac function is the inverse of the physical dimension of its argument. Accordingly, the physical dimension of $\delta(\tau - 0)$ is time^{-1}.

$$\frac{\partial c}{\partial t} = F\delta(\tau - 0) - m - \nabla\bullet(c\mathbf{u} - \mathbf{K}\bullet\nabla c) - \frac{\partial c}{\partial \tau} \tag{24}$$

Integrating (24) over the age, using (3), (17) and (19), one obtains the equation governing the evolution of the ice concentration:

$$\frac{\partial C}{\partial t} = F - M - \nabla\bullet(C\mathbf{u} - \mathbf{K}\bullet\nabla C) \tag{25}$$

This equation is in conservative form. To obtain the associated convective form, one must first introduce the material derivative operator,

$$D_t = \frac{\partial}{\partial t} + \mathbf{u}\bullet\nabla , \tag{26}$$

which leads to a measure of the time rate of change that can be estimated by an observer whose velocity is equal to that of the ice, i.e. $\mathbf{u}$. Then, combining (25) and (26) yields

$$D_t C = F - M - (\nabla\bullet\mathbf{u})C + \nabla\bullet(\mathbf{K}\bullet\nabla C) . \tag{27}$$

Thus, as expected, if the divergence of the ice velocity is positive (negative), the ice concentration will tend to decrease (increase).

Multiplying (24) by τ, integrating over the age, using (6), (18), (22) and (23), the equation for the age concentration in conservative form is arrived at:

$$\frac{\partial \alpha}{\partial t} = C - M a_m - \nabla\bullet(\alpha\mathbf{u} - \mathbf{K}\bullet\nabla\alpha) \tag{28}$$

It must be underscored that this relation contains no term directly related to the ice accretion rate. This is because it was assumed that the age of newly-formed ice is zero, implying that the associated ice area does not contribute to the global age content budget. However, as is seen below, the mean age is impacted by ice formation. The convective form of the age concentration equation reads

$$D_t\alpha = C - M a_m - (\nabla\bullet\mathbf{u})\alpha + \nabla\bullet(\mathbf{K}\bullet\nabla\alpha) . \tag{29}$$

If one is chiefly interested in the surface ice age distribution function, then equation (24) must be dealt with. This is unlikely to be straightforward. This is because there are four independent variables to be taken into account, i.e. the time, two space coordinates (x,y) and the age (τ). If, on the other hand, only the mean age needs be evaluated, then one should solve (25) and (28) and, then, derive the mean age by means of (7), i.e. $a = \alpha / C$. It must be pointed out that concentration and age concentration equations (25) and (28) are standard reactive transport equations, which exhibit similar transport terms and involve only three independent variables (t, x, y). Obviously, (25) and (28) are likely to be much easier to tackle than (24).

Combining (7), (25) and (28) the equation obeyed by the mean age may be derived. Unfortunately, the latter cannot be cast into a conservative form similar to (25) or (28), which is precisely why it is not advisable to solve this equation numerically. The mean age equation is as follows:

$$D_t\, a \;=\; 1 - \frac{Fa}{C} - \frac{M(a_m - a)}{C} + \frac{2}{C}\nabla C \bullet \mathbf{K} \bullet \nabla a + \nabla \bullet (\mathbf{K} \bullet \nabla a) \quad . \tag{30}$$

The first term in the right-hand member of (30) is related to ageing. According to the second term, ice accretion tends to diminish the mean age. This is because the age of the newly-produced ice is prescribed to be zero, thereby nudging the mean age toward zero. The impact of melting, which is associated with the third term in the right-hand member of (30), is a more subtle one. If the mean age of the ice being melted is greater (smaller) than the mean age of the ice, then melting tends to decrease (increase) the mean age. Finally, it must be pointed out that the ice velocity divergence has no influence on the mean age.

A sea ice age estimate to do what?

There are two important applications for sea ice age estimates: 1) the quantification of the changes occurring in the Arctic since about two decades and 2) sea ice type classification that is usually based on the distinction between first-year ice (FYI) and multi-year ice (MYI). The domain of interest is the whole Arctic basin and the required resolution is about 20-50 km. Note that in the following the time dimension is expressed in years for simplicity of notation.

In the following section, we describe which discretization could be used in the age, spatial and temporal dimensions and how to derived the sea ice age from sea ice drift and concentration coming from satellite observations or from a model.

Discretization of the age dimension into a FYI and MYI categories

Sea ice in the Arctic has a strong seasonal cycle, with the freezing season typically spanning from September to May. Any sea ice that is formed after the transition between the melting and freezing season is usually defined as FYI, whereas all the remaining ice at the end of the previous melting season is considered as MYI. If $t_0(t)$ is defined as the date of the previous melting/freezing transition, the concentration of FYI is given by

$$C_{FYI}(t,\mathbf{x}) \;=\; \int_0^{t-t_0(t)} c(t,\mathbf{x},\tau)\, d\tau \tag{31}$$

and the concentration of MYI is given by

$$C_{MYI}(t,\mathbf{x}) \;=\; \int_{t-t_0(t)}^{\infty} c(t,\mathbf{x},\tau)\, d\tau \;. \tag{32}$$

The age concentration for each class is defined as

$$\alpha_{FYI}(t,\mathbf{x}) \;=\; \int_0^{t-t_0(t)} \tau c(t,\mathbf{x},\tau)\, d\tau \tag{33}$$

and

$$\alpha_{MYI}(t,\mathbf{x}) \;=\; \int_{t-t_0(t)}^{\infty} \tau c(t,\mathbf{x},\tau)\, d\tau \;. \tag{34}$$

The evolution equations of the variables defined in (31-34) are obtained for $t_0(t) < t < t_0(t+1)$ with the same approach as for the total concentration and age concentration (the maths are not described here) and are

$$\frac{\partial C_{FYI}}{\partial t} = F - M_{FYI} - \nabla \bullet (C_{FYI}\,\mathbf{u} - \mathbf{K} \bullet \nabla C_{FYI}) \tag{35}$$

$$\frac{\partial C_{MYI}}{\partial t} = -M_{MYI} - \nabla \bullet (C_{MYI}\,\mathbf{u} - \mathbf{K} \bullet \nabla C_{MYI}) \tag{36}$$

$$\frac{\partial \alpha_{FYI}}{\partial t} = C_{FYI} - M_{FYI}\, a_{FYI\,m} - \nabla \bullet (\alpha_{FYI}\,\mathbf{u} - \mathbf{K} \bullet \nabla \alpha_{FYI}) \tag{37}$$

$$\frac{\partial \alpha_{MYI}}{\partial t} = C_{MYI} - M_{MYI}\, a_{MYIm} - \nabla \bullet (\alpha_{MYI}\,\mathbf{u} - \mathbf{K} \bullet \nabla \alpha_{MYI}) \tag{38}$$

At the end of the melting season when $t^- = t_0(t^- + 1)$, the FYI is transferred into the MYI category by doing

$$C_{FYI}(t) = 0 \tag{39}$$

$$C_{MYI}(t) = C_{MYI}(t^-) + C_{FYI}(t^-) \tag{40}$$

$$\alpha_{FYI}(t) = 0 \tag{41}$$

$$\alpha_{MYI}(t) = \alpha_{MYI}(t^-) + \alpha_{FYI}(t^-) \tag{42}$$

and $t_0(t)$ is set to $t_0(t^- + 1)$.

Discretization of the age dimension into classes of age

Having defined the FYI and MYI categories has the advantage of having a fixed and reduced number of classes but may not be detailed enough for some applications. A more general approach would be to define classes of age (similarly as in the education system). The class of indice $k \geq 1$ would contains the ice that was younger than k year(s) old but older than $k-1$ year(s) old at $t = t_0(t)$, and its concentration would be given by

$$C_k(t,\mathbf{x}) = \int_{t-t_0(t)+k-1}^{t-t_0(t)+k} c(t,\mathbf{x},\tau)\, d\tau\ . \tag{43}$$

For simplicity, we defined $t_0(t)$ as the previous 1[st] September, so that all bins have the same length in the age dimension.

The evolution equations are the same as equations (35-36), if the subscript FYI and MYI are replaced by 0 and $k \geq 1$, respectively.

At the end of the melting season when $t^- = t_0(t^- + 1)$, the transfer is made by doing

$$C_0(t) = 0 \tag{44}$$

$$C_k(t) = C_{k-1}(t^-) \tag{45}$$

$$\alpha_0(t) = 0 \tag{46}$$

$$\alpha_k(t) = \alpha_{k-1}(t^-) \tag{47}$$

and $t_0(t)$ is set to $t_0(t^- + 1)$.

Sea ice age estimate derived from a sea ice drift and concentration datasets

There exist several datasets providing satellite-derived estimates of sea ice drift and concentration over the whole Arctic basin and for the whole satellite-era period (1979-now). These data are provided at a temporal resolution Δt of typically 1 day, and a spatial resolution Δx of typically 25 km. We here assume that the data are provided everywhere (no gap) and on the same regular grid.

The sea ice drift data will directly provide $\mathbf{u}(t,x)$ in equations (35-36) whereas the sea ice concentration data will be used to constrain the remaining terms. In this exercise and due to the lack of sufficient constraints, we assume that the diffusivity K is negligible. We use the two categories (FYI and MYI) discretization, but a similar approach can be followed with the classes of age. We also assume that $a_{FYIm} = a_{FYI}$ and $a_{km} = a_k$. We assume that M_{MYI} can only be positive when no more FYI is present (Note that the melting terms could also include other processes than just melting, such as the ridging processes that limit the total concentration to 100%). In other words, the FYI is always melted or ridged before the MYI. We assume that the freezing term F and the melting terms cannot be active at the same time.

If we use the Euler explicit time integration for equations (35) and (36), one describes the evolution of the sea ice concentration as the following two steps:

$$C'^{n+1} = C^n - \nabla\bullet(C^n \mathbf{u})\Delta t \quad (48)$$

$$\frac{C^{n+1} - C'^{n+1}}{\Delta t} = F - M_{FYI} - M_{MYI} \quad (49)$$

Lets note $\dot{C}$ the left hand side of equation (49).

$$\text{If } \dot{C} \geq 0 \text{, then } F = \dot{C}, M_{FYI} = M_{MYI} = 0 . \quad (50)$$

$$\text{If } \dot{C} < 0 \text{ then } F = 0, M_{FYI} = \min(C'^{n+1}_{FYI}\Delta t, \dot{C}), M_{MYI} = \dot{C} - M_{FYI} . \quad (51)$$

Once the terms F, M_{FYI}, M_{MYI} are computed, equations (35-38) can be integrated to get the new FYI and MYI concentration and age concentration, and then use equation (7) to get their respective mean ages. At the transition date, one simply applies equations (39-42) to transfer the FYI into the MYI category.

The algorithm could then be the following:

1) Compute C'^{n+1} , C'^{n+1}_{FYI} C'^{n+1}_{MYI} α'^{n+1}_{FYI} and α'^{n+1}_{MYI} with a conservative monotonous advection scheme.
2) Compute $\dot{C}$, and derive F, M_{FYI}, M_{MYI} from equation (50) and (51)
3) Compute

$$C^{n+1}_{FYI} = C'^{n+1}_{FYI} + (F - M_{FYI})\Delta t \quad (52)$$

$$C^{n+1}_{MYI} = C'^{n+1}_{MYI} + M_{MYI}\Delta t \quad (53)$$

$$\alpha^{n+1}_{FYI} = \alpha'^{n+1}_{FYI} + (C^n_{FYI} - M_{FYI}a^n_{FYI})\Delta t \quad (52)$$

$$\alpha^{n+1}_{MYI} = \alpha'^{n+1}_{MYI} + (C^n_{MYI} - M_{MYI}a^n_{MYI})\Delta t \quad (53)$$

4) Verify that $C^{n+1} = C^{n+1}_{FYI} + C^{n+1}_{MYI}$
5) Compute

$$a_{FYI}^{n+1} = \alpha_{FYI}^{n+1} / C_{FYI}^{n+1} \quad (54)$$

$$a_{MYI}^{n+1} = \alpha_{MYI}^{n+1} / C_{MYI}^{n+1} \quad (55)$$

6) Verify that $0 \le a_{FYI}^{n+1} \le t_0(t)$ and $t_0(t) \le a_{MYI}^{n+1}$

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