On split extensions of product hoops

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Introduction

- Internal actions of the objects of a category have been introduced by F. Borceux, G. Janelidze and G. M. Kelly in order to generalize the equivalence between actions and split extensions given by semidirect products from the context of groups to semi-abelian categories;
- In some cases, it is more convenient to describe internal actions in terms of external actions, i.e., a set of maps satisfying a set of identities.

Preliminaries

Definition

A *BL-algebra* is an algebra $A = (A, \vee, \wedge, \cdot, \to, 0, 1)$ such that

1. $(A, \vee, \wedge, \cdot, \to, 0, 1)$ is a bounded residuated lattice;
2. $x \wedge y = x \cdot (x \to y)$;
3. $(x \to y) \vee (y \to x) = 1$.

BL-algebras can be described in the reduced language of *hoops*.

Definition

A *hoop* is an algebra $H = (H, \cdot, \rightarrow, 1)$ of type $(2, 2, 0)$ such that

1. $(H, \cdot, 1)$ is a commutative monoid;
2. $x \rightarrow x = 1$;
3. $x \cdot (x \rightarrow y) = y \cdot (y \rightarrow x)$;
4. $(x \cdot y) \rightarrow z = x \rightarrow (y \rightarrow z)$,

for every $x, y, z \in H$.

A hoop subreduct of a BL-algebras is called *basic hoop*, i.e., a hoop satisfying the identity

$$((x \rightarrow y) \rightarrow z) \rightarrow (((y \rightarrow x) \rightarrow z) \rightarrow z) = 1.$$

Every hoop H is endowed with a partial order $\leq$, called the *natural order*, which is defined by the following equivalent conditions: for any $x, y \in H$

1. $x \leq y$;
2. $x \to y = 1$;
3. there exists z in H such that $x = z \cdot y$.

Remark (S. Lapenta, G. Metere, L. Spada)

The category **Hoops** is semi-abelian.

Product hoops

Definition

A product algebra is a BL-algebra satisfying

$$\neg x \vee ((x \to x \cdot y) \to y) = 1,$$

where $\neg x := x \to 0$.

Product algebras are term equivalent to the class of bounded *product hoops*.

Definition

A *product hoop* is a basic hoop satisfying the identity

$$(y \to z) \vee ((y \to (x \cdot y)) \to x) = 1.$$

We denote by **PAlg** the variety of product algebras and by **PHoops** the variety of product hoops.

Example

Let C be a *cancellative* hoop, that is, a hoop satisfying the identity

$$x \to (y \cdot x) = x.$$

The algebra $2 \oplus C$ whose underlying set is $C \cup \{0\}$, where $0 \notin C$, and whose constants are 0 and 1, and the operations $\cdot$ and $\to$ are defined by

$$x \cdot y = \begin{cases} x \cdot y, & \text{if } x, y \in C, \\ 0, & \text{otherwise,} \end{cases}$$

$$x \to y = \begin{cases} x \to y, & \text{if } x, y \in C, \\ 1, & \text{if } x = 0, \\ 0, & \text{if } x \in C,\ y = 0, \end{cases}$$

is a product algebra.

Example

An example of product algebra is given by the set $A = [0,1]$ endowed with the operations $x \cdot_{\prod} y = xy$ (the multiplication between real numbers) and

$$x \to_{\prod} y = \begin{cases} 1, & \text{if } x \leq y, \\ \frac{y}{x}, & \text{otherwise.} \end{cases}$$

We denote this algebra by $[0,1]_{\prod}$.

Example

The two-element Boolean algebra $L_2 = \{0,1\}$ is a product algebra. This corresponds to classical two-valued logic (true and false), making Boolean algebras special cases of product algebras.

Definition

A subset F of a product hoop H is a *filter* if for every $x, y \in H$,

- $(F, \cdot, 1)$ is a submonoid of $(H, \cdot, 1)$;
- $x \in F$ and $y \geq x$ implies $y \in F$.

An *homomorphism* of product hoops is a map $f \colon H \to H'$ such that

$$f(h \cdot h') = f(h) \cdot f(h'),\ f(h \to h') = f(h) \to f(h'),\ f(1) = 1$$

for any $h, h' \in H$.

- Given a homomorphism of product hoops $f : H \to H'$, its kernel $\ker f$ is a filter of H.
- Conversely, every filter F of a product hoop H can be seen as the kernel of the canonical projection $\pi : H \to H/F$.

Remark

The category **PHoops** of product hoops is semi-abelian.

Remark

The initial object of the category **PAlg** is the two-element product algebra $L_2 = \{0, 1\}$.

The variety **PAlg** is not semi-abelian, since it is not pointed, but it is protomodular. As a consequence, **PAlg** is *ideally exact* and the semi-abelian category $(\mathbf{PAlg} \downarrow L_2)$ is equivalent to the category **PHoops** of product hoops.

Split extensions in **PHoops**

Definition

Let B, X be product hoops. A *split extension* of B by X is a diagram

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

in **PHoops** such that $p \circ s = 1_B$ and (X, k) is a kernel of p.

Since the category of product hoops is semi-abelian, there is an equivalence between split extensions and internal actions given by semidirect products.

Characterization of semi-abelian varieties

Theorem (D. Bourn, G. Janelidze)

A variety of universal algebra is semi-abelian if, and only if it has, among its operations, a unique constant 0*,* n *binary operations* α_i*,* $i = 1, \ldots, n$*, and an* $(n+1)$*-ary operation* θ *satisfying the identities*

$$\alpha_i(x, x) = 0, \quad \textit{for each } i = 1, \ldots, n,$$

and

$$\theta(\underline{\alpha}(x, y), y) = x,$$

where $\underline{\alpha}(x, y)$ *denotes* $(\alpha_1(x, y), \ldots, \alpha_n(x, y))$*.*

In the semi-abelian variety **Hoops**, we have $n = 2$, $\alpha_1(x, y) = (x \to y)$, $\alpha_2(x, y) = (((x \to y) \to y) \to x)$ and $\theta(x, y, z) = ((x \to z) \cdot y)$.

Semidirect product in semi-abelian varieties

Theorem (M. M. Clementino, A. Montoli, L. Sousa)

Given a semi-abelian variety V *and a split extension*

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

in V *and let* $\xi : B\flat X \to X$ *be the corresponding internal action of* B *on* X*. Then the semidirect product* $X \ltimes_\xi B$ *of* X *and* B *with respect to the action* ξ *is the set* $Y = \{(\underline{x}, b) \in X^n \times B : \ \underline{\alpha}(\theta(\underline{x}, s(b)), s(b)) = \underline{x}\}$ *equipped with the following structure: if* ω *is an* m*-ary operation of the variety, then in* Y *we have*

$$\omega_Y((\underline{x}_1, b_1), ..., (\underline{x}_m, b_m)) =$$
$$(\xi^n \underline{\alpha}_{B\flat X}(\omega_{B\flat X}(\theta_{B\flat X}(\underline{x}_1, b_1), ..., \theta_{B\flat X}(\underline{x}_m, b_m)), \omega_{B\flat X}(\underline{b})), \omega_B(\underline{b})).$$

Semidirect product in **PHoops**

Let

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

be a split epimorphism in **PHoops** and let $\xi \colon B\flat X \to X$ be the corresponding internal action of B on X. The semidirect product $X \rtimes_\xi B$ of X and B with respect to the action ξ is the set

$$\begin{aligned} Y = &\{(x_1, x_2, b) \in X^2 \times B \mid ((x_1 \to s(b)) \cdot x_2) \to s(b) = x_1, \\ &\quad ((((x_1 \to s(b)) \cdot x_2) \to s(b)) \to s(b)) \to ((x_1 \to s(b)) \cdot x_2) = x_2\} \end{aligned}$$

equipped with the operations

$$\begin{aligned} (x_1,x_2,b_1) \to (y_1,y_2,b_2) = &((((x_1 \to s(b_1)) \cdot x_2) \to ((y_1 \to s(b_2)) \cdot y_2)) \to (s(b_1) \to s(b_2)), \\ &(((((x_1 \to s(b_1)) \cdot x_2) \to ((y_1 \to s(b_2) \cdot y_2))) \to s(b_1 \to b_2)) \to s(b_1 \to b_2)) \to \\ &(((x_1 \to s(b_1)) \cdot x_2) \to ((y_1 \to s(b_2) \cdot y_2))), b_1 \to b_2), \end{aligned}$$

$$\begin{aligned} (x_1,x_2,b_1) \cdot (y_1,y_2,b_2) = &((((x_1 \to s(b_1)) \cdot x_2) \cdot ((y_1 \to s(b_2)) \cdot y_2)) \to (s(b_1) \cdot s(b_2)), \\ &((((((x_1 \to s(b_1)) \cdot x_2) \cdot ((y_1 \to s(b_2) \cdot y_2))) \to s(b_1 \cdot b_2)) \to s(b_1 \cdot b_2)) \to \\ &(((x_1 \to s(b_1)) \cdot x_2)) \cdot ((y_1 \to s(b_2) \cdot y_2))), b_1 \cdot b_2). \end{aligned}$$

Split extensions with strong section

Definition

A split extensions

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

strongly splits, or has a *strong section*, if the morphism s is a *strong section* of p, i.e., if the equation $a \to s(b) = sp(a) \to s(b)$ holds for every $a \in A$ and $b \in B$.

Let $\mathrm{SplExt}_{\mathrm{ss}}(-, X)\colon \mathbf{PHoops}^{\mathrm{op}} \to \mathbf{Set}$ be the functor which assigns to any product hoop B the set $\mathrm{SplExt}_{\mathrm{ss}}(B, X)$ of isomorphism classes of split extensions of B by X in **PHoops** that strongly splits.

Example (Trivial)

Let $p : A \to B$ be an isomorphism of product hoops. Then its inverse $s : B \to A$ is a strong section of p.

Example

Let A be a product algebra. Let

$$B(A) = \{x \in A | \neg\neg x = x\}$$

be the set of *regular elements* of A and let

$$\mathrm{D}(A) = \{x \in A | \neg\neg x = 1\}$$

be the set of *dense elements* of A. Then $\mathrm{B}(A)$ is the greatest Boolean subalgebra of A, $\mathrm{D}(A)$ is a filter of A and the split extension

$$\begin{tikzcd} D(A) \arrow[r, "i"] & A \arrow[r, shift left, "p"] & B(A) \arrow[l, shift left, "j"] \end{tikzcd}$$

where $p(a) = \neg\neg a$, for every $a \in A$, i and j are the canonical inclusions, has a strong section, since the identity $x \to y = \neg\neg x \to y$ holds in chains.

Semidirect product in **PHoops** (strong section)

Theorem

Let

$$X \xrightarrow{k} A \underset{s}{\overset{p}{\rightleftarrows}} B$$

be a split extension in **PHoops** *that strongly splits and let* $\xi\colon B\flat X \to X$ *be the corresponding internal action of* B *on* X*. The semidirect product* $X \rtimes_\xi B$ *of* X *and* B *w.r.t. the action* ξ *is given by the set*

$$Y = \{(x,b) \in X \times B \mid s(b) \to (s(b)\cdot x) = x\}$$

together with the operations

$$(x,b_1) \to (y,b_2) = (s(b_2 \to b_1) \to (x \to y),\ b_1 \to b_2),$$

$$(x,b_1)\cdot(y,b_1) = (s(b_1\cdot b_2) \to (s(b_1\cdot b_2)\cdot x\cdot y),\ b_1\cdot b_2).$$

Definition

Let B, X be product hoops. A *(strong) external action* of B on X consists of a pair of maps

$$f\colon B \times X \to X\colon (b, x) \mapsto f_b(x),$$

$$g\colon B \times X \to X\colon (b, x) \mapsto g_b(x)$$

such that

1. $f_b(1) = g_b(1) = 1$;
2. $f_1 = g_1 = \mathrm{id}_X$;
3. $f_{b_1 \cdot b_2}(x \cdot g_{b_1}(x \to y)) = f_{b_1 \cdot b_2}(x \cdot (x \to y))$;
4. $g_{(b_3 \to (b_1 \cdot b_2))}(f_{b_1 \cdot b_2}(x \cdot y) \to z) = g_{(b_2 \to b_3) \to b_1}(x \to g_{b_3 \to b_2}(y \to z))$;
5. $g_{\tilde{b}}(g_{b_3 \to (b_1 \to b_2)}(g_{b_2 \to b_1}(x \to y) \to z) \to (g_{b_3 \to (b_2 \to b_1)}(g_{b_1 \to b_2}(y \to x) \to z) \to z)) = 1$, where $\tilde{b} = (((b_2 \to b_1) \to b_3) \to b_3) \to ((b_1 \to b_2) \to b_3)$;
6. $$g_{((b_2 \to (b_1 b_2)) \to b_1) \to (b_2 \to b_3)}(g_{b_3 \to b_2}(y \to z) \to ((y \to f_{b_1 b_2}(xy)) \to x))$$
$$\to ((y \to f_{b_1 b_2}(xy)) \to x)) \wedge (g_{(b_2 \to b_3) \to ((b_2 \to (b_1 b_2)) \to b_1)}$$
$$(((y \to f_{b_1 b_2}(xy)) \to x) \to g_{b_3 \to b_2}(y \to z)) \to g_{b_3 \to b_2}(y \to z)) = 1,$$

for any $b, b_1, b_2, b_3 \in B$ and $x, y, z \in X$.

We denote by $\mathrm{EAct}_{\mathrm{ss}}(B, X)$ the set of strong external actions of B on X.

Remark

Let $f, g\colon B \times X \to X$ be a strong external action of product hoops. The map g satisfies the following additional properties:

(i) g_b is monotone for any $b \in B$, i.e., if $x \leq y$, then $g_b(x) \leq g_b(y)$;

(ii) $g_{b_1 \cdot b_2} = g_{b_1} \circ g_{b_2}$, for any $b_1, b_2 \in B$;

(iii) $g_b(x \to y) = g_b(x) \to g_b(y)$, for any $b \in B$ and $x, y \in X$.

External actions of product hoops

Let

$$\mathrm{EAct}_{\mathrm{ss}}(-, X)\colon \mathbf{PHoops}^{\mathrm{op}} \to \mathbf{Set}$$

be the functor which maps every product hoop B to $\mathrm{EAct}_{\mathrm{ss}}(B, X)$, and every morphism $\phi\colon B' \to B$ in **PHoops** to the function

$$\mathrm{EAct}_{\mathrm{ss}}(\phi, X)\colon\ \mathrm{EAct}_{\mathrm{ss}}(B, X) \to \mathrm{EAct}_{\mathrm{ss}}(B', X)$$

which maps an external action $f, g\colon B \times X \to X$ to the external action $f', g'\colon B' \times X \to X$ defined by

$$f'(b', x) = f(\phi(b'), x) \ \text{ and } \ g'(b', x) = g(\phi(b'), x).$$

Proposition

Let B, X be product hoops. There is a bijection τ_B between $\mathrm{SplExt}_{\mathrm{ss}}(B, X)$ and $\mathrm{EAct}_{\mathrm{ss}}(B, X)$.

Proof.

We define $\tau_B : \mathrm{SplExt}_{\mathrm{ss}}(B, X) \to \mathrm{EAct}_{\mathrm{ss}}(B, X)$ as follows: given any split extensions in **PHoops** that strongly splits

$$\begin{tikzcd} X \arrow[r, "k"] & A \arrow[r, shift left, "p"] & B \arrow[l, shift left, "s"] \end{tikzcd}$$

we associate the pair of maps $f, g \colon B \times X \to X$ defined by

$$f_b(x) = s(b) \to (s(b) \cdot x) \quad \text{and} \quad g_b(x) = s(b) \to x.$$

It is possibile to show that (f, g) defines a strong external action of B on X. $\square$

Proof.

Now, consider the map $\mu_B : \mathrm{EAct}_{\mathrm{ss}}(B, X) \to \mathrm{SplExt}_{\mathrm{ss}}(B, X)$ which sends a strong external action $f, g \colon B \times X \to X$ to the split extension of B by X

$$
\begin{tikzcd}
X \arrow[r, "\iota_1"] & Y \arrow[r, shift left, "\pi_2"] & B \arrow[l, shift left, "\iota_2"]
\end{tikzcd}
$$

where

$$Y = \{(x, b) \in X \times B \mid f_b(x) = x\}$$

and

$$(x, b) \to (y, b') = (g_{b' \to b}(x \to y),\ b \to b'), \tag{1}$$

$$(x, b) \cdot (y, b') = (f_{b \cdot b'}(x \cdot y),\ b \cdot b'). \tag{2}$$

This split extensions strongly splits and μ_B is the inverse of the map τ_B. $\square$

Theorem

There is a natural isomorphism

$$\tau\colon \mathrm{SplExt}_{\mathrm{ss}}(-,X) \cong \mathrm{EAct}_{\mathrm{ss}}(-,X).$$

Proof.

The bijection τ_B is natural in B, that is, for every morphism $\varphi\colon B' \to B$ in **PHoops**, the following diagram commutes

$$\begin{tikzcd} \mathrm{SplExt}_{\mathrm{ss}}(B,X) \arrow[r, "\tau_B"] \arrow[d, "\varphi^*"'] & \mathrm{EAct}_{ss}(B,X) \arrow[d, "\mathrm{EAct}_{ss}(\varphi,X)"] \\ \mathrm{SplExt}_{\mathrm{ss}}(B',X) \arrow[r, "\tau_{B'}"] & \mathrm{EAct}_{ss}(B',X) \end{tikzcd}$$

□

References

1. P. Agliano, I. Ferreirim, F. Montagna, “Basic Hoops: an Algebraic Study of Continuous t-norms", Studia Logica vol. 87, 2007, pp. 73-98.
2. F. Borceux, G. Janelidze and G. M. Kelly, “Internal object actions”, Commentationes Mathematicae Universitatis Carolinae vol. 46, 2005, no. 2, pp. 235–255.
3. D. Bourn, G. Janelidze, “Characterization of protomodular varieties of universal algebras”, Theory and Applications of Categories vol. 11, 2003, no. 6, pp. 143-147.
4. C. C. Chang, “Algebraic analysis of many-valued logics”, Transactions of the American Mathematical Society vol. 88, 1958, pp. 476–490.
5. C. C. Chang, “Algebraic Analysis of Many Valued Logics”, Transactions of the American Mathematical Society vol. 88, 1958, pp. 467–490.
6. R. Cignoli, A. Torrens, “An algebraic analysis of product logic”, Multiple Valued Logic vol. 5, 2000, pp. 45–65.
7. M. M. Clementino, A. Montoli, L. Sousa, “Semidirect products of (topological) semi-abelian algebras”, Journal of Pure and Applied Algebra vol. 219, 2015, no. 1, pp. 183–197.
8. P. Hàjek, “Metamathematics of fuzzy logic", Kluwer, 1998.
9. G. Janelidze, L. Márki and W. Tholen, “Semi-abelian categories”, Journal of Pure and Applied Algebra vol. 168, 2002, no. 2, pp. 367–386.
10. W. Rump, “A general Glivenko theorem”, Algebra Universalis vol. 61, 2009, no. 455.
11. W. Rump, “The category of L-algebras”, Theory and Applications of Categories vol. 39, 2023, no. 21, pp. 598–624.

Thank you for your attention.