# The Synchronizing Probability Function for Primitive Sets of Matrices

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**Abstract.** Motivated by recent results relating synchronizing automata and primitive sets, we tackle the synchronization process and the related longstanding Černý conjecture by studying the primitivity phenomenon for sets of nonnegative matrices having neither zero-rows nor zero-columns. We formulate the primitivity process in the setting of a two-player probabilistic game and we make use of convex optimization techniques to describe its behavior. We report numerical results and supported by them we state a conjecture that, if true, would imply an upper bound of $n(n-1)$ on the reset threshold of a certain class of automata.

## 1 Introduction

A (complete deterministic finite) *automaton* $\mathcal{A}$ on $n$ states is a set of $m$ binary row-stochastic[3] matrices of dimension $n \times n$ that are called the *letters* of the automaton. An automaton $\mathcal{A}$ is *synchronizing* if there exist a product of its letters, with repetitions allowed, that has an all-ones column[4]; the length of the shortest of these products is called the *reset threshold* (denoted by $rt(\mathcal{A})$) of the automaton. Synchronizing automata appear in different research fields; for example, they are often used as models of error-resistant systems [10,6] and in symbolic dynamics [16]. For a brief account on synchronizing automata and their other applications we refer the reader to [24]. The interest for automata arises also from one of the most longstanding open problems in this field:

*Conjecture 1 (The Černý conjecture [23]).* Any synchronizing automaton on $n$ states has reset threshold $\leq (n-1)^2$.

If the conjecture is true, the bound is sharp due to the existence of a family of automata attaining that value [23]. Despite many efforts, the best upper bound

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[3] A *binary* matrix is a matrix with entries in $\{0, 1\}$. A *row-stochastic* matrix is a matrix with nonnegative entries where the entries of each row sum up to 1. Therefore a matrix is *binary* and *row-stochastic* if each row has exactly one 1.

[4] A column whose entries are all equal to 1.

for the reset threshold known so far is $(15617n^3+7500n^2+9375n-31250)/93750$, recently obtained by Szykuła in [22] and thereby beating the 30 years-standing upper bound of $(n^3-n)/6$ found by Pin and Frankl [11,18]; exhaustive search confirmed the conjecture for small values of $n$ (see [3,8]). Automata with reset thresholds close to $(n-1)^2$, called *extremal* automata, are also hard to detect and not so many families are known (see [3,8,9,15] for examples). Partially due to the fact that the reset threshold is hard to compute (see [10,17]), the synchronization process is still far to be fully understood. In 2012 the second author built up in [14] a new tool in order to study this phenomenon: by looking at synchronization as a two-player probabilistic game, he developed the concept of *sychronizing probability function for automata* (SPFA), a function that describes the speed at which an automaton synchronizes. By reformulating the game as a linear programming problem and making use of convex optimization techniques, he shows that the behaviour of the SPFA is closely related with properties of the synchronizing automaton. In [13] Gonze and Jungers use this tool to prove a quadratic upper bound on the length of the shortest word of an automaton mapping three states into one.
Meantime, another novel approach to the study of synchronizing automata was being developed by Blondel et. al. in [4] and then further exploited by Gerencsér et. al. in [12]: this approach is based on the concept of *primitive set of matrices*, i.e. a finite set of nonnegative[5] matrices that admits a product, with repetitions allowed, with all positive entries; a product of this kind is called a *positive* product. This notion was introduced for the first time by Protasov and Voynov in [20] and has found application in different fields as in stochastic switching systems [19] and consensus for discrete-time multi-agent systems [7]. One of the connections between primitive sets and automata is established by Theorem 1 presented below, which summarizes two results proved by Blondel et. al. ([4], Theorems 16-17) and a result proved by Gerencsér et. al. ([12], Theorem 8); in the following we call the *exponent* of a primitive set $\mathcal{M}$ (denoted by $exp(\mathcal{M})$) the length of its shortest positive product and a nonnegative matrix is said to be *NZ* if it has neither zero-rows nor zero-columns[6].

**Theorem 1 (Theorems 16-17 in [4] and Theorem 8 in [12]).**
*Let $\mathcal{B} = \{B_1, \dots, B_m\}$ be a set of binary NZ-matrices of size $n \times n$, $\mathcal{A}$ be the automaton made of all the binary row-stochastic matrices that are entrywise smaller than at least one matrix in $\mathcal{B}$ and $\mathcal{A}'$ be the set of all the binary row-stochastic matrices that are entrywise smaller than at least one matrix in $\mathcal{B}^T = \{B_1^T, \dots, B_m^T\}$. Then it holds that $\mathcal{B}$ is primitive if and only if $\mathcal{A}$ (equiv. $\mathcal{A}'$) is synchronizing. If $\mathcal{B}$ is primitive, it also holds that:*

$$\max\{rt(\mathcal{A}), rt(\mathcal{A}')\} \leq exp(\mathcal{B}) \leq rt(\mathcal{A}) + rt(\mathcal{A}') + n - 1 \ .$$

[5] A matrix is *nonnegative* if it has nonnegative entries.
[6] Thus a NZ-matrix must have a positive entry in every row and in every column.

*Example 1.* A primitive set $\mathcal{M}_0$ and the corresponding automata $\mathcal{A}$ and $\mathcal{A}'$ described in Theorem 1, in both their matrix and graph representations (Fig. 1).

$$\mathcal{M}_0 = \left\{ \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&1\\ 0&0&1\\ 0&1&0 \end{pmatrix} \right\} \tag{1}$$

$\mathcal{A} = \left\{ \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}, \begin{pmatrix} 0&0&1\\ 0&0&1\\ 0&1&0 \end{pmatrix} \right\} = \{a, b, c\}$, $\mathcal{A}' = \left\{ \begin{pmatrix} 0&1&0\\ 1&0&0\\ 0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0\\ 0&0&1\\ 0&1&0 \end{pmatrix}, \begin{pmatrix} 1&0&0\\ 0&0&1\\ 1&0&0 \end{pmatrix} \right\} = \{a, b, c'\}$. $exp(\mathcal{M}) = 8$, $rt(\mathcal{A}) = 4$, $rt(\mathcal{A}') = 2$.


Fig. 1: The automata $\mathcal{A}$ (on the left) and $\mathcal{A}'$ (on the right) of Example 1.

The requirement in Theorem 1 that the set $\mathcal{B}$ has to be made of *binary* matrices is not restrictive, as in the definition of primitivity what counts is just the position of the positive entries within the matrices and not their actual values. Indeed, if given a nonnegative matrix $M$ we define $B_M$ as the binary matrix such that $B_M(i,j) = 1$ iff $M(i,j) > 0$, then for any matrix set $\mathcal{M} = \{M_1, \ldots, M_m\}$, $\mathcal{M}$ is primitive if and only if $\mathcal{B}_{\mathcal{M}} = \{B_{M_1}, \ldots, B_{M_m}\}$ is primitive and $exp(\mathcal{M}) = exp(\mathcal{B}_{\mathcal{M}})$. This fact will be further formalized in Sect. 2 and it will play a central role throughout the paper. Theorem 1 tells us that primitive sets can be used for generating automata and a primitive set with large exponent would translate into an automaton with large reset threshold. In particular, if the Černý conjecture is true, it would imply that $exp(\mathcal{M}) \leq 2n^2 - 3n + 1$ for any NZ-primitive set $\mathcal{M}$ of matrix size $n \times n$, while a primitive set with exponent larger than this value would immediately disprove the conjecture. The upper bounds on the automata reset threshold mentioned before imply that $exp(\mathcal{M}) = O(n^3)$. Improvements on the upper bound of the exponent and methods for approximating it are particularly of interest as, while testing the primitivity of a NZ-set of $m$ matrices is polynomial in $nm$ ([20], Proposition 2), finding its exponent is computationally hard ([12], Theorem 12). Protasov and Voynov also provide a characterization theorem for NZ-primitive sets ([20], Theorem 1); this result, combined with Theorem 1, has been used by Catalano and Jungers in [5] to construct a randomized procedure for finding automata with large reset threshold. Alpin and Alpina later extended the Protasov-Voynov characterization theorem to nonnegative matrix sets that are allowed to have zero-columns but not zero-rows ([2], Theorem 3), category in which automata fall. It naturally follows from all the above considerations that a better understanding of the primitivity phenomenon would give a further insight on the automata synchronization.

**Our Contribution.** In the present paper we study the primitivity phenomenon

by extending the concept of SPFA to primitive sets: our goal is to design a function that increases smoothly, representing the convergence of the primitivity process. In Sect. 3 we present primitivity as a two-player game and we define the *synchronizing probability function for primitive sets* (SPF) as the probability that the first player wins if both players play optimally. We then reformulate the game as a linear programming problem and we provide an analysis of some theoretical properties of the SPF in Sect. 4: this function captures the speed at which a primitive set reaches its first positive product and we show that it must increase regularly in some sense. Some numerical experiments are reported. The SPF is partly inspired from the *smoothed analysis* in combinatorial optimization, where probabilities are used in order to analyze the convergence of iterative algorithms on combinatorial structures (see e.g. [21]). In Sect. 5 we define another function, $\bar{K}(t)$, which is an upper bound of the SPF: we show that stronger theoretical properties hold for this function and we state a conjecture on $\bar{K}(t)$ that, if true, would imply an upper bound of $2n(n-1)$ on the exponent of any NZ-primitive set and an upper bound of $n(n-1)$ on the reset threshold of any synchronizing automaton $\mathcal{A}$ for which there exists a binary NZ-set $\mathcal{B}$ such that $\mathcal{A}$ is made of all the binary row-stochastic matrices that are entrywise smaller than at least one matrix in $\mathcal{B}$.
Due to length restriction, the proofs are omitted in this paper.

## 2 Preliminaries

Given a matrix $M$, we indicate with $M(i,:)$ its $i$-th row and with $M(:,j)$ its $j$-th column. We define the *directed graph associated to a $n \times n$ nonnegative matrix $M$* as the digraph $D_M$ on $n$ vertices with a directed edge from $i$ to $j$ if $M(i,j) > 0$. A matrix $M$ is *irreducible* if and only if $D_M$ is *strongly connected*, i.e. if there exists a directed path between any two given vertices in $D_M$. A set of nonnegative matrices $\{M_1, \ldots, M_m\}$ is said to be *irreducible* iff the matrix $\sum_{i=1}^m M_i$ is irreducible. The *directed graph associated to the set $\mathcal{M}$* is the digraph $D_{\sum_{i=1}^m M_i}$ where each edge $(i,j)$ is labelled by the matrices of the set whose $(i,j)$-th entry is positive.

**Definition 1.** *A set of nonnegative matrices $\mathcal{M} = \{M_1, \ldots, M_m\}$ is* primitive *if there exist a product $M_{i_1} \cdots M_{i_l} > 0$ entrywise, for $i_1, \ldots, i_l \in \{1, \ldots, m\}$. In this case, we call $M_{i_1} \cdots M_{i_l}$ a* positive *product.*

Irreducibility is a necessary but not sufficient condition for a matrix set to be primitive (see [20], Section 1). A *permutation* matrix is a binary matrix having exactly one 1 in every row and in every column.

**Definition 2.** *A matrix $A$* dominates *a matrix $B$ ($A \geq B$) if $A(i,j) \geq B(i,j)$ for all $i$ and $j$.*

As already mentioned in the Introduction, the property of being primitive concerns just about the position of the positive entries within the matrices of the set and not their actual values. We can then assume without loss of generality that the matrices and all their products have entries in $\{0,1\}$ by using the so-called *boolean product*; we remind its definition here below.

**Definition 3.** *Let $\mathbb{B}_n$ denote the set of all the binary matrices of size $n \times n$ and let $B_1, B_2 \in \mathbb{B}_n$. The* boolean product $B_1 \odot B_2$ *is defined as*
$$B_1 \odot B_2(i,j) = \begin{cases} 1 & \text{if } \sum_{k=1}^{n} B_1(i,k)B_2(k,j) > 0 \\ 0 & \text{otherwise} \end{cases}.$$
*To ease the notation we will simply write $B_1B_2$ for $B_1 \odot B_2$. Given a vector $v$, the product $B_1B_2v$ is to be understood as $(B_1 \odot B_2) \cdot v$ with $\cdot$ the standard matrix-vector product.*

In the rest of the paper every matrix product has to be read as a boolean product. The boolean product will play a crucial role in the definition of the *synchronizing probability function for primitive sets*, even if it will add some difficulties in the study of its behavior due to its non-linear nature.
Lastly, to increase the readability, we will denote with $e$ *any* all-ones vector: its length, case by case, will be clear from the context.

## 3 The Synchronizing Probability Function for Primitive Sets

### 3.1 Definition

We here introduce primitivity as a two-player probabilistic game. We fix a binary NZ-set $\mathcal{M} = \{M_1, \ldots, M_m\}$ of matrix size $n \times n$ and an integer $t \in \mathbb{N}$, and we set $D$ to be the directed graph associated to $\mathcal{M}$ (see Sect. 2). We see the $n$ vertices of $D$ as reservoirs that can be filled with water and the edges of $D$ as pipes in which the water can flow: an edge can be *open* and let the water flow, or *closed*. Applying a matrix $M_k$ to $D$ means opening all the edges $(i,j)$ such that $M_k(i,j) = 1$, in which case the water flows from reservoir $i$ to reservoir $j$ (if there is water in reservoir $i$), and keeping all the other edges closed. Applying a sequence of matrices $M_{i_1} \ldots M_{i_l}$ to $D$ means applying matrix $M_{i_s}$ at time $s$ for $s=1, \ldots, l$. The game is the following:

1. Player B secretly chooses a vertex $v$, and pours water in it.
2. Player A chooses a sequence $M_{i_1} \ldots M_{i_l}$ of at most $t$ matrices in $\mathcal{M}$ and applies them to $D$. The water flows from $v$ according to $M_{i_1} \ldots M_{i_l}$.
3. A final vertex is chosen uniformly at random. If there is water in it then Player A wins, otherwise player B wins.

If $\mathcal{M}$ is primitive and has a positive product of length $\leq t$, then such product is an optimal strategy for Player A as by playing it he will certainly win. The opposite is also true: if Player A is sure to win, then $\mathcal{M}$ must have a positive product of length $\leq t$. In the following $\mathcal{M}^{\leq t}$ represents the set of the products of elements in $\mathcal{M}$ of length $\leq t$.
We consider that both players can choose probabilistic strategies. The *policy* of player B is defined as a probability distribution over the vertices of $D$, that is any

vector $p \in \mathbb{R}_+^n$ such that $p^T e = 1$, also called a *stochastic* vector; he chooses the vertex $v$ with probability $p(v)$. The vector $e/n \in \mathbb{R}_+^n$ represents the final uniform distribution on the vertices of the graph. For $M \in \mathcal{M}^{\leq t}$, $p^T M(e/n)$ represents the probability that there will be water in the final vertex given the initial distribution $p$: Player A wants to maximize it over all her choices of the product $M$ while Player B wants to minimize it over all his choices of the distribution $p$.

**Definition 4.** *Let $\mathcal{M}$ be a binary NZ-set of $n \times n$ matrices. The* Synchronizing Probability Function (SPF) *for the set $\mathcal{M}$ is the function $K_{\mathcal{M}} : \mathbb{N} \to \mathbb{R}_+$:*

$$K_{\mathcal{M}}(t) = \min_{p \in \mathbb{R}_+^n,\, p^T e = 1} \left\{ \max_{M \in \mathcal{M}^{\leq t}} p^T M(e/n) \right\} \quad . \tag{2}$$

The SPF thus represents the probability that Player A wins if both players play optimally. By convention we assume that the product of length zero $\mathcal{M}^0$ is the identity matrix. When the matrix set will be clear from the context, we will indicate the SPF just with $K(t)$. The following proposition formalizes the fact, mentioned before, that Player A can win surely if and only if the set is primitive.

**Proposition 1.** *The function $K_{\mathcal{M}}(t)$ takes values in $[0, 1]$ and is nondecreasing in $t$. Moreover, there exists a $t \in \mathbb{N}$ such that $K_{\mathcal{M}}(t) = 1$ if and only if $\mathcal{M}$ is primitive. In this case, $exp(\mathcal{M}) = \min\{t : K_{\mathcal{M}}(t) = 1\}$.*

The main motivation for studying this particular function is that the SPF measures how fast the set reaches a positive product by taking into account the evolution of the matrix semigroup generated by the set. The function seems to increase quite regularly as shown in Fig. 2, where we report the SPF of the set $\mathcal{M}_0$ from Equation (1) and of the following examples:

$$\mathcal{M}_1 = \left\{ \begin{pmatrix} 0&0&0&1&0 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \\ 0&0&0&0&1 \end{pmatrix}, \begin{pmatrix} 1&0&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 0&1&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&1&0&0 \\ 0&0&1&0&1 \\ 0&0&1&1&0 \\ 1&0&0&0&0 \\ 0&1&0&0&0 \end{pmatrix} \right\}, \tag{3}$$

$$\mathcal{M}_2 = \left\{ \begin{pmatrix} 0&1&0&0&0 \\ 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \end{pmatrix}, \begin{pmatrix} 0&0&0&1&0 \\ 0&0&1&0&0 \\ 0&0&0&0&1 \\ 1&0&0&0&0 \\ 0&1&0&0&1 \end{pmatrix}, \begin{pmatrix} 0&1&0&0&0 \\ 0&0&0&0&1 \\ 0&0&1&0&0 \\ 1&0&0&0&0 \\ 0&0&0&1&0 \end{pmatrix} \right\}. \tag{4}$$

In the next section we see how to reformulate the SPF as the solution of a linear programming problem, which will enable us to prove some interesting properties on the behavior of the SPF in Sect. 4.

### 3.2 A Linear Programming Formulation for the SPF

**Theorem 2.** *The synchronizing probability function $K_{\mathcal{M}}(t)$ is given by:*

$$\min_{p,k} \quad \frac{k}{n} \qquad s.t. \qquad \begin{cases} p^T H(t) \leq k e^T \\ p^T e = 1 \\ p \geq 0 \end{cases} \quad . \tag{5}$$

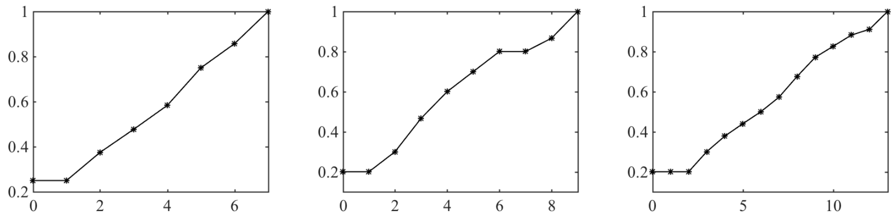


Fig. 2: The function $K(t)$ for the set $\mathcal{M}_0$ in Eq. (1) (left picture), for the set $\mathcal{M}_1$ in Eq. (3) (central picture) and for the set $\mathcal{M}_2$ in Eq. (4) (right picture).

*It is also given by the solution of*

$$\max_{q,k} \quad \frac{k}{n} \qquad s.t. \qquad \begin{cases} H(t)q \geq ke \\ e^T q = 1 \\ q \geq 0 \end{cases} , \tag{6}$$

*where $p$ is a $n \times 1$ vector, $q$ is a $h(t) \times 1$ vector with $h(t)$ the cardinality of $\mathcal{M}^{\leq t}$ and $H(t)$ is the $n \times h(t)$ matrix whose $i$-th column is equal to $A_i e$, with $A_i$ the $i$-th element of $\mathcal{M}^{\leq t}$.*

The matrix $H(t)$ has entries in $\{1, \dots, n\}$ due to the boolean product and $H(0) = e$; in particular, if $h_i$ is the $i$-th column of $H(t)$ and $A_i$ the $i$-th element of $\mathcal{M}^{\leq t}$, $h_i(l)$ is the number of positive entries in the $l$-th row of $A_i$. We notice that, if the vector $ne$ is a column of $H(t)$, then $K(t) = 1$.
The dual formulation (6) represents the point of view of Player A and her *policy* is described by the stochastic vector $q$, where $q(j)$ corresponds to the probability that Player A chooses to play the $j$-th element of $\mathcal{M}^{\leq t}$. Theorem 2 shows also that Player B can make his policy $p$ public without changing the outcome of the game if both players play optimally, as Player A can as well play before Player B.

# 4 Study of the SPF

## 4.1 General Behavior and Computability

In this section we analyze the above described game. The first result characterizes the behavior of $K(t)$ for small and big $t$: it shows that the SPF presents an initial stagnation at the value $1/n$ of length at most $n-1$ and it has to leave it with high discrete derivative; with high discrete derivative it also leaves the last step before hitting the value 1. This is formalized in the following proposition:

**Proposition 2.** *For any binary NZ-primitive set $\mathcal{M}$:*

1. $K(0) = 1/n$,
2. $K(n) > 1/n$,
3. $K(t) > 1/n \quad \Rightarrow \quad K(t) \geq (n+1)/n^2$,

*4.* $K(t) < 1 \qquad \Rightarrow \qquad K(t) \leq (n^2 - 1)/n^2$.

The fact that $K(t)$ could be constantly equal to $1/n$ until $t = n-1$ (there exist indeed some cases where it happens) suggests that, in particular when we are looking for primitive sets with large exponent, we could start computing the SPF directly from $t = n$. This, combined with the fact that the SPF seems to have a quite regular behavior after the initial stagnation (see Fig. 2), could be leveraged to guess the magnitude of the exponent of a primitive set without explicitly computing it (problem that, as already mentioned, is NP-hard [12]). For example, we could compute the first $s$ values of the SPF after $t = n-1$, let $r$ be the straight line passing through the points $(n, K(n))$ and $(s, K(s))$ and consider as approximation of the exponent the abscissa of the point at which $r = 1$.
The following result shows that, in order to compute the SPF, there is no need to use the full matrix $H(t)$ but we can always potentially find a much smaller submatrix that reaches the same optimal value. Furthermore, Proposition 4 tells that the set $\mathcal{M}^{\leq t}$ can be replaced by the set $\mathcal{M}^t$ of all the products of elements in $\mathcal{M}$ of length exactly $t$ in some cases.

**Proposition 3.** *For any binary NZ-set $\mathcal{M}$ of $n \times n$ matrices and any integer $t$, there always exists a submatrix $H'(t)$ of $H(t)$ of size $n \times s$, $s \leq n$, s.t. we can replace $H(t)$ with $H'(t)$ in the program (6) without changing the optimal value.*

**Proposition 4.** *For any integer $t$ and for any binary NZ-set $\mathcal{M}$ in which there exists at least a matrix that dominates a permutation matrix, $K_{\mathcal{M}}(t)$ remains the same if $\mathcal{M}^{\leq t}$ is replaced by $\mathcal{M}^t$ in (2).*

Proposition 4 may fail for a set in which all the matrices do *not* dominate a permutation matrix, as showed in Example 2. In this case we can observe that, if $s$ is the first time such that $\mathcal{M}^{\leq s}$ contains a matrix that dominates a permutation matrix ($s$ must exist if the set is primitive), then for every $t > s$ it holds that $K(t) \geq K^{=}(t) \geq K(t-s)$, with $K^{=}(t)$ the optimal solution of (5) with $\mathcal{M}^{\leq t}$ replaced by $\mathcal{M}^t$. This means that if $s$ is small enough, then $K^{=}$ still provides an accurate approximation of $K$ as $K^{=}(t+s) < 1$ implies $K(t) < 1$.

*Example 2.* $\mathcal{M} = \left\{ \begin{pmatrix} 1&0&0&0&0 \\ 1&0&0&0&0 \\ 1&0&0&0&0 \\ 0&1&1&0&0 \\ 0&0&0&1&1 \end{pmatrix}, \begin{pmatrix} 0&0&0&0&1 \\ 0&0&0&0&1 \\ 0&0&0&0&1 \\ 1&0&1&0&0 \\ 0&1&0&1&0 \end{pmatrix} \right\}$. It is easy to see that $H(1) = \begin{pmatrix} 1&1&1&2&2 \\ 1&1&1&2&2 \end{pmatrix}^T$ and $H(2) = H(1) \cup \begin{pmatrix} 1&1&1&1&4 \\ 2&2&2&1&3 \end{pmatrix}^T = H(1) \cup H^2$, where $H^2$ is the matrix whose columns are the set $\{Me : M \in \mathcal{M}^2\}$. The solution of program (5) for $t = 2$ is $K_{\mathcal{M}}(t) = 0.3$ for $p = (0, 0, 1/2, 1/2, 0)^T$, while if we substitute $H(2)$ with $H^2$ we have that $K^{=}_{\mathcal{M}}(t) = 0.2$ for $p = (0, 0, 0, 1, 0)^T$.

**Computability of the SPF.** The SPF can be hard to compute, due to the possible exponential growth of the matrix $H(t)$ as it requires to calculate all the possible products of length $\leq t$. One may think that $H(t+1)$ could be computed from $H(t)$ simply by setting $H(t+1) = \{Mh : h \text{ column of } H(t) \text{ and } M \in \mathcal{M}\}$. This equality is unfortunately false due to the boolean product that we are using;

indeed it is easy to find two binary matrices $B_1$ and $B_2$ such that $(B_1 \odot B_2)e \neq B_1(B_2 e)$. Here below we list some strategies that can be implemented to decrease the complexity of the problem, by avoiding to compute some products in $\mathcal{M}^{\leq t}$ or by reducing the size of $H(t)$:

- If $A_1, A_2 \in \mathcal{M}^{\leq t}$ and $A_2 \leq A_1$, then $A_2 e \leq A_1 e$. Hence for any stochastic vector $p$ such that $p^T A_1 e \leq k$, it also holds $p^T A_2 e \leq k$, and thus we can erase $A_2 e$ from $H(t)$ without changing the optimal value. Furthermore, for any binary NZ-matrix $B$, $A_2 \leq A_1$ implies $BA_2 \leq BA_1$ which again implies $BA_2 e \leq BA_1 e$. In conclusion, the product $A_2$ can be permanently erased from $\mathcal{M}^{\leq t}$ as every product of type $BA_2$ with $B \in \mathcal{M}^{\leq s}$ will *not* play a role in the solution of program (5) at time $t+s$, for any $s \geq 1$.
- Let $c_1$ and $c_2$ be two columns of $H(t)$. If $c_1 \leq c_2$, then column $c_1$ can be erased from $H(t)$ as the constraint $p^T c_1 \leq k$ in (5) is automatically fulfilled.
- Let $r_1$ and $r_2$ be two rows of $H(t)$. If $r_1 \geq r_2$ then row $r_1$ can be erased from $H(t)$ as, for any stochatic vector $q$, the constraint $r_1 q \geq k$ in (6) is automatically fulfilled by the constraint $r_2 q \geq k$.

### 4.2 Stagnations

Figure 2 shows that the function $K(t)$ can have *stagnations*, i.e. constant parts where $K(t) = K(t+1) = \cdots = K(t+s)$ for an $s > 0$. Suppose that one could prove that there exists an $a = a(n)$ such that $K(t+a) > K(t)$ for all $t$ s.t. $K(t) < 1$, and suppose furthermore that one could prove that there exists a $b = b(n)$ such that $K(t_1) > K(t_2)$ implies $K(t_1) - K(t_2) \geq 1/b$. These two results would provide an upper bound on the exponent of any NZ-primitive set: indeed, since $exp(\mathcal{M}) = \min\{t : K_{\mathcal{M}}(t) = 1\}$, it would hold that $exp(\mathcal{M}) \leq ab(n-1)/n$. In addition, if both $a$ and $b$ were linear in $n$, we would have a quadratic upper bound. We have not been able to derive such results for $K(t)$ yet and what we can say about its stagnations is summarized in Proposition 5. Before stating this proposition we need the following definition:

**Definition 5.** *Given a binary NZ-set $\mathcal{M}$ and an integer $t$, the polytope $P_t$ represents the set of optimal solutions of problem (5).*

Since $H(t)$ has always rank $\geq 1$, then $dim(P_t) \leq n-1$. In the following, given $V$ a set of vectors and $M$ a matrix, $M^T V$ represents the set $\{M^T v : v \in V\}$.

**Proposition 5.** *If $K_{\mathcal{M}}(t) = K_{\mathcal{M}}(t+1)$, then $P_{t+1} \subseteq P_t$ and for any binary row-stochastic matrix $R$ such that $R \leq M$ with $M \in \mathcal{M}$, it holds that $R^T P_{t+1} \subseteq P_t$.*

We remark that, if we proved that $P_{t+1} \subsetneq P_t$ at any time $t$ such that $K(t) = K(t+1)$, then it would hold that $K(t+n) > K(t)$ for any $t$ such that $K(t) < 1$. This can be proved in the case we define the SPF in the same way but using the standard matrix-product instead of the boolean product; the drawback is that now $K(t) = 1$ does not guarantee anymore the presence of a positive product of length $t$. Indeed, the use of the standard matrix-product implies that $H(t)$ does not count anymore the number of positive entries in the rows of each element of $\mathcal{M}^{\leq t}$. In the next section we show that we can define a function $\bar{K}(t) \geq K(t)$ where we can bound the length of its stagnations by a function $a(n) = O(n^2)$.

## 5 Approximation of the SPF: the Function $\bar{K}_{\mathcal{M}}$

We can simplify the SPF by requiring Player B to consider just *deterministic* distributions, i.e. to choose his policy $p$ among the vectors of the canonical basis $\mathcal{E} = \{e_1, e_2, \dots, e_n\}$; this means that we are minimizing Equation (2) and the linear programming (5) just on $\mathcal{E}$. Cleary this new function is, at each time $t$, an upper bound on $K(t)$.

**Definition 6.** *Given a binary NZ-primitive set $\mathcal{M}$, we define the* approximated synchronizing probability function $\bar{K}_{\mathcal{M}}(t)$ *as the optimal value of the following linear programming problem:*

$$\min_{e_i \in \mathcal{E}, k} \frac{k}{n} \quad s.t. \quad e_i^T H(t) \leq k e^T \ . \tag{7}$$

*The set of its optimal solutions is denoted by $\bar{P}_t \subseteq \mathcal{E}$.*

**Proposition 6.** *The approximated SPF is given by:*

$$\bar{K}_{\mathcal{M}}(t) = \frac{1}{n} \min_i \{\max\{H(i,:)\}\} \ . \tag{8}$$

*Furthermore, $\bar{K}_{\mathcal{M}}(t) \geq K_{\mathcal{M}}(t)$ and $\min\{t : \bar{K}_{\mathcal{M}}(t) = 1\} \leq exp(\mathcal{M})$.*

Note that $\bar{K}(t)$ takes values in the set $\{j/n : j = 1, \dots, n\}$ and so

$$\bar{K}(t_1) > \bar{K}(t_2) \quad \Rightarrow \quad \bar{K}(t_1) - \bar{K}(t_2) \geq 1/n \ . \tag{9}$$

Figure 3 shows, for each matrix set $\mathcal{M}_0$, $\mathcal{M}_1$, $\mathcal{M}_2$ respectively from Eq. (1), (3), (4), both the functions $K(t)$ and $\bar{K}(t)$. We now state a result on the length of the stagnations of $\bar{K}(t)$ and we conclude with a conjecture (Conjecture 2) that, if true, would lead to a quadratic upper bound on the exponent of any NZ-primitive set (Proposition 7) and on the reset threshold of a certain class of automata (Proposition 8).

**Theorem 3.** *Let $\mathcal{M}$ be a binary NZ-primitive set and $t \in \mathbb{N}$ such that $\bar{K}(t) < 1$ and $\bar{K}(t) = \bar{K}(t+1)$. Then:*

1. *if $|\bar{P}_t| < n$, $\bar{K}(t+n-1) > \bar{K}(t)$.*
2. *if $|\bar{P}_t| = n$, $\bar{K}\big(t + \frac{n(n+1)}{2} - 1\big) > \bar{K}(t)$.*

Our numerical simulations make us believe that the stagnation length when $|\bar{P}_t| = n$ should be much shorter:

*Conjecture 2.* Let $\mathcal{M}$ be a binary NZ-primitive set and $t \in \mathbb{N}$ such that $\bar{K}(t) < 1$, $\bar{K}(t) = \bar{K}(t+1)$ and $|\bar{P}_t| = n$. Then, $\bar{K}(t+n) > \bar{K}(t)$.

**Proposition 7.** *If Conjecture 2 is true, then for any NZ-primitive set $\mathcal{M}$, it holds that:*

$$exp(\mathcal{M}) \leq 2n(n-1) \ .$$

**Proposition 8.** *If Conjecture 2 is true, then any synchronizing automaton $\mathcal{A}$ for which there exists a binary NZ-set $\mathcal{M}$ such that $\mathcal{A}$ is made of all the binary row-stochastic matrices that are dominated by at least one matrix of $\mathcal{M}$ has reset threhsold $\leq n(n-1)$.*

We remark that any result of type $\bar{K}(t+O(n)) > \bar{K}(t)$ when $\bar{K}(t) < 1$, $\bar{K}(t) = \bar{K}(t+1)$ and $|\bar{P}_t| = n$ would imply that $exp(\mathcal{M}) = O(n^2)$ for any NZ-primitive set $\mathcal{M}$ and that $rt(\mathcal{A})=O(n^2)$ for any automaton $\mathcal{A}$ as in Proposition 8.

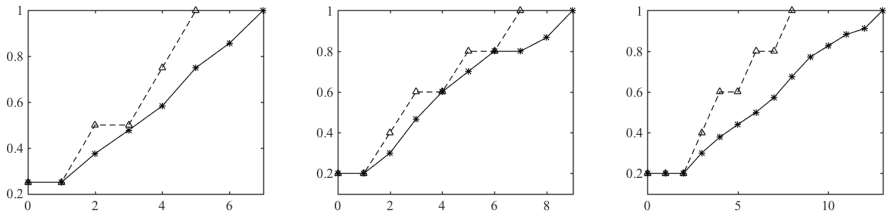


Fig. 3: The functions $K(t)$ (solid line) and $\bar{K}(t)$ (dashed line) of the sets $\mathcal{M}_0$ in Eq.(1) (left picture), $\mathcal{M}_1$ in Eq.(3) (central picture) and $\mathcal{M}_2$ in Eq.(4) (right picture).

## 6 Conclusions

Synchronizing automata are deeply connected to primitive sets of matrices; in this paper we addressed the primitivity phenomenon from a probabilistic game point of view by developing a new tool, the synchronizing probability function for primitive sets, whose aim is to bring more understanding to the primitivity process, which is not yet well understood. We believe that this tool would lead to a better insight on the synchronization phenomenon and provide new possibilities to prove Černý's conjecture. Indeed, the SPF takes into account the speed at which a primitive set reaches its first positive product: numerical experiments have shown that its behavior seems smooth and regular (after a potential stagnation phase of length smaller than n). We have then introduced the function $\bar{K}(t)$, which is an upper bound on the SPF, and we have showed that it cannot remain constant for too long. Supported by numerical experiments, we finally stated a conjecture that, if true, would lead to an upper bound of $2n(n-1)$ on the exponent of any NZ-primitive set and to an upper bound of $n(n-1)$ on the reset threshold of any synchronizing automaton made of all the binary row-stochastic matrices dominated by at least a matrix of a binary NZ-matrix set. Lastly, an upper bound on the length of the stagnations of $K(t)$ could also translate into a new upper bound on the exponent of any NZ-primitive set.

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# Appendix

*Proof. of Proposition 1.*
$K_{\mathcal{M}}(t)$ takes values in $[0,1]$ as it represents a probability and it is nondecreasing in $t$ since $\mathcal{M}^{\leq t} \subseteq \mathcal{M}^{\leq t+1}$ for every $t$. By (2), $K_{\mathcal{M}}(t)$ is equal to 1 if and only if for any stochastic vector $p$ there exist $M \in \mathcal{M}^{\leq t}$ such that $p^T M(e/n) = 1$. Then by taking $p = e/n$ it follows that the corresponding matrix $M$ is the all-ones matrix. □

*Proof. of Theorem 2.*
Programs (5) and (6) are the dual of each other. Since they both admit feasible solutions, their optima must be equal by the duality theorem of linear programming. □

*Proof. of Proposition 2.*

1. Since $H(0) = e$, $k = 1$ and $p = e/n$ is a feasible solution for (5), so $K(0) \leq 1/n$. On the other hand, $q = 1$ and $k = 1$ is a feasible solution for (6), so $K(0) \geq 1/n$.
2. We claim that $K(t) = 1/n$ if and only if $H(t)$ has an all-ones row. In fact, if $i$ is the index of an all-ones row of $H(t)$, $p = e_i$ and $k = 1$ is an optimal solution for (5). On the contrary, if every row of $H(t)$ has at least one entry $> 1$, then for every stochastic vector $p$ there exists a $1 \leq j \leq n$ such that $p(j) > 0$ and $\max\{p^T H(t)\} \geq 2p(j) + (1 - p(j)) = 1 + p(j) > 1$. This implies that $k > 1$ and so $K(t) > 1/n$, which proves the claim. Since the set is primitive, $H(1)$ must have a column $M_i e$, $M_i \in \mathcal{M}$, with an entry $\geq 2$. Suppose this entry is in row $s$. As $\mathcal{M}$ is irreducible, its associated digraph is strongly connected (see Sect. 2), and so given $l \neq s$ there exists a product $P$ of at most $n-1$ matrices in $\mathcal{M}$ such that $P(l,s) > 0$. Since $\mathcal{M}$ is NZ, this implies that $PM_i e(l) \geq M_i e(s) \geq 2$; but $PM_i \in \mathcal{M}^{\leq n}$, so for every $l = 1, \ldots, n$, $H(n)$ has a column whose $l$-th entry is greater or equal than 2. By what we have proven before, this implies that $K(n) > 1/n$.
3. If $p \in \mathbb{R}^n_+$ and $p^T e = 1$, then there exists a index $1 \leq j \leq n$ such that $p(j) \geq 1/n$. By hypothesis and what proved in item 2., for every stochastic vector $p$ it holds that $\max\{p^T H(t)\} \geq 1 + 1/n$ and so $K(t) \geq (n+1)/n^2$.
4. If $K(t) < 1$, then every column of $H(t)$ has at least one entry $\leq n-1$. Thus, $k = (e/n)^T ke \leq (e/n)^T Hq = ((e/n)^T H)q \leq ((n^2-1)/n)e^T q = (n^2-1)/n$ and so $K(t) \leq (n^2-1)/n^2$. □

*Proof.* *of Proposition 3.*
We indicate with $(6)'$ the program (6) where $H(t)$ is replaced by a submatrix $H'(t)$ of size $n \times s$ and with $K'(t)$ its optimum. Let $q'^*$ be one of the optimal solutions of $(6)'$; $q'^*$ is a feasible solution also for program (6) so $K'(t) \geq K(t)$. We now show that for an appropriate submatrix $H'$, we have $K'(t) \leq K(t)$. Let $q^*$ be an optimal solution of (6) with all positive entries: if this is not the case, we can just remove its zero entries and the corresponding columns of $H(t)$ without changing the optimum. If $H(t)$ has more than $n$ columns, the system $Hx = 0$ has a nonzero solution. We can suppose wlog that

$$e^T x \leq 0. \tag{10}$$

Then, setting

$$\lambda = \min_{x(i)<0} \{q^*(i)/\,(-x(i))\} \tag{11}$$

we obtain that $q^* + \lambda x$ is a feasible solution for (6). Indeed $q^* + \lambda x \geq 0$ by (11) and $H(q^* + \lambda x) = Hq^* \geq ke$. Furthermore (10) implies that $e^T(q^* + \lambda x) \leq 1$ as $\lambda > 0$. In the case $e^T(q^* + \lambda x) < 1$ we can increase a nonzero entry of $q^* + \lambda x$ until the sum is equal to one without losing optimality. By construction, $q^* + \lambda x$ has a zero entry so we can remove the corresponding column in $H$ without changing the optimum. We conclude by iteratively applying the above argument until there are no more than $n$ columns in $H$. □

*Proof.* *of Proposition 4.*
We prove it for program (5). As $\mathcal{M}^t \subset \mathcal{M}^{\leq t}$, it is clear that the optimal value decreases; we show that it actually remains the same. Let $A_i \in \mathcal{M}^{\leq t_i}$ with $t_i < t$: we claim that there exists a product $L \in \mathcal{M}^t$ such that $A_i e \leq Le$ and so $A_i e$ can be erased from $H(t)$ as for any optimal solution $p$ of (5), $p^T A_i e \leq p^T Le \leq k$. Let $M \in \mathcal{M}^{t-t_i}$ be a product that dominates a permutation matrix (it always exists by hypothesis); then for every column $a$ of $A_i$ there exist a column $l$ of $L = A_i M$ such that $l \geq a$, which implies $Le \geq A_i e$. Since $L$ is a product of length $t$, the claim is proven. □

*Proof.* *of Proposition 5.*
The fact that $P_{t+1} \subseteq P_t$ is trivial. Let now $p \in P_{t+1}$, $R$ be a binary row-stochastic matrix such that $R \leq M$ with $M \in \mathcal{M}$, and $A \in \mathcal{M}^{\leq t}$. By hypothesis, $nK(t) = k \geq p^T(MA)e \geq p^T(RA)e = p^T R(Ae)$, where the last two passages hold because $R$ is binary and row-stochastic (see Sect. 2). Since $(p^T R)^T = R^T p$ is a stochastic vector, then $R^T p \in P_t$. □

*Proof.* *of Theorem 3.*
*1.* If $\bar{K}(t) = \bar{K}(t+1)$, then $\bar{P}_{t+1} \subseteq \bar{P}_t$. By the same reasoning used in the proof of Proposition 5, it holds that for any binary row-stochastic matrix $R$ s.t. $R \leq M$ for a matrix $M \in \mathcal{M}$, $R^T \bar{P}_{t+1} \subseteq \bar{P}_t$. We now claim that $\bar{P}_{t+1} \subsetneq \bar{P}_t$. Indeed, suppose by contrary that $\bar{P}_{t+1} = \bar{P}_t$. This means that $R^T \bar{P}_t \subseteq \bar{P}_t$ for any binary row-stochastic matrix $R$ dominated by an element of $\mathcal{M}$, which in turn implies that for any product $R_1 \cdots R_l$ of binary row-stochastic matrices

dominated by matrices in $\mathcal{M}$, it holds that $(R_1 \cdots R_l)^T \bar{P}_t \subseteq \bar{P}_t$. The set of all the binary row-stochastic matrices dominated by a least a matrix in $\mathcal{M}$ is indeed the automaton $\mathcal{A}$ described in Theorem 1: by the same theorem, since $\mathcal{M}$ is primitive then $\mathcal{A}$ is synchronizing and so there exists a product $\bar{R} = R_{i_1} \cdots R_{i_s}$ of such binary row-stochastic matrices that has an all-ones column, say in position $j$. Since $\bar{R}^T \bar{P}_t = \{e_j\}$, we have that $e_j \in \bar{P}_t$. The set $\mathcal{M}$ is irreducible and so the automaton $\mathcal{A}$ is strongly connected (as $\mathcal{M}$ and $\mathcal{A}$ have the same associated digraph just with a different labelling); therefore, for any $l \neq j$ there exists a product $W_l$ of the matrices in $\mathcal{A}$ such that $W_l(j,l) = 1$ and so $\bar{R}W_l$ has an all-ones column in position $l$. As $(\bar{R}W_l)^T \bar{P}_t = \{e_l\}$ and $(\bar{R}W_l)^T \bar{P}_t \subseteq \bar{P}_t$, it holds that $e_l \in \bar{P}_t$ for every $l = 1, \ldots, n$, which contradicts the hypothesis. This means that $\bar{P}_{t+1} \subsetneq \bar{P}_t$ and so at each time step, if there is a stagnation, the cardinality of $\bar{P}_t$ has to decrease: as $1 \leq |\bar{P}_t| \leq n-1$, we conclude.
*2.* We rely on the fact that an NZ-set is primitive if and only if for any $i, j, k \in \{1, \ldots, n\}$ there exists a product $B$ of matrices of $\mathcal{M}$ such that $B(k,i) = 1 = B(k,j)$ ([1], Theorem 1). We claim that the length of the product $B$ is not greater than $n(n+1)/2 - 1$: indeed, let $D$ be the directed graph of vertex set $V = \{(i,j) : 1 \leq i \leq j \leq n\}$ and with a directed edge from vertex $(i,j)$ to vertex $(i',j')$ if there exists a matrix $M \in \mathcal{M}$ such that $M(i,i') > 0$ and $M(j,j') > 0$, or $M(i,j') > 0$ and $M(j,i') > 0$; in this case we label the edge with $M$ (multiple labels are allowed). A directed path from $(i,j)$ to $(i',j')$ whose edges are sequentially labelled by $M_{l_1} \ldots M_{l_u}$ means that $M_{l_1} \cdots M_{l_u}(i,i') > 0$ and $M_{l_1} \cdots M_{l_u}(j,j') > 0$, or $M_{l_1} \cdots M_{l_u}(i,j') > 0$ and $M_{l_1} \cdots M_{l_u}(j,i') > 0$. As $V$ has cardinality $n(n+1)/2$, the shortest path from a vertex $(i,j)$ to a vertex $(k,k)$ has length at most $n(n+1)/2 - 1$. Now, the fact that $|\bar{P}_t| = n$ implies that every row of $H(t)$ has an entry equal to $k = n\bar{K}(t)$ and $\bar{K}(t+1) = \bar{K}(t)$ implies that the same must hold for $H(t+1)$. Fix $i \in \{1, \ldots, n\}$ and choose a product $M \in \mathcal{M}^{\leq t}$ that has $k$ ones in row $i$: since $k < n$ by hypothesis and the set is NZ, there must exist two indices $j$ and $l$ such that $M(i,l) = 0$ and $M(j,l) = 1$. Let $B \in \mathcal{M}^{\leq \frac{n(n+1)}{2} - 1}$ such that $B(i,i) = 1 = B(i,j)$: then $BM(i,:) \geq M(i,:) + M(j,:)$ and so $BMe(i) > Me(i) = k$. Since this holds for every $i$, it implies that every row of $H(t + \frac{n(n+1)}{2} - 1)$ has an entry $> k$ and so $\bar{K}\big(t + \frac{n(n+1)}{2} - 1\big) > \bar{K}(t)$. □

*Proof. of Proposition 7.*
Let $\mathcal{M} = \{M_1, \ldots, M_m\}$ be a NZ-primitive set; we can suppose wlog that the matrices in $\mathcal{M}$ are binary. Let $t_1 = \min\{t : \bar{K}_{\mathcal{M}}(t) = 1\}$. If Conjecture 2 is true, then by item 1. of Theorem 3 it holds that $\bar{K}(t+n) > \bar{K}(t)$ for every $t \in \mathbb{N}$ such that $\bar{K}(t) < 1$. This, combined with Equation (9), implies that

$$t_1 \leq n(n-1).$$

We set $H = H(t_1)$. Since the matrix $H$ has entries in $\{1, \ldots, n\}$, the fact that $\bar{K}_{\mathcal{M}}(t_1) = 1$ implies that for every $i = 1, \ldots, n$, $\max\{H(i,:)\} = n$, i.e. every row of $H$ has an entry equal to $n$. This in turn implies that for every $i = 1, \ldots, n$ there exists a product $A_i \in \mathcal{M}^{\leq t_1}$ such that $A_i(i,:) = (1, 1, \ldots, 1)$ i.e. $A_i$ has an all-ones row in position $i$. We can apply the same reasoning to the transpose set

$\mathcal{M}^T = \{M_1^T, \ldots, M_m^T\}$, as it is still a NZ-primitive set: by setting $t_2 = \min\{t : \bar{K}_{\mathcal{M}^T}(t) = 1\}$, we have that $t_2 \le n(n-1)$ and for every $i = 1, \ldots, n$ there exists a product $B_i \in (\mathcal{M}^T)^{\le t_2}$ such that $B_i$ has an all-ones row in position $i$. Let's fix $i \in \{1, \ldots n\}$. Then $B_i^T$ has an all-ones column in position $i$ and $B_i^T \in \mathcal{M}^{\le t_2}$, which implies that $B_i^T A_i$ is a positive product of matrices in $\mathcal{M}$. Therefore,

$$exp(\mathcal{M}) \le t_1 + t_2 \le 2n(n-1).$$

□

*Proof. of Proposition 8.*
Let $\mathcal{A}$ be a synchronizing automaton whose letters are all the binary row-stochastic matrices that are dominated by at least one matrix of the binary NZ-set $\mathcal{M} = \{M_1, \ldots, M_m\}$. By Theorem 1, $\mathcal{M}$ is primitive. As shown in the proof of Proposition 7, Conjecture 2 implies that for any $l = 1, \ldots, n$, there exist a product in $\mathcal{M}^{\le n(n-1)}$ that has an all-ones column in position $l$; we fix $l = 1$ and we let $M = M_{i_1} \cdots M_{i_u} \in \mathcal{M}^{\le n(n-1)}$ be the product with $M(:,1) = (1, \ldots, 1)^T$. We claim that for every $r = 1, \ldots, u$ we can safely set to zero some entries of $M_{i_r}$ in order to make it row-stochastic while making sure that the final product has still all the entries in the first column equal to 1. In other words, we claim that for every $r = 1, \ldots, u$ we can select a binary row-stochastic matrix $A_r \le M_{i_r}$ such that $A_1 \cdots A_u(:,1) = (1, \ldots, 1)^T$. If this is true, then by hypothesis $A_r$ belongs to $\mathcal{A}$ for every $r$, so $A_1 \cdots A_u$ is a synchronizing word for the automaton $\mathcal{A}$ and therefore $rt(\mathcal{A}) \le n(n-1)$, which would conclude the proof. To prove the claim we apply the same argument used in the proof of Theorem 16 in [4]: let $D_r$ be the digraph associated to the matrix $M_{i_r}$ (see Sect. 2) and $E_r$ its edge set. The fact that $M_{i_1} \cdots M_{i_u}(:,1) = (1, \ldots, 1)^T$ means that for every $j = 1, \ldots, n$ there exists a sequence of vertices $v_j^1, \ldots, v_j^{u+1} \in \{1, \ldots, n\}$ such that:

$$v_j^1 = j \ , \tag{12}$$

$$v_j^{u+1} = 1 \ , \tag{13}$$

$$(v_j^r, v_j^{r+1}) \in E_r \quad \forall r = 1, \ldots, u \ . \tag{14}$$

Furthermore, we can impose an additional property on these sequences: if at step $s$ two sequences share the same vertex, then they have to coincinde for all the steps $s' > s$. More formally, if at step $s$ we have $v_j^s = v_{j'}^s$ for $j \ne j'$, then we set $v_{j'}^{s'} = v_j^{s'}$ for all $s' > s$ as the new sequence $v_{j'}^1, \ldots, v_{j'}^s, v_j^{s+1}, \ldots, v_j^{u+1}$ for vertex $j'$ fulfils all the requirements (12),(13),(14). We now remove, for every $r = 1, \ldots, u$, all the edges from $E_r$ that are not of type (14); we call this new edge set $\tilde{E}_r$ and let $\tilde{D}_r$ be the subgraph of $D_r$ with edge set $\tilde{E}_r$. Then, for every $r = 1, \ldots, u$, we set $A_r$ to be the binary matrix whose associated digraph is $\tilde{D}_r$; by construction $A_r$ has at most one positive entry in each row. If it happens that $A_r$ has a row with all zero-entries, we set one of them to 1. The final matrix $A_r$ is binary row-stochastic and $A_r \le M_{i_r}$ for all $r = 1, \ldots, u$. Finally, $A_1 \cdots A_u(:,1) = (1, \ldots, 1)^T$ by construction. □