Frames and semi-frames in mathematical physics

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Given a total sequence in a Hilbert space, we speak of an upper (resp. lower) semi-frame if the sequence satisfies only the upper (resp. lower) frame bound. Equivalently, for an upper semi-frame, the frame operator is bounded, but has an unbounded inverse, whereas a lower semi-frame has an unbounded frame operator, with bounded inverse. Both concepts are extended to the continuous case. For upper semi-frames, in the discrete and the continuous case, we build two natural Hilbert scales which may yield a novel characterization of certain function spaces of interest in signal processing or mathematical physics. We present some results concerning the duality between lower and upper semi-frames, as well as two examples involving wavelets/coherent states.

1 Introduction

Given a separable Hilbert space $\mathcal{H}$, it is often convenient to expand an arbitrary element $f \in \mathcal{H}$ in a sequence of simple, basic elements (atoms) $\Psi = (\psi_k)$, $k \in \Gamma$, $f = \sum_{k\in\Gamma} c_k \psi_k$, where the sum converges in an adequate fashion (e.g. strongly and unconditionally) and the coefficients c_k are (preferably) unique and easy to compute. There are several possibilities for obtaining that result. In order of increasing generality, we can require that Ψ be:

(i) an orthonormal basis;

(ii) a Riesz basis, necessarily of the form $\psi_k = V e_k$, where V is a bounded bijective operator and (e_k) is an orthonormal basis;

(iii) a frame, *i.e.*, there exist constants $\mathsf{m} > 0,\ \mathsf{M} < \infty$ such that

$$\mathsf{m}\, \|f\|^2 \leq \sum_{k\in\Gamma} |\langle \psi_k | f\rangle|^2 \leq \mathsf{M}\, \|f\|^2, \forall\, f \in \mathcal{H}. \tag{1.1}$$

Uniqueness is achieved in the first two cases, but lost in the third one. However, even a frame may be too restrictive, in the sense that it may impossible to satisfy the two frame bounds simultaneously. Accordingly, we define Ψ to be an *upper (resp. lower) semi-frame* if it is a total set and satisfies the upper (resp. lower) frame inequality. Then the question is to find whether the signal can still be reconstructed from its expansion coefficients. Furthermore, we generalize the concept to the continuous case. A detailed discussion may be found in two recent articles [4, 5], that we follow closely.

We start with the so-called continuous generalized frames, introduced some time ago [1, 2], and studied further by a number of authors (see [4] for references).

Let $\mathcal{H}$ be a Hilbert space and X a locally compact space with measure ν. Then a *generalized frame* for $\mathcal{H}$ is a family of vectors $\Psi := \{\psi_x,\, x \in X\}$, $\psi_x \in \mathcal{H}$, indexed by points of X, such that the map $x \mapsto \langle f|\psi_x\rangle$ is measurable, $\forall\, f \in \mathcal{H}$, and

$$\int_X \langle f|\psi_x\rangle\langle\psi_x|f'\rangle \;\mathrm{d}\nu(x) = \langle f|Sf'\rangle,\ \forall\, f, f' \in \mathcal{H},$$

where S is a bounded, positive, self-adjoint, invertible operator on $\mathcal{H}$, called the *frame operator*.

The operator S is invertible, but its inverse S^{-1}, while self-adjoint and positive, need not be bounded. We say that Ψ is a (continuous) *frame* if S^{-1} is bounded or, equivalently, if there exist constants $\mathsf{m} > 0$ and $\mathsf{M} < \infty$ (the frame bounds) such that

$$\mathsf{m} \|f\|^2 \leqslant \langle f|Sf\rangle = \int_X |\langle\psi_x|f\rangle|^2 \,\mathrm{d}\nu(x) \leqslant \mathsf{M} \|f\|^2, \forall f \in \mathcal{H}. \tag{1.2}$$

First, Ψ is a total set in $\mathcal{H}$. Next define the *analysis operator*. $C_\Psi : \mathcal{H} \to L^2(X, \mathrm{d}\nu)$ by $(C_\Psi f)(x) = \langle\psi_x|f\rangle,\ f \in \mathcal{H}$, and denote its range by $R_C := \mathsf{Ran}(C_\Psi)$. Its adjoint, called the *synthesis operator*, reads (the integral being understood in the weak sense)

$$C_\Psi^* F = \int_X F(x)\,\psi_x \,\mathrm{d}\nu(x), \text{ for } F \in L^2(X, \mathrm{d}\nu). \tag{1.3}$$

Then $C_\Psi^* C_\Psi = S$ and $\|C_\Psi f\|^2_{L^2(X)} = \|S^{1/2}f\|^2_{\mathcal{H}} = \langle f|Sf\rangle$. Furthermore, C_Ψ is injective, since $S > 0$, so that $C_\Psi^{-1} : R_C \to \mathcal{H}$ is well-defined.

If X is a discrete set and ν a counting measure, we go back to the familiar (discrete) frame (1.1). The Hilbert space $L^2(X, \mathrm{d}\nu)$ becomes ℓ^2 and the analysis operator $C : \mathcal{H} \to \ell^2$, the synthesis operator $D : \ell^2 \to \mathcal{H}$, and the frame operator $S : \mathcal{H} \to \mathcal{H}$ take their usual form:

$$\begin{aligned} Cf &= \{\langle\psi_k|f\rangle, k \in \Gamma\}; \\ Dc &= \sum_{k\in\Gamma} c_k\psi_k, \text{ where } c = (c_k); \\ Sf &= \sum_{k\in\Gamma} \langle\psi_k|f\rangle\,\psi_k, \text{ so that } \langle f|Sf\rangle \ = \sum_{k\in\Gamma} |\langle\psi_k|f\rangle|^2 . \end{aligned}$$

In the case of a Bessel sequence (*i.e.*, when the upper frame inequality is satisfied), *a fortiori* for a frame, all three operators are bounded and one has $D = C^*$, $C = D^*$ and $S = C^*C$.

For a discrete frame Ψ, we have two reconstruction formulas, both unconditionally norm convergent,

$$\begin{aligned} f = S^{-1}Sf &= \sum_{k\in\Gamma} \langle\psi_k|f\rangle S^{-1}\psi_k, \text{ for every } f \in \mathcal{H}, \\ f = SS^{-1}f &= \sum_{k\in\Gamma} \langle S^{-1}\psi_k|f\rangle\psi_k, \text{ for every } f \in \mathcal{H}. \end{aligned}$$

The sequence $(S^{-1}\psi_k)$ is also a frame, called the *canonical dual* frame.

Next, define on $R_C := \mathsf{Ran}\, C$ a new inner product

$$\langle c|d\rangle_\Psi = \langle c|CS^{-1}C^{-1}d\rangle_{\ell^2} = \langle c|G^{-1}d\rangle_{\ell^2},\ c, d \in R_C, \tag{1.4}$$

where we have defined the operator $G^{-1} := CS^{-1}C^{-1}$. It turns out that R_C complete in $\langle\cdot|\cdot\rangle_\Psi$. Thus $(R_C, \langle\cdot|\cdot\rangle_\Psi) =: \mathfrak{H}_\Psi$ is a Hilbert space.

2 Discrete upper semi-frames

Let now $\Psi = (\psi_k,\ k \in \Gamma)$ be an upper semi-frame, equivalently, a total (Bessel) sequence satisfying the upper frame bound, that is, there is a constant $\mathsf{M} < \infty$ such that

$$0 < \sum_{k\in\Gamma} |\langle\psi_k|f\rangle|^2 \leqslant \mathsf{M} \|f\|^2, \forall f \in \mathcal{H},\ f \neq 0.$$

We say that $\Psi = (\psi_k)$ is *regular* if every ψ_k belongs to $\mathsf{Dom}(S^{-1}) = \mathsf{Ran}(S) =: R_S$.

Lemma 2.1 *Let Ψ be an upper semi-frame. Then one has:*

(1) The analysis operator C is injective and bounded, the synthesis operator D is bounded with dense range. The frame operator S is bounded, self-adjoint, positive, with dense range. Its inverse S^{-1} is densely defined and self-adjoint.

(2) $C(R_S) \subseteq R_C \subseteq \overline{R_C}$*, with dense inclusions, where* $\overline{R_C}$ *is the closure of* R_C *in* ℓ^2.

In this case, we have reconstruction formulas as before, but with restrictions:

$$f = SS^{-1}f = \sum_{k\in\Gamma} \langle \psi_k | S^{-1} f\rangle \psi_k, \; \forall \, f \in R_S \, ;$$

$$f = SS^{-1}f = \sum_{k\in\Gamma} \langle S^{-1}\psi_k | f\rangle \psi_k, \; \forall f \in R_S, \text{ if } \Psi \text{ is regular.}$$

Next, we define the following operators, acting in the Hilbert space $\overline{R_C}$:

$$\begin{aligned} G_S = C\,S\,C^{-1} &: R_C \to C(R_S), \text{ which is bounded, positive and symmetric,} \\ G_S^{-1} = C\,S^{-1}C^{-1} &: C(R_S) \to R_C, \text{ which is positive and essentially self-adjoint [1].} \end{aligned}$$

Let $G = \overline{G_S}$ be the closure of G_S and let G^{-1} be the self-adjoint extension of G_S^{-1}, with domain $\mathsf{Dom}(G^{-1}) = C(R_D)$. Then G and G^{-1} are self-adjoint and positive, and inverse of each other, *i.e.*,

$$G^{-1}G = I|_{\overline{R_C}}, \quad GG^{-1} = I|_{C(R_D)}.$$

As in (1.4), we define on $C(R_S)$ the new inner product $\langle c|d\rangle_\Psi = \langle c|G^{-1}d\rangle_{\ell^2}$. Taking the completion of $C(R_S)$ in the corresponding norm, we obtain a new Hilbert space, that we call again $\mathfrak{H}_\Psi$ Then one shows the following

Theorem 2.2 *One has :*

(1) $\mathfrak{H}_\Psi$ *coincides with* R_C *(as sets) and* C *is a unitary map (isomorphism) between* $\mathcal{H}$ *and* $\mathfrak{H}_\Psi$.

(2) The norm $\|.\|_\Psi$ *is equivalent to the graph norm of* $G^{-1/2}$ *and, therefore,* $\mathsf{Dom}(G^{-1/2}) = \mathfrak{H}_\Psi$.

(3) $C^{*(\Psi)} = \big(S^{-1}D\big)|_{\mathfrak{H}_\Psi}$*, where* $C^{*(\Psi)} : \mathfrak{H}_\Psi \to \mathcal{H}$ *is the adjoint of* $C : \mathcal{H} \to \mathfrak{H}_\Psi$*. Moreover, for every* $f \in \mathcal{H}$*, one has*

$$f = C^{*(\Psi)}Cf = \big(S^{-1}D\big)\,Cf.$$

Putting everything together, we obtain the following commutative diagram:

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{H} \arrow[r, shift left, "C"] \arrow[d, shift right, "S"'] & \mathfrak{H}_\Psi = R_C \subseteq \overline{R_C} \subseteq \ell^2 \arrow[l, shift left, "C^{-1}"] \arrow[d, shift right, "G_S"'] \arrow[dl, "D"'] \\
\mathcal{H} \supseteq \mathsf{Dom}(S^{-1}) = R_S \arrow[u, shift right, "S^{-1}"'] \arrow[r, shift left, "C"] & C(R_S) \subseteq \ell^2 \arrow[l, shift left, "C^{-1}"] \arrow[u, shift right, "G_S^{-1}"']
\end{tikzcd} \qquad (2.1)$$

Corollary 2.3 $G^{1/2} : \overline{R_C} \to \mathfrak{H}_\Psi$ *is an isomorphism and so is its inverse* $G^{-1/2} : \mathfrak{H}_\Psi \to \overline{R_C}$.

In order to get a nice reproducing kernel, assume Ψ to be regular :

Theorem 2.4 *Let* (ψ_k) *be a regular upper semi-frame. Then* $\mathfrak{H}_\Psi$ *is a reproducing kernel Hilbert space, with kernel given by the (matrix) operator* $S^{-1}D$*, namely, the matrix* $\mathcal{G}$*, where*

$$\mathcal{G}_{k,l} = \langle \psi_k | S^{-1}\psi_l\rangle = \langle \psi_k | C^{-1}G^{-1}C\psi_l\rangle.$$

In that case, we get a reconstruction formulas, with unconditional convergence, but restricted to R_S, namely, for Ψ regular and for all $f \in R_S$:

$$f = \sum_{k\in\Gamma} \langle S^{-1}\psi_k | f\rangle \psi_k$$

If Ψ is not regular, we have to use the language of distributions, *i.e.*, a Gel'fand triplet. This runs as follows.

When Ψ is regular, one has

$$\sum_{k,l\in\Gamma} \overline{d_k}\, \mathcal{G}_{k,l}\, d'_l = \langle C^{-1}d|SC^{-1}d'\rangle_{\mathcal{H}},\ \forall\, d, d' \in \mathfrak{H}_\Psi.$$

Since C is isometric and S is bounded, this relation defines a bounded sesquilinear form over $\mathfrak{H}_\Psi$:

$$K^\Psi(d,d') = \langle C^{-1}d|SC^{-1}d'\rangle_{\mathcal{H}} = \langle d|CSC^{-1}d'\rangle_\Psi = \langle d|d'\rangle_{\ell^2}.$$

The point is that all this remains true even if Ψ is not regular. Denote by $\mathfrak{H}_\Psi^\times$ the completion of $\mathfrak{H}_\Psi$ in the norm given by K^Ψ. In this way one obtains, with continuous and dense range embeddings,

$$\mathfrak{H}_\Psi \ \subset\ \mathfrak{H}_0 \ \subset\ \mathfrak{H}_\Psi^\times\,, \tag{2.2}$$

where

- $\mathfrak{H}_\Psi = R_C$, which is a Hilbert space for the norm $\|\cdot\|_\Psi = \langle\cdot|G^{-1}\cdot\rangle_{\ell^2}^{1/2}$;
- $\mathfrak{H}_0 = \overline{\mathfrak{H}_\Psi} = \overline{R_C}$ is the closure of $\mathfrak{H}_\Psi$ in ℓ^2;
- $\mathfrak{H}_\Psi^\times$ is the completion of $\mathfrak{H}_0$ (or $\mathfrak{H}_\Psi$) in the norm $\|\cdot\|_{\Psi^\times} := \langle\cdot|G\cdot\rangle_{\ell^2}^{1/2}$ and the conjugate dual of $\mathfrak{H}_\Psi$.

If Ψ regular, all three spaces $\mathfrak{H}_\Psi, \mathfrak{H}_0, \mathfrak{H}_\Psi^\times$ are *reproducing kernel Hilbert spaces*, with the same (matrix) kernel $\mathcal{G}(k,l) = \langle\psi_k|S^{-1}\psi_l\rangle$. If S^{-1} is bounded (*i.e.*, Ψ is a frame), the three Hilbert spaces coincide as sets, with equivalent norms, since $S, S^{-1} \in \mathrm{GL}(\mathcal{H})$. Note, however, that no explicit reconstruction formulas are obtained!

In the triplet (2.2), the three spaces may be described as follows:

- $\mathfrak{H}_0$ is a sequence space contained in ℓ^2 (possibly ℓ^2 itself),
- $\mathfrak{H}_\Psi$ is a smaller sequence space, e.g. a space of *decreasing* sequences,
- $\mathfrak{H}_\Psi^\times$ is the (Köthe) dual of $\mathfrak{H}_\Psi$, normally not contained in ℓ^2, typically a space of *increasing* sequences.

Now, on the side of $\mathcal{H}$, the 'small space' is the form domain of S^{-1}, with norm $\|\cdot\|_\Psi^\sim = \langle\cdot|S^{-1}\cdot\rangle_{\mathcal{H}}^{1/2}$. Consider now the smaller space $R_S = \mathsf{Dom}(S^{-1})$, with norm $\|\cdot\|_{\mathfrak{S}} = \langle S^{-1}\cdot|S^{-1}\cdot\rangle_{\mathcal{H}}^{1/2}$. Thus we obtain a new Hilbert space $\mathfrak{S}$ and a new triplet

$$\mathfrak{S} \subset \mathcal{H} \subset \mathfrak{S}^\times. \tag{2.3}$$

In that way, one gets a reconstruction formula in the sense of distributions:

$$\langle f|f'\rangle = \sum_{k\in\Gamma} \overline{Cf} \langle S^{-1}\psi_k|f'\rangle\psi_k,\ \forall\, f \in \mathcal{H},\ \forall\, f' \in \mathfrak{S} = R_S.$$

Mapping everything into ℓ^2 by C, we obtain the following scale of Hilbert spaces:

$$\mathfrak{H}_{\mathfrak{S}} \subset \mathfrak{H}_\Psi \subset \mathfrak{H}_0 = \overline{R_C} \subset \mathfrak{H}_\Psi^\times \subset \mathfrak{H}_{\mathfrak{S}}^\times, \tag{2.4}$$

where $\mathfrak{H}_{\mathfrak{S}} := C\mathfrak{S} = C(R_S)$, with norm $\|\cdot\|_{\mathfrak{S}} = \langle\cdot|G^{-3}\cdot\rangle_{\ell^2}^{1/2}$.

Now we observe that the multiplet (2.4) is the central part of the scale of Hilbert spaces built on the powers of $G^{-1/2}$: $\mathfrak{H}_n = D(G^{-n/2}), n \in \mathbb{Z}$, with the graph norm:

$$\mathfrak{H}_\Psi \equiv \mathfrak{H}_1, \quad C(R_D) \equiv \mathfrak{H}_2, \quad \mathfrak{H}_{\mathfrak{S}} \equiv \mathfrak{H}_3, \quad \mathfrak{H}_{\mathfrak{S}^\times} \equiv \mathfrak{H}_{-1}, \quad \ldots$$

The same holds on the side of of $\mathcal{H}$: $\mathcal{H}_n := D(S^{-n/2}), n \in \mathbb{Z}$.

Iterating, one gets the following diagram, that generalizes (2.1):

$$
\begin{tikzcd}
 & & \mathcal{H}_2 \arrow[d, equal] & \mathcal{H}_1 \arrow[d, equal] & \mathcal{H}_0 \arrow[d, equal] & \mathcal{H}_{-1} \arrow[d, equal] & \mathcal{H}_{-2} \arrow[d, equal] & \\
 & \cdots \arrow[r, "S^{-1/2}"] & R_S = \mathfrak{S} \arrow[r, "S^{-1/2}"] \arrow[dl, "C"'] & R_D \arrow[r, "S^{-1/2}"] \arrow[dl, "C"'] & \mathcal{H} \arrow[r, "S^{-1/2}"] \arrow[dl, "C"'] & R_{S^{-1/2}} \arrow[r, "S^{-1/2}"] \arrow[dl, "C"'] & R_{S^{-1}} = \mathfrak{S}^{\times} \arrow[r, "S^{-1/2}"] \arrow[dl, "C"'] & \cdots \\
\cdots \arrow[r, "G^{-1/2}"] & C(R_S) \arrow[r, "G^{-1/2}"] \arrow[d, equal] & C(R_D) \arrow[r, "G^{-1/2}"] \arrow[d, equal] & R_C \arrow[r, "G^{-1/2}"] \arrow[ul, "D"] \arrow[d, equal] & \overline{R_C} \arrow[r, "G^{-1/2}"] \arrow[ul, "D"] \arrow[d, equal] & C(\mathfrak{S}^{\times}) \arrow[r, "G^{-1/2}"] \arrow[ul, "D"] \arrow[d, equal] & \cdots & \\
 & \mathfrak{H}_{\mathfrak{S}} \equiv \mathfrak{H}_3 & \mathfrak{H}_2 & \mathfrak{H}_\Psi \equiv \mathfrak{H}_1 & \mathfrak{H}_0 & \mathfrak{H}_\Psi^{\times} \equiv \mathfrak{H}_{-1} & &
\end{tikzcd}
$$

In the upper row, the operator $S^{-1/2}$ is unitary from each space onto the next one, and the same is true in the lower row for the operator $G^{-1/2}$.

In both cases, we get a Hilbert scale, the simplest example of partial inner product space [3]. The natural question is to identify the end spaces of both scales,

$$\mathfrak{H}_\infty(G^{-1/2}) := \bigcap_{n\in\mathbb{Z}} \mathfrak{H}_n, \qquad \mathfrak{H}_{-\infty}(G^{-1/2}) := \bigcup_{n\in\mathbb{Z}} \mathfrak{H}_n,$$

and similarly for the scale $\mathcal{H}_n = D(S^{-n/2}), n \in \mathbb{Z}$.

On the left of the central spaces of the scales, the operators C and D are defined as usual, then on the right they are defined by duality. However, we can say more. Since dual spaces of sequence spaces are sequence spaces again, with the duality relation $\langle c|d\rangle = \sum_k \overline{c_k} d_k$, inherited from the space ω of all sequences, we obtain the following result.

Theorem 2.5 *Let $\psi_k \in \mathcal{H}_{n_0}$ for $n_0 \geqslant 0$. Then, for all $n \leqslant n_0$, we have:*

- *$C : \mathcal{H}_{-n} \to \mathfrak{H}_{-(n-1)}$ is given by $Cf = \{\langle\psi_k|f\rangle_{\mathcal{H}_n,\mathcal{H}_{-n}}, k \in \Gamma\}$*
- *$D : \mathfrak{H}_{-n} \to \mathcal{H}_{-(n-1)}$ is given by $Dc = \sum_k c_k\psi_k$, with weak convergence.*
- *Let $m \leqslant n_0 - 2$. Then, for all $f \in \mathcal{H}_{-m}$, we have the reconstruction formula (in a weak sense):*

$$f = \sum_k \langle S\psi_k|f\rangle_{\mathcal{H}_m,\mathcal{H}_{-m}}\psi_k\,, \quad \textit{with } \psi_k \in \mathcal{H}_{m+2}.$$

Let us turn now to lower semi-frames and duality considerations. We say that $\Phi = \{\phi_k, k \in \Gamma\}$ is a *lower semi-frame* if there exist a constant $\mathsf{m} > 0$ such that

$$\mathsf{m}\,\|f\|^2 \leq \sum_{k\in\Gamma} |\langle\phi_k|f\rangle|^2\,, \ \ \forall\, f \in \mathcal{H}.$$

Clearly, the lower bound implies that Φ is total in $\mathcal{H}$.

To give a simple example, take an orthonormal basis (e_k), $k \in \mathbb{N}$, in $\mathcal{H}$ and define $\psi_k = \frac{1}{k}e_k$. Then (ψ_k) is an upper semi-frame :

$$0 < \sum_k |\langle\psi_k|f\rangle|^2 \leqslant \sum_k |\langle e_k|f\rangle|^2 = \|f\|^2.$$

There is no lower bound, since for $f = e_p$, one has $\sum_k |\langle\psi_k|f\rangle|^2 = p^{-2}$. Let now $\phi_k = k\,e_k$. The sequence (ϕ_k) is dual to (ψ_k), since

$$\sum_k \langle\psi_k|f\rangle\phi_k = f.$$

In addition, we have

$$\sum_k |\langle e_k|f\rangle|^2 = \|f\|^2 \leqslant \sum_k |\langle\phi_k|f\rangle|^2,$$

and this is unbounded since $\sum_k |\langle \phi_k | f\rangle|^2 = p^2$ for $f = e_p$. Hence, (ϕ_k) is a lower semi-frame, dual to (ψ_k) and living in $\mathfrak{H}_\Psi$.

In this case, the frame operator is $S = \mathsf{diag}(\frac{1}{n^2})$, thus $S^{-1} = \mathsf{diag}(n^2)$, which is clearly unbounded. The inner products are, respectively

- For $\mathfrak{H}_\Psi$: $\langle c|d\rangle_\Psi = \sum_n n^2\, \overline{c_n}\, d_n$,
- For $\mathfrak{H}_0$: $\langle c|d\rangle_0 = \sum_n \overline{c_n}\, d_n$,
- For $\mathfrak{H}_\Psi^\times$: $\langle c|d\rangle_\Psi^\times = \displaystyle\sum_k \frac{1}{n^2}\, \overline{c_n}\, d_n$.

Hence (ϕ_k) is the canonical dual of (ψ_k) : $\phi_k = S^{-1}\psi_k$. Both $C\Psi = (C\psi_k)$ and $C\Phi = (C\phi_k)$ live in $\mathfrak{H}_\Psi$, since $\{\psi_k\}_n = \frac{1}{k}\,\delta_{kn}$ and $\{\phi_k\}_n = k\,\delta_{kn}$.

In the Hilbert scale : $\mathfrak{H}_r = D(G^{-r/2}), r \in \mathbb{Z}$, with inner product $\langle c|d\rangle_r = \sum_n n^r\, \overline{c_n}\, d_n$, one has

$$\mathfrak{H}_\infty := \bigcap_{r\in\mathbb{Z}} \mathfrak{H}_r = s, \;\; \text{the space of fast decreasing sequences,}$$

$$\mathfrak{H}_{-\infty} := \bigcup_{r\in\mathbb{Z}} \mathfrak{H}_r = s^\times, \;\; \text{the space of slowly increasing sequences,}$$

i.e., the Hermite representation of tempered distributions [9]. It is interesting to note that the sequence used by Gabor in his original IEE-paper, a Gabor system with a Gaussian window, $a = 1$ and $b = 1$, is exactly such an upper semi-frame.

The example $(\frac{1}{k}e_k), (ke_k)$ can be generalized to weighted sequences $(\psi_k), (\phi_k)$, with $\psi_k := m_k e_k$, $\phi_k := {m_k}^{-1} e_k$, where $m \in \ell^\infty$ has a subsequence converging to zero and $m_k \neq 0, \forall\, k$. The frame operator is still diagonal : $S = \mathrm{diag}(m_n^2)$, so that $S^{-1} = \mathrm{diag}(m_n^{-2})$, which is clearly unbounded, and $\psi_k = S^{-1}\phi_k$. The inner products read:

- For $\mathfrak{H}_\Psi$: $\langle c|d\rangle_\Psi = \sum_n m_n^{-2}\, \overline{c_n}\, d_n$,
- For $\mathfrak{H}_0$: $\langle c|d\rangle_0 = \sum_n \overline{c_n}\, d_n$,
- For $\mathfrak{H}_\Psi^\times$: $\langle c|d\rangle_\Psi^\times = \sum_n m_n^2\, \overline{c_n}\, d_n$.

The question, of course, is whether one can identify the end spaces of the scale. For instance, if $(1/m_k)$ grows polynomially, one gets the same result :

$$\mathfrak{H}_\infty(G^{-1/2}) = \bigcap_{r\in\mathbb{Z}} \mathfrak{H}_r = s, \quad \mathfrak{H}_{-\infty}(G^{-1/2}) = \bigcup_{r\in\mathbb{Z}} \mathfrak{H}_r = s^\times ,$$

but, in general, the question is open.

The next step is to consider sequences of the form $\Psi = (\psi_k) := (Ve_k)$, where V is a nice, but nondiagonal operator. The possible situations are the following [6]:

(i) Ψ is a *Riesz basis* if and only if $V : \mathcal{H} \to \mathcal{H}$ is a bounded bijective operator.

(ii) Ψ is a *frame* if and only if $V : \mathcal{H} \to \mathcal{H}$ is a bounded surjective operator.

(iii) Ψ is an *upper semi-frame* if and only if $V : \mathcal{H} \to \mathcal{H}$ is a bounded operator.

(iv) Ψ is a *lower semi-frame* if and only if $V : \mathsf{Dom}(V) \to \mathcal{H}$ is a densely defined operator. such that $e_k \in \mathsf{Dom}(V), \forall\, k \in \Gamma$, V^* is injective with bounded inverse on $\mathsf{Ran}(V^*)$, and $V(\sum_{k=1}^n c_k e_k) \to V(\sum_{k=1}^\infty c_k e_k)$ as $n \to \infty$ for every $\sum_{k=1}^\infty c_k e_k \in \mathsf{Dom}(V)$.

If V is block-diagonal, with finite dimensional blocks, one is lead to fusion (semi-)frames, for which we refer to [5], as well as for other generalizations, such as rank-n frames, weighted (semi-)frames or (semi-)frames in Banach spaces.

We conclude this section by quoting some results about duality properties. For conciseness, we put all of them in a single proposition [7].

Proposition 2.6 *(1) If two Bessel sequences $(\psi_k), (\phi_k)$ are dual to each other, then both of them are frames.*

(2) Given any total family $\Phi = \{\phi_k\}$, the associated analysis operator C is closed. Then Φ satisfies the lower frame condition, i.e.*, it is a lower semi-frame, if and only if C has closed range and is injective.*

(3) Let $\Phi = \{\phi_k\}$ be any total family in $\mathcal{H}$. Then Φ is a lower semi-frame if and only if there exists an upper semi-frame Ψ dual to Φ, in the sense that

$$f = \sum_k \langle \phi_k | f \rangle \, \psi_k, \ \ \forall\, f \in \mathsf{Dom}(C).$$

3 Continuous upper semi-frames

Let us go back to the general (continuous) case. The main difference will be that, instead of strongly convergent sequences, we will obtain only weak integrals. Otherwise, the treament is entirely parallel to the discrete case.

Let first Ψ be a frame, as defined in (1.2). The lower frame bound implies that R_C is a *closed* subspace of $L^2(X, \, \mathrm{d}\nu)$. The corresponding projection is an integral operator with (reproducing) kernel $K(x,y) = \langle \psi_x | S^{-1} \psi_y \rangle$, thus R_C is a reproducing kernel Hilbert space. In addition, the subspace R_C is also complete in the norm $\| \cdot \|_\Psi$, associated to the inner product

$$\langle F | F' \rangle_\Psi := \langle F | C_\Psi \, S^{-1} \, C_\Psi^{-1} F' \rangle_{L^2(X)}, \ \text{for} \ \ F, F' \in R_C. \tag{3.1}$$

Hence $(R_C, \| \cdot \|_\Psi)$ is a Hilbert space, denoted by $\mathfrak{H}_\Psi$, and the map $C_\Psi : \mathcal{H} \to \mathfrak{H}_\Psi$ is unitary. Therefore, it can be inverted on its range by the adjoint operator $C_\Psi^{*(\Psi)} : \mathfrak{H}_\Psi \to \mathcal{H}$, which is nothing but the pseudo-inverse $C_\Psi^{+} = S^{-1} C_\Psi^{*}$. Thus one gets, for every $f \in \mathcal{H}$, a *reconstruction formula*:

$$f = C_\Psi^{*(\Psi)} = \int_X F(x) \, S^{-1} \, \psi_x \ \mathrm{d}\nu(x), \ \text{for} \ \ F = C_\Psi f \in \mathfrak{H}_\Psi \quad \text{(weak integral)}. \tag{3.2}$$

Let now Ψ be a (continuous) *upper semi-frame*, that is, there exists $\mathsf{M} < \infty$ such that

$$0 < \int_X |\langle \psi_x | f \rangle|^2 \ \mathrm{d}\nu(x) \leqslant \mathsf{M} \, \|f\|^2 \, , \ \forall\, f \in \mathcal{H}, \ f \neq 0. \tag{3.3}$$

In this case, Ψ is a total set in $\mathcal{H}$, the operators C_Ψ and S are bounded, S is injective and self-adjoint. Therefore $R_S := \mathsf{Ran}(S)$ is dense in $\mathcal{H}$ and S^{-1} is also self-adjoint. S^{-1} is unbounded, with dense domain $\mathsf{Dom}(S^{-1}) = R_S$.

Define the Hilbert space $\mathfrak{H}_\Psi$ as the completion of $C_\Psi(R_S)$ with respect to the norm $\| \cdot \|_\Psi$. defined in (3.1). Then, one shows that the map C_Ψ, restricted to the dense domain $\mathsf{Dom}(S^{-1}) = R_S$, is an isometry from $\mathsf{Dom}(S^{-1}) = R_S$ onto $C_\Psi(R_S) \subset \mathfrak{H}_\Psi$, *i.e.*,

$$\langle C_\Psi f | C_\Psi f' \rangle_\Psi = \langle f | f' \rangle_{\mathcal{H}}, \ \forall\, f, g \in R_S,$$

thus it extends by continuity to a *unitary* map between the respective completions, that is, from $\mathcal{H}$ onto $\mathfrak{H}_\Psi$. Therefore, $\mathfrak{H}_\Psi$ and R_C coincide as sets, so that $\mathfrak{H}_\Psi$ is a vector subspace (though not necessarily closed) of $L^2(X, \, \mathrm{d}\nu)$.

Consider now, in the Hilbert space $\overline{R_C}$, the operators $G = \overline{C_\Psi \, S \, C_\Psi^{-1}}$ and G^{-1}, the self-adjoint extension of $G_S^{-1} = C_\Psi \, S^{-1} \, C_\Psi^{-1}$, the latter being essentially self-adjoint. Both operators are self-adjoint and positive, G is bounded and G^{-1} is densely defined in $\overline{R_C}$. Furthermore, they are inverse of each other on appropriate domains. Moreover, the norm $\|\cdot\|_\Psi$ is equivalent to the graph norm of $G^{-1/2}$. Thus one gets gets the following commutative diagram, analogous to (2.1):

$$
\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{H} \arrow[r, "C_\Psi"] \arrow[d, shift right, "S"'] & \mathfrak{H}_\Psi = R_C \subseteq \overline{R_C} \subseteq L^2(X,\,\mathrm{d}\nu) \arrow[dl, "C_\Psi^*"'] \arrow[d, shift right, "G_S"'] \\
\mathcal{H} \supseteq \mathsf{Dom}(S^{-1}) = R_S \arrow[u, shift right, "S^{-1}"'] \arrow[r, "C_\Psi"] & C_\Psi(R_S) \subseteq L^2(X,\,\mathrm{d}\nu) \arrow[u, shift right, "G_S^{-1}"']
\end{tikzcd}
\tag{3.4}
$$

We will say that the upper semi-frame $\Psi = \{\psi_x,\, x \in X\}$ is *regular* if all the vectors ψ_x, $x \in X$, belong to $\mathsf{Dom}(S^{-1})$. In that case, the discussion proceeds exactly as in the bounded case. In particular, the reproducing kernel $K(x, y) = \langle \psi_x | S^{-1} \psi_y \rangle$ is a *bona fide* function on $X \times X$. One obtains the same weak reconstruction formula, but restricted to the subspace $R_S = \mathsf{Dom}(S^{-1})$:

$$f = C_\Psi^{*(\Psi)} F = \int_X F(x)\, S^{-1}\, \psi_x \;\,\mathrm{d}\nu(x), \forall\, f \in R_S,\; F = C_\Psi f \in \mathfrak{H}_\Psi. \tag{3.5}$$

On the other hand, if Ψ is not regular, one has to treat the kernel $K(x, y)$ as a bounded sesquilinear form over $\mathfrak{H}_\Psi$ and use the language of distributions, for instance, in terms of a Gel'fand triplet [4], exactly as in the discrete case.

For Ψ regular, the relation

$$K^\Psi(F, F') = \langle C_\Psi^{-1} F | S C_\Psi^{-1} F' \rangle_{\mathcal{H}} = \langle F | C_\Psi S C_\Psi^{-1} F' \rangle_\Psi = \langle F | F' \rangle_{L^2}$$

defines a bounded sesquilinear form over $\mathfrak{H}_\Psi$, and the same is true even if Ψ is not regular. Let $\mathfrak{H}_\Psi^\times$ be the completion of $\mathfrak{H}_\Psi$ in the norm given by K^Ψ. Thus one gets the following triplet, with continuous and dense range embeddings,

$$\mathfrak{H}_\Psi \;\subset\; \mathfrak{H}_0 \;\subset\; \mathfrak{H}_\Psi^\times, \tag{3.6}$$

where

- $\mathfrak{H}_\Psi = R_C$, which is a Hilbert space for the norm $\|\cdot\|_\Psi = \langle \cdot | G^{-1} \cdot \rangle_{L^2}^{1/2}$,
- $\mathfrak{H}_0 = \overline{\mathfrak{H}_\Psi} = \overline{R_C}$ is the closure of $\mathfrak{H}_\Psi$ in $L^2(X,\,\mathrm{d}\nu)$,
- $\mathfrak{H}_\Psi^\times$ is the completion of $\mathfrak{H}_0$ (or $\mathfrak{H}_\Psi$) in the norm $\|\cdot\|_{\Psi^\times} := \langle \cdot | G \cdot \rangle_{L^2}^{1/2}$ and the conjugate dual of $\mathfrak{H}_\Psi$.

Clearly (3.6) is the central triplet of the scale of Hilbert spaces generated by the powers of $G^{-1/2}$, $\mathfrak{H}_n := D(G^{-n/2}), n \in \mathbb{Z}$. Then again, the question is to identify the end spaces of the scale, $\mathfrak{H}_{\pm\infty}(G^{-1/2})$.

If Ψ is regular, all three spaces $\mathfrak{H}_\Psi, \mathfrak{H}_0, \mathfrak{H}_\Psi^\times$ are *reproducing kernel Hilbert spaces*, with the same kernel $K(x, y) = \langle \psi_x | S^{-1} \psi_y \rangle$. If S^{-1} is bounded (frame), the three Hilbert spaces coincide as sets, with equivalent norms, since $S, S^{-1} \in \mathrm{GL}(\mathcal{H})$.

We conclude this section by duality considerations. Given a frame $\Psi = \{\psi_x\}$, one says that a frame $\{\chi_x\}$ is *dual* to the frame $\{\psi_x\}$ if one has, in the weak sense, $f = \int_X \langle \chi_x | f \rangle\, \psi_x \;\mathrm{d}\nu(x)$, $\forall\, f \in \mathcal{H}$. Then the frame $\{\psi_x\}$ is dual to the frame $\{\chi_x\}$. We want to extend this notion to semi-frames.

Let first $\Psi = \{\psi_x\}$ be an arbitrary total family in $\mathcal{H}$. As usual, we define the analysis operator as $C_\Psi f(x) = \langle \psi_x | f \rangle$ and the synthesis operator $D_\Psi F = \int_X F(x)\, \psi_x \;\mathrm{d}\nu(x)$, on natural domains:

$$\mathsf{Dom}(C_\Psi) = \{f \in \mathcal{H} : \int_X |\langle \psi_x | f \rangle|^2 \;\mathrm{d}\nu(x) < \infty\},$$
$$\mathsf{Dom}(D_\Psi) = \{F \in L^2(X,\,\mathrm{d}\nu) : \int_X F(x)\, \psi_x \;\mathrm{d}\nu(x) \text{ converges weakly in } \mathcal{H}\}.$$

They may be both unbounded.

Lemma 3.1 *(i) Given any total family Ψ, the analysis operator C_Ψ is closed. Then Ψ satisfies the lower frame condition if and only if C_Ψ has closed range and is injective.*

(ii) If the function $x \mapsto \langle\psi_x|f\rangle$ is locally integrable for all $f \in \mathcal{H}$, then the operator D_Ψ is densely defined and one has $C_\Psi = D_\Psi^*$.

The condition of local integrability is satisfied for all $f \in \mathsf{Dom}(C_\Psi)$, but not necessarily for all $f \in \mathcal{H}$, unless Ψ is an upper semi-frame, since then $\mathsf{Dom}(C_\Psi) = \mathcal{H}$.

Finally, one defines the frame operator as $S = D_\Psi C_\Psi$ on the obvious domain $\mathsf{Dom}(S) \subset \mathsf{Dom}(C_\Psi)$. However, if the upper frame inequality is not satisfied, S and C_Ψ could have nondense domains, in which case one cannot define a unique adjoint C_Ψ^* and S may not be self-adjoint. However, if $\psi_y \in \mathsf{Dom}(C_\Psi)$, $\forall\, y \in X$, then C_Ψ is densely defined, $D_\Psi \subseteq C_\Psi^*$ and D_Ψ is closable. Finally, D_Ψ is closed if and only if $D_\Psi = C_\Psi^*$. Then $S = C_\Psi^* C_\Psi$ is self-adjoint.

Next, we say that a family $\Phi = \{\phi_x\}$ is a *lower semi-frame* if it satisfies the lower frame condition, that is, there exists a constant $\mathsf{m} > 0$ such that

$$\mathsf{m}\,\|f\|^2 \leq \int_X |\langle\phi_x|f\rangle|^2 \,\mathrm{d}\nu(x), \ \ \forall\, f \in \mathcal{H}. \tag{3.7}$$

Clearly, (3.7) implies that the family Φ is total in $\mathcal{H}$. With these definitions, we obtain a nice duality property between upper and lower semi-frames.

Proposition 3.2 *(i) Let $\Psi = \{\psi_x\}$ be an upper semi-frame, with upper frame bound M and let $\Phi = \{\phi_x\}$ be a total family dual to Ψ. Then Φ is a lower semi-frame, with lower frame bound M^{-1}.*

(ii) Conversely, if $\Phi = \{\phi_x\}$ is a lower semi-frame, there exists an upper semi-frame $\Psi = \{\psi_x\}$ dual to Φ, that is, one has, in the weak sense,

$$f = \int_X \langle\phi_x|f\rangle\, \psi_x \,\mathrm{d}\nu(x), \ \ \forall\, f \in \mathsf{Dom}(C_\Phi).$$

4 Examples

We conclude with two concrete examples from mathematical physics, both from the context of coherent states, more precisely, wavelets.

4.1 A non-regular upper semi-frame

A non-regular upper semi-frame may be found among affine coherent states [4, 5]. Consider the following unitary irreducible representation of the $ax + b$ group:

$$(U_n(a,b)f)(r) = a^{\frac{n}{2}} e^{-ibr} f(ar),\ a > 0,\ b \in \mathbb{R},\ f \in \mathcal{H}^{(n)} := L^2(\mathbb{R}^+, r^{n-1}\,\mathrm{d}r), n \geqslant 1.$$

For coherent states take: $\psi_x(r) = e^{-ixr}\psi(r),\ r \in \mathbb{R}^+$ [8]. The corresponding admissibility condition is

$$(i)\ \sup_{r\in\mathbb{R}^+} h(r) = 1, \text{where } h(r) := \pi r^{n-1}|\psi(r)|^2;$$
$$(ii)\ |\psi(r)|^2 \neq 0, \text{ except perhaps at isolated points } r \in \mathbb{R}^+.$$

The frame operator S and its inverse S^{-1} are multiplication operators on $\mathcal{H}^{(n)}$:

$$(S^{\pm 1} f)(r) = [h(r)]^{\pm 1} f(r).$$

Since $h(r) \leqslant 1$, the inverse S^{-1} is unbounded and no ψ_x is in the domain of S^{-1}, so that we have indeed a non-regular upper semi-frame. The coherent state map $C_\Psi : \mathcal{H}^{(n)} \to L^2(\mathbb{R}, \mathrm{d}x)$ is

$$(C_\Psi f)(x) = \langle\psi_x|f\rangle = \int_{\mathbb{R}^+} e^{ixr}\overline{\psi(r)} f(r) r^{n-1}\,\mathrm{d}r.$$

Finally, the reproducing kernel reads

$$K(x,y) = \text{“}\langle\psi_x|S^{-1}\psi_y\rangle\text{”} = \frac{1}{2\pi}\int_{\mathbb{R}^+} e^{i(x-y)r}\,\mathrm{d}r = \frac{i}{2\pi(x-y)} + \frac{1}{2}\delta(x-y),$$

a well-defined distribution which defines a bounded sesquilinear form on $\mathfrak{H}_\Psi$.

The associated Gel'fand triplet consists of three Hilbert spaces, with norms

- For $\mathfrak{H}_\Psi : \|F\|_\Psi^2 = \langle F|G^{-1}F\rangle_{L^2} = 2\pi\int_0^\infty |\widehat{F}(r)|^2\,[h(r)]^{-1}\,r^2\,\mathrm{d}r$.
- For $\mathfrak{H}_0 : \|F\|_0^2 = \|F\|_{L^2}^2$.
- For $\mathfrak{H}_\Psi^\times : \|F\|_{\Psi^\times}^2 = \langle F|GF\rangle_{L^2} = 2\pi\int_0^\infty |\widehat{F}(r)|^2\,h(r)\,r^2\,\mathrm{d}r$.

Finally, we can build explicitly the corresponding Hilbert scales.

(i) On the side of $\mathcal{H} :\ (S^m f)(r) = [h(r)]^m f(r),\ m\in\mathbb{Z}$:
The corresponding scale is $\{\mathcal{H}_m = \mathsf{Dom}(S^{-m/2}), m\in\mathbb{Z}\}$, with norms

$$\|f\|_m = \langle f|S^{-m}f\rangle^{1/2} = \left[\int_0^\infty |f(r)|^2\,[h(r)]^{-m}\,r^{n-1}\,\mathrm{d}r\right]^{1/2}$$

and the end spaces

$$\mathcal{H}_\infty(S^{-1/2}) := \bigcap_{m\in\mathbb{Z}} \mathcal{H}_m,\quad \mathcal{H}_{-\infty}(S^{-1/2}) := \bigcup_{m\in\mathbb{Z}} \mathcal{H}_m.$$

The problem is to identify those spaces, we have been unable to do it, even in concrete examples.

(ii) In $L^2(\mathbb{R}, \mathrm{d}x)$:

$$(G^m F)(x) = \int_{\mathbb{R}^+} e^{ixr}\,\widehat{F}(r)\,[h(r)]^m\,r^2\,\mathrm{d}r,\ \ m\in\mathbb{Z}$$

The scale is $\{\mathfrak{H}_m = \mathsf{Dom}(G^{-m/2}), m\in\mathbb{Z}\}$, with norms

$$\|F\|'_m = \langle F|G^{-m}F\rangle^{1/2} = \left[2\pi\int_0^\infty |\widehat{F}(r)|^2\,[h(r)]^{-m}\,r^2\,\mathrm{d}r\right]^{1/2}$$

and end spaces

$$\mathfrak{H}_\infty(G^{-1/2}) := \bigcap_{m\in\mathbb{Z}} \mathfrak{H}_m,\quad \mathfrak{H}_{-\infty}(G^{-1/2}) := \bigcup_{m\in\mathbb{Z}} \mathfrak{H}_m,$$

which could also not be identified.

This example can easily be discretized. Choose an infinite sequence of points $\{x_k, k=1,2,\ldots\}$. Then the coherent state map C_Ψ becomes $C : \mathcal{H}^{(n)} \to \ell^2$:

$$(Cf)_k = \langle\psi_{x_k}|f\rangle = \int_{\mathbb{R}^+} e^{ix_k r}\overline{\psi(r)} f(r) r^{n-1}\,\mathrm{d}r,$$

which is bounded, with range in ℓ^2 closed in the new norm

$$\|d\|_\Psi^2 = \langle d|C\,S^{-1}\,C^{-1}d\rangle_{\ell^2} = \langle d|G^{-1}\,d\rangle_{\ell^2}\,.$$

The reproducing matrix reads

$$K_{kl} = \text{“}\langle\psi_k|S^{-1}\psi_l\rangle\text{”} = \frac{1}{2\pi}\int_{\mathbb{R}^+} e^{i(x_k-y_l)r}\,\mathrm{d}r = \frac{i}{2\pi(x_k-y_l)} + \frac{1}{2}\delta(x_k-y_l),$$

a well-defined distribution.

4.2 A lower semi-frame

A concrete example of lower semi-frame may be found among wavelets on the two-sphere [2, Sec. 15.1.1]. Given an axisymmetric mother wavelet $\psi \in \mathcal{H} = L^2(\mathbb{S}^2, \mathrm{d}\mu)$, one builds the full set of spherical wavelets $\{\psi_{a,\varrho} := R_\varrho\, D_a \psi,\ (\varrho, a) \in X = \mathrm{SO}(3) \times \mathbb{R}^*_+\}$. The corresponding frame operator S is diagonal in Fourier space (Fourier analysis on $\mathbb{S}^2$ amounts to expansions in spherical harmonics Y_l^m):

$$\widehat{Sf}(l,m) = s(l)\widehat{f}(l,m), \quad \widehat{f}(l,m) := \langle Y_l^m | f\rangle$$
$$s(l) := \frac{8\pi^2}{2l+1} \sum_{|m|\leqslant l} \int_0^\infty |\langle Y_l^m | D_a\psi\rangle|^2 \,\frac{da}{a^3}, \quad \forall\, l \in \mathbb{N}$$

Then $\psi \in L^2(\mathbb{S}^2, \mathrm{d}\mu)$ is admissible (in the sense of group theory) whenever there exists a constant $c > 0$ such that $s(l) \leqslant c,\ \forall\, l \in \mathbb{N}$, that is, when the frame operator S is bounded and invertible. As a consequence, there exists $d > 0$ such that $d \leqslant s(l) \leqslant c,\ \forall\, l \in \mathbb{N}$, that is, S and S^{-1} are both bounded. In other words, $\{\psi_{a,\varrho},\ (\varrho, a) \in X = \mathrm{SO}(3) \times \mathbb{R}^*_+\}$ is a continuous frame.

The wavelet transform of $f \in \mathcal{H}$ is $W_f(\varrho, a) = \langle \psi_{a,\varrho} | f\rangle$ and there is a reconstruction formula :

$$f(\omega) = \int_{\mathbb{R}^*_+} \int_{\mathrm{SO}(3)} W_f(\varrho, a)\, [S^{-1} R_\varrho D_a \psi](\omega)\, \frac{da}{a^3}\, \mathrm{d}\varrho,\ \ f \in L^2(\mathbb{S}^2).$$

However, this reconstruction formula remains valid for $0 < s(l) < \infty,\ \forall\, l \in \mathbb{N}$, which means that S may be unbounded, but S^{-1} stays bounded [10] (the lower frame bound is independent of ψ, since it results from the asymptotic behavior of the spherical harmonics Y_l^m near the origin). In other words, we have an example of lower semi-frame.

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