2011/54

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November 2011

Abstract

In this paper we extend the basic model of Cournot competition to the case where both the demand function and the cost functions of each firm depend on the amounts produced by competitors. In this modified setting, proving existence of equilibria becomes harder. We develop a generalization of the theory of supermodular games in the context where individual decision variables take values in a totally ordered set to prove existence of equilibria in this generalized Cournot setting.

Keywords: Cournot oligopoly, complementarity, generalized modularity, generalized increasing differences, supermodular games.

JEL Classification: C60, C70, C72

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I express my thankfulness to Jean-François Mertens, Enrico Minelli and in particular to Pierre Dehez for their numerous suggestions and remarks. All errors are of course mine. Financial support from the National Bank of Belgium is gratefully acknowledged.

This paper presents research results of the Belgian Program on Interuniversity Poles of Attraction initiated by the Belgian State, Prime Minister's Office, Science Policy Programming. The scientific responsibility is assumed by the author.

Contents

1. Introduction: generalized Cournot model

The Cournot model of oligopolistic competition is a cornerstone in Industrial Organization. In the basic version of the model, there is a small number of firms producing an homogeneous product. The firms compete by choosing simultaneously the quantities of production to bring to the marketplace. Consumers are passive and interact with the firms through an aggregate demand function, with the mechanism of market clearing remaining unspecified. In this version the Cournot model is a one-shot game of complete and imperfect information. A Cournot equilibrium is a Nash equilibrium of the Cournot game, and a Cournot market outcome is an outcome induced by a Cournot equilibrium.

The standard payoff of firm i, $i = 1, \ldots, n$, in the model of Cournot has the form:

$$\Pi_i \left(q_i, Q_{-i}\right) = q_i P \left(Q\right) - C_i \left(q_i\right),$$

where q_i denotes the output of firm i and takes values in some set of nonnegative real numbers, $Q := \sum_i q_i$ is the aggregate output level, $Q_{-i} := \sum_{j \neq i} q_j$ is the aggregate output level of i's competitors, $P\left(Q\right)$ is the inverse demand function, mapping nonzero reals into nonzero reals, and $C_i\left(q_i\right)$ is the cost function of firm i, mapping again nonzero reals into nonzero reals.

A usual set of assumption in the Cournot context (see for example Vives 1999) consists in posing that $P\left(Q\right)$ is upper semicontinuous and decreasing, and that $C_i\left(q_i\right)$ is lower semicontinuous and increasing.

The first, fundamental problem consists in assessing whether Cournot competition is determinate, that is whether a Nash equilibrium exists in a Cournot game. In this respect, the Cournot model is a model of (indirect) price formation. Indeed, there are relevant instances of Cournot competition where no equilibrium exists in pure strategies, for example in the famous models of Roberts and Sonnenschein (1977) and Friedman (1983), where firms can produce at zero cost. On the other hand, the standard assumptions that allow to prove existence by Kakutani's fixpoint theorem entail quasiconcavity of profits Π_i in q_i, and this property is usually seen as disturbing because cost functions may well be nonconvex.

In the last decade, the approach of supermodular games has been fruitfully applied to the analysis of Cournot competition in order to obtain existence of equilibria without requiring quasiconcavity of payoffs. The theory of supermodular games is based on the early work of Topkis (1978, 1979) and Veinott (1992). It relies on the finding that complementarity is leading property of a game's payoffs in order to obtain existence of Nash equilibria as well as important properties of the Nash set, such as existence of maximal and minimal equilibria, learning properties, convergence properties, rationalizability (See Milgrom and Roberts, 1990, Milgrom and Shannon 1994, Topkis 1979 and 1998, Vives 1999 and 2005). The notion of complementarity used in supermodular games is that of Edgeworth and Pareto: increasing the level of one activity increases the marginal utility of all its complement activities (see Samuelson 1974 for a review of various notions of complementarity in economics).

The conditions ensuring that a homogeneous Cournot game is a supermodular game are conditions ensuring that each payer's payoff $\Pi_i(q_i, Q_{-i})$ has increasing differences in (q_i, Q_{-i}) or, in the duopoly case, decreasing differences in (q_i, Q_{-i}) (these notions will be defined in the next section). The first condition means that the variables (q_i, Q_{-i}) are complements. The latter means that they are substitutes. Various applications of supermodular games to Cournot oligopoly, covering the ordinal class of quasisupermodular games as well, can be found in Vives (1999) and Amir (1996, 2005). A seminal application of complementarity to oligopolistic competition can be found in Bulow et al. (1985).

In this paper, we make a step forward in modeling and studying Cournot's oligopolistic competition and a step forward in developing the theory of supermodular games. Recall that in the standard model of Cournot, the strategic interaction among the competing firms is due only to the fact that the inverse demand function depends on the aggregate output. The modification to the Cournot setting that we introduce in this paper consists in extending the strategic part of the Cournot model to the cost functions as well. More precisely, we extend the Cournot setting to the case where both the demand function and the cost functions of each firm depend on the amounts produced by competitors. In our context, the quantities brought to the market by the competitors of a firm affect this firm's profits not only through revenues, but also through costs.

Our generalized-Cournot payoff functions take then the following form, for each firm i, $i = 1, \ldots, n$:

$$\Pi_i(q_i, q_{-i}) = q_i P(Q) - C_i(q_i, q_{-i}).$$

Our payoff functions depend on (q_i, q_{-i}), not on (q_i, Q_{-i}) as in the standard model. In particular, q_{-i} is simply the vector of quantities produced by firm i's opponents, while $Q_{-i} := \sum_{j \neq i} q_j$ is the sum of such quantities. While we maintain that the inverse demand function depends on aggregate production, we do not pose this restriction on the cost function: costs depends on competitors' production quantities but not necessarily on the sum of such quantities. In this way, we keep our model as general as possible.

We envisage various justifications for introducing competitors' production decisions in the cost function of a firm. For example, suppose that production requires using a production factor available in a limited amount. The more competitors produce, the more costly it may be for the firm at stake to obtain the necessary quantities of the production factor.

"More costly" here does not necessarily mean more expensive in terms of price of the production factor. The production factor may well be a public good provided at fixed price independent on demand. More costly, in this paper, could well stand for more expensive in terms of transaction costs, in terms of litigation costs, in terms of costs paid for dealing with anticompetitive and exclusionary behavior on part of the opponents.

In other words, our idea is that as competitors decide to produce more and hence to acquire more of the limited production factor, our firm must deal with more fierce competition on the market of the production factor, and this renewed competition must be somehow costly to be dealt with. The cost function in this setting plays the role of a reduced form which embodies these strategic features of the production process without specifying them in the model.

Of course, in this limited-production-factor scenario, the marginal cost of production for firm i must increase with the level of production of at least one of its competitors; that is, in the differentiable case:

$$\frac{\partial^2 C_i\left(q_i, q_{-i}\right)}{\partial q_i \partial q_j} \geq 0$$

for some $j \neq i$.

Another justification to make costs depending on opponents' production decision relies on information transmission. Expertise can be gained by simply observing how competitors operate on the market, including how much they produce. This expertise could be about product quality, about market practices of the competitors, about their logistic strategy, about the advertising etc. The general idea, well-known in game theory literature, is that the more a firm operates on the market the more it reveals about itself and its organization, and this will typically entail a competitive advantage to competitors.

Thinking for example in terms of technology, more production on part of firm i's opponents reveals more to firm i about opponents' production technology, thus easing the eventual imitation, or rejection, of this technology by firm i, in so making somehow more efficient the production technology of i.

While this information-revelation scenario could and should certainly be modeled in the Cournot game by introducing incomplete information, we stick to cost functions as reduced forms embodying these informational features of the interaction.

This information-revelation effect should lead to a reduction of i's marginal cost as a competitor produces more; that is, in the differentiable case:

$$\frac{\partial^2 C_i\left(q_i, q_{-i}\right)}{\partial q_i \partial q_j} \leq 0$$

for some $j \neq i$.

Once cost functions depending on competitors' production decision have been justified and introduced in the Cournot model, it is easy to provide examples of reasonable functional forms where the standard conditions entailing that the Cournot game has an equilibrium fail. In the last section of this paper we will provide an instance of this.

In general, the Bamon-Fraysee-Novshek classical donwnward-sloping marginal revenue condition (Vives, 1999), which is equivalent in a differentiable setting to submodularity of payoffs, will fail as soon as the mixed partials of the generalized cost function makes the game supermodular on some region of payoff's domain, and submodular on the complement region. Or else, as soon as the cost function is supermodular with respect to production of some competitor but submodular with respect to production of a different competitor. Amir (1996, 2005) conditions ensuring that the Cournot game is log-supermodular may fail as well. Furthermore, even outside the realm of supermodular games, the condition that costs are identical and convex, as in the Roberts-Sonnenschein approach, fails since we have posed no conditions about the convexity of costs.

Thus, to deal with a Cournot model with costs depending on competitors' production decisions, in this paper we develop new tools which allow to study this generalized Cournot model. In order to construct these new tools, we generalize the existing theory of supermodular games in the context where individual decision variables take values in totally ordered sets (for example on the real line), as it is the case for the Cournot model presented in this paper.

Our generalization of supermodular games is presented and discussed in Section 2, in particular in Subsection 2.3, and consists in introducing a form of complementarity among payoffs' arguments more general than the Pareto-Edgeworth complementarity required in supermodular games.

To outline our approach, consider some utility function $u(x,t)$ defined on the real plane. Pareto-Edgeworth complementarity among x and t prescribes that any fixed increase in activity x becomes more profitable, in terms of the associated increase in utility, as the level of activity t increases.

The notion of complementarity that we introduce in Section 2, more generally prescribes that there is a complete reallocation of activities, say increase x and decrease t, which becomes more profitable as t increases (we can of course interchange the roles of x and t). A special case of such profitable reallocation would be the one where only x increases, and hence our notion of complementarity includes Pareto-Edgeworth complementarity as a special case.

Technically, our form of complementarity will be called generalized increasing differences, and is a generalization of increasing differences, which in turn is the mathematical form of Pareto-Edgeworth complementarity (see the next section for all the definitions).

We point out for completeness that there exists an ordinal extension of Pareto-Edgeworth complementarity, studied by Milgrom and Shannon (1994). Our notion of general complementarity can be easily extended to the ordinal setting as well, but we do not pursue this further extension in this paper.

The paper is organized as follows. In Section 2 we introduce and study our notion of complementarity. In Section 3 we show that if payoffs satisfy our general form of complementarity, then individual best replies are increasing exactly in the same way as they are in supermodular games. This results represents a generalization of the celebrated Topkis (1978)'s monotonicity theorem in the context of decision variables taking values in totally ordered sets. With the results of Section 3 in hands, we can apply Zhou (1994)'s fixpoint theorem to the joint best reply of our games in order to prove existence of Nash equilibria[1].

In Section 4 we finally applies the tools that we have developed thus far to a Cournot problem with costs depending on competitors' production decisions, and where due to this dependence the standard conditions ensuring existence fail. Section 5 concludes.

2. Generalized modularity and generalized increasing differences

2.1. **Supermodularity and increasing differences.** In order to facilitate comparison with our result, we recall and discuss here the notion of complementarity for payoffs used in supermodular games. We introduce in this subsection the notions of supermodularity and increasing differences.

To the scope of the paper we could have restricted ourself to introducing increasing differences only. However, increasing differences is fully characterized in terms of supermodularity, and we will show that the generalization that we introduce in the next subsection, g-increasing-differences, is fully characterized in terms of a generalization of supermodularity, that we will call g-modularity. Hence we introduce supermodularity here to set up a self-contained reference-framework for the generalizations of next subsection.

A partially ordered set X is a *lattice* if for every two points $a, b \in X$, the supremum $a \vee b$ and the infimum $a \wedge b$ are both in X. Given a lattice X, a function

[1]Zhou fixpoint theorem is based on that of Tarski (1955), and has been generalized in Calciano (2007, 2009). We do not need these more powerful results in this paper.

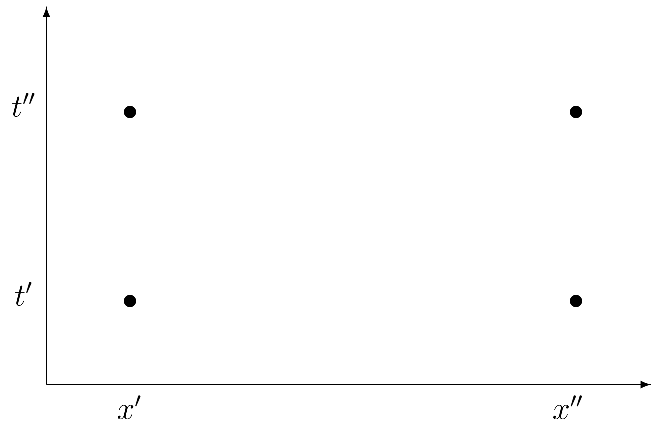
$u(x) : X \to \mathbb{R}$ is *supermodular* in x on X if for every $a, b \in X$, it holds that:

$$u(a) + u(b) \leq u(a \wedge b) + u(a \vee b).$$

If X and T are partially ordered sets, a function $u(x, t) : X \times T \to \mathbb{R}$ has *increasing differences* in (x, t) on $X \times T$ if for every $(x', t') \leq (x'', t'')$ in $X \times T$, it holds that:

$$u(x'', t') - u(x', t') \leq u(x'', t'') - u(x', t'').$$

Increasing differences represents Pareto-Edgeworth complementarity between the activities (x, t) according to the preferences of the player (her payoff $u(.)$), and is the fundamental notion of complementarity used in supermodular games. To illustrate increasing differences, consider the picture below:



According to increasing differences, the same increase in the level of activity x, from x' to x'', is more rewarding in terms of the associated increase of utility, when the level of activity t increases from t' to t''.

It is well known (see for example Topkis 1998) that increasing difference on the product $X \times T$ of totally ordered sets is equivalent to supermodularity in the vector variable (x, t) on $X \times T$. Hence, in such a context, supermodularity and increasing differences express in exactly the same way the notion of complementarity between the activities (x, t).

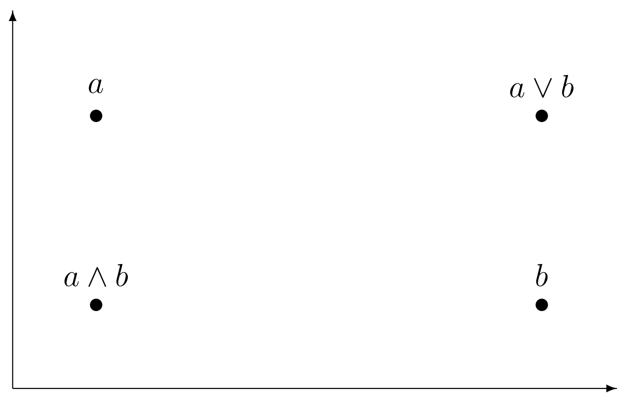
Another important notion lies at the heart of supermodular games, Veinott-increasingness (Veinott 1992). Let Y be a partially ordered set and X be a lattice. Consider a point-to-set mapping $B : Y \to X$. We say that B is *Veinott-increasing* if for every $x < y$ in Y, for every $a \in B(x)$ and every $b \in B(y)$, we have that $a \wedge b \in B(x)$ and $a \vee b \in B(y)$.

Topkis (1978) and Veinott (1992) have shown that if payoffs $u(x, t)$ are supermodular in own actions x and satisfies increasing differences in (x, t), then the joint best reply of a game is Veinott-increasing. Then, using the fixpoint theorem of Zhou (1994), it can be shown that the supermodular game has a nonempty Nash sets, plus many other interesting properties. We do not need to go into any more detail in this paper as far as supermodular games are concerned. The interested reader is referred to the wide surveys contained in Vives (1999) and Topkis (1998).

2.2. **Generalized modularity.** Now we introduce a first form of our generalized notion of complementarity for payoffs. The meaning of this form of complementarity will become clearer in the next subsection. Consider a lattice X in the real plane (see the picture below). Any two of its unordered elements a, b determines a (closed) rectangle with vertices $\{a, a \vee b, b, a \wedge b\}$, and with the two pairs of parallel sides given by the intervals

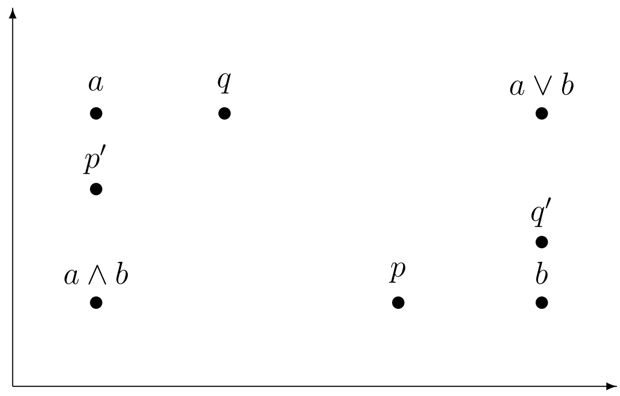
$$[a \wedge b, b], [a, a \vee b]$$

$$[a \wedge b, a], [b, a \vee b].$$



Supermodularity is a relation defined for the four vertices $a, b, a \wedge b,\ a \vee b$ of the rectangle generated by a, b. If the initial a, b are ordered the the rectangle is degenerate, it reduces to a line segment, and supermodularity always hold in this case.

Now, for the same rectangle generated by a and b, consider the two quadrilaterals depicted below, inscribed into the rectangle and having as vertex sets the sets $Q = \{a, q, b, p\}$ and $Q' = \{a, q', b, p'\}$ respectively.



Given the rectangle generated by a, b, we introduce in Definition 1 a form of our fundamental complementarity property for payoffs. It consists in a relation defined for the vertices of inscribed quadrilaterals such as Q and Q', and not necessarily for the vertex of the rectangle itself. Our property is a generalization of supermodularity, and its ordinal version, immediately deducible, is a generalization of the single crossing property. As we remarked in the introductory section, we do not pursue the ordinal extension of our approach in this paper. It should not create difficulties to work out the ordinal case however, in the same way as Milgrom and Shannon (1994) did for supermodularity.

Definition 1. (GENERALIZED MODULARITY, OR G-MODULARITY). Let X be a lattice. A function $u(x) : X \to \mathbb{R}$ is g-modular in x on X if for every $a, b \in X$,

$$b \nleq a \Rightarrow \exists p \in [a \wedge b, b), \exists q \in (a, a \vee b] : u(a) + u(b) \leq u(p) + u(q),$$
$$a \nleq b \Rightarrow \exists p' \in [a \wedge b, a), \exists q' \in (b, a \vee b] : u(a) + u(b) \leq u(p') + u(q').$$

Note that we have deleted in Definition 1 the initial points a and b from the intervals in which p, q, p', q' can take values. Otherwise, every function would be modular.

Every function is modular on X if X is a chain (a totally ordered set), exactly as it happens for supermodularity. Lemma 1 shows this.

Lemma 1. *Let X be a totally ordered set. Every function $u(x) : X \to \mathbb{R}$ is g-modular in x on X.*

Proof: Pick any function $u(x) : X \to \mathbb{R}$ and any x, y in X. Let first x, y be distinct. Let, without loosing generality, $x < y$. Set $x = a$ and $y = b$. Then $a \nleq b$ is false while $b \nleq a$ is true, and so function u needs to satisfy only the condition implied by $b \nleq a$, because that implied by $a \nleq b$ holds vacuously. Hence, we only need points

$$p \in [a \wedge b, b) \equiv [x, y),$$

$$q \in (a, a \vee b] \equiv (x, y]$$

such that

$$u(x) + u(y) \leq u(p) + u(q),$$

which is immediately accomplished by taking $p = x$ and $q = y$.

Set now $x = b$ and $y = a$. Then $b \not\leq a$ is false while $a \not\leq b$ is true. Thus we need only to find points

$$p' \in [a \wedge b, a) \equiv [x, y),$$

$$q' \in (b, a \vee b] \equiv (x, y]$$

such that

$$u(x) + u(y) \leq u(p') + u(q'),$$

which is immediately accomplished by taking $p' = x$ and $q' = y$.

If on the other hand $x = y$, then both $a \not\leq b$ and $b \not\leq a$ are false, and g-modularity holds vacuously. ■

Clearly every supermodular function is g-modular, just pick for every $a, b \in X$, $p = p' = a \wedge b$ and $q = q' = x \vee y$. Furthermore, if the lattice X is finite, a g-modular function on X is supermodular on X, as the following lemma states.

Lemma 2. *Let X be a finite lattice. A function $u(x) : X \to \mathbb{R}$ is g-modular in x on X if and only if it is supermodular in x on X.*

Proof: The proof is trivial for four-point lattices of the form

$$\{a, b, a \vee b, a \wedge b\}.$$

Hence it is trivial for any finite lattice. ■

We now exhibit an example of a function which is g-modular, but which is neither supermodular not quasisupermodular. Hence modularity is a genuine generalization of supermodularity. The function fails upper semi continuity, but this is not the reason why it fails supermodularity. In the next subsection we will show an example of a C^∞ function which is g-modular but non supermodular. This latter example will involve furthermore a function having more economic appeal.

Example 1: A g-modular function which is neither supermodular nor quasisupermodular. Consider the real function $u : [0,1] \times [0,1] \to \mathbb{R}$ such that $u(x, y) = 1$ everywhere but at the point $(1, 1)$, where $u(1, 1) = 0$. Take points $a = (0, 1)$ and $b = (1, 0)$. We see immediately that the function fails both supermodularity and the single crossing property on the lattice $\{a, b, a \wedge b, a \vee b\}$.

The function is however g-modular on its domain. For any two unordered points

$$a = (a', a'')$$
$$b = (b', b'')$$

in $[0,1] \times [0,1]$, on setting $u = |a' - b'|/2$ and $v = |a'' - b''|/2$, one can take:

$$p = (u, b''), q = (u, a'')$$
$$p' = (a', v), q' = (b', v).$$

Note that the argmax of the previous function is a 'complete lattice minus its greatest element'.

2.3. **Generalized increasing differences.** We are now ready to introduce our notion of general complementarity for payoffs, to illustrate it, and to study its relation with g-modularity.

Definition 2. (Generalized increasing differences, or g-increasing-differences). Let X and T be posets. A function $u(x,t) : X \times T \to \mathbb{R}$ has generalized increasing differences in (x,t) on $X \times T$ if for every $(x', t') < (x'', t'')$ in $X \times T$, there exists $(\tilde{x}, \tilde{t})$ and $(\hat{x}, \hat{t})$ in $X \times T$, with:

$$x' \leq \tilde{x} < x'',$$
$$t' \leq \tilde{t} < t'',$$
$$x' < \hat{x} \leq x'',$$
$$t' < \hat{t} \leq t'',$$

such that:

$$u(x'', t') - u(\tilde{x}, t') \leq u(\hat{x}, t'') - u(x', t'');$$
$$u(x', t'') - u(x', \tilde{t}) \leq u(x'', \hat{t}) - u(x'', t').$$

The complementarity notion entailed by generalized increasing differences. g-increasing-differences expresses a form of complementarity among 'reallocations of activities', that we will call *general complementarity*. The following configuration of points in the plane will help to understand this, and will help to follow the proof of Theorem 1 below.

Consider the first inequality in Definition 2:

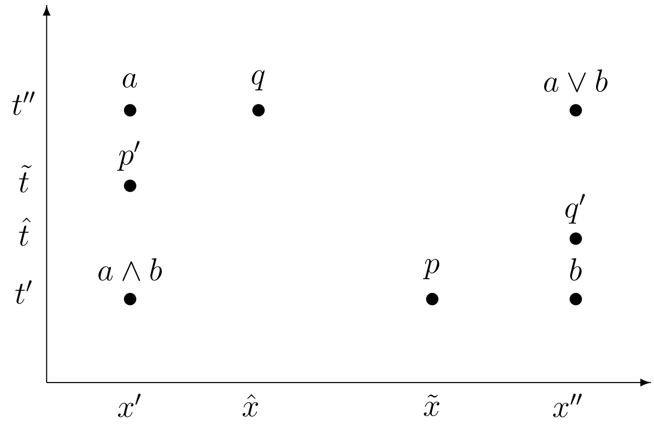
$$u(x'', t') - u(\tilde{x}, t') \leq u(\hat{x}, t'') - u(x', t'').$$

The inequality says that there is *some* increase in x, namely from x' to $\hat{x}$, which as t increases becomes more profitable than *some other* increase in x, namely that from $\tilde{x}$ to x''. This is different from increasing differences, where it is a *fixed* increase of x that becomes more profitable as t increases.

Rearranging terms in the first inequality of Definition 2 we obtain some more insights: g-increasing-differences says that:

$$u(x', t'') - u(\tilde{x}, t') \leq u(\hat{x}, t'') - u(x'', t').$$

Hence, starting from initial bundle $(\tilde{x}, t')$, there is a certain *qualitative reallocation of activities*, namely increase t and decrease x, which becomes more profitable as activity x increases, i.e. becomes more profitable if the initial bundle is (x'', t'). Quite more restrictively, increasing differences imposes that the reallocation of activities boils down to vary only one activity, and in fixed amount.



Definition 3. (Generalized complementarity for payoffs). Let X and T be posets and $u(x,t) : X \times T \to \mathbb{R}$. Activities x, t are general complements according to function u, if u has generalized increasing differences in (x, t) on $X \times T$.

It is well known, and it is an important result, that supermodularity can be completely characterized in terms of increasing differences. We are now ready to characterize completely, and under exactly the same assumptions, g-modularity in terms of g-increasing-differences. Hence, by Theorem 1, we understand that the notion of generalized increasing differences is a genuine extension of that of increasing differences, because so it is for g-modularity with respect to supermodularity.

Theorem 1. (Characterization of g-modularity in terms of g-increasing differences). *Let X and T be nonempty posets and $u(x,t) : X \times T \to \mathbb{R}$.*

(i) If X and T are lattices and u is g-modular in (x,t) on $X \times T$, it has g-increasing-differences in (x,t) on $X \times T$.

(ii) If X and T are chains and u has g-increasing-differences in (x,t) on $X \times T$, then it is g-modular in (x,t) on $X \times T$.

Proof: (i). Pick any two points $(x', t') < (x'', t'')$ in $X \times T$. Consider the two unordered points $a := (x', t'')$ and $b := (x'', t')$.
Since $b \nleq a$, by g-modularity there exists points

$$p := (\tilde{x}, t') \in [(x', t'), (x'', t')),$$
$$q := (\hat{x}, t'') \in ((x', t''), (x'', t'')],$$

such that

$$u(x', t'') + u(x'', t') \leq u(\tilde{x}, t') + \leq u(\hat{x}, t'').$$

Since $a \nleq b$, by g-modularity there exists points

$$p' := \left(x', \tilde{t}\right) \in [(x', t'), (x', t'')),$$
$$q' = \left(x'', \hat{t}\right) \in ((x'', t'), (x'', t'')];$$

such that

$$u(x', t'') + u(x'', t') \leq u\left(x', \tilde{t}\right) + \leq u\left(x'', \hat{t}\right).$$

These are the desired inequalities.

(ii). Take any points a, b in $X \times T$. If a and b are ordered, then g-modularity holds trivially. So let them be unordered. Without loss of generality, let $a = (x', t'')$ and $b = (x'', t')$, with $(x', t') < (x'', t'')$. We can write the latter inequality because X and T are chains. Hence $b \nleq a$ and $a \nleq b$, and furthermore by g-increasing-differences there exist points as in the proof of (i) such that the two inequalities in in the proof of (i) are satisfied. Hence we have proved that $u(x, t)$ is g-modular in (x, t). ■

Example 2. A function of class C^∞ which is g-modular but not supermodular. Let $m(y)$ and $q(y)$ be C^∞ on $[0, 1]$, and consider the function:

$$f(x, y) = m(y)x + q(y).$$

It is C^∞ on lattice $[0, 1] \times [0, 1]$. Consider a set of (compatible) properties:

$$m(0) > m(1),$$

$$f(x, y) \text{ strictly decreasing in } y \text{ for each } x,$$

$$m(.) > 0.$$

If $m(0) > m(1)$, the function fails supermodularity on the corners of $[0, 1] \times [0, 1]$. Indeed,

$$f(0, 1) + f(1, 0) = q(1) + m(0) + q(0)$$
$$>$$
$$f(0, 0) + f(1, 1) = q(0) + m(1) + q(1)$$

If the last two properties hold as well, then the function if g-modular. We prove the second inequality in the definition of g-modularity. The first one can be proved immediately using the assumption $m(.) > 0$.

Pick indeed any unordered $a := (x', y'')$, $b := (x'', y')$ in $[0,1] \times [0,1]$. We use Theorem 1, and hence we want to find some $y' \leq \hat{y} < y''$ and some $y' < \tilde{y} \leq y''$ such that:

$$g(\hat{y}, \tilde{y}) := f(x', y'') + f(x'', y') - f(x', \hat{y}) - f(x'', \tilde{y}) \leq 0.$$

Fix $y' = \hat{y}$ (i.e. use the inf). For function $g'(\tilde{y}) := g(y', \tilde{y})$, evaluated at $\tilde{y} = y'$, because $m(y)x + q(y)$ is strictly decreasing in y, we have that:

$$g'(y') = m(y'')x' + q(y'') - m(y')x' - q(y') < 0.$$

Hence by continuity the same is true everywhere in some neighborhood of y'. Thus we can find the required $\tilde{y}$ in $(y', y'']$

3. Increasing optimal solutions

In this section we develop the main result of the paper, Theorem 2. It is a monotone comparative statics theorem for functions having generalized complementarity. The context of the theorem is one where a function $u(x,t)$ has it decision-variable x taking values in a totally ordered and compact space (for example a compact subset of the real line) and is upper-semicontinuous in x for each value of the parameter t.

We state and prove Theorem 2 in its most general context, that of a topological compact chain[2]. We recall, without giving definitions to avoid making the paper heavier, that a topological chain is a totally ordered set endowed with its order topology. To those not acquainted with these order-theoretic notions we suggest, while reading the theorem, to refer simply to a compact subset of the real line.

Let X and T be partially ordered sets. For a function $u(x,t) : X \times T \to \mathbb{R}$, consider the correspondence

$$B(t) : t \in T \mapsto \{z \in X : \forall x \in X, u(z,t) \geq u(x,t)\}.$$

In other words, $B(t)$ is the argmax function of $u(x,t)$ over T.

Theorem 2 shows that a function that has generalized increasing differences has its $B(t)$ correspondence which is Veinott-increasing in the parameter t. As such, the theorem generalizes the celebrated Topkis (1978)'s monotonicity theorem in the context of upper-semicontinuous functions on a compact chain.

[2]A chain is a totally ordered set.

Theorem 2. *Let X be a topological, compact chain (for example, X is a compact subset of $\mathbb{R}$). Let T be a poset. Let $u(x,t) : X \times T \to \mathbb{R}$ be upper-semicontinuous in x on X for each t, and have generalized increasing differences in (x,t) on $X \times T$. Then the argmax $B(t)$ is Veinott-increasing in t on T.*

Remark. The assumption that $u(x,t)$ is upper-semicontinuous in x for each t cannot be avoided in this theorem. It is made not only to assure that the argamx $B(t)$ is nonempty, but also to assure that it is closed, and the latter cannot be dispensed off in the proof of the theorem. To see this, recall the function

$$u(x,t) : [0,1] \times [0,1] \to \mathbb{R}$$

of Example 1, defined as $u(x,t) = 1$ everywhere but at $(1,1)$, where $u(1,1) = 0$. We have seen in Example 1 that this function has g-increasing-differences (is g-modular). However, it fails upper semicontinuity at point $(1,1)$. Its argmax correspondence is everywhere nonempty and takes the form: $B(t) = [0,1]$ if $t \in [0,1)$, $B(t) = [0,1)$ if $t = 1$. This correspondence fails to be Veinott-increasing. Indeed, take any $t' \in [0,1)$. Then $1 \in B(t')$ but, for $t = 1$ and any $b \in B(1)$, clearly $1 \vee b \equiv 1 \notin B(1)$. The problem, as it will become clear after proving Theorem 2, is that $B(1)$ is not closed.

Proof of Theorem 2: $B(t)$ is nonempty for every t. Pick any $t' < t''$ in T. Pick any $x'' \in B(t')$ and any $x' \in B(t'')$. If $x'' \leq x'$ we are done. So let $x' < x''$. We need to show that $x'' \in B(t'')$. Since $(x', t') < (x'', t'')$, then by generalized increasing differences there exist in X some $\tilde{x}_1$ and $\hat{x}_1$, with

$$x' \leq \tilde{x}_1 < x''$$

$$x' < \hat{x}_1 \leq x''$$

such that:

$$0 \leq u(x'', t') - u(\tilde{x}_1, t') \leq u(\hat{x}_1, t'') - u(x', t''),$$

where the first inequality follows from optimality of x'' at t' and the second by generalized increasing differences. Hence, we must have that $\hat{x}_1 \in B(t'')$.

If $\hat{x}_1 = x''$ we are done. Otherwise we must have that $\hat{x}_1 < x''$. Hence, $(\hat{x}_1, t') < (x'', t')$, and again by generalized increasing differences there are in X some $\tilde{x}_2$ and $\hat{x}_2$, with

$$\hat{x}_1 \leq \tilde{x}_2 < x''$$

$$\hat{x}_1 < \hat{x}_2 \leq x''$$

such that:

$$0 \leq u(x'', t') - u(\tilde{x}_2, t') \leq u(\hat{x}_2, t'') - u(\hat{x}_1, t''),$$

where the first inequality follows from optimality of x'' at t' and the second one from generalized increasing differences. But since in the previous step we have proved that $\hat{x}_1 \in B(t'')$, then $\hat{x}_2 \in B(t'')$.

By continuing to apply iteratively generalized increasing differences in this way, we construct a sequence $(\hat{x}_n)$ in $B(t'')$ such that, either for some n we have that $\hat{x}_n = x''$, and we are done, or else $(\hat{x}_n)$ is strictly increasing and bounded from above by x''. Hence $(\hat{x}_n)$ must converge to some $y^{h'}$, were in order to denote that the sequence, hence its limit, depends on the initial points x', x'' that we have chosen, we have set

$$h' := (x', x'') \in B(t'') \times B(t')$$

Since $u(x,t)$ is upper semicontinuous in x for each t, then $B(t'')$ is closed, and so $y^{h'} \in B(t'')$.

For the given h', let $\Gamma_{h'}$ be the set of all the sequences of elements of $B(t'')$ that can be constructed from h' by iterative application of generalized increasing differences, as we have done above. Suppose that no element of any sequence in $\Gamma_{h'}$ coincides with x'', otherwise we are done. Now consider the set

$$U := \{h = (z, x'') \in B(t'') \times \{x''\} : z \leq x''\}.$$

For each point $h \in U$, let Γ_h be the set of all sequences of elements of $B(t'')$ that can be constructed from h by iterative application of generalized increasing differences. Still assume that no element of any sequence in Γ_h, for any h, coincides with x'', otherwise we are done.

Let

$$V := \{\cup_{h \in U} \Gamma_h\}.$$

Clearly every sequence in V is strictly increasing and bounded from above by x'', because we have assumed that for each sequence in V no one of its terms coincides with x'' and that the sequence has been constructed by iterative application of generalized increasing difference. Hence every sequence in V must converge, and since $B(t'')$ is closed, its limit belongs to $B(t'')$.

Let Y be the set of all and only the limits of sequences in V. By closeness of $B(t'')$, $Y \subseteq B(t'')$. Since $B(t'')$ is a closed subset of a compact space, it is compact, and since it is a lattice (is a chain), then it is a complete lattice. Hence $y := \vee Y \in B(t'')$.

Clearly we must have that $y \leq x''$, since x'' majorizes all limits of sequences in V. If equality holds we are done because $y \in B(t'')$. Let then $y < x''$. Since $y \in B(t'')$ and $(y, t') < (x'', t'')$, then we can construct by generalized increasing differences a strictly increasing sequence $y < y_1 < y_2 \dots$ in $B(t'')$ converging to some $k \in B(t'')$ such that $y < k \leq x''$. But $(y, x'') \in U$. Hence we must have that

$k \in Y$, a contradiction since y is the sup of Y. Thus it cannot be that $y < x''$, hence $y = x''$ and we are done.

The other part of Veinott-increasingness, that is the fact that $x' \in B(t')$, can be proved in exactly the same way, but using the second inequality in Definition 2 and using decreasing sequences and infima instead of increasing sequences and suprema. ■

4. Existence of equilibria for a generalized Cournot model

We are finally ready to apply the tools that we have developed so far to our initial, motivating Cournot problem. It is clear that, even if a generalized Cournot model fails to be a supermodular game, it can still be a game with general complementarities in the sense of Section 2 and hence it can have a nonempty Nash set. We show in this last section a generalized Cournot model which behaves exactly like this.

We have chosen a meaningful but simple model, to the scope of showing that even in a simple case the introduction of opponents' production decisions into the cost functions makes the model not amenable to be studied with standard supermodularity tools.

Consider a duopoly model with the following payoffs (we write the payoff for firm i):

$$\Pi_i(q_i, q_j) = [a - b(q_i + q_j)]\, q_i - q_i q_j^{-1}.$$

We assume that q_i, q_j take values in compact sets of strictly positive real numbers, and that a, b are strictly positive as well. The payoffs functions are continuous and then the individual decision problems of the firms have solutions.

The inverse demand function is linear, and the production cost of firm i, $C_i(q_i, q_j) = q_i q_j^{-i}$ decreases as firm j produces more. Furthermore i's marginal cost decreases as j increases its production level:

$$\frac{\partial^2 C_i(q_i, q_j)}{\partial q_i \partial q_j} = -q_j^{-2} < 0.$$

Let us check the supermodularity condition:

$$\frac{\partial^2 \Pi_i(q_i, q_j)}{\partial q_i \partial q_j} = -b + q_j^{-2} \geq 0 \Leftrightarrow q_j^2 \leq b^{-1}.$$

Hence this oligopoly game is supermodular for q_j small enough, in particular such that $q_j^2 < b^{-1}$, but become submodular as soon as $q_j^2 > b^{-1}$. The interpretation is that, for low levels of j production, an increase in the production of j induces a reduction in i's marginal revenues which is more than compensated by the induced reduction in i's marginal cost. This will cease to hold for higher levels of j's production.

Hence, for low levels of j's production the production decisions q_i, q_j are complements, but they become substitutes for higher levels of j's production.

This Cournot game cannot be studied with the theory of supermodular games, due exactly to the shift of the production decisions of the two firms from being complements to being substitutes. Note that this impossibility of using supermodular games has risen exactly because we have made the costs depending on opponents' production decisions. If the costs were functions of own production alone, then for any specification of the cost functions we would have had that:

$$\frac{\partial^2 \Pi_i (q_i, q_j)}{\partial q_i \partial q_j} = -b < 0,$$

hence we would have obtained a standard duopoly submodular game, perfectly treatable with the tools of supermodular games.

The last step consists in showing that this Cournot game has general complementarities, or in other words that the payoff functions have g-increasing-differences in (q_i, q_j).

To this end, using Definition 2 we need to check that, for every $\left(q_i', q_j'\right) < \left(q_i'', q_j''\right)$, there exist points $\tilde{q}_i$ and $\hat{q}_i$, with:

$$q_i' \leq \tilde{q}_i < q_i'',$$
$$q_i' < \hat{q}_i \leq q_i'',$$

such that:

$$\Pi_i \left(q_i'', q_j'\right) - \Pi_i \left(\tilde{q}_i, q_j'\right) \leq \Pi_i \left(\hat{q}_i, q_j''\right) - \Pi_i \left(q_i', q_j''\right).$$

We omit for the sake of brevity to check g-increasing-differences for the variable q_j. The argument would be exactly the same as below.

Define the following functions:

$$L\left(\tilde{q}_i\right) = \Pi_i \left(q_i'', q_j'\right) - \Pi_i \left(\tilde{q}_i, q_j'\right),$$

$$M\left(\hat{q}_i\right) = \Pi_i \left(\hat{q}_i, q_j''\right) - \Pi_i \left(q_i', q_j''\right).$$

Suppose that there is some point $q_i' \leq z < q_i''$ at which $L(z) := k < 0$. See that, by continuity,

$$\lim_{\hat{q}_i \to q_i'} M\left(\hat{q}_i\right) = 0.$$

Hence there must exist a full neighborhood I of q_i' such that at each point x of this neighborhood we have that $M(x) > k$. Thus we can pick the required $\hat{q}_i$ in the intersection between neighborhood I and interval $(q_i', q_i'']$. We finally pick the required $\tilde{q}_i$ on setting $\tilde{q}_i = z$.

Suppose now that at every point $q_i' \leq \tilde{q}_i < q_i''$, $L(\tilde{q}_i) \geq 0$. Then it can be shown by direct computation that $M(\hat{q}_i) \geq 0$ for each $q_i' < \hat{q}_i \leq q_i''$. Suppose now that for every $q_i' \leq \tilde{q}_i < q_i''$ and every $q_i' < \hat{q}_i \leq q_i''$ we would have that $L(\tilde{q}_i) > M(\hat{q}_i)$. Hence, for any fixed $\hat{q}_i{}'$, we would have by continuity that:

$$\lim_{\tilde{q}_i \to q_i''} L(\tilde{q}_i) = 0 \geq M(\hat{q}_i{}') .$$

Thus for every $q_i' < \hat{q}_i \leq q_i''$ we would have that $M(x) = 0$. A contradiction.

Hence there exist $q_i' \leq \tilde{q}_i < q_i''$ and $q_i' < \hat{q}_i \leq q_i''$ such that $L(\tilde{q}_i) \leq M(\hat{q}_i)$, and we have shown that payoffs satisfy g-increasing-differences in (q_i, q_j).

We are finally in the position to apply Theorem 2 of Section 3 to our Cournot game. By Theorem 2, the joint best reply of our Cournot game is Veinott-increasing, and this result is a fundamental step in order to follow, from now on, the standard route of existence proofs in supermodular games (see for example Topkis, 1998).

Namely, by continuity of payoffs and by the fact that the sets in which q_i, q_j take values are compact and totally ordered, we know that the joint best reply of our game is subcomplete-sublattice-valued. Since it is also Veinott-increasing, by Zhou (1994)'s fixpoint theorem we can conclude that this joint best reply has a fixed point and so that our game has a Cournot equilibrium.

5. Conclusion

In this paper, we have argued that it is reasonable to extend the standard homogeneous-product Cournot setting to the case where the cost function of each firm, besides own production decisions, depends on its opponents' production decisions. We have shown that with such generalized of cost functions, even a simple Cournot game typically treatable with the theory of supermodular games, turns out to be no longer treatable with such a theory.

We have shown existence of equilibria for this game by developing, and then applying to it, a generalization of the theory supermodular games for a context where individual decisions take values in a totally ordered set (for example a subset of the real line). The generalization has consisted in relaxing the notion of complementarity that payoffs are required to satisfy in supermodular games.

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