A MODEL OF PROMOTIONAL COMPETITION IN OLIGOPOLY

by

Richard Schmalensee*

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*University of California, San Diego. This paper was written while I was Visiting Associate Professor, Institut des Sciences Economiques, Université Catholique de Louvain.
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UNITE ANALYSE ECONOMIQUE
INSTITUT DES SCIENCES ECONOMIQUES
121 Parkstraat, B-3000 LOUVAIN (Belgium)

A B S T R A C T

In concentrated markets, price competition is often largely replaced by promotional competition. The Cournot behavioral assumption seems quite plausible in such cases. This paper presents and analyzes a symmetric Cournot model of promotional competition with fixed price and a particular demand structure. Entry generally reduces profits in this model, but it may not eliminate them. Local stability is analyzed, and it is noted that stability does not imply that profits are eliminated by entry. Increasing marginal promotion cost is always stabilizing, but increasing marginal production cost may be destabilizing. Boundedness and global stability are demonstrated under special assumptions.

I. INTRODUCTION

This paper is concerned with markets in which products are differentiated and sellers are few. It is a generally accepted "stylized fact" of industrial organization that prices competition is relatively rare in such markets. Prices tend to change infrequently, while sellers compete, if at all, mainly through product variation and promotional expenditures. It thus seems of some interest to attempt to model rigorously markets in which the only competition is of this sort, and this essay does so. The model is short-run, in the sense that only promotional competition is studied ; the products offered by the firms in the market are assumed fixed.

In order to construct analytical models of oligopoly markets, it has been necessary to make simple assumptions about firm bahevior. In most of the formal literature, firms are assumed to set price or output in a Cournot -- Bertrand fashion.[1] That is, they are assumed to suppose that their decisions will have no impact on their rivals' actions. Unfortunately, it is generally acknowledged that this assumption does not provide a realistic description of price or output decisions under oligopoly.

A case can be made, however, that the assumption of naive behavior is reasonable in the case of non-price competition.[2] First, it is generally difficult for firms to monitor competitors' promotional spending, at least without substantial lags. Second, it takes time to change promotional budgets, so that any firm could at least expect a substantial lag before competitors respond to its actions. Third, the impact of any promotional campaign depends on the nature of the "pitch" as well as the level of spending. This both complicates the monitoring problem and makes retaliation more difficult to define or effect. Finally, there is considerable evidence that firms in fact act independently in deciding promotional outlays.[3] Thus, both because it is convenient and because it may be reasonably realistic in this context, it is assumed in this essay that firms fix their promotional budgets on the assumption that their spending will not affect the spending of their rivals.

The next section presents the notation and the demand and cost assumptions of the model. Section III is concerned with the nature of equilibrium, in particular with the impact of changes in the number of competitors. Sections IV and V deal with the local and global stability of dynamic versions of the basic model.

There exists a substantial literature concerned with price-quantity competition under the Cournot -- Bertrand behavioral assumption. Sections III - V study many of the questions that have been considered in that literature. It has not been possible to answer them all, however, and a number of topics for future research are mentioned.

II. THE MODEL

Let N, an integer greater than one, be the number of firms in the industry, and let A_i be the real promotional spending of the i^{th} firm. It is assumed that the A_i are non-negative. One unit of real promotion might correspond to one thirty-second television commercial seen by one viewer ; the A_i are thus measured in "messages" or "exposures".[4] We assume that all firms are equally efficient in the production of promotion ; relaxing this symmetry condition and the others imposed below would greatly complicate the analysis without yielding much additional insight.

As the discussion in the introduction suggests, we assume that price, P, is fixed. The sales of the i^{th} firm, q_i, can thus depend only on the A_i. In fact, this dependence may not generally be expressible by means of a simple smooth function, as the negative econometric results discussed by Schmalensee [14, ch.4] and the theoretical arguments of Kuehn [5] suggest. Still, as a convenient first approximation, it is assumed that such a function does exist for each firm. It is further assumed that firms' products are equally attractive, so that this function is the same for all firms. Finally, we make the plausible assumption that the demand function can be written in the following form :

$$q_i = q(A_i,\overline{A}_i) = S(A_i,\overline{A}_i)Q(\sum_{j=1}^{N} A_j), \quad i = 1,\ldots,N, \quad (2.1)$$

where

$$\overline{A}_i \equiv (A_1, \ldots, A_{i-1}, A_{i+1}, \ldots, A_N), \quad i = 1, \ldots, N. \quad (2.2)$$

It will be convenient to define (A) as the N-vector $(A_1, \overline{A}_1)$.

In (2.1), Q is total industry sales and $S(A_i, \overline{A}_i)$ is the i^{th} firm's market share. Q is written as a function of total promotion because of symmetry ; it is logically non-decreasing. A convenient special case is

$$Q(\sum_{j=1}^{N} A_j) = Q_0(\sum_{j=1}^{N} A_j)^a, \qquad Q_0 > 0, \ a \geqslant 0. \qquad (2.3)$$

Because of our symmetry assumptions, it seems reasonable to require the S function to satisfy at least the following restrictions when at least one firm engages in positive promotion :

(a.) $0 \leqslant S(A_i, \overline{A}_i) \leqslant 1.$

(b.) $\sum_{i=1}^{N} S(A_i, \overline{A}_i) = 1.$

(c.) $\partial S(A_i, \overline{A}_i)/\partial A_i \geqslant 0.$

(d.) $\partial S(A_i, \overline{A}_i)/\partial A_j \leqslant 0$, for $j \neq i$.

(e.) $S(A_i, 0) = 1$ if $A_i > 0$.

(f.) $S(0, \overline{A}_i) = 0$ if $A_j > 0$ for some $j \neq i$.

(g.) $\lim_{A_i \to \infty} S(A_i, \overline{A}_i) = 1.$

(h.) S homogeneous of degree zero.

(i.) $S(A_i, \overline{A}_i) = S(A_i, \overline{A}_i')$ if $\overline{A}_i'$ is obtained from $\overline{A}_i$ by re-numbering firms.

There may be many functional forms that satisfy these requirements ; a simple one that does is due to Mills [6] :[5]

$$S(A_i, \overline{A}_i) = (A_i)^e \Big/ \sum_{j=1}^{N} (A_j)^e, \quad e > 0. \tag{2.4}$$

Combining (2.3) and (2.4), we obtain the demand assumption that is employed in all of the formal analysis that follows :

<u>Assumption 1</u> : Whenever at least one of the A_i is positive, the sales of the i^{th} firm, for $i = 1,\ldots,N$, are given by

$$q_i = q(A_i, \overline{A}_i) = [(A_i)^e \Big/ \sum_{j=1}^{N} (A_j)^e][Q_0(\sum_{j=1}^{N} A_j)^a],$$

where $a > 0$, $e > 0$, and $Q_0 > 0$ are constants.

Note that if $a = 0$, this function cannot be defined at the origin in any way that preserves continuity. Unless total sales fall to zero when total promotion is zero, this same problem would arise for any S function satisfying conditions (a.) - (i.) above. Away from the origin, we always assume that q is differentiable.

The excuse for working with this special demand function here is exactly that given for considering Cobb-Douglas or CES production functions in other contexts ; it has the right properties and is relatively simple to work with.

We also assume symmetry in production and promotion costs, so that the profit of the i^{th} firm is in general given by

$$\pi_i = \pi(A_i, \bar{A}_i) = Pq(A_i, \bar{A}_i) - C[q(A_i, \bar{A}_i)] - T(A_i)A_i, \quad (2.5)$$

where C is total production cost, and T is the average cost of promotion. Both C and T are always assumed to be differentiable and non-decreasing. In most of what follows, the following stronger assumptions are made :

<u>Assumption 2</u> : $T(A_i) = T$, a constant.

<u>Assumption 3</u> : Marginal production cost, dC/dq, is constant. The price -- marginal cost difference (also constant) is written as m.

III. EQUILIBRIUM

Under the Cournot -- Bertrand behavioral assumption, a vector of promotions (A^*) will be an equilibrium if A_i^* maximizes π_i, with $\overline{A}_i^*$ treated as fixed, for $i = 1,\ldots,N$. It might appear that one could follow Frank and Quandt [2], who considered quantity -- setting firms, and use the Kakutani fixed point theorem to prove the existence of such an equilibrium, even without assuming identical profit functions. Attempts to follow this approach, even with all our symmetry conditions, have not been successful, however, and I have not been able to establish a general existence theorem. [6]

If an internal equilibrium does exist, it is clearly necessary that

$$\partial\pi_i/\partial A_i = [P - dC/dq_i](\partial q_i/\partial A_i) - d[A_i T(A_i)]/dA_i = 0, \qquad (3.1)$$

$$i = 1,\ldots,N \;,$$

where all terms are evaluated at $(A) = (A^*)$. Further, for (3.1) to signal a maximum of each firm's profit, it is clearly sufficient to require that at this same point

$$\partial^2\pi_i / \partial A_i^2 < 0, \quad i = 1,\ldots,N. \qquad (3.2)$$

It is necessary that this quantity be non-positive.

Under Assumptions 1 - 3, equations (3.1) become

$$mQ_0G(1)^{a-1}[eA_i^{e-1}G(1) + aA_i^e - eA_i^{2e-1}G(1)/G(e)]/\,G(e) = T, \quad i = 1,\ldots,N, \tag{3.3}$$

where the function G is defined by

$$G(x) \equiv \sum_{j=1}^{N}(A_j)^x. \tag{3.4}$$

If and only if $a \neq 1$, equations (3.3) have the solution

$$A_i = A^E = (1/N)[mQ_0(Ne + a - e)/NT]^{1/(1-a)}, \quad i = 1,\ldots,N. \tag{3.5}$$

Letting q^E denote the sales of any firm at this point, the ratio of promotional spending to gross profit at this equilibrium can be shown to be

$$\frac{A^E T}{mq^E} = \frac{Ne + a - e}{N}. \tag{3.6}$$

Multiplication by (m/P), the Lerner measure of the degree of monopoly, yields the equilibrium promotion -- sales ratio. Note that while A^E depends on T, these two ratios do not.

Unless the right-hand side of (3.6) is less than or equal to one, all firms will be loosing money, and the equilibrium will not be viable. For finite N, this obviously requires

$$N(e - 1) + (a - e) \leqslant 0. \tag{3.7}$$

When the A_i are all equal, the second-order condition (3.2) becomes

$$a(a - 1) + e(N - 1)[N(e - 1) + 2(a - e)] < 0. \quad (3.8)$$

It is necessary that this quantity be non-positive. For convenience, we collect the sufficient conditions for existence and viability of equilibrium under Assumptions 1 - 3 in

> Assumption 4 : Conditions (3.7) and (3.8) are satisfied and $a \neq 1$.

Condition (3.7) will be satisfied for all N ($\geqslant 2$) if and only if $e \leqslant 1$ and $(e + a) \leqslant 2$. Condition (3.8) will be satisfied for all N if and only if $a < 1$ and $e \leqslant 1$. Combining these, it is clearly necessary and sufficient for (3.7) and (3.8) to be satisfied for all N that $a < 1$ and $e \leqslant 1$. In what follows, we use the term normal case to denote satisfaction of these inequalities. Since it is easy to show that $\partial q_i/\partial A_j$, for $j \neq i$, has the sign of $(a - e)$ when the A_i are equal, it is a somewhat surprising feature of the equilibrium given by (3.5) that this derivative may be positive, even in the normal case.

Unfortunately, I have not been able to produce any general conditions that would guarantee the uniqueness of equilibrium in this model.[7] Even under Assumptions 1 - 3, it has not been possible to prove that (3.5) is the unique equilibrium. When $e = 1$, however, so that a must be less than one by (3.8), the bracketed expression in (3.3) reduces to a linear function of A_i, so that this condition cannot be satisfied by two firms with different A_i, and (3.5) is the unique solution.

The positive results thus far can be summarized in

<u>Proposition 1</u> : Under Assumptions 1 - 3, the system has an equilibrium given by (3.5) if and only if Assumption 4 is satisfied. In the normal case this Assumption is satisfied for all N. If, in addition, e = 1, (3.5) is the unique equilibrium solution.

In the literature on formal models of oligopolistic price -- quantity competition, attention has been focused on the quasi-competitiveness and convergence of the equilibrium solution.[8] In the standard Cournot model without product differentiation, the solution is said to be quasi-competitive if equilibrium output rises (and equilibrium price falls) with N. The solution is said to converge if output and price approach their competitive values in the limit as N increases.

Several problems arise when one attempts to carry over these concerns directly to the present model. First, without specific assumptions about the demand function, it is not clear how one should represent the entry of new firms when products are differentiated. Second, with price fixed one can hardly hope to observe convergence to anything resembling a competitive equilibrium. Third, an increase in total sales in this model as N rises, which can only be brought about by increases in industry promotion, does not have unambiguous welfare implications.

Still, some effects of changes in N under Assumptions 1 - 3 are of interest. Straightforward differentiation of (3.5) and (3.6) indicates how promotion and the promotion -- sales ratio are affected by entry :

<u>Proposition 2</u> : Under Assumptions 1 - 3, at the equilibrium given by (3.5) :

(a.) $\partial(NA^E)/\partial N$ has the sign of $(e - a)/(1 - a)$,

(b.) $\partial(A^E T/Pq^E)/\partial N$ has the sign of $(e - a)$,

(c.) $\partial A^E/\partial N$ has the sign of

$$- Ne - [\cancel{2(e - a) - a(1 - e)}]/(1 - a).$$

$$(e - a)(e - a)$$

The implication that total promotion and the promotion -- sales ratio may rise with N is clearly somewhat artificial and results from the assumption that P does not depend on N. If markups fall with N, it becomes more likely that these quantities will also, at least for large N.[9]

Following Stigler [17], we can explore the effects of entry on excess profits. At (3.5), total excess profit in the industry is given by

$$\sum_{i=1}^{N} \pi_i = N\pi^E = N[mQ_0(NA^E)^a / N - A^E T] \qquad (3.9)$$

$$= mQ_0(NA^E)^a - (NA^E)T.$$

If this quantity falls with N, the solution can be said to possess something like the quasi-competitive property. Differentiating (3.9), making use of (3.5), and substituting for T from the first-order conditions yields

Proposition 3 : Under Assumptions 1 - 3, at the equilibrium given by (3.5), $\partial(N\pi^E)/\partial N$ has the sign of $(a - 1)$, which is negative in the normal case.

So, unless demand conditions are such that there are increasing or constant returns to industry promotion, the increased promotional competition brought about by entry serves to reduce total excess profits. It also follows, of course, that the profits of individual firms decline.

Let us now see when entry serves to eliminate profits in this model. If it does so, the model may be said to possess a close analog of the usual convergence property. Clearly, unless $a < 1$, we cannot hope that this will occur, so we make this assumption. When $e \leqslant 1$, conditions (3.7) and (3.8) are then satisfied for all N, and equation (3.6) shows that all excess profit will be eliminated in the limit if and only if $e = 1$.

The case $e > 1$ is a bit more complicated, since both (3.7) and (3.8) will be violated for large N. They will be satisfied for some $N \geqslant 2$ if and only if $e \leqslant (2 - a)$. If this condition is satisfied, a simple substitution shows that at the N for which (3.7) holds with equality, (3.8) will still be satisfied. Thus profits are eliminated by entry at finite N ; the equilibrium does not first break down because of violation of the second-order conditions. Hence we have established

<u>Proposition 4</u> : Under Assumptions 1 - 3, if $a < 1$ and attention is restricted to the equilibrium given by (3.5) :

(a.) If $e > (2 - a)$, no viable equilibrium exists.

(b.) If $(2 - a) \geq e > 1$, profits are eliminated by entry at finite N.

(c.) If $e = 1$, profits are eliminated by entry in the limit.

(d.) If $e < 1$, all firms earn positive profit, even in the limit.

When $e < 1$, there are diminishing returns to market share, in the sense that if $A_i = 2A_j$, firm i's market share will be less than twice firm j's. This serves to restrain promotional competition sufficiently so that excess profits remain for any number of firms.

IV. LOCAL STABILITY

For stability analysis, we must assume that for any $\overline{A}_i$ and for all i, the A_i^* that maximizes π_i is unique. In this section, we consider two dynamic versions of the basic model. Both follow the voluminous literature on the stability of price -- quantity competition and posit gradual adjustment of A_i to A_i^*. Letting t be time, the two adjustment mechanisms are described in

<u>Assumption 5</u> : Adjustment is continuous, according to

$$dA_i/dt = k_i\{A_i^*[\overline{A}_i(t)] - A_i(t)\}, \quad i = 1,\ldots,N ,$$

where the k_i are positive constants ; and

<u>Assumption 6</u> : Adjustment is discrete, according to

$$A_i(t) - A_i(t-1) = k_i\{A_i^*[\overline{A}_i(t)] - A_i(t-1)\},$$
$$i = 1,\ldots,N ,$$

where the k_i are positive constants.

The key quantity in local stability analysis is

$$r \equiv \partial A_i^* / \partial A_j, \quad j \neq i, \tag{4.1}$$

where the derivative is evaluated at equilibrium. Let K be an N x N diagonal matrix with i^{th} diagonal element equal to k_i, let I be the N x N identity matrix, and let R be an N x N matrix with zeros on the diagonal and all off-diagonal elements equal to r. Then the continuous system of Assumption 5 will be locally stable if and only if the characteristic roots of $[K(R - I)]$ have negative real parts. Fisher's [1, pp.130-1] Theorem 1 implies

<u>Lemma 1</u> : The characteristic roots of $[K(R - I)]$ are all real. They are negative if and only if $r > -1$.

Similarly, the discrete system of Assumption 6 will be locally stable if and only if the real parts of the characteristic roots of $[K(R - I) + I]$ are less than unity in absolute value. As is usually the case in difference equation analysis, stability cannot be guaranteed for arbitrary positive speeds of adjustment. A slight modification of Fisher's [1, p.131] Theorem 3 yields

<u>Lemma 2</u> : The characteristic roots of $[K(R - I) + I]$ are all real. There exists a set of positive k_i such that they are all less than unity in absolute value if and only if $1 / (N - 1) > r > -1$.

Since only symmetric equilibria are considered, we can simplify notation in the remainder of this section by dropping firm subscripts, so that A is now just the promotion of a typical firm. Let A' be the promotion of any one of its competitors, let marginal production cost (dC/dq) be MC, and let marginal promotion cost (d(AT) / dA) be MA. Then, using subscripts to indicate derivatives evaluated at the point of equilibrium, differentiation of (3.1) yields

$$r = -\frac{\pi_{AA'}}{\pi_{AA}} = \frac{(P - MC)q_{AA'} - MC_q\, q_A\, q_{A'}}{MA_A + MC_q(q_A)^2 - (P - MC)q_{AA}} . \qquad (4.2)$$

The denominator must be positive by virtue of (3.2).

This expression permits consideration of the stability implications of relaxing Assumptions 2 and 3. The larger is MA_A, that is, the more rapidly marginal costs of promotion rise, the smaller in absolute value is r. From Lemmas 1 and 2, driving r toward zero is stabilizing under either adjustment mechanism.

The implications of increasing marginal production cost are less reassuring. Differentiation of (4.2) establishes that $\partial r/\partial MC_q$ has the sign of

$$- [\, q_{A'} + r\, q_A \,].$$

Since the sign of $q_{A'}$ cannot be determined even under Assumptions 1 - 3 in the normal case, it is clear that increases in MC_q may be destabilizing under either adjustment mechanism. This contrasts sharply with the effect of rising marginal production cost under price -- quantity competition, where it is generally stabilizing.[10]

Under Assumptions 1 - 3, at the equilibrium given by (3.5),

$$r = -q_{AA'}/q_{AA} = -Z/(Z+Y), \text{ where} \tag{4.3}$$

$$Z = a(a-1) + e(N-2)(a-e), \tag{4.4}$$

$$Y = Ne[(a-e) + (N-1)(e-1)], \tag{4.5}$$

and $(Z + Y)$ is negative by (3.8).

Under Assumption 5, by Lemma 1, local stability requires $Y < 0$, or

$$(a-e) + (N-1)(e-1) < 0. \tag{4.6}$$

Assumption 4 does not ensure that this condition will be satisfied, so the result is

> <u>Proposition 5</u> : Under Assumptions 1 - 5, the equilibrium given by (3.5) will be locally stable if and only if condition (4.6) is satisfied. This condition is satisfied for all N in the normal case. Under this condition, if Assumption 2 is replaced by the requirement that marginal promotion cost be non-decreasing, the symmetric equilibrium (if it exists) will be locally stable.

Similarly, under Assumption 6, by Lemma 2, we need $Y < 0$, which implies condition (4.6), and $(NZ + Y) < 0$. This implies

$$(a-1)[a + e(N-1)] < 0, \tag{4.7}$$

which will be satisfied if and only if a is less than one. Since Assumption 4 does not require a to be less than one, the result is

> <u>Proposition 6</u> : Under Assumptions 1 - 4 and 6, there will exist positive k_i such that the equilibrium given by (3.5) is locally stable if and only if (4.6) is satisfied and $a < 1$. These conditions are satisfied for all N in the normal case. Under these conditions, if Assumption 2 is replaced by the requirement that marginal promotion cost be non-decreasing, there will exist positive k_i such that the symmetric equilibrium (if it exists) will be locally stable.

Note that the normal case ensures stability under Assumption 5 and the possibility of stability under Assumption 6. The stability conditions of Proposition 5 establish nothing about the change in industry profit when N rises. Oddly enough, however, stability of the difference equation system of Assumption 6 is sufficient to establish $a < 1$, which implies that entry reduces profits. Neither stability result requires $e = 1$, so neither casts any doubt on the possibility that entry may fail to eliminate excess profits.[11]

V. GLOBAL DYNAMICS

The analysis that follows is less general than that of Section IV in several respects. Only the continuous adjustment mechanism of Assumption 5 is considered. Further, it has not been possible to say anything about the effects of relaxing Assumptions 2 and 3 on behavior outside a small neighborhood of equilibrium. Finally, even under these Assumptions, the difficulty of determining the sign of dA_i/dt away from equilibrium when $e \neq 1$ has required me to impose the condition $e = 1$ in order to obtain any results at all ; I can only offer the conjecture that both Propositions of this Section are true for $e < 1$ in the normal case.

When $e = 1$, the derivative of any firm's profit with respect to its own promotion is simply

$$\partial\pi_i/\partial A_i = mQ_0[(A_i + D_i)^{a-1} + (a-1)A_i(A_i + D_i)^{a-2}] - T, \quad i = 1,\ldots,N, \tag{5.1}$$

where D_i is defined by

$$D_i \equiv \sum_{j=1}^{N} A_j - A_i, \quad i = 1,\ldots,N. \tag{5.2}$$

Also, $e = 1$ and condition (3.8) require $a < 1$. Using this and (5.1), it is easy to show that $\partial^2\pi_i / \partial A_i^2$ is negative as long as any firm is engaged in promotion. Hence, away from the origin, the sign of dA_i/dt under Assumption 5 is the same as that of $\partial\pi_i / \partial A_i$. If we define the quantity

$$U \equiv (mQ_0/T)^{1/(1-a)}, \tag{5.3}$$

this fact can be used to show that the system is bounded, by establishing

<u>Proposition 7</u> : Under Assumptions 1 - 5, for any N, if e = 1 and if at some time t_0

(a.) $A_i(t_0) > 0, \quad i = 1,\ldots,N$, and

(b.) $D_i(t_0) < U, \quad i = 1,\ldots,N$,

these conditions will be satisfied for all $t > t_0$.

Let B be the set defined by these conditions. The proposition is proven by first showing that the system cannot leave B by reaching the origin, then showing that it cannot reach points at which (b.) is violated, and finally showing that it cannot reach points at which (a.) is violated.

Consider any (A^0) such that the A_i^0 are non-negative and sum to unity, and let $A_i = \varepsilon A_i$, where $\varepsilon > 0$, for $i = 1,\ldots,N$. Then, from (5.1) a d (5.2), dA_i/dt has the sign of

$$U^{1-a}[(\varepsilon)^{a-1} + (a-1)\varepsilon A_i^0(\varepsilon)^{a-2}] - 1,$$

which is positive when

$$\varepsilon < \varepsilon_i = U[1 + (a-1)A_i^0]^{1/(1-a)}$$

Since at least one of the A_i^0 must be less than one,

$$0 \leqslant Ua^{1/(1-a)} < \max(\varepsilon_i),$$

so that whenever the sum of the A_i drops below some positive number, at least one of the A_i must be increasing, and the system clearly cannot reach the origin. So we may assume in what follows that at least one firm engages in positive promotion.

Now, without loss of generality, let A_1 be the smallest of the A_i, so that D_1 is the largest of the D_i. At points where $D_1 = U$, dA_i/dt, for $i \neq 1$, has the sign of

$$U^{1-a}\left[(A_i + D_i)^{a-1} + (a-1)A_i(A_i + D_i)^{a-2}\right] - 1$$

$$= U^{1-a}\left[(A_1 + U)^{a-1} + (a-1)A_i(A_1 + U)^{a-2}\right] - 1$$

$$\leq U^{1-a}\left[(A_1 + U)^{a-1} + (a-1)A_1(A_1 + U)^{a-2}\right] - 1$$

where strict inequality holds if $A_i > A_1$. Since this expression is a decreasing function of A_1, it attains its maximum when $A_1 = 0$, at which point it is equal to zero. So at points where $D_1 = U$, all the positive A_i are decreasing, and all zero A_i are constant. By the continuity of all functions involved, there thus exists a neighborhood, N, of any such point such that if $(A) \in N \cap B$, all the A_i are falling, so that dD_1/dt is negative, and points where $D_1 = U$ cannot be reached. Hence we need only consider points at which condition (b.) of the proposition is satisfied.

Finally, if $A_i = 0$, dA_i/dt has the sign of

$$(U/D_i)^{1-a} - 1,$$

which is always positive, since we have just shown that $D_i < U$. Reasoning as above, the system thus cannot reach points at which any of the A_i are zero, so that condition (a.) of the proposition must always be satisfied, and the proof is complete.

Condition (a.) of Proposition 7 amounts to the Assumption that all firms in the industry are actually present at time t_0. From the preceeding paragraph, condition (b.) implies that some promotion is in fact profitable for each firm, so that at t_0 all will in fact want to remain in the industry. Proposition 7 shows that if these conditions are once satisfied, they will always be satisfied thereafter.

It has proven even more difficult to establish the global stability of equilibrium. It normally seems to suffice in studies of this sort to know the signs of the time derivatives of the variables.[12] Here this does not seem to be the case, as even when e = 1 the derivatives often have the wrong signs away from equilibrium. When $a = 0$, however, an explicit expression for $A_i^*(D_i)$ can be obtained :

$$A_i^*(D_i) = (UD_i)^{1/2} - D_i, \quad i = 1,\ldots,N, \quad (5.4)$$

so that the time derivatives are simply

$$dA_i/dt = k_i[(UD_i)^{1/2} - (A_i + D_i)], \; i = 1,\ldots,N. \quad (5.5)$$

Using this expression, one can prove the following global stability result :

Proposition 8 : Under Assumptions 1 - 3 and 5, when a = 0 and e = 1, the unique equilibrium point is approached by all trajectories starting at points satisfying the conditions of Proposition 7.

For N = 2, a simple graphical proof of this Proposition is possible ; see Figure 1. By the boundedness of the system, trajectories beginning in region I must go to equilibrium or to region II or to region III. But trajectories beginning in on entering into either of these regions must go to equilibrium or to region IV. Finally, trajectories beginning in or entering into region IV must go to equilibrium. Hence global stability is assured.

For $N \geqslant 3$, a rather long proof based on Lypunov's second method must be used. Within the open set of admissable points, the system of differential equations (5.5) clearly satisfies all the necessary continuity conditions. Consider the function

$$V = (1/2U^2) \sum_{i=1}^{N} (A_i - A^E)^2 / k_i ,$$

where A^E is given by (3.5). This function is differentiable with respect to time. It is non-negative, and positive except at equilibrium. Global stability is proven by showing that this function has a negative time derivative away from equilibrium, where it is constant.

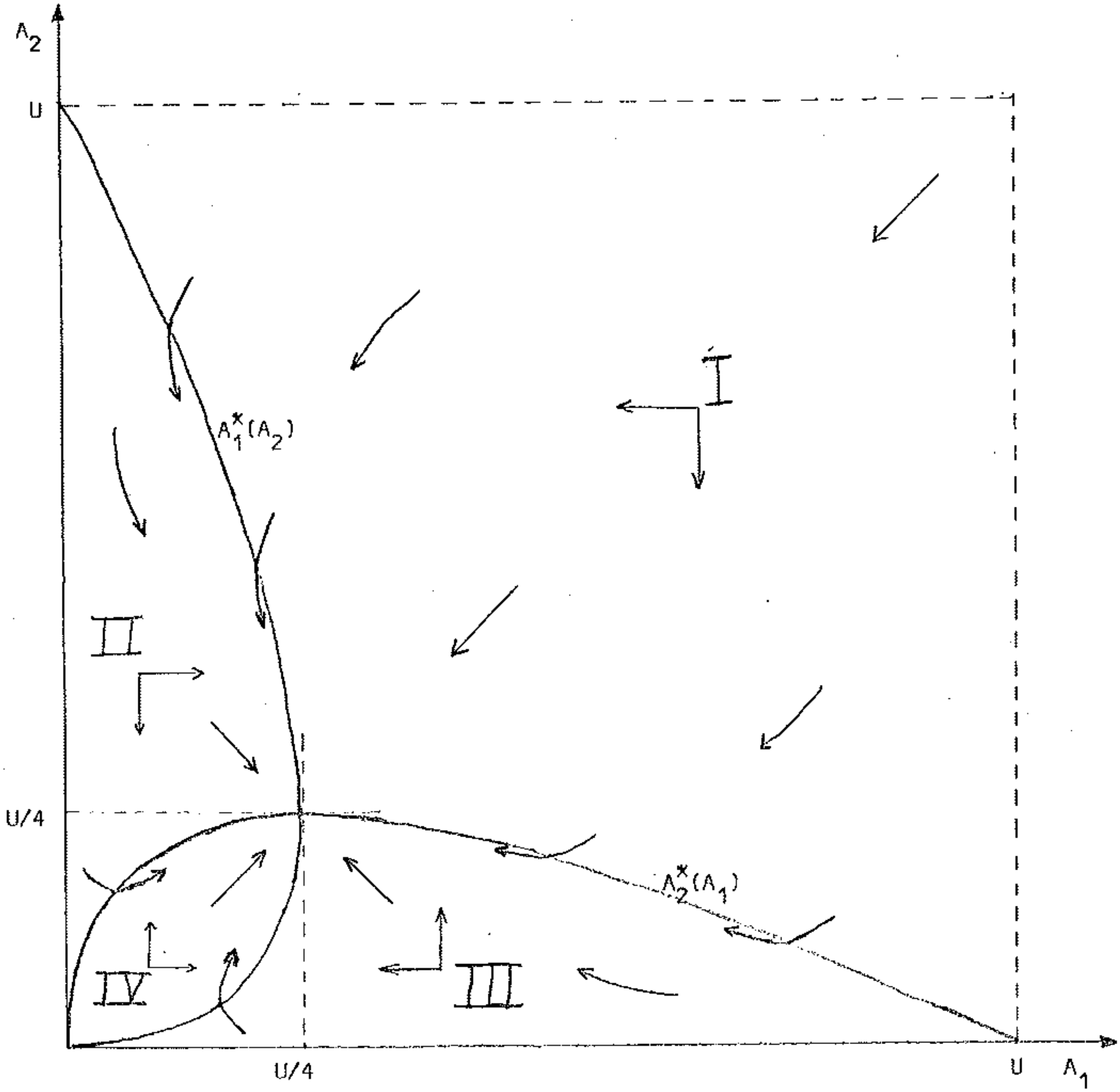

Figure 1

Phase Diagram: a=0, e=1, N=2

For convenience, let $x_i = A_i/U$, let x be the sum of the x_i, and let $E = A^E/U = (N-1)/N^2$. Then, from (5.5),

$$dx_i/dt = k_i[(x - x_i)^{1/2} - x]U,$$

and

$$dV/dt = (1/U^2)\sum_{i=1}^{N}(A_i - A^E)(dA_i/dt)/k_i = \sum_{i=1}^{N}(x_i - E)(x - x_i)^{1/2} - x(x - NE).$$

Note that this derivative is zero at equilibrium. Suppose now that the x_i are all equal. Then substitution of x/N for x_i yields

$$dV/dt = - x^{1/2}[x - NE][(x)^{1/2} - (NE)^{1/2}] < 0$$

away from equilibrium. So we need to show that this derivative is negative when the x_i are not all equal.

Under this assumption,

$$\sum_{i=1}^{N}(x_i - E)(x - x_i)^{1/2} / \sum_{i=1}^{N}(x - x_i)^{1/2} < (1/N)\sum_{i=1}^{N}(x_i - E) = (x - NE)/N,$$

since larger x_i receive smaller weight in the weighted average on the left than in the unweighted average on the right. Substituting,

$$dV/dt < (x - NE)[(1/N)\sum_{i=1}^{N}(x - x_i)^{1/2} - x].$$

Given x, it is easy to show that the average in brackets has a constrained maximum for non-negative x_i when all the x_i equal x/N, at which point it equals $x^{1/2}(NE)^{1/2}$. Similarly, this average attains its minimum value when one x_i is equal to x and the others are zero, at which point it equals $x^{1/2}(NE)$. (Note that NE < 1.) Hence,

$$x \geqslant NE \Longrightarrow dV/dt < -x^{1/2}[x - NE][(x)^{1/2} - (NE)^{1/2}] \leqslant 0,$$

and

$$x \leqslant NE \Longrightarrow dV/dt < -x^{1/2}[x - NE][(x)^{1/2} - (NE)],$$

which is non-positive for $x \leqslant (NE)^2$. This paragraph has thus established that

$$x \geqslant NE, \text{ or } x \leqslant (NE)^2 \Longrightarrow dV/dt < 0.$$

Now assume that $x \geqslant E$. Then, still assuming that the x_i are not all equal,

$$\sum_{i=1}^{N}(x_i - E)(x - x_i)^{1/2} < \sum_{i=1}^{N}(x_i - E)(x - E)^{1/2} = (x - NE)(x - E)^{1/2},$$

since if $x_i > E$, the first component of the i^{th} term in both summations is positive, but the second component is larger on the right, while if $x_i < E$, the i^{th} term is smaller in absolute value on the right than on the left. Substituting, for $x \geqslant E$,

$$dV/dt < (x - NE)[(x - E)^{1/2} - x].$$

Using the definition of E, it can be shown that the term in brackets is non-negative for $NE/(N-1) \leq x \leq NE$ and non-positive for $x \geq NE$. From this it follows that

$$x \geq NE/(N-1) \Longrightarrow dV/dt < 0.$$

Since for $N \geq 3$,

$$NE/(N-1) = 1/N < [(N-1)/N]^2 = (NE)^2,$$

the results of the last two paragraphs combine to show that dV/dt is negative away from equilibrium for all positive x when the x_i are not all equal. Since we showed above that this quantity is also negative when the x_i are all equal, the proof is complete.

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FOOTNOTES

[1]For attempts to relax this assumption in the stability analysis of price-quantity competition, see Sato and Naganani [12], Hosomatsu [4], and Okuguchi [8].

[2]For more detailed discussions, see Simon [15, pp. 622-3] and Scherer [13, pp. 335-6].

[3]See Simon [15, pp. 621-2] and Scherer [13, pp. 336-7]. On the other hand, Stern and Morganroth [16] have suggested that some coordination does occur.

[4]On the distinction between advertising spending and advertising messages or real advertising, see Schmalensee [14, pp. 20-5].

[5]See Schmalensee [14, pp. 34-5] for a discussion of related work.

[6]Following Frank and Quandt [2], one would be led to consider the mapping (A) B, where $(A') \in B$ implies that A_i' maximizes $\pi_i(A_i', \bar{A}_i)$ for all i. In order to employ the Kakutani theorem, we would need to show the existence of a compact convex set D such that $(A) \in D$ implies $B \subset D$. I have been unable to devise plausible assumptions that guarantee the existence of such a D, even when profit functions are identical. The key difficulty here is that a firm's optimal promotion is not generally a monotone function of its competitors' spending. The second basic problem is that the upper semicontinuity of this mapping, which is

also required for the Kakutani theorem, requires that the π_i functions be continuous in D. Since, as section II notes, under reasonable assumptions the demand functions are not continuous at the origin, establishing continuity in D involves either ruling out this discontinuity by (strong) assumption or producing plausible conditions under which the origin is not in D. This second problem also persists when profit functions are assumed identical.

[7]Okuguchi and Suzumura [10] have studied the uniqueness of equilibrium for a quantity-setting oligopoly using the Gale-Nikaido theorem on differentiable mappings. For the reasons given in footnote 6, the natural mapping is not well-behaved in this case. In particular, it is not generally everywhere differentiable. Further, even under Assumptions 1 - 3, it does not appear possible to sign the elements of the Jacobian away from equilibrium.

[8]See, for instance, Ruffin [11] and Okuguchi [9]. These authors stress the relation between these properties and various stability conditions ; this relationship is examined under Assumptions 1 - 3 in the next section.

[9]Schmalensee [14, pp.38-9] examines the dependence of these quantities on N when e = 1 and the price elasticity of demand is $(-\eta)$, a constant, under the ad hoc assumption that (m/P) equals $(1/N\eta)$, as it does in Cournot output -- setting models. He finds that the promotion -- sales ratio is a declining function of N, while total promotion may be greater for three firms than for two but always declines with N thereafter.

[10]See, for instance, Fisher [1].

[11]In fact, for N > 2 falls in e lower the left-hand side of (4.7), thus tending to stabilize the system.

[12]See, for instance, Hahn [3] and Okuguchi [7].