ON THE IMPACT OF INSURANCE ON HOUSEHOLDS SUSCEPTIBLE TO RANDOM PROPORTIONAL LOSSES: AN ANALYSIS OF POVERTY TRAPPING

Kira Henshaw, Jorge Ramirez, José Miguel Flores-Contró, Sooie-Hoe Loke, Enrique A. Thomann, Corina Constantinescu

LIDAM Discussion Paper ISBA
2026 / 22

UCLouvain

ISBA | LIDAM
Louvain Institute of Data Analysis and Modeling in economics and statistics

ISBA

Voie du Roman Pays 20 - L1.04.01

B-1348 Louvain-la-Neuve

Email : lidam-library@uclouvain.be

https://uclouvain.be/en/research-institutes/lidam/isba/publication

On the Impact of Insurance on Households Susceptible to Random Proportional Losses: An Analysis of Poverty Trapping

Kira Henshaw[1] Jorge Ramirez[2] José Miguel Flores-Contró[3]
Sooie-Hoe Loke[4] Enrique A. Thomann[5] Corina Constantinescu[1]

[1]Institute for Financial and Actuarial Mathematics, University of Liverpool, Liverpool, United Kingdom
[2]Departamento de Matemáticas, Universidad Nacional de Colombia, Sede Medellín, Medellín, Colombia
[3]Institute of Statistics, Biostatistics and Actuarial Science (ISBA), Louvain Institute of Data Analysis and Modeling (LIDAM), Catholic University of Louvain, Louvain-la-Neuve, Belgium
[4]Department of Mathematical Sciences, Middle Tennessee State University, Murfreesboro, Tennessee, United States of America
[5]Department of Mathematics, Oregon State University, Corvallis, Oregon, United States of America

Abstract

The trapping probability is studied for households' capital assuming that losses are proportional to the accumulated capital. We consider households without insurance and those purchasing proportional insurance. Parameter conditions under which poverty trapping is not certain in both cases are derived. For the uninsured model, a new closed-form expression for the trapping probability is obtained when the remaining proportion of capital after a loss has a beta distribution, showing that the trapping probability decays algebraically as the initial capital increases. When insurance is purchased and the remaining proportion is uniformly distributed, the trapping probability is computed by solving a non-local integro-differential equation. Our results show that insurance can prevent otherwise certain trapping in some cases, although it may increase trapping risk for households with initial capital barely above the poverty line, while reducing trapping risk for households with higher initial capital. JEL classification: G220; G520; O120.

Keywords— Microinsurance; poverty traps; trapping probability; risk processes; proportional losses; proportional microinsurance.

1 Introduction

1.1 Background

Vulnerable non-poor households, i.e., those living just above the poverty line, are highly susceptible to falling into poverty, particularly following a financial loss. Understanding this vulnerability and the nature of their loss experience is essential for designing effective poverty-reduction policies and advancing Sustainable Development Goal 1 (SDG 1): No Poverty (United Nations, 2015). An important indicator of financial stability is capital. In the vulnerable non-poor household setting, where monetary wealth is often limited, capital should be understood broadly to include the resources that enable production, whether for trade or self-sustaining purposes. This may

include land, property, physical and human capital, with health a form of capital in extreme cases where sufficient health services and food accessibility are not guaranteed (Dasgupta, 1997). The threat of catastrophic loss events is of great concern, particularly under this broad definition of capital. For example, vulnerable non-poor households are predominantly engaged in agricultural work and are exposed, among many other things, to natural disasters in the form of floods and droughts. In contrast to losses relating to health, life or death, agricultural losses can immediately eliminate a high proportion of a household's ability to produce through loss of land and livestock, irrespective of their level of capital.

In this article, we study the behaviour of households' capital under the assumption of proportional capital loss experience. Proportionality in loss experience captures the exposure of households of all capital levels to both catastrophic and low severity loss events. This is particularly relevant in the vulnerable non-poor household setting, where, in addition to infrequent but serious events such as natural disasters, more common events such as hospital admissions and household deaths, can be detrimental. To do this, we adopt the ruin-theoretic approach proposed in Kovacevic and Pflug (2011), by using a risk process with deterministic growth and multiplicative losses to model the capital of a household. At capital loss events, accumulated capital is reduced by a random proportion of itself, rather than by an amount of random value, as in Flores-Contró et al. (2025). Processes of this structure are typically referred to as growth-fragmentation or growth-collapse processes, characterised by their growth in between the random collapse times at which downwards jumps occur. In these models, the randomly occurring jumps have a random size which is dependent on the state of the process immediately before the jump.

Our aim is to derive the probability that a household ever falls below the poverty line, where this probability mimics an insurer's infinite-time ruin probability. To the best of our knowledge, Kovacevic and Pflug (2011) and Flores-Contró et al. (2025) are the only previous studies to examine this problem from a ruin-theoretic perspective. As in these earlier works, in this article, we consider the probability under two frameworks, one in which the household has no insurance coverage, and the other in which they are proportionally insured. We introduce insurance to assess its effectiveness as a measure of poverty reduction. Aligning with the low-income environment, proportional coverage is assumed to be provided through an *inclusive insurance* product, specifically designed to cater for those excluded from traditional insurance services or without access to alternative effective risk management strategies. This type of product, targeted towards low-income populations, is commonly referred to as *microinsurance* (Churchill, 2006; Churchill and Matul, 2012). In Flores-Contró et al. (2025), the risk process with deterministic growth and random-valued losses is instead used to assess the impact of government premium subsidy schemes on the probability of falling below the poverty line.

1.2 Literature Review

Although important, we do not consider the behaviour of a household's capital below the poverty line. Households that live or fall below the poverty line are said to be in a poverty trap, where a poverty trap is a state of poverty from which it is difficult to escape without external help. Poverty trapping is a well-studied topic in development economics (the interested reader may refer to Azariadis and Stachurski (2005), Bowles et al. (2006), Kraay and McKenzie (2014), Barrett et al. (2016) and references therein for further discussion; see Matsuyama (2008) for a detailed description of the mechanics of poverty traps), however, for the purpose of this study, we use the

term "trapping" only to describe the event that a household falls into poverty, focusing our interest on vulnerable non-poor households with initial capital levels slightly above the poverty line.

In Kovacevic and Pflug (2011), estimates of the infinite-time trapping probability (the probability that a household eventually falls into the poverty trap) are obtained numerically. The authors also argue, by drawing an analogy with classical ruin theory, for the existence of a Lundberg coefficient and the resulting exponential decay of the trapping probability as the initial capital increases. Azaïs and Genadot (2015) perform further numerical analysis of the same model, discussing applications to the capital setting of Kovacevic and Pflug (2011) and to population dynamics, where the poverty line denotes extinction. In neither study is an analytical expression for the trapping probability obtained.

Our analysis leads to a different conclusion regarding the asymptotic behaviour of the model in Kovacevic and Pflug (2011). When the remaining proportion of capital after each loss event follows a $\mathrm{Beta}(\alpha, 1)$ distribution, we derive a closed-form expression for the infinite-time trapping probability and show that it decays algebraically, rather than exponentially, as the initial capital tends to infinity. Thus, the Lundberg-type exponential decay proposed in Kovacevic and Pflug (2011), which relies on the existence of a positive solution to the corresponding Lundberg equation, does not hold for this model under the $\mathrm{Beta}(\alpha, 1)$ specification. This distinction is important for the proportional-loss model: unlike under additive losses, increasing initial capital does not reduce the trapping probability at an exponential rate.

A related issue concerns the conditions under which the infinite-time trapping probability is non-trivial. As in classical ruin theory, a meaningful analysis requires identifying parameter regimes in which trapping is not certain (see Remark 1.1 in Mandjes and Boxma (2023)). Using an embedded process at successive loss times, we derive a corresponding *net profit condition* for both the uninsured and insured capital models. The condition separates the parameter regimes in which trapping occurs with probability one from those in which the trapping probability vanishes as initial capital increases.

The insured model presents a different analytical difficulty. When proportional insurance coverage is purchased by the household, the trapping probability satisfies a non-local integro-differential equation. For uniformly distributed remaining proportions of capital, we transform this equation into a delay differential equation and develop a recursive layer construction based on Sturm–Liouville methods. The constructive nature of the method allows for an analytic representation of the trapping probability that can be evaluated numerically.

The proportional nature of the capital losses distinguishes the resulting process from classical insurance surplus models with additive claims (see, e.g., Asmussen and Albrecher (2010) and Mandjes and Boxma (2023)). Although the proportional loss factors are independent, the corresponding absolute losses depend on the capital level immediately before each loss event and therefore inherit the state dependence of the process. In addition, only the portion of household capital above the poverty line grows exponentially. These features lead to infinitesimal generators for the risk process that differ substantially from those arising in classical ruin models.

A separate literature that examines insurance and poverty trapping from a non-ruin perspective has been undertaken through application of multi-equilibrium models and dynamic stochastic programming (Chantarat et al., 2017; Ikegami et al., 2018; Carter and Janzen, 2018; Liao et al., 2020; Janzen et al., 2021; Kovacevic and Semmler, 2021). With the exception of the latter, each of these studies

considers the impact of subsidisation and the associated cost to the subsidy provider. Will et al. (2021) and Henshaw et al. (2023) extend the problem to the group-setting, assessing the impact of risk-sharing on the trapping probability. Will et al. (2021) undertake a simulation-based study and Henshaw et al. (2023) propose a Markov modulated stochastic dissemination model of group wealth interactions, using a bivariate normal approximation to calculate the trapping probability.

Notably, Kovacevic and Pflug (2011), Liao et al. (2020) and Flores-Contró et al. (2025) suggest that purchase of insurance and the associated need for premium payment increases the risk of falling below the poverty line for the most vulnerable. Barriers to microinsurance penetration that exist due to constraints on product affordability resulting from fundamental features of the microinsurance environment likely contribute to such observations; see Eling et al. (2014) for a detailed overview on the determinants of microinsurance demand. Limited consumer financial literacy and experience, product accessibility and data availability, are examples of the unique characteristics that must be accounted for when designing effective and affordable microinsurance products. Through our analysis, we further investigate the case of proportional loss experience to assess the associated implications on the affordability of insurance.

Janzen et al. (2021) optimise the level of insurance coverage across the population, observing that those in the neighbourhood of the poverty line do not optimally purchase insurance (without subsidies), instead suppressing their consumption and mitigating the probability of falling into poverty. This aligns with the increase in the trapping probability observed in the aforementioned studies, when those closest to the poverty line purchase insurance. Similarly, Kovacevic and Semmler (2021) derive the retention rate process that maximises the expected discounted capital, by allowing adjustments in the retention rate of the policyholder after each capital loss throughout the lifetime of the insurance contract. In this article, however, the proportion of insurance coverage and the choice to insure is fixed across the population, as in Kovacevic and Pflug (2011), Chantarat et al. (2017) and Flores-Contró et al. (2025). Our comparison is therefore conditional on a given insurance contract; we do not model optimal insurance demand or household welfare.

1.3 Contribution

Our contribution is fourfold. First, we derive a random recurrence relation for an auxiliary capital process associated with the original formulation of Kovacevic and Pflug (2011). This representation provides an alternative characterisation of the trapping event and leads naturally to a net profit condition (NPC) analogous to that of classical ruin theory. The condition separates parameter regimes in which trapping occurs with probability one from those in which the trapping probability converges to zero as initial capital increases, without imposing a distributional form for the remaining proportion of capital.

Second, we specialize to the uninsured model with $\text{Beta}(\alpha, 1)$-distributed remaining proportions of capital and derive a closed-form expression for the trapping probability. Unlike the preceding probabilistic characterization and NPC, this distributional assumption is introduced to obtain an explicit analytical solution, which then allows us to determine the asymptotic behaviour of the trapping probability.

Third, whereas Kovacevic and Pflug (2011) argue for exponential decay through the existence of a Lundberg coefficient, we show that the trapping probability instead decays algebraically, provided the NPC is satisfied. Thus, the asymptotic behaviour of the proportional-loss model differs

fundamentally from that of classical additive-loss risk processes.

Finally, for the proportionally insured model with uniformly distributed remaining proportions of capital, we transform the non-local integro-differential equation into a delay differential equation and develop a layer method that constructs its solution recursively over successive intervals. The resulting solution also allows us to compare the insured and uninsured households across initial capital levels. We find that proportional insurance can increase trapping risk close to the poverty threshold, but reduce it at higher capital levels and, in some parameter regimes, change the process from certain eventual trapping to a non-trivial trapping probability.

The remainder of this paper is structured as follows. Section 2 introduces the capital growth model under both the uninsured and insured settings. Section 3 develops an alternative characterisation of the trapping event and derives the corresponding net profit conditions. Section 4 derives the trapping probabilities, including the closed-form solution for the uninsured model and the recursive layer method for the insured model. Section 5 compares additive and multiplicative losses and examines the impact of proportional insurance. Concluding remarks are provided in Section 6. Most proofs of the results in this paper are provided in Appendix B.

2 The Capital Model

This section presents the stochastic capital models considered throughout the paper. For completeness, we describe both the uninsured and insured formulations that form the basis of our subsequent analysis. The notation and assumptions established here will be used throughout the remainder of the article.

2.1 Uninsured Model

We begin by recalling the uninsured capital model introduced by Kovacevic and Pflug (2011).

Consider a household with accumulated capital $\{X_{0,t}\}_{t\geq 0}$, with the subindex "0" connoting here the absence of insurance. Under the basic assumption that the household has no loss experience, its growth in accumulated capital is given by

$$\frac{dX_{0,t}}{dt} = r_0 \cdot [X_{0,t} - x_0^*]^+ , \tag{2.1}$$

where $r_0 > 0$ is the capital growth rate and $x_0^* > 0$ is the threshold below which a household lives in poverty. Here, we denote $[x]^+ = \max(x, 0)$.

The dynamics in (2.1) are built on the assumption that a household's income I_t is split into consumption C_t and savings S_t (or investments), such that at time t,

$$I_t = C_t + S_t, \tag{2.2}$$

where consumption is an increasing function of income:

$$C_t = \begin{cases} I_t & \textit{if} \quad I_t \leq I^*, \\ I^* + a\,(I_t - I^*) & \textit{if} \quad I_t > I^*, \end{cases} \tag{2.3}$$

for some household rate of consumption $0 < a < 1$. The critical point below which a household consumes all of its income, with no facility for savings or investment, is denoted by I^*. The accumulated capital is assumed to grow proportionally to the level of savings, such that

$$\frac{dX_{0,t}}{dt} = cS_t, \tag{2.4}$$

where $0 < c < 1$ is the household's rate of savings. Income is generated through the accumulated capital, such that $I_t = bX_{0,t}$, with $b > 0$ denoting the household's income generation rate. Combining (2.2), (2.3) and (2.4) gives exactly the dynamics in (2.1) with

$$r_0 = (1 - a) \cdot b \cdot c \quad \text{and} \quad x_0^* = \frac{I^*}{b}.$$

The notion of a household in this model setting may be extended for consideration of poverty trapping within economic units such as community groups, villages and tribes, in addition to the traditional household structure. Reflecting the ability of a household to produce, the level of accumulated capital of a household up to time t, $X_{0,t}$, is composed of land, property, physical and human capital. The poverty threshold x_0^* represents the amount of capital required to forever attain a critical level of income below which a household would not be able to sustain their basic needs, facing elementary problems relating to health and food security. We refer to this threshold as the *critical capital* or the *poverty line*. Since (2.1) is positive for all levels of capital greater than the critical capital, all points less than or equal to x_0^* are stationary, the level of capital remains constant if the critical capital is not met. In this basic model, stationary points below the critical capital are not attractors of the system if the initial capital exceeds x_0^*, in which case the capital process grows exponentially with rate r_0.

In line with Kovacevic and Pflug (2011), we expand the dynamics of (2.1) under the assumption that households are susceptible to the occurrence of capital losses such as those highlighted in Section 1, including severe illness, the death of a household member or breadwinner and catastrophic events such as droughts, floods and earthquakes. The occurrence of loss events is assumed to follow a Poisson process with intensity λ, where the capital process follows the dynamics of (2.1) in between events.

On the occurrence of the ith loss, the capital process experiences an instantaneous downwards jump from X_{0,T_i} to $X_{0,T_i} \cdot Z_i$, where $Z_i \in [0, 1]$ is the random proportion determining the remaining capital after loss i and X_{0,T_i} is the level of capital accumulated up to the time of loss T_i. The sequence $\{Z_i\}_{i=1}^{\infty}$ is a sequence of independent and identically distributed random variables with common distribution function G_Z, independent of the Poisson process. In contrast to Flores-Contró et al. (2025) where random-valued losses are considered, the dynamics of the model do not allow for the possibility of negative capital due to the proportionality of loss experience.

The structure of the process in-between loss events is derived through solution of the first order ordinary differential equation (ODE) in (2.1). The stochastic capital process with deterministic exponential growth and multiplicative losses is then formally defined as follows:

Definition 2.1. *Let T_i be the ith event time of a Poisson process $\{N_t\}_{t\geq 0}$ with parameter λ, where $T_0 = 0$. Let $0 \leq Z_i \leq 1$ be a sequence of i.i.d. random variables with distribution function G_Z, independent of the process N_t. For $T_{i-1} \leq t < T_i$, the stochastic growth process of the accumulated capital without insurance coverage $X_{0,t}$ is defined as*

$$X_{0,t} = \begin{cases} \left(X_{0,T_{i-1}} - x_0^*\right) e^{r_0(t-T_{i-1})} + x_0^* & \text{if} \quad X_{0,T_{i-1}} > x_0^*, \\ X_{0,T_{i-1}} & \text{otherwise.} \end{cases} \tag{2.5}$$

At the jump times $t = T_i$, the capital process without insurance coverage is given by

$$X_{0,T_i} = \begin{cases} \left[\left(X_{0,T_{i-1}} - x_0^*\right) e^{r_0(T_i-T_{i-1})} + x_0^*\right] \cdot Z_i & \text{if} \quad X_{0,T_{i-1}} > x_0^*, \\ X_{0,T_{i-1}} \cdot Z_i & \text{otherwise.} \end{cases}$$

Definition 2.1 indicates that the capital level of the household follows a piecewise deterministic Markov process (Davis, 1984, 1993) of compound Poisson-type. Namely, the capital evolves deterministically in-between the randomly occurring jump times at which capital losses occur. Consequently, the infinitesimal generator of the capital process without insurance coverage is given by

$$\mathcal{A}_0[h](x) = r_0(x - x_0^*)h'(x) + \lambda \int_0^1 [h(x \cdot z) - h(x)]\mathrm{d}G_Z(z), \quad x \geq x_0^*, \tag{2.6}$$

where h is any function in the domain of the generator $\mathcal{A}_0$.

2.2 Insured Model

In line with Kovacevic and Pflug (2011) and Flores-Contró et al. (2025), we now extend the model by assuming that households purchase insurance coverage.

Let $\mathcal{R} : [0,1] \to [0,1]$ denote the retained loss function, which satisfies $0 \leq \mathcal{R}(l) \leq 1$ and $\mathcal{R}(0) = 0$. This function specifies the portion of a loss borne by the household when a loss of size $l = 1-z \in [0,1]$ occurs per unit of capital. Accordingly, after a loss event, the household's capital changes from X_{T_i} to $(1 - \mathcal{R}(l)) \cdot X_{T_i}$ (note that we now omit the subscript "0" when referring to the accumulated capital with insurance). The insurer determines the premium rate p that the household must pay for the insurance coverage using the *expected value principle*. That is,

$$p = (1 + \theta) \cdot \lambda \cdot \mathbb{E}[1 - Z_i - \mathcal{R}(1 - Z_i)],$$

where $\theta > 0$ is the safety loading per unit of capital. Under this insurance coverage, the capital growth rate r of an insured household must account for the premium payments, which are deducted from the income generation rate. Naturally, we assume that $b > p$ and this yields

$$r = (1 - a) \cdot (b - p) \cdot c = r_0 \left(\frac{b - p}{b}\right) \leq r_0. \tag{2.7}$$

Moreover, the critical capital (or poverty line) is now given by

$$x^* = \left(\frac{b}{b-p}\right) x_0^* \geq x_0^*,$$

since $x_0^* = I^*/b$ and $x^* = I^*/(b-p)$, where I^* denotes the critical income.

Clearly, the capital model of an insured household has an analogous structure to that of Definition 2.1, with the remaining proportion of capital after each loss event instead denoted Y_i, where

$$Y_i := 1 - \mathcal{R}(1 - Z_i).$$

As such, in between loss events, where $T_{i-1} \leq t < T_i$, the capital growth process follows (2.5) with r and x^* replacing r_0 and x_0^*, respectively. At event times $t = T_i$, the process is given by

$$X_{T_i} = \begin{cases} \left[\left(X_{T_{i-1}} - x^*\right) e^{r(T_i - T_{i-1})} + x^*\right] \cdot Y_i & \textit{if} \quad X_{T_{i-1}} > x^*, \\ X_{T_{i-1}} \cdot Y_i & \textit{otherwise.} \end{cases} \tag{2.8}$$

Motivated by the importance of designing simple, transparent, and easy-to-understand insurance products in the microinsurance context (Churchill, 2006; Churchill and Matul, 2012; Eling et al., 2014), and by the fact that proportional insurance is generally considered to possess these characteristics (Lampaert and Walhin, 2005), we assume throughout this paper that the household acquires a quota-share proportional insurance coverage. Let $\kappa \in [0, 1]$ denote the proportion of the loss l that is retained by the household so that the retained loss function takes the form

$$\mathcal{R}(l) = \kappa \cdot l, \qquad \text{so that,} \qquad Y_i = 1 - \kappa(1 - Z_i). \tag{2.9}$$

In this setting, the premium rate per unit of capital is given by

$$p = (1 + \theta) \cdot \lambda \cdot (1 - \kappa) \cdot (1 - \mathbb{E}[Z_i]).$$

Note that $\kappa = 0$ corresponds to the full insurance case, where the insurer bears all of the household's losses. On the other hand, $\kappa = 1$ represents the absence of insurance, in which case the capital model in (2.8), together with the parameters r and x^*, coincides exactly with the uninsured household model introduced earlier in Section 2.1.

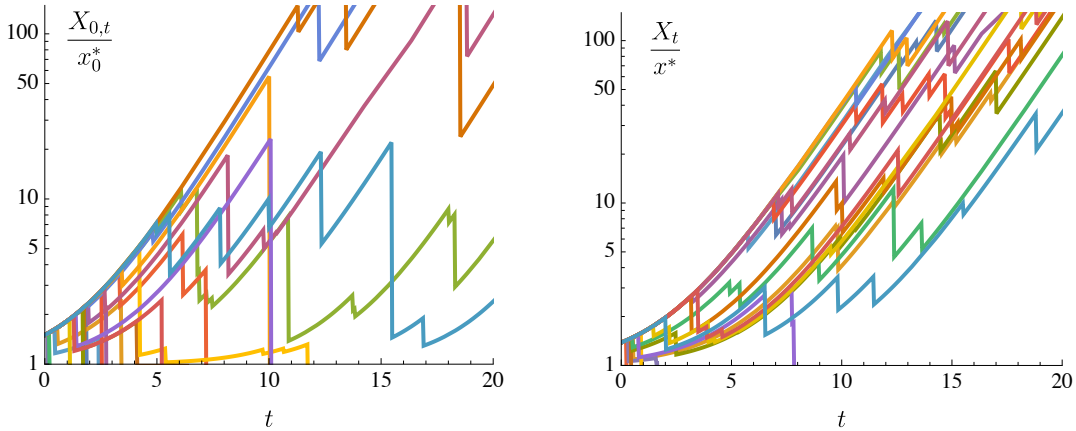
Thus, in the case of proportional insurance considered in this paper, the infinitesimal generator for the insured capital process as defined in this section is given by

$$\mathcal{A}[h](x) = r(x - x^*) h'(x) - \lambda h(x) + \lambda \int_{1-\kappa}^{1} h(x \cdot y) \mathrm{d}G_Y(y), \tag{2.10}$$

where

$$G_Y(y) = G_Z(1 - (1 - y)/\kappa)$$

is the cumulative distribution function of Y_i, whose support is $[1-\kappa, 1]$, and h is any function which lies in the domain of the generator $\mathcal{A}$.



(a) (b)

Figure 1: Examples of paths of the capital process X_t with initial capital $X_0 = x = 1.5$, $\lambda = 0.4$ and uniformly distributed Z_i. Note the logarithmic scale in the vertical axes. Trapping occurs for paths that cross the horizontal axes. Both plots have parameters $a = 0.1$, $b = 1.4$ and $c = 0.4$. Plot (a) shows the uninsured case with trapping probability $\psi(x) = 0.88$ as per Proposition 4.1. Plot (b) shows the insured case with $\theta = 0.1$ and $\kappa = 0.25$. The trapping probability can be estimated to be $\psi(x) \approx 0.35$ with the techniques of Section 4.2.

Remark 2.1. *Up to this point, we have denoted the uninsured capital process and its associated parameters by $\{X_{0,t}\}_{t\geq 0}$, r_0, and x_0^* in order to distinguish them from their counterparts $\{X_t\}_{t\geq 0}$, r, and x^*, arising in the insured capital model. To streamline the notation and improve expositional clarity, we henceforth suppress the subscript "0" distinction and use a common notation for both models, except when an explicit comparison is required.*

Figure 1 illustrates the capital dynamics both without and with insurance coverage. The parameters r and x^* are different for each setting, reflecting the presence or absence of insurance. In the insured case, the exponential growth rate r is lower and losses are less severe; conversely, the uninsured case exhibits faster capital accumulation but more damaging shocks. Moreover, the poverty trap region is wider under insurance, as reflected by a larger critical capital x^*.

3 Probabilistic Characterisation of the Capital Process

With the stochastic capital models now fully specified, we turn to the main objective of this paper, which is to study the probability that a household eventually falls below the poverty line under both uninsured and insured capital models. As in Kovacevic and Pflug (2011) and Flores-Contró et al. (2025), we refer to the first time at which the capital falls below the poverty line as the *trapping time*, defined by

$$\tau := \inf\{t \geq 0 : X_t < x^*\},$$

and define the corresponding (infinite-time) *trapping probability* as a function of the initial capital by

$$\psi(x) := \mathbb{P}\left(\tau < \infty \mid X_0 = x\right).$$

The infinite-time criterion is intended as a measure of long-run vulnerability rather than a short-term poverty forecast. Before studying the trapping probability analytically, it is useful to first understand the probabilistic structure of the capital process. In this section, we characterise the trapping time using an embedded discrete-time process and establish net profit conditions (analogous to that in classical ruin theory; see, e.g., Asmussen and Albrecher (2010) and Mandjes and Boxma (2023)) for both the uninsured and insured models. These results identify when the infinite-time trapping probability is non-trivial (equal to one) and motivate the analytical treatment developed in Section 4.

3.1 The Net Profit Condition in the Uninsured Model

An alternative characterisation of the trapping time is developed in this section. Recall that, as noted in Remark 2.1, x^*, r, and X_t denote the poverty line, capital growth rate, and capital model, respectively, in both the uninsured or insured case. Likewise, in either case, we let Y_i denote the remaining fraction of capital after the ith event.

First, observe that the trapping time (if finite) occurs at exactly one of the jump times T_i. Denote the capital process discretised at loss event times by

$$\tilde{X}_i := X_{T_i}.$$

Take $\tilde{X}_0 = x$ and assume that $x > x^* > 0$. Let $V_i := T_i - T_{i-1}$ be the ith inter-arrival time, and rewrite (2.8) as

$$\tilde{X}_i = \begin{cases} \left[\left(\tilde{X}_{i-1} - x^*\right) e^{rV_i} + x^*\right] \cdot Y_i & \textit{if} \quad \tilde{X}_{i-1} > x^*, \\ \tilde{X}_{i-1} \cdot Y_i & \textit{otherwise}. \end{cases}$$

Note that $V_i \sim \mathrm{Exp}(\lambda)$. The discretised process is sufficient to characterise the trapping time. However, the following auxiliary process admits a particularly convenient recursive representation. On the same probability space as $\{\tilde{X}_i\}_{i\in\mathbb{N}}$, define the auxiliary process $\{\widehat{X}_i\}_{i\in\mathbb{N}_0}$ by

$$\widehat{X}_i = \left[\left(\widehat{X}_{i-1} - x^*\right) e^{rV_i} + x^*\right] \cdot Y_i, \tag{3.1}$$

with $\widehat{X}_0 = x > x^* > 0$. Note that the auxiliary process $\{\widehat{X}_i\}_{i\in\mathbb{N}_0}$ generalises the Markov chain introduced in Equation (18) of Kovacevic and Pflug (2011), which is subsequently used there to argue for the existence of a Lundberg coefficient. We take a different approach. By exploiting the recursive structure of $\widehat{X}_i$ and applying the strong law of large numbers, we obtain directly the net profit condition that separates the regime of certain trapping from that in which the trapping probability vanishes as the initial capital increases.

Remark 3.1. *Equation (3.1) admits a simple financial interpretation: the household's capital at time T_i consists of two components: first, a fraction Y_i of the excess of capital over the critical capital level held at time T_{i-1}, accumulated over the period $[T_{i-1}, T_i]$ at a constant force of interest r; and a second, a deposit at time T_i equal to a fraction Y_i of the critical capital, i.e., $Y_i x^*$.*

Figure 2 shows examples of sample paths for both of these processes. For every realisation, the two processes $\{\tilde{X}_i\}_{i\in\mathbb{N}_0}$ and $\{\widehat{X}_i\}_{i\in\mathbb{N}_0}$ are identical as long as they do not cross x^*. After crossing, the entire path of $\widehat{X}_i$ lies below the path of $\tilde{X}_i$ for all future time steps. Therefore, the two processes have the same trapping time and so we have established the following equivalencies for the trapping time τ in terms of the continuous capital process, discretised process, and auxiliary process:

$$\tau = \inf\{t \geq 0 : X_t < x^*\} = \inf\left\{T_i : \tilde{X}_i < x^*\right\} = \inf\left\{T_i : \widehat{X}_i < x^*\right\}.$$

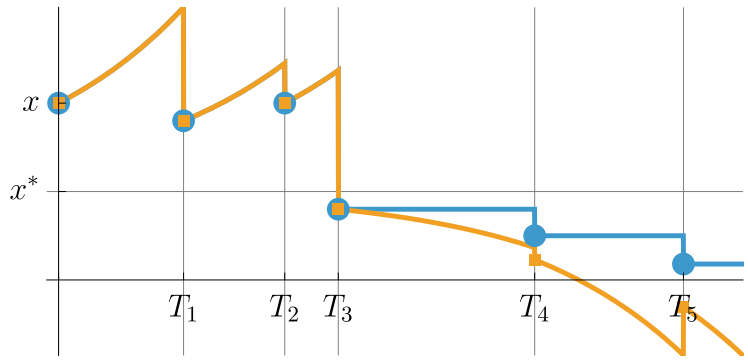


Figure 2: Sample paths of the discretised process $\{\tilde{X}_t\}_{t\geq 0}$ labeled by blue dots and the auxiliary process $\{\widehat{X}_t\}_{t\geq 0}$ labeled by orange squares.

The trapping time can be characterised as follows:

Proposition 3.1. *Let $\{\widehat{X}_i\}_{i\in\mathbb{N}_0}$ be defined by (3.1). For $k \geq 1$, let $\Pi_k = \prod_{j=1}^{k} Y_j$ and set*

$$M = \sum_{k=1}^{\infty} \frac{1 - Y_k}{\Pi_k} e^{-rT_k}.$$

Then,

$$\mathbb{P}(\tau < \infty \mid \widehat{X}_0 = x) = \mathbb{P}\left(\frac{x - x^*}{x^*} < M\right).$$

Proof. Let $R_i := \widehat{X}_i - x^*$, be the excess above the critical level at time T_i. If $R_{i-1} > 0$, we can rewrite (3.1) as

$$R_i = Y_i e^{rV_i} R_{i-1} - (1 - Y_i)x^*. \tag{3.2}$$

Iterating the recursion (3.2) yields

$$R_i = e^{rT_i}\Pi_i \left(R_0 - x^* \sum_{k=1}^{i} \frac{1 - Y_k}{\Pi_k} e^{-rT_k}\right).$$

Since $\Pi_i > 0$, the trapping time may be equivalently written as

$$\tau = \inf\{T_i : R_i < 0\} = \inf\left\{T_i : R_0 < x^* \sum_{k=1}^{i} \frac{1 - Y_k}{\Pi_k} e^{-rT_k}\right\}.$$

It follows that

$$\mathbb{P}(\tau < \infty \mid \widehat{X}_0 = x) = \mathbb{P}\left(\frac{R_0}{x^*} < \sup_{i \geq 1} \sum_{k=1}^{i} \frac{1 - Y_k}{\Pi_k} e^{-rT_k}\right).$$

Since the summands are positive,

$$\sup_{i \geq 1} \sum_{k=1}^{i} \frac{1 - Y_k}{\Pi_k} e^{-rT_k} = \sum_{k=1}^{\infty} \frac{1 - Y_k}{\Pi_k} e^{-rT_k} = M,$$

completing the proof. □

Remark 3.2. *The quantity M introduced in Proposition 3.1 has a natural interpretation: it represents the present value, when the force of interest is r, of a unit capital being available for investments at time $T_j, j = 1, 2, \ldots,$ when the investment is subject to proportional losses at time $T_i, 1 \leq i \leq j$ leaving fraction Y_i of the capital available for investment. Proposition 3.1 shows that trapping is avoided, if the initial excess capital $x - x^*$ is large enough to endow a fund that makes x^* available at each time $T_j, j = 1, 2, \ldots$.*

Proposition 3.1 shows that the long-run behaviour of the capital process is governed by the multiplicative random factors $Y_j e^{rV_j}$. We therefore introduce the following notation. Let $A_j = Y_j e^{rV_j}, j = 1, 2, \ldots$ be the i.i.d. random variables representing the remaining accumulation of a unit of capital under a force of interest r over the inter-arrival time V_i. Then,

$$\frac{e^{-rT_k}}{\Pi_k} = \frac{e^{-rT_k}}{\prod_{j=1}^{k} Y_j} = \frac{1}{\prod_{j=1}^{k} A_j}.$$

We are now in a position to state the following proposition:

Proposition 3.2. *(i) If $\mathbb{E}[\ln(A_1)] > 0$, then $\psi(x) \to 0$ as $x \to \infty$.*

(ii) If $\mathbb{E}[\ln(A_1)] < 0$, then $\psi(x) = 1$ for all $x > x^*$.

Proof. By the strong law of large numbers, $\frac{1}{n}\sum_{j=1}^{n} \ln(A_j) \to \mathbb{E}[\ln(A_1)]$ almost surely. If $\mathbb{E}[\ln(A_1)] > 0$, then $\ln\left(\prod_{j=1}^{n} A_j\right) \to +\infty$, i.e. $\prod_{j=1}^{n} A_j$ grows exponentially fast a.s. As such, M is finite a.s. and it follows that

$$\mathbb{P}(\tau < \infty \mid X_0 = x) = \mathbb{P}\left(M \geq \frac{x - x^*}{x^*}\right) \to 0, \qquad \text{as } x \to \infty.$$

If instead $\mathbb{E}[\ln(A_1)] < 0$, then $\ln\left(\prod_{j=1}^{n} A_j\right) \to -\infty$ and so $\prod_{j=1}^{n} A_j \to 0$ a.s. The quantities $1 - Y_k$ are positive infinitely often a.s. which forces M to diverge to $+\infty$ a.s. Thus $\mathbb{P}(\tau < \infty \mid X_0 = x) = 1$ in this case. □

Proposition 3.2 identifies the balance between deterministic capital growth and stochastic proportional losses that governs the long-run behaviour of the capital process. When the average logarithmic accumulation is positive, sufficiently large initial capital makes trapping increasingly unlikely. Conversely, when the logarithmic accumulation is negative, every household eventually falls below the poverty threshold regardless of its initial capital.

In terms of the parameters of the capital process, we have $\mathbb{E}\left[\ln\left(A_1\right)\right] = \mathbb{E}\left[\ln\left(Y_1\right)\right] + \frac{r}{\lambda}$ so that the condition for obtaining the non-trivial trapping probability becomes

$$\lambda \mathbb{E}\left[\ln\left(Y_1\right)\right] + r > 0. \tag{3.3}$$

Remark 3.3. *There is a close structural connection between the discretised model described above and the classical Cramér-Lundberg model, originally introduced by Lundberg (1903) and Cramér (1930). In the latter framework, an insurer receives premium income continuously while incurring claims of random size at stochastic arrival times. A central object of study in this setting is the infinite-time ruin probability, defined as the probability that the insurer's surplus ever becomes negative, leading to insolvency. This notion is strongly related to the trapping probability considered in the present work. In classical ruin theory, the analysis of the infinite-time ruin probability begins with the formulation of the net profit condition, which ensures that the insurer's expected premium income exceeds its expected claim outflows. Condition (3.3) plays an analogous role in our setting; accordingly, we adopt the same terminology and refer to it as the net profit condition.*

The preceding result simplifies considerably when the remaining proportions of capital follow a $\text{Beta}(\alpha, 1)$ distribution, a case that is analysed in greater detail later in the paper.

Proposition 3.3. *Consider the capital process of a household without insurance coverage as proposed in Definition 2.1 with initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital Z_i with distribution Beta$(\alpha, 1)$. If the following net profit condition holds,*

$$\rho := \frac{\lambda}{r} < \alpha, \tag{3.4}$$

then $\lim_{x\to\infty} \psi(x) = 0$.

Proof. With $Z_i \sim \text{Beta}(\alpha, 1)$, one has $\mathbb{E}[\ln(Z_i)] = \alpha \int_0^1 \ln(z) z^{\alpha-1} \mathrm{d}z = -\frac{1}{\alpha}$. Thus, (3.4) holds if and only if (3.3) is satisfied. Applying Proposition 3.2 gives the desired result. □

Since households face trapping with probability one if the net profit condition is violated, our subsequent analysis in Section 4 focuses only on the region for which (3.4) holds. In particular, the assumptions made in Proposition 3.3 allow for a fully explicit expression of the trapping probability as demonstrated in Section 4.1. In the statement of this result, as in others presented throughout this article, the non-dimensional quantity $\rho = \frac{\lambda}{r}$ naturally appears. For clarity and consistency, we express our results in terms of ρ whenever appropriate.

3.2 The Net Profit Condition Under Proportional Insurance

We now extend the analysis of Section 3.1 to households purchasing quota-share proportional insurance coverage as the one introduced in Section 2.2. Although the framework remains unchanged,

insurance modifies the effective capital accumulation through both premium payments and retained losses. Consequently, the corresponding net profit condition differs from that of the uninsured model, as given by the following proposition:

Proposition 3.4. *Consider the capital process of a household with proportional insurance coverage and proportionality factor $\kappa \in [0,1]$, defined by (2.8). Furthermore, consider an initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital Z_i with distribution Beta$(\alpha, 1)$. The net profit condition in Proposition 3.2 is,*

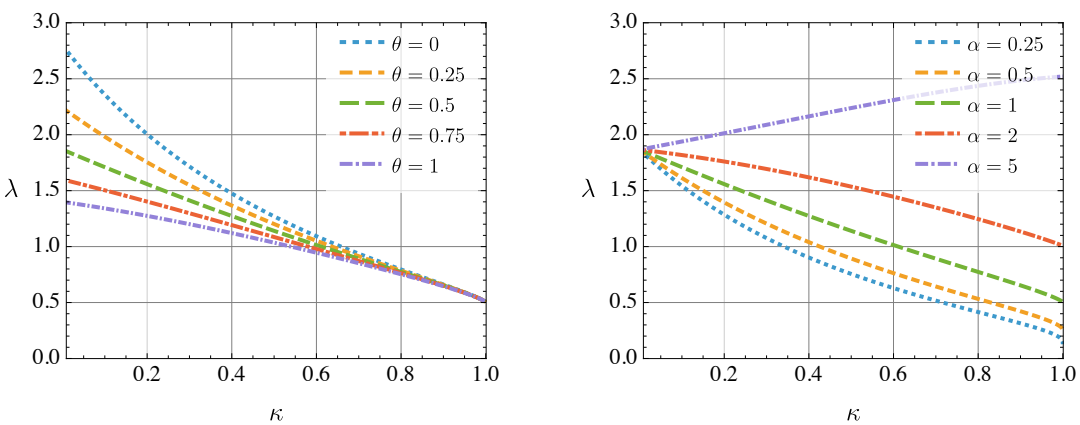
$$\frac{1}{\rho} = \frac{r}{\lambda} > \frac{\kappa}{(\alpha+1)(1-\kappa)} {}_2F_1\left(1, \alpha+1; \alpha+2; -\frac{\kappa}{1-\kappa}\right), \tag{3.5}$$

where ${}_2F_1(\cdot)$ is the Gauss hypergeometric function as defined in (A.1.1). If (3.5) holds, then $\lim_{x\to\infty} \psi(x) = 0$, otherwise trapping is certain.

Proof. By Proposition 3.2, the net profit condition for the capital process with proportional insurance coverage can be written as $\mathbb{E}\left[\ln\left(Y_i e^{rV_i}\right)\right] > 0$, where $Y_i = 1 - \kappa(1 - Z_i)$ as per (2.9). By independence of Y_i and V_i, the above inequality reduces to $\frac{r}{\lambda} > -\mathbb{E}\left[\ln\left(Y_i\right)\right]$. For $Z_i \sim \text{Beta}(\alpha, 1)$, using integration by parts, we have

$$\mathbb{E}[\ln(1-\kappa(1-Z_i))] = -\kappa \int_0^1 (1-\kappa+\kappa z)^{-1} z^{\alpha} \mathrm{d}z.$$

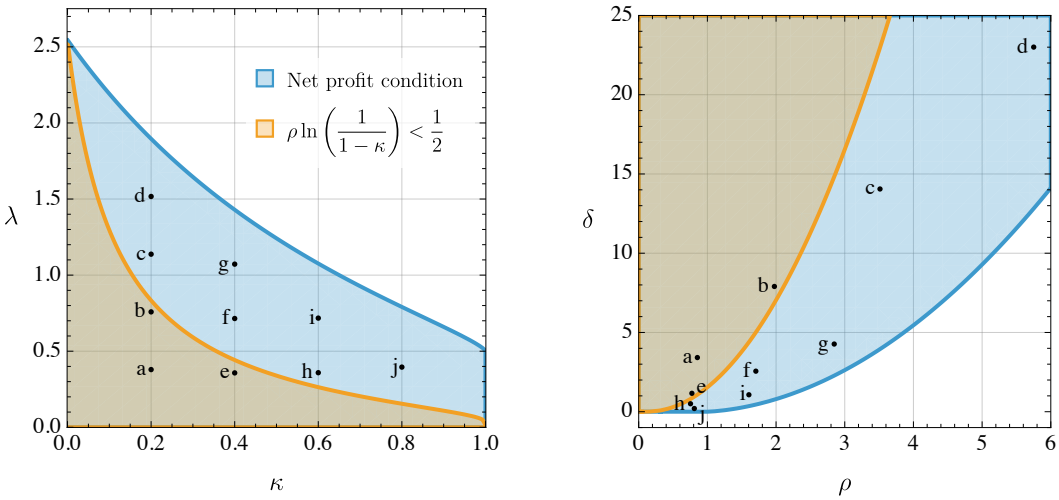
The above is an integral representation of a Gauss hypergeometric function (A.1.2), giving exactly (3.5), as required. □



(a) (b)

Figure 3: Upper boundary of the region defined by the net profit condition of (3.5) in terms of κ and λ for $a = 0.1$, $b = 1.4$, $c = 0.4$ with (a) fixed $\alpha = 1$ and different values of θ and (b) fixed $\theta = 0.5$ and different values of α. Note that $r = r(\kappa, \lambda)$ is given in Equation (2.7).

Condition (3.5) reflects the competing effects of proportional insurance on the household's capital process. Insurance reduces the proportion of each loss retained by the household, while the associated premium payments reduce the effective capital growth rate r. Moreover, r is not an independent parameter, but is related to parameters modeling the household income growth, consumption and saving rates, as noted in (2.7). Therefore, the net profit condition (3.5) may be interpreted as defining the admissible combinations of λ and κ for fixed values of the remaining parameters. In particular, it implies that $r > 0$, and hence automatically satisfies the requirement $b > p$. The dependence of the net profit condition on the model parameters is shown in Figure 3.



(a) (b)

Figure 4: Regions in the (κ, λ) plane corresponding to the various conditions of interest for an insured household under proportional losses with $Z_i \sim \text{Beta}(1,1)$. The parameters are set to $a = 0.1$, $b = 1.4$, $c = 0.4$ and $\theta = 0.1$. Note that the condition under which h is monotone in Lemma 4.1 implies the net profit condition (3.6). Here, $\rho = \frac{\lambda}{r}$ and $\delta = \rho\left(\frac{1-\kappa}{\kappa}\right)$, as defined in Remark 4.2.

Figure 3a illustrates the dependence of the net profit condition on θ for $\alpha = 1$, that is, when Z_i is uniformly distributed on $[0, 1]$. This case will be treated in detail in Section 4.2 and for it, the net profit condition (3.5) reduces to:

$$\frac{1}{\rho} = \frac{r}{\lambda} > 1 + \frac{1-\kappa}{\kappa} \ln(1-\kappa). \tag{3.6}$$

Observe that lowering the loading factor θ increases the region in which certain trapping is prevented when remaining proportions are uniformly distributed. This is intuitive, since a lower value of θ corresponds to more affordable insurance coverage for households. Furthermore, note that the sensitivity of the constraint to the loading factor θ increases for decreasing κ and thus increasing insurance coverage.

In contrast, Figure 3b varies α and shows that, as the shape parameter α increases, the region in which certain trapping is prevented becomes wider. This is due to the fact that higher values of α

imply a larger expected proportion of capital remaining after a household experiences a loss. It is worth noting that the sensitivity of the constraint to the shape parameter α decreases for decreasing κ and thus increasing insurance coverage. Indeed, decreasing κ mitigates the impact of losses on the household's capital (see, e.g., Figure 1b). As a result, when insurance coverage is sufficiently high, differences in the shape parameter α have a progressively smaller effect on the household's post-loss capital and, consequently, on the boundary of the net profit condition.

The net profit condition established here provides the parameter regime under which the analytical characterisation of the insured trapping probability in Section 4.2 is meaningful. Further interactions among the various parameter restrictions are summarised in Figure 4. In particular, the net profit condition in the uninsured model corresponds to $\kappa = 1$. Thus, as shown in Figure 4a, in the absence of insurance coverage, the net profit condition is satisfied only for cases a, e, h, and j, whereas it is violated for cases b, c, d, f, g, and i. On the other hand, once insurance coverage is introduced, the net profit condition is satisfied for all cases considered, as all points lie within the blue region corresponding to the insured framework. Indeed, as it will be further discussed in Section 5, it follows from (3.4) and (3.6) that some households that were facing certain trapping without insurance, have a positive probability of avoiding trapping if insurance is acquired, even though the critical capital x_0^* increases. For completeness, Figure 4b also presents the regions of the net profit condition in the (ρ, δ) plane.

4 Analytical Approach to Trapping Probabilities

Having established the parameter regimes under which the trapping probability is non-trivial, we now turn to its analytical characterisation. Our objective in this section is to derive explicit and recursive representations of the trapping probability for the uninsured and insured capital models, respectively.

The approach taken here differs from that of Kovacevic and Pflug (2011). Their analysis proceeds in two stages: an integral equation is first solved for the transition probability of the capital process, which is then used in a second integral equation for the trapping probability. In contrast, we work directly with the trapping probability through the infinitesimal generator of the capital process. This leads to a boundary value problem for the trapping probability itself, allowing us to bypass the intermediate calculation of the transition probabilities.

Our generator approach can be formulated using a classical result due to Paulsen and Gjessing (1997), which shows that the trapping probability is a bounded function on $[x^*, \infty)$, twice continuously differentiable with bounded first derivative on (x^*, ∞), and satisfies the following boundary value problem:

$$\mathcal{A}\left[\psi\right](x) = 0 \text{ for all } x \in (x^*, \infty) \quad \text{and} \quad \psi(x^*) = 1,$$

where $\mathcal{A}$ is the infinitesimal generator of the stochastic process $\{X_t\}_{t\geq 0}$ given by either (2.6) or (2.10). Furthermore, if the net profit condition holds (as detailed in Sections 3.1 and 3.2), we have $\lim_{x\to\infty} \psi(x) = 0$.

The remainder of this section develops analytical solutions of this boundary value problem. We begin with the uninsured model in Section 4.1, where a closed-form expression for the trapping

probability is available. We then turn to the proportionally insured model in Section 4.2, where the non-local structure of the resulting equation leads to the recursive layer method developed in this paper. All proofs of the results in this section are provided in Appendix B.

4.1 Trapping Probability Under No Insurance Coverage

We show that in the uninsured capital model, the boundary value problem admits a closed-form solution. Besides serving as a benchmark for the insured case, the explicit expression reveals the asymptotic behaviour of the trapping probability.

The following proposition presents the closed-form expression of the trapping probability for the uninsured setting when $Z_i \sim \text{Beta}(\alpha, 1)$. In this case, ψ satisfies:

$$\mathcal{A}[\psi](x) = r(x - x^*)\psi'(x) - \lambda\psi(x) + \lambda \int_0^1 \psi(x \cdot z)\alpha z^{\alpha-1}\mathrm{d}z = 0, \quad x \geq x^*, \tag{4.1}$$

with boundary conditions $\psi(x^*) = 1$ and $\lim_{x\to\infty} \psi(x) = 0$.

Proposition 4.1. *Consider an uninsured household capital process as proposed in Definition 2.1 with initial capital $x \geq x^*$, capital growth rate r, loss intensity $\lambda > 0$ and remaining proportions of capital Z_i with distribution Beta$(\alpha, 1)$. Assume the net profit condition $\rho = \frac{\lambda}{r} < \alpha$ in Proposition 3.3 holds. Then, the trapping probability is given by*

$$\psi(x) = \frac{\Gamma(\alpha)}{\Gamma(\rho)\Gamma(\alpha-\rho)} \left(\frac{x^*}{x}\right)^{\alpha-\rho} \int_0^1 t^{\alpha-\rho-1} \left(1 - \frac{x^*}{x}t\right)^{\rho-1} \mathrm{d}t. \tag{4.2}$$

Remark 4.1. *Although our primary interest is vulnerable non-poor households with capital levels relatively close to the poverty threshold, the asymptotic behavior of the trapping probability is useful for understanding the long-run effect of proportional losses and for comparison with classical additive-loss models. It can be obtained directly from the explicit form of the solution. Indeed, the integral term in (4.2) satisfies*

$$\lim_{x\to\infty} \int_0^1 t^{\alpha-\rho-1} \left(1 - t\frac{x^*}{x}\right)^{\rho-1} \mathrm{d}t = \int_0^1 t^{\alpha-\rho-1}\mathrm{d}t = \frac{1}{\alpha-\rho}.$$

Thus

$$\psi(x) \sim \left(\frac{x^*}{x}\right)^{\alpha-\rho} \frac{\Gamma(\alpha)}{\Gamma(\rho)\Gamma(\alpha-\rho)} \frac{1}{\alpha-\rho} = \left(\frac{x^*}{x}\right)^{\alpha-\rho} \frac{\Gamma(\alpha)}{\Gamma(\rho)\Gamma(\alpha-\rho+1)}, \tag{4.3}$$

as established in Equation (3.3.8) of Henshaw (2022). Equation (4.3) shows that the trapping probability decays algebraically with exponent $\alpha - \rho$. This differs with the exponential decay observed in the classical Cramér–Lundberg model with light-tailed claims (see, e.g., Asmussen and Albrecher (2010); Mandjes and Boxma (2023)) and highlights a fundamental qualitative difference between the present capital process and the classical surplus process.

Figure 5a illustrates the effect that the varying initial capital x and shape parameter α have on the trapping probability determined in Proposition 4.1. As highlighted before, higher values of α increase the mean of the distribution of the remaining proportion of capital. Observation of a decreasing trapping probability with increasing α is therefore intuitive and aligns with the reduction in loss.

Alternatively, Figure 5b presents the trapping probability (4.2) for varying loss frequency λ and fixed $\alpha = 1$. In this case, the remaining proportions of capital are uniformly distributed, a scenario that is examined in Section 4.2 for insured households. Note that the trapping probability tends to one as ρ tends to α in line with the net profit condition found in Proposition 3.3. Increasing the frequency of loss events increases the trapping probability, as is to be expected.

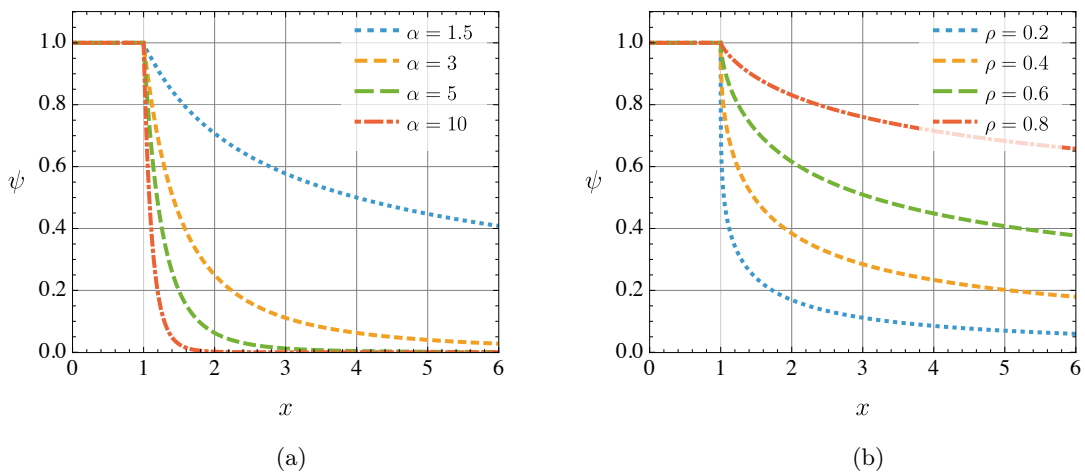


Figure 5: Trapping probability (4.2) under no insurance for $Z_i \sim \text{Beta}(\alpha, 1)$, $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$ and $x^* = 1$. (a) Fixed $\rho = 1$ and varying α (b) Fixed $\alpha = 1$ and varying ρ. The parameter values satisfy the net profit condition $\rho < \alpha$ in Proposition 3.3.

Unlike the uninsured model, no closed-form solution is available under proportional insurance. We therefore develop a recursive analytical method for solving the corresponding boundary value problem.

4.2 Trapping Probability Under Proportional Insurance Coverage

Before describing the method used to solve for the trapping probability in the proportional insured model, we establish a maximum principle for solutions of the equation $\mathcal{A}[h](x) = 0$, where $\mathcal{A}$ is the infinitesimal generator of the insured capital process defined by (2.10) in Section 2.2. Under some conditions on the parameters ρ and κ, we also show that these solutions are bounded. Specifically, we have the following result:

Lemma 4.1. *Assume that h is a function with the following properties: (i) continuous on $[(1-\kappa)x^*, \infty)$; (ii) constant on $[(1-\kappa)x^*, x^*]$; (iii) differentiable on (x^*, ∞); (iv) there exists $\epsilon > 0$ such that $h'(x) \neq 0$ for all $x \in (x^*, x^* + \epsilon)$; and (v) $\mathcal{A}[h](x) = 0$ for $x > x^*$. Then, if G_Y is absolutely*

continuous with respect to the Lebesgue measure, h is strictly monotone on $[x^*, \infty)$. If furthermore,

$$\rho \ln\left(\frac{1}{1-\kappa}\right) < \frac{1}{2}, \tag{4.4}$$

then h is bounded and thus $\lim_{x\to\infty} h(x)$ exists and is finite.

In Section 4.2.1, our analysis relies on the assumptions of Lemma 4.1. In Lemma 4.2, we show that condition (4.4) implies that the net profit condition is satisfied. Consequently, the analysis in Section 4.2.1 is restricted to the region defined by (3.6), thus excluding the cases in which households are trapped with probability one due to a failure of the net profit condition.

For the rest of the paper, we focus on the scenario in which $Z_i \sim \text{Beta}(1,1)$ in the case of proportional insurance, i.e., the remaining proportions of capital are uniformly distributed. In this case, the integral equation satisfied by the trapping probability, $\mathcal{A}[\psi](x) = 0$, where $\mathcal{A}$ is given by (2.10), can be replaced by an equivalent second order differential equation, albeit non-local as shown in Proposition 4.2. Despite its non-locality, the resulting equation can be solved using a layer method described in this section resulting in Proposition 4.3. This section finalises with the proof of the main result in this paper, Theorem 4.1, and details the construction of the fundamental solution set needed in the layer method. Although the case of uniform Z_i serves as a natural benchmark in which no particular remaining proportion within $[0,1]$ is favored, a solution for general distributions would be desirable. However, if for example, Z_i has a general Beta distribution, Equation (2.10) cannot be written as a differential equation, and thus requires a different treatment from the one we propose.

From Section 2.2, we know that the remaining proportion of capital after each loss event Y_i, is given, in the case of proportional insurance, by $Y_i = 1 - \kappa(1 - Z_i)$; see equation (2.9). Since we are assuming that Z_i is uniformly distributed on $[0,1]$, Y_i is uniformly distributed on the interval $[1-\kappa, 1]$. Accordingly, $\mathrm{d}G_Y(y) = \mathrm{d}y/\kappa$ and from Equation (2.10), the infinitesimal generator of the insured capital process can be rewritten as:

$$\mathcal{A}[h](x) = r(x - x^*)h'(x) - \lambda h(x) + \frac{\lambda}{\kappa x}\int_{(1-\kappa)x}^{x} h(y)\mathrm{d}y. \tag{4.5}$$

Remark 4.2. *Since capital is measured in units of the critical capital x^*, we henceforth set $x^* = 1$. On the other hand, time is measured in units of $\frac{1}{\lambda}$. Recalling that $\rho = \frac{\lambda}{r}$, we introduce the following non-dimensional quantity: $\delta := \rho\left(\frac{1-\kappa}{\kappa}\right)$. From (2.7), the parameters κ, λ, and r can be written in terms of ρ and δ as follows:*

$$\kappa(\rho,\delta) = \frac{\rho}{\rho+\delta}, \quad \lambda(\rho,\delta) = \frac{2bc\rho(a-1)(\delta+\rho)}{\delta((a-1)c(\theta+1)\rho - 2) - 2\rho}, \quad \textit{and} \quad r(\rho,\delta) = \frac{\lambda(\rho,\delta)}{\rho}.$$

In terms of ρ and δ the net profit condition in Equation (3.6), reads

$$\rho - \delta \ln\left(1 + \frac{\rho}{\delta}\right) < 1. \tag{4.6}$$

To find a solution to $\mathcal{A}[h](x) = 0$, where h is continuous on $[0, \infty)$ and twice differentiable on $(1, \infty)$, we manipulate (4.5) to obtain a related differential equation. Specifically, we have that

$$\frac{1}{r}\frac{\mathrm{d}(x\mathcal{A}[h](x))}{\mathrm{d}x} = \mathcal{L}_\delta[h](x) - \delta h((1-\kappa)x),$$

where $\mathcal{L}_\delta$ is the Sturm-Liouville operator defined by

$$\mathcal{L}_\delta[h](x) = x(x-1)h''(x) + [(2-\rho)x - 1]h'(x) + \delta h(x). \tag{4.7}$$

For ease of presentation a slight abuse of notation takes place. The dependence on ρ and δ of the infinitesimal generator $\mathcal{A}$ is suppressed. However, we explicitly keep the dependence on δ of the differential operator $\mathcal{L}_\delta$. This will be used later to relate solutions of these operators for different values of δ. In Proposition 4.2 this abuse of notation results in stating a relation between the parameterised $\mathcal{L}_\delta$ with the not explicitly parameterised $\mathcal{A}$.

Proposition 4.2. *Let h be a continuous function on $[0, \infty)$ that is constant on $[0, 1]$ and twice continuously differentiable for $x > 1$. Then,*

$$\mathcal{A}[h](x) = 0, \quad \textit{for } x \geq 1, \tag{4.8}$$

if and only if,

$$\mathcal{L}_\delta[h](x) = \delta h((1-\kappa)x), \quad \textit{for } x \geq 1. \tag{4.9}$$

Proposition 4.2 establishes the equivalence between (4.8) and (4.9). Therefore, the introduction of proportional insurance results in a second order differential equation with multiplicative delay, which is notably different from the uninsured case discussed in Section 4.1. The second-order part of $\mathcal{L}_\delta$ is also degenerate at the poverty threshold $x = 1$, since the coefficient $x(x-1)$ of $h''(x)$ vanishes there. This motivates expressing the local differential operator in the canonical speed-scale form used in the theory of one-dimensional diffusions. In particular, the speed and scale measures absorb the singular behavior at $x = 1$ and provide the Sturm-Liouville structure needed to construct the corresponding Green's function and, subsequently, the recursive layer solution as documented in Section 4.2.1. To proceed, we write the operator $\mathcal{L}_\delta$ in Sturm-Liouville form:

$$\mathcal{L}_\delta[h](x) = \frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}}{\mathrm{d}s}h(x) + \delta h(x),$$

where the speed and scale measures are $\mathrm{d}m(x) = \frac{\mathrm{d}x}{(x-1)^\rho}$ and $\mathrm{d}s(x) = \frac{\mathrm{d}x}{x(x-1)^{1-\rho}}$, respectively.

4.2.1 The Layer Method

We seek the trapping probability ψ which satisfies (4.8) or (4.9) in Proposition 4.2 with the boundary conditions $\psi(x) = 1$ for $x \in [0, 1]$ and $\lim_{x\to\infty} \psi(x) = 0$. The Ansatz for the trapping probability is:

$$\psi(x) = 1 - \frac{f(x)}{L}, \tag{4.10}$$

where the *auxiliary function* f satisfies (4.9) with boundary conditions

$$f(x) = 0 \text{ for } x \in [0,1] \quad \text{and} \quad \exists\, L = \lim_{x\to\infty} f(x) > 0. \tag{4.11}$$

Observe that the function $\frac{1}{L}f(x)$ is the non-trapping probability, i.e., the probability that a household never falls below the poverty line. To obtain f, we solve the differential equation in each layer $\left[1/(1-\kappa)^{n-1}, 1/(1-\kappa)^n\right] := [x_{n-1}, x_n]$ of $[1,\infty) = \bigcup_{n=1}^{\infty}[x_{n-1}, x_n]$ by iteratively using classical Sturm-Liouville theory. A schematic illustration of our layer method is shown in Figure 6.

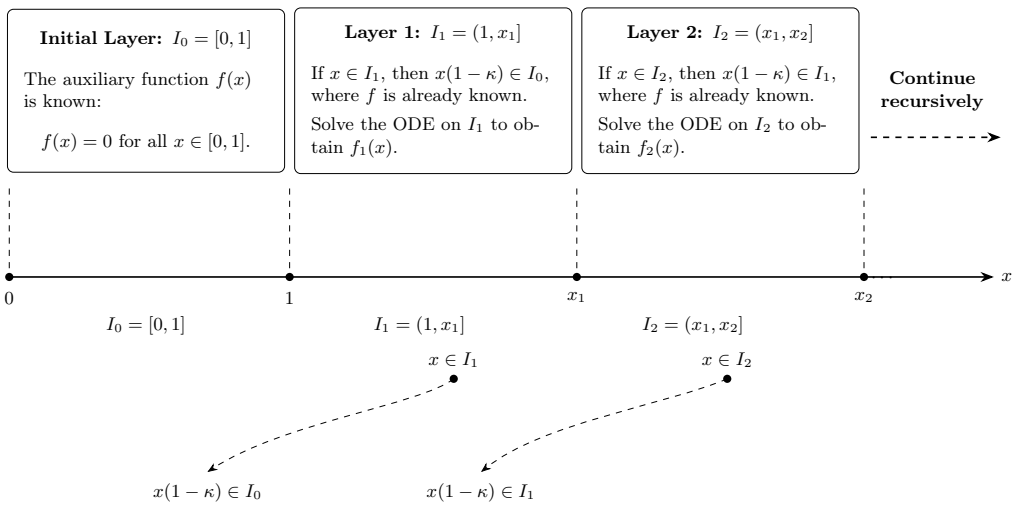


Figure 6: The positive real number line is partitioned into successive layers $I_0 = [0,1]$, $I_1 = (1, x_1]$, $I_2 = (x_1, x_2]$, and so on. On each new layer, the delayed argument $x(1-\kappa)$ lies in a previously constructed layer, allowing the corresponding differential equation to be solved recursively.

Proposition 4.3. *Let u and w be linearly independent solutions to the homogeneous problem $\mathcal{L}_\delta[\cdot](x) = 0$, with $u(1) = 1$, $w(1) = 0$, and $\frac{\mathrm{d}w}{\mathrm{d}s}(1^+) > 0$. Define the Green's function, G, for the initial value problem and the corresponding Wronskian, W, by*

$$G(x,x') = \frac{w(x)u(x') - u(x)w(x')}{W} \quad \textit{and} \quad W = u(x)\frac{\mathrm{d}w}{\mathrm{d}s}(x) - w(x)\frac{\mathrm{d}u}{\mathrm{d}s}(x) > 0, \tag{4.12}$$

respectively. For $n = 1, 2, \ldots,$ let $x_n = \frac{1}{(1-\kappa)^n}$ and recursively define the sequence of functions $\{z_n\}_{n=1}^{\infty}$ by

$$z_1(x) = \begin{cases} 0, & 0 < x < 1, \\ w(x), & 1 \le x, \end{cases}$$

$$z_n(x) = \begin{cases} 0, & 0 < x < x_{n-1}, \\ \delta \displaystyle\int_{x_{n-1}}^{x} z_{n-1}((1-\kappa)x')\ G(x,x')\ \mathrm{d}m(x'), & x_{n-1} \le x. \end{cases}$$

Furthermore, define $\{f_n\}_{n=1}^{\infty}$ *as*

$$f_n(x) = z_1(x) + \cdots + z_n(x), \quad 0 \le x \le x_n. \tag{4.13}$$

Then, the point-wise limit $f = \lim_{n\to\infty} f_n$ *solves* (4.8) *and* (4.9) *on* $[1,\infty)$*. Moreover, if* $0 < \rho < 1$ *or* $\rho \ln(1/(1-\kappa)) < 1/2$*,* f *is bounded and has a finite limit* $L = \lim_{x\to\infty} f(x) > 0$*.*

We make two additional observations about the conditions under which Proposition 4.3 ensures boundedness of f. First, as Lemma 4.2 states, they guarantee that the net profit condition in (4.6) holds; see Figure 4. This ensures, by Proposition 3.4, that $\lim_{x\to\infty} \psi(x) = 0$, which is consistent with the behavior of the Ansatz in (4.10).

Lemma 4.2. *If* $0 < \rho < 1$ *or* $\rho \ln(1/(1-\kappa)) < 1/2$ *then the net profit condition* $\frac{1}{\rho} > 1 + \frac{1-\kappa}{\kappa}\ln(1-\kappa)$ *holds for all* $\kappa \in (0,1)$*.*

Second, the condition if $0 < \rho < 1$ or $\rho \ln(1/(1-\kappa)) < 1/2$ seems to be more restrictive than necessary for the existence of the limit L. Indeed, the numerical simulations presented in Section 5 include cases of parameters outside the region identified in the statement of the proposition in which boundedness can be established. Specifically, the values of κ and λ corresponding to the points labeled c, d, f, g and i in Figure 4, fall outside of this region. Nevertheless, the numerical calculations in Section 5 illustrate that the auxiliary function f is bounded and thus it can be used to determine the trapping probability via the Ansatz in (4.10).

Theorem 4.1 gives the construction of the trapping probability:

Theorem 4.1. *Let* f *be constructed as in Proposition 4.3 and suppose that* $\psi(x')$ *is known for some* $x' > 1$*. Then, the trapping probability is given by*

$$\psi(x) = \begin{cases} 1, & 0 \le x \le 1, \\ 1 - \frac{1-\psi(x')}{f(x')} f(x), & 1 \le x. \end{cases} \tag{4.14}$$

Thus, the layer construction determines the functional form of the trapping probability up to a single normalizing quantity, L (or equivalently, the trapping probability at one reference point). To summarise, computing the trapping probability by the layer method and Theorem 4.1, requires linearly independent solutions $\{u, w\}$ as in Proposition 4.3, and an estimate for $\psi(x')$ at some $x' > 1$, or equivalently, an estimate of the limiting value L of the auxiliary function f.

For completeness, we give a full construction for the functions $\{u, w\}$ in Appendix C.1, which are then used to compute f via the layer method. This is illustrated in Figure 10. The method we use to estimate L is as follows: we consider five values of $x' \in [1, x_1)$, where $f(x')$ equals the analytically known $w(x')$. For each, we estimate the value of $\psi(x')$ by simulating 20,000 realisations of the process X_t with $X_0 = x'$ and counting the number of trapped trajectories by time $t = 100$. The value of L is then taken by averaging the expression (B.6.1) over the five cases. This method is used for the computation of ψ in Figures 7 and 9 used in the comparisons, graphs and general discussion presented in Section 5.

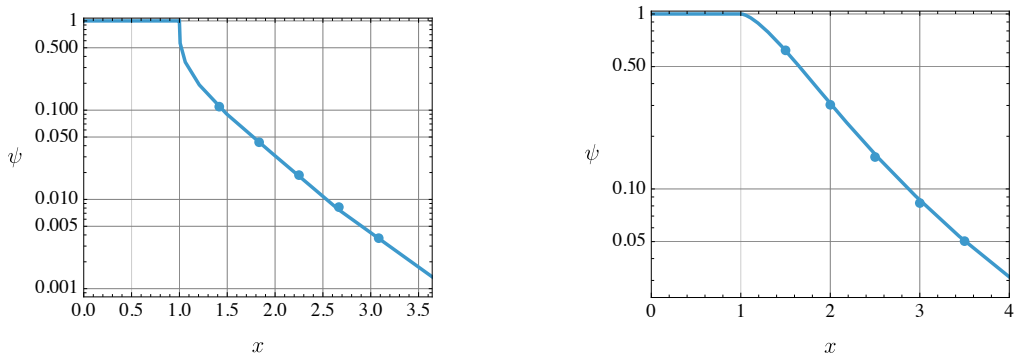


Figure 7: Comparison between simulated values of the trapping probability and our method. The left graph shows $\psi(x)$ for the case $(\rho, \delta) = (0.2, 0.1)$ which corresponds to $(\kappa, \lambda) = (0.67, 0.1)$. The right graph corresponds to the case labeled with the letter f in Figure 4. Solid lines show the function $\psi(x)$ computed via the layer method described in Section 4.2.1. Dots show the estimation of $\psi(x)$ obtained by directly simulating $250,000$ trajectories of the process X_t with $X_0 = x$, and counting the number of realisations that are trapped by $t = 50$. These estimates are performed independently from those used to estimate L.

5 Model Comparisons and the Impact of Insurance

The analytical results of Section 4 allow us to examine two features of the model that are of particular interest. First, we compare the asymptotic behavior of the trapping probability under proportional losses with that obtained under the additive loss model of Flores-Contró et al. (2025). This comparison highlights a fundamental difference between the two loss mechanisms. Second, we use the recursive solution developed in Section 4.2 to examine how proportional insurance affects the trapping probability across different initial capital levels. In particular, we identify the conditions under which this coverage is beneficial, in the sense that it reduces the trapping probability relative to the case without insurance.

5.1 Additive versus Multiplicative Losses

We first compare the asymptotic decay of the uninsured household trapping probability under proportional (multiplicative) losses with that obtained under the random-valued (additive) losses. For exponentially distributed additive losses, the uninsured trapping probability for $x \geq x^*$ is given by (see Equation (3.3) in Flores-Contró et al. (2025)):

$$\psi^{\text{EXP}}(x) = \frac{\Gamma\left(\rho; \mu(x - x^*)\right)}{\Gamma\left(\rho\right)}, \tag{5.1}$$

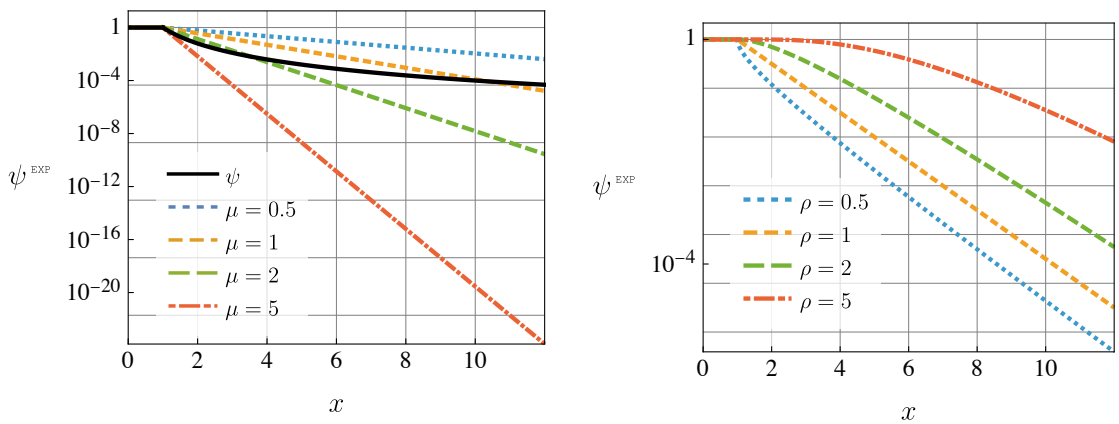
where $\rho = \frac{\lambda}{r}$ and $\Gamma(a; z) := \int_z^\infty e^{-t} t^{a-1} \mathrm{d}t$ is the upper incomplete gamma function and μ is the exponential loss parameter. The asymptotic behaviour as $x \to \infty$ of the trapping probability in (5.1) is given by

$$\psi^{\text{EXP}}(x) \sim \mu^{\rho-1}(x - x^*)^{\rho-1} e^{-\mu(x-x^*)}(1 + O(|\mu(x - x^*)|^{-1})), \tag{5.2}$$

so the limiting behavior as $x \to \infty$ of the ratio of (5.2) to (4.3) is

$$\frac{\psi^{\text{EXP}}(x)}{\psi(x)} \sim C x^{\alpha-\rho}(x-x^*)^{\rho-1} e^{-\mu(x-x^*)}(1+O(|\mu(x-x^*)|^{-1})),$$

for some constant C. Thus, the trapping probability under additive exponential losses decays strictly faster than under multiplicative losses, with the severity of this difference dependent on the parameters of the loss distributions. This reflects the scale dependence of the two loss mechanisms. Under additive losses, a random loss becomes relatively less important as initial capital increases. Under multiplicative losses, by contrast, the distribution of the fraction of capital lost does not change with the capital level. Consequently, even households with large initial capital remain exposed to losses representing a substantial proportion of their wealth.



(a) (b)

Figure 8: (a) Comparison of the trapping probability $\psi^{\text{EXP}}(x)$ given in (5.1) for random-valued losses with $Z_i \sim \text{Exponential}(\mu)$ and different values of μ, against the trapping probability $\psi(x)$ in (4.2) for proportional losses with $Z_i \sim \text{Beta}(5,1)$ (black solid line). Here, the parameters are given by $a = 0.1$, $b = 1.4$, $c = 0.4$, $\rho = 1$ and $x^* = 1$ (b) The trapping probability $\psi^{\text{EXP}}(x)$ in (5.1) for different values of ρ.

A comparison of the decay of the trapping probability under proportional losses against that of random-valued losses is provided in Figure 8. Here, the slower rate of decay under proportional losses is clearly observable. Figure 8a compares the trapping probabilities under the case in which the remaining proportions of capital are $\text{Beta}(5,1)-$distributed given by (4.2) and exponentially distributed additive losses given by (5.1) for different values of μ. For random-valued claim sizes with an expected value of 0.5 ($\mu = 2$) the trapping probability is greater than for proportional losses for the most vulnerable (households with initial capital levels close to the poverty line), however, as capital increases the trapping probability under proportional losses exceeds the random-valued case. If the expected claim size increases to one ($\mu = 1$) the trapping probability for proportional losses is significantly lower than in the random-valued case for a higher portion of the most vulnerable. Compared to the mean loss associated with $\text{Beta}(5,1)-$distributed remaining

proportions, an expected claim size of one is low with respect to high levels of initial capital. Figure 8b illustrates the trapping probability under the random-valued loss model (5.1) for different values of ρ. Note that the net profit condition holds here, so the trapping probability decreases to zero as the initial capital increases.

5.2 Impact of Proportional Insurance on Trapping Probability

Figure 9 compares the trapping probabilities of uninsured and insured households for different values of λ and κ. The probabilities for uninsured households are computed using the closed-form expression (4.2), whereas those for insured households are obtained from the recursive solution presented in Section 4.2. In line with the developments in Section 4.2, we consider the case in which the remaining proportions of capital are $\text{Beta}(1,1)-$distributed.

A particularly important effect of insurance is worth highlighting. For several parameter regimes considered in Figure 9, specifically plots labeled b, c, d, f, g and i, the uninsured model violates the net profit condition $\rho < 1$ determined in Proposition 3.3. Hence, trapping occurs with certainty for all uninsured households, regardless of their initial capital level, yielding a constant trapping probability of one. As explained in Section 2.2, insurance modifies both the retained-loss distribution and the capital growth rate. Consequently, it can shift a household for which escape from the poverty trap is impossible into a parameter regime satisfying the insured net profit condition, thereby creating a positive possibility of escape. In other words, proportional insurance fundamentally changes the qualitative behaviour of the process: whereas trapping is certain in the uninsured model, insurance reduces the trapping probability to a value strictly below one, with this probability decreasing as the initial capital increases.

Furthermore, we also observe in Figure 9 that for some households with initial capital levels barely above the poverty threshold x_0^*, purchasing proportional insurance coverage is detrimental as their trapping probabilities increase after acquiring this protection (see, e.g., plots labeled a, e, h and j). The increase for this particular group is consistent with previous findings in Kovacevic and Pflug (2011) and Flores-Contró et al. (2025), and arises because paying the insurance premium reduces a household's income generation rate, thereby raising the critical capital threshold to a higher level, $x^* \geq x_0^*$, as explained in Section 2.2.

Despite being disadvantageous for some of the most vulnerable households, Figure 9 shows that proportional insurance coverage is beneficial for the great majority of households. Specifically, the figure suggests the existence of a threshold $x_\bullet > x_0^*$ such that, for $x_0^* < x < x_\bullet$, the uninsured household has the lower trapping probability. Hence, vulnerable households with capital levels just above the poverty line may require support to make insurance beneficial for them (see, for instance, Flores-Contró et al. (2025), which introduces two different frameworks for offering subsidies to households, thereby reducing the impact of premium payments on their income-generating capacity).

The comparisons presented in Sections 5.1 and 5.2 reveal two distinct effects of proportional losses and proportional insurance. Multiplicative losses fundamentally alter the tail behavior of the trapping probability, producing algebraic rather than exponential decay. Insurance, meanwhile, has two distinguishable effects. On the one hand, it shows that households that can afford insurance can avoid the certainty of trapping since the NPC (3.5) holds for a larger set of parameters than the case of no-insurance (3.4). Insurance, however, does not uniformly reduce trapping risk. In-

deed, households with capital barely above the trapping probability would not be able to afford the purchase of an insurance and thus remain under the poverty level. In particular, while premium payments may increase the likelihood of falling into the poverty trap for households with initial capital levels that are close to the poverty line, insurance can eliminate the certainty of trapping for households with higher initial capital.

6 Conclusion

This article studies the capital process introduced in Kovacevic and Pflug (2011), where households are susceptible to losses that are proportional to their accumulated capital. Under the assumption of proportional losses, households at all capital levels are exposed to events that can result in substantial reductions in capital. This feature is particularly relevant for vulnerable non-poor households, i.e., those living just above the poverty line, but also distinguishes the model from classical risk processes with additive losses, since even households with relatively high levels of capital remain exposed to losses representing a significant proportion of their wealth.

Focusing on the probability that a household ever falls below the poverty line, referred to as the trapping probability, we consider two cases: (i) households with no insurance coverage and (ii) households with proportional insurance coverage. We first develop an alternative characterisation of the trapping time through an embedded discrete-time process. This representation leads naturally to a net profit condition, analogous to that in classical ruin theory, which identifies the parameter regimes under which the infinite-time trapping probability is non-trivial. In particular, when the expected logarithmic accumulation is negative, trapping occurs with probability one, whereas under the net profit condition the trapping probability converges to zero as the initial capital tends to infinity.

For households with no insurance coverage, explicit infinite-time trapping probabilities are obtained when the remaining proportions of capital are $\mathrm{Beta}(\alpha, 1)-$distributed. The resulting expression also provides the asymptotic behaviour of the trapping probability, which decays algebraically as the initial capital increases. This differs from the exponential decay arising in classical additive-loss models and, in particular, rules out exponential decay for the proportional-loss model considered here. Comparison with the additive (random-valued) loss framework of Flores-Contró et al. (2025) further illustrates the slower decay produced by multiplicative losses.

Incorporating proportional insurance coverage modifies both the capital growth rate and the distribution of the remaining proportion of capital after a loss. The corresponding infinitesimal generator therefore contains non-local terms, and the resulting integro-differential equation does not admit the same explicit treatment as in the uninsured case. To address this, we transform the problem into a delay differential equation and develop a recursive layer method, based on Sturm–Liouville theory and the special functions that arise in this setting. This provides a constructive solution of the insured problem and allows the trapping probability to be evaluated numerically.

Comparisons between trapping probabilities under no insurance coverage and proportional insurance coverage show that there are parameter regimes in which an uninsured household faces certain eventual trapping because the net profit condition is violated, while the corresponding insured household has a trapping probability strictly below one. This highlights the importance of insurance as a risk mitigation tool, as it can turn certain trapping into a positive possibility of escape

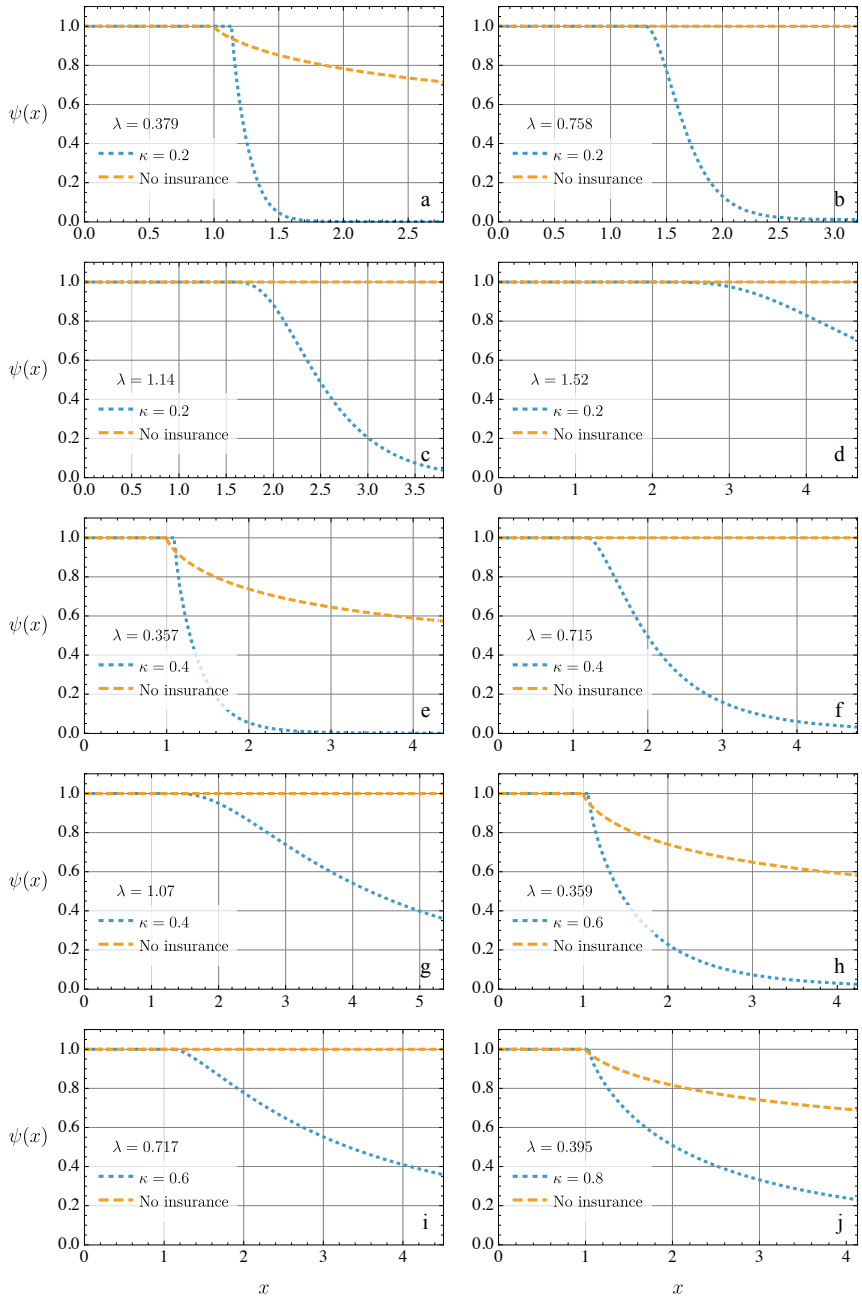


Figure 9: Comparison of the trapping probabilities for uninsured and insured households (with proportional insurance coverage) under proportional losses with $Z_i \sim \text{Beta}(1, 1)$. The parameters are set to $a = 0.1$, $b = 1.4$, $c = 0.4$, $\lambda = 1$ and $x_0^* = 1$. Each subplot corresponds to one of the cases illustrated in Figure 4 (refer to the label in the bottom-right corner of each subplot). The trapping probabilities for uninsured households (orange dashed curves) and insured households (blue dotted curves) are computed using (4.2) and (4.14), respectively.

for households that would otherwise be trapped with certainty. Our analyses also show that for some households with initial capital levels just above the poverty line, insurance and the associated premium payments can increase the probability of falling below the poverty line, consistent with the findings of earlier studies. Nevertheless, our comparisons indicate that proportional insurance coverage reduces the trapping probability over a broad range of higher initial capital levels.

There are several natural directions in which the present model could be extended. In particular, the proportion of insurance coverage is fixed, premiums are calculated using the expected value principle, and the insured analysis is developed for uniformly distributed remaining proportions of capital. We also focus only on the event that a household falls below the poverty line, rather than on its subsequent behaviour after trapping. Future work could consider endogenous insurance decisions, premium subsidies, and more general proportional-loss distributions. Even under the present assumptions, however, the model helps identify how premium payments and proportional loss protection interact to affect the long-run probability of poverty trapping.

Funding

The work of K.H. and C.C. was supported by Engineering and Physical Sciences Research Council (EPSRC) [grant number EP/W522399/1]; and the EPSRC and ESRC Centre for Doctoral Training on Quantification and Management of Risk and Uncertainty in Complex Systems Environments [grant number EP/L015927/1]. José Miguel Flores-Contró gratefully acknowledges funding from the FWO and F.R.S.-FNRS under the Excellence of Science (EOS) programme, project ASTeRISK (40007517).

Acknowledgements

E.T. thanks the colleagues in the Department of Mathematical Sciences at the University of Liverpool for their hospitality during his Fall 2023 visit, when part of this work was conducted.

Appendices

A The Gauss Hypergeometric Function

In this section, we collect the definition and selected properties of the Gauss hypergeometric function that are used throughout the article. The material presented here is not intended to be exhaustive; rather, it provides the specific results and identities required for our analysis.

For a comprehensive treatment of hypergeometric functions, including their relations, transformation formulas, and domains of convergence, see Abramowitz and Stegun (1972) and Kristensson (2010).

A.1 Definition

The Gauss hypergeometric function is defined as:

$$ {}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n}{(c)_n} \frac{z^n}{n!}, \tag{A.1.1} $$

where $(a)_n = \frac{\Gamma(a+n)}{\Gamma(a)}$ denotes the Pochhammer symbol (Seaborn, 1991). If $\mathrm{Re}(c) > \mathrm{Re}(b) > 0$ and $z \in \mathbb{C}$, $z \notin [1, \infty)$ the Gauss hypergeometric function has the integral representation (see, for instance, (Abramowitz and Stegun, 1972, p. 558)):

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{-a}\,\mathrm{d}t. \tag{A.1.2} $$

A.2 Selected Properties

A number of transformation formulas for ${}_2F_1\left(a,b;c;z\right)$ can be derived. We present below those that are used in this manuscript.

$$ \begin{aligned} {}_2F_1\left(a,b;c;z\right) &= \frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\, {}_2F_1\left(a,b;a+b-c+1;1-z\right) \\ &+ (1-z)^{c-a-b}\frac{\Gamma(c)\Gamma(a+b-c)}{\Gamma(a)\Gamma(b)}\, {}_2F_1\left(c-a,c-b;c-a-b+1;1-z\right) \quad |\arg(1-z)| < \pi. \end{aligned} \tag{A.2.1} $$

$$ {}_2F_1\left(a,b;c;z\right) = (1-z)^{c-a-b}\, {}_2F_1\left(c-a,c-b;c;z\right) \tag{A.2.2} $$

In particular, (A.2.2) is also known as Euler's transformation.

B Mathematical Proofs

In this section, we present the proofs of all the results in Section 4.

B.1 Proof of Proposition 4.1

Proof. The trapping probability satisfies $\mathcal{A}\left[\psi\right](x) = 0$ where $\mathcal{A}$ is given by (4.1). This yields

$$ (x - x^*)\psi'(x) = \rho\psi(x) - \rho\int_0^1 \psi(x\cdot z)\mathrm{d}z^\alpha. \tag{B.1.1} $$

Since

$$ \frac{\mathrm{d}}{\mathrm{d}x}\int_0^1 \psi(x\cdot z)\mathrm{d}z^\alpha = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\alpha}{x^\alpha}\int_0^x \psi(t)t^{\alpha-1}\mathrm{d}t\right) = \frac{\alpha}{x}\left[\psi(x) - \int_0^1 \psi(x\cdot z)\mathrm{d}z^\alpha\right], $$

we have that

$$\rho \frac{\mathrm{d}}{\mathrm{d}x} \int_0^1 \psi(x \cdot z) \mathrm{d}z^\alpha = \frac{\alpha}{x}(x - x^*)\psi'(x).$$

Differentiation of (B.1.1) leads to

$$(x - x^*)\psi'' + \psi' = \rho\psi' - \frac{\alpha}{x}(x - x^*)\psi'.$$

Upon rearranging, we arrive at

$$0 = x(x - x^*)\psi'' + \left[(1 + \alpha - \rho)x - \alpha x^*\right]\psi'. \tag{B.1.2}$$

Since (B.1.2) is first order in ψ', separation of variables leads to

$$\frac{\psi''}{\psi'} = -\left(\frac{\alpha}{x} + \frac{1 - \rho}{x - x^*}\right),$$

so that $\psi'(x) = Cx^{-\alpha}(x - x^*)^{\rho-1}$ for some positive constant C. Note that ψ' is integrable at infinity since $\rho < \alpha$. So integrating on the interval $[x, \infty)$ and using the fact that $\lim_{x\to\infty} \psi(x) = 0$, we get

$$\psi(x) = C \int_x^\infty v^{-\alpha}(v - x^*)^{\rho-1} \mathrm{d}v = C \int_x^\infty v^{-\alpha+\rho} \left(1 - \frac{x^*}{v}\right)^{\rho-1} \frac{\mathrm{d}v}{v}.$$

With the substitution $v = x/t$, we obtain

$$\psi(x) = C \int_0^1 \left(\frac{x}{t}\right)^{-\alpha+\rho} \left(1 - \frac{x^*}{x}t\right)^{\rho-1} \frac{\mathrm{d}t}{t} = \frac{C}{x^{\alpha-\rho}} \int_0^1 t^{\alpha-\rho-1} \left(1 - \frac{x^*}{x}t\right)^{\rho-1} \mathrm{d}t.$$

The boundary condition at x^* gives

$$C = \frac{(x^*)^{\alpha-\rho}}{\int_0^1 t^{\alpha-\rho-1}(1-t)^{\rho-1}\, dt} = (x^*)^{\alpha-\rho} \frac{\Gamma(\alpha)}{\Gamma(\rho)\Gamma(\alpha - \rho)},$$

as claimed. Note that the integral representation of the trapping probability can be identified as a Gauss hypergeometric function as shown in Proposition 3.3.3 of Henshaw (2022). Using (A.1.2), we can rewrite the integral term as

$$\int_0^1 t^{\alpha-\rho-1} \left(1 - \frac{x^*}{x}t\right)^{\rho-1} \mathrm{d}t = \frac{\Gamma(\alpha - \rho)\Gamma(1)}{\Gamma(\alpha - \rho + 1)} {}_2F_1\left(1 - \rho, \alpha - \rho; \alpha - \rho + 1; \frac{x^*}{x}\right).$$

This yields

$$\psi(x) = \frac{\Gamma(\alpha)}{\Gamma(\rho)\Gamma(\alpha - \rho)} \left(\frac{x^*}{x}\right)^{\alpha-\rho} \frac{1}{\alpha - \rho} {}_2F_1\left(1 - \rho, \alpha - \rho; \alpha - \rho + 1; \frac{x^*}{x}\right).$$

which is exactly Equation (3.3.17) in Henshaw (2022). □

B.2 Proof of Lemma 4.1

Proof. For $x > x^*$, rewrite $\mathcal{A}[h](x) = 0$ as

$$h'(x) = \frac{\rho}{x - x^*} \int_{1-\kappa}^{1} (h(x) - h(x \cdot y))\, \mathrm{d}G_Y(y).$$

Since h is continuous and G_Y is absolutely continuous, the right-hand side is continuous as a function of x. Hence h' is continuous on (x^*, ∞).

Assumption (iv) ensures that h does not have an accumulation of critical points on a neighborhood to the right of x^*, and thus we can define

$$x_c = \inf\{x > x^* : h'(x) = 0\} \geq x^* + \epsilon.$$

If $x_c < \infty$, then $|h'(x)| > 0$ for all $x^* < x < x_c$; and since h' is continuous on (x^*, ∞), we also have $h'(x_c) = 0$. Thus, evaluating $\mathcal{A}[h](x_c) = 0$ gives

$$0 = \lambda h(x_c) - \lambda \int_{1-\kappa}^{1} h(x_c \cdot y) \mathrm{d}G_Y(y) = \lambda \mathbb{E}\left[h(x_c) - h(x_c \cdot Y)\right]. \tag{B.2.1}$$

Since h is assumed constant on $[(1-\kappa)x^*, x^*]$ and strictly monotone on (x^*, x_c), and $x_c \cdot Y < x_c$ almost surely, the random variable inside the expectation is either positive or negative with probability one, which contradicts (B.2.1). Hence $x_c = \infty$, and h is strictly monotone on $[x^*, \infty)$.

We now suppose h is strictly increasing and prove that condition (4.4) implies the boundedness of h. The proof in the case of decreasing h is similar.

We write $\mathcal{A}[h](x) = 0$ as

$$h'(x) = \frac{\rho}{x - x^*} \mathbb{E}\left[h(x) - h(x \cdot Y)\right],$$

and use layers $x_n = \frac{x^*}{(1-\kappa)^n}$ as in Section 4.2.1. For $n \geq 2$, integrating from x_{n-1} to x_n, we define

$$\Delta_n \equiv h(x_n) - h(x_{n-1}) = \rho \int_{x_{n-1}}^{x_n} \frac{\mathbb{E}\left[h(x) - h(x \cdot Y)\right]}{x - x^*} \mathrm{d}x.$$

Now, since h is increasing and $(1-\kappa) < Y < 1$ almost surely, one has that for $x_{n-1} < x < x_n$,

$$h(x) - h(x \cdot Y) \leq h(x_n) - h(x_{n-2}) = \Delta_n + \Delta_{n-1}, \tag{B.2.2}$$

with probability one. We thus define for $n \geq 2$,

$$\gamma_n = \rho \int_{x_{n-1}}^{x_n} \frac{1}{x - x^*} \mathrm{d}x = \rho \ln\left(\frac{1 - (1-\kappa)^n}{1 - (1-\kappa)^{n-1}} \frac{1}{1-\kappa}\right). \tag{B.2.3}$$

Then, combining (B.2.2) and (B.2.3),

$$\Delta_n \leq \gamma_n(\Delta_n + \Delta_{n-1}), \quad n = 2, 3, \ldots \tag{B.2.4}$$

Note that

$$\lim_{n\to\infty} \gamma_n = \rho \ln\left(\frac{1}{1-\kappa}\right) < \frac{1}{2},$$

by hypothesis. Then, we can take n_0 large enough so that $\gamma_n < 1$ for all $n \geq n_0$, and write (B.2.4) as

$$\Delta_n \leq \frac{\gamma_n}{1-\gamma_n}\Delta_{n-1} \leq p_n \Delta_{n_0}, \quad p_n := \left(\prod_{j=n_0+1}^{n} \frac{\gamma_j}{1-\gamma_j}\right), \quad n > n_0.$$

Let $N > n_0$, and sum over $n_0 + 1 \leq n \leq N$ to obtain

$$h(x_N) - h(x_{n_0}) \leq \Delta_{n_0} \sum_{n=n_0+1}^{N} p_n.$$

By the ratio test, the series $\sum_n p_n$ converges if $\lim_{n\to\infty} \frac{\gamma_n}{1-\gamma_n} < 1$ which is equivalent to the assumption $\lim_{n\to\infty} \gamma_n < 1/2$. Thus $\lim_{N\to\infty} h(x_N) < \infty$, which implies that h is bounded. The existence of the limit of $h(x)$ as x tends to ∞ follows from the fact that h is bounded and increasing. □

B.3 Proof of Proposition 4.2

Proof. Clearly, (4.8) implies (4.9). Now, suppose that h satisfies (4.9), and define $A(x) := \mathcal{A}[h](x)$. Then $\frac{\mathrm{d}xA(x)}{\mathrm{d}x} = 0$ so $A(x) = C/x$ for some $C \in \mathbb{R}$. Evaluating (4.5) as $x \downarrow 1$ gives

$$A(1) = \lim_{x\downarrow 1}(x-1)\frac{\mathrm{d}h}{\mathrm{d}x} - \lambda h(1) + \lambda h(0). \tag{B.3.1}$$

Since $h(x) = \tilde{a}$ on $[0,1]$, $h(x) = \tilde{a} + \tilde{h}(x)$, where $\tilde{h}$ satisfies, for $x > 1$,

$$\mathcal{L}_\delta[\tilde{h}](x) = 0. \tag{B.3.2}$$

In Proposition 6.1 and Proposition 6.2 we identify a fundamental solution set for the operator $\mathcal{L}_\delta$ so that any solution of (B.3.2) has the property that for any $\rho > 0$, $\lim_{x\downarrow 1}(x-1)^{1-\rho}\frac{\mathrm{d}h}{\mathrm{d}x} > 0$. Consequently, the limit term in (B.3.1) vanishes. Using that h is constant on $[0,1]$, we obtain $C = A(1) = 0$ and thus $\mathcal{A}[h](x) = 0$ for all $x \geq 1$, as claimed. □

B.4 Proof of Lemma 4.2

Proof. We need to show that, for $\kappa \in (0,1)$,

$$0 < \rho < 1 \quad \text{or} \quad \rho \ln\left(\frac{1}{1-\kappa}\right) < \frac{1}{2} \quad \Rightarrow \quad \frac{1}{\rho} > 1 + \frac{1-\kappa}{\kappa}\ln(1-\kappa).$$

Clearly, if $0 < \rho < 1$ the implication follows. For the second case, note that it suffices to show that

$$2\ln\left(\frac{1}{1-\kappa}\right) > 1 + \frac{1-\kappa}{\kappa}\ln(1-\kappa),$$

which can be equivalently written as

$$\nu(\kappa) := -\ln(1-\kappa) - \frac{\kappa}{\kappa+1} > 0. \tag{B.4.1}$$

To establish (B.4.1) for $\kappa \in (0,1)$, it is enough to note that $\nu(0) = 0$ and $\nu'(\kappa) = \frac{\kappa(\kappa+3)}{(1+\kappa)^2(1-\kappa)} > 0$, so $\nu(\kappa)$ is strictly increasing, and thus positive, for $\kappa \in (0,1)$. □

B.5 Proof of Proposition 4.3

Proof. Before we provide the proof, several observations are in order. First, we assume that $\{u, w\}$ is a set of linearly independent, solutions to the homogeneous problem (B.5.2) with boundary conditions $u(1) = 1$, $w(1) = 0$, and $\frac{\mathrm{d}w}{\mathrm{d}s}|_{1^+} > 0$. We will later show in Appendix C.1 how to construct such solutions. Second, we have defined the Wronskian using derivatives with respect to the scale measure s to avoid singularities at $x = 1$. With this definition, $W(x)$ is constant as shown in Itô (2006), and positive since $W(1^+) = \frac{\mathrm{d}w}{\mathrm{d}s}(1^+) > 0$. Third, by properties of the Green's function, for any g sufficiently smooth and any $\xi \geq 1$, the function h given by

$$h(x) = \int_{\xi}^{x} g(x')G(x, x')\mathrm{d}m(x') \quad \text{solves} \quad \mathcal{L}_\delta[h](x) = g(x) \text{ for } x \in [\xi, \infty), \tag{B.5.1}$$

with initial condition $h(\xi) = h'(\xi) = 0$. In the first layer $\left[1, \frac{1}{1-\kappa}\right]$, the function f that we seek must solve the homogeneous problem

$$\mathcal{L}_\delta[f](x) = 0, \quad \text{for} \quad x \in \left(1, \frac{1}{1-\kappa}\right), \tag{B.5.2}$$

so we take

$$f(x) = \begin{cases} 0, & 0 < x < 1, \\ w(x), & 1 \leq x \leq \frac{1}{1-\kappa}. \end{cases}$$

Proceeding to the second layer, i.e., for $x \in \left[1/(1-\kappa), 1/(1-\kappa)^2\right]$, our solution satisfies $\mathcal{L}_\delta[f](x) = \delta w((1-\kappa)x)$. Thus, we utilise the Green's function to obtain f in that layer as given by:

$$f(x) = w(x) + \delta \int_{\frac{1}{1-\kappa}}^{x} w((1-\kappa)x')G(x, x')\mathrm{d}m(x'), \quad \text{for } \frac{1}{1-\kappa} \leq x \leq \frac{1}{(1-\kappa)^2}.$$

Sequentially, by adding integrals with respect to the Green's function, we find solutions on each layer, as detailed by the following arguments.

Each z_n is, by construction and (B.5.1), a solution to

$$\mathcal{L}_\delta[z_n] = \delta z_{n-1}((1-\kappa)x), \quad x \geq x_{n-1}, \quad z_n(x_{n-1}) = 0.$$

Let f_n be as in (4.13) and note that, if $x \leq x_{n-1}$, then $f_{n-1}(x) = f_n(x)$. Now, for $x \in [0, x_n]$, we have the following,

$$\begin{aligned} \mathcal{L}_\delta[f_n](x) &= \mathcal{L}_\delta[z_1](x) + \mathcal{L}_\delta[z_2](x) + \cdots + \mathcal{L}_\delta[z_n](x) \\ &= \delta z_1((1-\kappa)x) + \delta z_2((1-\kappa)x) + \cdots + \delta z_{n-1}((1-\kappa)x) \\ &= \delta f_{n-1}((1-\kappa)x) \end{aligned}$$

$$= \delta f_n((1-\kappa)x). \tag{B.5.3}$$

Each function f_n is constant on $[0,1]$, continuous $[0,x_n]$, twice-differentiable on $(1,x_n]$, and by Proposition 4.2, solves $\mathcal{A}[f_n]=0$. Moreover, from the definition of the Wronskian in (4.12), we have $\frac{\mathrm{d}f_n}{\mathrm{d}s}(1^+) = \frac{\mathrm{d}w}{\mathrm{d}s}(1^+) = W > 0$. Therefore, there exists $\epsilon > 0$ such that $f_n'(x) \neq 0$ for all $x \in (1, 1+\epsilon)$. The point-wise limit $f = \lim_{n\to\infty} f_n$ inherits all these properties, and by Lemma 4.1 is a positive and strictly increasing solution to $\mathcal{A}[f]=0$ on $(1,\infty)$.

If $\rho \ln(1/(1-\kappa)) < 1/2$, the boundedness of f follows from Lemma 4.1. To establish that f is bounded in the case $\rho < 1$, write Equation (B.5.3) as

$$\frac{\mathrm{d}}{\mathrm{d}m}\frac{\mathrm{d}f_n}{\mathrm{d}s}(x) + \delta f_n(x) = \delta f_n((1-\kappa)x), \quad 1 < x \leq x_n,$$

and integrate with respect to $\mathrm{d}m$ and $\mathrm{d}s$ to get

$$f_n(x) = \frac{\mathrm{d}f_n}{\mathrm{d}s}(1^+)\int_1^x \mathrm{d}s(x') + \delta \int_1^x \int_1^{x'} (f_n((1-\kappa)x'') - f_n(x''))\mathrm{d}m(x'')\mathrm{d}s(x'). \tag{B.5.4}$$

Since f_n is increasing, the second integral on the right side of Equation (B.5.4) is negative and we can bound $f_n(x), x \in [1, x_n]$ uniformly in n by

$$f_n(x) < \frac{\mathrm{d}f_n}{\mathrm{d}s}(1^+)\int_1^\infty \mathrm{d}s(x') = W\int_1^\infty \frac{1}{x'(x'-1)^{1-\rho}}\mathrm{d}x' = W\pi\csc(\pi\rho).$$

Since on $[1, x_n]$, $f(x) = f_n(x)$, f is bounded on $[1,\infty)$, as claimed. □

B.6 Proof of Theorem 4.1

Proof. Proposition 4.3 establishes the existence of a monotone increasing function f satisfying $\mathcal{L}_\delta[f](x) = \delta f((1-\kappa)x)$ for $x > 1$, $f(1) = 0$ and such that there exists $L = \lim_{x\to\infty} f(x) > 0$. Use the Ansatz (4.10) to define $\psi(x) = 1 - \frac{f(x)}{L}$. Then ψ solves (4.8) and (4.9) in Proposition 4.2 along with the boundary conditions in (4.11). The function ψ in (4.14) is thus the trapping probability, and

$$L = \frac{f(x')}{1-\psi(x')}, \tag{B.6.1}$$

for any $x' > 1$, which completes the proof. □

C Trapping Probability Under Proportional Insurance Coverage

C.1 Analytic Solutions

In this section, we find explicit forms for a pair u, w of linearly independent, real valued, solutions to $\mathcal{L}_\delta[h] = 0$ with $\mathcal{L}_\delta$ given by (4.7) that satisfy the conditions of Proposition 4.3. The solutions are obtained in terms of hypergeometric functions. Indeed, recall that the hypergeometric differential

equation with parameters A, B, and $C = 1$ is given by (see, e.g., Equations (2.26) and (15.5.1) from Seaborn (1991) and Abramowitz and Stegun (1972), respectively):

$$x(x-1)\frac{\mathrm{d}^2 h}{\mathrm{d}x^2} + ((A+B+1)x-1)\frac{\mathrm{d}h}{\mathrm{d}x} + ABh = 0. \tag{C.1.1}$$

Its solutions have the form

$$U(x) = x^{-A}\,{}_2F_1\left(A, A, A-B+1, \frac{1}{x}\right), \quad V(x) = x^{-B}\,{}_2F_1\left(B, B, B-A+1, \frac{1}{x}\right) \tag{C.1.2}$$

(see, e.g., Equations (15.5.7) and (15.5.8) from Abramowitz and Stegun (1972)).

Clearly, the homogeneous equation $\mathcal{L}_\delta[h](x) = 0$ with $\mathcal{L}_\delta$ as in Equation (4.7) is a hypergeometric differential equation with parameters A, B, satisfying $A+B+1 = 2-\rho$, $AB = \delta$. Hence, we define

$$A = \frac{1}{2}\left((1-\rho) + \sqrt{(1-\rho)^2 - 4\delta}\right) \quad \text{and} \quad B = \frac{1}{2}\left((1-\rho) - \sqrt{(1-\rho)^2 - 4\delta}\right). \tag{C.1.3}$$

Our task now is to use the general solutions U, V to construct real-valued solutions u, w such that $u(1) = 1$, $w(1) = 0$ and $\frac{\mathrm{d}w}{\mathrm{d}s}(1^+) > 0$, as per Proposition 4.3. The main tool for our derivation is Gauss' connection formula relating the behavior of U at $x \approx 1$ to the values of hypergeometric functions near zero, in particular Equation (A.2.1) (see NIST Digital Library of Mathematical Functions, §15.8, Equation (15.8.1)),

$$\begin{aligned} U(x) &= U_0 x^{-A}\,{}_2F_1\left(A, A, 1-\rho, 1-\frac{1}{x}\right) + U_\rho x^{-A}\left(1-\frac{1}{x}\right)^{\rho}\,{}_2F_1\left(1-B, 1-B, 1+\rho, 1-\frac{1}{x}\right) \\ &= U_0 u_0(x) + U_\rho u_\rho(x), \end{aligned} \tag{C.1.4}$$

where we have used that $A + B = 1 - \rho$, and the coefficients are,

$$U_0 = U(1) = \frac{\Gamma(1+A-B)\Gamma(\rho)}{\Gamma(1-B)^2} \quad \text{and} \quad U_\rho = \frac{\Gamma(1+A-B)\Gamma(-\rho)}{\Gamma(A)^2}. \tag{C.1.5}$$

The notation u_0 and u_ρ used in Equation (C.1.4) reflects the fact that these functions are precisely the two Frobenius branch solutions about the regular singular point $x = 1$ of Equation (C.1.1), whose indicial exponents are 0 and ρ.

The function V can be similarly decomposed with $V_0 = V(1)$ and V_ρ obtained by exchanging A by B in Equation (C.1.5). Further, note that we can apply Euler's transformation formula (A.2.2) to u_0 and obtain

$$\begin{aligned} u_0(x) &= x^{-A}\,{}_2F_1\left(A, A, A+B, 1-\frac{1}{x}\right) = x^{-A}\left(\frac{1}{x}\right)^{B-A}\,{}_2F_1\left(B, B, A+B, 1-\frac{1}{x}\right) \\ &= x^{-B}\,{}_2F_1\left(B, B, 1-\rho, 1-\frac{1}{x}\right), \end{aligned} \tag{C.1.6}$$

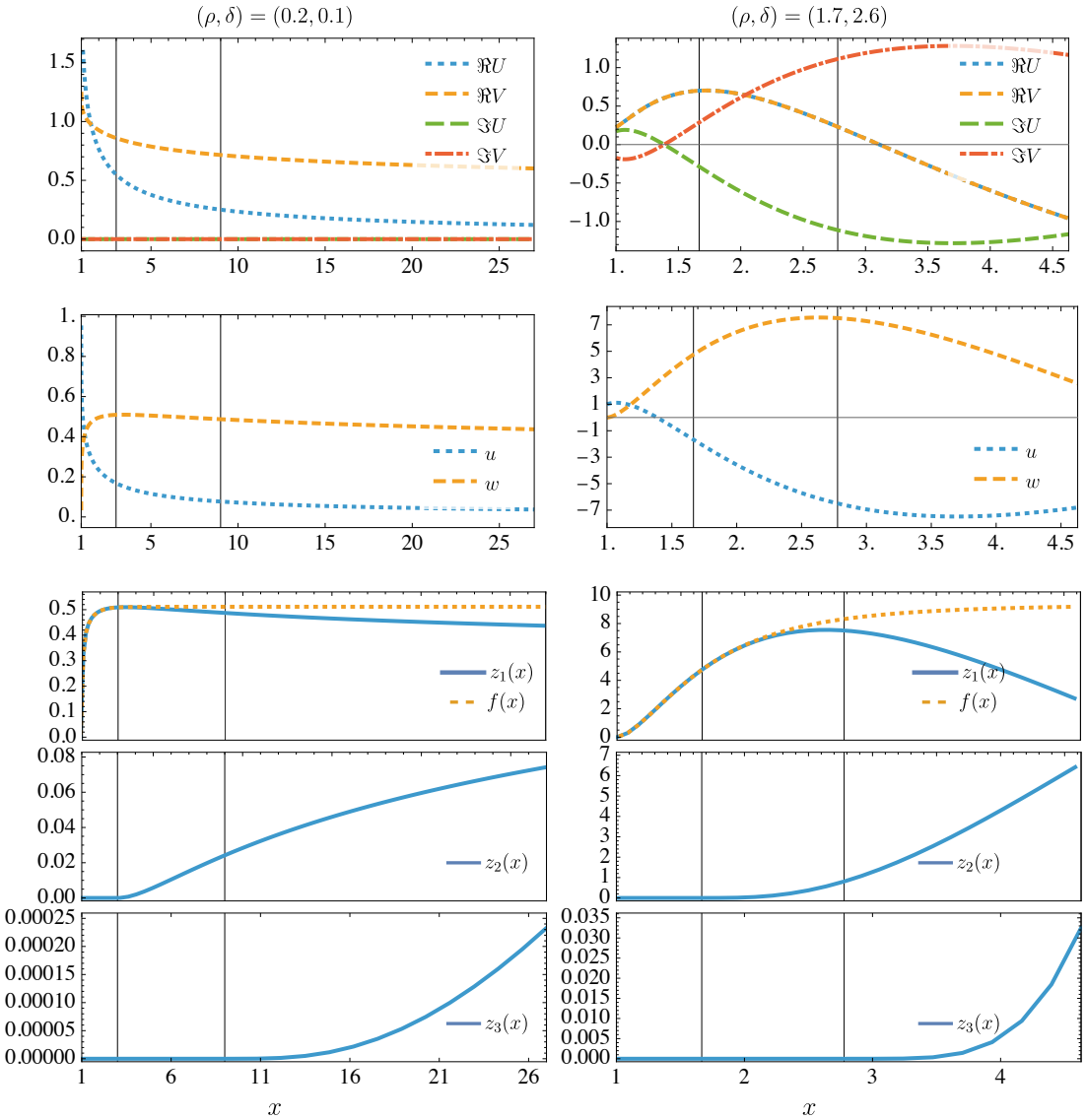


Figure 10: Illustration of the analytical method for the calculation of the trapping probability under proportional insurance as described in Section 4.2. The left column shows calculations for $(\rho, \delta) = (0.2, 0.1)$ which corresponds to $(\kappa, \lambda) = (0.67, 0.1)$ and A, B being real-valued. The right column corresponds to the case labeled with the letter f in Figure 4. All plots have $x \in [1, x_3]$ as the independent variable, with the values of x_1 and x_2 marked with vertical lines. The functions U, V in the first row are computed via Equation (C.1.2). The second row shows the corresponding real solutions u, w constructed in Propositions 6.1 and 6.2, respectively. The bottom three rows show the functions z_1, z_2, z_3 and f constructed as per Proposition 4.3, and computed via numerical integration.

and similarly for u_ρ. This establishes two things. First, the function $V(x)$ can be decomposed as $V(x) = V_0 u_0(x) + V_\rho u_\rho(x)$. Secondly, in the case of imaginary A and B, since $A = \bar{B}$, we have, by properties of the hypergeometric function and Equation (C.1.6),

$$\overline{u_0(x)} = x^{-\bar{A}}\,{}_2F_1\left(\bar{A}, \bar{A}, 1-\rho, 1-\frac{1}{x}\right) = x^{-B}\,{}_2F_1\left(B, B, 1-\rho, 1-\frac{1}{x}\right) = u_0(x).$$

Namely, u_0 is real-valued. And so is u_ρ by an analogous calculation. Real-valued and complex cases of U and V are plotted in the first row of Figure 10.

From the definition of A, B in Equation (C.1.3) and U, V in Equation (C.1.2) we can see that the method of construction of the final set $\{u, w\}$ of linearly independent solutions to the homogeneous problem depends on the value of the discriminant $(1-\rho)^2 - 4\delta$. Also, the qualitative behaviour of the solution depends on the sign of $\rho - 1$. These regimes need to be compounded with the net profit condition in Equation (4.6).

Proposition 6.1 constructs the solutions u and w in the case where U and V are real.

Proposition 6.1. *Suppose $(1-\rho)^2 > 4\delta$, corresponding to real solutions U and V. Then the net profit condition holds only if $0 < \rho < 1$. In this regime, we define*

$$u(x) = \frac{U(x)}{U(1)}, \quad v(x) = \frac{V(x)}{V(1)}, \quad \text{and} \quad w(x) = v(x) - u(x).$$

Then u, w are real valued, linearly independent solutions to the homogeneous problem (B.5.2) satisfying $u(1) = 1, w(1) = 0$, $\frac{\mathrm{d}w}{\mathrm{d}s}|_{1^+} > 0$, and $\frac{\mathrm{d}u}{\mathrm{d}s}|_{1^+}$ is finite.

Proof. We first prove that if $\rho > 1$, the net profit condition forces the discriminant to be negative. Write the left hand side of Equation (4.6) as $N_\rho(\delta)$ for $\delta > 0$ and note that N_ρ is strictly decreasing on $(0, \infty)$ for any $\rho > 0$. Evaluating at $\delta_0 := \frac{1}{4}(\rho-1)^2$ and using that $\ln(1+x) \le x$ for all $x \ge 0$, we can bound

$$N_\rho(\delta_0) = \rho - \frac{(\rho-1)^2}{2}\ln\left(1 + \frac{2}{\rho-1}\right) \ge \rho - (\rho - 1) = 1.$$

Since N_ρ is decreasing, we have that $N_\rho(\delta) > 1$ for $\delta < \delta_0$. Namely, the net profit condition is violated whenever $\rho > 1$ and $(1-\rho)^2 > 4\delta$, as claimed.

Regarding u, v and w, we note that clearly $u(1) = v(1) = 1$ and $w(1) = 0$. It suffices to prove $\frac{\mathrm{d}w}{\mathrm{d}s}(1^+) > 0$. From continuation formula in Equation (C.1.4),

$$w(x) = \left(\frac{V_\rho}{V_0} - \frac{U_\rho}{U_0}\right) u_\rho(x) = \left(\frac{V_\rho}{V_0} - \frac{U_\rho}{U_0}\right)\left(1 - \frac{1}{x}\right)^\rho (1 + O(x-1)) \quad \text{as } x \downarrow 1,$$

which, since $0 < \rho < 1$, has infinite right derivative at $x = 1$. However, differentiating by $\mathrm{d}s = \frac{\mathrm{d}x}{x(x-1)^{1-\rho}}$, we obtain:

$$\frac{\mathrm{d}w}{\mathrm{d}s}(x) = \frac{\rho}{x^\rho}\left(\frac{V_\rho}{V_0} - \frac{U_\rho}{U_0}\right) + O(x-1), \quad \text{as } x \downarrow 1.$$

Therefore, taking the limit and using Euler's reflection formula yields,

$$\begin{aligned}\frac{\mathrm{d}w}{\mathrm{d}s}(1^+) =& \rho\left(\frac{V_\rho}{V_0} - \frac{U_\rho}{U_0}\right) = \frac{\rho\Gamma(-\rho)}{\Gamma(\rho)}\left(\frac{\Gamma(1-A)^2}{\Gamma(B)^2} - \frac{\Gamma(1-B)^2}{\Gamma(A)^2}\right)\\ =& \frac{-\Gamma(1-\rho)}{\Gamma(\rho)}\frac{(\Gamma(1-A)\Gamma(A))^2 - (\Gamma(1-B)\Gamma(B))^2}{(\Gamma(A)\Gamma(B))^2}\\ =& \frac{\pi^3}{(\Gamma(A)\Gamma(B)\Gamma(\rho))^2}\frac{\sin^2(\pi B) - \sin^2(\pi A)}{(\sin(\pi A)\sin(\pi B))^2\sin(\pi\rho)}\\ =& \frac{\pi^3}{(\Gamma(A)\Gamma(B)\Gamma(\rho))^2}\frac{\sin(\pi(A+B))\sin(\pi(A-B))}{(\sin(\pi A)\sin(\pi B))^2\sin(\pi\rho)}.\end{aligned}$$

Since $\sin(\pi(A+B)) = \sin(\pi(1-\rho)) = \sin(\pi\rho)$ we obtain:

$$\frac{\mathrm{d}w}{\mathrm{d}s}(1^+) = \frac{\pi^3}{(\Gamma(A)\Gamma(B)\Gamma(\rho)\sin(\pi A)\sin(\pi B))^2}\sin(\pi\sqrt{(1-\rho)^2 - 4\delta}),$$

which is positive because $0 < \rho < 1$.

The finiteness of $\frac{\mathrm{d}u}{\mathrm{d}s}(1^+)$ follows similar arguments since, by (C.1.4), $u(x) = U_0 u_0(x) + U_\rho u_\rho(x)$, where $u_0(x)$ is clearly differentiable at 1. □

In the case with $(1-\rho)^2 < 4\delta$, both U and V take complex values and $U(x) = \overline{V(x)}$. Then, the linearly independent solutions $\{u, w\}$ can be obtained by normalising the real and imaginary parts of U. This is detailed in Proposition 6.2. The contrast between the cases with real and complex solution is illustrated in Figure 10.

Proposition 6.2. *Suppose $(1-\rho)^2 < 4\delta$, let U as in Equation (C.1.2), and define*

$$v(x) = \begin{cases}\dfrac{\Re U(x)}{\Re U(1)}, & \text{if } \Re U(1)\Im U(1) > 0\\ \dfrac{\Im U(x)}{\Im U(1)}, & \text{if } \Re U(1)\Im U(1) < 0\end{cases}, \quad u(x) = \begin{cases}\dfrac{\Im U(x)}{\Im U(1)}, & \text{if } \Re U(1)\Im U(1) > 0\\ \dfrac{\Re U(x)}{\Re U(1)}, & \text{if } \Re U(1)\Im U(1) < 0\end{cases},$$

and $w(x) = v(x) - u(x)$. Then u, w are real valued, linearly independent solutions to the homogeneous problem (B.5.2) satisfying $u(1) = 1, w(1) = 0$, $\frac{\mathrm{d}w}{\mathrm{d}s}|_{1^+} > 0$, and $\frac{\mathrm{d}u}{\mathrm{d}s}|_{1^+}$ is finite.

Proof. Again, it suffices to prove $\frac{\mathrm{d}w}{\mathrm{d}s}(1^+) > 0$. From the representation of $U(x)$ in Equation (C.1.4) and recalling that u_0 and u_ρ are real-valued, and that $U_0 = U(1)$, we can write

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Re U(x)}{\Re U(1)} - \frac{\Im U(x)}{\Im U(1)}\right) = \left(\frac{\Re U_\rho}{\Re U_0} - \frac{\Im U_\rho}{\Im U_0}\right)\frac{\mathrm{d}u_\rho}{\mathrm{d}s}(x).$$

But $u_\rho(x) = (1-x)^\rho(1+O(x-1))$ as $x \downarrow 1$, then

$$\frac{\mathrm{d}u_\rho}{\mathrm{d}s}(x) = x(x-1)^{1-\rho}\left(\rho(x-1)^{\rho-1} + O((x-1)^\rho)\right) = \rho + O(x-1) \quad \text{as } x \downarrow 1.$$

We thus have established that,

$$\frac{\mathrm{d}}{\mathrm{d}s}\left(\frac{\Re U(x)}{\Re U(1)} - \frac{\Im U(x)}{\Im U(1)}\right)\bigg|_{x=1^+} = \rho\left(\frac{\Re U_\rho}{\Re U_0} - \frac{\Im U_\rho}{\Im U_0}\right). \tag{C.1.7}$$

Expanding U_ρ and U_0 from Equation (C.1.3) in polar form, we can write

$$\begin{aligned}\frac{\rho U_\rho}{U_0} = \rho\frac{\Gamma(-\rho)}{\Gamma(\rho)}\left|\frac{\Gamma(A+\rho)}{\Gamma(A)}\right|^2 e^{i\theta} &= -\frac{\Gamma(1-\rho)}{\Gamma(\rho)}\left|\frac{\Gamma(A+\rho)}{\Gamma(A)}\right|^2 e^{i\theta} \\ &= -\frac{\pi}{\Gamma(\rho)^2\sin(\pi\rho)}\left|\frac{\Gamma(A+\rho)}{\Gamma(A)}\right|^2 e^{i\theta},\end{aligned}$$

where $\theta = 2(\arg\Gamma(A+\rho) - \arg\Gamma(A))$. It follows that the derivative in Equation (C.1.7) is

$$\frac{\Re(\rho U_\rho)}{\Re U_0} - \frac{\Im(\rho U_\rho)}{\Im U_0} = \left|\frac{\Gamma(A+\rho)}{\Gamma(A)\Gamma(\rho)}\right|^2 \frac{\pi|U_0|^2}{\Re U_0 \Im U_0}\frac{\sin(\theta)}{\sin(\pi\rho)}.$$

Hence, in order to prove $\frac{\mathrm{d}w}{\mathrm{d}s}(1^+) > 0$ for all cases, we just need to prove that $\frac{\sin(\theta)}{\sin(\pi\rho)} > 0$. First, note that $A + \rho = 1 - \bar{A}$, therefore

$$\begin{aligned}\theta &= 2(\arg\Gamma(1-\bar{A}) - \arg\Gamma(A)) = 2(\arg\overline{\Gamma(1-A)} - \arg\Gamma(A)) \\ &= -2(\arg(\Gamma(1-A)\Gamma(A)) = -2\arg\left(\frac{\pi}{\sin(\pi A)}\right) \\ &= 2\arg(\sin(\pi A)).\end{aligned}$$

Writing $\pi A = \alpha + i\beta$ with $\alpha = \frac{\pi}{2}(1-\rho)$, $\beta = \frac{\pi}{2}\sqrt{4\delta - (1-\rho)^2} > 0$, we can express

$$\sin(\pi A) = \sin\alpha\cosh\beta + i\cos\alpha\sinh\beta,$$

and using properties of trigonometric functions, we derive

$$\begin{aligned}\sin(\theta) = \sin(2\arg(\sin(\pi A)) &= \frac{2\sin\alpha\cos\alpha\cosh\beta\sinh\beta}{\sin^2\alpha\cosh^2\beta + \cos^2\alpha\sinh^2\beta} \\ &= \frac{\sin(2\alpha)\sinh(2\beta)}{\cosh(2\beta) - \cos(2\alpha)}.\end{aligned}$$

Noting that by definition of α, $\sin(2\alpha) = \sin(\pi\rho)$ and $\cos(2\alpha) = -\cos(\pi\rho)$, then

$$\frac{\sin(\theta)}{\sin(\pi\rho)} = \frac{\sinh(2\beta)}{\cosh(2\beta) + \cos(\pi\rho)},$$

which is always positive since $\beta > 0$ implies $\sinh(2\beta) > 0$ and $\cosh(2\beta) > 1$.

The same argument used in the proof of Proposition 6.2 to show that $\frac{\mathrm{d}u}{\mathrm{d}s}(1^+)$ is finite can be applied here since u is, also in this case, given by (C.1.4). □

References

[1] Abramowitz, M. and I. A. Stegun (1972). *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*. Washington, D.C.: U.S. Department of Commerce.

[2] Asmussen, S. and H. Albrecher (2010). *Ruin Probabilities*. Singapore: World Scientific.

[3] Azaïs, R. and A. Genadot (2015). Semi-Parametric Inference for the Absorption Features of a Growth-Fragmentation Model. *TEST 24*(2), 341–360.

[4] Azariadis, C. and J. Stachurski (2005). Poverty Traps. In P. Aghion and S. N. Durlauf (Eds.), *Handbook of Economic Growth*, Volume 22 of *Handbooks in Economics*, Chapter 5, pp. 295–384. North-Holland: Elsevier.

[5] Barrett, C. B., T. Garg, and L. McBride (2016). Well-Being Dynamics and Poverty Traps. *Annual Review of Resource Economics 8*(1), 303–327.

[6] Bowles, S., S. N. Durlauf, and K. Hoff (2006). *Poverty Traps*. New Jersey: Princeton University Press.

[7] Carter, M. R. and S. A. Janzen (2018). Social Protection in the Face of Climate Change: Targeting Principles and Financing Mechanisms. *Environment and Development Economics 23*(3), 369–389.

[8] Chantarat, S., A. G. Mude, C. B. Barrett, and C. G. Turvey (2017). Welfare Impacts of Index Insurance in the Presence of a Poverty Trap. *World Development 94*, 119–138.

[9] Churchill, C. (2006). *Protecting the Poor: A Microinsurance Compendium — Volume I*. Geneva, Switzerland: International Labour Organization (ILO).

[10] Churchill, C. and M. Matul (2012). *Protecting the Poor: A Microinsurance Compendium — Volume II*. Geneva, Switzerland: International Labour Organization (ILO).

[11] Cramér, H. (1930). *On the Mathematical Theory of Risk*. Stockholm: Centraltryckeriet.

[12] Dasgupta, P. (1997). Nutritional Status, the Capacity for Work, and Poverty Traps. *Journal of Econometrics 77*(1), 5–37.

[13] Davis, M. H. A. (1984). Piecewise-Deterministic Markov Processes: A General Class of Non-Diffusion Stochastic Models. *Journal of the Royal Statistical Society: Series B (Methodological) 46*(3), 353–388.

[14] Davis, M. H. A. (1993). *Markov Models & Optimization*, Volume 49 of *Monographs on Statistics and Applied Probability*. New Delhi: Chapman & Hall.

[15] Eling, M., S. Pradhan, and J. T. Schmit (2014). The Determinants of Microinsurance Demand. *The Geneva Papers on Risk and Insurance. Issues and Practice 39*(2), 224–263.

[16] Flores-Contró, J. M., K. Henshaw, S.-H. Loke, S. Arnold, and C. Constantinescu (2025). Subsidizing Inclusive Insurance to Reduce Poverty. *North American Actuarial Journal 29*(1), 44–73.

[17] Henshaw, K. (2022). *Mathematical Perspectives on Insurance for Low-Income Populations*. The University of Liverpool (United Kingdom).

[18] Henshaw, K., M. Mandjes, and C. D. Constantinescu (2023). A Stochastic Model of Group Wealth Responses to Insurance Mechanisms in Low-Income Communities. *Scandinavian Actuarial Journal*, 1–28.

[19] Ikegami, M., M. R. Carter, C. B. Barrett, and S. A. Janzen (2018). Poverty Traps and the Social Protection Paradox. In C. B. Barrett, M. R. Carter, and J. P. Chavas (Eds.), *The Economics of Poverty Traps*, Chapter 6, pp. 223–256. Chicago: University of Chicago Press.

[20] Itô, K. (2006). *Essentials of Stochastic Processes*. Providence, Rhode Island: American Mathematical Society.

[21] Janzen, S. A., M. R. Carter, and M. Ikegami (2021). Can Insurance Alter Poverty Dynamics and Reduce the Cost of Social Protection in Developing Countries? *Journal of Risk and Insurance 88*(2), 293–324.

[22] Kovacevic, R. M. and G. C. Pflug (2011). Does Insurance Help to Escape the Poverty Trap? — A Ruin Theoretic Approach. *Journal of Risk and Insurance 78*(4), 1003–1027.

[23] Kovacevic, R. M. and W. Semmler (2021). Poverty Traps and Disaster Insurance in a Bi-level Decision Framework. In J. L. Haunschmied, R. M. Kovacevic, W. Semmler, and V. M. Veliov (Eds.), *Dynamic Economic Problems with Regime Switches*, Volume 25 of *Dynamic Modeling and Econometrics in Economics and Finance*, Chapter 3, pp. 57–83. Switzerland: Springer Nature.

[24] Kraay, A. and D. McKenzie (2014). Do Poverty Traps Exist? Assessing the Evidence. *Journal of Economic Perspectives 28*(3), 127–148.

[25] Kristensson, G. (2010). *Second Order Differential Equations: Special Functions and their Classification*. Springer Science & Business Media.

[26] Lampaert, I. and J. Walhin (2005). On the Optimality of Proportional Reinsurance. *Scandinavian Actuarial Journal 2005*(3), 225–239.

[27] Liao, P., X. Zhou, and Q. Fan (2020). Does Agricultural Insurance Help Farmers Escape the Poverty Trap? Research Based on Multiple Equilibrium Models. *The Geneva Papers on Risk and Insurance - Issues and Practice 45*(1), 203–223.

[28] Lundberg, F. (1903). *I. Approximerad Framställning af Sannolikhetsfunktionen II. Återförsäkring af Kollektivrisker. Akademisk Afhandling*. Uppsala: Almqvist & Wiksell.

[29] Mandjes, M. and O. Boxma (2023). *The Cramér–Lundberg Model and Its Variants: A Queueing Perspective*. Cham, Switzerland: Springer Nature Switzerland AG.

[30] Matsuyama, K. (2008). Poverty Traps. In S. N. Durlauf and L. E. Blume (Eds.), *The New Palgrave Dictionary of Economics*. London: Palgrave Macmillan.

[31] Paulsen, J. and H. K. Gjessing (1997). Ruin Theory with Stochastic Return on Investments. *Advances in Applied Probability 29*(4), 965–985.

[32] Seaborn, J. B. (1991). *Hypergeometric Functions and Their Applications*. New York: Springer-Verlag.

[33] United Nations (2015). *Transforming Our World: The 2030 Agenda for Sustainable Development*. Number A/RES/70/1. New York: United Nations.

[34] Will, M., J. Groeneveld, K. Frank, and B. Müller (2021). Informal Risk-Sharing Between Smallholders May Be Threatened By Formal Insurance: Lessons From a Stylized Agent-Based Model. *PLOS ONE 16*(3), 1–18.

Kira Henshaw
Institute for Financial and Actuarial Mathematics
Department of Mathematical Sciences
University of Liverpool
Liverpool, United Kingdom
E-Mail: khenshaw@liverpool.ac.uk
ORCiD: https://orcid.org/0000-0002-9018-7993

Jorge Ramirez
Departamento de Matemáticas
Universidad Nacional de Colombia
Sede Medellín
Medellín, Colombia
E-Mail: jmramirezo@unal.edu.co
ORCiD: https://orcid.org/0000-0002-0402-7484

José Miguel Flores-Contró
Institute of Statistics, Biostatistics and Actuarial Science – ISBA
Louvain Institute of Data Analysis and Modeling – LIDAM
Catholic University of Louvain
Louvain-la-Neuve, Belgium
E-Mail: jose.flores@uclouvain.be
ORCiD: https://orcid.org/0000-0001-9332-8811

Sooie-Hoe Loke
Department of Mathematical Sciences
Middle Tennessee State University
Murfreesboro, Tennessee
United States of America
E-Mail: sooiehoe.loke@mtsu.edu
ORCiD: https://orcid.org/0000-0001-9021-1562

Enrique A. Thomann
Department of Mathematics
Oregon State University
Corvallis, Oregon
United States of America
E-Mail: thomann@math.oregonstate.edu
ORCiD: https://orcid.org/0000-0003-0972-1659

Corina Constantinescu
Institute for Financial and Actuarial Mathematics
Department of Mathematical Sciences
University of Liverpool
Liverpool, United Kingdom
E-Mail: c.constantinescu@liverpool.ac.uk
ORCiD: https://orcid.org/0000-0002-5219-3022