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LFIN

Voie du Roman Pays 34, L1.03.01 B-1348 Louvain-la-Neuve

Tel (32 10) 47 43 04

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On the Combination of Naive and Mean-Variance Portfolio Strategies*

Nathan Lassance[1] Rodolphe Vanderveken[1] Frédéric Vrins[1,2]

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Abstract

We study how to best combine the sample mean-variance portfolio with the naive equally weighted portfolio to optimize out-of-sample performance. We show that the seemingly natural convexity constraint that Tu and Zhou (2011) impose—the two combination coefficients must sum to one—is undesirable because it severely constrains the allocation to the risk-free asset relative to the unconstrained portfolio combination. However, we demonstrate that relaxing the convexity constraint inflates estimation errors in combination coefficients, which we alleviate using a shrinkage estimator of the unconstrained combination scheme. Empirically, the constrained combination outperforms the unconstrained one in a range of generally small degrees of risk aversion, but severely deteriorates otherwise. In contrast, the shrinkage unconstrained combination enjoys the best of both strategies and performs consistently well for all levels of risk aversion.

Keywords: portfolio optimization, parameter uncertainty, estimation risk, equally weighted portfolio, portfolio constraints.
JEL Classification: G11, G12.

*R. Vanderveken is corresponding author. Affiliations: (1) LIDAM/LFIN, UCLouvain, Belgium, (2) Decision science department, HEC Montréal, Canada. Email addresses: `nathan.lassance@uclouvain.be`, `rodolphe.vanderveken@uclouvain.be`, `frederic.vrins@uclouvain.be`. The authors thank Victor DeMiguel, Alberto Martìn-Utrera, Majeed Simaan, Valeri Sokolovski, Xiaolu Wang, Morten Wilke, Guofu Zhou, and participants at the Belgian Financial Research Forum, 16th International Conference on Computational and Financial Econometrics, IFABS 2022 conference, CORE brown bag seminar, HEC Montréal seminar, and Louvain Finance seminar for helpful comments. This work was supported by the Fonds de la Recherche Scientifique (F.R.S.-FNRS) under Grant Number J.0115.22 and T.0221.22, and by the Belgian Federal Science Policy Office under Grant Number ARC 18-23/089.

1 Introduction

The sample mean-variance (SMV) portfolio of Markowitz (1952) performs notoriously poorly out of sample because it is highly sensitive to estimation errors in the mean and covariance matrix of asset returns (Barroso and Saxena, 2021). One solution to alleviate this problem is to *combine* the SMV portfolio with another portfolio less sensitive to estimation risk.

In this paper, we study how to best combine the SMV portfolio with the naive equally weighted (EW) portfolio so as to optimize out-of-sample performance. Specifically, we consider the portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}^\star + \kappa_2 \boldsymbol{w}_{ew}$, where $\hat{\boldsymbol{w}}^\star = \frac{1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$ is the SMV portfolio obtained from a sample of size T, $\boldsymbol{w}_{ew} = \mathbf{1}/N$ is the EW portfolio, and $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ are the combination coefficients. As in Kan and Zhou (2007), we calibrate $\boldsymbol{\kappa}$ to optimize the performance, measured by the expected out-of-sample utility, $\mathbb{E}\Big[\hat{\boldsymbol{w}}'(\boldsymbol{\kappa})\boldsymbol{\mu} - \frac{\gamma}{2}\hat{\boldsymbol{w}}'(\boldsymbol{\kappa})\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})\Big]$. Several combinations with naive strategies can be used to improve the performance of the SMV portfolio, the most famous ones being the EW and the sample global-minimum-variance (SGMV) portfolios. Although SGMV is used in many studies such as Kan and Zhou (2007), DeMiguel et al. (2015), and Kan et al. (2022), we focus on EW in this paper for several reasons.[1] First, DeMiguel et al. (2009) show that, albeit naive, the EW portfolio often compares favourably to mean-variance models. Second, unlike the SGMV portfolio that depends on the inverse covariance matrix, the EW portfolio is not subject to any estimation error. Thus, our results indicate that when the number of assets is sizable, combining with EW delivers superior performance and lower turnover than combining with SGMV. Third, we demonstrate theoretically and empirically that the SMV and EW portfolios have a smaller out-of-sample return correlation than SMV and SGMV, and thus deliver larger diversification gains when combined together. Finally, in addition to these performance-related arguments, when combining the SMV and EW portfolios the relevant literature is the paper by Tu and Zhou (2011). However, we show that Tu and Zhou (2011) combine the SMV and EW portfolios in

[1] Although we focus on combining the SMV and EW portfolios, in Section C.3 of the Supplementary Material we also derive counterparts of our main results for the combination of the SMV and SGMV portfolios.

a suboptimal way, which calls for further analysis of this particular combination.

We contribute to the portfolio combination literature in four ways. First, we analyze the impact of the *convexity constraint* that Tu and Zhou (2011) impose on combination coefficients, i.e., $\kappa_1 + \kappa_2 = 1$, which gives rise to the *constrained strategy.* This constraint suffices for the constrained strategy to outperform the SMV and EW portfolios, due to the concavity of expected utility. However, we show that the constrained strategy *underperforms* the third underlying fund—the risk-free asset—once the risk-aversion coefficient γ exceeds a given threshold. This can be explained because the convexity constraint prevents the constrained strategy to invest in the risk-free asset directly; it can only do so via the SMV portfolio.[2] In this case, a higher exposure to the risk-free asset comes at the cost of a higher exposure to the tangent portfolio, which is not desirable for highly risk-averse investors.

To address this issue, we propose an *unconstrained strategy* that combines the SMV and EW portfolios without imposing the convexity constraint. We theoretically compare the properties of the portfolios $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^c)$ and $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)$, which are the *oracle* constrained and unconstrained strategies, i.e., the combination coefficients $\boldsymbol{\kappa}^c$ and $\boldsymbol{\kappa}^u$ use the true values of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$. We show that the unconstrained strategy *always outperforms* not just the SMV and EW portfolios, but also the risk-free asset and the constrained strategy of Tu and Zhou (2011). One striking shortcoming of the constrained strategy $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^c)$ is that *it always overinvests in the SMV portfolio* relative to $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)$, and this effect is substantial for highly risk-averse investors.[3] Another drawback we prove theoretically is that for most degrees of risk aversion, the weights $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^c)$ are more extreme (i.e., have higher L_2 norm) than $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)$.

Our second contribution is to study the impact of the convexity constraint when the combination coefficients must be estimated and become $\hat{\boldsymbol{\kappa}}^c$ and $\hat{\boldsymbol{\kappa}}^u$. Although relaxing the convexity constraint helps improve the performance of the oracle portfolio combination, it inflates the sensitivity of combination coefficients to estimation errors in mean returns. We

[2] In contrast with EW that never invests in the risk-free asset, SMV does so whenever $\gamma \neq \mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$.

[3] For the dataset of 25 size-and-book-to-market portfolios (25SBTM) spanning July 1926 to December 2022, a sample size $T = 120$ months, and a risk aversion $\gamma = 10$, the constrained and unconstrained strategies allocate 57% and 20% to SMV, respectively.

show analytically and via simulations that this larger sensitivity makes the unconstrained strategy $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^u)$ *underperform* the constrained strategy $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^c)$ in an interval of risk aversion γ. As a remedy, we introduce a *shrinkage estimator* of the unconstrained strategy. We derive the shrinkage intensity minimizing the mean-squared error and show that it shrinks intensively exactly for those levels of risk aversion where estimation errors hurt performance most. This estimator offers an appealing tradeoff between the constrained and unconstrained strategies.

Our third contribution is to demonstrate *theoretically* the robustness of our results to the i.i.d. normality assumption that is used to derive the optimal portfolio combinations. Specifically, we consider a model in which asset returns are *elliptically* distributed. We exploit results in El Karoui (2010, 2013) to derive the expected out-of-sample utility loss incurred when asset returns are elliptically distributed but the investor relies on the normal combination coefficients, and we show that this loss is typically very small.[4] These results give theoretical foundation to the empirical findings of Tu and Zhou (2004), who conclude that "[...] the certainty–equivalent losses associated with ignoring fat tails are small. This suggests that the normality assumption works well in evaluating portfolio performance for a mean-variance investor."

Our fourth contribution is to compare empirically the performance of the three competing approaches to combine the SMV and EW portfolios: constrained, unconstrained, and shrinkage unconstrained. Moreover, we benchmark them against relevant combination strategies in Kan and Zhou (2007), Kan et al. (2022), and Bodnar et al. (2023), and against the EW and SGMV portfolios. We compare the strategies in terms of out-of-sample utility net of transaction costs, across three datasets of characteristic-sorted portfolios, one dataset of industry-sorted portfolios, and two datasets of 50 and 100 U.S. individual stocks.

The empirical observations match our theory. The constrained strategy can outperform the unconstrained strategy for a range of generally small γ's but quickly deteriorates other-

[4] Suppose asset returns are t distributed with six degrees of freedom, i.e., a realistic excess kurtosis of three for monthly returns. For the 25SBTM dataset, $N/T = 0.2$, and a risk aversion $\gamma = 5$, the unconstrained combination coefficients are $(\kappa_1, \kappa_2) = (0.19, 0.29)$, while those under normality are $(\kappa_1, \kappa_2) = (0.20, 0.29)$. The expected out-of-sample utility loss by relying on the normal coefficients is a mere 0.007% per month.

wise, in which case it often underperforms the risk-free asset and the two-fund rule of Kan and Zhou (2007) that does not invest in the EW portfolio. In contrast, the unconstrained strategy performs consistently well, always outperforms the risk-free asset, and also the two-fund rule in most cases. Moreover, it delivers less turnover and transaction costs. Finally, the shrinkage unconstrained strategy closely matches the performance of either the constrained or the unconstrained strategy depending on which one outperforms given the risk aversion γ.

The two larger datasets of 50 and 100 individual stocks uncover additional insights on portfolio combination. First, combining the SMV portfolio with EW rather than with SGMV delivers much better performance and lower turnover, because EW suffers no estimation error unlike SGMV. This is true even if one uses shrinkage estimates of the covariance matrix. Second, one might deem it difficult to outperform the EW portfolio for such datasets, particularly when $N = 100$. Still, our proposed unconstrained strategies do not lose much relative to EW when $T = 120$, and actually outperform EW when $T = 240$, unlike the constrained strategy. Third, the value of the risk-aversion coefficient γ for which the constrained and unconstrained strategies coincide is larger than that for the characteristic and industry-sorted datasets, and thus the unconstrained and shrinkage unconstrained strategies also deliver a large improvement relative to the constrained strategy when γ is small and equal to one.[5]

2 The portfolio combination problem

Let $\boldsymbol{r}_t$ be the $N \times 1$ vector of asset excess returns at time t. As in Kan and Zhou (2007), Tu and Zhou (2011), and Ao et al. (2019), we assume that $\boldsymbol{r}_t \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and $\boldsymbol{\Sigma}$ is positive definite. In Section 5, we relax the normality assumption and only assume that $\boldsymbol{r}_t$ is elliptically distributed. At time T, the investor collects a sample of T i.i.d. returns, $(\boldsymbol{r}_1, \dots, \boldsymbol{r}_T)$. The investor then chooses her portfolio $\boldsymbol{w} = (w_1, ..., w_N)'$ on the risky assets, the weight on the risk-free asset being $w_0 = 1 - \mathbf{1}'\boldsymbol{w}$ with $\mathbf{1}$ a vector of ones. We assume that $N + 4 < T < \infty$.[6]

[5] A detailed literature review is available in Section A of the Supplementary Material.

[6] In Section C.2 of the Supplementary Material, we derive theoretical results in the high-dimensional asymptotic regime, i.e., $N, T \to \infty$ and $N/T \to \rho \in (0, 1)$, which is often considered in recent literature; see,

When $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are known, the optimal *mean-variance portfolio* maximizes utility,

$$U(\boldsymbol{w}) = \boldsymbol{w}'\boldsymbol{\mu} - \frac{\gamma}{2}\boldsymbol{w}'\boldsymbol{\Sigma}\boldsymbol{w}, \tag{1}$$

where $\gamma > 0$ is the risk-aversion coefficient. Note that the risk-free asset has a utility of zero. The well-known solution to (1) and its utility are

$$\boldsymbol{w}^\star = \frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} \quad \text{and} \quad U(\boldsymbol{w}^\star) = U^\star = \frac{\theta^2}{2\gamma}, \tag{2}$$

where $\theta = \sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$ is the maximum Sharpe ratio. Note that by investing in the mean-variance portfolio $\boldsymbol{w}^\star$, one invests in two funds: the fully invested tangent portfolio $\boldsymbol{w}_{tan} = \boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/(\boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu})$ and the risk-free asset. Specifically,

$$\boldsymbol{w}^\star = \frac{\gamma_{tan}}{\gamma}\boldsymbol{w}_{tan} \quad \text{and} \quad w_0^\star = 1 - \boldsymbol{1}'\boldsymbol{w}^\star = 1 - \frac{\gamma_{tan}}{\gamma} \quad \text{with} \quad \gamma_{tan} = \boldsymbol{1}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}. \tag{3}$$

2.1 Parameter uncertainty and expected out-of-sample utility

The moments $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ are unknown in practice. As in Kan and Zhou (2007), Tu and Zhou (2011), and Kan et al. (2022), we rely on sample estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, which are

$$\hat{\boldsymbol{\mu}} = \frac{1}{T}\sum_{t=1}^{T}\boldsymbol{r}_t \quad \text{and} \quad \widehat{\boldsymbol{\Sigma}} = \frac{1}{T-N-2}\sum_{t=1}^{T}(\boldsymbol{r}_t - \hat{\boldsymbol{\mu}})(\boldsymbol{r}_t - \hat{\boldsymbol{\mu}})'. \tag{4}$$

The *sample mean-variance (SMV)* portfolio, which exploits $\hat{\boldsymbol{\mu}}$ and $\widehat{\boldsymbol{\Sigma}}$, is defined as

$$\hat{\boldsymbol{w}}^\star = \frac{1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}. \tag{5}$$

Under i.i.d. normality, $\widehat{\boldsymbol{\Sigma}}^{-1}$ is unbiased and independent of $\hat{\boldsymbol{\mu}}$, and thus the SMV portfolio is unbiased: $\mathbb{E}[\hat{\boldsymbol{w}}^\star] = \boldsymbol{w}^\star$. To account for estimation errors, we measure the performance of a sample portfolio $\hat{\boldsymbol{w}}$ via its *expected out-of-sample utility* (EU) as in Kan and Zhou (2007): [7]

$$EU(\hat{\boldsymbol{w}}) = \mathbb{E}[U(\hat{\boldsymbol{w}})] = \mathbb{E}\Big[\hat{\boldsymbol{w}}'\boldsymbol{\mu} - \frac{\gamma}{2}\hat{\boldsymbol{w}}'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}\Big]. \tag{6}$$

e.g., Bodnar et al. (2018), Ao et al. (2019), Ledoit and Wolf (2020), and Bodnar et al. (2023).

[7] In Section C.7 of the Supplementary Material, we also study the expected out-of-sample Sharpe ratio.

2.2 Combining the SMV and EW portfolios

We study the combination of the SMV portfolio $\hat{\boldsymbol{w}}^\star$ with the EW portfolio $\boldsymbol{w}_{ew} = \mathbf{1}/N$:

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}^\star + \kappa_2 \boldsymbol{w}_{ew}, \tag{7}$$

where $\boldsymbol{\kappa} = (\kappa_1, \kappa_2)$ is the vector of *combination coefficients*. We denote $\mu_{ew} = \boldsymbol{w}_{ew}'\boldsymbol{\mu}$ and $\sigma_{ew}^2 = \boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\boldsymbol{w}_{ew}$. We make the mild assumption that $\mu_{ew} \geq 0$.[8] Moreover, we define γ_{ew} as[9]

$$\gamma_{ew} = \mu_{ew}/\sigma_{ew}^2, \tag{8}$$

and ψ^2 as the difference between the maximum squared Sharpe ratio and that of EW:

$$\psi^2 = \theta^2 - \theta_{ew}^2 \geq 0. \tag{9}$$

Our objective is to determine $\boldsymbol{\kappa}$ that maximizes the EU. If the investor wants to outperform the SMV and EW portfolios, it is sufficient to impose a convexity constraint,

$$\kappa_1 + \kappa_2 = 1, \tag{10}$$

and optimize a single combination coefficient. This is what Tu and Zhou (2011) consider, and they show that the resulting portfolio combination outperforms both the SMV and EW portfolios, due to the concavity of the EU. In the next section, we discuss why imposing the convexity constraint (10) is actually unnecessary and study the *unconstrained* combination.

In Section C.3 of the Supplementary Material we also consider the combination of the SMV and SGMV portfolios in Kan and Zhou (2007). Moreover, in Section C.5, we consider the combination of the SMV, EW, and SGMV portfolios, but find that adding the SGMV portfolio to the mix only delivers a small EU gain in theory when the optimal combination coefficients are known, and thus often underperforms empirically when they are estimated.

[8] We make this assumption to simplify the exposition of our theory. The results that need this assumption can also be extended to the case in which $\mu_{ew} < 0$.

[9] The parameter γ_{ew} has the following interpretation: an investor with risk aversion γ investing in the EW portfolio and the risk-free asset fully allocates her wealth to the EW portfolio if and only if $\gamma = \gamma_{ew}$.

3 The convexity constraint and oracle estimators

The convexity constraint (10) is necessary when combining fully invested portfolios as in Bodnar et al. (2018), Kan et al. (2022), and Bodnar et al. (2023), as it ensures that the combination is fully invested too. However, the SMV portfolio is *not* fully invested: $\mathbf{1}'\hat{\boldsymbol{w}}^\star = \mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}/\gamma \neq 1$, in general. Therefore, combining the SMV and EW portfolios amounts to investing in three funds: the sample tangent portfolio, the EW portfolio, *and* the risk-free asset. But, under constraint (10), a *single* coefficient κ_1 controls the weights on the three funds, the second one being $\kappa_2 = 1 - \kappa_1$. Reducing the weight κ_1 on the SMV portfolio to reduce exposure to estimation risk can only be done by increasing the weight $\kappa_2 = 1 - \kappa_1$ on the EW portfolio, which increases the exposure to risky assets because EW is fully invested. Thus, the investor cannot disentangle her exposure to financial risk and estimation risk. To optimally balance the three funds, we propose to relax the convexity constraint (10).

3.1 Combination coefficients and underlying funds

In the next proposition, we derive the oracle constrained combination coefficients in Tu and Zhou (2011). Our presentation has two benefits relative to Tu and Zhou: we provide a closed-form expression for the coefficients, which makes their implementation easier and more efficient, and we also provide an explicit expression for the EU they deliver, which is useful for future results. We then derive similar results for the unconstrained combination.[10]

Proposition 1. *Concerning the combination of the SMV and EW portfolios in* (7)*:*

1. *The expected out-of-sample utility is*

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = \frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_{ew} - \frac{\gamma}{2}\left(\frac{\kappa_1^2}{\gamma^2}\left(\theta^2 + d\right) + \kappa_2^2\sigma_{ew}^2 + \frac{2\kappa_1\kappa_2}{\gamma}\mu_{ew}\right), \tag{11}$$

   where

$$d = cN/T + (c-1)\theta^2 \quad \textit{with} \quad c = \frac{(T-N-2)(T-2)}{(T-N-1)(T-N-4)} \geq 1. \tag{12}$$

[10] All the proofs are available in Section D of the Supplementary Material.

2. *The oracle combination coefficients* $\boldsymbol{\kappa}^c$ *maximizing* (11) *under the convexity constraint* (10) *are*

$$\kappa_1^c = \frac{\psi^2 + \ell^2}{\psi^2 + \ell^2 + d} \in [0,1] \quad \textit{and} \quad \kappa_2^c = 1 - \kappa_1^c \in [0,1], \tag{13}$$

where $\ell = \sigma_{ew}(\gamma - \gamma_{ew})$*. Moreover, the oracle unconstrained combination coefficients* $\boldsymbol{\kappa}^u$ *maximizing* (11) *are*[11]

$$\kappa_1^u = \frac{\psi^2}{\psi^2 + d} \in [0,1] \quad \textit{and} \quad \kappa_2^u = \frac{\gamma_{ew}}{\gamma}(1 - \kappa_1^u) \in \mathbb{R}, \tag{14}$$

which are equal to $\boldsymbol{\kappa}^c$ *in* (13) *if and only if* $\gamma = \gamma_{ew}$ *in* (8).

3. *The expected out-of-sample utility of the constrained and unconstrained portfolio combinations, denoted* $\hat{\boldsymbol{w}}^c = \hat{\boldsymbol{w}}(\boldsymbol{\kappa}^c)$ *and* $\hat{\boldsymbol{w}}^u = \hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)$*, are*

$$EU(\hat{\boldsymbol{w}}^c) = U^\star - \frac{d}{2\gamma}\kappa_1^c, \tag{15}$$

$$EU(\hat{\boldsymbol{w}}^u) = U^\star - \frac{d}{2\gamma}\kappa_1^u. \tag{16}$$

Both combinations outperform the SMV and EW portfolios, the unconstrained combination always outperforms the risk-free asset, but the constrained combination underperforms the risk-free asset if and only if $d > \theta^2$ *and*

$$\gamma > \gamma_{ew}\left(1 + \sqrt{1 + \frac{\theta^4/\theta_{ew}^2}{d - \theta^2}}\right) = \gamma_{neg}. \tag{17}$$

Moreover, γ_{neg} *decreases with* d *defined in* (12).

Several comments are in order. First, the parameter d measures estimation risk: it increases with N, decreases with T, and is equal to zero as $T \to \infty$, in which case both the constrained and unconstrained strategies correspond to the SMV portfolio.

Second, the convexity constraint impacts the combination coefficients on the SMV and EW portfolios: $\boldsymbol{\kappa}^c \neq \boldsymbol{\kappa}^u$ except if $\gamma = \gamma_{ew}$. We compare these coefficients in detail in Section

[11] Kan and Wang (2023, Section 2.3.2) use similar coefficients in a different context, that of combining a benchmark factor model with a set of test assets.

C.1 of the Supplementary Material. In summary, the constrained strategy *always overinvests in the SMV portfolio* in the sense that $\kappa_1^c \geq \kappa_1^u$. Moreover, it also overinvests in the EW portfolio, $\kappa_2^c \geq \kappa_2^u$, when $\gamma \in [\gamma_{ew}, (\theta^2 + d)/\mu_{ew}]$, which is typically a large interval. Finally, the constrained strategy underinvests in the risk-free asset relative to the unconstrained strategy when $\gamma \geq \gamma_{ew}$, and leverages less otherwise. These differences can be substantial. For the 25SBTM dataset, $\gamma = 5 > \gamma_{ew} = 1.86$, and $T = 120$, the constrained strategy allocates 22% to the tangent portfolio, 71% to the EW portfolio, and 7% to the risk-free asset. In contrast, the unconstrained strategy invests much more in the risk-free asset (55%) and much less in the tangent and EW portfolios (15% and 30%).

Third, because the convexity constraint severely constrains the allocation to the risk-free asset, the constrained strategy may underperform the risk-free asset if estimation risk is sufficient ($d > \theta^2$) and risk aversion too ($\gamma > \gamma_{neg}$). This problem does not arise in the unconstrained strategy that can fully invest in the risk-free asset if needed by setting $\boldsymbol{\kappa} = \mathbf{0}$. This deficiency of the constrained strategy is particularly notable because the two conditions under which it underperforms the risk-free asset can easily be met.[12]

Fourth, Proposition 1 suggests two cases in which outperforming the EW portfolio is likely difficult in practice for the unconstrained strategy. First, when ψ^2 is small, which happens when EW provides a Sharpe ratio close to the maximum, e.g., because there is not much cross-sectional variation in mean returns.[13] Second, when estimation risk d is large. In these two cases, the weight κ_1^u on the SMV portfolio is close to zero, and the unconstrained strategy is close to the optimal combination of EW and the risk-free asset given by $\boldsymbol{\kappa} = (0, \gamma_{ew}/\gamma)$. Therefore, the oracle unconstrained strategy delivers a small improvement:

$$EU(\hat{\boldsymbol{w}}^u) - EU(\gamma_{ew}/\gamma \boldsymbol{w}_{ew}) = \frac{\psi^2}{2\gamma}\kappa_1^u \tag{18}$$

decreases when d increases and ψ^2 decreases. Thus, it may even underperform when the combination coefficients are estimated.

[12]For the 25SBTM dataset and $T = 120$, $d > \theta^2$ holds as soon as $N > 8$, and $\gamma_{neg} = 5.27$.

[13]Provided $\boldsymbol{\Sigma}$ is a constant diagonal matrix, $\psi^2 = 0$ if and only if $\boldsymbol{\mu} = c\mathbf{1}$ for some $c \in \mathbb{R}$.

3.2 Outperformance in expected out-of-sample utility

By relaxing the convexity constraint, the unconstrained strategy delivers a larger EU than the constrained strategy. In the next proposition, we quantify and decompose this gain.

Proposition 2. *The unconstrained strategy always delivers a larger expected out-of sample utility (EU) than the constrained strategy:*

$$EU(\hat{\boldsymbol{w}}^u) - EU(\hat{\boldsymbol{w}}^c) = \frac{d}{2\gamma}(\kappa_1^c - \kappa_1^u) \geq 0, \tag{19}$$

with equality if and only if $\gamma = \gamma_{ew}$ *or* $\gamma = \infty$. *Moreover, the gain in* (19) *increases with estimation risk* d *and can be decomposed as* $\Delta_{smv} + \Delta_{ew} + \Delta_{cov}$, *where:*

1. Δ_{smv} *is the difference in EU coming from the exposure to the SMV portfolio:*[14]

$$\Delta_{smv} = EU(\kappa_1^u \hat{\boldsymbol{w}}^\star) - EU(\kappa_1^c \hat{\boldsymbol{w}}^\star) = \frac{d^2\ell^2\left[\ell^2(\psi^2 + d - \theta_{ew}^2) - 2\theta_{ew}^2(d+\psi^2)\right]}{2\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}. \tag{20}$$

2. Δ_{ew} *is the difference in EU coming from the exposure to the EW portfolio:*[15]

$$\begin{aligned}\Delta_{ew} &= EU(\kappa_2^u \boldsymbol{w}_{ew}) - EU(\kappa_2^c \boldsymbol{w}_{ew}) \\ &= \frac{d\ell(\theta^2 + d - \mu_{ew}\gamma)}{2\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\left[d\ell(\theta^2+d-\mu_{ew}\gamma) - 2\theta_{ew}\psi^2(\psi^2+\ell^2+d)\right].\end{aligned} \tag{21}$$

3. Δ_{cov} *is the difference in EU coming from the covariance between the return of the SMV and EW portfolios:*

$$\begin{aligned}\Delta_{cov} &= \mathbb{E}[(\kappa_1^c \hat{\boldsymbol{w}}^\star)'\boldsymbol{\Sigma}(\kappa_2^c \boldsymbol{w}_{ew})] - \mathbb{E}[(\kappa_1^u \hat{\boldsymbol{w}}^\star)'\boldsymbol{\Sigma}(\kappa_2^u \boldsymbol{w}_{ew})] \\ &= \frac{d\ell\theta_{ew}}{\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\left[(\psi^2+d)^2(\psi^2+\ell^2) - \ell\theta_{ew}(\psi^2(\psi^2+\ell^2) - d^2)\right].\end{aligned} \tag{22}$$

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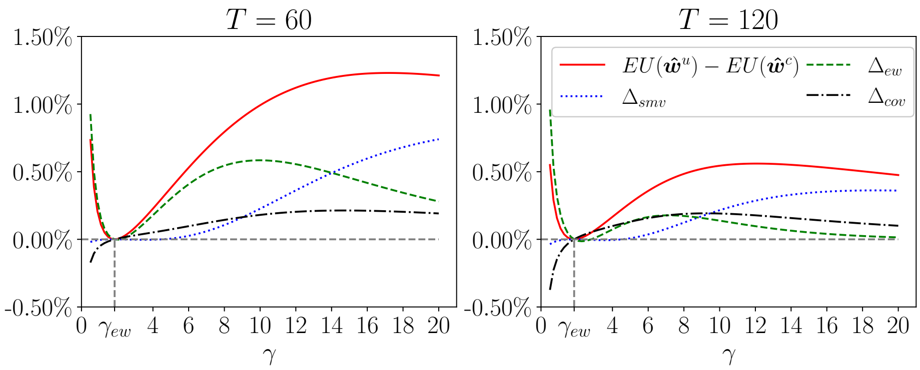
[14] $\Delta_{smv} \leq 0$ if $\psi^2 + d - \theta_{ew}^2 \leq 0$, and else $\Delta_{smv} \geq 0$ if $\gamma \geq \gamma_{ew}\left(1 + \sqrt{\frac{2(\psi^2+d)}{\psi^2+d-\theta_{ew}^2}}\right)$, and $\Delta_{smv} < 0$ otherwise.

[15] If d is small enough such that $d^2(\psi^2+d) - 8\theta_{ew}^2\psi^2(2\psi^2+d) \leq 0$, then $\Delta_{ew} \leq 0$ if $\gamma \in [\gamma_{ew}, (\theta^2+d)/\mu_{ew}]$, and else $\Delta_{ew} > 0$. Otherwise, $\Delta_{ew} \geq 0$ if either $\gamma \leq \gamma_{ew}$, $\gamma \geq (\theta^2+d)/\mu_{ew}$, or

$$\gamma \in \left[\frac{d^2 + d(\theta^2+\theta_{ew}^2) + 4\theta_{ew}^2\psi^2 \pm \sqrt{(\psi^2+d)(d^2(\psi^2+d) - 8\theta_{ew}^2\psi^2(2\psi^2+d))}}{2\mu_{ew}(2\psi^2+d)}\right].$$

Figure 1: Expected out-of-sample utility of constrained versus unconstrained strategy



Notes. This figure depicts the gain in expected out-of-sample utility (red solid) delivered by the unconstrained combination strategy $\hat{\boldsymbol{w}}^u$ relative to the constrained strategy $\hat{\boldsymbol{w}}^c$, and its decomposition in the three components given in Proposition 2: Δ_{smv} (dotted blue), Δ_{ew} (dashed green) and Δ_{cov} (dash-dotted black). The figure is constructed by calibrating the population vector of means and covariance matrix of excess asset returns from monthly returns on the 25 portfolios of firms sorted on size and book-to-market spanning July 1926 to December 2022. We consider sample sizes $T = 60$ (left panel) and $T = 120$ (right panel) and depict the utility gain as a function of the risk-aversion coefficient γ between 0.5 and 20.

Proposition 2 shows that the EU gain delivered by the unconstrained strategy increases with estimation risk d. This is because the constrained strategy overinvests in the SMV portfolio, which is the only portfolio in the combination that is exposed to estimation risk.

We illustrate Proposition 2 in Figure 1 using the 25SBTM dataset. The total EU gain delivered by the unconstrained strategy can be substantial when γ departs from $\gamma_{ew} = 1.86$. When $T = 120$ and γ varies between 0.5 and 20, the monthly EU gain in (19) goes up to around 0.5-0.6%, i.e., 6-7% annually, and it is even larger when $T = 60$. Moreover, when γ is around γ_{ew} and is of small or medium magnitude, most of the EU gain comes from taking an optimal exposure to the EW portfolio. This is because while the unconstrained coefficient κ_2^u in (14) is proportional to $1/\gamma$ and thus is large when γ is small, the constrained coefficient κ_2^c in (13) is bounded above by $\max_\gamma \kappa_2^c = d/(\psi^2 + d)$ and thus is too small. In contrast, when γ gets large, most of the EU gain comes from taking an optimal exposure to the SMV portfolio. This is because as γ increases, both (κ_2^u, κ_2^c) go to zero but $\kappa_1^u = \psi^2/(\psi^2 + d)$ is independent of γ while κ_1^c in (14) goes to one and thus is too large. The third component, Δ_{cov}, is such that the three components sum to (19). It is negative for $\gamma \leq \gamma_{ew}$ due to the larger exposure to the EW portfolio in the unconstrained strategy, and is positive otherwise.

3.3 Outperformance relative to the optimal two-fund strategy

We show that the constrained strategy does not fully benefit from investing in the EW portfolio. Indeed, it can underperform the unconstrained two-fund strategy of Kan and Zhou (2007) that drops the EW portfolio from the combination in (7), which is defined as

$$\hat{\boldsymbol{w}}^{2f} = \kappa^{2f} \hat{\boldsymbol{w}}^{\star} \quad \text{with} \quad \kappa^{2f} = \frac{\theta^2}{\theta^2 + d}. \tag{23}$$

Specifically, we show that while the unconstrained three-fund strategy $\hat{\boldsymbol{w}}^u$ always outperforms the two-fund strategy $\hat{\boldsymbol{w}}^{2f}$, the constrained three-fund strategy $\hat{\boldsymbol{w}}^c$ does so only if $\gamma \leq 2\gamma_{ew}$.

Proposition 3. *The unconstrained three-fund strategy $\hat{\boldsymbol{w}}^u$ always delivers a larger expected out-of-sample utility than the unconstrained two-fund strategy $\hat{\boldsymbol{w}}^{2f}$. In contrast, the constrained three-fund strategy $\hat{\boldsymbol{w}}^c$ does so if and only if $\gamma \leq 2\gamma_{ew}$.*

3.4 Less extreme portfolio weights

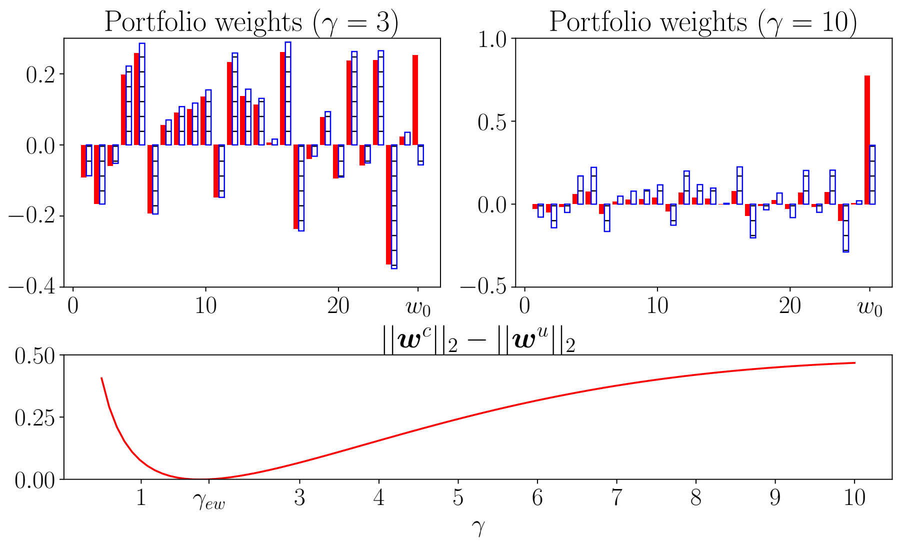
Finally, another benefit of the unconstrained strategy is that it delivers on average less extreme weights on risky assets, i.e., with smaller norm. Large weights are undesirable because they often result in larger turnover and transaction costs, which we confirm in Section 6.

Proposition 4. *Let $\gamma \in [\gamma_{ew}, (\theta^2 + d)/\mu_{ew}]$. Then, the expected portfolio weights of the unconstrained strategy have a smaller L_2-norm than those of the constrained strategy: $||\boldsymbol{w}^u||_2 \leq ||\boldsymbol{w}^c||_2$, where $\boldsymbol{w}^u = \mathbb{E}[\hat{\boldsymbol{w}}^u]$ and $\boldsymbol{w}^c = \mathbb{E}[\hat{\boldsymbol{w}}^c]$.*

This result can be explained because, as discussed in Section 3.1, the constrained strategy always allocates more weight to SMV relative to the unconstrained strategy, and also to EW when $\gamma \in [\gamma_{ew}, (\theta^2 + d)/\mu_{ew}]$. This interval is wide in practice; it is equal to $[1.86, 43.1]$ for the 25SBTM dataset and $T = 120$. Moreover, this is only a sufficient condition: the unconstrained strategy may deliver a smaller L_2-norm when γ is outside this interval, too.

We illustrate Proposition 4 in Figure 2 for the 25SBTM dataset and $T = 120$. When $\gamma = 3$, the unconstrained strategy allocates a positive weight to the risk-free asset, whereas

Figure 2: Portfolio weights and L_2-norms of constrained versus unconstrained strategy



Notes. The two upper panels depict the expected portfolio weights allocated to the 25 risky assets and the risk-free asset (w_0), for risk-aversion coefficients of $\gamma = 3$ and $\gamma = 10$. The red solid bars depict the expected weights in the unconstrained strategy, and the blue striped bars those in the constrained strategy. The bottom panel depicts the difference between the L_2-norm of the unconstrained and constrained portfolio weights on the risky assets as a function of γ. The figure is constructed by calibrating the population vector of means and covariance matrix of excess asset returns from monthly returns on the 25 portfolios of firms sorted on size and book-to-market spanning July 1926 to December 2022, and using a sample size $T = 120$.

the constrained strategy allocates a negative weight. When $\gamma = 10$, the L_2-norm is much smaller for the unconstrained strategy, $0.25 = ||\boldsymbol{w}^u||_2 < ||\boldsymbol{w}^c||_2 = 0.72$, and consequently it allocates much more weight to the risk-free asset, $0.77 = w_0^u > w_0^c = 0.35$. Finally, the constrained strategy delivers a larger L_2-norm for any γ in the considered range $\gamma \leq 10$.

4 Shrinkage estimator of unconstrained combination

The results in Section 3 hold for the *oracle* combination coefficients. To assess the impact of the convexity constraint in practice, we need to compare the EU of the constrained and unconstrained strategies when $\boldsymbol{\kappa}^c$ and $\boldsymbol{\kappa}^u$ are fully estimated. However, they are complex functions of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, and thus, this is not feasible analytically. Still, what we want to capture is the *difference* in the way estimation errors impact the constrained and unconstrained coefficients, and we show that the main part of this difference is treatable analytically.

We proceed as follows. First, we show that the estimation errors are essentially driven by the scaling coefficient μ_{ew} in κ_2^u.Second, we study analytically the impact of estimation errors resulting from μ_{ew} in κ_2^u on the EU of the unconstrained strategy, and propose a shrinkage estimator of the latter. Third, we compare via simulations the EU of the fully estimated constrained, unconstrained, and shrinkage unconstrained strategies.

4.1 Main impact of estimation errors in combination coefficients

In this section, we provide simulation and theoretical evidence that the main impact of dropping the convexity constraint on estimation risk is that κ_2^u becomes much more sensitive to the errors in combination coefficients. This is because the constrained coefficients in (13) are both between zero and one whereas, for the unconstrained coefficients in (14), it is only the case for κ_1^u while κ_2^u is unbounded.

In Section B of the Supplementary Material, we assess this difference using simulations. We simulate normal asset returns with parameters $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ and repeatedly estimate the combination coefficients. We find first that errors in $\boldsymbol{\mu}$ have a much larger impact on the $\boldsymbol{\kappa}$'s than errors in $\boldsymbol{\Sigma}$. And second, while κ_1^c, κ_2^c, and κ_1^u are not very sensitive to estimation errors in $\boldsymbol{\mu}$, κ_2^u is highly impacted by these errors because it is directly proportional to $\boldsymbol{\mu}$ via μ_{ew}.

In Section B of the Supplementary Material, we also give theoretical complement to these simulation results. Specifically, we derive in closed form the estimation errors in combination coefficients as a function of the estimation errors in mean returns, $\boldsymbol{\varepsilon} = \hat{\boldsymbol{\mu}} - \boldsymbol{\mu}$. These expressions allow us to confirm that $\boldsymbol{\varepsilon}$ has a much larger impact on $\hat{\kappa}_2^u$ than on the other coefficients, and that the impact of $\boldsymbol{\varepsilon}$ on $\hat{\kappa}_2^u$ comes mostly from the proportionality to $\hat{\mu}_{ew}$.

4.2 Alleviating errors in unconstrained combination coefficients

Following Section 4.1, we assess the impact of estimation risk in the combination coefficients by computing the EU loss of the unconstrained strategy when only μ_{ew} is estimated in κ_2^u:

$$\tilde{\kappa}_2^u = \frac{\hat{\mu}_{ew}}{\gamma \sigma_{ew}^2}(1 - \kappa_1^u) \quad \text{with} \quad \hat{\mu}_{ew} = \boldsymbol{w}_{ew}' \hat{\boldsymbol{\mu}}. \tag{24}$$

Proposition 5. *Let the combination coefficient κ_2^u be estimated by $\tilde{\kappa}_2^u$ in* (24)*. Then,*

1. *The expected out-of-sample utility of the estimated unconstrained strategy satisfies*

$$EU(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)) - EU(\hat{\boldsymbol{w}}^u) = -\frac{1-(\kappa_1^u)^2}{2\gamma T} \leq 0, \tag{25}$$

where $EU(\hat{\boldsymbol{w}}^u) = EU(\hat{\boldsymbol{w}}(\kappa_1^u, \kappa_2^u))$ is the utility of the oracle strategy in (16)*.*

2. *Let $N \geq 2$. Then, the oracle constrained strategy delivers a larger expected out-of-sample utility than the estimated unconstrained strategy, $EU(\hat{\boldsymbol{w}}^c) \geq EU(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u))$, if and only if γ belongs to the following interval:*

$$\gamma \in \left[\gamma_{ew} \pm \sigma_{ew}^{-1}\sqrt{\frac{(\psi^2+d)(2\psi^2+d)}{dT(\psi^2+d)-(2\psi^2+d)}}\right]. \tag{26}$$

Moreover, the length of this interval decreases with estimation risk d.

Proposition 5 shows that when γ is sufficiently close to γ_{ew} and thus the EU gain provided by the oracle unconstrained strategy in (19) is not large, the constrained strategy can deliver a larger EU due to the impact of estimation errors in μ_{ew} on κ_2^u. However, as d increases and the EU loss of the oracle constrained strategy in (19) is larger, the interval (26) is narrower.

The length of the interval in (26) is not negligible; for the 25SBTM dataset, the interval is $\gamma \in [1.86 \pm 1.64]$ when $T = 120$. Therefore, the constrained strategy is more desirable than the unconstrained one for most small values of γ, but the unconstrained strategy remains preferable when γ is medium or large. This result is expected because when γ increases, the constrained strategy overinvests in risky assets, as we discuss in Section 3.1.

To alleviate estimation errors, we propose to shrink $\tilde{\kappa}_2^u$ in (24) toward the target $1-\kappa_1^u$,

$$\tilde{\kappa}_2^u(\delta) = (1-\delta)\tilde{\kappa}_2^u + \delta(1-\kappa_1^u). \tag{27}$$

This target is justified because $1-\kappa_1^u$ is less sensitive to estimation errors since it is bounded between zero and one, and because it is equal to the true κ_2^u when $\gamma = \gamma_{ew}$, which is the γ for which estimation errors hurt most relative to $\boldsymbol{\kappa}^c$ as shown in Proposition 5.

Proposition 6. *The following holds about $\tilde{\kappa}_2^u(\delta)$ in (27):*

1. *The shrinkage intensity δ minimizing the mean squared error of $\tilde{\kappa}_2^u(\delta)$ is*

$$\delta^\star = \frac{1}{1+T\ell^2} \in [0,1]. \tag{28}$$

Moreover, $\delta^\star = 1$ for $\gamma = \gamma_{ew}$ and $\delta^\star \to 0$ as $T \to \infty$ and $\gamma \to \infty$.

2. *The optimal shrinkage estimator of κ_2^u is*

$$\tilde{\kappa}_2^u(\delta^\star) = \frac{1 + \frac{\hat{\mu}_{ew}}{\gamma\sigma_{ew}^2} T\ell^2}{1+T\ell^2}(1-\kappa_1^u), \tag{29}$$

and is less sensitive to $\hat{\mu}_{ew}$ than $\tilde{\kappa}_2^u$:

$$\frac{\frac{\partial}{\partial\hat{\mu}_{ew}}\tilde{\kappa}_2^u(\delta^\star)}{\frac{\partial}{\partial\hat{\mu}_{ew}}\tilde{\kappa}_2^u} = 1 - \delta^\star \leq 1. \tag{30}$$

3. *$\tilde{\kappa}_2^u(\delta^\star)$ delivers a larger expected out-of-sample utility than $\tilde{\kappa}_2^u$:*

$$EU(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u(\delta^\star))) - EU(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)) = \frac{\delta^\star(1-(\kappa_1^u)^2)}{2\gamma T} \geq 0. \tag{31}$$

The optimal shrinkage intensity $\delta^\star$ behaves as desired, i.e., it shrinks $\tilde{\kappa}_2^u$ more intensively when γ is closer to γ_{ew} and when T decreases. As a result, $\tilde{\kappa}_2^u(\delta^\star)$ is less sensitive to $\hat{\mu}_{ew}$ than $\tilde{\kappa}_2^u$. Moreover, when $\delta^\star$ is known, $\tilde{\kappa}_2^u(\delta^\star)$ delivers a larger EU than $\tilde{\kappa}_2^u$.

In Section C.3 of the Supplementary Material, we derive a similar shrinkage estimator for the unconstrained combination of the SMV and SGMV portfolios in Kan and Zhou (2007). We compare the performance of this shrinkage unconstrained strategy to that of the plain unconstrained strategy and find that it delivers an improved EU.

The theoretical results in this section only account for the impact of estimation error in μ_{ew} on κ_2^u for analytical tractability. Nonetheless, in order to *evaluate* the EU of the novel shrinkage unconstrained strategy and compare it to that of the constrained and unconstrained strategies, it is necessary to consider the *feasible* estimators of these strategies, i.e., with fully estimated combination coefficients. This is what we do in the next section.

4.3 Out-of-sample performance of feasible portfolio combinations

We now compare the EU of the three different combinations of the SMV and EW portfolios: constrained, unconstrained, and shrinkage unconstrained. The corresponding oracle combination coefficients are defined in part 2 of Proposition 1 and part 2 of Proposition 6, and they depend on the following functions of the population $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$: μ_{ew}, σ^2_{ew}, θ^2, ψ^2.

As in Tu and Zhou (2011), we estimate μ_{ew} and σ^2_{ew} by plugging the maximum-likelihood estimates of $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, i.e.,

$$\hat{\mu}_{ew} = \boldsymbol{w}'_{ew}\hat{\boldsymbol{\mu}} \quad \text{and} \quad \hat{\sigma}^2_{ew} = \boldsymbol{w}'_{ew}\widehat{\boldsymbol{\Sigma}}_{ml}\boldsymbol{w}_{ew}, \quad \text{where} \quad \widehat{\boldsymbol{\Sigma}}_{ml} = \frac{T-N-2}{T}\widehat{\boldsymbol{\Sigma}}. \tag{32}$$

Moreover, we estimate θ^2 and ψ^2 using the adjusted estimators in Kan and Zhou (2007) and Kan and Wang (2023) because the plug-in estimators are highly biased, and we denote them as $\hat{\theta}^2$ and $\hat{\psi}^2$. The feasible constrained, unconstrained, and shrinkage unconstrained combination strategies are obtained by plugging the estimates $(\hat{\mu}_{ew}, \hat{\sigma}^2_{ew}, \hat{\theta}^2, \hat{\psi}^2)$ in (13), (14), (27) and (28) to compute $\hat{\kappa}^c_1, \hat{\kappa}^u_1, \hat{\kappa}^u_2$ and $\hat{\kappa}^u_2(\hat{\delta}^\star)$:

$$\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^c) = \hat{\kappa}^c_1\hat{\boldsymbol{w}}^\star + (1-\hat{\kappa}^c_1)\boldsymbol{w}_{ew}, \tag{33}$$

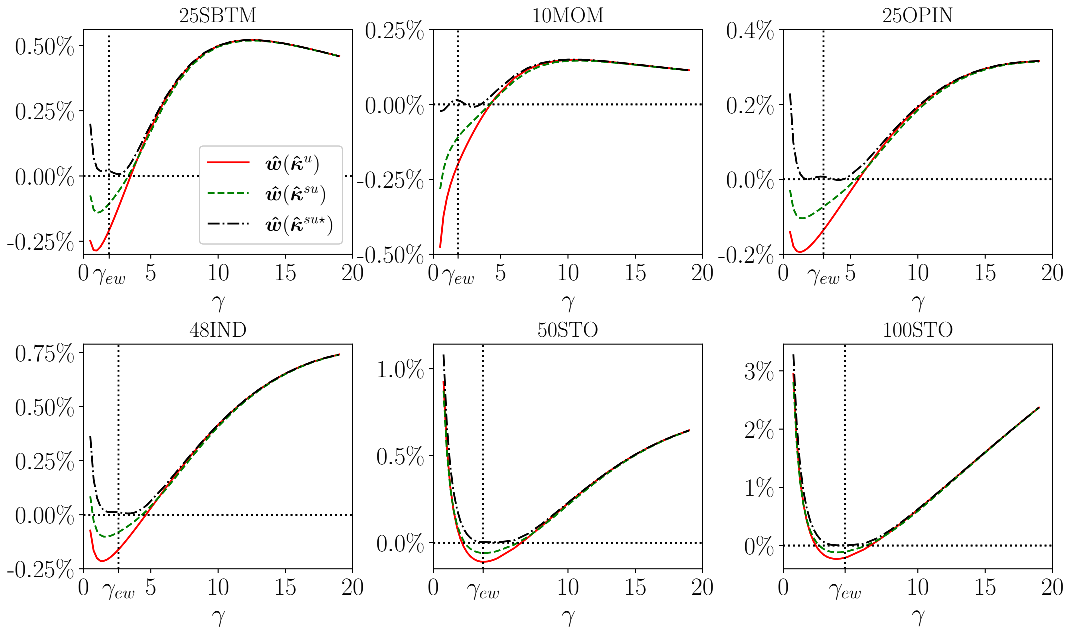
$$\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^u) = \hat{\kappa}^u_1\hat{\boldsymbol{w}}^\star + \hat{\kappa}^u_2\boldsymbol{w}_{ew}, \tag{34}$$

$$\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su}) = \hat{\kappa}^u_1\hat{\boldsymbol{w}}^\star + \hat{\kappa}^u_2(\hat{\delta}^\star)\boldsymbol{w}_{ew}. \tag{35}$$

We evaluate the EU of the three feasible portfolio strategies in (33)–(35) via Monte Carlo simulations. Specifically, we calibrate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ to the six datasets of monthly excess returns we consider in the empirical analysis; see Section 6.1. Then, for each dataset, we simulate $M = 100{,}000$ times T monthly return observations for each of the N assets, from which we compute the feasible strategies $\hat{\boldsymbol{w}}_m(\hat{\boldsymbol{\kappa}}^c_m)$, $\hat{\boldsymbol{w}}_m(\hat{\boldsymbol{\kappa}}^u_m)$, and $\hat{\boldsymbol{w}}_m(\hat{\boldsymbol{\kappa}}^{su}_m)$, with $m = 1, \ldots, M$. Finally, the EU of the three feasible strategies is approximated by

$$EU(\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}})) \approx \frac{1}{M}\sum_{m=1}^{M}\hat{\boldsymbol{w}}_m(\hat{\boldsymbol{\kappa}}_m)'\boldsymbol{\mu} - \frac{\gamma}{2}\hat{\boldsymbol{w}}_m(\hat{\boldsymbol{\kappa}}_m)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_m(\hat{\boldsymbol{\kappa}}_m). \tag{36}$$

Figure 3: Expected out-of-sample utility relative to the feasible constrained combination with simulated i.i.d. normal data



Notes. This figure depicts the difference between the EU of three different strategies relative to the EU of the feasible constrained strategy $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{c})$ in (33): (i) the feasible unconstrained strategy $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{u})$ in (34) (solid red), (ii) the feasible shrinkage unconstrained strategy $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su})$ in (35) (dashed green), and (iii) the unfeasible shrinkage unconstrained strategy $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su\star})$ in (36) that uses the oracle shrinkage intensity $\delta^{\star}$ (dash-dotted black). We consider a sample size $T = 120$ and report the EU difference as a function of the risk-aversion coefficient γ for all six datasets described in Section 6.1. 50STO and 100STO are 500 randomly drawn investment sets of 50 and 100 U.S. individual stocks from a total pool of 235 stocks. For these two datasets, we report the average EU across the 500 random selections. We obtain the EU using 100,000 Monte Carlo simulations assuming i.i.d. normal returns whose population moments are calibrated from the whole sample of each dataset.

In Figure 3, we set $T = 120$ and depict as a function of γ the difference between the EU of $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{u})$ and $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su})$ and the EU of $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{c})$. We also report the difference for the *unfeasible* shrinkage unconstrained strategy that uses the oracle $\delta^{\star}$ in (28), i.e.,

$$\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su\star}) = \hat{\kappa}_1^{u}\hat{\boldsymbol{w}}^{\star} + \hat{\kappa}_2^{u}(\delta^{\star})\boldsymbol{w}_{ew}. \tag{37}$$

We make three main observations. First, $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su\star})$ that uses the oracle $\delta^{\star}$ essentially outperforms the constrained strategy $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{c})$ for all γ, because $\delta^{\star}$ shrinks intensively when estimation errors hurt performance the most, i.e., when γ is close to γ_{ew}. Second, $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su})$ that uses the plug-in estimator $\hat{\delta}^{\star}$ has smaller EU than $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su\star})$, and underperforms $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{c})$ for a range of γ's around γ_{ew}. However, this EU loss is quite small, while the EU gain outside of this range can be substantial. Third, $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^u)$ significantly underperforms the shrinkage unconstrained strategy $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su})$ for γ close to γ_{ew}, while the two strategies are close to one another otherwise. Therefore, it is largely preferable to rely on $\hat{\boldsymbol{w}}(\hat{\boldsymbol{\kappa}}^{su})$ over the plain unconstrained strategy.

5 Robustness to normality

So far, our theory assumes that asset returns are i.i.d. normally distributed. This assumption being typically rejected in empirical data, an important question is whether the optimality of the portfolio combination strategies we study is robust to dropping this assumption. The literature addresses this question empirically or via simulations, and finds that assuming normality delivers accurate results even if the true distribution is not normal; see, e.g., Kan and Smith (2008, Section 3.5) and Tu and Zhou (2004). Comparing the results under t distributions with various degrees of freedom, the latter conclude that "[...] the certainty–equivalent losses associated with ignoring fat tails are small. This suggests that the normality assumption works well in evaluating portfolio performance for a mean-variance investor."

In this section, we give *theoretical* foundation to the findings of Tu and Zhou (2004). Specifically, we derive the constrained and unconstrained combinations of the SMV and EW portfolios when asset returns are *elliptically* (not just t) distributed. We derive the EU loss incurred by the investor who relies on the combination coefficients calibrated assuming normality while returns are elliptical, and we show that this EU loss is typically very small.

5.1 Elliptical model for asset returns

We consider the elliptical model for asset returns handled in El Karoui (2010, 2013), i.e.,

$$\boldsymbol{r}_t = \boldsymbol{\mu} + \lambda_t \boldsymbol{\Sigma}^{1/2} \boldsymbol{z}_t, \tag{38}$$

where $\{\lambda_t\}_{t=1}^T$ are identically distributed, $\boldsymbol{z}_t \overset{\text{i.i.d.}}{\sim} \mathcal{N}(\boldsymbol{0}, \boldsymbol{I})$ is a standard multivariate normal, and $\{\lambda_t\}_{t=1}^T$ is independent of $\{\boldsymbol{z}_t\}_{t=1}^T$. We impose $\mathbb{E}[\lambda_t^2] = 1$ so that $\boldsymbol{\Sigma}$ is the covariance matrix of $\boldsymbol{r}_t$. For example, we obtain the t distribution with $\nu > 2$ degrees of freedom when

$$\lambda_t \sim \sqrt{\frac{\nu - 2}{\chi^2_\nu}}. \tag{39}$$

We only require that the second moment of $\boldsymbol{r}_t$ exists; moments of higher order may not exist.

As in the i.i.d. normal model, which we recover when $\nu \to \infty$ in (39), the distribution of $\boldsymbol{r}_t$ is identical for all t. However, this elliptical model is less restrictive than the i.i.d. normal model in two main ways. First, the distribution of returns is elliptical, which is a more general family of distributions than the normal. Second, we do not require $\{\lambda_t\}_{t=1}^T$ to be independent, in which case there can be serial dependence in the asset returns.

5.2 Optimal portfolio combinations under elliptical returns

Recent literature shows that in the high-dimensional regime in which $N, T \to \infty$ and $N/T \to \rho \in (0, 1)$, one can rely on less restrictive distributional assumptions than the normal; see, e.g., Bodnar et al. (2018) and Bodnar et al. (2023) who only require the existence of the fourth moment. In the next proposition, we consider this asymptotic regime and exploit results in El Karoui (2010, 2013) to derive the out-of-sample utility of the portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$, and the oracle constrained and unconstrained combination coefficients, $\boldsymbol{\kappa}^c$ and $\boldsymbol{\kappa}^u$.

For comparison purposes, note that as $N, T \to \infty$ and $N/T \to \rho \in (0, 1)$, the constrained and unconstrained combination coefficients under normally distributed returns in (13) and (14) converge to

$$\bar{\boldsymbol{\kappa}}^c_n = (\bar{\kappa}^c_{1,n}, \bar{\kappa}^c_{2,n}) \quad \text{where} \quad \bar{\kappa}^c_{1,n} = \frac{\psi^2 + \ell^2}{\psi^2 + \ell^2 + \frac{\rho}{1-\rho}(1 + \theta^2)} \quad \text{and} \quad \bar{\kappa}^c_{2,n} = 1 - \bar{\kappa}^c_{1,n}, \tag{40}$$

$$\bar{\boldsymbol{\kappa}}^u_n = (\bar{\kappa}^u_{1,n}, \bar{\kappa}^u_{2,n}) \quad \text{where} \quad \bar{\kappa}^u_{1,n} = \frac{\psi^2}{\psi^2 + \frac{\rho}{1-\rho}(1 + \theta^2)} \quad \text{and} \quad \bar{\kappa}^u_{2,n} = \frac{\gamma_{ew}}{\gamma}(1 - \bar{\kappa}^u_{1,n}). \tag{41}$$

Proposition 7. *Let asset returns $\boldsymbol{r}_t$ follow the elliptical model in* (38)*. Let $\alpha \geq 1$ be the unique positive solution to*

$$\mathbb{E}\left[\frac{1}{1 - \rho + \alpha\rho\lambda_t^2}\right] = 1, \tag{42}$$

and $\beta \geq \alpha^2$ be defined as

$$\beta = \frac{1-\rho}{\alpha^{-2} - \rho \mathbb{E}\Big[\frac{\lambda_t^4}{(1-\rho+\alpha\rho\lambda_t^2)^2}\Big]}. \tag{43}$$

Let $N, T \to \infty$ and $N/T \to \rho \in (0,1)$ while $(\mu_{ew}, \sigma_{ew}^2, \theta^2)$ stay bounded, and (Assumption-BB) in El Karoui (2013) holds.[16] *Then, the following holds:*

1. *The out-of-sample utility of the portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ in (7) converges to*

$$U(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) \xrightarrow{p} \bar{U}(\boldsymbol{\kappa}) = \frac{\kappa_1}{\gamma}\alpha\theta^2 + \kappa_2\mu_{ew} - \frac{\gamma}{2}\left(\frac{\kappa_1^2}{\gamma^2}\frac{\beta\theta^2 + \alpha\rho}{1-\rho} + \kappa_2^2\sigma_{ew}^2 + \frac{2\kappa_1\kappa_2}{\gamma}\alpha\mu_{ew}\right). \tag{44}$$

2. *The oracle constrained combination coefficients maximizing (44) under the convexity constraint (10), $\bar{\boldsymbol{\kappa}}^c = (\bar{\kappa}_1^c, \bar{\kappa}_2^c)$, are*

$$\bar{\kappa}_1^c = \frac{\alpha\psi^2 + \sigma_{ew}\ell(\gamma - \alpha\gamma_{ew})}{\alpha\psi^2 + \sigma_{ew}\ell(\gamma - \alpha\gamma_{ew}) + (1-\alpha)\gamma\mu_{ew} + \frac{(\beta-\alpha)\theta^2 + \alpha\rho(1+\theta^2)}{1-\rho}} \tag{45}$$

and $\bar{\kappa}_2^c = 1 - \bar{\kappa}_1^c$. Moreover, the asymptotic loss in out-of-sample utility when using the normal coefficients $\bar{\boldsymbol{\kappa}}_n^c$ instead of the elliptical ones $\bar{\boldsymbol{\kappa}}^c$ is

$$\bar{U}(\bar{\boldsymbol{\kappa}}_n^c) - \bar{U}(\bar{\boldsymbol{\kappa}}^c) = -\frac{1}{2\gamma}\Big(\alpha\psi^2 + \sigma_{ew}\ell(\gamma - \alpha\gamma_{ew})\Big)\frac{(\bar{\kappa}_1^c - \bar{\kappa}_{1,n}^c)^2}{\bar{\kappa}_1^c}. \tag{46}$$

3. *The oracle unconstrained combination coefficients maximizing (44), $\bar{\boldsymbol{\kappa}}^u = (\bar{\kappa}_1^u, \bar{\kappa}_2^u)$, are*

$$\bar{\kappa}_1^u = \frac{\alpha\psi^2}{\alpha^2\psi^2 + \frac{(\beta-\alpha^2)\theta^2 + \alpha\rho(1+\alpha\theta^2)}{1-\rho}} \quad \text{and} \quad \bar{\kappa}_2^u = \frac{\gamma_{ew}}{\gamma}(1 - \alpha\bar{\kappa}_1^u). \tag{47}$$

Moreover, the asymptotic loss in out-of-sample utility when using the normal coefficients $\bar{\boldsymbol{\kappa}}_n^u$ instead of the elliptical ones $\bar{\boldsymbol{\kappa}}^u$ is

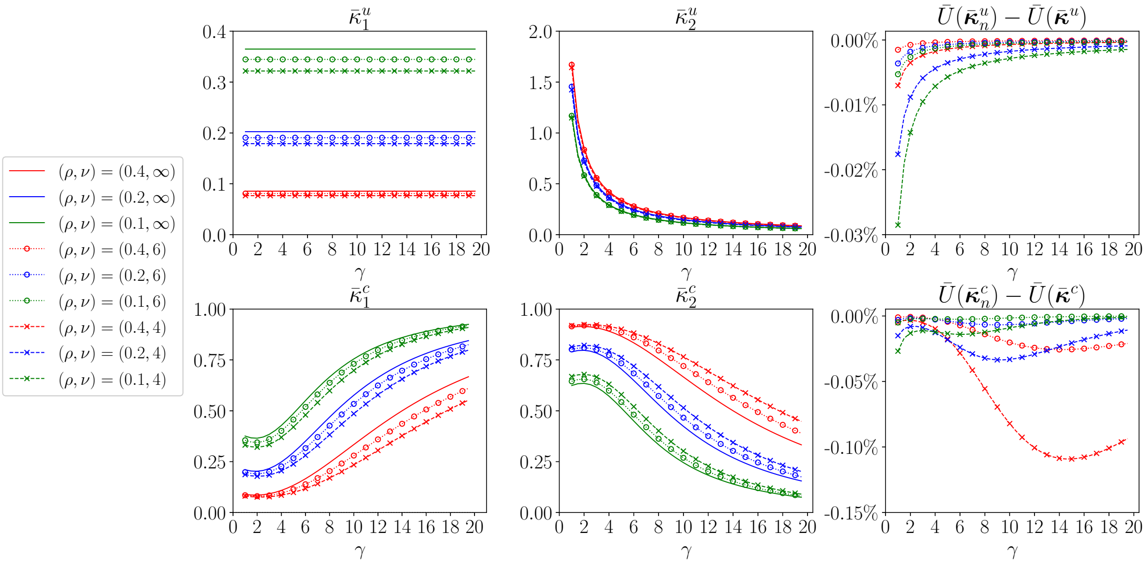
$$\bar{U}(\bar{\boldsymbol{\kappa}}_n^u) - \bar{U}(\bar{\boldsymbol{\kappa}}^u) = -\frac{1}{2\gamma}\left(\alpha\psi^2\frac{(\bar{\kappa}_1^u - \bar{\kappa}_{1,n}^u)^2}{\bar{\kappa}_1^u} + (\alpha-1)^2\theta_{ew}^2(\bar{\kappa}_{1,n}^u)^2\right). \tag{48}$$

In Figure 4, we calibrate α and β to the t distribution with ν degrees of freedom in (39).[17]

[16] (Assumption-BB) ensures that the distribution of the λ_t's does not put too much mass near zero.

[17] We recover the case of i.i.d. normal asset returns when $\nu \to \infty$, i.e., $\lambda_t \to 1$. This situation leads to $\alpha = \beta = 1$, and thus, $\bar{\boldsymbol{\kappa}}_n^c = \bar{\boldsymbol{\kappa}}^c$ and $\bar{\boldsymbol{\kappa}}_u^c = \bar{\boldsymbol{\kappa}}^u$.

Figure 4: Combination coefficients and utility loss under normal versus elliptical returns



Notes. The left and middle panels depict the constrained and unconstrained combination coefficients when returns are normally distributed (solid, no markers), as in (40)–(41), and elliptically distributed (dashed with cross markers and dotted with circle markers), as in (45) and (47). The right panels depict the expected out-of-sample utility loss when using the normal coefficients instead of the elliptical ones, as in (46) and (48). We consider the high-dimensional asymptotic regime in which $N, T \to \infty$ and $N/T \to \rho \in (0, 1)$. The elliptical distribution is a t distribution with ν degrees of freedom. We depict the combination coefficients and utility losses as a function of the risk-aversion coefficient γ for different (ρ, ν). The figure is constructed by calibrating the population vector of means and covariance matrix of asset excess returns from monthly returns on the 25 portfolios of firms sorted on size and book-to-market spanning July 1926 to December 2022.

First, we compare the normal combination coefficients in (40)–(41) with the elliptical ones in (45) and (47) for $\rho = (0.1, 0.2, 0.4)$ and $\nu = (4, 6)$, and find they are close to each other. For example, when $\gamma = 5$, $\rho = 0.2$, and $\nu = 6$, which corresponds to a realistic excess kurtosis of three for monthly returns, the normal coefficients are $(\bar{\kappa}^u_{1,n}, \bar{\kappa}^u_{2,n}, \bar{\kappa}^c_{1,n}, \bar{\kappa}^c_{2,n}) = (0.20, 0.29, 0.30, 0.70)$, while the elliptical ones are $(\bar{\kappa}^u_1, \bar{\kappa}^u_2, \bar{\kappa}^c_1, \bar{\kappa}^c_2) = (0.19, 0.29, 0.27, 0.73)$. Second, we depict the EU loss in (46) and (48) and we observe that it is typically very small. For example, when $\gamma = 5$, $\rho = 0.2$, and $\nu = 6$, the EU loss is a mere 0.0007% and 0.0038% per month for the unconstrained and constrained strategies, respectively.

The theoretical results above suggest that we can safely use the normality assumption to derive combination strategies even if returns are non-normal. Moreover, in Section C.8 of the Supplementary Material we compare the *empirical* net out-of-sample utility of the constrained and unconstrained strategies calibrated under the normal and t distributions,

Table 1: List of datasets considered in the empirical analysis

Dataset	N	Time period	τ	Abbreviation
25 portfolios sorted on size and book-to-market	25	07/1926 - 12/2022	1158	25SBTM
10 portfolios sorted on momentum	10	01/1927 - 12/2022	1152	10MOM
25 portfolios sorted on operating profitability and investment	25	07/1963 - 12/2022	714	25OPIN
48 portfolios sorted on industry	48	07/1969 - 12/2022	642	48IND
500 random portfolios of 50 U.S. stocks	50	01/1998 - 03/2022	291	50STO
500 random portfolios of 100 U.S. stocks	100	01/1998 - 03/2022	291	100STO

Notes. This table lists the datasets of monthly excess returns we use in the empirical analysis of Section 6. τ is the total number of months in each dataset. The first four datasets are downloaded from Kenneth French's website, and the last two from CRSP.

and we also find that the loss caused by the normality assumption is small or even negative.

6 Empirical analysis

6.1 Datasets and portfolio strategies

Table 1 lists the six datasets of monthly excess returns we consider in this empirical analysis, which are three datasets of characteristic-sorted portfolios, one of industry-sorted portfolios, and two composed of 50 and 100 individual stocks. To construct the datasets of individual stocks, we collect adjusted returns from CRSP for the 235 stocks traded on the three major U.S. stock exchanges that have an history of returns between 1998 and 2022. To ensure the robustness of our results to stock selection, we report the average performance across 500 datasets of size $N = 50$ and $N = 100$, randomly drawn from the total pool of 235 stocks.

We evaluate the out-of-sample performance of 10 portfolio strategies that we list and describe in Table 2. The three most relevant ones are those we study in this paper that combine the SMV and EW portfolios. First and second, the unconstrained and shrinkage unconstrained strategies (UNC3F and SUNC3F). Third, the constrained strategy (CON3F). We also report the performance of the SMV portfolio, $(\kappa_1, \kappa_2) = (1, 0)$, to which CON3F, UNC3F, and SUNC3F tend when estimation risk $d \to 0$. Conversely, we also report the performance of the combination of the EW portfolio with the risk-free asset (EWRF),

Table 2: List of portfolio strategies considered in the empirical analysis

Name	Funds	Description
UNC3F	tan-ew-rf	Unconstrained combination of the sample tangent portfolio, EW portfolio, and risk-free asset. See Proposition 1.
SUNC3F	tan-ew-rf	Shrinkage estimator of the unconstrained combination UNC3F. See Proposition 6.
CON3F	tan-ew-rf	Constrained combination of the sample tangent portfolio, EW portfolio, and risk-free asset in Tu and Zhou (2011).
KZ2F	tan-rf	Combination of the sample tangent portfolio and risk-free asset in Kan and Zhou (2007).
KZ3F	tan-gmv-rf	Combination of the sample tangent portfolio, SGMV portfolio, and risk-free asset in Kan and Zhou (2007).
KWZ	tan-gmv	Combination of the sample tangent portfolio and SGMV portfolio with no-risk free asset in Kan et al. (2022).
BOP	tan-gmv-ew	Combination of the sample tangent portfolio, SGMV portfolio, and EW portfolio with no-risk free asset in Bodnar et al. (2023).
EWRF	ew-rf	Combination of the EW portfolio and the risk-free asset.
GMVRF	gmv-rf	Combination of the SGMV portfolio and the risk-free asset.
SMV	tan-rf	Sample mean-variance portfolio.

Notes. This table lists the portfolio strategies we use in the empirical analysis of Section 6, which are estimated with the sample covariance matrix in (4) or the linear shrinkage estimate of Ledoit and Wolf (2004).

$(\kappa_1, \kappa_2) = (0, \gamma_{ew}/\gamma)$, to which UNC3F tends when estimation risk $d \to \infty$.

The remaining five strategies are other combination rules. First and second, the two-fund and three-fund rules in Kan and Zhou (2007) (KZ2F and KZ3F). Third and fourth, the combination rules in Kan et al. (2022) and Bodnar et al. (2023) that do not invest in the risk-free asset (KWZ and BOP). Fifth, the combination of the SGMV portfolio with the risk-free asset (GMVRF): $(\mu_g/(c\gamma))\widehat{\mathbf{\Sigma}}^{-1}\mathbf{1}$.

We implement the portfolio strategies both with the sample covariance matrix estimate in (4) and the linear shrinkage estimate of Ledoit and Wolf (2004).[18] This is because Kan et al. (2022) and Lassance et al. (2023) find that using both a portfolio combination and a shrinkage covariance matrix delivers better performance than using each in isolation.

We need to estimate several parameters to obtain the combination coefficients in the 10 strategies. We estimate the parameters μ_{ew}, σ^2_{ew}, θ^2, and ψ^2 as in the simulation analysis of Section 4.3. We estimate μ_g and σ^2_g using the maximum-likelihood estimates of $\boldsymbol{\mu}$ and $\mathbf{\Sigma}$,

[18] We also consider the nonlinear shrinkage estimate of Ledoit and Wolf (2020) in Section C.9 of the Supplementary Material and find the conclusions are similar to those obtained with linear shrinkage.

and estimate $\psi_g^2 = \theta^2 - \mu_g^2/\sigma_g^2$ via the adjusted estimator of Kan and Zhou (2007).

Finally, consistent with DeMiguel et al. (2009) and Martellini and Ziemann (2010), we consider risk-aversion coefficients $\gamma = 1, 5, 10$, and 15.

6.2 Performance measure and statistical significance

We measure portfolio performance using rolling windows. At the end of month t, we estimate portfolio k using the T previous months, and we compute its out-of-sample return in month $t+1$. We consider $T = 120$ and 240 months. We repeat this iteratively, giving a time series of $\tau - T$ out-of-sample gross returns $r_{gross,k,t}$, where τ is the total number of months in the dataset. We then compute the net out-of-sample returns, $r_{net,k,t} = r_{gross,k,t}$ if $t = T+1$ and

$$r_{net,k,t} = (1 + r_{gross,k,t})(1 - p \times \text{turnover}_{k,t-1}) - 1 \quad \text{if} \quad t = T+2, \ldots, \tau, \tag{49}$$

where p is the proportional cost required to rebalance the portfolio and

$$\text{turnover}_{k,t} = \sum_{i=1}^{N} |w_{i,k,t} - w_{i,k,(t-1)^+}|, \quad t = T+1, \ldots, \tau, \tag{50}$$

with $w_{i,k,t}$ the weight of asset i in month t and $w_{i,k,(t-1)^+}$ the prior-month weight before rebalancing in month t. We set $p = 10$ basis points as in Ao et al. (2019). Finally, we compare the portfolio strategies in terms of *annualized out-of-sample utility net of transaction costs*,

$$U_k = 12 \times \left(\hat{\mu}_k - \frac{\gamma}{2}\hat{\sigma}_k^2\right), \tag{51}$$

where $\hat{\mu}_k$ and $\hat{\sigma}_k^2$ are the sample mean and variance of $r_{net,k,t}$. In Section C.7 of the Supplementary Material, we also report the net out-of-sample Sharpe ratio.

We test the statistical significance of the difference in net utilities of UNC3F and SUNC3F versus CON3F by using the stationary block bootstrap approach of Politis and Romano (1994) and Ledoit and Wolf (2008, Remark 3.2) to produce the resulting p-values, using 10,000 bootstrap samples and an average block size of five. For the 50STO and 100STO datasets, we compute the p-values by merging the out-of-sample returns across the 500 randomly drawn datasets. Moreover, we identify the superior set of models at a confidence level of 5% using the model confidence set (MCS) of Hansen et al. (2011).[19]

6.3 Discussion of results

The out-of-sample performance of the 10 portfolio strategies listed in Table 2, and their statistical significance, is reported in Table 3 for the sample covariance matrix and Table 4 for the linear shrinkage covariance matrix. In Section C.10 of the Supplementary Material, we also report the average monthly turnover. The sets of superior models obtained with the MCS test typically include many strategies, and thus in Tables 3 and 4 we underline those that are rejected. For the 50STO and 100STO datasets we obtain 500 sets of models, and we underline in the tables the three portfolio strategies that are most often rejected.

In the vast majority of cases, we find that exploiting the shrinkage covariance matrix of Ledoit and Wolf (2004) to estimate the portfolio strategies in Table 4 delivers better performance than exploiting the sample covariance matrix in Table 3. Therefore, we recommend investors to rely on a shrinkage covariance matrix, even if the theory builds on the sample one. However, the ranking of the different strategies is similar in both tables, and therefore the main conclusions we discuss below are applicable to both estimators.

In essentially all cases, the SMV portfolio delivers by far the worst out-of-sample performance. In particular, it is outperformed by UNC3F, SUNC3F, and CON3F that combine SMV with the EW portfolio, as we predict in part 3 of Proposition 1. As a result, the SMV portfolio is nearly consistently rejected by the MCS test.

The main comparison concerns the three-fund rules that combine tangent, EW, and the risk-free asset. When γ is close enough to the estimated values of γ_{ew}, the constrained strategy CON3F can outperform the unconstrained strategy UNC3F, in line with Proposition 5

[19]The MCS test needs as an input a performance difference for each pair of models and for all time t. However, our performance measure, the net out-of-sample utility, can only be computed over several out-of-sample monthly returns. Therefore, to provide a performance difference for all time t, we collect daily returns for all our datasets and evaluate the net out-of-sample utility of a given portfolio at time t from the next-month daily returns, using daily rebalancing.

Table 3: Annualized net out-of-sample utility (sample covariance matrix)

	$T = 120$						$T = 240$					
	25SBTM	10MOM	25OPIN	48IND	50STO	100STO	25SBTM	10MOM	25OPIN	48IND	50STO	100STO
$\hat{\gamma}_{ew}^{med}$	3.20	2.78	3.15	3.16	3.95	3.95	2.88	2.67	3.16	3.29	4.43	4.43
					Panel (a): $\gamma = 1$							
UNC3F	29.4	22.5	6.63	2.92	15.4***	10.7***	20.9	25.3	24.6	6.25	31.2***	30.0***
SUNC3F	30.1	22.5	7.67	3.43	13.9***	9.38***	20.8	25.2	23.9	5.11	30.4***	29.1***
CON3F	31.2	23.4	8.31	5.36	8.59	4.95	21.8	26.6	21.3	2.82	23.2	18.7
KZ2F	27.1	21.6	4.07	-0.98	-3.24	-8.13	19.1	25.0	17.0	-3.91	10.2	3.18
KZ3F	24.4	26.6	6.20	-7.33	-14.3	-21.4	22.2	30.0	30.8	2.57	18.7	5.85
KWZ	17.1	16.5	4.33	2.73	3.16	-2.45	16.0	24.0	15.4	0.66	10.3	9.07
BOP	17.0	7.12	-1.24	-18.4	-$\underline{31.8}$	-$\underline{25.7}$	15.6	20.9	13.6	-5.70	-$\underline{51.1}$	-$\underline{102}$
EWRF	8.32	5.72	-0.59	2.85	17.5	17.5	5.99	2.47	8.93	9.87	27.4	27.7
GMVRF	15.9	11.5	7.67	-7.49	-12.5	-18.1	16.1	8.33	28.1	4.11	18.6	5.77
SMV	-$\underline{140}$	-$\underline{21.1}$	-$\underline{158}$	-$\underline{492}$	-$\underline{583}$	-$\underline{3140}$	-$\underline{47.5}$	8.99	-$\underline{58.0}$	-$\underline{217}$	-$\underline{147}$	-$\underline{505}$
					Panel (b): $\gamma = 5$							
UNC3F	5.71	4.45	1.28	0.57	3.07***	2.12***	3.96	5.02*	4.90	1.25	6.24***	5.99***
SUNC3F	5.95	4.75	1.93	1.39	4.30***	3.61***	4.02	5.14**	5.02	1.30	6.79***	6.71***
CON3F	5.08	4.41	2.43	1.67	5.67	5.02	4.01	5.88	5.56	1.90	7.40	7.33
KZ2F	5.26	4.27	0.77	-0.21	-0.66	-1.65	3.59	4.96	3.37	-0.79	2.04	0.64
KZ3F	4.69	5.27	1.16	-1.51	-2.95	-4.39	4.24	5.95	6.13	0.50	3.71	1.12
KWZ	6.19	5.50	3.47	0.23	1.57	-10.5	7.22	8.15	8.24	0.61	6.78	4.91
BOP	4.35	3.05	1.73	-3.05	-4.00	-$\underline{2.96}$	5.31	7.21	6.27	-0.62	-5.95	-$\underline{20.3}$
EWRF	1.66	1.14	-0.12	0.57	3.49	3.49	1.20	0.49	1.79	1.97	5.48	5.55
GMVRF	3.07	2.29	1.49	-1.53	-2.59	-3.71	3.17	1.66	5.62	0.82	3.69	1.10
SMV	-$\underline{30.7}$	-$\underline{4.48}$	-$\underline{32.8}$	-$\underline{104}$	-$\underline{120}$	-$\underline{784}$	-$\underline{10.4}$	1.69	-$\underline{12.0}$	-$\underline{44.2}$	-$\underline{29.9}$	-$\underline{104}$
					Panel (c): $\gamma = 10$							
UNC3F	2.84***	2.22*	0.64*	0.28**	1.53***	1.06***	1.97**	2.51	2.45	0.62*	3.12***	3.00***
SUNC3F	2.76***	2.21*	0.52*	0.06**	1.30***	0.72***	1.92**	2.48	2.46	0.59**	3.05***	2.92***
CON3F	-0.74	1.06	-1.53	-4.01	-$\underline{1.59}$	-2.80	0.29	2.24	1.13	-$\underline{2.83}$	-0.16	-0.94
KZ2F	2.62	2.13	0.38	-0.11	-0.33	-0.83	1.78	2.48	1.68	-0.39	1.02	0.32
KZ3F	2.33	2.63	0.58	-0.76	-1.48	-2.20	2.11	2.98	3.07	0.25	$\underline{1.85}$	$\underline{0.56}$
KWZ	0.32	-1.06	-1.43	-4.60	-3.58	-24.1	2.30	1.63	2.84	-$\underline{3.06}$	$\underline{1.71}$	-$\underline{0.69}$
BOP	-3.03	-$\underline{2.74}$	-2.84	-6.31	-5.79	-$\underline{7.84}$	0.27	1.04	1.05	-$\underline{4.21}$	-2.65	-12.1
EWRF	0.83	0.57	-0.06	0.28	1.75	1.75	0.60	0.25	0.89	0.99	2.74	2.77
GMVRF	1.53	1.14	0.74	-0.77	-1.30	-1.86	1.58	0.83	2.81	0.41	$\underline{1.84}$	$\underline{0.55}$
SMV	-$\underline{15.5}$	-2.26	-$\underline{16.5}$	-$\underline{52.3}$	-$\underline{60.5}$	-$\underline{403}$	-$\underline{5.28}$	0.84	-$\underline{6.01}$	-$\underline{22.2}$	-$\underline{15.0}$	-$\underline{52.3}$
					Panel (d): $\gamma = 15$							
UNC3F	1.89***	1.48***	0.42***	0.19***	1.02***	0.70***	1.31***	1.67	1.63**	0.42***	2.08***	2.00***
SUNC3F	1.84***	1.46***	0.38***	0.12***	0.94***	0.59***	1.29***	1.66	1.64**	0.41***	2.06***	1.98***
CON3F	-$\underline{2.93}$	-$\underline{0.11}$	-$\underline{3.55}$	-$\underline{7.85}$	-$\underline{6.76}$	-$\underline{10.5}$	-1.04	1.07	-$\underline{0.63}$	-$\underline{5.18}$	-$\underline{3.39}$	-6.20
KZ2F	1.75	1.42	0.25	-0.07	-0.22	-0.55	1.18	1.65	1.12	-0.26	0.68	0.21
KZ3F	1.55	1.75	0.38	-0.51	-0.99	-1.47	1.40	1.98	2.04	0.17	1.23	$\underline{0.37}$
KWZ	-$\underline{4.96}$	-$\underline{7.09}$	-$\underline{6.56}$	-$\underline{9.51}$	-8.94	-37.9	-2.18	-$\underline{3.90}$	-$\underline{2.29}$	-$\underline{6.96}$	-$\underline{3.39}$	-$\underline{6.32}$
BOP	-$\underline{8.55}$	-$\underline{8.35}$	-$\underline{7.59}$	-$\underline{11.2}$	-$\underline{10.1}$	-$\underline{15.1}$	-4.05	-$\underline{4.22}$	-$\underline{3.73}$	-$\underline{8.23}$	-6.36	-12.2
EWRF	0.55	0.38	-0.04	0.19	1.16	1.16	0.40	0.16	0.60	0.66	1.83	1.85
GMVRF	1.02	0.76	0.50	-0.51	-0.87	-1.24	1.05	0.55	1.87	0.27	1.23	$\underline{0.36}$
SMV	-$\underline{10.4}$	-$\underline{1.51}$	-$\underline{11.0}$	-$\underline{35.0}$	-$\underline{40.4}$	-$\underline{271}$	-$\underline{3.54}$	0.56	-$\underline{4.02}$	-$\underline{14.8}$	-$\underline{10.0}$	-$\underline{34.9}$

Notes. This table reports the annualized net out-of-sample utility in percentage points for the portfolio strategies in Table 2 and the datasets in Table 1 when using the sample covariance matrix. The net out-of-sample utility is computed using proportional transaction costs of 10 basis points. For 50 and 100STO, we report the average performance across 500 randomly drawn datasets from a total pool of 235 stocks. We evaluate the statistical significance of the difference in net utilities of UNC3F vs CON3F and SUNC3F vs CON3F using the stationary block bootstrap approach of Politis and Romano (1994) and Ledoit and Wolf (2008) to produce p-values. The stars *, **, and *** indicate that the p-value is less than 10%, 5%, and 1%, respectively. The underlined numbers indicate the strategies that do not belong to the superior set of models of Hansen et al. (2011) at a confidence level of 5%. For 50STO and 100STO, we underline the three strategies that are most often rejected, or more in case of equalities.

Table 4: Annualized net out-of-sample utility (linear shrinkage covariance matrix)

	$T = 120$						$T = 240$					
	25SBTM	10MOM	25OPIN	48IND	50STO	100STO	25SBTM	10MOM	25OPIN	48IND	50STO	100STO
$\hat{\gamma}_{ew}^{med}$	3.20	2.78	3.15	3.16	3.95	3.95	2.88	2.67	3.16	3.29	4.43	4.43
	Panel (a): $\gamma = 1$											
UNC3F	37.3	31.6	14.3	3.70	15.9***	16.7***	34.1	30.5	26.4	4.37	30.5***	30.5***
SUNC3F	38.4	31.7	15.8	4.60	14.7***	15.5***	34.1	30.5	25.8	3.27	29.9***	29.8***
CON3F	42.2	34.5	18.8	6.93	11.5	13.2	36.1	32.4	23.7	0.91	24.3	21.8
KZ2F	38.6	32.6	15.1	1.28	-0.07	-0.51	34.0	31.0	19.7	-5.86	11.7	5.78
KZ3F	33.3	34.7	14.9	-3.33	-4.31	8.87	31.9	34.5	31.2	3.01	12.0	0.55
KWZ	23.9	25.7	10.3	4.67	4.14	4.82	24.4	28.6	16.6	-1.36	10.1	9.88
BOP	16.8	7.23	-1.24	-18.5	-31.9	-25.7	15.7	20.9	13.6	-5.67	-51.0	-102
EWRF	8.32	5.72	-0.59	2.85	17.5	17.5	5.99	2.47	8.93	9.87	27.4	27.7
GMVRF	14.9	10.0	10.7	-4.64	-3.09	9.55	11.8	7.77	27.1	8.41	11.9	0.21
SMV	-26.2	13.6	-112	-508	-539	-1838	1.39	21.4	-53.7	-236	-165	-626
	Panel (b): $\gamma = 5$											
UNC3F	7.44	6.31	2.83	0.73	3.18***	3.34***	6.76	6.08**	5.26	0.87	6.10***	6.11***
SUNC3F	7.68	6.64	3.49	1.52	4.43***	4.85***	6.82	6.20**	5.34	0.92	6.61***	6.80***
CON3F	8.05	6.96	4.15	1.85	5.88	6.34	7.24	7.02	5.86	1.50	7.19	7.40
KZ2F	7.72	6.50	2.98	0.25	-0.02	-0.10	6.74	6.18	3.91	-1.18	2.34	1.16
KZ3F	6.59	6.93	2.92	-0.71	-0.93	1.76	6.31	6.88	6.21	0.59	2.37	0.04
KWZ	8.17	7.67	5.57	3.07	4.05	3.21	8.97	9.13	8.53	1.36	7.05	6.63
BOP	4.09	3.14	1.72	-3.02	-3.86	-2.94	5.28	7.12	6.22	-0.51	-5.69	-20.1
EWRF	1.66	1.14	-0.12	0.57	3.49	3.49	1.20	0.49	1.79	1.97	5.48	5.55
GMVRF	2.91	1.99	2.11	-0.97	-0.69	1.90	2.33	1.55	5.41	1.68	2.35	-0.03
SMV	-5.85	2.61	-23.3	-106	-110	-388	-0.07	4.22	-11.1	-48.1	-33.4	-128
	Panel (c): $\gamma = 10$											
UNC3F	3.72**	3.15	1.41	0.36**	1.59***	1.67***	3.37	3.04	2.63	0.44**	3.05***	3.05***
SUNC3F	3.60**	3.13	1.26	0.11**	1.32***	1.32***	3.32	3.02	2.63	0.40**	2.96***	2.96***
CON3F	1.98	2.76	-0.46	-3.96	-1.44	-1.94	2.53	3.04	1.21	-3.37	-0.95	-1.36
KZ2F	3.86	3.25	1.49	0.12	-0.01	-0.05	3.37	3.09	1.95	-0.59	1.17	0.58
KZ3F	3.29	3.46	1.46	-0.36	-0.47	0.88	3.15	3.44	3.10	0.29	1.18	0.01
KWZ	2.28	0.56	0.72	-0.57	0.39	-0.59	3.37	2.31	3.18	-1.55	2.29	1.98
BOP	-3.26	-2.59	-2.82	-6.27	-5.67	-7.80	0.18	1.02	1.04	-4.10	-2.66	-12.0
EWRF	0.83	0.57	-0.06	0.28	1.75	1.75	0.60	0.25	0.89	0.99	2.74	2.77
GMVRF	1.45	0.99	1.05	-0.49	-0.35	0.95	1.16	0.77	2.71	0.84	1.17	-0.02
SMV	-2.97	1.30	-11.7	-53.1	-55.3	-195	-0.06	2.10	-5.56	-24.1	-16.7	-64.4
	Panel (d): $\gamma = 15$											
UNC3F	2.48***	2.10	0.94***	0.24***	1.06***	1.11***	2.25*	2.03	1.75**	0.29***	2.03***	2.04***
SUNC3F	2.41***	2.08	0.89***	0.16***	0.97***	1.00***	2.23**	2.02	1.75**	0.28***	2.01***	2.01***
CON3F	-0.14	1.34	-2.57	-8.04	-6.62	-9.49	0.93	1.72	-0.59	-5.90	-4.68	-7.76
KZ2F	2.57	2.17	0.99	0.08	-0.01	-0.03	2.24	2.06	1.30	-0.39	0.78	0.39
KZ3F	2.19	2.31	0.97	-0.24	-0.31	0.59	2.10	2.29	2.07	0.20	0.79	0.01
KWZ	-2.58	-5.41	-4.03	-4.30	-3.51	-4.52	-1.21	-3.20	-1.80	-4.92	-2.52	-2.72
BOP	-8.59	-8.02	-7.46	-11.2	-10.2	-15.1	-4.02	-4.14	-3.66	-8.14	-6.41	-12.2
EWRF	0.55	0.38	-0.04	0.19	1.16	1.16	0.40	0.16	0.60	0.66	1.83	1.85
GMVRF	0.97	0.66	0.70	-0.33	-0.23	0.63	0.77	0.52	1.80	0.56	0.78	-0.01
SMV	-1.99	0.86	-7.80	-35.4	-36.9	-131	-0.04	1.40	-3.71	-16.1	-11.2	-43.0

Notes. This table reports the annualized net out-of-sample utility in percentage points for the portfolio strategies in Table 2 and the datasets in Table 1 when using the shrinkage covariance matrix of Ledoit and Wolf (2004). The net out-of-sample utility is computed using proportional transaction costs of 10 basis points. For 50 and 100STO, we report the average performance across 500 randomly drawn datasets from a total pool of 235 stocks. We evaluate the statistical significance of the difference in net utilities of UNC3F vs CON3F and SUNC3F vs CON3F using the stationary block bootstrap approach of Politis and Romano (1994) and Ledoit and Wolf (2008) to produce p-values. The stars *, **, and *** indicate that the p-value is less than 10%, 5%, and 1%, respectively. The underlined numbers indicate the strategies that do not belong to the superior set of models of Hansen et al. (2011) at a confidence level of 5%. For 50STO and 100STO, we underline the three strategies that are most often rejected, or more in case of equalities.

and Figure 3. For the 25SBTM, 10MOM, 25OPIN, and 48IND datasets where the median estimated γ_{ew} is around three, this happens when $\gamma = 1$ and 5. For the 50STO and 100STO where the median estimated γ_{ew} is larger and around four, this happens when $\gamma = 5$. This result shows that the convexity constraint can help performance by alleviating the impact of estimation errors on combination coefficients.

However, when $\gamma = 1$, the CON3F strategy can sometimes largely underperform the UNC3F strategy. A first case in which this happens is for the 48IND dataset when $T = 240$. For example, when using the sample covariance matrix, UNC3F and CON3F deliver a net utility of 6.25 and 2.82, respectively. We can explain this theoretically because, as discussed in Section 3.1, CON3F overinvests in the SMV portfolio relative to UNC3F, and this exposure to the SMV portfolio is larger for $T = 240$ than 120. This overinvestment is problematic for the 48IND dataset because SMV performs notoriously poorly for industry portfolios that present little cross-sectional variation in sample mean returns (Kirby and Ostdiek, 2012).

A second case in which UNC3F largely outperforms CON3F for $\gamma = 1$ is for the 50STO and 100STO datasets, because the median estimated γ_{ew} is further from one relative to the other datasets, and thus there is more room for UNC3F to outperform CON3F for a small γ. For example, when $T = 120$ in Table 3, UNC3F delivers a net utility of 15.4 and 10.7 for the 50STO and 100STO datasets, respectively, versus 8.59 and 4.95 delivered by CON3F.

Although CON3F can perform well when $\gamma = 1$ or 5, it quickly deteriorates as γ increases, is outperformed by UNC3F in a statistically significant way, and is often out of the MCS set of models. Take the sample covariance matrix, $T = 120$, and $\gamma = 10$. Then, CON3F delivers net utilities of -0.74, 1.06, -1.53, -4.01, -1.59, -2.80 for the six datasets, versus 2.84, 2.22, 0.64, 0.28, 1.53, 1.06 for UNC3F. These negative utilities are consistent with Proposition 1, in which we show that CON3F underperforms the risk-free asset once γ is too large. Similarly, Proposition 3 predicts that CON3F underperforms the two-fund rule of Kan and Zhou (2007) (KZ2F) once γ is too large, and this is what we observe empirically too. The out-of-sample utility of UNC3F is on the contrary *always* positive, and in most cases larger than that of KZ2F. UNC3F is also *never* rejected by the MCS test.

Moreover, we find in Section C.10 of the Supplementary Material that UNC3F systematically delivers a smaller turnover than that of CON3F, which is in line with Proposition 4.

We turn next to the *shrinkage* unconstrained strategy (SUNC3F), which alleviates the sensitivity to estimation errors of UNC3F, particularly for γ's close to γ_{ew}. Our results show that this objective is generally accomplished: SUNC3F outperforms UNC3F for all datasets when $\gamma = 5$, and for all datasets except 50STO and 100STO when $\gamma = 1$. For a larger $\gamma = 10$ and 15, and when $\gamma = 1$ for 50STO and 100STO, SUNC3F remains close to UNC3F, and thus largely outperforms CON3F. In Section C.11 of the Supplementary Material, we replicate Figure 3 empirically and find that the results closely mimic the simulation findings. Finally, just like UNC3F, the SUNC3F strategy is never rejected by the MCS test.

Next, we compare our strategies with the EWRF and GMVRF strategies that do not invest in the SMV. In the majority of cases, the two novel strategies we introduce, UNC3F and SUNC3F, largely outperform. Let us concentrate on the results using the linear shrinkage covariance matrix in Table 4. Then, UNC3F and SUNC3F outperform GMVRF and EWRF in all cases except for the 48IND dataset when $T = 240$, where they underperform both, and the 50STO and 100STO datasets when $T = 120$, where they underperform EWRF. These two cases in which EWRF is difficult to beat are exactly those we predict theoretically in Section 3.1. The first case is when ψ^2 is small, which happens for the 48IND dataset because there is only little cross-sectional variation in sample mean returns for industry-sorted portfolios (Kirby and Ostdiek, 2012). The second case is when estimation risk d is large, which happens in the 50STO and 100STO datasets when $T = 120$.

It is notable that even for the 100STO dataset and $T = 120$, a case of high estimation risk, UNC3F and SUNC3F lose little relative to EWRF in Table 4, while they outperform when $T = 240$. This is reassuring because it is not known a priori for which sample size EWRF will outperform, and thus investors can safely rely on UNC3F or SUNC3F without fearing to lose much. In contrast, CON3F can deliver large performance losses relative to EWRF.

We now compare the three-fund rule of Kan and Zhou (2007) (KZ3F) and its fully invested counterpart in Kan et al. (2022) (KWZ). When γ is small ($\gamma = 1$) and large ($\gamma = 10$ and 15), KZ3F generally outperforms KWZ because KWZ cannot invest in the risk-free asset. However, the contrary nearly systematically happens when $\gamma = 5$. We demonstrate in Section C.4 of the Supplementary Material that this result can be explained with a similar reasoning to that in Section 4: Kan et al. (2022) impose the convexity constraint so that KWZ is fully invested in risky assets, which alleviates the impact of estimation errors on combination coefficients relative to KZ3F and can help improve performance for some γ's. As a result, our SUNC3F strategy largely outperforms KWZ when $\gamma = 1$, 10, and 15, but not $\gamma = 5$. Note also that KWZ and our two proposed strategies, UNC3F and SUNC3F, are superior to the fully invested combination of Bodnar et al. (2023) (BOP).

Finally, the comparison of the unconstrained combination of SMV and EW (UNC3F) with the combination of SMV and SGMV in Kan and Zhou (2007) (KZ3F) shows that it is more preferable to combine with EW than SGMV, for three main reasons. First, EWRF has a much lower turnover than GMVRF, and thus UNC3F has a smaller turnover than KZ3F too. Second, while EW has no estimation error, SGMV depends on the inverse covariance matrix. Therefore, when N is rather large in the 48IND, 50STO, and 100STO datasets, EWRF largely outperforms GMVRF, and UNC3F outperforms KZ3F too. The third reason concerns diversification. Consider the 25SBTM, 10MOM, and 25OPIN datasets. GMVRF largely outperforms EWRF, and thus, one would expect KZ3F to also outperform UNC3F. However, this is not always the case: they perform similarly for the 25OPIN dataset, and UNC3F outperforms for 25SBTM. This result can be explained because of diversification. Specifically, in Section C.6 of the Supplementary Material we derive a closed-form expression for the correlation between the out-of-sample return of SMV and that of either SGMV or EW. By calibrating this correlation to our datasets, we find that the correlation with the EW portfolio is systematically much smaller, which is in line with the empirical correlations.[20]

[20]In Section C.5 of the Supplementary Material, we also derive a four-fund strategy that combines SMV, EW, and SGMV. However, we find that it is generally outperformed by UNC3F that does not invest in

7 Conclusion: How to combine?

We conclude with a summary of insights on the portfolio combination problem. First, which portfolios to combine? The SMV portfolio is optimal without estimation risk, and thus is a natural choice. Our results favor combining it with the EW portfolio over the SGMV portfolio because EW has lower turnover, suffers no estimation risk, offers larger diversification gains, and leads to portfolio combinations that perform largely better when N/T is sizable. Moreover, investing in three funds — the sample tangent portfolio, the EW portfolio, and the risk-free asset — hits the sweet spot as it outperforms two-fund and four-fund strategies.

Second, how to estimate the combination coefficients? The constrained coefficients are less sensitive to estimation errors than the unconstrained ones, and thus outperform for a range of risk-aversion coefficients γ. However, the constrained combination can severely underperform outside of this range. The shrinkage unconstrained combination offers an appealing tradeoff between the two approaches by closely matching the one that performs best for the chosen γ.

Third, what is the impact of estimation risk? When N/T is sizable it becomes even more necessary to combine SMV with a more robust portfolio, and combining with EW largely outperforms combining with SGMV. The range of γ for which the constrained combination outperforms the unconstrained one decreases with estimation risk, and indeed, the unconstrained combination is particularly superior for the two datasets of 50 and 100 individual stocks. It can also be difficult to surpass the combination of the EW portfolio and the risk-free asset when N/T is large, but this combination is largely suboptimal otherwise.

Fourth, when can we expect to outperform the combined portfolios? Both the constrained and unconstrained combinations easily outperform the SMV portfolio. While the unconstrained combination consistently adds value to the risk-free asset, the constrained combination is inferior to the risk-free asset once γ is large enough. Compared to the combination of the EW portfolio and the risk-free asset, the unconstrained and shrinkage unconstrained

SGMV. This can be explained because the theoretical gain from adding SGMV is small and thus is offset by the estimation risk coming from the additional combination coefficient to estimate.

combinations deliver large gains in most settings, but can underperform in two cases: when there is little cross-sectional variation in mean returns and when estimation risk is large. However, the shrinkage unconstrained strategy estimated with a shrinkage covariance matrix only loses little performance in such cases, and thus can be safely relied upon.

Fifth, how does the distributional assumption matter? Given that empirical data are typically not i.i.d. normal, one can wonder if we can safely exploit our portfolio combinations calibrated under this assumption. Our answer is positive because in a rather general elliptical model for asset returns, the expected out-of-sample utility loss incurred by an investor who wrongly assumes that the data is i.i.d. normal is very small. The satisfying performance of our portfolio strategies in the six empirical datasets we consider reinforces this conclusion.

Overall, the shrinkage unconstrained combination of the SMV and EW portfolios, calibrated under the i.i.d. normal assumption, offers the best all-around out-of-sample performance across different datasets, levels of estimation risk, and degrees of risk aversion.

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Supplementary Material to

On the Combination of Naive and Mean-Variance Portfolio Strategies

Nathan Lassance Rodolphe Vanderveken Frédéric Vrins

This Supplementary Material is divided in four sections. In Section A, we provide a detailed literature review. In Section B, we give simulation and theoretical evidence regarding the impact of estimation errors on constrained and unconstrained combination coefficients. In Section C, we provide additional theoretical results and corresponding empirical results. In Section D, we detail the proofs of all theoretical results in the main body of the paper and in this Supplementary Material.

A Literature review

Our work is related to three strands of the portfolio selection literature. First, we contribute to the literature on portfolio combination, which was pioneered by Kan and Zhou (2007) and extended in, e.g., Frahm and Memmel (2010), Tu and Zhou (2011), DeMiguel et al. (2013, 2015), Branger et al. (2019), Kan et al. (2022), Kazak and Pohlmeier (2022), Yuan and Zhou (2022), Kan and Wang (2023), Lassance et al. (2023), and Bodnar et al. (2023). We consider the setting of Tu and Zhou (2011) who combine the SMV and EW portfolios to optimize expected out-of-sample utility under a convexity constraint on the combination coefficients. We derive the *unconstrained* combination and demonstrate several of its benefits relative to the constrained strategy. We also introduce a shrinkage methodology to alleviate the impact of estimation errors on the unconstrained combination coefficients, which can be applied to other types of portfolio combination considered in the above literature.

Second, our work is related to other papers that study the value of the naive EW portfolio. DeMiguel et al. (2009) find that the SMV portfolio and 13 robust extensions cannot outperform EW consistently. Pflug et al. (2012), Yan and Zhang (2017), and Yuan and Zhou (2022) explain that EW is not so naive because it is nearly optimal under high model ambiguity, in the absence of mispricing, and under a one-factor model, respectively. Tu and Zhou (2011) and Yuan and Zhou (2022) combine the EW and SMV portfolios to optimize out-of-sample utility and Sharpe ratio, respectively. While they allow for a risk-free asset, Frahm and Memmel (2010), Simaan and Simaan (2019), and Bodnar et al. (2023) combine EW with SMV or SGMV portfolios in the setting with no risk-free asset. In this paper, we develop two novel estimators to optimally combine the SMV and EW portfolios. Our results affirm the value of the EW portfolio because we show theoretically and empirically that our unconstrained strategy consistently outperforms the two-fund rule of Kan and Zhou (2007) that does not invest in EW. We also demonstrate that combining the SMV portfolio with EW rather than SGMV delivers larger diversification gains, lower turnover, and largely superior performance when the number of assets is sizable relative to the sample size.

Third, we contribute to the literature on the role of constraints in portfolio selection. Frost and Savarino (1988), Jagannathan and Ma (2003), and Behr et al. (2013) show the performance benefits of imposing lower and upper bound constraints on portfolio weights. Levy and Levy (2014) introduce a class of differential variance-based constraints that significantly outperforms homogeneous bound constraints. DeMiguel et al. (2009a), Yen (2016), Callot et al. (2021), and Zhao et al. (2021) show the large benefits of norm constraints. Li (2015) and Ao et al. (2019) introduce constrained linear regression approaches to estimate mean-variance portfolios. A key differentiating feature of our work is that we show the benefits of constraints for combining *sample portfolios*, instead of assets. In particular, we demonstrate in several settings that imposing a convexity constraint on combination coefficients, although unnecessary in theory when a risk-free asset is available, acts as a bound constraint that decreases their sensitivity to estimation errors. We exploit this result to construct an improved shrinkage estimator of unconstrained portfolio combination strategies.

B Estimation errors in combination coefficients

To establish more precisely the effect of estimation errors in mean returns $\boldsymbol{\mu}$ on combination coefficients discussed in Section 4, consider the experiment in Figure SM.1. We calibrate $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$ to the 25SBTM dataset and simulate 10,000 times $T = 120$ normal returns. For each simulation, we estimate the constrained and unconstrained combination coefficients with a risk-aversion coefficient $\gamma = 3$, in two different cases. First, all parameters in the coefficients are estimated. We estimate the parameters as in the simulation analysis of Section 4.3. Second, mean returns $\boldsymbol{\mu}$ are known, and we only plug in the sample estimate of $\boldsymbol{\Sigma}$.

We make two main observations from Figure SM.1. First, mostly errors in $\boldsymbol{\mu}$ matter, as the boxplots are much thinner when only $\boldsymbol{\Sigma}$ is estimated. Second, the boxplots are similar and quite thin for κ_1^c, κ_2^c, and κ_1^u, because they are bounded between zero and one and estimation errors compensate in their numerator and denominator. In contrast, when $\boldsymbol{\mu}$ is estimated, the boxplot for κ_2^u is substantially wider. Given that $\kappa_2^u = \frac{1}{\gamma}\frac{\mu_{ew}}{\sigma_{ew}^2}(1-\kappa_1^u)$ while κ_1^u is moderately sensitive to $\boldsymbol{\mu}$ and errors in $\boldsymbol{\mu}$ are more impactful than errors in $\boldsymbol{\Sigma}$, this difference is mostly explained by the proportionality to $\mu_{ew} = \boldsymbol{w}_{ew}'\boldsymbol{\mu}$. Indeed, in Figure SM.1 we depict scatterplots of estimation errors in κ_1^u and κ_2^u as a function of estimation errors in μ_{ew} and we find a near-zero correlation for κ_1^u versus a near-perfect one for κ_2^u. Moreover, the best linear fit for estimation errors in κ_2^u is nearly identical to the derivative of κ_2^u with respect to μ_{ew} assuming that κ_1^u is independent of μ_{ew}.

In the next proposition, we give theoretical complement to these simulation results by deriving in closed form the estimation error in combination coefficients as a function of estimation errors in mean returns, $\boldsymbol{\varepsilon} = \hat{\boldsymbol{\mu}} - \boldsymbol{\mu}$.

Proposition SM.1. *Let $\boldsymbol{\Sigma}$ be known and denote $\boldsymbol{\varepsilon} = \hat{\boldsymbol{\mu}} - \boldsymbol{\mu}$ and $\bar{\varepsilon} = \boldsymbol{w}_{ew}'\boldsymbol{\varepsilon}$. Then, the estimation errors in combination coefficients are given by*

$$\hat{\kappa}_1^u - \kappa_1^u = \frac{\psi^2 + 2\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma_{ew} - \bar{\varepsilon}^2/\sigma_{ew}^2}{\psi^2 + d + 2c\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + c\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma_{ew} - \bar{\varepsilon}^2/\sigma_{ew}^2} - \frac{\psi^2}{\psi^2 + d}, \tag{SM1}$$

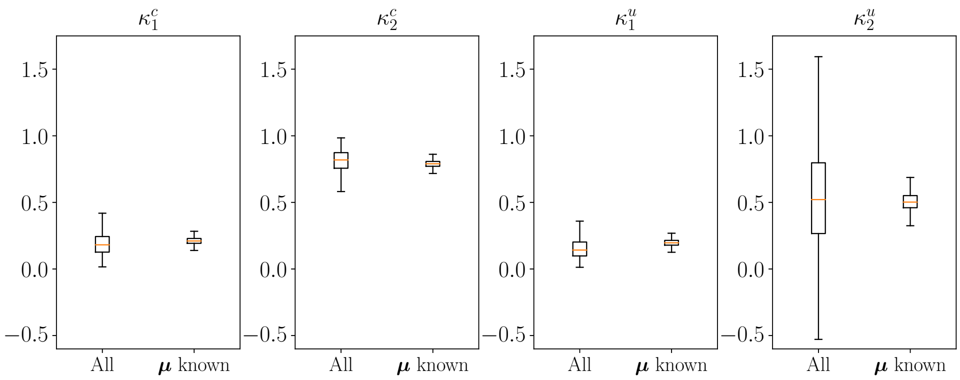
$$\hat{\kappa}_2^u - \kappa_2^u = \frac{\gamma_{ew}}{\gamma}(\kappa_1^u - \hat{\kappa}_1^u) + \frac{\bar{\varepsilon}}{\gamma\sigma_{ew}^2}(1 - \hat{\kappa}_1^u), \tag{SM2}$$

$$\hat{\kappa}_1^c - \kappa_1^c = -(\hat{\kappa}_2^c - \kappa_2^c) = \frac{\psi^2 + \ell^2 + 2\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma}{\psi^2 + \ell^2 + d + 2c\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + c\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma} - \frac{\psi^2 + \ell^2}{\psi^2 + \ell^2 + d}. \tag{SM3}$$

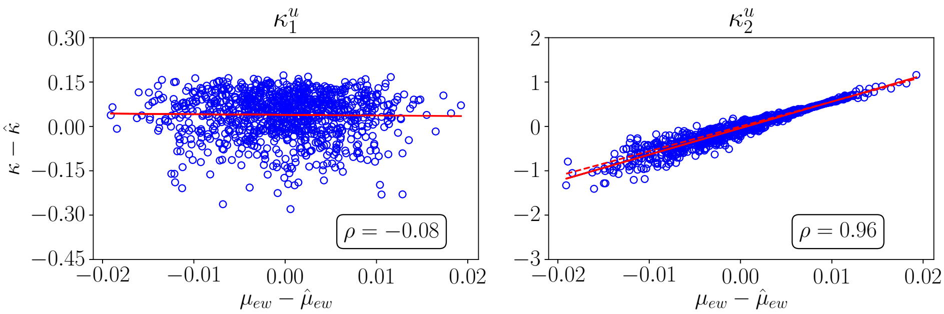
Moreover, the estimation error for $\tilde{\kappa}_2^u = \frac{\hat{\mu}_{ew}}{\gamma\sigma_{ew}^2}(1-\kappa_1^u)$ is

$$\tilde{\kappa}_2^u - \kappa_2^u = \frac{\bar{\varepsilon}d}{\gamma\sigma_{ew}^2(\psi^2 + d)}. \tag{SM4}$$

Figure SM.1: Estimation errors in constrained and unconstrained combination coefficients



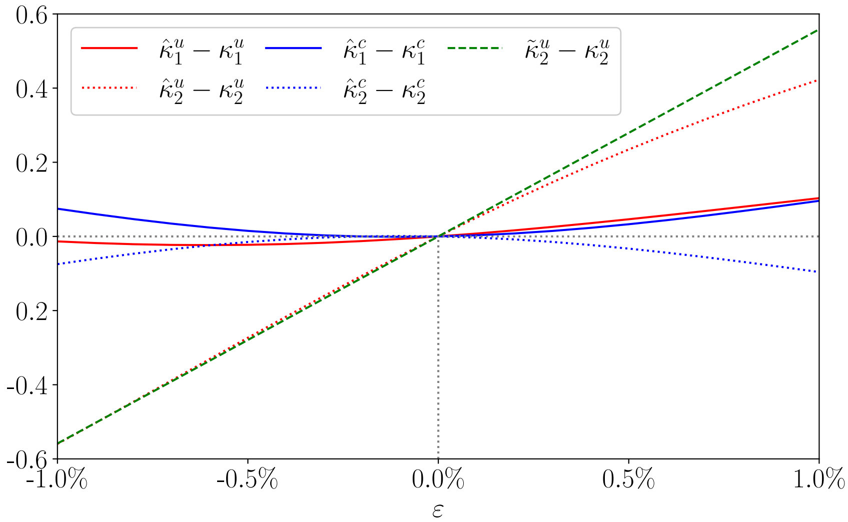
(a) Boxplots of estimation errors in all combination coefficients



(b) Scatterplots of estimation errors in unconstrained combination coefficients

Notes. This figure depicts estimation errors in constrained combination coefficients, κ_1^c and κ_2^c in (13), and unconstrained combination coefficients, κ_1^u and κ_2^u in (14), when the risk-aversion coefficient $\gamma = 3$. The results are obtained by simulating 10,000 times $T = 120$ asset returns from a multivariate normal distribution whose moments are calibrated from a dataset of 25 portfolios of firms sorted on size and book-to-market spanning July 1926 to December 2022. In the boxplots of Panel (a), we consider two cases to estimate the coefficients: 1) all parameters are estimated, 2) mean returns $\boldsymbol{\mu}$ are known and we only plug in the sample estimate of $\boldsymbol{\Sigma}$. In Panel (b), we depict how estimation errors in unconstrained combination coefficients depend on estimation errors in the parameter μ_{ew}. The best linear fit in each scatterplot is depicted in solid red. The dashed red line in the right plot of Panel (b) is the derivative of κ_2^u with respect to μ_{ew} assuming that κ_1^u is independent of μ_{ew}.

Figure SM.2: Estimation errors in combination coefficients as a function of estimation errors in mean returns



Notes. This figure depicts the estimation errors in combination coefficients given by Proposition SM.1 when errors in sample mean returns are of the form $\boldsymbol{\varepsilon} = \hat{\boldsymbol{\mu}} - \boldsymbol{\mu} = \varepsilon \mathbf{1}$, as a function of ε. We use a sample size $T = 120$ and a risk-aversion coefficient $\gamma = 3$. The figure is constructed by calibrating the population vector of means and covariance matrix of asset excess returns from monthly returns on the 25 portfolios of firms sorted on size and book-to-market spanning July 1926 to December 2022.

In Figure SM.2, we consider the 25SBTM dataset, $T = 120$ and $\gamma = 3$, as in Figure SM.1. We depict the estimation errors in combination coefficients derived in Proposition SM.1 in the specific case where $\boldsymbol{\varepsilon} = \varepsilon\mathbf{1}$, as a function of $\varepsilon \in [-1\%, 1\%]$. Consistent with the simulation results in Figure SM.1, it is the estimation error in $\hat{\kappa}_2^u$ that is most sensitive to ε. Moreover, the estimation errors for $\hat{\kappa}_2^u$ and $\tilde{\kappa}_2^u$ are similar, which confirms that it is mostly the proportionality to $\hat{\mu}_{ew}$ that matters.

In conclusion, the results presented in this section confirm that we can focus on the impact of estimation errors in μ_{ew} on κ_2^u to capture most of the difference in the way estimation errors impact the constrained and unconstrained combination coefficients.

C Additional theoretical and empirical results

In this section, we provide theoretical and empirical results that complement those in the main body of the paper. First, we compare the exposure to three underlying funds in the constrained and unconstrained strategies. Second, we provide theoretical results for the high-dimensional asymptotic regime. Third, we provide a shrinkage estimator for the unconstrained combination of the SMV and SGMV portfolios. Fourth, we show that the convexity constraint helps explain why combining portfolios with the risk-free asset can hurt performance relative to fully invested combinations. Fifth, we derive an unconstrained four-fund strategy that combines the SMV portfolio with both the SGMV and EW portfolios. Sixth, we compare the correlation between the out-of-sample return of the SMV portfolio and that of the SGMV and EW portfolios. Seventh, we compare the expected out-of-sample Sharpe ratio of the constrained and unconstrained strategies. Eighth, we compare the empirical performance under the normal versus the elliptical distribution. Ninth, we replicate our empirical results with a nonlinear shrinkage estimator of the covariance matrix. Tenth, we report the average monthly turnover of the considered portfolio strategies. Eleventh, we depict the empirical outperformance relative to the constrained strategy.

C.1 Comparison of exposure to the three funds

The portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ invests in three funds: the EW portfolio, the sample tangent portfolio, and the risk-free asset. In this section, we compare the allocation to these three funds implied by the constrained and unconstrained combination coefficients, $\boldsymbol{\kappa}^c$ in (13) and $\boldsymbol{\kappa}^u$ in (14). Note that these two combination strategies are not always different: (i) they fully invest in the SMV portfolio in the absence of estimation risk ($d \to 0$), (ii) they fully invest in the risk-free asset as $\gamma \to \infty$, and (iii) they coincide when $\gamma = \gamma_{ew}$.

Specifically, in the next proposition, we compare the expected value of the weight allocated to the three funds:[SM1]

$$\pi_{tan} = \frac{\gamma_{tan}}{\gamma} \kappa_1, \tag{SM5}$$

$$\pi_{ew} = \kappa_2, \tag{SM6}$$

[SM1] We study the expected value of because the actual weights on the sample tangent portfolio and the risk-free asset depend on $\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$, which is sample-dependent.

$$\pi_{rf} = 1 - \pi_{tan} - \pi_{ew}, \tag{SM7}$$

Proposition SM.2. *The following holds regarding the allocation to the three funds in the constrained and unconstrained strategies:*

1. *The constrained strategy overinvests in the SMV portfolio:* $0 \leq \kappa_1^u \leq \kappa_1^c \leq 1$. *Similarly, for the weight on the sample tangent portfolio,* $0 \leq \pi_{tan}^u \leq \pi_{tan}^c$ *if* $\gamma_{tan} \geq 0$.[SM2]

2. *The constrained strategy overinvests in the EW portfolio,* $0 \leq \kappa_2^u \leq \kappa_2^c$, *if* $\gamma \in \left[\gamma_{ew}, \frac{\theta^2+d}{\mu_{ew}}\right]$, *and* $\kappa_2^u \geq \kappa_2^c$ *otherwise.*[SM3]

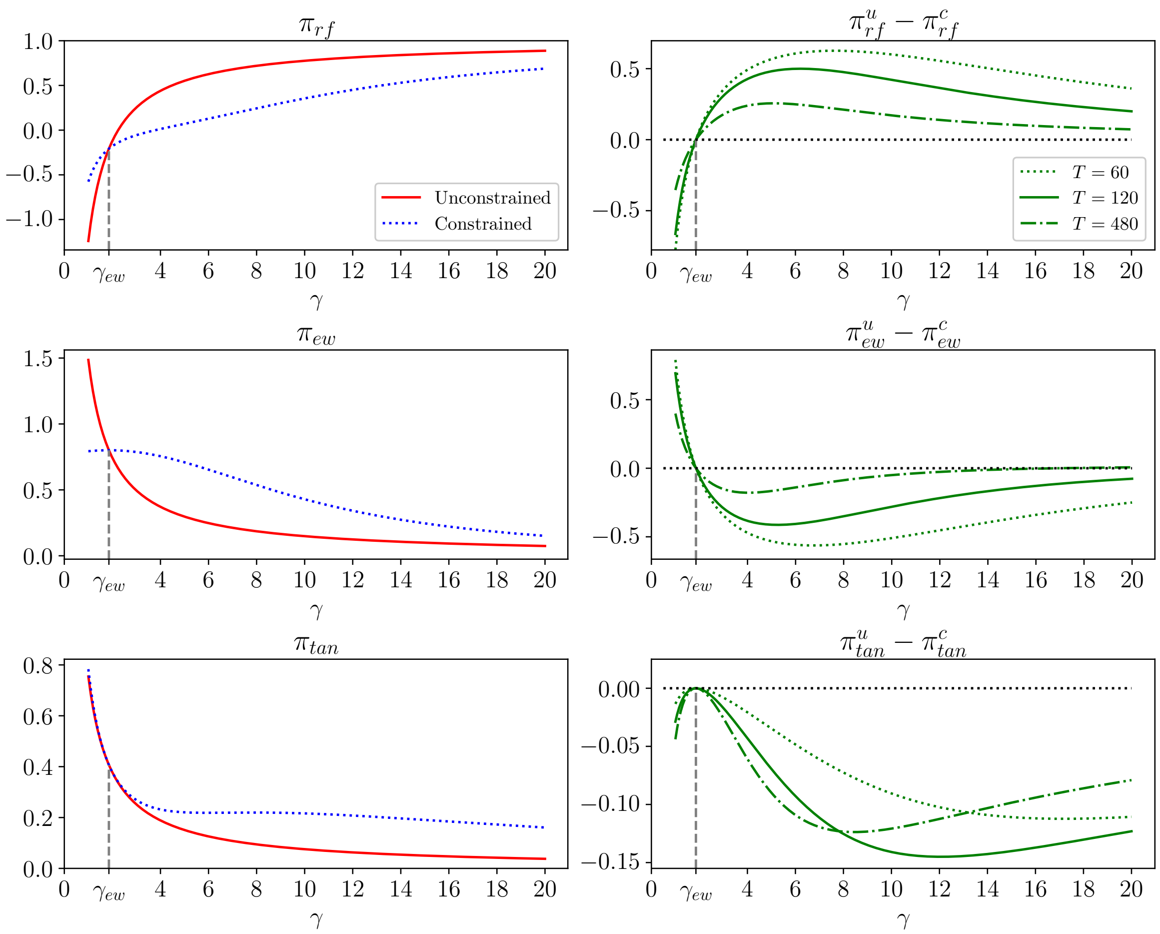
3. *Let* $\gamma_{ew} < \gamma_{tan} < (\theta^2 + d)/\mu_{ew}$.[SM4] *Then, the constrained strategy underexploits the risk-free asset:* $\pi_{rf}^u \leq \pi_{rf}^c \leq 0$ *if* $\gamma \leq \gamma_{ew}$ *and* $\pi_{rf}^u \geq \pi_{rf}^c$ *otherwise.*

We illustrate Proposition SM.2 in Figure SM.3 by calibrating the moments of asset excess returns, $\boldsymbol{\mu}$ and $\boldsymbol{\Sigma}$, to the 25SBTM dataset. The left panels use $T = 120$, and the right panels use $T = 60, 120$, and 480. The figure shows that the allocation to the three funds in the constrained strategy is substantially different from that in the unconstrained strategy. Consider an investor with $\gamma = 5$, which is larger than $\gamma_{ew} = 1.86$. When $T = 120$, the constrained strategy commands a weight of 22% on the tangent portfolio, 71% on the EW portfolio, and 7% on the risk-free asset. In contrast, the unconstrained strategy invests much more in the risk-free asset (55%) and much less in the tangent and EW portfolios (15% and 30%, respectively). When γ increases to 10, the difference is even more striking: $(\pi_{rf}^u, \pi_{tan}^u, \pi_{ew}^u) = (77\%, 8\%, 15\%)$ while $(\pi_{rf}^c, \pi_{tan}^c, \pi_{ew}^c) = (35\%, 22\%, 43\%)$.

Finally, the right panels show that the difference between the constrained and unconstrained strategies is larger when T is smaller. That is, it is when combining portfolios is most needed because estimation errors are large that the convexity constraint hurts most.

[SM2] $\gamma_{tan} \geq 0$ if the GMV portfolio has a positive mean return, which is a mild assumption.
[SM3] This interval is typically wide. For the data used in Figure SM.3, it is equal to $[1.86, 43.06]$ when $T = 120$.
[SM4] This is a mild assumption fulfilled in all datasets we use in the empirical analysis of Section 6.

Figure SM.3: Weights on the three funds in the constrained and unconstrained strategies



Notes. This figure compares the expected weight allocated to the three funds by the constrained and unconstrained combination strategies defined in (13) and (14), respectively. The three funds (weights) are the risk-free asset (π_{rf}), the equally weighted portfolio (π_{ew}), and the sample tangent portfolio (π_{tan}). The formulas for the expected weights are given by (SM5)–(SM7). The figure is constructed by calibrating the population vector of means and covariance matrix of asset excess returns from monthly returns on the 25 portfolios of firms sorted on size and book-to-market spanning July 1926 to December 2022. The left panels depict the weight allocated to the three funds as a function of γ when the sample size $T = 120$. The right panels depict the difference in weights between the unconstrained and constrained strategies as a function of γ when $T = 60, 120,$ and 480.

C.2 High-dimensional asymptotic regime

A recent literature has extensively studied the behavior of sample portfolios in the high-dimensional asymptotic regime, i.e., when $N, T \to \infty$ and $N/T \to \rho \in (0,1)$, where ρ is called the concentration ratio; see Bodnar et al. (2018), Ao et al. (2019), Ledoit and Wolf (2020), Bodnar et al. (2022), Kan et al. (2022a), and Bodnar et al. (2023), among others. This asymptotic regime is useful for evaluating portfolio performance in high dimension, and also because it is often analytically more tractable than the finite-sample setting.

In the next proposition, we derive the asymptotic joint distribution of the out-of-sample mean return and variance of the portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ in (7), and also the asymptotic oracle constrained and unconstrained combination coefficients.

Proposition SM.3. *Let $N, T \to \infty$ and $N/T \to \rho \in (0,1)$ while $(\mu_{ew}, \sigma_{ew}^2, \theta^2)$ stay bounded. Then, the following holds:*

1. *The joint distribution of the out-of-sample mean return and variance of the portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ in (7) is*

$$\sqrt{T-N}\begin{bmatrix} \hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu} - \left(\frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_{ew}\right) \\ \hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) - \left(\frac{\kappa_1^2}{\gamma^2}\frac{\theta^2+\rho}{1-\rho} + \kappa_2^2\sigma_{ew}^2 + \frac{2\kappa_1\kappa_2}{\gamma}\mu_{ew}\right) \end{bmatrix} \xrightarrow{d} \mathcal{N}(\mathbf{0}_2, \boldsymbol{V}), \tag{SM8}$$

where $\mathbf{0}_2$ is a 2×1 vector of zeros and $\boldsymbol{V}$ is a 2×2 matrix with entries

$$\boldsymbol{V}_{11} = \frac{\kappa_1^2}{\gamma^2}\theta^2(1+2\theta^2), \tag{SM9}$$

$$\boldsymbol{V}_{12} = \boldsymbol{V}_{21} = \frac{2\kappa_1^2}{\gamma^2}(1+2\theta^2)\left(\frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_{ew}\right), \tag{SM10}$$

$$\begin{aligned}\boldsymbol{V}_{22} = \frac{2\kappa_1^2}{\gamma^2}\Bigg(&\frac{\kappa_1^2}{\gamma^2}\Big(\theta^4(7+\rho) + 2\theta^2(2+\rho) + \rho\Big) + 2\kappa_2^2\sigma_{ew}^2(1+\theta_{ew}^2+\theta^2) \\ &+ \frac{4\kappa_1\kappa_2}{\gamma}\mu_{ew}(1+2\theta^2)\Bigg).\end{aligned} \tag{SM11}$$

2. The asymptotic oracle constrained combination coefficients $\boldsymbol{\kappa}^c$ in (13) *are*

$$\kappa_1^c \xrightarrow{p} \bar{\kappa}_1^c = \frac{\psi^2 + \ell^2}{\psi^2 + \ell^2 + \frac{\rho}{1-\rho}(1+\theta^2)} \quad \textit{and} \quad \kappa_2^c \xrightarrow{p} 1 - \bar{\kappa}_1^c. \tag{SM12}$$

3. The asymptotic oracle unconstrained combination coefficients $\boldsymbol{\kappa}^u$ in (14) *are*

$$\kappa_1^u \xrightarrow{p} \bar{\kappa}_1^u = \frac{\psi^2}{\psi^2 + \frac{\rho}{1-\rho}(1+\theta^2)} \quad \textit{and} \quad \kappa_2^u \xrightarrow{p} \frac{\gamma_{ew}}{\gamma}(1 - \bar{\kappa}_1^u). \tag{SM13}$$

Using the joint asymptotic distribution of the out-of-sample mean return and variance in (SM8), we can also obtain the asymptotic distribution of out-of-sample utility as

$$\begin{aligned}&\sqrt{T-N}\left[U(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) - \left(\frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_{ew} - \frac{\gamma}{2}\left(\frac{\kappa_1^2}{\gamma^2}\frac{\theta^2+\rho}{1-\rho} + \kappa_2^2\sigma_{ew}^2 + \frac{2\kappa_1\kappa_2}{\gamma}\mu_{ew}\right)\right)\right] \\ &\xrightarrow{d} \mathcal{N}\left(0, \boldsymbol{V}_{11} + \frac{\gamma^2}{4}\boldsymbol{V}_{22} - \gamma\boldsymbol{V}_{12}\right),\end{aligned} \tag{SM14}$$

where

$$\begin{aligned}\boldsymbol{V}_{11} + \frac{\gamma^2}{4}\boldsymbol{V}_{22} - \gamma\boldsymbol{V}_{12} = &\frac{\kappa_1^2}{2\gamma^2}\Big(\kappa_1^2\Big(\theta^4(7+\rho) + 2\theta^2(2+\rho) + \rho\Big) + 2(1-2\kappa_1)\theta^2(1+2\theta^2)\Big) \\ &- \frac{2}{\gamma}\kappa_1^2(1-\kappa_1)\kappa_2\mu_{ew}(1+2\theta^2) + \kappa_1^2\kappa_2^2\sigma_{ew}^2(1+\theta_{ew}^2+\theta^2).\end{aligned} \tag{SM15}$$

C.3 Shrinkage estimator of Kan and Zhou three-fund rule

In Section 4, we introduce a shrinkage estimator of the unconstrained combination of the SMV and EW portfolios. In this section, we derive a similar shrinkage estimator for the three-fund rule of Kan and Zhou (2007) that combines the SMV and SGMV portfolios. Specifically, we consider the three-fund portfolio combination

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1\hat{\boldsymbol{w}}^\star + \kappa_2\hat{\boldsymbol{w}}_g, \tag{SM16}$$

where $\hat{\boldsymbol{w}}_g = \widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}/(\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1})$ is the SGMV portfolio and $\hat{\boldsymbol{w}}^\star$ is the SMV portfolio in (5). This combination is comparable to that in Tu and Zhou (2011) because, just like $\boldsymbol{w}_{ew}$, $\hat{\boldsymbol{w}}_g$ is fully invested in risky assets. Kan and Zhou (2007) consider a similar portfolio combination but with unnormalized weights for the SGMV portfolio. We use the notation

$$\mu_g = \boldsymbol{w}_g'\boldsymbol{\mu}, \quad \sigma_g^2 = \boldsymbol{w}_g'\boldsymbol{\Sigma}\boldsymbol{w}_g, \quad \theta_g = \mu_g/\sigma_g, \quad \text{and} \quad \psi_g^2 = \theta^2 - \theta_g^2 \geq 0. \tag{SM17}$$

In the following proposition, we derive the EU of the portfolio combination in (SM16) as well as the unconstrained combination coefficients.

Proposition SM.4. *The following holds concerning the three-fund portfolio in* (SM16)*:*

1. *The expected out-of-sample utility is*

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = \frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_g - \frac{\gamma}{2}\left(\frac{\kappa_1^2}{\gamma^2}(\theta^2 + d) + c_1\kappa_2^2\sigma_g^2 + \frac{2c_1\kappa_1\kappa_2}{\gamma}\mu_g\right), \tag{SM18}$$

where

$$c_1 = \frac{T-2}{T-N-1} \geq 1. \tag{SM19}$$

2. *The unconstrained combination coefficients maximizing* (SM18) *are*

$$\kappa_1^u = \frac{1}{c}\frac{\psi_g^2}{\psi_g^2 + N/T + \frac{2\theta_g^2}{T-N-2}} \in [0,1] \quad \textit{and} \quad \kappa_2^u = \frac{\gamma_{tan}}{\gamma}(1/c_1 - \kappa_1^u). \tag{SM20}$$

3. *The expected out-of-sample utility delivered by the unconstrained combinations coefficients is*

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)) = U^\star - \frac{d}{2\gamma}\kappa_1^u - \frac{1}{2}(c_1 - 1)\mu_g\kappa_2^u. \tag{SM21}$$

Similar to the combination with EW in Section 4, κ_2^u is unbounded and proportional to γ_{tan}, and thus is sensitive to estimation errors in mean returns $\boldsymbol{\mu}$ via μ_g. Therefore, in the next proposition we derive the EU of the unconstrained strategy when κ_2^u is estimated as

$$\tilde{\kappa}_2^u = \frac{\hat{\mu}_g}{\gamma\sigma_g^2}(1/c_1 - \kappa_1^u) = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}{\gamma}(1/c_1 - \kappa_1^u). \tag{SM22}$$

Proposition SM.5. *Let the combination coefficient κ_2^u be estimated by $\tilde{\kappa}_2^u$ in* (SM22)*. Then,* *the expected out-of-sample utility of the estimated unconstrained strategy is*

$$EU(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)) = EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)) - \frac{c_1}{2\gamma T}(1/c_1^2 - (\kappa_1^u)^2), \tag{SM23}$$

where $EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u))$ *in* (SM21) *is the utility of the oracle unconstrained strategy.*

We find numerically that the EU loss in (SM23) can be substantial. To alleviate the impact of estimation errors on the performance of unconstrained strategy, we proceed as in Section 4 and shrink $\tilde{\kappa}_2^u$ in (SM22) toward the target $1/c_1 - \kappa_1^u$,

$$\tilde{\kappa}_2^u(\delta) = (1-\delta)\tilde{\kappa}_2^u + \delta(1/c_1 - \kappa_1^u), \tag{SM24}$$

where δ is the shrinkage intensity. We now calibrate δ to minimize the mean squared error of $\tilde{\kappa}_2^u(\delta)$.

Proposition SM.6. *The shrinkage intensity δ minimizing the mean squared error of $\tilde{\kappa}_2^u(\delta)$ in* (SM24) *is*

$$\delta^\star = \frac{1}{1 + \sigma_g^2 T(\gamma - \gamma_{tan})^2}. \tag{SM25}$$

In Table SM.1, we compare the out-of-sample performance of the shrinkage estimator of the unconstrained three-fund rule (SUNC3F), which we compare with its non-shrunk version (UNC3F). The table shows that SUNC3F greatly improves upon UNC3F for small values of $\gamma = 1$ and 5, while they perform similarly when γ increases to 10 and 15.

C.4 Investing in the risk-free asset can hurt performance

We observe in our empirical results in Tables 3 and 4 that the combination of the SMV and SGMV portfolios without a risk-free asset in Kan et al. (2022) often outperforms the combination with a risk-free asset in Kan and Zhou (2007) when the risk-aversion coefficient is $\gamma = 5$. In this section, we show that this can be explained due to the convexity constraint, similar to the results in Section 4.

Table SM.1: Annualized net out-of-sample utility of shrinkage estimator of Kan and Zhou three-fund rule

	$T = 120$						$T = 240$					
	25SBTM	10MOM	25OPIN	48IND	50STO	100STO	25SBTM	10MOM	25OPIN	48IND	50STO	100STO
Panel (a): $\gamma = 1$												
SUNC3F	14.1	25.2	-2.77	-31.6	-104	-939	19.8	29.3	28.2	0.45	-1.80	-79.7
UNC3F	11.9	25.3	-4.83	-37.3	-113	-949	19.7	29.8	28.4	-0.77	-5.61	-85.0
Panel (b): $\gamma = 5$												
SUNC3F	2.54	5.31	-0.28	-5.87	-20.8	-219	4.04	6.17	6.07	0.38	-0.18	-16.1
UNC3F	2.05	5.00	-1.10	-7.68	-23.2	-221	3.71	5.93	5.66	-0.17	-1.20	-17.4
Panel (c): $\gamma = 10$												
SUNC3F	1.41	2.68	-0.14	-3.44	-10.2	-111	2.01	3.00	2.91	-0.21	0.21	-7.91
UNC3F	1.00	2.50	-0.56	-3.85	-11.6	-113	1.84	2.96	2.83	-0.09	-0.60	-8.74
Panel (d): $\gamma = 15$												
SUNC3F	0.87	1.63	-0.24	-2.49	-6.76	-74.7	1.23	1.95	1.78	-0.22	-0.22	-5.06
UNC3F	0.66	1.66	-0.38	-2.57	-7.75	-75.6	1.22	1.97	1.88	-0.06	-0.40	-5.83

Notes. This table reports the annualized net out-of-sample utility in percentage points for two portfolio strategies derived in Section C.3 when using the sample covariance matrix. The first strategy (UNC3F) is the unconstrained combination of the sample tangent portfolio, the sample global-minimum-variance portfolio, and the risk-free asset. The second strategy (SUNC3F) is the shrinkage estimator of the unconstrained combination. The net out-of-sample utility is computed using proportional transaction costs of 10 basis points. We consider sample sizes $T = 120$ and 240, and risk-aversion coefficients $\gamma = 1, 5, 10,$ and 15.

Kan et al. (2022) combine the fully invested SMV and SGMV portfolios,

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1 \hat{\boldsymbol{w}}_g + \kappa_2(\hat{\boldsymbol{w}}_g + \hat{\boldsymbol{w}}_z), \tag{SM26}$$

where $\hat{\boldsymbol{w}}_g = \widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}/(\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1})$ is the SGMV portfolio and

$$\hat{\boldsymbol{w}}_z = \frac{1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}(\hat{\boldsymbol{\mu}} - \hat{\mu}_g \mathbf{1}) \tag{SM27}$$

is a zero-cost portfolio (i.e., $\mathbf{1}'\hat{\boldsymbol{w}}_z = 0$). To ensure that $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ is fully invested in risky assets, the authors impose the convexity constraint $\kappa_1 + \kappa_2 = 1$, so that the combination depends on a single combination coefficient κ:

$$\hat{\boldsymbol{w}}(\kappa) = \hat{\boldsymbol{w}}_g + \kappa \hat{\boldsymbol{w}}_z. \tag{SM28}$$

Kan et al. (2022) show that the combination coefficient κ maximizing the EU is

$$\kappa^{kwz} = \frac{(T-N)(T-N-3)}{(T-2)(T-N-2)} \frac{\psi_g^2}{\psi_g^2 + (N-1)/T}. \tag{SM29}$$

Relaxing the convexity constraint would amount to combine the SMV and SGMV portfolios with the risk-free asset, which is what the three-fund rule of Kan and Zhou (2007) is designed to do. Their optimal unconstrained portfolio combination is

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz}) = \frac{\kappa_1^{kz}}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} + \frac{\kappa_2^{kz}}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}, \tag{SM30}$$

where

$$\kappa_1^{kz} = \frac{1}{c}\frac{\psi_g^2}{\psi_g^2 + N/T} \quad \text{and} \quad \kappa_2^{kz} = \mu_g(1/c - \kappa_1^{kz}). \tag{SM31}$$

In the next proposition, we derive the EU of the portfolio combinations of Kan and Zhou (2007) and Kan et al. (2022).

Proposition SM.7. *The expected out-of-sample utility of the optimal portfolio combinations of Kan and Zhou (2007) and Kan et al. (2022) are, respectively,*

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz})) = U^\star - \frac{d}{2\gamma}\kappa_1^{kz} - \frac{c-1}{2\gamma}\gamma_{tan}\kappa_2^{kz}, \tag{SM32}$$

$$EU(\hat{\boldsymbol{w}}(\kappa^{kwz})) = \mu_g - \frac{\gamma}{2}c_1\sigma_g^2 + \frac{T-N-2}{T-N-1}\frac{\psi_g^2}{2\gamma}\kappa^{kwz}, \tag{SM33}$$

where c_1 is defined in (SM19).

Similar to the combination with the EW portfolio in Section 4, κ_2^{kz} in (SM31) is unbounded and proportional to μ_g, and therefore is more sensitive to estimation errors in mean returns $\boldsymbol{\mu}$ relative to the other coefficients. Therefore, in the next proposition we derive the EU of the three-fund rule of Kan and Zhou (2007) when κ_2^{kz} is estimated as

$$\tilde{\kappa}_2^{kz} = \hat{\mu}_g(1/c - \kappa_1^{kz}) = \frac{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}(1/c - \kappa_1^{kz}). \tag{SM34}$$

Moreover, we identify the range of risk aversion γ for which the fully invested rule of Kan et al. (2022) delivers a larger EU than that of the estimated Kan-Zhou three-fund rule that also invests in the risk-free asset.

Proposition SM.8. *Let the combination coefficient κ_2^{kz} be estimated by $\tilde{\kappa}_2^{kz}$ in* (SM34).

Then,

1. *The expected out-of-sample utility of the estimated unconstrained three-fund strategy of Kan and Zhou (2007) is*

$$EU(\hat{\boldsymbol{w}}(\kappa_1^{kz}, \tilde{\kappa}_2^{kz})) = EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz})) - \frac{c}{2\gamma T}(1/c^2 - (\kappa_1^{kz})^2), \tag{SM35}$$

where $EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz}))$ *in* (SM32) *is the utility of the oracle unconstrained strategy.*

2. *The fully invested strategy of Kan et al. (2022) delivers a larger expected out-of-sample utility than the estimated unconstrained three-fund strategy of Kan and Zhou (2007),* $EU(\hat{\boldsymbol{w}}(\kappa^{kwz})) \geq EU(\hat{\boldsymbol{w}}(\kappa_1^{kz}, \tilde{\kappa}_2^{kz}))$*, if and only if* $b \geq 0$ *and* γ *belongs to the following interval:*

$$\gamma \in \left[\frac{\gamma_{tan}}{c_1} \pm \frac{\sqrt{b}}{c_1 \sigma_g}\right], \tag{SM36}$$

where

$$b = \theta_g^2 + c_1\Bigg(- \theta^2 + \psi_g^2 \frac{T-N-2}{T-N-1}\kappa^{kwz} + d\kappa_1^{kz} + (c-1)\gamma_{tan}\kappa_2^{kz} + \frac{c}{T}(1/c^2 - (\kappa_1^{kz})^2)\Bigg). \tag{SM37}$$

For the 25SBTM dataset and $T = 120$, we find that the interval is $\gamma \in [3.18 \pm 1.73]$. This explains why for $\gamma = 5$ in Tables 3 and 4 the fully invested rule of Kan et al. (2022) outperforms the three-fund rule of Kan and Zhou (2007), while the opposite happens when $\gamma = 1$, 10, and 15.

C.5 The optimal four-fund combination strategy

Kan and Zhou (2007) combine the SMV and SGMV portfolios, while as in Tu and Zhou (2011) we combine the SMV and EW portfolios. In this section, we derive the unconstrained four-fund portfolio that combines SMV with both SGMV and EW, and we assess empirically whether it is beneficial to include SGMV in addition to EW.

The four-fund portfolio combination is

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \frac{\kappa_1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} + \kappa_2 \boldsymbol{w}_{ew} + \frac{\kappa_3}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}. \tag{SM38}$$

In the next proposition, we derive the optimal unconstrained combination coefficients and the resulting EU for this four-fund combination, which are novel results in the literature.

Proposition SM.9. *The following holds concerning the four-fund portfolio combination in* (SM38)*:*

1. *The expected out-of-sample utility is*

$$\begin{aligned} EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) &= \frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_{ew} + \frac{\kappa_3}{\gamma}\gamma_{tan} \\ &- \frac{\gamma}{2}\left(c\left(\frac{\kappa_1^2}{\gamma^2}\left(\theta^2 + \frac{N}{T}\right) + \frac{\kappa_3^2}{\gamma^2}\frac{1}{\sigma_g^2} + \frac{2\kappa_1\kappa_3}{\gamma^2}\gamma_{tan}\right) + \kappa_2^2\sigma_{ew}^2 + \frac{2\kappa_1\kappa_2}{\gamma}\mu_{ew} + \frac{2\kappa_2\kappa_3}{\gamma}\right). \end{aligned} \tag{SM39}$$

2. *The unconstrained combination coefficients maximizing* (SM39) *are*

$$\kappa_1^u = \frac{1}{f}\left[\psi^2 - \theta_g^2 - \frac{\theta^2}{c}\frac{\sigma_g^2}{\sigma_{ew}^2} + \left(1 + \frac{1}{c}\right)\mu_g\gamma_{ew}\right], \tag{SM40}$$

$$\kappa_2^u = \frac{\gamma_{ew}}{\gamma}\frac{1}{f}\left[\frac{N}{T}\left(c - \frac{\mu_g}{\mu_{ew}}\right) + (c-1)\psi_g^2\right], \tag{SM41}$$

$$\kappa_3^u = \frac{\mu_g}{f}\left[\frac{d}{c}\left(1 - \frac{\gamma_{ew}}{\gamma_{tan}}\right) - \left(1 - \frac{1}{c}\right)\psi^2\right], \tag{SM42}$$

where

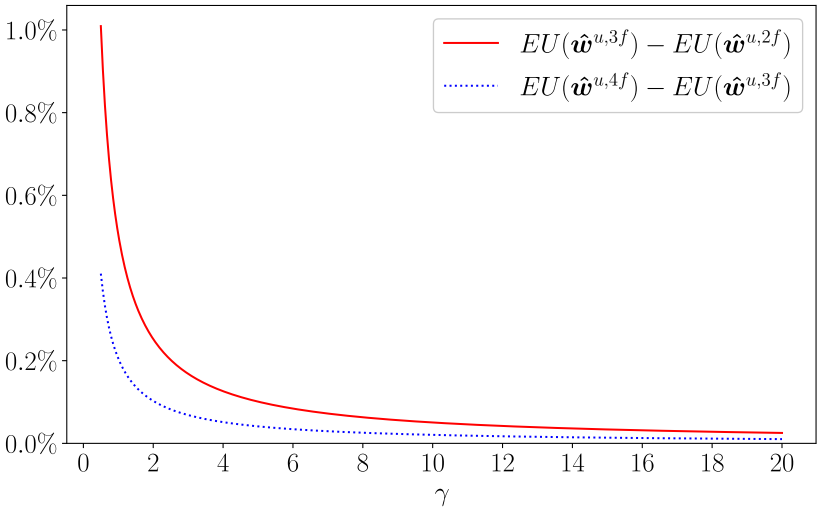
$$f = \psi^2 - c\theta_g^2 + d - \frac{\sigma_g^2}{\sigma_{ew}^2}\left(\theta^2 + \frac{N}{T}\right) + 2\mu_g\gamma_{ew}. \tag{SM43}$$

3. *The expected out-of-sample utility of the unconstrained four-fund portfolio is*

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)) = U^\star - \frac{d}{2\gamma}\kappa_1^u - \frac{c-1}{2\gamma}\gamma_{tan}\kappa_3^u. \tag{SM44}$$

In Figure SM.4, we evaluate how much EU can be gained in theory by opting for the unconstrained four-fund strategy rather than the unconstrained three-fund strategy we con-

Figure SM.4: Theoretical gain in expected out-of-sample utility with four-fund strategy



Notes. This figure depicts the difference between the expected out-of-sample utility of: (i) the unconstrained three-fund and two-fund rules (solid red), and (ii) the unconstrained four-fund and three-fund rules (dotted blue). The two-fund strategy is that in Kan and Zhou (2007) that combines the sample tangent portfolio with the risk-free asset. The three-fund strategy adds the equally weighted portfolio, and the four-fund strategy adds the sample global-minimum-variance portfolio. The figure is constructed by calibrating the population vector of means and covariance matrix of stock returns from monthly returns on the 25 portfolios of stocks sorted on size and book-to-market spanning July 1926 to December 2022, and using a sample size $T = 120$. The differences are depicted as a function of the risk-aversion coefficient γ between 0.5 and 20.

sider in the main body of the paper. That is, how much can be gained by adding the SGMV portfolio in the portfolio mix besides the SMV and EW portfolios. We depict this theoretical gain for the 25SBTM dataset and $T = 120$, and we compare it to the theoretical gain when going from the unconstrained two-fund strategy of Kan and Zhou (2007) to the unconstrained three-fund strategy that also invests in the EW portfolio. The figure shows that the incremental gain from adding the SGMV portfolio and relying on the four-fund portfolio is quite small relative to the gain obtained when going from two-fund to three-fund. Combined with the additional coefficient to estimate in the four-fund strategy relative to the three-fund strategy, this means that the unconstrained four-fund strategy may not outperform in practical settings.

We now evaluate the empirical performance of the unconstrained four-fund combination strategy. We compare its out-of-sample performance with that of the unconstrained three-fund strategy in the main body of the paper that does not invest in the SGMV portfolio.

Table SM.2: Annualized net out-of-sample utility of three-fund and four-fund strategies

	$T = 120$						$T = 240$					
	25SBTM	10MOM	25OPIN	48IND	50STO	100STO	25SBTM	10MOM	25OPIN	48IND	50STO	100STO
	Panel (a): $\gamma = 1$											
UNC3F	29.4	22.5	6.63	2.92	15.4	10.7	20.9	25.3	24.6	6.25	31.2	30.0
UNC4F	23.2	21.1	-0.74	-10.7	-20.6	-13.8	22.4	27.4	25.1	2.83	16.2	2.89
	Panel (b): $\gamma = 5$											
UNC3F	5.71	4.45	1.28	0.57	3.07	2.12	3.96	5.02	4.90	1.25	6.24	5.99
UNC4F	4.39	4.14	-0.25	-2.21	-4.24	-2.92	4.26	5.45	4.97	0.55	3.20	0.51
	Panel (c): $\gamma = 10$											
UNC3F	2.84	2.22	0.64	0.28	1.53	1.06	1.97	2.51	2.45	0.62	3.12	3.00
UNC4F	2.18	2.07	-0.13	-1.11	-2.13	-1.47	2.11	2.72	2.48	0.28	1.60	0.25
	Panel (d): $\gamma = 15$											
UNC3F	1.89	1.48	0.42	0.19	1.02	0.70	1.31	1.67	1.63	0.42	2.08	2.00
UNC4F	1.45	1.38	-0.09	-0.74	-1.42	-0.98	1.41	1.81	1.66	0.18	1.06	0.16

Notes. This table reports the annualized net out-of-sample utility in percentage points for two portfolio strategies when using the sample covariance matrix. The first strategy is the unconstrained three-fund strategy in the main body of the paper, which combines the sample tangent portfolio, the equally weighted portfolio, and the risk-free asset. The second strategy is the unconstrained four-fund strategy in Section C.5, which adds the sample global-minimum-variance portfolio. The net out-of-sample utility is computed using proportional transaction costs of 10 basis points. We consider sample sizes $T = 120$ and 240, and risk-aversion coefficients $\gamma = 1, 5, 10$, and 15.

We report the net out-of-sample utility of the two strategies in Table SM.2.

The table shows that the three-fund strategy outperforms the four-fund strategy, with the exception of the 25SBTM and 10MOM datasets when $T = 240$, in which case the four-fund portfolio is slightly superior. This result is consistent with the previous discussion that the theoretical gain from adding the SGMV portfolio is small and thus may disappear in practical settings due to estimation errors in the additional combination coefficient to estimate.

C.6 Out-of-sample return correlations

Tables 3 and 4 introduce the puzzling result that even in those cases where the GMVRF portfolio largely outperforms the EWRF portfolio, combining the SMV portfolio with the EW portfolio can deliver better performance than combining with the SGMV portfolio in Kan and Zhou (2007). Specifically, consider the 25SBTM, 10MOM, and 25OPIN datasets. GMVRF largely outperforms EWRF for these datasets, and thus, one would expect KZ3F to also outperform UNC3F. However, this is not always the case. Indeed, they perform similarly for the 25OPIN dataset, and UNC3F outperforms for 25SBTM. For instance, consider the shrinkage covariance matrix, $T = 120$, $\gamma = 1$, and the 25SBTM dataset. Then, GMVRF delivers a net utility of 14.9 and EWRF only 8.32, but still, UNC3F delivers a net utility of 37.3 while KZ3F delivers 33.3. This suggests that more favorable diversification properties arise when combining with the EW portfolio. To see this, in the next proposition we derive the correlation between the out-of-sample return of the SMV portfolio and that of either the SGMV portfolio or the EW portfolio.

Proposition SM.10. *Let $\boldsymbol{r}_{T+1} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$ be the out-of-sample asset excess returns, $\hat{\boldsymbol{w}}^\star = \frac{1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$ the sample mean-variance portfolio, $\boldsymbol{w}_{ew} = \boldsymbol{1}/N$ the equally weighted portfolio, and $\hat{\boldsymbol{w}}_g = \frac{1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{1}$ the sample global-minimum-variance portfolio. Then,*

$$\mathbb{C}\text{orr}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}] = \frac{\theta_{ew}}{\sqrt{\frac{c}{T}(\theta^2(T+1)+N) + \frac{2\theta^4}{T-N-4}}} \xrightarrow{T\to\infty} \frac{\theta_{ew}}{\theta}, \tag{SM45}$$

$$\mathbb{C}\text{orr}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}] = \frac{\theta_g(c+(c-1)\theta^2)}{\sqrt{\left(c+(c-1)\theta_g^2\right)\left(\frac{c}{T}(\theta^2(T+1)+N) + \frac{2\theta^4}{T-N-4}\right)}} \xrightarrow{T\to\infty} \frac{\theta_g}{\theta}. \tag{SM46}$$

Assuming that θ_{ew} and θ_g are positive, it holds that $\mathbb{C}\text{orr}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}]$ is smaller than $\mathbb{C}\text{orr}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}]$ if

$$\theta_{ew} \leq \theta_g \frac{c+(c-1)\theta^2}{\sqrt{c+(c-1)\theta_g^2}} \approx \theta_g\sqrt{c}. \tag{SM47}$$

In Figure SM.5, we calibrate Equations (SM45)–(SM46) to the four datasets of characteristic and industry-sorted portfolios we use in our empirical analysis and we depict the two correlations as a function of the sample size. Moreover, we also depict the empirical value of the correlations obtained from our empirical analysis in Section 6 for $T = 120$ months, and we find that these empirical correlations are overall in line with the theoretical ones.[SM5] The figure shows indeed that, for all datasets and sample size, the correlation is much smaller between the SMV portfolio and the EW portfolio than between the SMV portfolio and the

[SM5] We depict the empirical correlations using net out-of-sample returns, which we find are essentially the same as those obtained using gross returns.

SGMV portfolio. For example, for the 25SBTM dataset, the empirical correlation between the out-of-sample returns of the SMV and EW portfolios is 0.18 when $T = 120$. In contrast, the empirical correlation between SMV and SGMV is twice larger: 0.37 for $T = 120$. Therefore, there are larger out-of-sample diversification gains from combining the SMV and EW portfolios, which is beneficial to out-of-sample performance.

C.7 Expected out-of-sample Sharpe ratio

In Proposition 2, we show that the oracle unconstrained strategy delivers a larger EU than the oracle constrained strategy. In this section, we show moreover that the oracle unconstrained strategy delivers the largest expected out-of-sample Sharpe ratio (ESR), and thus, a larger one than the constrained strategy. However, this theoretical gain is relatively small and mostly disappears when accounting for the fact that unconstrained combination coefficients are more sensitive to estimation errors in mean returns, as discussed in Section 4.

Just like the maximum utility in (2) is unattainable in the presence of parameter uncertainty, the maximum Sharpe ratio $\theta = \sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$ is also unattainable. To quantify the impact of parameter uncertainty on the out-of-sample Sharpe ratio of an estimated portfolio $\hat{\boldsymbol{w}}$, we define the expected out-of-sample Sharpe ratio (ESR) as in DeMiguel et al. (2013):

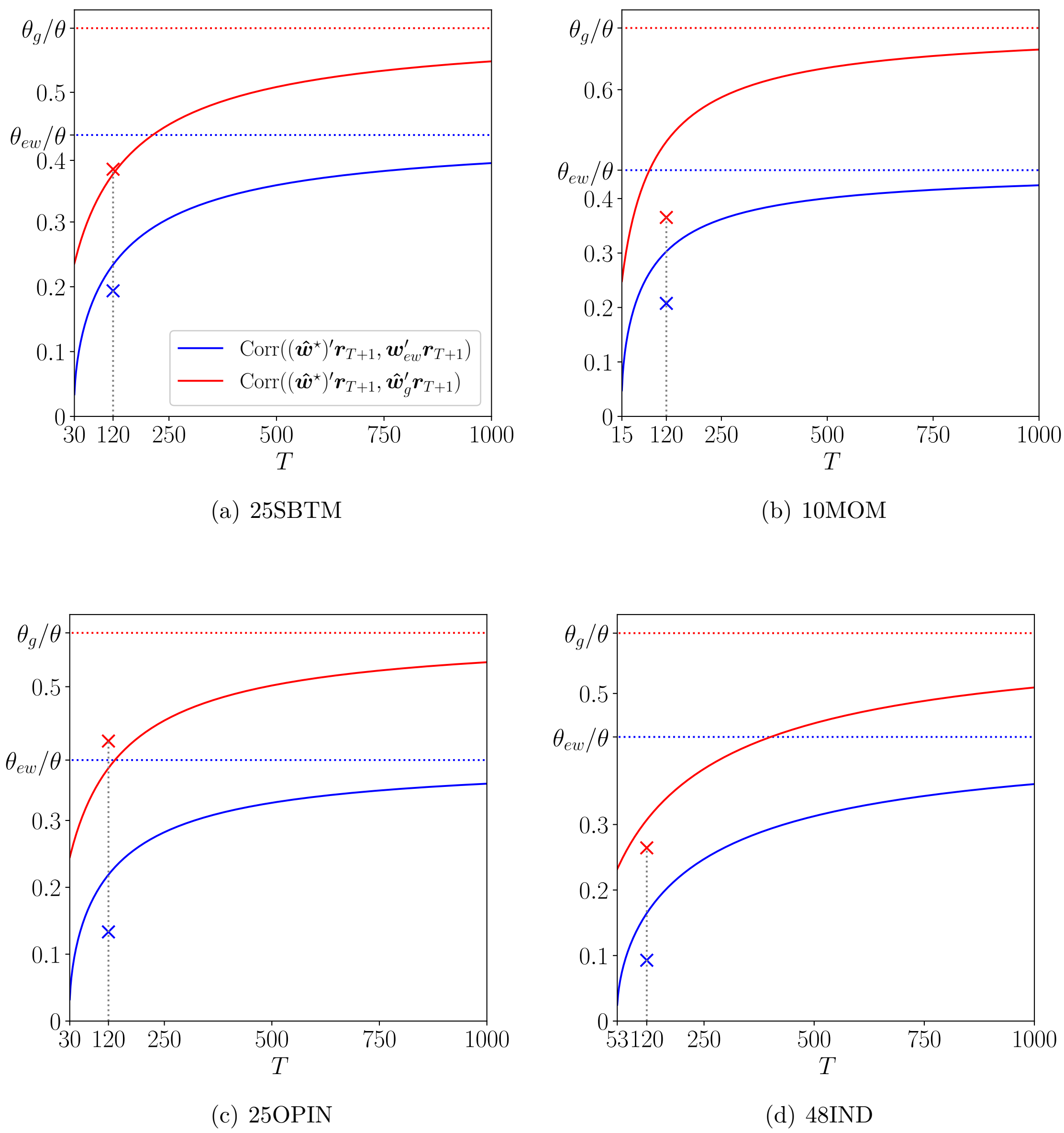
$$ESR(\hat{\boldsymbol{w}}) = \frac{\mathbb{E}[\hat{\boldsymbol{w}}'\boldsymbol{\mu}]}{\sqrt{\mathbb{E}[\hat{\boldsymbol{w}}'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}]}}. \tag{SM48}$$

From the proof of Proposition 1, we have that the ESR of the combination between the SMV and EW portfolios in (7) is

$$ESR(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = \frac{\frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_{ew}}{\sqrt{\frac{\kappa_1^2}{\gamma^2}(\theta^2 + d) + \kappa_2^2\sigma_{ew}^2 + \frac{2\kappa_1\kappa_2}{\gamma}\mu_{ew}}}. \tag{SM49}$$

In the next proposition, we show that the oracle unconstrained combination strategy $\hat{\boldsymbol{w}}^u$ delivers the maximum ESR, which is independent of the risk-aversion coefficient γ. In contrast, the oracle constrained strategy $\hat{\boldsymbol{w}}^c$ does not deliver the maximum ESR, except for two

Figure SM.5: Out-of-sample return correlations



(a) 25SBTM

(b) 10MOM

(c) 25OPIN

(d) 48IND

Notes. This figure depicts the correlation between the out-of-sample return of the sample mean-variance portfolio and that of the equally weighted portfolio (in blue) and sample global-minimum-variance portfolio (in red). The correlation is depicted for four datasets listed in Table 1 as a function of the sample size T from $N + 5$ up to 1,000 months. The solid lines depict the theoretical value of the correlations obtained from Proposition SM.10. The dotted horizontal lines depict the value as the sample size T goes to infinity. The crosses depict the empirical value of the correlations, which we obtain from the time series of net out-of-sample portfolio returns in the empirical analysis of Section 6 for $T = 120$.

specific values of γ, and generally performs worst in ESR as γ tends to zero and infinity.

Proposition SM.11. *The following holds concerning the expected out-of-sample Sharpe ratio of combination strategies:*

1. *The oracle unconstrained strategy $\hat{\boldsymbol{w}}^u$ achieves the maximum ESR of all combination strategies, which is bounded by the Sharpe ratios of the EW and optimal mean-variance portfolios:*

$$\theta_{ew} \leq \max_{\boldsymbol{\kappa}} ESR(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = ESR(\hat{\boldsymbol{w}}^u) = \sqrt{\theta_{ew}^2 + (\theta^2 - \theta_{ew}^2)\kappa_1^u} \leq \theta. \tag{SM50}$$

2. *The constrained combination strategy $\hat{\boldsymbol{w}}^c$ delivers the maximum ESR if and only if $\gamma = \gamma_{ew}$ or $\gamma = \theta^2/\mu_{ew}$. Moreover, if and only if $d > \frac{\theta^2}{\theta_{ew}^2}(\theta - 3\theta_{ew})(\theta - \theta_{ew})$, $\hat{\boldsymbol{w}}^c$ achieves its lowest ESR when $\gamma = 0$ or ∞:*[SM6]

$$\min_{\gamma} ESR(\hat{\boldsymbol{w}}^c) = \lim_{\gamma \to 0} ESR(\hat{\boldsymbol{w}}^c) = \lim_{\gamma \to \infty} ESR(\hat{\boldsymbol{w}}^c) = \frac{\theta^2}{\sqrt{\theta^2 + d}}. \tag{SM51}$$

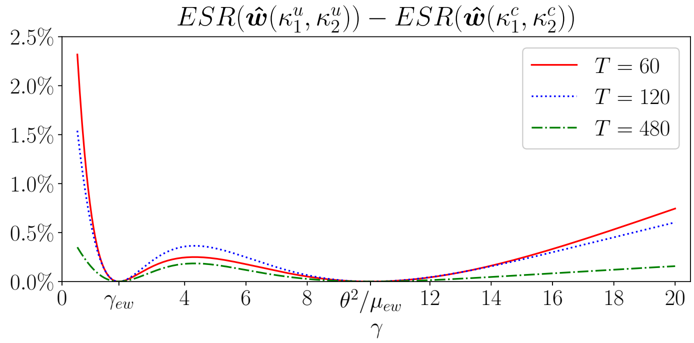
The intuition behind part 1 of Proposition SM.11 is similar to the classical result from portfolio theory that any portfolio maximizing utility in (2) also achieves the maximum Sharpe ratio. Similarly, any unconstrained portfolio combination maximizing the EU also achieves the maximum ESR. However, it is no longer the case under the convexity constraint (10), apart from two specific values of γ, and the loss in ESR is generally largest for investors with small and large degrees of risk aversion.

We illustrate Proposition SM.11 in Panel (a) of Figure SM.6 for the 25SBTM dataset. We depict the difference between the ESR of the unconstrained strategy $\hat{\boldsymbol{w}}^u$ and that of the constrained strategy $\hat{\boldsymbol{w}}^c$ as a function of γ for a samples size $T = 60$, 120, and 480. The figure shows that the difference is typically not large, expect for small and large values of γ. Nonetheless, it is always positive and gets larger as estimation risk, N/T, increases.

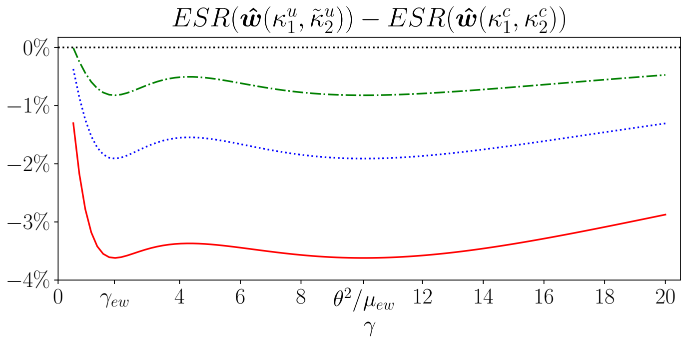
The consistent outperformance in ESR delivered by the unconstrained strategy in Proposition SM.11 assumes however that combination coefficients are known. As discussed in Sec-

[SM6]Because $d > 0$, a sufficient condition for $d > \frac{\theta^2}{\theta_{ew}^2}(\theta - 3\theta_{ew})(\theta - \theta_{ew})$ is $\theta_{ew} > \theta/3$, which often holds.

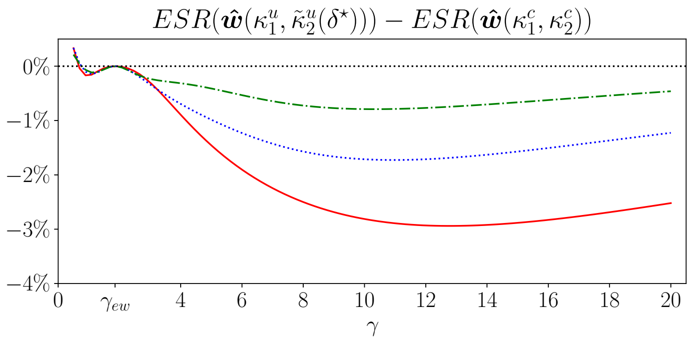
Figure SM.6: Expected out-of-sample Sharpe ratio of constrained and unconstrained strategies



(a) Known combination coefficients



(b) κ_2^u is estimated by $\tilde{\kappa}_2^u$



(c) κ_2^u is estimated by $\tilde{\kappa}_2^u(\delta^\star)$

Notes. This figure depicts the difference between the expected out-of-sample Sharpe ratio (ESR) of the unconstrained combination strategy $\hat{\boldsymbol{w}}^u$ and that of the constrained combination strategy $\hat{\boldsymbol{w}}^c$ in three different settings. In panel (a), all combination coefficients are known without errors. In panel (b), κ_2^u is estimated by $\tilde{\kappa}_2^u$ in (24). In panel (c), κ_2^u is estimated by $\tilde{\kappa}_2^u(\delta^\star)$ in (29). The figure is constructed by calibrating the population vector of means and covariance matrix of asset excess returns from monthly returns on the 25 portfolios of stocks sorted on size and book-to-market spanning July 1926 to December 2022. The ESR difference is depicted as a function of the risk-aversion coefficient γ between 0.5 and 20 for a sample size $T = 60$ (red solid), $T = 120$ (blue dotted), and $T = 480$ (green dash-dotted).

tion 4, the main difference in estimation errors between the constrained and unconstrained coefficients is that while the constrained ones are bounded, κ_2^u is very sensitive to errors in mean returns because it is unbounded and proportional to μ_{ew}. When κ_2^u is estimated by $\tilde{\kappa}_2^u$ in (24), we have from the proof of Proposition 5 that the ESR of the estimated unconstrained strategy becomes

$$ESR(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)) = \frac{\mathbb{E}[(\hat{\boldsymbol{w}}^u)'\boldsymbol{\mu}]}{\sqrt{\mathbb{E}[(\hat{\boldsymbol{w}}^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^u] + \frac{1-(\kappa_1^u)^2}{\gamma^2 T}}}, \tag{SM52}$$

where

$$\mathbb{E}[(\hat{\boldsymbol{w}}^u)'\boldsymbol{\mu}] = \frac{\kappa_1^u}{\gamma}\theta^2 + \kappa_2^u\mu_{ew}, \tag{SM53}$$

$$\mathbb{E}[(\hat{\boldsymbol{w}}^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^u] = \frac{(\kappa_1^u)^2}{\gamma^2}(\theta^2 + d) + (\kappa_2^u)^2\sigma_{ew}^2 + \frac{2\kappa_1^u\kappa_2^u}{\gamma}\mu_{ew}, \tag{SM54}$$

are the expected out-of-sample mean return and variance of the unconstrained strategy when κ_2^u is known, respectively. We can also obtain the ESR of the shrinkage estimator of the unconstrained strategy proposed in Section 4 by plugging $\delta^\star$ in (28) into (SM100)–(SM101).

In panels b) and c) of Figure SM.6, we depict the difference between the ESR of the two unconstrained strategies with that of the constrained strategy. When the unconstrained strategy is not shrunk, panel b) shows that the theoretical gain in ESR completely disappears. When the unconstrained strategy is shrunk, panel c) shows that the ESR difference is reduced and is negligible around $\gamma = \gamma_{ew}$.

Finally, we report the empirical out-of-sample Sharpe ratio of the 10 portfolio strategies we consider in the main body of the paper, which are listed in Table 2. We follow the methodology of Section 6 and define the annualized out-of-sample net Sharpe ratio of portfolio strategy k as

$$SR_k = \sqrt{12} \times \frac{\hat{\mu}_k}{\hat{\sigma}_k}, \tag{SM55}$$

where $\hat{\mu}_k$ and $\hat{\sigma}_k$ are the sample mean and standard deviation of the out-of-sample portfolio returns net of proportional transaction costs of 10 basis points for strategy k.

Table SM.3: Annualized net out-of-sample Sharpe Ratio (sample covariance matrix)

	$T = 120$						$T = 240$					
	25SBTM	10MOM	25OPIN	48IND	50STO	100STO	25SBTM	10MOM	25OPIN	48IND	50STO	100STO
$\hat{\gamma}_{ew}^{med}$	3.40	2.91	3.28	3.40	4.51	4.51	2.95	2.76	3.23	3.42	4.69	4.69
Panel (a): $\gamma = 1$												
UNC3F	0.88	0.77	0.54	0.39	0.61***	0.56***	0.85	0.80	0.76	0.40	0.80***	0.78***
SUNC3F	0.88	0.77	0.54	0.36	0.57***	0.51***	0.84	0.79	0.75	0.36	0.78***	0.77***
CON3F	0.87	0.77	0.53	0.35	0.43	0.36	0.84	0.80	0.72	0.28	0.72	0.72
KZ2F	0.83	0.75	0.47	0.20	0.15	-0.13	0.82	0.79	0.68	0.11	0.44	0.26
KZ3F	0.87	0.82	0.61	0.28	0.47	0.05	0.86	0.84	0.84	0.37	0.80	0.66
KWZ	0.61	0.65	0.39	0.26	0.25	0.08	0.64	0.74	0.56	0.21	0.72	0.65
BOP	0.61	0.55	0.28	0.01	0.22	0.12	0.63	0.70	0.52	0.13	0.04	-0.02
EWRF	0.46	0.41	0.31	0.35	0.62	0.63	0.40	0.31	0.43	0.45	0.75	0.75
GMVRF	0.70	0.55	0.55	0.23	0.48	0.08	0.68	0.49	0.77	0.36	0.80	0.67
SMV	0.81	0.79	0.53	0.19	0.17	-0.12	0.86	0.83	0.71	0.15	0.56	0.39
Panel (b): $\gamma = 5$												
UNC3F	0.88	0.77	0.54*	0.39**	0.61***	0.56***	0.84	0.80***	0.76**	0.40***	0.80***	0.78***
SUNC3F	0.89	0.78	0.58*	0.46	0.69***	0.65***	0.85	0.80***	0.77**	0.44***	0.83***	0.83***
CON3F	0.90	0.80	0.65	0.54	0.78	0.75	0.88	0.85	0.82	0.53	0.87	0.87
KZ2F	0.83	0.75	0.47	0.20	0.15	-0.13	0.82	0.79	0.68	0.11	0.44	0.26
KZ3F	0.87	0.82	0.61	0.28	0.47	0.05	0.86	0.84	0.84	0.37	0.80	0.67
KWZ	0.82	0.83	0.66	0.40	0.49	0.16	0.89	0.94	0.91	0.40	0.84	0.71
BOP	0.79	0.74	0.58	0.36	0.60	0.55	0.84	0.90	0.81	0.44	0.53	0.38
EWRF	0.46	0.40	0.31	0.35	0.62	0.63	0.40	0.31	0.43	0.45	0.75	0.75
GMVRF	0.70	0.55	0.55	0.23	0.48	0.08	0.68	0.49	0.77	0.36	0.80	0.67
SMV	0.81	0.79	0.53	0.19	0.17	-0.11	0.85	0.83	0.71	0.15	0.56	0.39
Panel (c): $\gamma = 10$												
UNC3F	0.88	0.77	0.54*	0.39*	0.61***	0.56***	0.84	0.80***	0.76*	0.40***	0.80***	0.78***
SUNC3F	0.88	0.78	0.55*	0.39*	0.62***	0.57***	0.84	0.80***	0.76*	0.41**	0.79***	0.78***
CON3F	0.86	0.80	0.66	0.52	0.77	0.77	0.89	0.85	0.82	0.49	0.87	0.87
KZ2F	0.83	0.75	0.47	0.20	0.15	-0.13	0.82	0.79	0.68	0.11	0.44	0.26
KZ3F	0.87	0.82	0.61	0.28	0.47	0.05	0.86	0.84	0.84	0.37	0.80	0.67
KWZ	0.83	0.82	0.71	0.40	0.51	0.17	0.95	0.95	0.95	0.41	0.85	0.71
BOP	0.80	0.77	0.66	0.49	0.74	0.70	0.93	0.93	0.88	0.52	0.80	0.66
EWRF	0.46	0.40	0.31	0.35	0.62	0.63	0.40	0.31	0.43	0.45	0.75	0.75
GMVRF	0.70	0.55	0.55	0.23	0.48	0.08	0.68	0.49	0.77	0.36	0.80	0.67
SMV	0.81	0.79	0.53	0.19	0.17	-0.11	0.85	0.83	0.71	0.15	0.56	0.39
Panel (d): $\gamma = 15$												
UNC3F	0.88	0.77	0.54*	0.39	0.61***	0.56***	0.84	0.80***	0.76	0.40	0.80***	0.78***
SUNC3F	0.88	0.78	0.55*	0.39	0.62***	0.56***	0.84	0.80***	0.76	0.41	0.80***	0.78***
CON3F	0.85	0.80	0.65	0.49	0.71	0.76	0.89	0.85	0.81	0.43	0.84	0.86
KZ2F	0.83	0.75	0.47	0.20	0.15	-0.13	0.82	0.79	0.68	0.11	0.44	0.26
KZ3F	0.87	0.82	0.61	0.28	0.47	0.05	0.86	0.84	0.84	0.37	0.80	0.67
KWZ	0.80	0.78	0.70	0.39	0.51	0.17	0.93	0.91	0.95	0.40	0.85	0.71
BOP	0.81	0.76	0.69	0.53	0.76	0.75	0.96	0.91	0.91	0.52	0.87	0.80
EWRF	0.46	0.40	0.31	0.35	0.62	0.63	0.40	0.31	0.43	0.45	0.75	0.75
GMVRF	0.70	0.55	0.55	0.23	0.48	0.08	0.68	0.49	0.77	0.36	0.80	0.67
SMV	0.81	0.79	0.53	0.19	0.17	-0.11	0.85	0.83	0.71	0.15	0.56	0.39

Notes. This table reports the annualized net out-of-sample Sharpe ratio for the portfolio strategies in Table 2 and the datasets in Table 1 when using the sample covariance matrix. The net out-of-sample Sharpe ratio is computed using proportional transaction costs of 10 basis points. For 50 and 100STO, we report the average performance across 500 randomly drawn datasets from a total pool of 235 stocks. We evaluate the statistical significance of the difference in net Sharpe ratios of UNC3F vs CON3F and SUNC3F vs CON3F using the stationary block bootstrap approach of Politis and Romano (1994) and Ledoit and Wolf (2008) to produce p-values. The stars *, **, and *** indicate that the p-value is less than 10%, 5%, and 1%, respectively.

Table SM.3 reports the annualized net out-of-sample Sharpe ratios. We only consider the sample covariance matrix for conciseness; the conclusions are robust to using the shrinkage covariance matrices of Ledoit and Wolf (2004, 2020). In line with the theoretical predictions above, the constrained strategy CON3F outperforms the unconstrained strategy UNC3F in most cases, although the difference is small, and the shrinkage unconstrained strategy SUNC3F approaches the Sharpe ratio of the constrained strategy more closely.

C.8 Empirical performance under the elliptical distribution

In Table SM.4, we report the annualized net out-of-sample utility of the unconstrained and constrained strategies when the combination coefficients are calibrated in three different ways. First, assuming normality and a finite sample, as in part 2 of Proposition 1 (UNC3F, CON3F). Second, assuming normality and the high-dimensional asymptotic regime, as in Equations (40)–(41) (AsyUNC3F, AsyCON3F). Third, assuming a t distribution and the high-dimensional asymptotic regime, as in Equation (39) and parts 2 and 3 of Proposition 7 (EllUNC3F, EllCON3F). In the latter case, we estimate the number of degrees of freedom ν via maximum likelihood. We only report the results for a sample size $T = 120$ for conciseness, and we consider both the sample estimate of the covariance matrix and the shrinkage estimate of Ledoit and Wolf (2004).

We make two main observations from the results in Table SM.4. First, AsyUNC3F and AsyCON3F underperform UNC3F and CON3F. This is expected because the high-dimensional asymptotic regime only provides an approximation to the finite-sample EU. However, the difference in performance is small, which suggests that this approximation is accurate. Second, UNC3F and CON3F generally underperform EllUNC3F and EllCON3F. This is expected because the latter two account for the effect of fat tails and also allocate a smaller weight to the SMV portfolio, which reduces turnover and thus transaction costs. However, the difference in performance is small, which is consistent with the theoretical results presented in Section 5.2. For the 100STO dataset, we even observe that UNC3F and

CON3F outperform EllUNC3F and EllCON3F, which can be explained because SMV performs very poorly and has a very large turnover when $T = 120$ and $N = 100$, and thus the extra variation of the coefficient κ_1 on the SMV portfolio caused by the estimation of the number of degrees of freedom ν hurts the net out-of-sample utility.

C.9 Nonlinear shrinkage of the covariance matrix

In Table SM.5, we replicate Tables 3 and 4 in the main body of the paper using the nonlinear shrinkage estimator of the covariance matrix of Ledoit and Wolf (2020) to estimate the SMV portfolio. The results are overall similar to those obtained using the linear shrinkage estimator of Ledoit and Wolf (2004) in Table 4, and therefore the conclusions drawn in the main paper are robust to using a nonlinear shrinkage estimator.

C.10 Portfolio turnover

In Table SM.6, we report the average monthly turnover of the 10 portfolio strategies we consider, when the sample size is $T = 120$. The average turnover is computed as the average of the monthly turnover values in (50). We observe that UNC3F systematically delivers a smaller turnover than CON3F, which is in line with Proposition 4. Similarly, SUNC3F also delivers a smaller turnover than CON3F.

C.11 Empirical outperformance relative to constrained strategy

Figure SM.7 replicates the simulation results in Figure 3 empirically. Specifically, we depict, for $T = 120$ and the sample covariance matrix, the difference between the net utility of UNC3F and SUNC3F and the net utility of CON3F. The figure shows that SUNC3F trades off between the CON3F and UNC3F strategies. Indeed, for γ around the estimated γ_{ew}, SUNC3F outperforms UNC3F and does not lose much relative to CON3F, and when γ moves away from γ_{ew}, SUNC3F is close to UNC3F and largely outperforms CON3F.

Table SM.4: Annualized net out-of-sample utility (elliptical and asymptotic coefficients)

	Sample $\boldsymbol{\Sigma}$						Shrinkage $\boldsymbol{\Sigma}$					
	25SBTM	10MOM	25OPIN	48IND	50STO	100STO	25SBTM	10MOM	25OPIN	48IND	50STO	100STO
						Panel (a): $\gamma = 1$						
UNC3F	29.4	22.5	6.63	2.92	15.4	10.7	37.3	31.6	14.3	3.70	15.9	16.7
AsyUNC3F	28.7	22.1	6.24	2.69	14.6	8.88	37.3	31.5	14.1	3.49	14.9	16.7
EllUNC3F	31.5	24.7	8.32	3.08	15.4	1.85	37.9	32.8	16.0	3.89	15.6	16.1
CON3F	31.2	23.4	8.31	5.36	8.59	4.95	42.2	34.5	18.8	6.93	11.5	13.2
AsyCON3F	30.5	22.9	7.76	5.03	8.88	2.40	42.1	34.3	18.4	6.62	11.4	13.1
EllCON3F	33.4	25.8	10.0	5.43	9.65	-5.96	42.5	35.7	20.3	7.03	12.0	12.2
						Panel (b): $\gamma = 5$						
UNC3F	5.71	4.45	1.28	0.57	3.07	2.12	7.44	6.31	2.83	0.73	3.18	3.34
AsyUNC3F	5.56	4.37	1.20	0.52	2.91	1.74	7.43	6.29	2.79	0.68	2.98	3.34
EllUNC3F	6.16	4.91	1.63	0.60	3.06	0.14	7.55	6.54	3.16	0.77	3.11	3.22
CON3F	5.08	4.41	2.43	1.67	5.67	5.02	8.05	6.96	4.15	1.85	5.88	6.34
AsyCON3F	4.91	4.32	2.33	1.60	5.50	4.61	8.04	6.93	4.09	1.79	5.65	6.35
EllCON3F	5.59	4.81	2.75	1.72	5.69	2.95	8.03	7.10	4.43	1.89	5.79	6.21
						Panel (c): $\gamma = 10$						
UNC3F	2.84	2.22	0.64	0.28	1.53	1.06	3.72	3.15	1.41	0.36	1.59	1.67
AsyUNC3F	2.77	2.18	0.60	0.26	1.45	0.87	3.71	3.14	1.39	0.34	1.49	1.67
EllUNC3F	3.07	2.45	0.81	0.30	1.53	0.05	3.77	3.27	1.58	0.38	1.55	1.61
CON3F	-0.74	1.06	-1.53	-4.01	-1.59	-2.80	1.98	2.76	-0.46	-3.96	-1.44	-1.94
AsyCON3F	-0.83	1.02	-1.58	-4.06	-1.85	-3.00	1.99	2.75	-0.48	-4.01	-1.73	-1.83
EllCON3F	-0.48	1.18	-1.38	-3.93	-1.70	-3.92	1.91	2.75	-0.35	-3.87	-1.67	-1.91
						Panel (d): $\gamma = 15$						
UNC3F	1.89	1.48	0.42	0.19	1.02	0.70	2.48	2.10	0.94	0.24	1.06	1.11
AsyUNC3F	1.84	1.45	0.40	0.17	0.97	0.58	2.48	2.10	0.93	0.23	0.99	1.11
EllUNC3F	2.05	1.63	0.54	0.20	1.02	0.03	2.51	2.18	1.05	0.25	1.04	1.07
CON3F	-2.93	-0.11	-3.55	-7.85	-6.76	-10.5	-0.14	1.34	-2.57	-8.04	-6.62	-9.49
AsyCON3F	-2.98	-0.12	-3.57	-7.90	-6.96	-10.7	-0.12	1.34	-2.57	-8.09	-6.88	-9.23
EllCON3F	-2.81	-0.08	-3.54	-7.78	-6.84	-11.6	-0.28	1.30	-2.61	-7.92	-6.90	-9.33

Notes. This table reports the annualized net out-of-sample utility in percentage points for the datasets in Table 1 and the constrained unconstrained and constrained portfolio strategies when the combination coefficients are estimated in three different ways. First, assuming normality and a finite sample, as in part 2 of Proposition 1 (UNC3F, CON3F). Second, assuming normality and the high-dimensional asymptotic regime, as in Equations (40)–(41) (AsyUNC3F, AsyCON3F). Third, assuming a t distribution and the high-dimensional asymptotic regime, as in Equation (39) and parts 2 and 3 of Proposition 7 (EllUNC3F, EllCON3F). In the latter case, we estimate the number of degrees of freedom ν via maximum likelihood. The net out-of-sample utility is computed using proportional transaction costs of 10 basis points. We report the results for the sample estimate of the covariance matrix and the shrinkage estimate of Ledoit and Wolf (2004). For 50 and 100STO, we report the average performance across 500 randomly drawn datasets from a total pool of 235 stocks. We consider a sample size $T = 120$ and risk-aversion coefficients $\gamma = 1$, 5, 10, and 15.

Table SM.5: Annualized net out-of-sample utility (nonlinear shrinkage covariance matrix)

	$T = 120$						$T = 240$					
	25SBTM	10MOM	25OPIN	48IND	50STO	100STO	25SBTM	10MOM	25OPIN	48IND	50STO	100STO
$\hat{\gamma}_{ew}^{med}$	3.40	2.91	3.28	3.40	4.51	4.51	2.95	2.76	3.23	3.42	4.69	4.69
						Panel (a): $\gamma = 1$						
UNC3F	36.5	24.9	14.6	3.23	15.9***	16.7***	21.1	25.9	25.1	3.85	30.3***	30.4***
SUNC3F	37.4	24.9	16.0	4.09	14.7***	15.5***	21.0	25.8	24.5	2.73	29.7***	29.8***
CON3F	39.8	26.5	18.4	6.51	11.3	13.0	22.4	27.4	21.9	0.30	23.9	22.0
KZ2F	36.1	24.7	14.6	0.71	-0.31	-0.61	19.9	25.8	17.6	-6.52	11.3	5.98
KZ3F	28.6	28.5	12.8	-2.70	-2.93	11.7	20.6	30.5	30.9	3.19	11.3	8.54
KWZ	23.4	19.0	10.3	4.45	3.60	4.57	17.6	24.8	16.8	-2.08	9.93	9.99
BOP	16.9	7.07	-1.22	-18.5	-<u>31.8</u>	-<u>25.7</u>	15.6	20.9	13.6	-5.68	-<u>51.0</u>	-<u>102</u>
EWRF	8.32	5.72	-0.59	2.85	17.5	17.5	5.99	2.47	8.93	9.87	27.4	27.7
GMVRF	12.5	9.20	9.17	-3.26	-1.07	12.8	12.4	7.75	28.3	9.93	11.5	8.50
SMV	-<u>127</u>	-<u>19.0</u>	-<u>140</u>	-<u>503</u>	-<u>548</u>	-<u>1479</u>	-<u>51.0</u>	8.42	-<u>66.8</u>	-<u>237</u>	-<u>169</u>	-<u>523</u>
						Panel (b): $\gamma = 5$						
UNC3F	7.17	4.93	2.88	0.64	3.18***	3.33***	3.97	5.13**	5.00	0.77	6.07***	6.09***
SUNC3F	7.39	5.24	3.54	1.43	4.44***	4.84***	4.02	5.25**	5.09	0.81	6.59***	6.77***
CON3F	6.82	4.94	4.20	1.84	5.89	6.32	4.03	6.00	5.63	1.41	7.17	7.38
KZ2F	7.11	4.90	2.88	0.13	-0.07	-0.12	3.74	5.12	3.48	-1.31	2.26	1.20
KZ3F	5.55	5.64	2.48	-0.58	-0.65	2.34	3.90	6.06	6.16	0.63	2.23	1.66
KWZ	7.78	6.10	5.37	3.31	3.75	3.22	7.52	8.33	8.58	1.41	6.96	6.89
BOP	4.27	3.06	1.76	-3.09	-3.90	-<u>2.94</u>	5.31	7.23	6.25	-0.51	-5.76	-<u>20.2</u>
EWRF	1.66	1.14	-0.12	0.57	3.49	3.49	1.20	0.49	1.79	1.97	5.48	5.55
GMVRF	2.39	1.82	1.79	-0.69	-0.27	2.55	2.43	1.54	5.64	1.98	2.26	1.65
SMV	-<u>27.6</u>	-<u>4.08</u>	-<u>29.2</u>	-<u>104</u>	-<u>112</u>	-<u>312</u>	-<u>11.2</u>	1.57	-<u>13.8</u>	-<u>48.1</u>	-<u>34.2</u>	-<u>107</u>
						Panel (c): $\gamma = 10$						
UNC3F	3.58*	2.46	1.44	0.32**	1.59***	1.67***	1.97	2.56	2.50	0.38**	3.03***	3.05***
SUNC3F	3.46**	2.45	1.29	0.07**	1.33***	1.32***	1.92	2.54	2.50	0.35**	2.95***	2.95***
CON3F	0.19	1.29	-0.54	-3.84	-<u>1.40</u>	-1.93	0.21	2.26	0.99	-<u>3.41</u>	-0.98	-1.34
KZ2F	3.55	2.45	1.44	0.07	-0.03	-0.06	1.86	2.56	1.74	-0.65	1.13	0.60
KZ3F	2.76	2.82	1.23	-0.29	-0.33	1.17	1.93	3.03	3.08	0.31	<u>1.11</u>	<u>0.82</u>
KWZ	1.72	-0.60	0.41	-0.22	0.16	-0.36	2.54	1.77	3.15	-1.34	<u>2.27</u>	<u>2.43</u>
BOP	-2.94	-<u>2.66</u>	-2.72	-6.31	-5.70	-<u>7.81</u>	0.30	1.08	1.07	-<u>4.04</u>	-2.65	-12.0
EWRF	0.83	0.57	-0.06	0.28	1.75	1.75	0.60	0.25	0.89	0.99	2.74	2.77
GMVRF	1.19	0.91	0.89	-0.35	-0.14	1.27	1.21	0.77	2.82	0.99	<u>1.13</u>	<u>0.82</u>
SMV	-<u>13.9</u>	-<u>2.05</u>	-<u>14.7</u>	-<u>52.4</u>	-<u>56.2</u>	-<u>157</u>	-<u>5.64</u>	0.78	-<u>6.91</u>	-<u>24.1</u>	-<u>17.1</u>	-<u>53.6</u>
						Panel (d): $\gamma = 15$						
UNC3F	2.38***	1.64	0.96***	0.21***	1.06***	1.11***	1.31*	1.71	1.66**	0.26***	2.02***	2.03***
SUNC3F	2.32***	1.62	0.91***	0.13***	0.97***	0.99***	1.30**	1.70	1.67**	0.25***	2.00***	2.00***
CON3F	-<u>2.30</u>	<u>0.03</u>	-<u>2.84</u>	-<u>7.88</u>	-<u>6.60</u>	-<u>9.44</u>	-1.17	1.06	-<u>0.90</u>	-<u>5.93</u>	-<u>4.72</u>	-7.44
KZ2F	2.36	1.63	0.96	0.04	-0.02	-0.04	1.23	1.70	1.16	-0.44	0.75	0.40
KZ3F	1.84	1.88	0.82	-0.20	-0.22	0.78	1.29	2.02	2.05	0.21	<u>0.74</u>	<u>0.55</u>
KWZ	-<u>3.32</u>	-<u>6.59</u>	-<u>4.45</u>	-<u>3.88</u>	-3.70	-4.07	-1.89	-<u>3.77</u>	-<u>1.92</u>	-<u>4.59</u>	-<u>2.48</u>	-<u>2.06</u>
BOP	-<u>8.23</u>	-<u>8.18</u>	-<u>7.30</u>	-<u>11.2</u>	-<u>10.1</u>	-<u>15.1</u>	-3.93	-<u>4.17</u>	-<u>3.67</u>	-<u>8.03</u>	-6.33	-12.2
EWRF	0.55	0.38	-0.04	0.19	1.16	1.16	0.40	0.16	0.60	0.66	1.83	1.85
GMVRF	<u>0.79</u>	0.61	0.60	-0.23	-0.09	0.85	0.81	0.51	1.88	0.66	0.75	<u>0.55</u>
SMV	-<u>9.31</u>	-<u>1.37</u>	-<u>9.79</u>	-<u>35.0</u>	-<u>37.5</u>	-<u>105</u>	-<u>3.77</u>	0.52	-<u>4.61</u>	-<u>16.1</u>	-<u>11.4</u>	-<u>35.8</u>

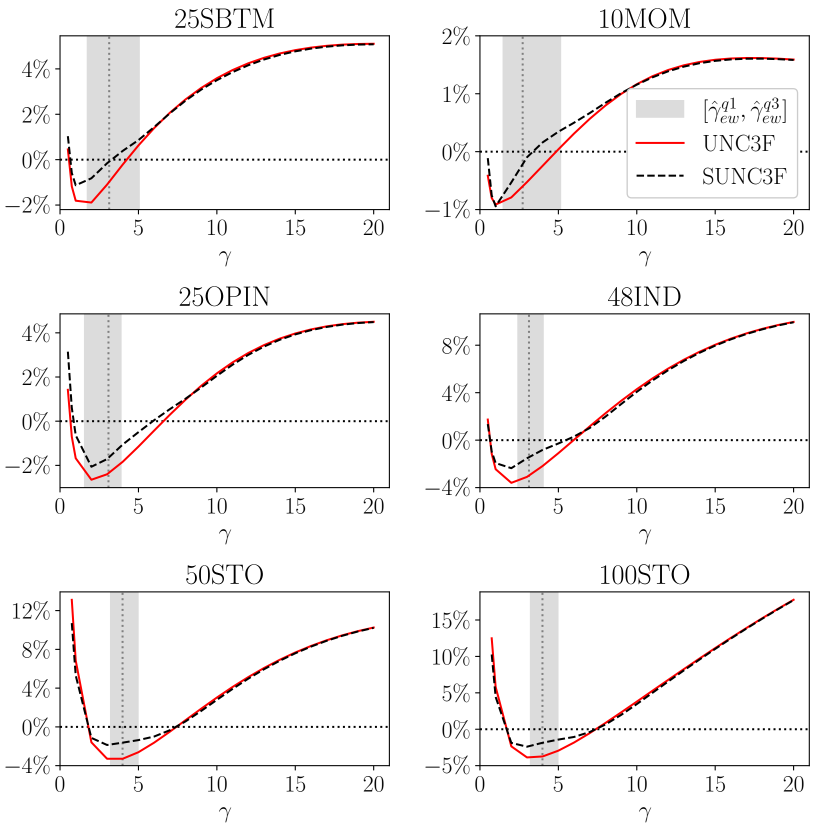
Notes. This table reports the annualized net out-of-sample utility in percentage points for the portfolio strategies in Table 2 and the datasets in Table 1 when using the nonlinear shrinkage covariance matrix of Ledoit and Wolf (2020). The net out-of-sample utility is computed using proportional transaction costs of 10 basis points. For 50 and 100STO, we report the average performance across 500 randomly drawn datasets from a total pool of 235 stocks. We evaluate the statistical significance of the difference in net utilities of UNC3F vs CON3F and SUNC3F vs CON3F using the stationary block bootstrap approach of Politis and Romano (1994) and Ledoit and Wolf (2008) to produce p-values. The stars *, **, and *** indicate that the p-value is less than 10%, 5%, and 1%, respectively. The underlined numbers indicate the strategies that do not belong to the superior set of models of Hansen et al. (2011) at a confidence level of 5%. For 50STO and 100STO, we underline the three strategies that are most often rejected, or more in case of equalities.

Table SM.6: Average monthly turnover ($T = 120$)

	Sample $\mathbf{\Sigma}$						Shrinkage $\mathbf{\Sigma}$					
	25SBTM	10MOM	25OPIN	48IND	50STO	100STO	25SBTM	10MOM	25OPIN	48IND	50STO	100STO
Panel (a): $\gamma = 1$												
UNC3F	9.38	4.06	5.48	2.79	1.62	2.92	9.38	4.06	4.29	1.96	1.04	0.76
SUNC3F	9.38	4.06	5.48	2.79	1.62	2.92	9.38	4.06	4.29	1.96	1.05	0.76
CON3F	9.93	4.31	5.79	3.12	2.05	3.29	9.93	4.31	4.54	2.20	1.32	0.81
KZ2F	9.96	4.30	5.76	3.00	1.91	3.06	9.96	4.30	4.52	2.13	1.25	0.77
KZ3F	10.2	4.15	6.07	4.76	4.10	7.98	10.2	4.15	4.68	3.24	2.38	1.95
KWZ	7.90	3.87	4.60	2.69	1.08	3.51	7.90	3.87	3.67	1.84	0.65	0.64
BOP	8.55	5.23	5.24	6.40	5.92	11.0	8.55	5.23	5.24	6.40	5.92	11.0
EWRF	0.17	0.19	0.20	0.21	0.30	0.30	0.17	0.19	0.20	0.21	0.30	0.30
GMVRF	6.12	1.67	3.73	3.63	3.94	7.51	6.12	1.67	2.84	2.50	2.29	1.92
SMV	25.2	7.54	16.9	27.1	16.6	106	25.2	7.54	13.3	19.0	10.5	26.5
Panel (b): $\gamma = 5$												
UNC3F	1.88	0.81	1.10	0.56	0.32	0.58	1.88	0.81	0.86	0.39	0.21	0.15
SUNC3F	1.88	0.81	1.10	0.56	0.32	0.58	1.88	0.81	0.86	0.39	0.21	0.15
CON3F	2.10	0.91	1.20	0.65	0.35	0.61	2.10	0.91	0.94	0.46	0.23	0.16
KZ2F	1.99	0.86	1.15	0.60	0.38	0.61	1.99	0.86	0.90	0.43	0.25	0.15
KZ3F	2.04	0.83	1.21	0.95	0.82	1.60	2.04	0.83	0.94	0.65	0.48	0.39
KWZ	1.81	0.85	1.10	0.98	0.54	2.75	1.81	0.85	0.83	0.54	0.26	0.38
BOP	1.92	0.91	1.14	1.21	1.08	2.20	1.92	0.91	1.13	1.20	1.07	2.20
EWRF	0.03	0.04	0.04	0.04	0.06	0.06	0.03	0.04	0.04	0.04	0.06	0.06
GMVRF	1.22	0.33	0.75	0.73	0.79	1.50	1.22	0.33	0.57	0.50	0.46	0.38
SMV	5.03	1.51	3.37	5.42	3.31	21.2	5.03	1.51	2.67	3.80	2.10	5.31
Panel (c): $\gamma = 10$												
UNC3F	0.94	0.41	0.55	0.28	0.16	0.29	0.94	0.41	0.43	0.20	0.10	0.08
SUNC3F	0.94	0.41	0.55	0.28	0.16	0.29	0.94	0.41	0.43	0.20	0.10	0.08
CON3F	1.32	0.53	0.78	0.59	0.31	0.43	1.32	0.53	0.62	0.41	0.19	0.11
KZ2F	1.00	0.43	0.58	0.30	0.19	0.31	1.00	0.43	0.45	0.21	0.13	0.08
KZ3F	1.02	0.42	0.61	0.48	0.41	0.80	1.02	0.42	0.47	0.32	0.24	0.19
KWZ	1.16	0.51	0.75	0.85	0.50	2.71	1.16	0.51	0.53	0.43	0.23	0.36
BOP	1.24	0.53	0.74	0.51	0.38	1.08	1.24	0.53	0.72	0.51	0.38	1.08
EWRF	0.02	0.02	0.02	0.02	0.03	0.03	0.02	0.02	0.02	0.02	0.03	0.03
GMVRF	0.61	0.17	0.37	0.36	0.39	0.75	0.61	0.17	0.28	0.25	0.23	0.19
SMV	2.52	0.75	1.69	2.71	1.66	10.6	2.52	0.75	1.33	1.90	1.05	2.65
Panel (d): $\gamma = 15$												
UNC3F	0.63	0.27	0.37	0.19	0.11	0.19	0.63	0.27	0.29	0.13	0.07	0.05
SUNC3F	0.63	0.27	0.37	0.19	0.11	0.20	0.63	0.27	0.29	0.13	0.07	0.05
CON3F	1.07	0.40	0.68	0.65	0.36	0.47	1.07	0.40	0.53	0.45	0.22	0.12
KZ2F	0.66	0.29	0.38	0.20	0.13	0.20	0.66	0.29	0.30	0.14	0.08	0.05
KZ3F	0.68	0.28	0.40	0.32	0.27	0.53	0.68	0.28	0.31	0.22	0.16	0.13
KWZ	0.98	0.41	0.66	0.82	0.49	2.70	0.98	0.41	0.45	0.41	0.22	0.35
BOP	1.07	0.42	0.65	0.43	0.25	0.68	1.07	0.42	0.62	0.42	0.24	0.69
EWRF	0.01	0.01	0.01	0.01	0.02	0.02	0.01	0.01	0.01	0.01	0.02	0.02
GMVRF	0.41	0.11	0.25	0.24	0.26	0.50	0.41	0.11	0.19	0.17	0.15	0.13
SMV	1.68	0.50	1.12	1.81	1.10	7.05	1.68	0.50	0.89	1.27	0.70	1.77

Notes. This table reports the average monthly turnover for the portfolio strategies in Table 2 and the datasets in Table 1 when using either the sample covariance matrix or the shrinkage covariance matrix of Ledoit and Wolf (2004). The average turnover is computed as the average of the monthly turnover values in (50). We consider a sample size $T = 120$ and risk-aversion coefficients $\gamma = 1$, 5, 10, and 15.

Figure SM.7: Net annualized out-of-sample utility relative to constrained strategy



Notes. This figure depicts for all six datasets the difference between the annualized net out-of-sample utility of the unconstrained strategy (UNC3F, solid red) and the shrinkage estimator of the unconstrained strategy (SUNC3F, dashed black) relative to the constrained strategy (CON3F). The difference is depicted as a function of the risk-aversion coefficient γ. We use a sample size $T = 120$ and the sample covariance matrix. The net out-of-sample utility is computed using proportional transaction costs of 10 basis points. The gray region depicts the first, second, and third quartiles of the estimated parameter γ_{ew} defined in (8).

D Proofs of all results

We often use the following three properties throughout the proofs. First, because returns are i.i.d. normal, the sample mean $\hat{\boldsymbol{\mu}} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma}/T)$, the sample covariance matrix $(T-N-2)\widehat{\boldsymbol{\Sigma}} \sim \mathcal{W}_N(T-1, \boldsymbol{\Sigma})$, and they are independent of each other. Second, the inverse sample covariance matrix is unbiased because of the $1/(T-N-2)$ coefficient in (4), $\mathbb{E}[\widehat{\boldsymbol{\Sigma}}^{-1}] = \boldsymbol{\Sigma}^{-1}$. Third, the expectation of a quadratic form in the random variable $\boldsymbol{x}$ is

$$\mathbb{E}[\boldsymbol{x}'\boldsymbol{A}\boldsymbol{x}] = \mathbb{E}[\boldsymbol{x}]'\boldsymbol{A}\mathbb{E}[\boldsymbol{x}] + \operatorname{Trace}(\boldsymbol{A}\mathbb{V}[\boldsymbol{x}]), \tag{SM56}$$

where $\boldsymbol{A}$ is a constant matrix (Rencher and Schaalje, 2008).

Proof of Proposition 1

Part 1. Because the inverse sample covariance matrix is unbiased, the SMV portfolio $\hat{\boldsymbol{w}}^\star$ is unbiased as well, and the expected out-of-sample mean return of $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ in (7) is

$$\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}] = \frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_{ew}. \tag{SM57}$$

We can then obtain the expected out-of-sample variance, $\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})]$, from Equation (SM56). This requires knowing the covariance matrix of the SMV portfolio $\hat{\boldsymbol{w}}^\star$, which from Javed et al. (2021, Corollary 2.2) is[SM7]

$$\mathbb{V}[\hat{\boldsymbol{w}}^\star] = \frac{(T-N-2)(\theta^2 + (T-2)/T)\boldsymbol{\Sigma}^{-1} + (T-N)\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}}{\gamma^2(T-N-1)(T-N-4)}. \tag{SM58}$$

Using $\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = \boldsymbol{w}(\boldsymbol{\kappa})$ and $\mathbb{V}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = \kappa_1^2\mathbb{V}[\hat{\boldsymbol{w}}^\star]$, we find from (SM56) that the expected out-of-sample variance of $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ is

$$\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = \frac{\kappa_1^2}{\gamma^2}(\theta^2 + d) + \kappa_2^2\sigma_{ew}^2 + \frac{2\kappa_1\kappa_2}{\gamma}\mu_{ew}. \tag{SM59}$$

[SM7] Javed et al. (2021, Corollary 2.2) provide the covariance matrix of $\hat{\boldsymbol{w}}^\star$ for the case in which the sample covariance matrix is scaled by $T-1$, which we adapt in Equation (SM58) to our case in which it is scaled by $T-N-2$.

Combining (SM57) and (SM59) results in the EU formula in (11).

Part 2. We obtain the combination coefficients maximizing the EU under the convexity constraint (10) by setting the derivative of (11) with respect to κ_1 equal to zero, taking into account that $\kappa_2 = 1 - \kappa_1$. Similarly, the unconstrained combination coefficients are found by setting the derivative of (11) with respect to κ_1 and κ_2 equal to zero and solving the resulting linear system of two equations with two unknowns. It is clear by comparing $\boldsymbol{\kappa}^c$ in (13) and $\boldsymbol{\kappa}^u$ in (14) that they are equal if and only if $\gamma = \gamma_{ew}$.

Part 3. Plugging the constrained combination coefficients $\boldsymbol{\kappa}^c$ in (13) into the EU formula in (11), we obtain that the EU of the constrained strategy is

$$\begin{aligned} EU(\hat{\boldsymbol{w}}^c) &= \frac{\psi^2 + \ell^2}{\psi^2 + \ell^2 + d}\frac{\theta^2}{\gamma} + \frac{d\mu_{ew}}{\psi^2 + \ell^2 + d} - \frac{\gamma}{2}\Bigg[\left(\frac{\psi^2 + \ell^2}{\psi^2 + \ell^2 + d}\right)^2 \frac{\theta^2 + d}{\gamma^2} \\ &\quad + \frac{d^2\sigma_{ew}^2}{(\psi^2 + \ell^2 + d)^2} + \frac{2d(\psi^2 + \ell^2)}{(\psi^2 + \ell^2 + d)^2}\frac{\mu_{ew}}{\gamma}\Bigg] \\ &= U^\star - \frac{1}{2\gamma(\psi^2 + \ell^2 + d)^2}\Big[\theta^2(\psi^2 + \ell^2 + d)^2 + (\theta^2 + d)(\psi^2 + \ell^2)^2 + \gamma^2 d^2 \sigma_{ew}^2 \\ &\quad + 2\gamma d\mu_{ew}(\psi^2 + \ell^2) - 2\theta^2(\psi^2 + \ell^2)(\psi^2 + \ell^2 + d) - 2\gamma d\mu_{ew}(\psi^2 + \ell^2 + d)\Big]. \end{aligned} \tag{SM60}$$

The last term in brackets is equal to $d(\psi^2 + \ell^2)(\psi^2 + \ell^2 + d)$, and thus (SM60) simplifies to (15). We proceed similarly for the unconstrained strategy: plugging $\boldsymbol{\kappa}^u$ in (14) into the EU formula in (11), the EU of the unconstrained strategy is

$$\begin{aligned} EU(\hat{\boldsymbol{w}}^u) &= \frac{\psi^2}{\psi^2 + d}\frac{\theta^2}{\gamma} + \frac{d}{\psi^2 + d}\frac{\theta_{ew}^2}{\gamma} - \frac{\gamma}{2}\left[\frac{\psi^4}{(\psi^2 + d)^2}\frac{\theta^2 + d}{\gamma^2} + \frac{d^2}{(\psi^2 + d)^2}\frac{\theta_{ew}^2}{\gamma^2} + \frac{2d\psi^2}{(\psi^2 + d)^2}\frac{\theta_{ew}^2}{\gamma^2}\right] \\ &= U^\star - \frac{1}{2\gamma(\psi^2 + d)^2}\Big[\theta^2(\psi^2 + d)^2 + (\theta^2 + d)\psi^4 + d^2\theta_{ew}^2 + 2d\psi^2\theta_{ew}^2 \\ &\quad - 2\theta^2\psi^2(\psi^2 + d) - 2d\theta_{ew}^2(\psi^2 + d)\Big]. \end{aligned} \tag{SM61}$$

The last term in brackets is equal to $d\psi^2(\psi^2 + d)$, and thus (SM61) simplifies to (16). The unconstrained strategy $\hat{\boldsymbol{w}}^u$ delivers a larger EU than that of the SMV portfolio, the EW portfolio, and the risk-free asset because they are part of the search space by setting

$\boldsymbol{\kappa} = (1,0)$, $(0,1)$, and $(0,0)$, respectively. For the same reason, the constrained strategy delivers a larger EU than that of the SMV and EW portfolios. However, the risk-free asset is not part of the search space of the constrained because $\boldsymbol{\kappa} = (0,0)$ does not fulfill the convexity constraint (10). Using Equation (15), the constrained strategy $\hat{\boldsymbol{w}}^c$ delivers a negative EU, and thus underperforms the risk-free asset, when

$$U^\star - \frac{d}{2\gamma}\kappa_1^c < 0. \tag{SM62}$$

After some developments, we find that inequality (SM62) is equivalent to

$$\gamma^2[\sigma_{ew}^2(d-\theta^2)] + \gamma[2\mu_{ew}(\theta^2 - d)] - \theta^4 > 0. \tag{SM63}$$

The discriminant of this second-degree polynomial in γ is

$$\Delta = 4\sigma_{ew}^2(d-\theta^2)(\theta_{ew}^2 d + \theta^2\psi^2). \tag{SM64}$$

It follows from (SM64) that $\Delta \geq 0$ and real roots to the polynomial exist if and only if $\theta^2 < d$. In that case, (SM63) holds if and only if γ is not between the two roots:

$$\gamma \notin \left[\gamma_{ew}\left(1 \pm \sqrt{1 + \frac{\theta^4/\theta_{ew}^2}{d-\theta^2}}\right)\right]. \tag{SM65}$$

The negative sign gives a negative root, which we can ignore because any $\gamma > 0$. Therefore, the constrained strategy delivers a negative EU if and only if (17) holds. Finally, the threshold γ_{neg} decreases with d because

$$\frac{\partial \gamma_{neg}}{\partial d} = -\frac{\mu_{ew}\theta^2}{2(d-\theta^2)^2\sqrt{1+\frac{\theta^4/\theta_{ew}^2}{d-\theta^2}}} \leq 0. \tag{SM66}$$

Proof of Proposition 2

Using Equations (15) and (16), the EU gain of the unconstrained strategy relative to the constrained strategy is

$$EU(\hat{\boldsymbol{w}}^u) - EU(\hat{\boldsymbol{w}}^c) = \frac{d}{2\gamma}(\kappa_1^c - \kappa_1^u). \tag{SM67}$$

Since $d \geq 0$, this EU gain is positive if $\kappa_1^c \geq \kappa_1^u$, which is the case because

$$\kappa_1^c - \kappa_1^u = \frac{d\ell^2}{(\psi^2 + d)(\psi^2 + \ell^2 + d)} \geq 0. \tag{SM68}$$

The EU gain in (SM67) is zero if and only if $\gamma = \infty$ or γ_{ew}; the former case is obvious and the latter case is because $\ell^2 = 0$ when $\gamma = \gamma_{ew}$. Moreover, the EU gain in (SM67) increases with d if $d(\kappa_1^c - \kappa_1^u)$ increases with d, which is the case because

$$\begin{aligned}\frac{\partial}{\partial d} d(\kappa_1^c - \kappa_1^u) &= \frac{\partial}{\partial d} \frac{d^2\ell^2}{(\psi^2 + d)(\psi^2 + \ell^2 + d)} \\ &= \frac{d\ell^2(\ell^2(2\psi^2 + d) + 2\psi^2(\psi^2 + d))}{(\psi^2 + d)^2(\psi^2 + \ell^2 + d)^2} \geq 0. \end{aligned} \tag{SM69}$$

Next, we turn to the decomposition of the EU gain in (19). Given the EU formula in (11), it is clear that the EU of a portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ is

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = EU(\kappa_1\hat{\boldsymbol{w}}^\star) + EU(\kappa_2\boldsymbol{w}_{ew}) - \mathbb{E}[(\kappa_1\hat{\boldsymbol{w}}^\star)'\boldsymbol{\Sigma}(\kappa_2\boldsymbol{w}_{ew})], \tag{SM70}$$

where

$$EU(\kappa_1\hat{\boldsymbol{w}}^\star) = \frac{\kappa_1}{\gamma}\theta^2 - \frac{\gamma}{2}\frac{\kappa_1^2}{\gamma^2}(\theta^2 + d), \tag{SM71}$$

$$EU(\kappa_2\boldsymbol{w}_{ew}) = \kappa_2\mu_{ew} - \frac{\gamma}{2}\kappa_2^2\sigma_{ew}^2, \tag{SM72}$$

$$\mathbb{E}[(\kappa_1\hat{\boldsymbol{w}}^\star)'\boldsymbol{\Sigma}(\kappa_2\boldsymbol{w}_{ew})] = \kappa_1\kappa_2\mu_{ew}. \tag{SM73}$$

Therefore, the EU gain in (19) can be decomposed as $\Delta_{smv} + \Delta_{ew} + \Delta_{cov}$, where Δ_{smv} is defined in (20), Δ_{ew} in (21), and Δ_{cov} in (22). We conclude this proof by deriving a closed-form expression for these three quantities.

Expression for Δ_{smv}. Given (SM71) and the formula for κ_1^c in (13) and κ_1^u in (14), we have

$$\begin{aligned}
\Delta_{smv} &= \frac{\psi^2}{\psi^2+d}\frac{\theta^2}{\gamma} - \frac{\psi^4}{(\psi^2+d)^2}\frac{\theta^2+d}{2\gamma} - \frac{\psi^2+\ell^2}{\psi^2+\ell^2+d}\frac{\theta^2}{\gamma} + \frac{(\psi^2+\ell^2)^2}{(\psi^2+\ell^2+d)^2}\frac{\theta^2+d}{2\gamma} \\
&= \frac{1}{2\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\Big[2\theta^2\psi^2(\psi^2+d)(\psi^2+\ell^2+d)^2 - \psi^4(\theta^2+d)(\psi^2+\ell^2+d)^2 \\
&\quad - 2\theta^2(\psi^2+\ell^2)(\psi^2+\ell^2+d)(\psi^2+d)^2 + (\theta^2+d)(\psi^2+\ell^2)^2(\psi^2+d)^2\Big] \\
&= \frac{d^2\ell^2(\psi^2+d-\theta_{ew}^2)}{2\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\left[\sigma_{ew}^2\gamma^2 - 2\mu_{ew}\gamma - \theta_{ew}^2\frac{d+\psi^2+\theta_{ew}^2}{d+\psi^2-\theta_{ew}^2}\right], \qquad \text{(SM74)}
\end{aligned}$$

which is equivalent to (20). The term in brackets is a second-degree polynomial in γ whose discriminant is $8\mu_{ew}^2(\psi^2+d)/(\psi^2+d-\theta_{ew}^2)$. When $\psi^2+d-\theta_{ew}^2 \leq 0$, the polynomial is always positive, and thus $\Delta_{smv} \leq 0$ because of the term $\psi^2+d-\theta_{ew}^2$ in front. Otherwise, the two roots of the polynomial are

$$\gamma_{ew}\left(1 \pm \sqrt{\frac{2(\psi^2+d)}{\psi^2+d-\theta_{ew}^2}}\right). \qquad \text{(SM75)}$$

We have $\Delta_{smv} \geq 0$ when γ is not in between the two roots, and since the smaller root is negative while any $\gamma > 0$, this is the case if $\gamma \geq \gamma_{ew}\left(1 + \sqrt{\frac{2(\psi^2+d)}{\psi^2+d-\theta_{ew}^2}}\right)$. Otherwise, $\Delta_{smv} < 0$.

Expression for Δ_{ew}. Given (SM72) and the formula for κ_2^c in (13) and κ_2^u in (14), we have

$$\begin{aligned}
\Delta_{ew} &= \frac{d}{\psi^2+d}\frac{\theta_{ew}^2}{\gamma} - \frac{d^2}{(\psi^2+d)^2}\frac{\theta_{ew}^2}{2\gamma} - \frac{d\mu_{ew}}{\psi^2+\ell^2+d} + \frac{\gamma}{2}\frac{d^2\sigma_{ew}^2}{(\psi^2+\ell^2+d)^2}, \\
&= \frac{d\sigma_{ew}^2}{2\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\Big[2\gamma_{ew}^2(\psi^2+d)(\psi^2+\ell^2+d)^2 - d\gamma_{ew}^2(\psi^2+\ell^2+d)^2 \\
&\quad - 2\gamma_{ew}\gamma(\psi^2+\ell^2+d)(\psi^2+d)^2 + d\gamma^2(\psi^2+d)^2\Big] \\
&= \frac{d\sigma_{ew}^2(\gamma-\gamma_{ew})(\theta^2+d-\mu_{ew}\gamma)}{2\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\Big[d(\gamma-\gamma_{ew})(\theta^2+d-\mu_{ew}\gamma) - 2\gamma_{ew}\psi^2\Big(\psi^2+\ell^2+d\Big)\Big],
\end{aligned}$$
$$\text{(SM76)}$$

which corresponds to (21). The last term in brackets is the following second-degree polynomial in γ:

$$d(\gamma-\gamma_{ew})\theta^2+d-\mu_{ew}\gamma) - 2\gamma_{ew}\psi^2\Big(\psi^2+\ell^2+d\Big)$$

$$= -\gamma^2\mu_{ew}[d+2\psi^2] + \gamma[d(\theta^2+\theta_{ew}^2+d)+4\theta_{ew}^2\psi^2] - \gamma_{ew}[d(d+\theta_{ew}^2+3\psi^2)+2\psi^2\theta^2]. \tag{SM77}$$

The discriminant of this polynomial is $\Delta = (\psi^2+d)(d^2(\psi^2+d) - 8\theta_{ew}^2\psi^2(2\psi^2+d))$. If d is small enough such that $\Delta \leq 0$, then the polynomial in (SM77) is always negative, and thus $\Delta_{ew} \leq 0$ if $\gamma \in [\gamma_{ew}, (\theta^2+d)/\mu_{ew}]$, and else $\Delta_{ew} > 0$. Otherwise, if d is large enough such that $\Delta > 0$, then the numerator of Δ_{ew}, which is a fourth-degree polynomial in γ, has four roots: $\gamma = \gamma_{ew}$, $\gamma = (\theta^2+d)/\mu_{ew}$, and the two roots of (SM77), which are

$$\gamma = \frac{d^2 + d(\theta^2+\theta_{ew}^2) + 4\theta_{ew}^2\psi^2 \pm \sqrt{(\psi^2+d)(d^2(\psi^2+d) - 8\theta_{ew}^2\psi^2(2\psi^2+d))}}{2\mu_{ew}(2\psi^2+d)}. \tag{SM78}$$

It is easy to check that γ_{ew} is the smallest root, $(\theta^2+d)/\mu_{ew}$ is the largest root, and the two roots in (SM78) are in between. Therefore, given that the coefficient in front of γ^4 in the numerator of Δ_{ew} is positive, it holds that $\Delta_{ew} \geq 0$ if either $\gamma \leq \gamma_{ew}$, $\gamma \geq (\theta^2+d)/\mu_{ew}$, or γ is in between the two roots in (SM78).

Expression for Δ_{cov}. Given (SM73) and the formula for (κ_1^c, κ_2^c) in (13) and (κ_1^u, κ_2^u) in (14), we have

$$\begin{aligned}
\Delta_{cov} &= \mu_{ew}\left(\frac{d(\psi^2+\ell^2)}{(\psi^2+\ell^2+d)^2} - \frac{d\psi^2}{(\psi^2+d)^2}\frac{\gamma_{ew}}{\gamma}\right) \\
&= \frac{d\mu_{ew}}{\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\Big[\gamma(\psi^2+\ell^2)(\psi^2+d)^2 - \gamma_{ew}\psi^2(\psi^2+\ell^2+d)^2\Big] \\
&= \frac{d\mu_{ew}(\gamma-\gamma_{ew})}{\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\Big[\psi^2(\psi^2+d)^2 + \sigma_{ew}^2(\gamma-\gamma_{ew})\Big(\gamma(\psi^2+d)^2 \\
&\quad - \gamma_{ew}\psi^2(\ell^2+2(\psi^2+d))\Big)\Big] \\
&= \frac{d\mu_{ew}(\gamma-\gamma_{ew})}{\gamma(\psi^2+d)^2(\psi^2+\ell^2+d)^2}\Big[(\psi^2+d)^2(\psi^2+\ell^2) + \mu_{ew}(\gamma-\gamma_{ew})(d^2-\psi^2(\psi^2+\ell^2))\Big],
\end{aligned} \tag{SM79}$$

which is equivalent to (22).

Proof of Proposition 3

The unconstrained three-fund strategy always delivers a larger EU than that of the unconstrained two-fund strategy by construction. To determine when the constrained three-fund strategy outperforms the unconstrained two-fund one, we need to determine the EU of the two-fund strategy. Plugging $\kappa_1 = \kappa^{2f} = \theta^2/(\theta^2 + d)$ and $\kappa_2 = 0$ into Equation (11), we have

$$EU(\hat{\boldsymbol{w}}^{2f}) = U^\star - \frac{d}{2\gamma}\kappa^{2f}. \tag{SM80}$$

Therefore, given (15), the constrained three-fund strategy delivers a larger EU than that of the unconstrained two-fund strategy when $\kappa_1^c \leq \kappa^{2f}$, which is equivalent to $\gamma \leq 2\gamma_{ew}$.

Proof of Proposition 4

We want to find when the L_2-norm of the expected weights of unconstrained strategy is smaller than that of the expected weights of the constrained strategy, which is equivalent to showing the same result for the squared L_2-norm because the L_2-norm is positive. We define $f(\gamma)$ the function of γ corresponding to the difference between the squared L_2-norms:

$$f(\gamma) = g(\boldsymbol{\kappa}^c) - g(\boldsymbol{\kappa}^u), \tag{SM81}$$

where

$$g(\boldsymbol{\kappa}) = ||\kappa_1 \boldsymbol{w}^\star + \kappa_2 \boldsymbol{w}_{ew}||_2^2 = \frac{\kappa_1^2}{\gamma^2}\|\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\|_2^2 + \frac{\kappa_2^2}{N} + \frac{2\kappa_1\kappa_2}{N\gamma}\gamma_{tan}. \tag{SM82}$$

We want to find the values of γ for which $f(\gamma) \geq 0$ holds, which can be rewritten as

$$f(\gamma) = \frac{(\kappa_1^c)^2 - (\kappa_1^u)^2}{\gamma^2}\|\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\|_2^2 + \frac{(\kappa_2^c)^2 - (\kappa_1^u)^2}{N} + 2\frac{\gamma_{tan}}{N\gamma}(\kappa_1^c\kappa_2^c - \kappa_1^u\kappa_2^u) \geq 0. \tag{SM83}$$

Proposition SM.2 shows that $\kappa_1^c \geq \kappa_1^u \geq 0$, $\|\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\|_2^2 \geq 0$ by definition, and we assume $\gamma_{tan} \geq 0$. Therefore, a sufficient condition for (SM83) to hold is

$$(\kappa_2^c)^2 - (\kappa_2^u)^2 + \frac{2\gamma_{tan}}{\gamma}(\kappa_1^c\kappa_2^c - \kappa_1^u\kappa_2^u) \geq 0. \tag{SM84}$$

Inequality (SM84) holds for all values of γ_{tan} if the following two inequalities hold:

$$(\kappa_2^c)^2 \geq (\kappa_2^u)^2 \quad \text{and} \quad \kappa_1^c \kappa_2^c \geq \kappa_1^u \kappa_2^u. \tag{SM85}$$

Both inequalities hold whenever $0 \leq \kappa_2^u \leq \kappa_2^c$ which, from Proposition SM.2, holds when $\gamma \in \left[\gamma_{ew}, \frac{\theta^2+d}{\mu_{ew}}\right]$. Therefore, $f(\gamma) \geq 0$ in (SM81) if $\gamma \in [\gamma_{ew}, (\theta^2+d)/\mu_{ew}]$.

Proof of Proposition 5

Part 1. The two combination coefficients are κ_1^u in (14) and $\tilde{\kappa}_2^u$ in (24). Because $\hat{\mu}_{ew}$ is unbiased, $\tilde{\kappa}_2^u$ is unbiased as well, and the expected out-of-sample mean return of the estimated unconstrained strategy is the same as that when κ_2^u is known:

$$\mathbb{E}[\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)'\boldsymbol{\mu}] = \frac{\kappa_1^u}{\gamma}\theta^2 + \kappa_2^u \mu_{ew}. \tag{SM86}$$

In comparison, the expected out-of-sample variance is larger than that when κ_2^u is known. Specifically,

$$\begin{aligned}\mathbb{E}[\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)] &= \mathbb{E}\left[\left(\frac{\kappa_1^u}{\gamma}\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma} + \tilde{\kappa}_2^u \boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\right)'\left(\frac{\kappa_1^u}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} + \tilde{\kappa}_2^u \boldsymbol{w}_{ew}\right)\right] \\ &= \frac{(\kappa_1^u)^2}{\gamma^2}(\theta^2+d) + \mathbb{E}\Big[(\tilde{\kappa}_2^u)^2\Big]\sigma_{ew}^2 + \frac{2\kappa_1^u}{\gamma}\mathbb{E}\Big[\tilde{\kappa}_2^u \hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\boldsymbol{w}_{ew}\Big].\end{aligned} \tag{SM87}$$

From the definition of $\tilde{\kappa}_2^u$ in (24) and the identity $\mathbb{E}[\hat{\mu}_{ew}^2] = \mu_{ew}^2 + \sigma_{ew}^2/T$, we find that

$$\sigma_{ew}^2 \mathbb{E}\Big[(\tilde{\kappa}_2^u)^2\Big] = (\kappa_2^u)^2\sigma_{ew}^2 + \frac{(1-\kappa_1^u)^2}{\gamma^2 T} \tag{SM88}$$

and

$$\frac{2\kappa_1^u}{\gamma}\mathbb{E}\Big[\tilde{\kappa}_2^u \hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\boldsymbol{w}_{ew}\Big] = \frac{2\kappa_1^u\kappa_2^u}{\gamma}\mu_{ew} + \frac{2\kappa_1^u(1-\kappa_1^u)}{\gamma^2 T}. \tag{SM89}$$

Therefore, the expected out-of-sample variance is

$$\mathbb{E}[\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u)] = \mathbb{E}[(\hat{\boldsymbol{w}}^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^u] + \frac{1-(\kappa_1^u)^2}{\gamma^2 T}, \tag{SM90}$$

where $\mathbb{E}[(\hat{\boldsymbol{w}}^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^u] = \frac{(\kappa_1^u)^2}{\gamma^2}(\theta^2 + d) + (\kappa_2^u)^2\sigma_{ew}^2 + \frac{2\kappa_1^u\kappa_2^u}{\gamma}\mu_{ew}$ is the expected out-of-sample variance when κ_2^u is known. Finally, using Equation (16), the EU loss relative to $\hat{\boldsymbol{w}}^u$ when κ_2^u is estimated by $\hat{\kappa}_2^u$ is given by (25).

Part 2. Using Equation (16), the constrained strategy delivers a larger EU than that of the estimated unconstrained strategy, $EU(\hat{\boldsymbol{w}}^c) \geq EU(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u))$, when

$$U^\star - \frac{d}{2\gamma}\kappa_1^c \geq U^\star - \frac{d}{2\gamma}\kappa_1^u - \frac{1-(\kappa_1^u)^2}{2\gamma T}. \tag{SM91}$$

After some developments, we find that inequality (SM91) is equivalent to

$$\gamma^2[-\sigma_{ew}^2] + \gamma[2\mu_{ew}] + k \geq 0, \tag{SM92}$$

where

$$k = \frac{(\psi^2+d)(2\psi^2+d)}{dT(\psi^2+d) - (2\psi^2+d)} - \theta_{ew}^2. \tag{SM93}$$

The discriminant of the second-degree polynomial in (SM92) is $\Delta = 4\sigma_{ew}^2(k + \theta_{ew}^2)$, which is positive because we assume $N \geq 2$. Therefore, inequality (SM92) holds when the risk-aversion coefficient γ belongs to the interval in Equation (26). Finally, the length of the interval (26) decreases with d because

$$\frac{\partial}{\partial d}\left[\frac{(\psi^2+d)(2\psi^2+d)}{dT(\psi^2+d)-(2\psi^2+d)}\right] = -\frac{2\psi^6 T + 4\psi^4(dT+1) + 2\psi^2 d(dT+2) + d^2}{\left(dT(\psi^2+d)-(2\psi^2+d)\right)^2} \leq 0. \tag{SM94}$$

Proof of Proposition 6

Part 1. The mean and variance of $\tilde{\kappa}_2^u(\delta)$ are

$$\mathbb{E}[\tilde{\kappa}_2^u(\delta)] = (1-\delta)\kappa_2^u + \delta(1-\kappa_1^u), \tag{SM95}$$

$$\mathbb{V}[\tilde{\kappa}_2^u(\delta)] = \frac{(1-\delta)^2(1-\kappa_1^u)^2}{\gamma^2\sigma_{ew}^2 T}. \tag{SM96}$$

Therefore, the mean squared error of $\tilde{\kappa}_2^u(\delta)$ is

$$(\mathbb{E}[\tilde{\kappa}_2^u(\delta)] - \kappa_2^u)^2 + \mathbb{V}[\tilde{\kappa}_2^u(\delta)] = \frac{(1-\kappa_1^u)^2}{\gamma^2\sigma_{ew}^2}\left(\delta^2\ell^2 + \frac{(1-\delta)^2}{T}\right). \tag{SM97}$$

Setting the derivative of (SM97) equal to zero delivers the optimal $\delta^\star$ in (28).

Part 2. The formula for $\tilde{\kappa}_2^u(\delta^\star)$ in (29) is directly obtained by plugging $\delta = \delta^\star$ into (27). Moreover, the two derivatives in (30) are

$$\frac{\partial}{\partial\hat{\mu}_{ew}}\tilde{\kappa}_2^u(\delta^\star) = \frac{1-\kappa_1^u}{\gamma\sigma_{ew}^2}\frac{T\ell^2}{1+T\ell^2}, \tag{SM98}$$

$$\frac{\partial}{\partial\hat{\mu}_{ew}}\tilde{\kappa}_2^u = \frac{1-\kappa_1^u}{\gamma\sigma_{ew}^2}, \tag{SM99}$$

and thus their ratio is equal to $1-\delta^\star \le 1$, as desired.

Part 3. To prove the result in (31), we derive a formula for the EU delivered by $\tilde{\kappa}_2^u(\delta)$ for any δ, and then plug in $\delta^\star$. Using the mean and variance of $\tilde{\kappa}_2^u(\delta)$ in (SM95)–(SM96), we find that the expected out-of-sample mean return delivered by $\tilde{\kappa}_2^u(\delta)$ is

$$\mathbb{E}[\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u(\delta))'\boldsymbol{\mu}] = \frac{\kappa_1^u}{\gamma}\theta^2 + (1-\delta)\kappa_2^u\mu_{ew} + \delta(1-\kappa_1^u)\mu_{ew}. \tag{SM100}$$

Then, the expected out-of-sample variance is

$$\begin{aligned}&\mathbb{E}[\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u(\delta))'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u(\delta))]\\ &= \frac{(\kappa_1^u)^2}{\gamma^2}(\theta^2+d) + \sigma_{ew}^2\mathbb{E}\Big[\tilde{\kappa}_2^u(\delta)^2\Big] + \frac{2\kappa_1^u}{\gamma}\mathbb{E}\Big[\tilde{\kappa}_2^u(\delta)\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\boldsymbol{w}_{ew}\Big],\end{aligned} \tag{SM101}$$

where, using (SM88)–(SM89), the two expectations in (SM101) are

$$\mathbb{E}\Big[\tilde{\kappa}_2^u(\delta)^2\Big] = (1-\delta)^2(\kappa_2^u)^2 + \delta^2(1-\kappa_1^u)^2 + 2\delta(1-\delta)(1-\kappa_1^u)\kappa_2^u + \frac{(1-\delta)^2(1-\kappa_1^u)^2}{\gamma^2\sigma_{ew}^2 T}, \tag{SM102}$$

$$\mathbb{E}\Big[\tilde{\kappa}_2^u(\delta)\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\boldsymbol{w}_{ew}\Big] = (1-\delta)\kappa_2^u\mu_{ew} + \frac{(1-\delta)(1-\kappa_1^u)}{\gamma T} + \delta(1-\kappa_1^u)\mu_{ew}. \tag{SM103}$$

Next, using $\kappa_2^u = \frac{\gamma_{ew}}{\gamma}(1-\kappa_1^u)$ and the fact that the EU when κ_2^u is known, $EU(\hat{\boldsymbol{w}}^u)$ in (16) is obtained by plugging $(\kappa_1, \kappa_2) = (\kappa_1^u, \kappa_2^u)$ into (11), we obtain after some simplifications that the EU delivered by $\tilde{\kappa}_2^u(\delta)$ is

$$EU(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u(\delta))) = EU(\hat{\boldsymbol{w}}^u) - \frac{1-\kappa_1^u}{2\gamma T}\left(\frac{\delta^2(1-\kappa_1^u)}{\delta^\star} - 2\delta + 1 + \kappa_1^u\right). \tag{SM104}$$

Now, plugging $\delta = \delta^\star$ into (SM104), we obtain

$$EU(\hat{\boldsymbol{w}}(\kappa_1^u, \tilde{\kappa}_2^u(\delta^\star))) = EU(\hat{\boldsymbol{w}}^u) - \frac{(1-\delta^\star)(1-(\kappa_1^u)^2)}{2\gamma T}, \tag{SM105}$$

which, given (25), is equivalent to (31).

Proof of Proposition 7

Under the elliptical model (38) and in the high-dimensional regime as $N, T \to \infty$ and $N/T \to \rho \in (0,1)$, El Karoui (2010, 2013) show that under a set of technical assumptions that ensure that the distribution of the λ_t's does not put too much mass near zero (see (Assumption-BB) in El Karoui (2013) for details) we have, for any vector $\boldsymbol{v}$,

$$\boldsymbol{v}'\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} \xrightarrow{p} \alpha \boldsymbol{v}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}, \tag{SM106}$$

$$\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} \xrightarrow{p} \frac{\beta\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \alpha\rho}{1-\rho}, \tag{SM107}$$

provided $\boldsymbol{v}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ and $\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ are bounded as $N \to \infty$, where $\alpha \geq 1$ and $\beta \geq \alpha^2$ are defined in the proposition statement.[SM8] We exploit these results to prove the proposition.

Part 1. The out-of-sample utility of the portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ is

$$U(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = \frac{\kappa_1}{\gamma}\boldsymbol{\mu}'\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} + \kappa_2\mu_{ew} - \frac{\gamma}{2}\left(\frac{\kappa_1^2}{\gamma^2}\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}} + \kappa_2^2\sigma_{ew}^2 + \frac{2\kappa_1\kappa_2}{\gamma}\boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}\right). \tag{SM108}$$

Given (SM106)–(SM107), we directly find that the asymptotic limit of $U(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}))$ is equal to $\bar{U}(\boldsymbol{\kappa})$ in (44).

Part 2. We obtain the combination coefficients maximizing $\bar{U}(\boldsymbol{\kappa})$ under the convexity con-

[SM8] Our definition of α and β is such that $\alpha = (1-\rho)s$ and $\beta = (1-\rho)^3\xi$, where (s, ξ) are defined in Equations (3.1) and (3.2) in El Karoui (2013).

straint (10) by setting the derivative of (44) with respect to κ_1 equal to zero, taking into account that $\kappa_2 = 1 - \kappa_1$. Then, to evaluate the asymptotic loss in out-of-sample utility, we use the fact that when $\kappa_2 = 1 - \kappa_1$, $\bar{U}(\boldsymbol{\kappa})$ simplifies to

$$\bar{U}(\kappa_1, 1-\kappa_1) = \kappa_1^2\left(\alpha\mu_{ew} - \frac{\gamma}{2}\sigma_{ew}^2 - \frac{1}{2\gamma}\frac{\beta\theta^2 + \alpha\rho}{1-\rho}\right) + \kappa_1\left(\frac{\alpha\theta^2}{\gamma} + \gamma\sigma_{ew}^2 - (1+\alpha)\mu_{ew}\right)$$
$$+ \mu_{ew} - \frac{\gamma}{2}\sigma_{ew}^2. \tag{SM109}$$

Using (SM109), the asymptotic out-of-sample utility loss is

$$\bar{U}(\bar{\boldsymbol{\kappa}}_n^c) - \bar{U}(\bar{\boldsymbol{\kappa}}^c) = \Big((\bar{\kappa}_{1,n}^c)^2 - (\bar{\kappa}_1^c)^2\Big)\left(\alpha\mu_{ew} - \frac{\gamma}{2}\sigma_{ew}^2 - \frac{1}{2\gamma}\frac{\beta\theta^2+\alpha\rho}{1-\rho}\right)$$
$$+ \Big(\bar{\kappa}_{1,n}^c - \bar{\kappa}_1^c\Big)\left(\frac{\alpha\theta^2}{\gamma} + \gamma\sigma_{ew}^2 - (1+\alpha)\mu_{ew}\right). \tag{SM110}$$

Now, notice that

$$\begin{aligned}
&\alpha\mu_{ew} - \frac{\gamma}{2}\sigma_{ew}^2 - \frac{1}{2\gamma}\frac{\beta\theta^2+\alpha\rho}{1-\rho} \\
&= -\frac{1}{2\gamma}\left(\alpha\psi^2 + \sigma_{ew}\ell(\gamma - \alpha\gamma_{ew}) + (1-\alpha)\gamma\mu_{ew} + \frac{(\beta-\alpha)\theta^2 + \alpha\rho(1+\theta^2)}{1-\rho}\right) \\
&= -\frac{1}{2\gamma}\frac{\alpha\psi^2 + \sigma_{ew}\ell(\gamma-\alpha\gamma_{ew})}{\bar{\kappa}_1^c},
\end{aligned} \tag{SM111}$$

and

$$\frac{\alpha\theta^2}{\gamma} + \gamma\sigma_{ew}^2 - (1+\alpha)\mu_{ew} = \frac{1}{\gamma}\Big(\alpha\psi^2 + \sigma_{ew}\ell(\gamma-\alpha\gamma_{ew})\Big). \tag{SM112}$$

Plugging (SM111)–(SM112) into (SM110), we obtain after some developments that the asymptotic utility loss simplifies to (46).

Part 3. The unconstrained combination coefficients maximizing $\bar{U}(\boldsymbol{\kappa})$ are found by setting the derivative of (44) with respect to κ_1 and κ_2 equal to zero and solving the resulting linear system of two equations with two unknowns. Then, we evaluate the asymptotic loss in out-of-sample utility. Since $\bar{\kappa}_{2,n}^u = \frac{\gamma_{ew}}{\gamma}(1-\bar{\kappa}_{1,n}^u)$, the asymptotic out-of-sample utility delivered by the combination coefficients that are optimal under normally distributed returns is

$$\bar{U}(\bar{\boldsymbol{\kappa}}_n^u) = U^\star - \frac{1}{2\gamma}\left((\bar{\kappa}_{1,n}^u)^2\left(\frac{\beta\theta^2+\alpha\rho}{1-\rho} - (2\alpha-1)\theta_{ew}^2\right) - 2\alpha\bar{\kappa}_{1,n}^u\psi^2 + \psi^2\right). \tag{SM113}$$

For the combination coefficients that are optimal under elliptically distributed returns, we have $\bar{\kappa}_2^u = \frac{\gamma_{ew}}{\gamma}(1-\alpha\bar{\kappa}_1^u)$ and thus the asymptotic out-of-sample utility is

$$\bar{U}(\bar{\boldsymbol{\kappa}}^u) = U^\star - \frac{1}{2\gamma}\left((\bar{\kappa}_1^u)^2\left(\frac{\beta\theta^2+\alpha\rho}{1-\rho} - \alpha^2\theta_{ew}^2\right) - 2\alpha\bar{\kappa}_1^u\psi^2 + \psi^2\right). \tag{SM114}$$

Using (SM113)–(SM114), the asymptotic out-of-sample utility loss is

$$\begin{aligned}&\bar{U}(\bar{\kappa}_{1,n}^u, \bar{\kappa}_{2,n}^u) - \bar{U}(\bar{\kappa}_1^u, \bar{\kappa}_2^u) = \\ &-\frac{1}{2\gamma}\left((\bar{\kappa}_1^u)^2\left(\alpha^2\theta_{ew}^2 - \frac{\beta\theta^2+\alpha\rho}{1-\rho}\right) + (\bar{\kappa}_{1,n}^u)^2\left(\frac{\beta\theta^2+\alpha\rho}{1-\rho} - (2\alpha-1)\theta_{ew}^2\right) + 2\alpha\psi^2\left(\bar{\kappa}_1^u - \bar{\kappa}_{1,n}^u\right)\right).\end{aligned} \tag{SM115}$$

Now, notice that

$$\alpha^2\theta_{ew}^2 - \frac{\beta\theta^2+\alpha\rho}{1-\rho} = -\left(\alpha^2\psi^2 + \frac{(\beta-\alpha^2)\theta^2 + \alpha\rho(1+\alpha\theta^2)}{1-\rho}\right) = -\frac{\alpha\psi^2}{\bar{\kappa}_1^u}, \tag{SM116}$$

and

$$\begin{aligned}&\frac{\beta\theta^2+\alpha\rho}{1-\rho} - (2\alpha-1)\theta_{ew}^2 \\ &= \alpha^2\psi^2 + (\alpha-1)^2\theta_{ew}^2 + \frac{(\beta-\alpha^2)\theta^2 + \alpha\rho(1+\alpha\theta^2)}{1-\rho} = \frac{\alpha\psi^2}{\bar{\kappa}_1^u} + (\alpha-1)^2\theta_{ew}^2.\end{aligned} \tag{SM117}$$

Plugging (SM116)–(SM117) into (SM115), we obtain after some developments that the asymptotic utility loss simplifies to (48).

Proof of Proposition SM.1

Part 1. When $\hat{\boldsymbol{\mu}} = \boldsymbol{\mu} + \boldsymbol{\varepsilon}$ while $\boldsymbol{\Sigma}$ is known, we have

$$\hat{\mu}_{ew} = \boldsymbol{w}_{ew}'\hat{\boldsymbol{\mu}} = \mu_{ew} + \bar{\boldsymbol{\varepsilon}}, \tag{SM118}$$

$$\hat{\gamma}_{ew} = \hat{\mu}_{ew}/\sigma_{ew}^2 = \gamma_{ew} + \bar{\boldsymbol{\varepsilon}}/\sigma_{ew}^2, \tag{SM119}$$

$$\hat{\theta}^2 = \hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}} = \theta^2 + 2\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon}, \tag{SM120}$$

$$\hat{\psi}^2 = \hat{\theta}^2 - \frac{\hat{\mu}_{ew}^2}{\sigma_{ew}^2} = \psi^2 + 2\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma_{ew} - \bar{\varepsilon}^2/\sigma_{ew}^2. \tag{SM121}$$

Using (SM118)–(SM121), the estimated combination coefficients are

$$\hat{\kappa}_1^u = \frac{\hat{\psi}^2}{\hat{\psi}^2 + cN/T + (c-1)\hat{\theta}^2} = \frac{\psi^2 + 2\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma_{ew} - \bar{\varepsilon}^2/\sigma_{ew}^2}{\psi^2 + d + 2c\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + c\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma_{ew} - \bar{\varepsilon}^2/\sigma_{ew}^2}, \tag{SM122}$$

$$\hat{\kappa}_2^u = \frac{\hat{\gamma}_{ew}}{\gamma}(1-\hat{\kappa}_1^u) = \frac{\mu_{ew} + \bar{\varepsilon}}{\gamma\sigma_{ew}^2}(1-\hat{\kappa}_1^u), \tag{SM123}$$

$$\hat{\kappa}_1^c = \frac{\hat{\psi}^2 + \sigma_{ew}^2(\gamma - \hat{\gamma}_{ew})^2}{\hat{\psi}^2 + \sigma_{ew}^2(\gamma - \hat{\gamma}_{ew})^2 + cN/T + (c-1)\hat{\theta}^2} = \frac{\psi^2 + \ell^2 + 2\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + \boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma}{\psi^2 + \ell^2 + d + 2c\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu} + c\boldsymbol{\varepsilon}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\varepsilon} - 2\bar{\varepsilon}\gamma}, \tag{SM124}$$

$$\hat{\kappa}_2^c = 1 - \hat{\kappa}_1^c, \tag{SM125}$$

which yields the desired estimation errors in (SM1)–(SM3). Finally, $\tilde{\kappa}_2^u = \kappa_2^u + \frac{\bar{\varepsilon}}{\gamma\sigma_{ew}^2}(1-\kappa_1^u)$, which corresponds to (SM4).

Proof of Proposition SM.2

Part 1. In Proposition 2, we prove that $EU(\hat{\boldsymbol{w}}^u) - EU(\hat{\boldsymbol{w}}^c) = \frac{d}{2\gamma}(\kappa_1^c - \kappa_1^u) \geq 0$, which implies that $\kappa_1^c \geq \kappa_1^u$. Moreover, both κ_1^c and κ_1^u are between zero and one given their definition in (13) and (14). Finally, since $\pi_{tan} = \frac{\gamma_{tan}}{\gamma}\kappa_1$, we have $0 \leq \pi_{tan}^u \leq \pi_{tan}^c$ if $\gamma_{tan} \geq 0$.

Part 2. We can show that

$$\begin{aligned}
\kappa_2^u - \kappa_2^c &= \frac{\gamma_{ew}}{\gamma}\frac{d}{\psi^2 + d} - \frac{d}{\psi^2 + \ell^2 + d} \\
&= \frac{d}{\gamma(\psi^2+d)(\psi^2+\ell^2+d)}\Big(\gamma_{ew}(\psi^2+\ell^2+d) - \gamma(\psi^2+d)\Big) \\
&= \frac{d\mu_{ew}(\gamma - \gamma_{ew})}{\gamma(\psi^2+d)(\psi^2+\ell^2+d)}\left(\gamma - \frac{\theta^2+d}{\mu_{ew}}\right).
\end{aligned} \tag{SM126}$$

Therefore, the inequality $\kappa_2^u \geq \kappa_2^c$ is equivalent to

$$(\gamma - \gamma_{ew})\left(\gamma - \frac{\theta^2 + d}{\mu_{ew}}\right) \geq 0. \tag{SM127}$$

Since $(\theta^2 + d)/\mu_{ew} \geq \gamma_{ew}$ under the assumption that $\mu_{ew} \geq 0$, it directly follows that $0 \leq \kappa_2^u \leq \kappa_2^c$ if $\gamma \in \left[\gamma_{ew}, \frac{\theta^2+d}{\mu_{ew}}\right]$ and $\kappa_2^u \geq \kappa_2^c$ otherwise.

Part 3. After some developments, the inequality $\pi_{rf}^u \geq \pi_{rf}^c$ is equivalent to

$$\gamma^2\Big[\sigma_{ew}^2(\gamma_{tan} - \gamma_{ew})\Big] + \gamma\Big[2\mu_{ew}(\gamma_{ew} - \gamma_{tan}) + \psi^2 + d\Big] + \Big[\gamma_{tan}\theta_{ew}^2 - \gamma_{ew}(\theta^2 + d)\Big] \geq 0. \tag{SM128}$$

Because we assume $\gamma_{tan} > \gamma_{ew}$, this polynomial has two roots:

$$\gamma = \gamma_{ew} \quad \text{and} \quad \gamma = \frac{\gamma_{tan} - (\theta^2 + d)/\mu_{ew}}{\gamma_{tan}/\gamma_{ew} - 1}, \tag{SM129}$$

and we can then write the following about inequality (SM128):

$$\pi_{rf}^u \leq \pi_{rf}^c \quad \text{if} \quad \gamma \in \left[\frac{\gamma_{tan} - (\theta^2 + d)/\mu_{ew}}{\gamma_{tan}/\gamma_{ew} - 1}, \gamma_{ew}\right] \quad \text{and} \quad \pi_{rf}^u \geq \pi_{rf}^c \quad \text{otherwise.} \tag{SM130}$$

The lower bound of the interval is negative under the assumption $\gamma_{tan} < (\theta^2 + d)/\mu_{ew}$. Therefore, $\pi_{rf}^u \leq \pi_{rf}^c$ holds for $\gamma \leq \gamma_{ew}$ becaus any $\gamma > 0$. Finally, when $\gamma \leq \gamma_{ew}$, we have that $\pi_{rf}^c \leq 0$ because $\pi_{rf}^c = \frac{1}{\gamma}\kappa_1^c(\gamma - \gamma_{tan})$ and $\gamma_{tan} > \gamma_{ew}$ by assumption.

Proof of Proposition SM.3

The proof of parts 2 and 3 of the proposition is direct by taking the limit of $\boldsymbol{\kappa}^u$ and $\boldsymbol{\kappa}^c$ in (13) and (14) as $N, T \to \infty$ and $N/T \to \rho \in (0, 1)$. Therefore, we turn to the proof of part 1 of the proposition. The asymptotic expectation of the out-of-sample mean return and variance of $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ can also be directly obtained by taking the limit in (11). We are left with the asymptotic covariance matrix of the out-of-sample mean return and variance, which are

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu} = \kappa_1\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\mu} + \kappa_2\mu_{ew}, \tag{SM131}$$

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \kappa_1^2\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star + \kappa_2^2\sigma_{ew}^2 + 2\kappa_1\kappa_2\boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star, \tag{SM132}$$

where $\hat{\boldsymbol{w}}^\star = \frac{1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}$ in (5) is the SMV portfolio, which is unbiased and has a covariance matrix given by (SM58). Kan et al. (2023, Theorem 3.2) implies that the asymptotic joint distribution of $(\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\mu}, \hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star, \boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star)$ is normal. Therefore, using the delta method, the asymptotic joint distribution of $(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}, \hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa}))$ is normal as well. Using the formulas for moments of linear and quadratic forms in normal random variables (Rencher and Schaalje, 2008), we can then find the three entries of the asymptotic covariance matrix of $(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}, \hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa}))$.

First, the asymptotic variance of $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}$ as $N, T \to \infty$ and $N/T \to \rho \in (0, 1)$ is

$$\lim_{N/T\to\rho}(T-N)\mathbb{V}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}] = \kappa_1^2 \lim_{N/T\to\rho}(T-N)\boldsymbol{\mu}'\mathbb{V}[\hat{\boldsymbol{w}}^\star]\boldsymbol{\mu} = \frac{\kappa_1^2}{\gamma^2}\theta^2(1+2\theta^2), \tag{SM133}$$

which corresponds to $\boldsymbol{V}_{11}$ in (SM9).

Second, the asymptotic covariance between $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}$ and $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ as $N, T \to \infty$ and $N/T \to \rho \in (0, 1)$ is

$$\begin{aligned}&\lim_{N/T\to\rho}(T-N)\mathbb{C}\mathrm{ov}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}, \hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})]\\ &= \lim_{N/T\to\rho}(T-N)\Big(2\kappa_1^2\kappa_2\mathbb{C}\mathrm{ov}[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\mu}, \boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star] + \kappa_1^3\mathbb{C}\mathrm{ov}[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\mu}, \hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star]\Big),\end{aligned} \tag{SM134}$$

where, given $\mathbb{E}[\hat{\boldsymbol{w}}^\star]$ and $\mathbb{V}[\hat{\boldsymbol{w}}^\star]$,

$$\lim_{N/T\to\rho}(T-N)\mathbb{C}\mathrm{ov}[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\mu}, \boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star] = \lim_{N/T\to\rho}(T-N)\boldsymbol{\mu}'\mathbb{V}[\hat{\boldsymbol{w}}^\star]\boldsymbol{\Sigma}\boldsymbol{w}_{ew} = \frac{\mu_{ew}}{\gamma^2}(1+2\theta^2), \tag{SM135}$$

$$\lim_{N/T\to\rho}(T-N)\mathbb{C}\mathrm{ov}[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\mu}, \hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star] = \lim_{N/T\to\rho}2(T-N)\boldsymbol{\mu}'\mathbb{V}[\hat{\boldsymbol{w}}^\star]\boldsymbol{\Sigma}\mathbb{E}[\hat{\boldsymbol{w}}^\star] = \frac{2\theta^2}{\gamma^3}(1+2\theta^2). \tag{SM136}$$

Plugging (SM135)–(SM136) into (SM134) yields

$$\lim_{N/T\to\rho}(T-N)\mathbb{C}\mathrm{ov}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}, \hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = \frac{2\kappa_1^2}{\gamma^2}(1+2\theta^2)\left(\frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_{ew}\right), \tag{SM137}$$

which corresponds to $\boldsymbol{V}_{12} = \boldsymbol{V}_{21}$ in (SM10).

Third, the asymptotic variance of $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$ as $N, T \to \infty$ and $N/T \to \rho \in (0,1)$ is

$$\begin{aligned}
&\lim_{N/T\to\rho}(T-N)\mathbb{V}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] \\
&= \lim_{N/T\to\rho}(T-N)\Big(\kappa_1^4\mathbb{V}[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star] + 4\kappa_1^2\kappa_2^2\boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\mathbb{V}[\hat{\boldsymbol{w}}^\star]\boldsymbol{\Sigma}\boldsymbol{w}_{ew} + 4\kappa_1^3\kappa_2\mathbb{C}\mathrm{ov}[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star, \boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star]\Big),
\end{aligned} \tag{SM138}$$

where, given $\mathbb{E}[\hat{\boldsymbol{w}}^\star]$ and $\mathbb{V}[\hat{\boldsymbol{w}}^\star]$,

$$\begin{aligned}
&\lim_{N/T\to\rho}(T-N)\mathbb{V}[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star] = \lim_{N/T\to\rho}(T-N)\Big(2\mathrm{Tr}((\boldsymbol{\Sigma}\mathbb{V}[\hat{\boldsymbol{w}}^\star])^2) + 4\mathbb{E}[\hat{\boldsymbol{w}}^\star]'\boldsymbol{\Sigma}\mathbb{V}[\hat{\boldsymbol{w}}^\star]\boldsymbol{\Sigma}\mathbb{E}[\hat{\boldsymbol{w}}^\star]\Big) \\
&= \frac{2}{\gamma^4}(\theta^4(7+\rho) + 2\theta^2(2+\rho) + \rho), \qquad \text{(SM139)} \\
&\lim_{N/T\to\rho}(T-N)\boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\mathbb{V}[\hat{\boldsymbol{w}}^\star]\boldsymbol{\Sigma}\boldsymbol{w}_{ew} = \frac{\sigma_{ew}^2}{\gamma^2}(1+\theta_{ew}^2+\theta^2), \qquad \text{(SM140)} \\
&\lim_{N/T\to\rho}(T-N)\mathbb{C}\mathrm{ov}[\hat{\boldsymbol{w}}^{\star\prime}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star, \boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star] = \lim_{N/T\to\rho}2(T-N)\boldsymbol{w}_{ew}'\boldsymbol{\Sigma}\mathbb{V}[\hat{\boldsymbol{w}}^\star]\boldsymbol{\Sigma}\mathbb{E}[\hat{\boldsymbol{w}}^\star] = \frac{2\mu_{ew}}{\gamma^3}(1+2\theta^2). \qquad \text{(SM141)}
\end{aligned}$$

Plugging (SM139)–(SM141) into (SM138) yields

$$\begin{aligned}
&\lim_{N/T\to\rho}(T-N)\mathbb{V}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] \\
&= \frac{2\kappa_1^2}{\gamma^2}\left(\frac{\kappa_1^2}{\gamma^2}\Big(\theta^4(7+\rho) + 2\theta^2(2+\rho) + \rho\Big) + \frac{4\kappa_1\kappa_2}{\gamma}\mu_{ew}(1+2\theta^2) + 2\kappa_2^2\sigma_{ew}^2(1+\theta_{ew}^2+\theta^2)\right),
\end{aligned} \tag{SM142}$$

which corresponds to $\boldsymbol{V}_{22}$ in (SM11).

Proof of Proposition SM.4

Part 1. The SMV portfolio $\hat{\boldsymbol{w}}^\star$ is unbiased, and the SGMV portfolio $\hat{\boldsymbol{w}}_g$ too (Okhrin and Schmid, 2006). Therefore, the expected out-of-sample mean return of the portfolio combination (SM16) is

$$\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = \frac{\kappa_1}{\gamma}\theta^2 + \kappa_2\mu_g. \tag{SM143}$$

The expected out-of-sample variance decomposes as

$$\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = \frac{\kappa_1^2}{\gamma^2}(\theta^2 + d) + \kappa_2^2\mathbb{E}\Big[\hat{\boldsymbol{w}}_g'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big] + \frac{2\kappa_1\kappa_2}{\gamma}\mathbb{E}\Big[\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big]. \tag{SM144}$$

From Kan et al. (2022, Lemma 1), it holds that

$$\mathbb{E}\Big[\hat{\boldsymbol{w}}_g'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big] = c_1\sigma_g^2, \tag{SM145}$$

where the constant c_1 is defined in (SM19). Therefore, it remains to evaluate the expectation

$$\mathbb{E}\Big[\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big] = \mathbb{E}\left[\frac{\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\right]. \tag{SM146}$$

In the following lemma, we prove that this expectation is equal to $c_1\mu_g$.

Lemma 1. $\mathbb{E}\left[\frac{\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\right] = c_1\mu_g$, *where* c_1 *is defined in* (SM19).

We prove this lemma in the next section. Combining Equations (SM144)–(SM145) and Lemma 1, the expected out-of-sample variance is

$$\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = \frac{\kappa_1^2}{\gamma^2}(\theta^2 + d) + c_1\kappa_2^2\sigma_g^2 + \frac{2c_1\kappa_1\kappa_2}{\gamma}\mu_g, \tag{SM147}$$

which results in the EU formula in (SM18).

Part 2. The unconstrained combination coefficients are found by setting the derivative of (SM18) with respect to κ_1 and κ_2 equal to zero and solving the resulting linear system of two equations with two unknowns.

Part 4. The EU of the unconstrained strategies is found by plugging the unconstrained coefficients (κ_1, κ_2) in (SM20) into the EU formula (SM18) after some developments.

Proof of Lemma 1

Proving the lemma amounts to show that

$$\mathbb{E}\left[\frac{\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\mathbf{1}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\mathbf{1}}\right] = \frac{T(T-2)\mu_g}{(T-N-1)(T-N-2)}, \tag{SM148}$$

where $\widehat{\boldsymbol{\Sigma}}_{ml} = \frac{T-N-2}{T}\widehat{\boldsymbol{\Sigma}}$ is the maximum-likelihood estimator of $\boldsymbol{\Sigma}$.

Let $\boldsymbol{P}$ be an $N \times N$ orthonormal matrix whose first two columns are

$$\boldsymbol{\nu} = \sigma_g \boldsymbol{\Sigma}^{-\frac{1}{2}}\mathbf{1}, \tag{SM149}$$

$$\boldsymbol{\eta} = \frac{\boldsymbol{\Sigma}^{-\frac{1}{2}}(\boldsymbol{\mu} - \mu_g \mathbf{1})}{\psi_g}, \tag{SM150}$$

where ψ_g is defined in (SM17). Let

$$\boldsymbol{z} = \sqrt{T}\boldsymbol{P}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\hat{\boldsymbol{\mu}} \sim \mathcal{N}(\boldsymbol{\mu}_z, \boldsymbol{I}_N), \tag{SM151}$$

$$\boldsymbol{W} = T\boldsymbol{P}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\widehat{\boldsymbol{\Sigma}}_{ml}\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{P} \sim \mathcal{W}_N(T-1, \boldsymbol{I}_N), \tag{SM152}$$

and they are independent of each other. We can show that

$$\boldsymbol{\mu}_z = \sqrt{T}\boldsymbol{P}'\boldsymbol{\Sigma}^{-\frac{1}{2}}\boldsymbol{\mu} = \sqrt{T}\theta_g \boldsymbol{e}_1 + \sqrt{T}\psi \boldsymbol{e}_2, \tag{SM153}$$

where $\boldsymbol{e}_i$ is the ith column of the identity matrix $\boldsymbol{I}_N$. Using $\boldsymbol{W}$ and $\boldsymbol{z}$, we can write

$$\mathbf{1}'\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\hat{\boldsymbol{\mu}} = \frac{T^{\frac{3}{2}}}{\sigma_g}\boldsymbol{e}_1'\boldsymbol{W}^{-2}\boldsymbol{z}, \tag{SM154}$$

$$\mathbf{1}'\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\mathbf{1} = \frac{T\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1}{\sigma_g^2}. \tag{SM155}$$

It follows that

$$\frac{\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\mathbf{1}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}_{ml}^{-1}\mathbf{1}} = \frac{\sqrt{T}\sigma_g \boldsymbol{e}_1'\boldsymbol{W}^{-2}\boldsymbol{z}}{\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1} =: q. \tag{SM156}$$

We are interested in obtaining the exact mean of q, which is equal to

$$\mathbb{E}[q] = T\mu_g \mathbb{E}\left[\frac{\boldsymbol{e}_1'\boldsymbol{W}^{-2}\boldsymbol{e}_1}{\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1}\right] + T\sigma_g\psi_g \mathbb{E}\left[\frac{\boldsymbol{e}_1'\boldsymbol{W}^{-2}\boldsymbol{e}_2}{\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1}\right]. \tag{SM157}$$

Partition $\boldsymbol{W}$ into four blocks such that $\boldsymbol{W}_{11}$ is its (1,1)th element. Using Muirhead (1982, Theorem 3.2.10), we can show that

$$x := (\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1)^{-1} = \boldsymbol{W}_{11} - \boldsymbol{W}_{12}\boldsymbol{W}_{22}^{-1}\boldsymbol{W}_{21} \sim \chi^2_{T-N}, \tag{SM158}$$

$$\boldsymbol{y} := -\boldsymbol{W}_{22}^{-\frac{1}{2}}\boldsymbol{W}_{21} \sim \mathcal{N}(\mathbf{0}_{N-1}, \boldsymbol{I}_{N-1}), \tag{SM159}$$

$$\boldsymbol{W}_{22} \sim \mathcal{W}_{N-1}(T-1, \boldsymbol{I}_{N-1}), \tag{SM160}$$

and they are independent of each other. Let $\boldsymbol{Q} = [\boldsymbol{e}_2, \ldots, \boldsymbol{e}_N]$. From the partition matrix inverse formula, we can verify that

$$\boldsymbol{Q}'\boldsymbol{W}^{-1}\boldsymbol{e}_1 = \frac{\boldsymbol{W}_{22}^{-\frac{1}{2}}\boldsymbol{y}}{x}, \tag{SM161}$$

$$\boldsymbol{Q}'\boldsymbol{W}^{-1}\boldsymbol{Q} = \boldsymbol{W}_{22}^{-1} + \frac{\boldsymbol{W}_{22}^{-\frac{1}{2}}\boldsymbol{y}\boldsymbol{y}'\boldsymbol{W}_{22}^{-\frac{1}{2}}}{x}, \tag{SM162}$$

so we have

$$\boldsymbol{e}_2'\boldsymbol{W}^{-1}\boldsymbol{e}_1 = \frac{[1,\ \boldsymbol{0}_{N-2}']\boldsymbol{W}_{22}^{-\frac{1}{2}}\boldsymbol{y}}{x}. \tag{SM163}$$

It follows that

$$\begin{aligned}
\boldsymbol{e}_1'\boldsymbol{W}^{-2}\boldsymbol{e}_1 &= \boldsymbol{e}_1'\boldsymbol{W}^{-1}[\boldsymbol{e}_1,\ \boldsymbol{Q}][\boldsymbol{e}_1,\ \boldsymbol{Q}]'\boldsymbol{W}^{-1}\boldsymbol{e}_1 \\
&= (\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1)^2 + \boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{W}^{-1}\boldsymbol{e}_1 \\
&= \frac{1 + \boldsymbol{y}'\boldsymbol{W}_{22}^{-1}\boldsymbol{y}}{x^2}, \qquad \text{(SM164)} \\
\boldsymbol{e}_1'\boldsymbol{W}^{-2}\boldsymbol{e}_2 &= \boldsymbol{e}_1'\boldsymbol{W}^{-1}[\boldsymbol{e}_1,\ \boldsymbol{Q}][\boldsymbol{e}_1,\ \boldsymbol{Q}]'\boldsymbol{W}^{-1}\boldsymbol{e}_2 \\
&= (\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1)(\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_2) + \boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{Q}\boldsymbol{Q}'\boldsymbol{W}^{-1}\boldsymbol{e}_2 \\
&= \frac{[1,\ \boldsymbol{0}_{N-2}']\boldsymbol{W}_{22}^{-\frac{1}{2}}\boldsymbol{y}}{x^2} + \frac{\boldsymbol{y}'\boldsymbol{W}_{22}^{-\frac{1}{2}}}{x}\left(\boldsymbol{W}_{22}^{-1} + \frac{\boldsymbol{W}_{22}^{-\frac{1}{2}}\boldsymbol{y}\boldsymbol{y}'\boldsymbol{W}_{22}^{-\frac{1}{2}}}{x}\right)\begin{bmatrix} 1 \\ \boldsymbol{0}_{N-2} \end{bmatrix}. \qquad \text{(SM165)}
\end{aligned}$$

Using the above expressions, we obtain

$$\begin{aligned}
\mathbb{E}\left[\frac{\boldsymbol{e}_1'\boldsymbol{W}^{-2}\boldsymbol{e}_1}{\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1}\right] &= \mathbb{E}\left[\frac{1 + \boldsymbol{y}'\boldsymbol{W}_{22}^{-1}\boldsymbol{y}}{x}\right] \\
&= \mathbb{E}[1 + \boldsymbol{y}'\boldsymbol{W}_{22}^{-1}\boldsymbol{y}]\mathbb{E}[x^{-1}] \\
&= \frac{1 + \frac{N-1}{T-N-1}}{T-N-2} \\
&= \frac{T-2}{(T-N-1)(T-N-2)}, \qquad \text{(SM166)} \\
E\left[\frac{\boldsymbol{e}_1'\boldsymbol{W}^{-2}\boldsymbol{e}_2}{\boldsymbol{e}_1'\boldsymbol{W}^{-1}\boldsymbol{e}_1}\right] &= 0, \qquad \text{(SM167)}
\end{aligned}$$

and therefore we have from (SM157) that, as desired,

$$\mathbb{E}[q] = \frac{T(T-2)\mu_g}{(T-N-1)(T-N-2)}. \tag{SM168}$$

Proof of Proposition SM.5

The two combination coefficients are κ_1^u in (SM20) and $\hat{\kappa}_2^u$ in (SM22). Because $\mathbf{1}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}$ is unbiased, $\hat{\kappa}_2^u$ is unbiased as well, and the expected out-of-sample mean return of the estimated unconstrained strategy is the same as that when κ_2^u is known:

$$\mathbb{E}[\hat{\boldsymbol{w}}(\kappa_1^u, \hat{\kappa}_2^u)'\boldsymbol{\mu}] = \frac{\kappa_1^u}{\gamma}\theta^2 + \kappa_2^u\mu_g. \tag{SM169}$$

In comparison, the expected out-of-sample variance is larger than that when κ_2^u is known. Specifically,

$$\mathbb{E}[\hat{\boldsymbol{w}}(\kappa_1^u, \hat{\kappa}_2^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\kappa_1^u, \hat{\kappa}_2^u)] = \frac{(\kappa_1^u)^2}{\gamma^2}(\theta^2 + d) + \mathbb{E}\Big[(\hat{\kappa}_2^u)^2\hat{\boldsymbol{w}}_g'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big] + \frac{2\kappa_1^u}{\gamma}\mathbb{E}\Big[\hat{\kappa}_2^u\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big]. \tag{SM170}$$

Applying (SM56), $\mathbb{E}\Big[\hat{\boldsymbol{w}}_g'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big] = c_1\sigma_g^2$ in (SM145), and Lemma 1, we have that

$$\begin{aligned}\mathbb{E}\Big[(\hat{\kappa}_2^u)^2\hat{\boldsymbol{w}}_g'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big] &= \frac{(1/c_1-\kappa_1^u)^2}{\gamma^2}\mathbb{E}\left[\frac{\hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}{(\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1})^2}\right] \\ &= c_1\sigma_g^2(\kappa_2^u)^2 + \frac{c_1(1/c_1-\kappa_1^u)^2}{\gamma^2 T}\end{aligned} \tag{SM171}$$

and

$$\begin{aligned}\frac{2\kappa_1^u}{\gamma}\mathbb{E}\Big[\hat{\kappa}_2^u\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\hat{\boldsymbol{w}}_g\Big] &= \frac{2\kappa_1^u(1/c_1-\kappa_1^u)}{\gamma^2}\mathbb{E}\left[\frac{\hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}}\right] \\ &= \frac{2c_1\kappa_1^u\kappa_2^u}{\gamma}\mu_g + \frac{2c_1\kappa_1^u(1/c_1-\kappa_1^u)}{T\gamma^2}.\end{aligned} \tag{SM172}$$

Therefore, the expected out-of-sample variance is

$$\mathbb{E}[\hat{\boldsymbol{w}}(\kappa_1^u, \hat{\kappa}_2^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\kappa_1^u, \hat{\kappa}_2^u)] = \mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)] + \frac{c_1(1/c_1^2-(\kappa_1^u)^2)}{\gamma^2 T}, \tag{SM173}$$

where $\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)] = \frac{(\kappa_1^u)^2}{\gamma^2}(\theta^2+d)+c_1\sigma_g^2(\kappa_2^u)^2+\frac{2c_1\kappa_1^u\kappa_2^u}{\gamma}\mu_g$ is the expected out-of-sample variance when κ_2^u is known. Finally, the EU loss relative to $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)$ when κ_2^u is estimated by $\hat{\kappa}_2^u$ is given by (SM23).

Proof of Proposition SM.6

The mean and variance of $\tilde{\kappa}_2^u(\delta)$ are

$$\mathbb{E}[\tilde{\kappa}_2^u(\delta)] = (1-\delta)\kappa_2^u + \delta(1/c_1 - \kappa_1^u), \tag{SM174}$$

$$\mathbb{V}[\tilde{\kappa}_2^u(\delta)] = \frac{(1-\delta)^2(1/c_1-\kappa_1^u)^2}{\gamma^2\sigma_g^2 T}. \tag{SM175}$$

Therefore, the mean squared error of $\tilde{\kappa}_2^u(\delta)$ is

$$(\mathbb{E}[\tilde{\kappa}_2^u(\delta)] - \kappa_2^u)^2 + \mathbb{V}[\tilde{\kappa}_2^u(\delta)] = \frac{(1/c_1-\kappa_1^u)^2}{\gamma^2}\left(\delta^2(\gamma-\gamma_{tan})^2 + \frac{(1-\delta)^2}{\sigma_g^2 T}\right). \tag{SM176}$$

Setting the derivative of (SM176) equal to zero delivers the optimal $\delta^\star$ in (SM.6).

Proof of Proposition SM.7

The EU of the combination of the SMV and SGMV portfolios,

$$\hat{\boldsymbol{w}}(\boldsymbol{\kappa}) = \frac{\kappa_1}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\mu} + \frac{\kappa_2}{\gamma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}, \tag{SM177}$$

is obtained by plugging $\kappa_2 = 0$ into (SM39), which gives

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = \frac{\kappa_1}{\gamma}\theta^2 + \frac{\kappa_2}{\gamma}\gamma_{tan} - \frac{c\gamma}{2}\left(\frac{\kappa_1^2}{\gamma^2}\left(\theta^2+\frac{N}{T}\right) + \frac{\kappa_2^2}{\gamma^2}\frac{1}{\sigma_g^2} + \frac{2\kappa_1\kappa_2}{\gamma^2}\gamma_{tan}\right). \tag{SM178}$$

The EU of the portfolio combination of Kan and Zhou (2007) in (SM32) is then obtained by plugging κ_1^{kz} and κ_2^{kz} in (SM31) into the EU formula (SM178) after some developments.

Concerning the fully invested combination of the SMV and SGMV portfolios, $\hat{\boldsymbol{w}}(\kappa)$

in (SM28), we have from Kan et al. (2022) that the EU depends on κ as

$$EU(\hat{\boldsymbol{w}}(\kappa)) = \mu_g - \frac{\gamma}{2}c_1\sigma_g^2 + \frac{1}{\gamma}\frac{T-N-2}{T-N-1}\left(\kappa\psi_g^2 - \frac{\kappa^2}{2}\left(\psi_g^2 + \frac{N-1}{T}\right)\frac{(T-2)(T-N-2)}{(T-N)(T-N-3)}\right). \tag{SM179}$$

The EU of the optimal portfolio combination is then obtained by plugging κ^{kwz} in (SM29) into the EU formula (SM179).

Proof of Proposition SM.8

Part 1. The two combination coefficients are κ_1^{kz} in (SM31) and $\hat{\kappa}_2^{kz}$ in (SM34). Because $\hat{\mu}_g$ is unbiased, $\hat{\kappa}_2^{kz}$ is unbiased as well, and the expected out-of-sample mean return of the estimated unconstrained strategy is the same as that when κ_2^{kz} is known in (SM178):

$$\mathbb{E}\Big[\hat{\boldsymbol{w}}(\kappa_1^{kz}, \hat{\kappa}_2^{kz})'\boldsymbol{\mu}\Big] = \frac{\kappa_1^{kz}}{\gamma}\theta^2 + \kappa_2^{kz}\gamma_{tan}. \tag{SM180}$$

In comparison, the expected out-of-sample variance is larger than that when κ_2^{kz} is known. Specifically,

$$\begin{aligned}\mathbb{E}\Big[\hat{\boldsymbol{w}}(\kappa_1^{kz}, \hat{\kappa}_2^{kz})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\kappa_1^{kz}, \hat{\kappa}_2^{kz})\Big] &= \frac{(\kappa_1^{kz})^2}{\gamma^2}(\theta^2 + d) + \frac{1}{\gamma^2}\mathbb{E}\Big[(\hat{\kappa}_2^{kz})^2\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\Big] \\ &\quad + \frac{2\kappa_1^{kz}}{\gamma^2}\mathbb{E}\Big[\hat{\kappa}_2^{kz}\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}\Big].\end{aligned} \tag{SM181}$$

Applying (SM56) and $\mathbb{E}\Big[\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\Big] = c\boldsymbol{\Sigma}^{-1}$, we have that

$$\begin{aligned}\frac{1}{\gamma^2}\mathbb{E}\Big[(\hat{\kappa}_2^{kz})^2\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\Big] &= \frac{(1/c - \kappa_1^{kz})^2}{\gamma^2}\mathbb{E}\left[\frac{\hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}\hat{\boldsymbol{\mu}}}{(\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1})^2}\right] \\ &= \frac{c(\kappa_2^{kz})^2}{\gamma^2\sigma_g^2} + \frac{c(1/c - \kappa_1^{kz})^2}{\gamma^2 T}\end{aligned} \tag{SM182}$$

and

$$\begin{aligned}\frac{2\kappa_1^{kz}}{\gamma^2}\mathbb{E}\Big[\hat{\kappa}_2^{kz}\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}\Big] &= \frac{2\kappa_1^{kz}(1/c - \kappa_1^{kz})}{\gamma^2}\mathbb{E}\left[\frac{\hat{\boldsymbol{\mu}}'\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{\Sigma}\widehat{\boldsymbol{\Sigma}}^{-1}\hat{\boldsymbol{\mu}}}{\mathbf{1}'\boldsymbol{\Sigma}^{-1}\mathbf{1}}\right] \\ &= \frac{2c\kappa_1^{kz}\kappa_2^{kz}}{\gamma^2}\gamma_{tan} + \frac{2c\kappa_1^{kz}(1/c - \kappa_1^{kz})}{\gamma^2 T}.\end{aligned} \tag{SM183}$$

Therefore, the expected out-of-sample variance is

$$\mathbb{E}\Big[\hat{\boldsymbol{w}}(\kappa_1^{kz},\hat{\kappa}_2^{kz})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\kappa_1^{kz},\hat{\kappa}_2^{kz})\Big]=\mathbb{E}\Big[\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz})\Big]+\frac{c(1/c^2-(\kappa_1^{kz})^2)}{\gamma^2T}, \tag{SM184}$$

where $\mathbb{E}\Big[\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz})\Big]=c\Big(\frac{(\kappa_1^{kz})^2}{\gamma^2}(\theta^2+\frac{N}{T})+\frac{(\kappa_2^{kz})^2}{\gamma^2}\frac{1}{\sigma_g^2}+\frac{2\kappa_1^{kz}\kappa_2^{kz}}{\gamma^2}\gamma_{tan}\Big)$ is the expected out-of-sample variance when κ_2^{kz} is known. Finally, the loss in EU relative to $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^{kz})$ when κ_2^{kz} is estimated by $\hat{\kappa}_2^{kz}$ is given by (SM35).

Part 2. Using Equations (SM33) and (SM35), the fully invested strategy of Kan et al. (2022) delivers a larger EU than the estimated three-fund strategy of Kan and Zhou (2007), $EU(\hat{\boldsymbol{w}}(\kappa^{kwz}))\geq EU(\hat{\boldsymbol{w}}(\kappa_1^{kz},\hat{\kappa}_2^{kz}))$, when

$$\mu_g-\frac{\gamma}{2}c_1\sigma_g^2+\frac{T-N-2}{T-N-1}\frac{\psi_g^2}{2\gamma}\kappa^{kwz}\geq U^\star-\frac{d}{2\gamma}\kappa_1^{kz}-\frac{c-1}{2\gamma}\gamma_{tan}\kappa_2^{kz}-\frac{c(1/c^2-(\kappa_1^{kz})^2)}{2\gamma T}. \tag{SM185}$$

After some developments, we find that inequality (SM185) is equivalent to

$$\gamma^2[-c_1\sigma_g^2]+\gamma[2\mu_g]+\frac{b-\theta_g^2}{c_1}\geq 0, \tag{SM186}$$

where b is defined in (SM37). The discriminant of this second-degree polynomial is

$$\Delta=4\sigma_g^2 b. \tag{SM187}$$

If $b<0$, the polynomial (SM186) has no real roots and the Kan-Zhou three-fund strategy outperforms for any γ. Otherwise, the two real roots of the polynomial are those in (SM36).

Proof of Proposition SM.9

Part 1. Because the four-fund portfolio combination in (SM38) is unbiased, the out-of-sample mean return is unbiased as well,

$$\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\mu}]=\frac{\kappa_1}{\gamma}\theta^2+\kappa_2\mu_{ew}+\frac{\kappa_3}{\gamma}\gamma_{tan}. \tag{SM188}$$

To find the expected out-of-sample variance in (SM56), we need the covariance matrix of $\hat{\boldsymbol{w}}(\boldsymbol{\kappa})$. Using the result in Javed et al. (2021, Corollary 2.2), it is

$$\begin{aligned}\mathbb{V}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = c\frac{\kappa_1^2}{\gamma^2}\left(\frac{\theta^2}{T-2}+\frac{1}{T}\right)\boldsymbol{\Sigma}^{-1} \\ + c_2\left(\frac{\kappa_1^2}{\gamma^2}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}+\frac{\kappa_3^2}{\gamma^2}\boldsymbol{\Sigma}^{-1}\mathbf{1}\mathbf{1}'\boldsymbol{\Sigma}^{-1}+\frac{\kappa_1\kappa_3}{\gamma^2}\boldsymbol{\Sigma}^{-1}(\boldsymbol{\mu}\mathbf{1}'+\mathbf{1}\boldsymbol{\mu}')\boldsymbol{\Sigma}^{-1}\right),\end{aligned} \tag{SM189}$$

where

$$c_2 = \frac{c(T-N)}{(T-2)(T-N-2)} = \frac{T-N}{(T-N-1)(T-N-4)}. \tag{SM190}$$

The expected out-of-sample variance is then given by (SM56), where

$$\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})]'\boldsymbol{\Sigma}\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = \frac{\kappa_1^2}{\gamma^2}\theta^2+\kappa_2^2\sigma_{ew}^2+\frac{\kappa_3^2}{\gamma^2}\frac{1}{\sigma_g^2}+\frac{2\kappa_1\kappa_2}{\gamma}\mu_{ew}+\frac{2\kappa_1\kappa_3}{\gamma^2}\gamma_{tan}+\frac{2\kappa_2\kappa_3}{\gamma}, \tag{SM191}$$

$$\text{Trace}(\boldsymbol{\Sigma}\mathbb{V}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})]) = c\frac{\kappa_1^2}{\gamma^2}\frac{N}{T}+\left(c_2+\frac{cN}{T-2}\right)\left(\frac{\kappa_1^2}{\gamma^2}\theta^2+\frac{\kappa_3^2}{\gamma^2}\frac{1}{\sigma_g^2}+\frac{2\kappa_1\kappa_3}{\gamma^2}\gamma_{tan}\right). \tag{SM192}$$

Noticing that $1+c_2+\frac{cN}{T-2}=c$, the expected out-of-sample variance is

$$\mathbb{E}[\hat{\boldsymbol{w}}(\boldsymbol{\kappa})'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}(\boldsymbol{\kappa})] = c\left(\frac{\kappa_1^2}{\gamma^2}\left(\theta^2+\frac{N}{T}\right)+\frac{\kappa_3^2}{\gamma^2}\frac{1}{\sigma_g^2}+\frac{2\kappa_1\kappa_3}{\gamma^2}\gamma_{tan}\right)+\kappa_2^2\sigma_{ew}^2+\frac{2\kappa_1\kappa_2}{\gamma}\mu_{ew}+\frac{2\kappa_2\kappa_3}{\gamma}, \tag{SM193}$$

and thus the EU is given by (SM39).

Part 2. The EU of the four-fund portfolio in (SM39) can be rewritten in matrix format as

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = \boldsymbol{\kappa}'\boldsymbol{\eta}-\frac{\gamma}{2}\boldsymbol{\kappa}'\boldsymbol{S}\boldsymbol{\kappa}, \tag{SM194}$$

where

$$\boldsymbol{\eta} = \begin{pmatrix}\theta^2/\gamma \\ \mu_{ew} \\ \gamma_{tan}/\gamma\end{pmatrix} \quad \text{and} \quad \boldsymbol{S} = \begin{pmatrix}\frac{c}{\gamma^2}\left(\theta^2+\frac{N}{T}\right) & \frac{\mu_{ew}}{\gamma} & \frac{c\gamma_{tan}}{\gamma^2} \\ \frac{\mu_{ew}}{\gamma} & \sigma_{ew}^2 & \frac{1}{\gamma} \\ \frac{c\gamma_{tan}}{\gamma^2} & \frac{1}{\gamma} & \frac{c}{\gamma^2\sigma_g^2}\end{pmatrix}. \tag{SM195}$$

The matrix $\boldsymbol{S}$ is positive definite and thus invertible. This is because the covariance matrix $\boldsymbol{\Sigma}$ is assumed positive definite, and thus any expected out-of-sample variance is strictly positive,

$\boldsymbol{\kappa}'\boldsymbol{S}\boldsymbol{\kappa} > 0$ for any $\boldsymbol{\kappa} \neq \mathbf{0}$. As a result, the combination coefficients $\boldsymbol{\kappa}$ maximizing the EU in (SM39) are

$$\boldsymbol{\kappa}^u = \arg\max_{\boldsymbol{\kappa}} \ \boldsymbol{\kappa}'\boldsymbol{\eta} - \frac{\gamma}{2}\boldsymbol{\kappa}'\boldsymbol{S}\boldsymbol{\kappa} = \frac{1}{\gamma}\boldsymbol{S}^{-1}\boldsymbol{\eta}, \tag{SM196}$$

which from (SM195) simplify to (SM40)–(SM42) after some developments.

Part 3. Just like the utility of the optimal mean-variance portfolio $\boldsymbol{w}^\star$ is $U^\star = \theta^2/(2\gamma) = \boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}/(2\gamma)$, the EU achieved by the optimal four-fund portfolio combination $\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)$ obtained from the combination coefficients (SM196) is

$$EU(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)) = \frac{\boldsymbol{\eta}'\boldsymbol{S}^{-1}\boldsymbol{\eta}}{2\gamma}, \tag{SM197}$$

where $\boldsymbol{\eta}$ and $\boldsymbol{S}$ are defined in (SM195). After some developments, we can show that (SM197) corresponds to (SM44).

Proof of Proposition SM.10

We begin with the correlation between the out-of-sample return of the SMV and EW portfolios, which is

$$\mathbb{C}\text{orr}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}] = \frac{\mathbb{C}\text{ov}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}]}{\sqrt{\mathbb{V}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}]\mathbb{V}[\boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}]}}. \tag{SM198}$$

We must evaluate the three terms appearing in (SM198). It is clear that $\mathbb{V}[\boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}] = \sigma_{ew}^2$. Moreover, applying (SM56),

$$\mathbb{C}\text{ov}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}] = \frac{1}{\gamma}\mathbb{E}[\boldsymbol{r}_{T+1}'\boldsymbol{w}_{ew}\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{r}_{T+1}] - \mathbb{E}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}]\mathbb{E}[\boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}] = \mu_{ew}/\gamma. \tag{SM199}$$

Finally, from the law of total variance,

$$\mathbb{V}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}] = \mathbb{E}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star] + \boldsymbol{\mu}'\mathbb{V}[\hat{\boldsymbol{w}}^\star]\boldsymbol{\mu}, \tag{SM200}$$

where $\mathbb{E}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{\Sigma}\hat{\boldsymbol{w}}^\star] = \frac{\theta^2+d}{\gamma^2}$ from the proof of Proposition 1 and $\mathbb{V}[\hat{\boldsymbol{w}}^\star]$ is given by (SM58). This gives

$$\mathbb{V}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}] = \frac{1}{\gamma^2}\left(\frac{c}{T}(\theta^2(T+1)+N) + \frac{2\theta^4}{T-N-4}\right). \tag{SM201}$$

Plugging $\mathbb{C}\mathrm{ov}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \boldsymbol{w}_{ew}'\boldsymbol{r}_{T+1}]$, $\mathbb{V}[\boldsymbol{w}_{ew}'R_{T+1}]$, and $\mathbb{V}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}]$ into (SM198) gives (SM45). This correlation tends to θ_{ew}/θ as $T \to \infty$.

Let us now consider the correlation between the out-of-sample return of the SMV and SGMV portfolios, which is

$$\mathbb{C}\mathrm{orr}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}] = \frac{\mathbb{C}\mathrm{ov}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}]}{\sqrt{\mathbb{V}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}]\mathbb{V}[\hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}]}}. \tag{SM202}$$

Applying (SM56) and $\mathbb{E}[\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{A}\widehat{\boldsymbol{\Sigma}}^{-1}] = c\boldsymbol{\Sigma}^{-1}\boldsymbol{A}\boldsymbol{\Sigma}^{-1}$ for any constant matrix $\boldsymbol{A}$, we have that

$$\begin{aligned}\mathbb{C}\mathrm{ov}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}] &= \frac{1}{\gamma}\mathbb{E}[\boldsymbol{r}_{T+1}'\hat{\boldsymbol{w}}_g\hat{\boldsymbol{\mu}}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{r}_{T+1}] - \mathbb{E}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}]\mathbb{E}[\hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}] \\ &= \frac{\gamma_{tan}}{\gamma^2}(c+(c-1)\theta^2). \end{aligned}\tag{SM203}$$

Similarly, we can show that

$$\mathbb{V}[\hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}] = \frac{1}{\gamma^2}\Big(\mathbb{E}[\boldsymbol{r}_{T+1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{1}\boldsymbol{1}'\widehat{\boldsymbol{\Sigma}}^{-1}\boldsymbol{r}_{T+1}] - \gamma_{tan}^2\Big) = \frac{1}{\gamma^2}\Big(c/\sigma_g^2 + (c-1)\gamma_{tan}^2\Big). \tag{SM204}$$

Finally, plugging $\mathbb{C}\mathrm{ov}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}, \hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}]$, $\mathbb{V}[\hat{\boldsymbol{w}}_g'\boldsymbol{r}_{T+1}]$, and $\mathbb{V}[(\hat{\boldsymbol{w}}^\star)'\boldsymbol{r}_{T+1}]$ into (SM202) gives (SM46). This correlation tends to θ_g/θ as $T \to \infty$.

Proof of Proposition SM.11

Part 1. Given Equation (11), the ESR of the three-fund portfolio in (7) can be rewritten in matrix format as

$$ESR(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = \frac{\boldsymbol{\kappa}'\boldsymbol{\eta}}{\sqrt{\boldsymbol{\kappa}'\boldsymbol{S}\boldsymbol{\kappa}}}, \tag{SM205}$$

where

$$\boldsymbol{\eta} = \begin{pmatrix} \theta^2/\gamma \\ \\ \mu_{ew} \end{pmatrix} \quad \text{and} \quad \boldsymbol{S} = \begin{pmatrix} \frac{c}{\gamma^2}\left(\theta^2 + \frac{N}{T}\right) & \frac{\mu_{ew}}{\gamma} \\ \\ \frac{\mu_{ew}}{\gamma} & \sigma^2_{ew} \end{pmatrix}. \tag{SM206}$$

The matrix $\boldsymbol{S}$ is positive definite and thus invertible. This is because the covariance matrix $\boldsymbol{\Sigma}$ is assumed positive definite, and thus any expected out-of-sample variance is strictly positive, $\boldsymbol{\kappa}'\boldsymbol{S}\boldsymbol{\kappa} > 0$ for any $\boldsymbol{\kappa} \neq \mathbf{0}$. As a result, just like any maximum-utility portfolio $\frac{1}{\gamma}\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}$ provides the maximum Sharpe ratio $\sqrt{\boldsymbol{\mu}'\boldsymbol{\Sigma}^{-1}\boldsymbol{\mu}}$, any vector of unconstrained combination coefficients $\boldsymbol{\kappa}^u = \frac{1}{\gamma}\boldsymbol{S}^{-1}\boldsymbol{\eta}$ provides the maximum ESR:

$$\max_{\boldsymbol{\kappa}} ESR(\hat{\boldsymbol{w}}(\boldsymbol{\kappa})) = ESR(\hat{\boldsymbol{w}}(\boldsymbol{\kappa}^u)) = \sqrt{\boldsymbol{\eta}'\boldsymbol{S}^{-1}\boldsymbol{\eta}}. \tag{SM207}$$

After some developments, we find that

$$\sqrt{\boldsymbol{\eta}'\boldsymbol{S}^{-1}\boldsymbol{\eta}} = \sqrt{\theta^2_{ew} + (\theta^2 - \theta^2_{ew})\kappa^u_1}, \tag{SM208}$$

which is in between θ_{ew} and θ because $\kappa^u_1 \in [0, 1]$.

Part 2. The constrained combination coefficients maximizing the ESR in (SM49) are

$$\boldsymbol{\kappa} = \arg\max_{\boldsymbol{\kappa}'\mathbf{1}=1} \frac{\boldsymbol{\kappa}'\boldsymbol{\eta}}{\sqrt{\boldsymbol{\kappa}'\boldsymbol{S}\boldsymbol{\kappa}}} = \frac{\boldsymbol{S}^{-1}\boldsymbol{\eta}}{\mathbf{1}'\boldsymbol{S}^{-1}\boldsymbol{\eta}} = \left(\frac{\psi^2}{\psi^2 + \frac{\gamma_{ew}}{\gamma}d}, \frac{\frac{\gamma_{ew}}{\gamma}d}{\psi^2 + \frac{\gamma_{ew}}{\gamma}d} \right), \tag{SM209}$$

and they correspond to the unconstrained combination coefficients in (13) for $\gamma = \gamma_{ew}$ and $\gamma = \theta^2/\mu_{ew}$, in which case the constrained strategy delivers the maximum ESR.

To find when the ESR of the constrained strategy $\hat{\boldsymbol{w}}^c$ is minimized, we first simplify its ESR by plugging (13) into (SM49), which gives

$$ESR(\hat{\boldsymbol{w}}^c) = \frac{d\mu_{ew}\gamma + \theta^2(\psi^2 + \ell^2)}{\sqrt{(\psi^2 + \ell^2 + d)(d\sigma^2_{ew}\gamma^2 + \theta^2(\psi^2 + \ell^2))}}. \tag{SM210}$$

From (SM210), it is easy to check that

$$\lim_{\gamma \to 0} ESR(\hat{\boldsymbol{w}}^c) = \lim_{\gamma \to \infty} ESR(\hat{\boldsymbol{w}}^c) = \frac{\theta^2}{\sqrt{\theta^2 + d}}. \tag{SM211}$$

To see under which condition no value of γ between 0 and ∞ can give a smaller ESR than (SM211), we differentiate $ESR(\hat{\boldsymbol{w}}^c)$ with respect to γ:

$$\frac{\partial}{\partial\gamma}ESR(\hat{\boldsymbol{w}}^c) = \frac{d^2(\theta^2 - \mu_{ew}\gamma)(\mu_{ew} - \sigma_{ew}^2\gamma)(\theta^2 - \sigma_{ew}^2\gamma^2)}{\Big((\psi^2 + \ell^2 + d)(d\sigma_{ew}^2\gamma^2 + \theta^2(\psi^2 + \ell^2))\Big)^{3/2}}. \tag{SM212}$$

This derivative is equal to zero when $\gamma = \gamma_{ew}$ and $\gamma = \theta^2/\mu_{ew}$, which correspond to the two maxima identified above, and when $\gamma = \theta/\sigma_{ew}$, which corresponds to a local minimum. To conclude the proof, we must thus find when the value of $ESR(\hat{\boldsymbol{w}}^c)$ with $\gamma = \theta/\sigma_{ew}$ is larger than $\theta^2/\sqrt{\theta^2 + d}$ in (SM211). By plugging $\gamma = \theta/\sigma_{ew}$ into (SM210), we find that

$$ESR(\hat{\boldsymbol{w}}^c) = \theta\frac{\theta_{ew}d + 2\theta^2(\theta - \theta_{ew})}{\theta d + 2\theta^2(\theta - \theta_{ew})} \quad \text{when} \quad \gamma = \theta/\sigma_{ew}, \tag{SM213}$$

and after some developments this value is larger than $\theta^2/\sqrt{\theta^2 + d}$ when

$$d > \frac{\theta^2}{\theta_{ew}^2}(\theta - 3\theta_{ew})(\theta - \theta_{ew}). \tag{SM214}$$

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