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Games and Economic Behaviour, 110, 58-70, 2018

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Games and Economic Behavior 110 (2018) 58–70

Contents lists available at ScienceDirect

Games and Economic Behavior

www.elsevier.com/locate/geb

Note

Games for cautious players: The Equilibrium in Secure Strategies

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ARTICLE INFO

Article history:
Received 26 August 2013
Available online 4 April 2018

JEL classification:
C72
D03
D43
D72
L12
L13

Keywords:
Noncooperative games
Equilibrium existence
Equilibrium in secure strategies
Hotelling model
Tullock contest
Insurance market
Bertrand–Edgeworth duopoly

ABSTRACT

A non-cooperative solution, the Equilibrium in Secure Strategies (EinSS), is defined as an extension of the Nash equilibrium in pure strategies, and is meant to solve games where players are "cautious," *i.e.*, looking for secure positions and avoiding threats. This concept abstracts and unifies *ad hoc* solutions already formulated in various applied economic games that have been discussed extensively in the literature. A general existence theorem is provided and then applied to the price-setting game in the Hotelling location model, to Tullock's rent-seeking contests, and to Bertrand–Edgeworth duopoly. Finally, competition in the insurance market game is re-examined and the Rothschild–Stiglitz–Wilson contract is shown to be an EinSS even when the Nash equilibrium breaks down.
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1. Introduction

There are well-known economic games where a Nash–Cournot equilibrium does not exist. Examples include the Bertrand–Edgeworth duopoly model,[1] Hotelling's game of price competition on the line when the sellers locations are close,[2] Tullock's rent-seeking game with the success function parameter greater than two,[3] and Rothschild and Stiglitz's game of competitive insurance markets with adverse selection.[4] This existence problem was highlighted by Dasgupta and Maskin in their seminal paper (1986a). They proved the existence of mixed strategy Nash equilibria for a family of games with discontinuous payoff functions covering all mentioned models. However, these equilibria are not easy to characterize in most cases. Also, they are usually difficult to interpret when applied to specific economic contexts. On the other

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[1] Edgeworth (1925).
[2] Hotelling (1929), d'Aspremont et al. (1979).
[3] Tullock (1967, 1980) and Baye et al. (1993), for discussion and references.
[4] Rothschild and Stiglitz (1976), Wilson (1977).

https://doi.org/10.1016/j.geb.2018.03.011
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hand, in some contexts there is an intuitive expectation that a stable position in pure strategies should exist. In support of this hypothesis a variety of *ad hoc* equilibrium concepts have been developed to describe the specific behavior of players in particular models. For instance, Wilson (1977) and Riley (1979) suggested two different pseudo-equilibrium concepts for the insurance market model. Eaton and Lipsey (1978) proposed the 'zero conjectural variation solution' to restore the existence of equilibrium in Hotelling's model. In d'Aspremont and Gabszewicz (1985), a concept of quasi-monopoly is introduced that ensures the existence of a pseudo-equilibrium in the Bertrand–Edgeworth duopoly model when one capacity is small relative to the other. A common feature of these and many other concepts is that they describe a particular logic of rational behavior that takes into account the interaction of the players in a specific way. This raises the key question: is there a general rationale, independent of the specifics of a particular model, behind these intuitively perceived equilibrium positions?

In this paper we try to address this question by proposing an extension of the Nash concept called an Equilibrium in Secure Strategies (EinSS).[5] We reformulate the EinSS concept and prove its existence in the four well-known economic games mentioned above. For each game we provide a unique (or, with symmetry, unique up to a permutation of players) EinSS solution and its intuitive interpretation.[6] For the first three applications (including Hotelling's game), existence is derived from a general existence theorem. This theorem establishes the existence of an EinSS in a class of games that may be neither continuous nor quasi-concave. For the last application, the insurance game, a specific proof has to be derived. The main notion at the basis of the proposed concept is the notion of a *threat of one player against another*, defined as a unilateral deviation of the first player increasing his own payoff and simultaneously decreasing the payoff of the other player, and the main assumption is that players are "cautious" and looking for "secure" positions, namely positions where they cannot be "threatened" by another player. By this we mean that a player is not only avoiding threats from other players, as in a Cournot–Nash equilibrium, but also that, when considering the possibility of deviating, a player restricts this possibility to profitable deviations that are "secure": no other player should have a "counter-deviation", that is a threat making the deviator worse-off than before deviating. Hence, instead of defining an equilibrium as a (secure) payoff profile at which there is no profitable deviation from any of the players (Cournot–Nash), we define it as a secure strategy profile at which there is no profitable "secure deviation" from any of the players. By definition when a Nash Equilibrium exists, it is an Equilibrium in Secure Strategies. A way to look at this concept, suggested by Aumann (2008), is to use the distinction between act-rationality and rule-rationality. Instead of computing and choosing directly acts that maximize their payoffs (such as deviating whenever profitable), players adopt simple rules of behavior that may improve their payoff (say on average), but not always. As remarked by Aumann (2008), the rule might not be "consciously adopted" but is the result of "evolutionary forces" and is typically implemented via some mechanism. For the EinSS concept, the main rule could be for a player to avoid other players' threats (*i.e.*, to be cautious). To implement this rule the mechanism could have two simple instructions: (i) choose a strategy such that no other player has a threat against you and (ii) whenever possible, improve your payoff but only by a secure deviation.

Related Literature. The proposed equilibrium concept can be associated with several areas of game theory. First, in applying the deviation and counter-deviation rationale, it is related to the Bargaining Set concept introduced in cooperative game theory using the idea of objection and counter-objection (Aumann and Maschler, 1964) to deal with the vacuity of the Core in cooperative games with side payments (for more recent references, see Holzman, 2001). It is also related to the literature on farsighted solution concepts, defining profile stability by checking whether a player (or group of players) can deviate non-myopically, that is, anticipating a possible sequence of deviations by other players. The farsighted solution concepts originally arose as a result of the study of vNM stable set in coalitional games (von Neumann and Morgenstern, 1944) and the non-myopic alternative proposed by Harsanyi (1974) and formalized and extended by Chwe (1994).[7] Although these concepts can formally be used to also study noncooperative games (Suzuki and Muto, 2005; Nakanishi, 2007; Jamroga and Melissen, 2011), it raises questions concerning the tractability of the obtained results. The first two papers assume that in the chain of deviations there could be unilateral deviations that are not profitable for deviators. The third paper is closer to what we do, but it assumes that strategic thinking takes three or more steps ahead (instead of only two deviations ahead) and that the initiating deviator has the opportunity to make a last unilateral deviation in order to recover his original payoff. Finally, a substantial difference between the EinSS and all farsighted solution concepts is the requirement that an EinSS be a secure strategy profile. In particular, dropping this assumption may lead to the emergence of multiple equilibria. As we prove the existence of an EinSS in a class of discontinuous games, our work is also related to the literature on the existence of Nash equilibrium in discontinuous games. Developing the ideas of Dasgupta and Maskin (Dasgupta and Maskin, 1986b), Reny's (1999) important paper proposes a fairly simple existence criterion covering most of previous existence results.[8] What we propose is another approach for finding equilibria in discontinuous games. Rather than imposing a condition on the game, we relax the requirements at equilibrium, allowing for applications to games in which a pure Nash equilibrium does not exist.

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[5] The concept of EinSS was introduced in Iskakov (2005) and later applied to Hotelling's game (Iskakov and Iskakov, 2012).

[6] In general the set of EinSS is not large. From a theoretical point of view, this follows from the fact that the problem of finding EinSS can be formulated as an optimization problem for players' payoffs on the sets of their secure strategies.

[7] Harsanyi's definition has been recently modified by Ray and Vohra (2015) in order to respect coalitional sovereignty.

[8] Reny's conditions have been weakened or simplified in a number of ways. See, for example, Bagh and Jofre (2006), Bich (2009), McLennan et al. (2011), Barelli and Meneghel (2013).

In the following section, the concept of EinSS is defined and its basic properties are analyzed. In section 3, we state and prove an existence theorem for discontinuous games and apply it, successively, to Hotelling's model, Tullock's contest, and the Bertrand–Edgeworth duopoly model. Section 4 gives an *ad hoc* existence result for the competitive insurance market with adverse selection. Omitted proofs are given in the Appendix.

2. Equilibrium in secure strategies

2.1. Equilibrium concept

Consider a non-cooperative game $G = (S_i, u_i)_{i=1}^N$ of N players in normal form. We use the standard notation s_{-i} for strategies of all players other than i. If $s_i \in S_i$ is a deviation of player i from the profile s to profile s', we use the notation $s \xrightarrow{i} s'$ with the obvious constraint $s_{-i} = s'_{-i}$. The proposed equilibrium concept is based on the notions of threat and secure deviation.

Definition 2.1. A threat of player i against player j at strategy profile s is a deviation s'_i such that $u_i(s'_i, s_{-i}) > u_i(s)$ and $u_j(s'_i, s_{-i}) < u_j(s)$. A deviation of player i at strategy profile s is **a competitive deviation** if it is a threat against some player at s.

A threat (and a competitive deviation) indicates a situation, in which it is profitable for one player to worsen the situation of another. In that respect, an important property of a strategy is to avoid such threats.

Definition 2.2. A strategy s_i of player i is **a secure strategy** at strategy profile s if no player $j \neq i$ has a threat against player i at s. Otherwise it is an insecure strategy at s. A strategy profile s is **a secure profile**, if all players' strategies are secure.

Alternatively, a strategy profile is secure if no player has a competitive deviation. Let us now introduce a complementary notion to competitive deviation.

Definition 2.3. A non-competitive deviation of player i at strategy profile s is a deviation s'_i such that $u_i(s'_i, s_{-i}) > u_i(s)$ for player i and $u_j(s'_i, s_{-i}) \geq u_j(s)$ for all other players $j \neq i$.

In this way all profitable unilateral deviations of players can be divided into competitive and non-competitive deviations depending on their effect on the other players. Non-competitive deviations do not pose an immediate danger to other players. They could even be mutually beneficial. A Nash equilibrium can be characterized in the following manner.

Definition 2.4. A secure strategy profile is **a Nash Equilibrium** if no player has a non-competitive deviation.

Following the "objection and counter-objection rationale" (Aumann and Maschler, 1964), and reinforcing the security objective of the players, one can extend the applicability of a non-cooperative equilibrium. The idea is that, although a non-competitive deviation by some player does not worsen others' payoffs (and may even be mutually beneficial), it may give the opportunity to some other player to threaten the deviator at the new strategy profile. There are two possibilities after such a non-competitive deviation. Either the threat can make the deviator worse off than before the deviation, in which case we assume the deviation to be *insecure* for the deviator. This should induce the deviator, as a cautious player, to refrain from deviating. Or the threat makes the deviator as well off as or even better than initially; in this case the deviation can be considered *secure* for the deviator. Formally, we have:

Definition 2.5. A secure non-competitive deviation of player i at strategy profile s is a non-competitive deviation s'_i such that $u_i(s'_i, s'_j, s_{-ij}) \geq u_i(s)$ for any threat s'_j of player $j \neq i$ against player i at profile (s'_i, s_{-i}).

In order to formulate a concept of equilibrium in a game of cautious players, we may assume that such players avoid *insecure* non-competitive deviations. Together with the requirement that the equilibrium is a secure profile, this leads us to the proposed definition for an Equilibrium in Secure Strategies:

Definition 2.6. A secure strategy profile is **an Equilibrium in Secure Strategies (EinSS)** if no player has a secure non-competitive deviation.

2.2. Basic properties

Proposition 2.1. *Any Nash equilibrium is an Equilibrium in Secure Strategies.*

Whenever a Nash equilibrium exists, an EinSS also exists. However, for some practically important problems without Nash equilibrium (such as Hotelling's model and Bertrand–Edgeworth duopoly, which will be considered in this paper) the EinSS exists and provides an interesting interpretation. Let us now consider a simple matrix game example having no Nash equilibrium:

	t_1	t_2
s_1	$(1,1)$	$(2,0)$
s_2	$(2,2)$	$(0,3)$

The profile (s_1,t_1) is the only secure profile in the game. The t-player cannot increase his profit by any deviation from it. There is a profitable deviation for the s-player from this profile to the profile (s_2,t_1) with payoffs $(2,2)$. However, it is not a secure deviation since the s-player can lose more after the response deviation of t-player from the profile (s_2,t_1) with payoffs $(2,2)$ to the profile (s_2,t_2) with payoffs $(0,3)$. Therefore, in the profile (s_1,t_1), no player has a secure deviation, and this profile is an EinSS. This means that a cautious player would prefer the guaranteed payment of 1 in the profile (s_1,t_1) to the possibility of gaining 2 in (s_2,t_1) accompanied by a high risk of getting zero in (s_2,t_2).

For some games the reverse of Proposition 2.1 is true. For instance, for strictly competitive games (i.e., two-person games where all pairs of strategies are Pareto optimal), we get:

Proposition 2.2. *In a strictly competitive game, any Equilibrium in Secure Strategies is a Nash equilibrium.*

Proof. Consider a strictly competitive game and suppose there is an EinSS that is not a Nash equilibrium. Then, there is at least one player who has a profitable deviation. Yet, (by Pareto optimality) the other player will get a strictly lower payoff. Therefore, this is a competitive deviation so that the profile is not secure and cannot be EinSS. □

However, this property may not hold if the condition of strict competitiveness is weakened. For instance, it does not hold for almost strictly competitive games introduced by Aumann (1961) on the basis of the concept of twisted equilibrium. A twisted equilibrium point in a two-player game is a pair of strategies at which neither player can decrease the other player's payoff by a unilateral change in strategy. A game is called almost strictly competitive if (i) the set of Nash equilibrium payoff vectors is equal to the set of twisted equilibrium payoff vectors and (ii) if the set of Nash equilibrium strategy profiles and the set of twisted equilibrium strategy profiles intersect. Let us modify the previous matrix game in the following way:

	t_1	t_2	t_3
s_1	$(1,1)$	$(2,0)$	$(-1,1)$
s_2	$(2,2)$	$(0,3)$	$(-1,1)$
s_3	$(1,-1)$	$(1,-1)$	$(0,0)$

There is a unique Nash equilibrium profile (s_3,t_3) with payoffs $(0,0)$ that at the same time is a unique twisted equilibrium. So, the game is almost strictly competitive. The profile (s_3,t_3) is also an EinSS. But there is another EinSS. This is the profile (s_1,t_1) with payoffs $(1,1)$. It is not a Nash equilibrium. This example shows that cautious players (avoiding being threatened by others) may enforce (as an EinSS) a strategy profile that is not a Nash equilibrium but may dominate another EinSS, which is a Nash equilibrium. This illustrates how rule-rationality may sometimes dominate act-rationality (*i.e.*, improve players' payoffs).

However, the reverse can also happen. As the following example shows, a cautious player can even play a strictly dominated strategy at an EinSS, but the deviations offered to him at this equilibrium should be non-competitive and non-secure.

	t_1	t_2
s_1	$(9,6)$	$(4,5)$
s_2	$(10,7)$	$(8,8)$

The strategy s_2 dominates strategy s_1. There is a Nash equilibrium profile with payoffs $(8,8)$, which is an EinSS. But, the strategy profile with payoffs $(9,6)$ is another EinSS solution, in which the s-player plays the strictly dominated strategy s_1. If the s-player was using the deviation $(s_1,t_1) \xrightarrow{s} (s_2,t_1)$, both players would gain but the t-player would then be able to gain even more by deviating to the profile (s_2,t_2) where the s-player is worse-off than at the EinSS.

Suppose now a simpler rule: players are only looking for secure profiles. We get an interesting but weaker equilibrium concept than the concept of EinSS. However, it will be useful in applications (as we shall see later) as an intermediate step in finding an EinSS. Let $Q_i(s_{-i}) \subseteq S_i$ be the set of secure strategies of player i for given strategies s_{-i} of the other players. We can define a profile in which the secure strategy of each player is the best one.

Definition 2.7. A secure strategy $s_i^* \in Q_i(s_{-i})$ of player i is **a best secure response** to strategies s_{-i} of all other players if player i has no more profitable secure strategy in $Q_i(s_{-i})$. A profile s^* is **the Best Secure Profile (or BS-profile)** if the strategies of all players are best secure responses, i.e., $s_i^* \in Q_i(s_{-i}^*)$ and $u_i(s^*) = \max\limits_{s_i \in Q_i(s_{-i}^*)} u_i(s_i, s_{-i}^*)$ for all i.

Clearly, an EinSS is a secure profile by definition. In addition, it must be the best secure response for each player since otherwise there is a player who can increase his payoff by a secure non-competitive deviation. Therefore an EinSS is a BS-profile. The reverse is not true. Consider the following matrix game example:

	t_1	t_2	t_3
s_1	$(0,0)$	$(2,2)$	$(2,2)$
s_2	$(2,2)$	$(1,3)$	$(3,1)$
s_3	$(2,2)$	$(3,1)$	$(1,3)$

The profile (s_1, t_1) is the only secure profile in the game, and therefore, it is a BS-profile. However, it is not an EinSS because of secure non-competitive deviation $(s_1, t_1) \xrightarrow{s} (s_2, t_1)$.

Hence a BS-profile is an EinSS if no player has secure non-competitive deviation.

3. Existence

3.1. Existence result

Recall that a strategy s_i of player i is insecure at strategy profile s, *if there is another player j who has a threat against player i at s*. Pursuing on the BS-profile idea, and as a tool to obtain an existence theorem, we shall associate to every game $G = (S_i, u_i)_{i=1}^N$ a modified game $\widetilde{G} = (S_i, v_i)_{i=1}^N$ with the same strategies but payoffs adjusted for insecure strategies by taking into account the worst threat.

Definition 3.1. A secure payoff of player i at strategy profile s is the function:

$$v_i(s) = \begin{cases} \inf\limits_{j \neq i,\ s'_j:\ u_j(s'_j, s_{-j}) > u_j(s)} u_i(s'_j, s_{-j}), & \text{if } s_i \text{ is an insecure strategy for player } i \text{ in } G \\ u_i(s), & \text{if } s_i \text{ is a secure strategy for player } i \text{ in } G \end{cases}$$

The following proposition is an immediate but useful consequence:

Proposition 3.1. *If a secure strategy profile s^* is a strict Nash equilibrium in the modified game $\widetilde{G}$, then it is an EinSS. If s^* is an EinSS of the game G, then s^* is a Nash equilibrium of the modified game $\widetilde{G}$.*

On the basis of this observation, we now introduce a class of games for which the existence of an EinSS can be shown. Roughly speaking this is a class of games such that, whenever a profile s is insecure (in the game G) for some player i, i.e. $s_i \notin Q_i(s_{-i})$, this player can increase his secure payoff v_i by deviating to a profile (s'_i, s_{-i}), which is secure for him, i.e., $s'_i \in Q_i(s_{-i})$. Player i is then said to have *a better secure alternative* at profile s. However, this condition will be weakened by requiring its fulfillment only relative to some box $B = \times_{i=1}^N B_i$, where B_i is assumed to be a compact convex subset of S_i.

Definition 3.2. A player i has **a better secure alternative** in B, if for every $s_{-i} \in B_{-i}$ there exists a non-empty subset $\widetilde{Q}_i(s_{-i}) \subseteq Q_i(s_{-i}) \cap B_i$ such that, for every strategy $s_i \notin \widetilde{Q}_i(s_{-i})$ there exists a strategy $s'_i \in \widetilde{Q}_i(s_{-i})$ such that $u_i(s'_i, s_{-i}) = v_i(s'_i, s_{-i}) > v_i(s_i, s_{-i})$. A game $G = (S_i, u_i)_{i=1}^N$ is said to be a **BSA-game** relative to B if every player has a better secure alternative in B.

For any $s_{-i} \in B_{-i}$, the set $\widetilde{Q}_i(s_{-i})$ in a BSA-game is always understood to be non-empty. The graph of the multi-valued function $\widetilde{Q}_i(s_{-i})$, $\Gamma(\widetilde{Q}_i) = \{(s_i, s_{-i}) \mid s_i \in \widetilde{Q}_i(s_{-i}),\ s_{-i} \in B_{-i}\}$ is defined as a subset of B.

We shall use the Debreu (1952) theorem to prove the existence of an EinSS in BSA games in finite Euclidean spaces. For this class of games, we relax the standard conditions for the existence of a (pure) Nash equilibrium by requiring the fulfillment of these conditions only in the corresponding box B. In this case, a Nash equilibrium in the set B turns out to be an EinSS in the original game.

Theorem 3.1. *Let G be a BSA-game relative to B such that for all i, the graph $\Gamma(\widetilde{Q}_i)$ is closed, $u_i(s)$ is a continuous function from $\Gamma(\widetilde{Q}_i)$ to $\mathbb{R}$, and $\phi_i(s_{-i}) = \max\limits_{s_i \in \widetilde{Q}_i(s_{-i})} u_i(s_i, s_{-i})$ is continuous. If for every i and $s_{-i} \in B_{-i}$ the set $M_{s_{-i}} = \{s_i \in \widetilde{Q}_i(s_{-i}) \mid u_i(s_i, s_{-i}) = \phi_i(s_{-i})\}$ is contractible, then there exists an EinSS in B.*

Proof. As the basis of the proof we shall use the social equilibrium existence theorem of Debreu (1952). Since the sets $\widetilde{Q}_i(s_{-i})$ are assumed to be non-empty for all $s_{-i} \in B_{-i}$ in a BSA-game we can consider them as a multivalued function from B_{-i} to B_i. Next, following Debreu we define a profile s^* as a social equilibrium point, if for all $i = 1, ..., N$: $s^*_i \in \widetilde{Q}_i(s^*_{-i})$ and

$u_i(s^*) = \max_{s_i \in \widetilde{Q}_i(s^*_{-i})} u_i(s_i, s^*_{-i})$. By assumption, all conditions of Debreu (1952) existence theorem are satisfied and there exists a social equilibrium point $s^* \in B$. Consider a profitable deviation s'_i of any player i at profile s^*. It is obviously non-competitive, because s^* is a secure profile. If $s'_i \in \widetilde{Q}_i(s^*_{-i})$, then $u_i(s'_i, s^*_{-i}) \le u_i(s^*)$, and the deviation is not profitable. If $s'_i \notin \widetilde{Q}_i(s^*_{-i})$, then according to BSA condition there exists $s''_i \in \widetilde{Q}_i(s^*_{-i})$ such that $u_i(s''_i, s^*_{-i}) > v_i(s'_i, s^*_{-i})$. Since $u_i(s''_i, s^*_{-i}) \le u_i(s^*)$, we obtain $v_i(s'_i, s^*_{-i}) < u_i(s^*)$. By Definition 3.1 of the function v_i this implies that the non-competitive deviation s'_i is not secure. Therefore, in the secure profile s^* no player has a secure non-competitive deviation, i.e., s^* is an EinSS in the original game G. □

3.2. *Applications of Theorem 3.1*

3.2.1. *Hotelling's model*

In Hotelling's model (1929), two sellers of a homogeneous product are located on a closed line of length l at respective distances a and b from the ends of this line ($a + b \le l, a \ge 0, b \ge 0$). Customers are evenly distributed with a unit density along the line and consume a unit of this product per unit of time. A customer will buy from the seller who quotes the lowest delivery price, namely the mill price plus transportation cost, which is assumed linear with respect to the distance. In the price-setting subgame, sellers choose their mill prices p_1 and p_2. The payoff function of seller 1 is

$$u_1(p_1, p_2) = \begin{cases} u_1^I = p_1 l, & p_1 < p_2 - d \\ u_1^{II} = p_1(a + \frac{d+p_2-p_1}{2}), & |p_1 - p_2| \le d \\ 0, & p_1 > p_2 + d \end{cases}, \quad \text{where } d = l - a - b \tag{1}$$

The payoff function of seller 2 is symmetric. These functions are two-peak. The best response functions of player 2 and player 1 are discontinuous respectively at the points $\hat{p}_1$ and $\hat{p}_2$, where there is a jump from the right peak to the left peak corresponding to undercutting price. Two cases are possible. The best response functions can intersect in the strategy space (p_1, p_2) at one point $(p_1^*, p_2^*) = (l + \frac{a-b}{3}, l + \frac{b-a}{3})$, at which the price-setting game admits a Nash equilibrium. This case occurs when $p_1^* \le \hat{p}_1$ and $p_2^* \le \hat{p}_2$. If on the contrary $p_1^* > \hat{p}_1$ or $p_2^* > \hat{p}_2$, then the jump to the undercutting price occurs before the two best response functions can intersect in the diagonal domain $|p_1 - p_2| \le d$. In this case there is no Nash equilibrium in the game. These conditions can be written as (d'Aspremont et al., 1979):

$$\left(l + \frac{a-b}{3}\right)^2 < \frac{4}{3}l(a+2b) \quad \text{or} \quad \left(l + \frac{b-a}{3}\right)^2 < \frac{4}{3}l(b+2a) \tag{2}$$

To apply Theorems 3.1, it is necessary to examine the structure of the sets of secure strategies of the two players in the price-setting subgame. In the area $p_i < p_{-i} - d$, where seller i "undercuts" its competitor and takes away the entire market, all strategies of i are insecure, because the competitor can reduce its price and capture back a market share. In the area $p_i > p_{-i} + d$ seller i gets zero payoff. Therefore, all secure profiles with positive payoffs for the two players belong to the diagonal domain $|p_1 - p_2| \le d$ of the strategy space. In this area there are two types of threats against a player i. If $u^{II}_{-i}(p_i, p_{-i}) < u^I_{-i}(p_i - d)$, the competitor $-i$ can profitably "undercut" the price of seller i and take away the entire market. In the contrary case, the best response of i's competitor is simply $BR_{-i}(p_i) = \operatorname{argmax}_{|p_i - p'_{-i}| \le d} u^{II}_{-i}(p_i, p'_{-i})$. But, if $p_{-i} > BR_{-i}(p_i)$, the competitor $-i$ can also profitably reduce the price and capture a market share from player i. Therefore, the set of strategy profiles in the diagonal domain, which are secure for player i is defined by the condition

$$\begin{cases} u^{II}_{-i}(p_i, p_{-i}) \ge u^I_{-i}(p_i - d) \\ p_{-i} \le BR_{-i}(p_i) \end{cases} \tag{3}$$

It remains to choose the box B so that for every (p_1, p_2) in B, the inverse $BR_1^{-1}(p_1)$ and $BR_2^{-1}(p_2)$ of the best response functions are defined and continuous; then, the strategy of player $-i$ in the profile $(p_i, BR_i^{-1}(p_i))$ is clearly secure. The best response functions $BR_1(p_2)$ and $BR_2(p_1)$, strictly increase up to the points of their discontinuity, respectively at profiles $(BR_1(\hat{p}_2), \hat{p}_2)$ and $(\hat{p}_1, BR_2(\hat{p}_1))$. Therefore, these points determine the size of B:

$$B = B_1 \times B_2 = [0, p_1^B] \times [0, p_2^B], \text{with } p_1^B = \min\{\hat{p}_1, BR_1(\hat{p}_2)\}, \ p_2^B = \min\{\hat{p}_2, BR_2(\hat{p}_1)\}.$$

Proposition 3.2. *Under condition (2) the price-setting Hotelling's subgame, with payoffs given by (1), satisfies the conditions of Theorem 3.1 in B and admits an EinSS in B.*

For locations outside condition (2), a Nash equilibrium (hence an EinSS) exists, which coincides with the solution found by Hotelling. One can show in addition that at each location pair there is a unique EinSS (Iskakov and Iskakov, 2012). The

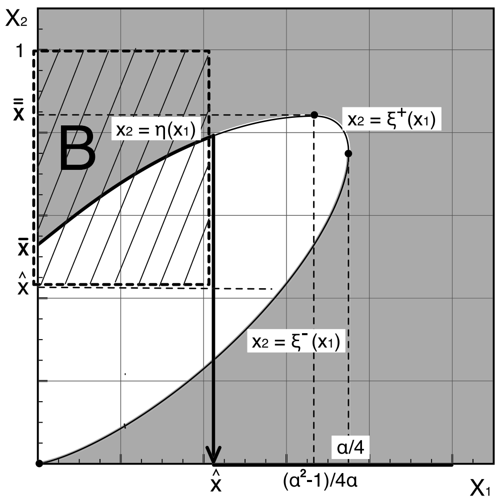

Fig. 1. Secure strategies (gray area) and BSR function of player 2.

existence and uniqueness of EinSS in the price-setting subgame allows for the correct solution (i.e. the *subgame perfect* EinSS) of the two-stage location-price Hotelling's game. In the first stage seller 1 chooses its location a and seller 2 its location b, anticipating the equilibrium prices $\left(p_1^*(a,b),\, p_2^*(a,b)\right)$ of the second stage subgame. As shown by Iskakov and Iskakov (2012), the Nash equilibrium in locations gives

$$a^* = 3l + b^* - 6\sqrt{lb^*},\ b^* \geq l/4, \text{ and } b^* = 3l + a^* - 6\sqrt{la^*},\ a^* \geq l/4 \tag{4}$$

There are no other EinSS equilibria in the first stage game. We see that the equilibrium locations lie on the boundary between the domain in which the Nash equilibrium in prices found by Hotelling exists and the domain in which it does not because of undercutting, i.e., the conditions (2) hold on this boundary as equalities. By symmetry, we obtain the (unique) equilibrium locations: $a^* = b^* = l/4$.

3.2.2. *Tullock contest of two players*

In the basic formulation of Tullock's rent-seeking contest, N players compete for a prize and each player exerts effort $x_i \in [0, 1]$ so as to increase his probability of winning $x_i / \sum_{j=1}^{N} x_j$ (Tullock, 1980). A more generalized form of the game has the expected profits of the players given by $x_i^\alpha / \sum_{j=1}^{N} x_j^\alpha - x_i,\ \alpha > 0$ (see, e.g., Skaperdas, 1996). Here we consider a contest of two players with payoff functions defined as:

$$u_1 = \frac{x_1^\alpha}{x_1^\alpha + x_2^\alpha} - x_1,\ u_2 = \frac{x_2^\alpha}{x_1^\alpha + x_2^\alpha} - x_2,\ \alpha > 0 \tag{5}$$

This game has a unique Nash equilibrium $(\alpha/4, \alpha/4)$ when $\alpha \leq 2$ and there is no Nash equilibrium when $\alpha > 2$. When $\alpha > 1$ the payoff functions u_i have a double peak in x_i. The left peak arises at $x^{1peak} = 0$ and the position of the right peak x^{2peak} is defined by

$$x^{2peak} = \xi^{-1}(x_{-i}) \equiv \begin{cases} \left(\xi^+\right)^{-1}(x_{-i}), & x_{-i} > \frac{\alpha}{4} \\ \left(\xi^-\right)^{-1}(x_{-i}), & x_{-i} \leq \frac{\alpha}{4} \end{cases}, \text{ where}$$

$$\xi^+(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2}\left(\alpha - 2x_i + \sqrt{\alpha^2 - 4\alpha x_i}\right)\right)^{1/\alpha}, \quad \max\left\{0, \frac{\alpha^2-1}{4\alpha}\right\} \leq x_i \leq \alpha/4 \tag{6}$$

$$\xi^-(x_i) \equiv \left(\frac{x_i^{\alpha-1}}{2}\left(\alpha - 2x_i - \sqrt{\alpha^2 - 4\alpha x_i}\right)\right)^{1/\alpha}, \quad 0 \leq x_i \leq \alpha/4$$

The player's strategy is secure in the contest if and only if the competitor cannot increase his payoff by increasing his effort. Following this criterion consider the set of secure strategies of player 2 when $\alpha > 2$. The payoff function $u_1(x_1, x_2)$ can be two-peak in x_1 depending on x_2. When $0 \leq x_2 < \bar{x} \equiv \frac{1}{\alpha}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$ the right peak is higher than the left one and player 1 has a threat at this profile against player 2 when $x_1 < x^{2peak} = \xi^{-1}(x_2)$ (see Fig. 1). When $\bar{x} < x_2 < \bar{\bar{x}} \equiv \frac{1}{4\alpha}(\alpha+1)^{\frac{\alpha+1}{\alpha}}(\alpha-1)^{\frac{\alpha-1}{\alpha}}$, the left peak is higher than the right one, and insecure strategies for player 2 lie in the interval $\tilde{x}_1 < x_1 < \xi^{-1}(x_2)$, where $\tilde{x}_1(x_2) : u_1(\tilde{x}_1, x_2) = u_1(\xi^{-1}(x_2), x_2), \tilde{x}_1 < \xi^{-1}(x_2)$. Finally, when $x_2 \geq \bar{\bar{x}}$, the function u_1 decreases monotonically in x_1, and all profiles are secure for player 2. Combining these three cases the set of profiles (x_1, x_2) that are secure for player 2 is shaded in gray in Fig. 1 and can be formally written as

$$\{x_1 \geq \alpha/4\} \cup \{x_1 < \alpha/4, x_2 \geq \eta(x_1) \text{ or } x_2 \leq \xi^-(x_1)\}, \tag{7}$$

$$\text{where } \eta(x_1) \equiv \begin{cases} \xi^+(x_1), & \frac{\alpha^2-1}{4\alpha} \leq x_1 \leq \frac{\alpha}{4} \\ x_2 : u_1(x_1, x_2) = u_1\left(\xi^{-1}(x_2), x_2\right), & x_1 \leq \frac{\alpha^2-1}{4\alpha} \end{cases} \tag{8}$$

As can be seen from Fig. 1 the set of secure strategies of player 2 at a given $x_1 < \alpha/4$ consists of two intervals separated by the right peak $x_2 = x^{2peak}$ of the function u_2. Therefore, the best secure response $BSR_2(x_1)$ of player 2 could be at either $x_2 = 0$, $x_2 = \xi^-(x_1)$, or $x_2 = \eta_2(x_1)$. The accurate estimation shows that when $\alpha > 2$ (see Fig. 1)

$$BSR_2(x_1) = \begin{cases} \eta_2(x_1), & x_1 \leq \hat{x} \\ 0, & x_1 \geq \hat{x} \end{cases}, \quad \text{where } \hat{x} : u_2(\hat{x}, \eta(\hat{x})) = 0$$

This implies that the connected subsets of secure strategies of player 2 for the BSA condition in Theorem 3.1 can be chosen for $x_1 < \hat{x}$ as $\widetilde{Q}_2(x_1) = \{\eta_2(x_1) \leq x_2 \leq 1\}$ and for $x_1 > \hat{x}$, as $\widetilde{Q}_2(x_1) = \{0 \leq x_2 \leq \delta\}$, with $\delta > 0$ small enough. When $x_1 = \hat{x}$, any BSA subset $\widetilde{Q}_2(x_1)$ must include the two points $x_2 = 0$ and $x_2 = \eta_2(\hat{x})$, and hence the corresponding set $M_{\hat{x}}$ from the Theorem 3.1 would be non-contractible. Thus, if we want to apply Theorem 3.1, the set B should not include strategy profiles at $x_1 = \hat{x}$ and $x_2 = \hat{x}$. However, the above analysis suggests that one can choose a strategy profile box B as (see Fig. 1):

$$B = B_1 \times B_2 = [0, \hat{x} - \epsilon] \times [\hat{x} + \epsilon, 1] \text{ with an arbitrary small } \epsilon > 0. \tag{9}$$

Proposition 3.3. *When $\alpha > 2$ the two-person Tullock contest (5) satisfies the conditions of Theorem 3.1 in B and admits an EinSS in B.*

A full solution of the Tullock contest for two players in secure strategies as well as some generalizations for many players were obtained in (Iskakov et al., 2014). In particular, when a contest admits no Nash equilibrium, the following proposition applies:

Proposition 3.4. *When $\alpha > 2$ the two-person Tullock contest (5) admits the following EinSS (which is unique up to a permutation of players):*

$$(0, x^*) \text{ or } (x^*, 0), \quad x^* = \frac{1}{\alpha}(\alpha - 1)^{\frac{\alpha-1}{\alpha}} \tag{10}$$

When there is no Nash equilibrium (i.e., a symmetric competition is unprofitable for players), the EinSS concept allows us to discover a different type of non-cooperative equilibrium in the Tullock contest, where one player exerts a high level of effort to keep the prize while the other player refuses to participate. In this situation the first player prefers to retain his secure monopolistic position and maintains a high level of effort to create an entry barrier to the contest for the competitor. The observed EinSS solution is unique up to a permutation of players. The efficiency of equilibrium in the contest is usually characterized by the rent dissipation, which is the ratio of the total effort of all players to the value of the prize. The higher the rent dissipation, the lower the efficiency of equilibrium. When a pure Nash equilibrium does not exist, the rent dissipation in mixed strategy equilibria is equal to one, so the rent is completely dissipated (Baye et al., 1993). However, in the discovered monopolistic EinSS the rent dissipation is significantly less. Thus, in this case the EinSS provides a more efficient solution than the Nash equilibrium in mixed strategies.

3.2.3. Bertrand–Edgeworth duopoly model

In this subsection we consider the Bertrand–Edgeworth model of price-setting duopolists, selling a homogeneous product, under capacity constraints and with identical marginal costs normalized to zero (Bertrand, 1883; Edgeworth, 1925). We assume a continuous strictly decreasing consumer demand function $D(p)$. Each firm $i = 1, 2$, has productive capacity S_i, and it is assumed that $D(0) \geq S_1 + S_2$. Firms compete in prices (p_1, p_2) and play non-cooperatively. The firm quoting the lower price serves the entire market up to its capacity and the residual demand is met by the other firm. Formally, we define the payoff function of player 1 as

$$u_1(p_1, p_2) = \begin{cases} p_1 \min\{S_1, D(p_1)\}, & p_1 < p_2 \\ p_1 \min\{S_1, \frac{S_1}{S_1+S_2} D(p_1)\}, & p_1 = p_2 \\ p_1 \min\{S_1, \frac{D(p_1)}{D(p_2)} \max\{0, D(p_2) - S_2\}\}, & p_1 > p_2 \end{cases} \tag{11}$$

The payoff function of player 2 is symmetric. It is well known since Edgeworth (1925) that this model may not possess a Nash equilibrium. We now show that, unless capacities are excessive (in some specific sense), there is an equilibrium in secure strategies. Let us denote by p^* the price such that demand equals capacity, i.e., $D(p^*) = S_1 + S_2$. We have:

Proposition 3.5. *Assume that the receipt function $pD(p)$ is strictly concave. If the Bertrand–Edgeworth game with payoff functions (11) satisfies the conditions:*

$$\begin{cases} \arg\max\limits_{p>0}\{p(D(p)-S_1)\} \le p^* \\ \arg\max\limits_{p>0}\{p(D(p)-S_2)\} \le p^* \end{cases}, \quad \text{where } D(p^*) = S_1 + S_2, \tag{12}$$

then it satisfies the conditions of Theorem 3.1 and admits an EinSS.

Proof. Let us take $B = [0, p^*] \times [0, p^*]$ as a strategy profile box. If $p \in B$, the payoff functions of each player, $u_1 = p_1 S_1$ and $u_2 = p_2 S_2$, are independent of the other player strategies. All strategy profiles in B are secure. Choose $\widetilde{Q}_1(p_2) = [0, p^*]$ and $\widetilde{Q}_2(p_1) = [0, p^*]$. Clearly, the graph of each function $\widetilde{Q}_i$, $\Gamma(\widetilde{Q}_i) = B$, is closed, the maximum payoff $\phi_i(p_{-i}) = p^* S_i$ is constant, and each set $M_{p_{-i}}$ consists of a unique point. It only remains to prove that G is a BSA-game in B, i.e., that player 1 with insecure strategy $p_1 > p^*$ always has a BSA in B when $p_2 \le p^*$. Observe that there exists a deviation $(p_1, p_2) \xrightarrow{2} (p_1, p^* + \varepsilon)$, with $\varepsilon > 0$ small enough so that player 2 strictly increases its payoff. Then, if $p^* D(p^*) > p_1 D(p_1)$, $v_1(p_1, p_2) \le u_1(p_1, p^* + \varepsilon) \le p_1 \frac{D(p_1)}{D(p^*+\varepsilon)}(D(p^*+\varepsilon) - S_2) < p^*(D(p^*) - S_2) = p^* S_1 = u_1(p^*, p_2)$, i.e. the deviation $(p_1, p_2) \xrightarrow{1} (p^*, p_2)$ provides for player 1 a BSA in B_1. Now suppose $p_1 D(p_1) \ge p^* D(p^*)$. Because of strict concavity of $pD(p)$ we have $(p_1 - \varepsilon) D(p_1 - \varepsilon) > p^* D(p^*)$, with $\varepsilon > 0$ arbitrarily small. Then there exists a deviation $(p_1, p_2) \xrightarrow{2} (p_1, p_1 - \varepsilon)$, at which player 2 strictly increases its payoff from $p_2 S_2$ to $u_2(p_1, p_1 - \varepsilon) > \min\{p^* D(p^*), p^* S_2\} \ge p_2 S_2$. From (12) we obtain $v_1(p_1, p_2) \le u_1(p_1, p_1 - \varepsilon) = \max\{0, p_1 \frac{D(p_1)}{D(p_1-\varepsilon)}(D(p_1 - \varepsilon) - S_2)\} < p^*(D(p^*) - S_2) = p^* S_1 = u_1(p^*, p_2)$, i.e., the deviation $(p_1, p_2) \xrightarrow{1} (p^*, p_2)$ provides for player 1 a BSA in B_1. □

Under strict concavity of the receipt function, it is known and easy to see that (p^*, p^*) is the unique Nash Equilibrium if $p^* \ge p_M = \arg\max\limits_{p>0}\{pD(p)\}$. In such a case, firm i, anticipating that its competitor chooses p^*, cannot do better than quoting price p^* and producing at capacity. For any larger price p_i, $p_i D(p_i) < p^* D(p^*)$, and its profit would only be $p_i \frac{D(p_i)}{D(p^*)}(D(p^*) - S_{-i}) < p^* S_i$. The following proposition goes further and provides a full characterization of EinSS in the Bertrand–Edgeworth game.

Proposition 3.6. *Let the receipt function $pD(p)$ be strictly concave and reach its maximum at p_M. Then, (p^*, p^*) is an EinSS (with $D(p^*) = S_1 + S_2$) if and only if*

$$\begin{cases} \arg\max\limits_{p>0}\{p(D(p)-S_1)\} \le p^* \\ \arg\max\limits_{p>0}\{p(D(p)-S_2)\} \le p^* \end{cases} \tag{13}$$

If $p^* \ge p_M$ it is a Nash equilibrium. There are no other EinSS in the game.

For the linear demand function $D(p) = 1 - p$ the conditions (13) take the form

$$S_1 + 2S_2 \le 1 \ \text{ and } \ S_2 + 2S_1 \le 1 \tag{14}$$

When $p^* \ge p_M$ (i.e. $1 - S_1 - S_2 \ge 1/2$), an EinSS coincides with the Nash equilibrium solution. When a Nash equilibrium does not exist ($p^* < p_M$) and (14) holds, there is a unique EinSS solution with equilibrium prices (p^*, p^*). Then p^* is less than the monopoly price (equal to $max\{1/2, 1 - S_1 - S_2\}$). The corresponding difference in price $S_1 + S_2 - \frac{1}{2}$ can be interpreted as an additional cost to maintain security when duopolists secure themselves against the threat of being undercut in industry competition. This example also illustrates the fact that, for an EinSS to exist when p^* is less than p_M, one firm should have a productive capacity that is not too large (less than 1/3 in the example).

4. Insurance market model of Rothschild, Stiglitz and Wilson

In this section we shall consider the insurance market model analyzed by Rothschild and Stiglitz (1976) and Wilson (1977), and show that it always has an equilibrium in secure strategies. The proof will not be an application of Theorem 3.1, but will follow the graphical procedure introduced by Rothschild and Stiglitz.

Two insurance companies sell insurance contracts to consumers, which fall into two classes: n_H high-risk consumers and n_L low-risk consumers. High-risk consumers have accidents with probability p_H, and low-risk consumers have accidents with probability $p_L < p_H$. All consumers have the same strictly positive initial endowment $w = (w_1, w_2) \in R^2$ representing their income in the two states of nature: w_2, if having an accident and w_1, if not. Preferences of all consumers are represented by the same strictly concave utility function u. Each insurance contract is a vector $c = (c_1, c_2) \in R^2$, where c_1 is the insurance premium and c_2 is the accident benefit net of premium. The endowment of a consumer with an insurance contract becomes $(w_1 - c_1, w_2 + c_2)$. Consumers of a given risk class j buy at most one insurance contract c (if they prefer it to initial endowment w) which maximizes their expected utility:

$$V_j(c) = p_j u(w_2 + c_2) + (1 - p_j)u(w_1 - c_1), \text{ where } j = H \text{ or } L \tag{15}$$

Each insurance company offers a pair of contracts (c^H, c^L), where without loss of generality one can assume that high risk consumers find c^H at least as desirable as c^L, and low risk consumers find c^L at least as desirable as c^H. The expected profit of the company from the contract $c^j = (c_1^j, c_2^j)$ sold to a customer of class j,

$$\pi_j(c^j) = -p_j c_2^j + (1 - p_j)c_1^j, \text{ where } j = H \text{ or } L \tag{16}$$

Suppose that company 1 offers contracts $(c^H(1), c^L(1))$ and company 2 $(c^H(2), c^L(2))$. Then, the expected profit of company 1 is

$$U_1 = \sum_{j=H,L} \begin{cases} n_j \pi_j(c^j(1)), & \text{if } V_j(c^j(1)) > V_j(c^j(2)) \\ \frac{1}{2} n_j \pi_j(c^j(1)), & \text{if } V_j(c^j(1)) = V_j(c^j(2)) \\ 0, & \text{otherwise} \end{cases} \tag{17}$$

Expected profit of company 2 is defined symmetrically.

The detailed interpretation and investigation of this model can be found in Rothschild and Stiglitz (1976) and Wilson (1977). In particular, it was shown that, if a pure strategy equilibrium exists, both companies must offer the same contract pair $c^* = (c^{*H}, c^{*L})$ satisfying $w_2 + c_2^{*H} = w_1 - c_1^{*H}$ (i.e., high risks are perfectly insured), $\pi_H(c^{*H}) = \pi_L(c^{*L}) = 0$ (i.e., customers of each risk class generate zero expected profits for companies), and $V_H(c^{*H}) = V_H(c^{*L})$ (i.e., high-risk customers are indifferent between the low-risk contract and their own). Following Dasgupta and Maskin (Dasgupta and Maskin, 1986b) we will call contract pair c^* a "Rothschild–Stiglitz–Wilson" or RSW contract pair. However, if there is a sufficiently high proportion of low-risk customers one company can deviate from c^* and earn positive profit. It can offer a "pooling" contract c^{**} that both high and low-risk customers prefer to c^*. It was shown that there is no Nash equilibrium in this case. We can show however that contract pair c^* is still an EinSS.

Proposition 4.1. *An RSW contract pair c^* is always an Equilibrium in Secure Strategies in the insurance market game.*

Proof. We will follow the graphical procedure introduced in Rothschild and Stiglitz (1976). In Fig. 2 the horizontal and vertical axis represent income of customers in the states of no accident and accident, respectively. The point E with coordinates $w = (w_1, w_2)$ is the uninsured state of customer. Purchasing the insurance contract $c = (c_1, c_2)$ moves the individual from E to the point $(w_1 - c_1, w_2 + c_2)$. The set of insurance contracts for low-risk customers that break even lies on the line EL. The set of contracts for high-risk customers lies on the line EH, respectively. If a company offers a "pooling" contract that is the same for both groups (such that $c^H = c^L$), it should, in case of equilibrium, lie on the market odds line EF. The pair of contracts (c^{*H}, c^{*L}) in Fig. 2 represents the RSW solution of the insurance market game. The indifference curves through c^{*H} and c^{*L} for high-risk and low-risk customers U_H and U_L are shown by broken lines. Suppose that c^* is not a Nash equilibrium. It is still a secure profile since any change in the insurance policies of one company will not bring losses to the other company. Its payoffs will remain zero. Suppose one company profitably deviates by offering a new insurance policy. It is either a "pooling" insurance contract γ, which must lie above the low-risk indifference curve U_L through c^{*L} and below the marketodds line EF (see Fig. 2 on the left), or a new separating contract (c'^H, c'^L) at which c'^H must lie above high-risk line EH and c'^L must lie below low-risk line EL (see Fig. 2 on the right). In both cases the second company as a response can offer a pooling contract γ' such that all low-risk customers would choose γ' and all high-risk customers would stay with the first company making it lose money (see Fig. 2). Therefore, neither γ nor (c'^H, c'^L) are secure deviations. No company can make a secure deviation from (c^{*H}, c^{*L}). Therefore, it is an EinSS. □

Conclusion

The non-cooperative equilibrium that we have analyzed, the EinSS, is meant to extend the Nash equilibrium concept to solve games where the Nash equilibrium does not exist and where it is not unreasonable to introduce, as a behavioral assumption, that players are cautious, namely that players are looking for secure positions and consider only secure deviations. In that respect, this concept abstracts and unifies *ad hoc* solutions already formulated in various applied economic games that have been discussed extensively in the literature. It usefully complements mixed-strategy Nash equilibria that are not explicit and are difficult to interpret in these games. Like the Nash equilibrium, the EinSS is a static concept, and the basic requirement of excluding at equilibrium competitive deviations remains. Yet it also appeals to dynamic intuitions, tolerating at equilibrium the possibility of a non-competitive deviation that would be blocked by some counter-deviation punishing the deviator. This is in line with the "objection-counterobjection" logic first introduced in cooperative games.

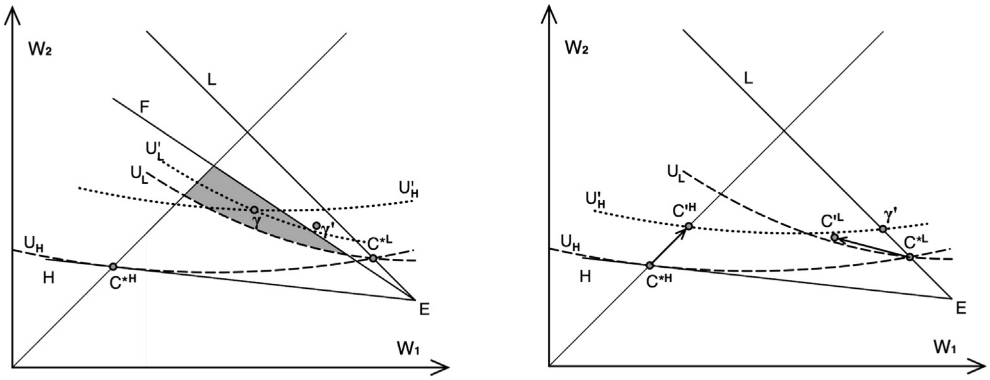

Fig. 2. Deviations from RSW solution (c^{*H}, c^{*L}) are not secure. Both "pooling" deviation γ (on the left) and separating deviation (c'^{H}, c'^{L}) (on the right) pose a threat γ' to receive negative payoff.

In the first application, the Hotelling price-setting game, there is no Nash equilibrium when sellers are too close to each other, but there is a unique EinSS for all location pairs; the subgame perfect EinSS solution selects one location pair. This is the location pair that minimizes differentiation on the domain where the Nash equilibrium in prices exists. It also minimizes transportation costs. In the second example, the symmetric two-player contest, a pure-strategy Nash–Cournot equilibrium does not exist when the success function parameter is greater than two, but there is an EinSS (unique up to permutations of players) providing a more efficient solution than a Nash equilibrium in mixed strategies. In the no-Nash-equilibrium cases (i.e., symmetric competition being unprofitable), one player exerts a high level of effort while the other player refuses to participate. In the third example, the Bertrand–Edgeworth duopoly with capacity constraints, we have shown that, in many cases, the Nash equilibrium does not exist but there is a unique EinSS with equilibrium prices lower than the monopoly price. The corresponding difference in prices can be interpreted as an additional cost supported by the firms to protect themselves from the threat of price undercutting. Existence of an EinSS in these three applications can be derived from a general existence theorem for discontinuous games (the payoff functions or the best response functions being discontinuous). Finally, we consider the Rothschild–Stiglitz–Wilson (RSW) insurance game, in which at a Nash equilibrium both companies must offer the same RSW-contract pair. However, if there is a sufficiently high proportion of low-risk customers, a single pooling contract will be preferred by every customer and will therefore upset the RSW-contract as a Nash equilibrium. But, an RSW-contract always remains an EinSS.

These four applications are only examples, but they illustrate well the kind of determinate solutions cautious players may reach in some classes of games. To confirm this analysis, it would be interesting to enlarge this set of applications and, as required, the conditions for existence of an EinSS.

Acknowledgments

We are especially thankful to F. Aleskerov and D. Novikov for regular discussions on the subject during seminars at Higher School of Economics and at V.A. Trapeznikov Institute of Control Sciences RAS. We wish to thank seminar leaders V. Burkov, V. Danilov, V. Polterovich and J.-F. Thisse. We wish to thank A. Chkhartishvili, M. Gubko, S. Izmalkov, V. Korepanov, N. Korgin, S. Kokovin, N. Kukushkin, S. Mishin, P. Pavlov, A. Savvateev, K. Sonin, A. Vasin, A. Zakharov and E. Zhelobodko for the fruitful discussions. Our work was supported by the research project No. 14-01-00131-a of the Russian Foundation for Basic Research.

Appendix A. Omitted proofs

Proof of Proposition 3.2. All secure strategies in the diagonal domain defined by (3) can be taken as the sets $\widetilde{Q}_i(p_{-i})$ in the BSA condition. These sets can be explicitly found as:

$$\begin{aligned}\widetilde{Q}_1(p_2) &= \left[\max\{p_2 - d, 2p_2 - 2b - d\}, \min\left\{p_2 + d, \frac{p_2(2b + d - p_2) + 2dl}{2l - p_2}\right\}\right] \cap B_1 \\ \widetilde{Q}_2(p_1) &= \left[\max\{p_1 - d, 2p_1 - 2a - d\}, \min\left\{p_1 + d, \frac{p_1(2a + d - p_1) + 2dl}{2l - p_1}\right\}\right] \cap B_2\end{aligned} \tag{A.1}$$

Clearly, the graphs $\Gamma(\widetilde{Q}_i)$ of functions $\widetilde{Q}_i(p_{-i})$ are closed sets. Payoff functions of players are continuous on these graphs. Maximum payoffs of players $\phi_i(p_{-i})$ on the sets $\widetilde{Q}_i(p_{-i})$ are continuous because the payoff functions are one-peak in the diagonal domain, and the boundaries of segments $\widetilde{Q}_i(p_{-i})$ change continuously with p_{-i}. Finally, each set $M_{p_{-i}}$ consists of a unique point and contractible. Let us prove that player 1, with insecure strategy p_1, always has a BSA in B_1 when $p_2 \in B_2$. Notice that the best response functions $BR_{-i}(p_i)$ in the chosen B are defined and continuously increase. If $p_1 > p_2 + d$, then any deviation to secure strategy in B_1 with a positive payoff provides BSA for player 1. Otherwise, it is provided by the deviation $p_1 \xrightarrow{1} BR_2^{-1}(p_2)$. If $|p_1 - p_2| \le d$, it follows from the strict quasiconcavity of $u_1(p)$ in p_1 and the fact that the strategy p_1 is insecure only if $p_2 > BR_2(p_1)$. Then we have $p_1 < BR_2^{-1}(p_2) \le BR_1(p_2)$ and $u_1(p_1, p_2) < u_1(p_1, BR_2^{-1}(p_2))$. If $p_1 < p_2 - d$, the BSA condition follows from the following estimate $0 < v_1(p_1, p_2) \le u_1(p_1, 0) \le |p_1(a + \frac{d-p_1}{2})| < (p_2 - d)(a + d) = u_1^{II}(p_2 - d, p_2) < u_1^{II}(BR_2^{-1}(p_2), p_2)$. Therefore, the game G is a BSA-game in B. By Theorem 3.1, G admits an EinSS in B. □

Proof of Proposition 3.3. As sets $\widetilde{Q}_i(x_{-i})$ in B for the BSA condition, we choose

$$\widetilde{Q}_2(x_1) = \{\eta_2(x_1) \le x_2 \le 1\}, \quad \widetilde{Q}_1(x_2) = \{0 \le x_1 \le \delta\}, \tag{A.2}$$

where $\delta > 0$ is sufficiently small so that $u_1(x_1)$ always decreases on the interval $\widetilde{Q}_1$. The graphs $\Gamma\left(\widetilde{Q}_i(x_{-i})\right)$ of the functions $\widetilde{Q}_i(x_{-i})$ are closed sets. The payoff functions of players are continuous on these graphs. The maximum payoffs $\phi_i(x_{-i})$ on the sets $\widetilde{Q}_i(x_{-i})$ are continuous, and the boundaries of segments $\widetilde{Q}_i(x_{-i})$ change continuously with x_{-i}. Finally, each set $M_{x_{-i}}$ in Theorem 3.1 consists of a unique point and contractible. By Theorem 3.1 G admits an EinSS in B, if we only prove the BSA condition for the chosen sets (A.2).

Let us prove the BSA condition for the set $\widetilde{Q}_2(x_1), x_1 < \hat{x}$. In the domain $D_1 = \{0 < x_1 < \hat{x},\ \ 0 \le x_2 \le \xi^-(x_1)\}$ with secure strategies of player 2, it is always profitable for player 2 to deviate $x_2 \xrightarrow{2} \eta(x_1)$. Since $u_2(x_1, x_2)$ is quasiconvex in D_1 with respect to x_2, it suffices to show that $\Delta u_2 \equiv u_2(x_1, \eta(x_1)) - u_2(x_1, \xi^-(x_1)) > 0$. Introduce notations y and $\tilde{y}$ such that $x_1 = \frac{\alpha y}{(1+y)^2}$ and $\tilde{x}_1 \equiv \xi^{-1}(\eta(x_1)) = \frac{\alpha \tilde{y}}{(1+\tilde{y})^2}$. When $x_1 \le \frac{\alpha^2-1}{4\alpha}$, we obtain $\Delta u_2 = f(\tilde{y}) - f(y)$, where the function $f(t) \equiv \frac{t}{t+1} + \frac{\alpha t(1-t^{1/\alpha})}{(t+1)^2}$ is quasiconcave at $0 \le t \le \frac{\alpha+1}{\alpha-1}$; $f(y) < \min\{f\left(\frac{1}{\alpha-1}\right), f\left(\frac{\alpha+1}{\alpha-1}\right)\}$, $\tilde{y} \in \left[\frac{1}{\alpha-1}, \frac{\alpha+1}{\alpha-1}\right]$ and hence $\Delta u_2 > 0$. When $x_1 \ge \frac{\alpha^2-1}{4\alpha}$, we have $\Delta u_2 = x_1\left((y^{1/\alpha} - y^{-1/\alpha}) - \frac{1}{\alpha}(y - y^{-1})\right)$, which is positive at $0 < y < 1$ and $\alpha > 2$. In the domain $D_2 = \{0 < x_1 < \hat{x},\ \ \xi^-(x_1) < x_2' < \eta(x_1)\}$ there is always a threat of player 1 against player 2 to deviate $x_1 \xrightarrow{1} x_1' = \xi^{-1}(x_2')$, and we prove that $\Delta u_2 = u_2(x_1, \eta(x_1)) - u_2(x_1', x_2') > 0$. When $x_1 \le \frac{\alpha^2-1}{4\alpha}$, we introduce y', such that $x_1' = \frac{\alpha y'}{(1+y')^2}$. If $1 \le y' < \tilde{y}$, the required inequality follows from $u_2(x_1', x_2') < u_2(\tilde{x}_1, \eta(x_1)) < u_2(x_1, \eta(x_1))$. If $\frac{1}{\alpha-1} \le y' \le 1$, it follows from $u_2(x_1', x_2') < 0 \le u_2(x_1, \eta(x_1))$. If $y < y' < \frac{1}{\alpha-1}$, let us introduce $U_2(t) \equiv \frac{t}{t+1} - \frac{\alpha t \cdot t^{1/\alpha}}{(t+1)^2}$ and $\hat{y} = \operatorname{argmax}_{t\in[0,1]} U_2(t)$. Then, one can verify that $\Delta u_2 \ge f(\tilde{y}) + \frac{\alpha \hat{y}}{(1+\hat{y})^2} - \frac{\alpha y}{(1+y)^2} - f(\hat{y}) > f(\tilde{y}) - f(\hat{y}) > 0$. The last inequality follows from the estimates $f(\hat{y}) < \min\left\{f\left(\frac{1}{\alpha-1}\right), f\left(\frac{\alpha+1}{\alpha-1}\right)\right\}$, $\tilde{y} \in [\frac{1}{\alpha-1}, \frac{\alpha+1}{\alpha-1}]$, and quasiconcavity of $f(t)$. When $x_1 \ge \frac{\alpha^2-1}{4\alpha}$, $U_2(t)$ is quasiconcave on the interval $\left[\frac{\alpha-1}{\alpha+1}, \frac{\alpha+1}{\alpha-1}\right]$ and $\frac{\alpha-1}{\alpha+1} \le y < y' < y^{-1} \le \frac{\alpha+1}{\alpha-1}$, it implies $U_2(y') < \max\{U_2(y), U_2(y^{-1})\} = U_2(y^{-1})$ $(y < 1)$ and $\Delta u_2 = u_2(x_1, \xi^+(x_1)) - u_2(x_1', x_2') = U_2(y^{-1}) - U_2(y') > 0$.

Let us now prove that the BSA condition holds for the set $\widetilde{Q}_1(x_2), x_2 > \hat{x}$. Consider an arbitrary $x_1' > \delta$. In strategy profiles where $u_1(x_1', x_2) < 0$ the BSA condition is met. Consider the domain $D_3 = \{(x_1', x_2) |\ x_1' > \delta,\ \ x_2 > \hat{x},\ \ u_1(x_1', x_2) \ge 0\}$. At any strategy profile in D_3 there is always a threat of player 2 against player 1 to deviate profitably $x_2 \xrightarrow{2} x_2' = \xi^{-1}(x_1')$ and $u_1(x_1', x_2') < u_1(0, x_2) = 0$. The proof repeats the proof for the domain D_2 with the replacement of indices. □

Proof of Proposition 3.6. One can immediately verify that (p_1, p_2) is a secure profile with positive payoffs in the game (11) if and only if it lies in the set $M = \{(p_1, p_2) : 0 < p_i \le p^*,\ \ i = 1, 2\}$. There is only one BS-profile (p^*, p^*) with positive payoffs in the set M (otherwise one player could securely slightly increase his price). Since any EinSS is a BS-profile, there are no other EinSS in the game except this profile. Let us now consider profile (p^*, p^*) and prove the conditions (13). Suppose that the conditions (13) do not hold, for example $p^* < \hat{p}(S_2) \equiv \arg\max_{p>0}\{p(D(p) - S_2)\}$. Then one can easily check that player 1 can deviate $p_1^* \to \hat{p}$ profitably and securely. Therefore profile (p^*, p^*) is not an EinSS. Hence follows the necessity of (13). Let us now assume that (13) holds (i.e., $\hat{p}(S_1) \le p^*$ and $\hat{p}(S_2) \le p^*$). Consider an arbitrary deviation $p^* \xrightarrow{1} p_1$. If $p_1 < p^*$, it is not profitable. If $p_1 > p^*$, it is profitable and only if $p^* D(p^*) < p_1 D(p_1)$. Then there is the retaliatory threat of player 2 to deviate from profile (p_1, p^*) to a profile $(p_1, p_1 - 0)$, with $p_1 - 0$ arbitrarily close to p_1. The payoff of player 1 in this profile is arbitrarily close to $u_1(p_1, p_1 - 0) = p_1(D(p_1) - S_2)$. Since $p(D(p) - S_2)$ is strictly decreasing at $p \ge \hat{p}(S_2)$ and $p_1 > p^* \ge \hat{p}(S_2)$, then $u_1(p^*, p^*) = p^*(D(p^*) - S_2) > p_1(D(p_1) - S_2) = u_1(p_1, p_1 - 0)$ and $p^* \xrightarrow{1} p_1$ is not a secure deviation. Player 2 is considered symmetrically. No player can make secure deviation in (p^*, p^*). By definition it is an EinSS. The sufficiency of (13) is proven. □

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