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Detecting changes in cross-sectional dependence in multivariate time series

KOJADINOVIC, I., ROHMER, T. and J. SEGERS

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Ivan Kojadinovic, Tom Rohmer
Laboratoire de mathématiques et applications, UMR CNRS 5142
Université de Pau et des Pays de l'Adour
B.P. 1155, 64013 Pau Cedex, France
{ivan.kojadinovic,tom.rohmer}@univ-pau.fr

Johan Segers
Institut de statistique, biostatistique et sciences actuarielles
Université catholique de Louvain
Voie du Roman Pays 20, B-1348 Louvain-la-Neuve, Belgium
johan.segers@uclouvain.be

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Abstract

Classical and more recent tests for detecting distributional changes in multivariate time series often lack power against alternatives that involve changes in the cross-sectional dependence structure. To be able to detect such changes better, a test is introduced based on a recently studied variant of the sequential empirical copula process. In contrast to earlier attempts, ranks are computed with respect to relevant subsamples, with beneficial consequences for the sensitivity of the test. For the computation of p-values we propose a multiplier resampling scheme that takes the serial dependence into account. The large-sample theory for the test statistic and the resampling scheme is developed. The finite-sample performance of the procedure is assessed by Monte Carlo simulations. Two case studies involving time series of financial returns are presented as well.

1 Introduction

Given a sequence $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ of d-dimensional observations, change-point detection aims at testing

$$H_0 : \exists\, F \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have c.d.f. } F \tag{1.1}$$

against alternatives involving the nonconstancy of the c.d.f. Under H_0 and the assumption that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ have continuous marginal c.d.f.s $F_1, \ldots, F_d$, we have from the work of

Sklar (1959) that the common multivariate c.d.f. F can be written in a unique way as

$$F(\boldsymbol{x}) = C\{F_1(x_1), \ldots, F_d(x_d)\}, \qquad \boldsymbol{x} \in \mathbb{R}^d,$$

where the function $C : [0,1]^d \to [0,1]$ is a *copula* and can be regarded as capturing the dependence between the components of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. It follows that H_0 can be rewritten as $H_{0,m} \cap H_{0,c}$, where

$$H_{0,m} : \exists\, F_1, \ldots, F_d \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have marginal c.d.f.s } F_1, \ldots, F_d, \tag{1.2}$$

$$H_{0,c} : \exists\, C \text{ such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_n \text{ have copula } C. \tag{1.3}$$

Classical tests for H_0 are based on sequential empirical processes; see Csörgő and Horváth (1997, Section 2.6) and Bai (1994). For moderate sample sizes, however, such tests appear to have little power against alternative hypotheses that leave the margins unchanged but that involve a change in the copula, i.e., when $H_{0,m} \cap (\neg H_{0,c})$ holds. Empirical evidence of the latter fact can be found in Holmes et al. (2013, Section 4). For that reason, nonparametric tests for change-point detection particularly sensitive to changes in the dependence structure are of practical interest.

Several tests designed to capture changes in cross-sectional dependence structure were proposed in the literature. Tests based on Kendall's tau were investigated by Gombay and Horváth (1999), Gombay and Horváth (2002), Quessy et al. (2013) and Dehling et al. (2013). Although these have good power when the copula changes in such a way that Kendall's tau changes as well, they are obviously useless when the copula changes but Kendall's tau does not change or only very little. Tests based on sequential empirical copula processes were considered in Rémillard (2010), Bücher and Ruppert (2013), van Kampen and Wied (2013) and Wied et al. (2013). However, the power of such tests is often disappointing; see Section 5 for some numerical evidence.

It is our aim to construct a new test for H_0 that is more powerful than its predecessors against alternatives that involve a change in the copula. The test is based on sequential empirical copula processes as well, but the crucial difference lies in the computation of the ranks. Whereas in Rémillard (2010) and subsequent papers, ranks are always computed with respect to the full sample, we propose to compute the ranks with respect to the relevant subsamples; see Section 2 for details. The intuition is that in this way, the copulas of those subsamples are estimated more accurately, so that differences between copulas of disjoint subsamples are detected more quickly. The phenomenon is akin to the one observed in Genest and Segers (2010) that the empirical copula, which is based on pseudo-observations, is often a better estimator of a copula than the empirical distribution function based on observations from the copula itself. For another illustration in the context of tail dependence functions, see Bücher (2013a).

The paper is organized as follows. The test statistic is presented in Section 2, and its asymptotic distribution under the null hypothesis is found in Section 3. Next, Section 4 contains a detailed description of the multiplier resampling scheme and its asymptotic validity under the null hypothesis. The results of a large-scale Monte Carlo simulation study are reported in Section 5, and two brief case studies are given in Section 6. Section 7 concludes. Proofs and details regarding the simulation study are deferred to the Appendices.

In the rest of the paper, the arrow '$\rightsquigarrow$' denotes weak convergence in the sense of Definition 1.3.3 in van der Vaart and Wellner (2000). Given a set T, let $\ell^\infty(T)$ denote the space of all bounded real-valued functions on T equipped with the uniform metric.

2 Test statistic

We now describe our test statistic and highlight the difference with the one in Rémillard (2010) and Bücher and Ruppert (2013). Let $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ be random vectors. For integers $1 \leq k \leq l \leq n$, let $C_{k:l}$ be the empirical copula of the sample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$. Specifically,

$$C_{k:l}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{k:l} \leq \boldsymbol{u}), \tag{2.1}$$

for $\boldsymbol{u} \in [0,1]^d$, where

$$\hat{\boldsymbol{U}}_i^{k:l} = \frac{1}{l-k+1}(R_{i1}^{k:l}, \ldots, R_{id}^{k:l}), \qquad i \in \{k, \ldots, l\}, \tag{2.2}$$

with $R_{ij}^{k:l}$ the rank of X_{ij} among $X_{kj}, \ldots, X_{lj}$ for all $j \in \{1, \ldots, d\}$. An important point is that the ranks are computed within the subsample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ and not within the whole sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$.

Write $\Delta = \{(s,t) \in [0,1]^2 : s \leq t\}$. Let $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$ for $(s,t) \in \Delta$. Our test statistic is based on the difference process, $\mathbb{D}_n$, defined by

$$\mathbb{D}_n(s, \boldsymbol{u}) = \sqrt{n}\, \lambda_n(0,s)\, \lambda_n(s,1) \left\{ C_{1:\lfloor ns \rfloor}(\boldsymbol{u}) - C_{\lfloor ns \rfloor + 1:n}(\boldsymbol{u}) \right\} \tag{2.3}$$

for $(s, \boldsymbol{u}) \in [0,1]^{d+1}$. For every $s \in [0,1]$, it gives a weighted difference between the empirical copulas at $\boldsymbol{u}$ of the first $\lfloor ns \rfloor$ and the last $n - \lfloor ns \rfloor$ points of the sample. Large absolute differences point in the direction of a change in the copula.

To aggregate over $\boldsymbol{u}$, we consider the Cramér–von Mises statistic

$$S_{n,k} = \int_{[0,1]^d} \left\{ \mathbb{D}_n\left(k/n, \boldsymbol{u}\right) \right\}^2 \mathrm{d}C_{1:n}(\boldsymbol{u}), \qquad k \in \{1, \ldots, n-1\}.$$

The test statistic for detecting changes in cross-sectional dependence is then

$$S_n = \max_{1 \leq k \leq n-1} S_{n,k} = \sup_{s \in [0,1]} \int_{[0,1]^d} \{\mathbb{D}_n(s, \boldsymbol{u})\}^2 \, \mathrm{d}C_{1:n}(\boldsymbol{u}). \tag{2.4}$$

Other aggregating functions can be thought of as well, leading for instance to Kolmogorov–Smirnov and Kuiper statistics. In numerical experiments, the resulting tests were found to be less powerful than the one based on the Cramér–von Mises statistic and hence are not considered further in this paper.

The null hypothesis of a constant distribution is rejected when S_n is large. Indeed, under the alternative hypothesis, the difference process $\mathbb{D}_n$ will be of the order $\sqrt{n}$. The p-values are determined by the null distribution of S_n, whose large-sample limit is derived

in Section 3. To estimate the p-values from the data, a multiplier bootstrap method is proposed in Section 4.

Finally, if H_0 is rejected, there could be one or several abrupt or smooth changes in the joint distribution. Moreover, the change(s) could concern one or more marginal distributions, the copula, or both. In the case where there is just a single (abrupt) change-point $k^* \in \{1, \ldots, n-1\}$, one can for instance estimate it by

$$k_n^\star = \underset{1 \le k \le n-1}{\arg\max}\, S_{n,k}. \tag{2.5}$$

We do not pursue the issue of single or multiple change-point estimation nor the diagnosis of the nature of the change-point.

Our test statistic S_n differs from the one considered in Rémillard (2010, Section 5.2) and Bücher and Ruppert (2013, Section 3.2) in the way the copulas of the subsamples $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ are estimated. Rather than the empirical copula $C_{k:l}$, these authors propose to use

$$C_{k:l,n}(\boldsymbol{u}) = \frac{1}{l-k+1} \sum_{i=k}^{l} \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \le \boldsymbol{u}), \qquad \boldsymbol{u} \in [0,1]^d. \tag{2.6}$$

In comparison with $C_{k:l}$ in (2.1), the ranks for the subsample $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ are computed relative to the complete sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$. The estimators $C_{k:l,n}$ yield the difference process

$$\mathbb{D}_n^R(s, \boldsymbol{u}) = \sqrt{n}\, \lambda_n(0, s)\, \lambda_n(s, 1)\, \{C_{1:\lfloor ns \rfloor, n}(\boldsymbol{u}) - C_{\lfloor ns \rfloor + 1:n, n}(\boldsymbol{u})\} \tag{2.7}$$

for $(s, \boldsymbol{u}) \in [0,1]^{d+1}$. The process $\mathbb{D}_n^R$ is to be compared with the process $\mathbb{D}_n$ in (2.3). The difference lies in the use of $C_{k:l,n}$ rather than $C_{k:l}$. From the process $\mathbb{D}_n^R$, one obtains the test statistic

$$S_n^R = \sup_{s \in [0,1]} \int_{[0,1]^d} \left\{ \mathbb{D}_n^R(s, \boldsymbol{u}) \right\}^2 \mathrm{d}C_{1:n}(\boldsymbol{u}), \tag{2.8}$$

which is the analogue of S_n in (2.4).

In the Monte Carlo simulation experiments (Section 5), we will see that S_n is usually more powerful than S_n^R for detecting changes in the cross-sectional copula. Intuitively, the reason is that the empirical copula $C_{k:l}$ in (2.1) is often a better copula estimator than $C_{k:l,n}$ in (2.6). Note that $C_{k:l}$ is not only the empirical copula of $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$, it is also equal to the empirical copula of $\hat{\boldsymbol{U}}_k^{1:n}, \ldots, \hat{\boldsymbol{U}}_l^{1:n}$, of which $C_{k:l,n}$ is the empirical distribution function.

In Genest and Segers (2010), situations are identified where the empirical copula of an independent random sample drawn from a given bivariate copula has a lower asymptotic variance than the empirical distribution function of that sample. Of course, the situation here is different from the one in the cited paper: multivariate rather than bivariate, serial dependence rather than independence. But still, we suspect the same mechanisms to be active.

3 Large-sample distribution

The asymptotic distribution under H_0 of our test statistic S_n in (2.4) can be obtained by writing it as a functional of the two-sided sequential empirical copula process studied in Bücher and Kojadinovic (2013). Let $\boldsymbol{X}_1, \boldsymbol{X}_2, \ldots$ be a strictly stationary d-variate time series with stationary c.d.f. F having continuous margins $F_1, \ldots, F_d$ and copula C. Recall $C_{k:l}$ in (2.1) and $\hat{\boldsymbol{U}}_i^{k:l}$ in (2.2). The two-sided sequential empirical copula process, $\mathbb{C}_n$, is defined by

$$\mathbb{C}_n(s,t,\boldsymbol{u}) = \sqrt{n}\,\lambda_n(s,t)\,\{C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})\} \tag{3.1}$$

$$= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \left\{ \mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns\rfloor+1:\lfloor nt\rfloor} \le \boldsymbol{u}) - C(\boldsymbol{u}) \right\}, \tag{3.2}$$

for $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$. The link of $\mathbb{C}_n$ to our test statistic S_n in (2.4) is that, under H_0, the difference process $\mathbb{D}_n$ in (2.3) can be written as

$$\mathbb{D}_n(s,\boldsymbol{u}) = \lambda_n(s,1)\,\mathbb{C}_n(0,s,\boldsymbol{u}) - \lambda_n(0,s)\,\mathbb{C}_n(s,1,\boldsymbol{u}), \tag{3.3}$$

for $(s,\boldsymbol{u}) \in [0,1]^{d+1}$.

In order to study $\mathbb{C}_n$, we need to be able to easily go back and forth between (normalized) ranks and empirical quantile functions. To this end, ties must not occur. In the case of serial independence, it is sufficient to assume that the marginal distributions are continuous. In the case of serial dependence, however, continuity of the marginal distributions is *not* sufficient to guarantee the absence of ties: think for instance of a moving maximum process. Therefore, we introduce the following condition.

Condition 3.1. *With probability one, there are no ties in each of the d component series $X_{1j}, X_{2j}, \ldots$, where $j \in \{1, \ldots, d\}$.*

For a sequence of d-dimensional random vectors $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$, the σ-field generated by $(Y_i)_{a\le i\le b}$, $a, b \in \mathbb{Z} \cup \{-\infty, +\infty\}$, is denoted by $\mathcal{F}_a^b$. The strong mixing coefficients corresponding to the sequence $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$ are defined by

$$\alpha_r = \sup_{p\in\mathbb{Z}} \sup_{A\in\mathcal{F}_{-\infty}^p, B\in\mathcal{F}_{p+r}^{+\infty}} |P(A\cap B) - P(A)P(B)|.$$

for positive integer r. The sequence $(\boldsymbol{Y}_i)_{i\in\mathbb{Z}}$ is said to be *strongly mixing* if $\alpha_r \to 0$ as $r \to \infty$.

The weak limit of the two-sided empirical copula process $\mathbb{C}_n$ defined in (3.2) under strong mixing was established in Bücher and Kojadinovic (2013) under the following smoothness condition on C proposed in Segers (2012):

Condition 3.2. *For any $j \in \{1, \ldots, d\}$, the partial derivatives $\dot{C}_j = \partial C/\partial u_j$ exist and are continuous on $V_j = \{\boldsymbol{u} \in [0,1]^d : u_j \in (0,1)\}$.*

As we continue, for any $j \in \{1, \ldots, d\}$, we define $\dot{C}_j$ to be zero on the set $\{\boldsymbol{u} \in [0,1]^d : u_j \in \{0,1\}\}$ (see also Segers, 2012; Bücher and Volgushev, 2013). Also, for any

$j \in \{1, \ldots, d\}$ and any $\boldsymbol{u} \in [0,1]^d$, $\boldsymbol{u}^{(j)}$ is the vector of $[0,1]^d$ defined by $u_i^{(j)} = u_j$ if $i = j$ and 1 otherwise.

The weak convergence of the process $\mathbb{C}_n$ defined in (3.2) actually follows from that of the process

$$\mathbb{B}_n(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \{\mathbf{1}(\boldsymbol{U}_i \le \boldsymbol{u}) - C(\boldsymbol{u})\}, \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d, \tag{3.4}$$

where $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ is the unobservable sample obtained from $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ by the probability integral transforms $U_{ij} = F_j(X_{ij})$, $i \in \{1, \ldots, n\}$, $j \in \{1, \ldots, d\}$, and with the convention that $\mathbb{B}_n(s,t,\cdot) = 0$ if $\lfloor nt \rfloor - \lfloor ns \rfloor = 0$.

If $\boldsymbol{U}_1, \ldots, \boldsymbol{U}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{U}_i)_{i \in \mathbb{Z}}$ whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$ with $a > 1$, we have from Bücher (2013b) that $\mathbb{B}_n(0,\cdot,\cdot)$ converges weakly in $\ell^\infty([0,1]^{d+1})$ to a tight centered Gaussian process $\mathbb{Z}_C$ with covariance function

$$\operatorname{cov}\{\mathbb{Z}_C(s,\boldsymbol{u}), \mathbb{Z}_C(t,\boldsymbol{v})\} = (s \wedge t) \sum_{k \in \mathbb{Z}} \operatorname{cov}\{\mathbf{1}(\boldsymbol{U}_0 \le \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_k \le \boldsymbol{v})\}. \tag{3.5}$$

As a consequence of the continuous mapping theorem, $\mathbb{B}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty(\Delta \times [0,1]^d)$, where

$$\mathbb{B}_C(s,t,\boldsymbol{u}) = \mathbb{Z}_C(t,\boldsymbol{u}) - \mathbb{Z}_C(s,\boldsymbol{u}), \qquad (s,t,\boldsymbol{u}) \in \Delta \times [0,1]^{d+1}. \tag{3.6}$$

The following result is a corollary of Theorem 1 of Bücher and Kojadinovic (2013).

Proposition 3.3. *Assume that $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ is drawn from a strictly stationary sequence $(\boldsymbol{X}_i)_{i \in \mathbb{Z}}$ with continuous margins and whose strong mixing coefficients satisfy $\alpha_r = O(r^{-a})$, $a > 1$. Then, provided Conditions 3.1 and 3.2 hold,*

$$\sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} \left| \mathbb{C}_n(s,t,\boldsymbol{u}) - \tilde{\mathbb{C}}_n(s,t,\boldsymbol{u}) \right| \xrightarrow{\mathrm{P}} 0,$$

where

$$\tilde{\mathbb{C}}_n(s,t,\boldsymbol{u}) = \mathbb{B}_n(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\, \mathbb{B}_n(s,t,\boldsymbol{u}^{(j)}). \tag{3.7}$$

Consequently, $\mathbb{C}_n \rightsquigarrow \mathbb{C}_C$ in $\ell^\infty(\Delta \times [0,1]^d)$, where, for $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$,

$$\mathbb{C}_C(s,t,\boldsymbol{u}) = \mathbb{B}_C(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\, \mathbb{B}_C(s,t,\boldsymbol{u}^{(j)}). \tag{3.8}$$

In view of (3.3), the weak limit of $\mathbb{D}_n$ under H_0 is a mere corollary of Proposition 3.3 and the continuous mapping theorem.

Corollary 3.4. *Under the conditions of Proposition 3.3, $\mathbb{D}_n \rightsquigarrow \mathbb{D}_C$ in $\ell^\infty([0,1]^{d+1})$, where, for any $(s,\boldsymbol{u}) \in [0,1]^{d+1}$,*

$$\mathbb{D}_C(s,\boldsymbol{u}) = \mathbb{C}_C(0,s,\boldsymbol{u}) - s\, \mathbb{C}_C(0,1,\boldsymbol{u}), \tag{3.9}$$

with $\mathbb{C}_C$ *defined in* (3.8)*. As a consequence,*

$$S_n \rightsquigarrow S = \sup_{s\in[0,1]} \int_{[0,1]^d} \{\mathbb{D}_C(s,\boldsymbol{u})\}^2 \,\mathrm{d}C(\boldsymbol{u}). \tag{3.10}$$

The covariance function of $\mathbb{D}_C$ can be expressed in terms of the one of $\mathbb{C}_C$ by

$$\mathrm{cov}\{\mathbb{D}_C(s,\boldsymbol{u}), \mathbb{D}_C(t,\boldsymbol{v})\} = \{\min(s,t) - st\}\ \mathrm{cov}\{\mathbb{C}_C(0,1,\boldsymbol{u}), \mathbb{C}_C(0,1,\boldsymbol{v})\}.$$

4 Resampling

In order to compute p-values for S_n based on (3.10), we propose to use resampling methods. Tracing back the definition of S_n via $\mathbb{D}_n$ to $\mathbb{C}_n$ in (3.1) and using the approximation via $\tilde{\mathbb{C}}_n$ in (3.7), we find that it suffices to construct a resampling scheme for $\mathbb{B}_n$ defined in (3.4) and to estimate the first-order partial derivatives, $\dot{C}_j$, of C.

4.1 Multiplier sequences

Because of the unwieldy nature of the weak limit $\mathbb{C}_C$ given in (3.8), the computation of approximate p-values for tests involving $\mathbb{C}_n$ needs typically to be based on some resampling scheme. In a nonsequential setting based on independent observations, Bücher and Dette (2010) compared the finite-sample behavior of the various resampling techniques proposed in the literature and concluded that the *multiplier bootstrap* of Rémillard and Scaillet (2009) has, overall, the best finite-sample properties. This technique was revisited theoretically by Segers (2012) who showed its asymptotic validity under Condition 3.2. A sequential generalization of the latter result will be stated later in this section. In the case of independent observations, the multiplier bootstrap is based on *i.i.d. multiplier sequences*. We say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is an *i.i.d. multiplier sequence* if:

(M0) $(\xi_{i,n})_{i\in\mathbb{Z}}$ is i.i.d., independent of $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$, with distribution not changing with n, having mean 0, variance 1, and being such that $\int_0^\infty \{\mathrm{P}(|\xi_{0,n}| > x)\}^{1/2} \mathrm{d}x < \infty$.

Starting from the seminal work of Bühlmann (1993, Section 3.3), Bücher and Kojadinovic (2013) have studied a *dependent* multiplier bootstrap for $\mathbb{C}_n$ which extends the multiplier bootstrap of Rémillard and Scaillet (2009) to the sequential and strongly mixing setting. The key idea in Bühlmann (1993) is to replace i.i.d. multipliers by suitably serially dependent multipliers that will capture the serial dependence in the data. In the rest of the paper, we say that a sequence of random variables $(\xi_{i,n})_{i\in\mathbb{Z}}$ is a *dependent multiplier sequence* if:

(M1) The sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is strictly stationary with $\mathrm{E}(\xi_{0,n}) = 0$, $\mathrm{E}(\xi_{0,n}^2) = 1$ and $\sup_{n\geq 1} \mathrm{E}(|\xi_{0,n}|^\nu) < \infty$ for all $\nu \geq 1$, and is independent of the available sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$.

(M2) There exists a sequence $\ell_n \to \infty$ of strictly positive constants such that $\ell_n = o(n)$ and the sequence $(\xi_{i,n})_{i\in\mathbb{Z}}$ is ℓ_n-dependent, i.e., $\xi_{i,n}$ is independent of $\xi_{i+h,n}$ for all $h > \ell_n$ and $i \in \mathbb{N}$.

(M3) There exists a function $\varphi : \mathbb{R} \to [0,1]$, symmetric around 0, continuous at 0, satisfying $\varphi(0) = 1$ and $\varphi(x) = 0$ for all $|x| > 1$ such that $\mathrm{E}(\xi_{0,n}\xi_{h,n}) = \varphi(h/\ell_n)$ for all $h \in \mathbb{Z}$.

Ways to generate dependent multiplier sequences are mentioned in Section 5 and Appendix B.

4.2 Computing p-values via resampling

Let M be a large integer and let $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}}, \ldots, (\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ be M independent copies of the same multiplier sequence. We will define two multiplier resampling schemes for the process $\mathbb{B}_n$ in (3.4). These will lead to two resampling schemes for the test statistic S_n in (2.4), on the basis of which approximate p-values can be computed.

Recall that $C_{k:l}$ in (2.1) is the empirical copula of $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$, which is the empirical distribution of the vectors of rescaled ranks $\hat{\boldsymbol{U}}_i^{k:l}$ in (2.2). For any $m \in \{1, \ldots, M\}$ and $(s, t, \boldsymbol{u}) \in \Delta \times [0,1]^d$, let

$$\hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \{ \mathbf{1}(\hat{\boldsymbol{U}}_i^{1:n} \le \boldsymbol{u}) - C_{1:n}(\boldsymbol{u}) \}, \tag{4.1}$$

and

$$\begin{aligned} \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \{ \mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns \rfloor + 1 : \lfloor nt \rfloor} \le \boldsymbol{u}) - C_{\lfloor ns \rfloor + 1 : \lfloor nt \rfloor}(\boldsymbol{u}) \} \\ &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} (\xi_{i,n}^{(m)} - \bar{\xi}_{\lfloor ns \rfloor + 1 : \lfloor nt \rfloor}^{(m)}) \mathbf{1}(\hat{\boldsymbol{U}}_i^{\lfloor ns \rfloor + 1 : \lfloor nt \rfloor} \le \boldsymbol{u}), \end{aligned} \tag{4.2}$$

where $\bar{\xi}_{k:l}^{(m)}$ is the arithmetic mean of $\xi_{i,n}^{(m)}$ for $i \in \{k, \ldots, l\}$. By convention, the sums are zero if $\lfloor ns \rfloor = \lfloor nt \rfloor$. Note that the ranks are computed relative to the complete sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ for $\hat{\mathbb{B}}_n^{(m)}(s,t,\cdot)$, whereas they are computed relative to the subsample $\boldsymbol{X}_{\lfloor ns \rfloor + 1}, \ldots, \boldsymbol{X}_{\lfloor nt \rfloor}$ for $\check{\mathbb{B}}_n^{(m)}(s,t,\cdot)$.

In order to get to resampling versions of $\tilde{\mathbb{C}}_n$ in (3.7), we need estimators of the first-order partial derivatives of C. A simple estimator based on $\boldsymbol{X}_k, \ldots, \boldsymbol{X}_l$ consists of finite differencing at a bandwidth of $h \equiv h(k,l) = \min\{(l-k+1)^{-1/2}, 1/2\}$. Varying slightly upon the definition in Rémillard and Scaillet (2009) and following Kojadinovic et al. (2011a, Section 3), we put

$$\dot{C}_{j,k:l}(\boldsymbol{u}) = \frac{C_{k:l}(\boldsymbol{u} + h\boldsymbol{e}_j) - C_{k:l}(\boldsymbol{u} - h\boldsymbol{e}_j)}{\min(u_j + h, 1) - \max(u_j - h, 0)} \tag{4.3}$$

for $\boldsymbol{u} \in [0,1]^d$, where $\boldsymbol{e}_j$ is the jth canonical unit vector in $\mathbb{R}^d$. Note that if $h \leq u_j \leq 1-h$, the denominator is just $2h$. The more general form of the denominator corrects for boundary effects (u_j close to 0 or 1) and ensures that the estimator is uniformly bounded (see Kojadinovic et al., 2011a, proof of Proposition 2).

The resampling versions $\hat{\mathbb{B}}_n^{(m)}$ and $\check{\mathbb{B}}_n^{(m)}$ of $\mathbb{B}_n$ then lead to the following resampling versions for $\tilde{\mathbb{C}}_n$: for $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$,

$$\hat{\mathbb{C}}_n^{(m)}(s,t,\boldsymbol{u}) = \hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_{j,1:n}(\boldsymbol{u})\, \hat{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}^{(j)}),$$

$$\check{\mathbb{C}}_n^{(m)}(s,t,\boldsymbol{u}) = \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_{j,\lfloor ns \rfloor + 1:\lfloor nt \rfloor}(\boldsymbol{u})\, \check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}^{(j)}). \tag{4.4}$$

Recall that $\lambda_n(s,t) = (\lfloor nt \rfloor - \lfloor ns \rfloor)/n$. The difference process $\mathbb{D}_n$ is to be resampled by one of the following two methods:

$$\begin{aligned}\hat{\mathbb{D}}_n^{(m)}(s,\boldsymbol{u}) &= \lambda_n(s,1)\, \hat{\mathbb{C}}_n^{(m)}(0,s,\boldsymbol{u}) - \lambda_n(0,s)\, \hat{\mathbb{C}}_n^{(m)}(s,1,\boldsymbol{u}) \\ &= \hat{\mathbb{C}}_n^{(m)}(0,s,\boldsymbol{u}) - \lambda_n(0,s)\, \hat{\mathbb{C}}_n^{(m)}(0,1,\boldsymbol{u}),\end{aligned}$$

$$\check{\mathbb{D}}_n^{(m)}(s,\boldsymbol{u}) = \lambda_n(s,1)\, \check{\mathbb{C}}_n^{(m)}(0,s,\boldsymbol{u}) - \lambda_n(0,s)\, \check{\mathbb{C}}_n^{(m)}(s,1,\boldsymbol{u}).$$

For resampling the test statistic, one has the choice between

$$\hat{S}_n^{(m)} = \sup_{s \in [0,1]} \int_{[0,1]^d} \{\hat{\mathbb{D}}_n^{(m)}(s,\boldsymbol{u})\}^2 \, \mathrm{d}C_{1:n}(\boldsymbol{u}), \tag{4.5}$$

$$\check{S}_n^{(m)} = \sup_{s \in [0,1]} \int_{[0,1]^d} \{\check{\mathbb{D}}_n^{(m)}(s,\boldsymbol{u})\}^2 \, \mathrm{d}C_{1:n}(\boldsymbol{u}). \tag{4.6}$$

Finally, approximate p-values of the observed test statistic S_n can be computed via either

$$\frac{1}{M} \sum_{m=1}^{M} \mathbf{1}\left(\hat{S}_n^{(m)} \geq S_n\right) \qquad \text{or} \qquad \frac{1}{M} \sum_{m=1}^{M} \mathbf{1}\left(\check{S}_n^{(m)} \geq S_n\right).$$

The null hypothesis is rejected if the estimated p-value is smaller than the desired significance level.

By comparison, note that for the test statistic S_n^R in (2.8) based on the process $\mathbb{D}_n^R$ in (2.7), an approximate p-value can be computed using the multiplier processes

$$\mathbb{D}_n^{R,(m)}(s,\boldsymbol{u}) = \hat{\mathbb{B}}_n^{(m)}(0,s,\boldsymbol{u}) - \lambda_n(0,s)\, \hat{\mathbb{B}}_n^{(m)}(0,1,\boldsymbol{u}), \tag{4.7}$$

where $\hat{\mathbb{B}}_n^{(m)}$ is defined in (4.1); see also Rémillard (2010, Section 5.2) and Bücher and Ruppert (2013, Section 3.2).

4.3 Asymptotic validity of the resampling scheme

We establish the asymptotic validity of the multiplier resampling schemes described above under the null hypothesis. First, we need to impose conditions on the data generating process $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ and the multiplier sequences $(\xi_{i,n}^{(m)})_{i \in \mathbb{Z}}$ for $m \in \{1, \ldots, M\}$.

Condition 4.1. *One of the following two conditions holds:*

(i) *The random vectors* $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ *are i.i.d. and* $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}}, \ldots, (\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ *are independent multiplier sequences satisfying (M0).*

(ii) *The random vectors* $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ *are drawn from a strictly stationary sequence* $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ *whose strong mixing coefficients satisfy* $\alpha_r = O(r^{-a})$ *for some* $a > 3+3d/2$*, and* $(\xi_{i,n}^{(1)})_{i\in\mathbb{Z}}, \ldots, (\xi_{i,n}^{(M)})_{i\in\mathbb{Z}}$ *are independent copies of a dependent multiplier sequence satisfying (M1)–(M3) with* $\ell_n = O(n^{1/2-\gamma})$ *for some* $0 < \gamma < 1/2$*.*

In both cases, the stationary distribution of $\boldsymbol{X}_i$ *has continuous margins and a copula* C *satisfying Condition 3.2.*

If the random vectors $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are i.i.d., they can also be considered to be drawn from a strongly mixing, strictly stationary sequence. Hence, for the multiplier sequences $(\xi_{i,n}^{(m)})_{i\in\mathbb{Z}}$, one could either assume (M0) or (M1)–(M3): both should work. However, if the process is i.i.d., Monte Carlo simulations show that working with i.i.d. multiplier sequences improves the finite-sample performance. In other words, in finite samples, if there is no evidence against serial independence, it appears more sensible to work under (M0).

We can now formulate the asymptotic distribution of the multiplier resampling schemes under the null hypothesis of a constant distribution. We provide two propositions, one for the resampling scheme based on $\hat{\mathbb{B}}_n^{(m)}$ in (4.1) and another one for the scheme based on $\check{\mathbb{B}}_n^{(m)}$ in (4.2).

Proposition 4.2. *If Conditions 3.1 and 4.1 hold, then*

$$\left(\mathbb{C}_n, \hat{\mathbb{C}}_n^{(1)}, \ldots, \hat{\mathbb{C}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \ldots, \mathbb{C}_C^{(M)}\right)$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$*, where* $\mathbb{C}_C$ *is defined in* (3.8)*, and* $\mathbb{C}_C^{(1)}, \ldots, \mathbb{C}_C^{(M)}$ *are independent copies of* $\mathbb{C}_C$*. As a consequence, also*

$$\left(\mathbb{D}_n, \hat{\mathbb{D}}_n^{(1)}, \ldots, \hat{\mathbb{D}}_n^{(M)}\right) \rightsquigarrow \left(\mathbb{D}_C, \mathbb{D}_C^{(1)}, \ldots, \mathbb{D}_C^{(M)}\right),$$

in $\{\ell^\infty([0,1]^{(d+1)})\}^{M+1}$*, where* $\mathbb{D}_C$ *is defined in* (3.9) *and* $\mathbb{D}_C^{(1)}, \ldots, \mathbb{D}_C^{(M)}$ *are independent copies of* $\mathbb{D}_C$*. Finally,*

$$\left(S_n, \hat{S}_n^{(1)}, \ldots, \hat{S}_n^{(M)}\right) \rightsquigarrow \left(S, S^{(1)}, \ldots, S^{(M)}\right)$$

where S *is defined in* (3.10) *and* $S^{(1)}, \ldots, S^{(M)}$ *are independent copies of* S*.*

Under Condition 4.1(i), the above result can be easily proved by starting from Theorem 1 of Holmes et al. (2013) and adapting the arguments used in Segers (2012, proof of Proposition 3.2). Under Condition 4.1(ii), the result was obtained in Bücher and Kojadinovic (2013, Proposition 4.2).

Proposition 4.3. *If Conditions 3.1 and 4.1 hold, then the conclusions of Proposition 4.2 also hold with* $\hat{\mathbb{C}}_n^{(m)}$ *replaced by* $\check{\mathbb{C}}_n^{(m)}$*,* $\hat{\mathbb{D}}_n^{(m)}$ *replaced by* $\check{\mathbb{D}}_n^{(m)}$*, and* $\hat{S}_n^{(m)}$ *replaced by* $\check{S}_n^{(m)}$*.*

The proof of Proposition 4.3 is somewhat involved and is given in detail in Appendix A.

5 Simulation study

Large-scale Monte Carlo experiments were carried out in order to study the finite-sample performance of the derived tests for detecting changes in cross-sectional dependence. The main questions addressed by the study are the following:

(i) How well do the tests hold their size under the null hypothesis H_0 in (1.1) of no change?

(ii) What is the power of the tests against the alternative $H_{1,c}$ of a single change in cross-sectional dependence at constant margins? Specifically, the alternative hypothesis is $H_{1,c} \cap H_{0,m}$ with $H_{0,m}$ in (1.2) and $H_{1,c}$ defined by

$$\begin{aligned} H_{1,c} : &\exists \text{ distinct } C_1 \text{ and } C_2, \text{ and } k^\star \in \{1, \ldots, n-1\} \text{ such that} \\ &\boldsymbol{X}_1, \ldots, \boldsymbol{X}_{k^\star} \text{ have copula } C_1 \text{ and } \boldsymbol{X}_{k^\star+1}, \ldots, \boldsymbol{X}_n \text{ have copula } C_2. \end{aligned} \tag{5.1}$$

(iii) What happens if the change in distribution is only due to a change in the margins, the copula remaining constant? Specifically, the alternative hypothesis is $H_{1,m} \cap H_{0,c}$ with $H_{0,c}$ given in (1.3) and $H_{1,m}$ defined by

$$\begin{aligned} H_{1,m} : &\exists \text{ distinct } F_{1,1}, F_{1,2} \text{ as well as } F_2, \ldots, F_d \text{ and } k_1^\star \in \{1, \ldots, n-1\} \\ &\text{such that } \boldsymbol{X}_1, \ldots, \boldsymbol{X}_{k_1^\star} \text{ have marginal c.d.f.s } F_{1,1}, F_2, \ldots, F_d \\ &\text{and } \boldsymbol{X}_{k_1^\star+1}, \ldots, \boldsymbol{X}_n \text{ have marginal c.d.f.s } F_{1,2}, F_2, \ldots, F_d. \end{aligned} \tag{5.2}$$

In addition to the three questions above, many others can be formulated, involving other alternative hypotheses for instance. The problem is complex and there are countless ways of combining factors in the experimental design. In our study, the settings were chosen to represent a wide and hopefully representative variety of situations, in function of the three questions above. The main factors of our experiments are summarized below:

- Test statistics:
  - Our statistic S_n in (2.4) with p-values computed via resampling using $\hat{S}_n^{(m)}$ or $\check{S}_n^{(m)}$ in (4.5) and (4.6), respectively. As we continue, we shall simply talk about the test based on $\hat{S}_n$ or $\check{S}_n$, respectively, to distinguish between these two situations.
  - The statistic S_n^R in (2.8) of Bücher and Ruppert (2013), with p-values computed according to the resampling method for $\mathbb{D}_n^R$ in (4.7).
- Sample size: $n \in \{50, 100, 200\}$.
- Number of samples per setting: 1 000.
- Cross-sectional dimension: $d \in \{2, 3\}$.
- Significance level: $\alpha = 5\%$.
- Serial dependence: The data were generated either as being serially independent or via two time-series models, an autoregressive process and a multivariate version of the exponential autoregressive model considered in Auestad and Tjøstheim (1990) and Paparoditis and Politis (2001, Section 3.3). The dependent multiplier sequences were generated using the "moving average approach" proposed initially in Bühlmann (1993) and revisited in some detail in Bücher and Kojadinovic (2013,

Section 6.1). The value of the bandwidth parameter ℓ_n defined in Condition (M2) was chosen automatically using the approach described in Bücher and Kojadinovic (2013, Section 5). See Appendix C for details.

- Margins: since all tests and resampling schemes are rank-based, margins have been taken to be uniform on $(0, 1)$, except for the scenario $H_{1,m}$ in (5.2) of changing margins.
- Copulas: Clayton, Gumbel–Hougaard, Normal, Frank, with positive or negative (insofar possible) association, as well as asymmetric versions obtained via Khoudraji's device (Khoudraji, 1995; Genest et al., 1998; Liebscher, 2008).
- Alternative hypotheses involving a single change-point occurring at time $k^\star = \lfloor nt \rfloor$ with $t \in \{0.1, 0.25, 0.5, 0.75\}$:
  - $H_{0,m} \cap H_{1,c}$ with a change of the parameter within a copula family.
  - $H_{0,m} \cap H_{1,c}$ with a change of the copula family at constant Kendall's tau.
  - $H_{1,m} \cap H_{0,c}$, i.e., a change of one of the *margins* rather than of the copula.
  - For the serially dependent case, a change in the copula of the innovations, leading to a gradual change of the copula of the marginal distributions of the observables.

The experiments were carried out in the R statistical system (R Development Core Team, 2013) using the `copula` package (Hofert et al., 2013). Code will be made available in an R package whose tentative name is `npcp`.

For the sake of brevity, only a representative subset of the results is reported here. Specifically, the following tables are provided in Appendix C:

- Size of the tests under the null hypothesis H_0:
  - Table 1: Percentage of false rejections when data are serially independent.
  - Table 2: Percentage of false rejections when data are serially dependent.
- Power of the tests against specific alternatives:
  - Table 3: Power against $H_{0,m} \cap H_{1,c}$ involving a change of the copula parameter within a copula family and at serial independence.
  - Table 4: Power against $H_{0,m} \cap H_{1,c}$ involving a change of copula family at a constant value of Kendall's tau and at serial independence.
  - Table 5: Power against $H_{1,m} \cap H_{0,c}$ involving a change in one of the margins and at serial independence..
  - Table 6: Power against $\neg H_0$ when data are serially dependent and the change occurs in the copula of the innovations.

Besides findings of a more anecdotical nature, the following conclusions may be drawn from the results:

- All tests hold their level reasonably well in the case of serial independence (Table 1), with minor fluctuations depending on sample size, test statistic, copula parameter and copula family.
- In case of serial dependence, the test based on $\hat{S}_n$ is too conservative for the sample sizes under consideration (Table 2). In line with this observation, the test based on $\check{S}_n$ appears to be more powerful than the one based on $\hat{S}_n$ (Table 6).

- For alternative hypotheses involving a change in the *copula*, the tests based on $\hat{S}_n$ and $\check{S}_n$ have a higher power than S_n^R (Tables 3 and 4). When the copula changes in such a way that Kendall's tau remains constant, the power of S_n^R is especially low. With respect to that last setting, note that distinguishing copulas on the basis of low amounts of data is known to be difficult (Genest et al., 2009; Kojadinovic et al., 2011b). The fact that the change-point is unknown makes the problem even harder.
- For alternative hypotheses involving a change in one of the *margins*, it is the test statistic S_n^R that is substantially more powerful than S_n (Table 5). The weak power of S_n can be explained by the fact that it is designed for detecting changes in the copula. Another tentative reading of the results is that the test based on S_n, regarded as a procedure for testing $H_{0,c}$, is relatively robust against small changes in one margin. In contrast, the test based on S_n^R behaves as an all-purpose test for the hypothesis of a constant distribution rather than as a test for a constant copula.

6 Case studies

As an illustration, we first applied the test based on $\check{S}_n$ to bivariate financial data consisting of daily logreturns computed from the DAX and the Standard and Poor 500 indexes. Following Dehling et al. (2013, Section 7), attention was restricted to the years 2006–2009. The corresponding closing quotes were obtained from `http://quote.yahoo.com` using the `get.hist.quote` function of the `tseries` R package (Trapletti and Hornik, 2013), which resulted in $n = 993$ bivariate logreturns. Dependent multiplier sequences were generated as explained in Appendix B. An approximate p-value of 0.039 was obtained, providing some evidence against H_0. The conclusion is in line with the results reported in Dehling et al. (2013). Of course, as discussed earlier, it is only under the assumption that $H_{0,m}$ in (1.2) holds that it would be fully justified to decide to reject $H_{0,c}$ in (5.1) on the basis of the previous approximate p-value. The value of the change-point estimator $k_n^\star$ in (2.5) is 529, corresponding to February 22nd, 2008.

As a second illustration, we followed again Dehling et al. (2013) and considered $n = 504$ bivariate logreturns computed from closing daily quotes of the Dow Jones Industrial Average and the Nasdaq Composite for the years 1987 and 1988. The former quotes, not being available on `http://quote.yahoo.com` anymore, were taken from the R package `QRM` (Pfaff and McNeil, 2013). This two-year period is of interest because it contains October 19th, 1987, known as "black Monday" (see Dehling et al., 2013, Figure 4). An approximate p-value of 0.63 was obtained. Hence, despite the extreme events that occured during the period under consideration, the test based on $\check{S}_n$ detects no evidence against H_0 in the data, which is in line with the results reported in Dehling et al. (2013).

7 Conclusion

We have demonstrated that the sensitivity of rank-based tests for the null hypothesis of a constant distribution against changes in cross-sectional dependence can be improved if

ranks are computed with respect to relevant subsamples. In this way, the test we propose achieves in many cases a higher power than the one proposed in Bücher and Ruppert (2013). The limit distribution of the test statistic under the null hypothesis is unwieldy, but approximate p-values can still be computed via a multiplier resampling scheme. To deal with potential serial dependence, we make use of dependent multiplier sequences, an idea going back to Bühlmann (1993) and revisited in Bücher and Kojadinovic (2013).

Here are some potential avenues for further research:

– Once the null hypothesis has been rejected, the nature of the nonstationary needs to be investigated further: is there a single change-point or is there more than one? Or maybe the change is gradual rather than sudden? And does the change concern the margins or the copula?
– Can one detect a change in the copula without the hypothesis that the margins are constant?
– The procedure is computationally intensive because the ranks have to be recomputed for every $k \in \{1, \ldots, n-1\}$. Efficient algorithms for reutilising calculations from one value of k to the next one might speed up the computations.

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A Proof of Proposition 4.3

We shall only prove the result in the case of strongly mixing observations, that is, when Condition 4.1(ii) is assumed. The proof is similar but simpler when Condition 4.1(i) is assumed instead.

It is sufficient to show the statement involving $\check{\mathbb{C}}_n^{(m)}$. The statements for $\check{\mathbb{D}}_n^{(m)}$ and $\check{S}_n^{(m)}$ then follow from the continuous mapping theorem.

For any $m \in \{1, \ldots, M\}$ and $(s, t, \boldsymbol{u}) \in \Delta \times [0,1]^d$, put

$$\mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}) = \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\boldsymbol{U}_i \le \boldsymbol{u}) - C(\boldsymbol{u})\}, \tag{A.1}$$

$$\mathbb{C}_n^{(m)}(s,t,\boldsymbol{u}) = \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}) - \sum_{j=1}^{d} \dot{C}_j(\boldsymbol{u})\, \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}^{(j)}).$$

[Recall that $\boldsymbol{u}^{(j)} = (1, \ldots, 1, u_j, 1, \ldots, 1) \in [0,1]^d$, with u_j appearing at the j-th coordinate.] From Theorem 3.1 in Bücher and Kojadinovic (2013), the continuous mapping theorem and Proposition 3.3, we find

$$\big(\mathbb{B}_n, \mathbb{B}_n^{(1)}, \ldots, \mathbb{B}_n^{(M)}\big) \rightsquigarrow \big(\mathbb{B}_C, \mathbb{B}_C^{(1)}, \ldots, \mathbb{B}_C^{(M)}\big)$$

and thus

$$\big(\mathbb{C}_n, \mathbb{C}_n^{(1)}, \ldots, \mathbb{C}_n^{(M)}\big) \rightsquigarrow \big(\mathbb{C}_C, \mathbb{C}_C^{(1)}, \ldots, \mathbb{C}_C^{(M)}\big),$$

in $\{\ell^\infty(\Delta \times [0,1]^d)\}^{M+1}$, where $\mathbb{B}_C^{(1)}, \ldots, \mathbb{B}_C^{(M)}$ are independent copies of $\mathbb{B}_C$ in (3.6). It is therefore sufficient to show that

$$\sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} \big|(\check{\mathbb{C}}_n^{(m)} - \mathbb{C}_n^{(m)})(s,t,\boldsymbol{u})\big| \xrightarrow{\mathrm{P}} 0 \tag{A.2}$$

for every $m \in \{1, \ldots, M\}$. Below, we will show the following two assertions: first,

$$\sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} \big|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u})\big| \xrightarrow{\mathrm{P}} 0, \tag{A.3}$$

and second, for every $\delta \in (0, 1/2)$ and every $\varepsilon \in (0,1)$,

$$\sup_{\substack{\boldsymbol{u} \in [0,1]^d \\ \delta \le u_j \le 1-\delta}} \sup_{\substack{(s,t) \in [0,1]^2 \\ t-s \ge \varepsilon}} \big|\dot{C}_{j,\lfloor ns \rfloor + 1 : \lfloor nt \rfloor}(\boldsymbol{u}) - \dot{C}_j(\boldsymbol{u})\big| \xrightarrow{\mathrm{P}} 0. \tag{A.4}$$

In view of the structure of $\check{\mathbb{C}}_n^{(m)}$ in (4.4), the assertions (A.3) and (A.4) imply (A.2), as we show next. Clearly,

$$\begin{aligned}
&\big|(\check{\mathbb{C}}_n^{(m)} - \mathbb{C}_n^{(m)})(s,t,\boldsymbol{u})\big| \\
&\qquad \le \big|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u})\big| + \sum_{j=1}^{d} \big|\dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u})\big| \, \big|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u}^{(j)})\big| \\
&\qquad\qquad + \sum_{j=1}^{d} \big|\dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) - \dot{C}_j(\boldsymbol{u})\big| \, \big|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}^{(j)})\big|. \quad \text{(A.5)}
\end{aligned}$$

Taking suprema over $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$, the first and the second term on the right-hand side of (A.5) converge to zero in probability because of assertion (A.3) and uniform boundedness of $\dot{C}_{j,k:l}$ (see Kojadinovic et al., 2011a, proof of Proposition 2). The third term on the right-hand side of (A.5) converges to zero in probability because of assertion (A.4) and the fact that the weak limit of the process $(s,t,\boldsymbol{u}) \mapsto \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u}^{(j)})$ in $\ell^\infty(\Delta \times [0,1]^d)$ is the process $(s,t,\boldsymbol{u}) \mapsto \mathbb{B}_C(s,t,\boldsymbol{u}^{(j)})$, which has continuous trajectories and which vanishes as soon as $s = t$ or $u_j \in \{0,1\}$; these properties follow from (3.5) and (3.6).

It remains to show (A.3) and (A.4). The proof of the latter assertion is simplest and is given first.

Proof of (A.4). Observe that

$$C_{\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) = C(\boldsymbol{u}) + \frac{1}{\sqrt{n}\,\lambda_n(s,t)}\,\mathbb{C}_n(s,t,\boldsymbol{u}).$$

Fix $\delta \in (0,1/2)$ and $\varepsilon \in (0,1)$. Without loss of generality, assume that n is large enough so that the bandwidth $h = h_n(s,t) = 1/\sqrt{\lfloor nt\rfloor - \lfloor ns\rfloor}$ is less than δ whenever $t - s \ge \varepsilon$. Then, for $\boldsymbol{u} \in [0,1]^d$ with $\delta \le u_j \le 1-\delta$, we have

$$\begin{aligned}
\dot{C}_{j,\lfloor ns\rfloor+1:\lfloor nt\rfloor}(\boldsymbol{u}) &= \frac{1}{2h}\{C(\boldsymbol{u}+h\boldsymbol{e}_j) - C(\boldsymbol{u}-h\boldsymbol{e}_j)\} \\
&\qquad + \frac{1}{2h\sqrt{n}\,\lambda_n(s,t)}\{\mathbb{C}_n(s,t,\boldsymbol{u}+h\boldsymbol{e}_j) - \mathbb{C}_n(s,t,\boldsymbol{u}-h\boldsymbol{e}_j)\}.
\end{aligned}$$

By the assumption of existence and continuity of $\dot{C}_j$ on V_j (see Condition 3.2), and since $0 \le \dot{C}_j \le 1$, it follows from the mean-value theorem that

$$\sup_{\substack{\boldsymbol{u}\in[0,1]^d \\ \delta\le u_j\le 1-\delta}} \left|\frac{1}{2h}\{C(\boldsymbol{u}+h\boldsymbol{e}_j) - C(\boldsymbol{u}-h\boldsymbol{e}_j)\} - \dot{C}_j(\boldsymbol{u})\right| \to 0, \qquad h \to 0.$$

Further, since $\mathbb{C}_n$ is uniformly equicontinuous in probability (by weak convergence to a process with continuous trajectories), it follows that

$$\sup_{\substack{\boldsymbol{u}\in[0,1]^d \\ \delta\le u_j\le 1-\delta}} \sup_{\substack{(s,t)\in[0,1]^2 \\ t-s\ge\varepsilon}} \big|\mathbb{C}_n(s,t,\boldsymbol{u}+h\boldsymbol{e}_j) - \mathbb{C}_n(s,t,\boldsymbol{u}-h\boldsymbol{e}_j)\big| \xrightarrow{\mathrm{P}} 0.$$

Finally,

$$\frac{1}{2h\sqrt{n}\,\lambda_n(s,t)} = \frac{1}{2\sqrt{\lambda_n(s,t)}} \leq \frac{1}{2\sqrt{\varepsilon - 1/n}}.$$

Combine the four previous displays to arrive at the desired conclusion. ■

The proof of (A.3) is more complicated and requires some additional notation. For integers $1 \leq k \leq l$, let $H_{k:l}$ denote the empirical c.d.f. of the unobservable sample $\boldsymbol{U}_k, \ldots, \boldsymbol{U}_l$ and let $H_{k:l,j}$, for $j \in \{1, \ldots, d\}$, denote its margins. The empirical quantile functions are

$$H_{k:l,j}^{-1}(u) = \inf\{v \in [0,1] : H_{k:l,j}(v) \geq u\}, \qquad u \in [0,1],$$

which are collected in a vector via

$$\boldsymbol{H}_{k:l}^{-1}(\boldsymbol{u}) = \big(H_{k:l,1}^{-1}(u_1), \ldots, H_{k:l,d}^{-1}(u_d)\big), \qquad \boldsymbol{u} \in [0,1]^d.$$

The auto-centered version of the process $\mathbb{B}_n^{(m)}$ in (A.1) is the process

$$\begin{aligned}
\mathring{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) &= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \{\mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}) - H_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}(\boldsymbol{u})\} \\
&= \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} (\xi_{i,n}^{(m)} - \bar{\xi}_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}^{(m)})\, \mathbf{1}(\boldsymbol{U}_i \leq \boldsymbol{u}),
\end{aligned} \tag{A.6}$$

with the usual convention that empty sums are zero.

Proof of (A.3). Consider the decomposition

$$\begin{aligned}
\big|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u})\big| \leq{}& \big|\check{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{u}) - \mathring{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}^{-1}(\boldsymbol{u}))\big| \\
&+ \big|\mathring{\mathbb{B}}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}^{-1}(\boldsymbol{u})) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}^{-1}(\boldsymbol{u}))\big| \\
&+ \big|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}^{-1}(\boldsymbol{u})) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u})\big|.
\end{aligned}$$

Write out the definitions of the processes $\mathbb{B}_n^{(m)}$, $\mathring{\mathbb{B}}_n^{(m)}$ and $\check{\mathbb{B}}_n^{(m)}$ in (A.1), (A.6) and (4.2), respectively, and take suprema over $(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d$ to obtain

$$\begin{aligned}
&\sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} \big|(\check{\mathbb{B}}_n^{(m)} - \mathbb{B}_n^{(m)})(s,t,\boldsymbol{u})\big| \\
&\leq \frac{2}{\sqrt{n}} \max_{1 \leq i \leq n} |\xi_{i,n}^{(m)}| \sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \big|\mathbf{1}\big(\hat{\boldsymbol{U}}_i^{\lfloor ns \rfloor + 1:\lfloor nt \rfloor} \leq \boldsymbol{u}\big) - \mathbf{1}\big(\boldsymbol{U}_i \leq \boldsymbol{H}_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}^{-1}(\boldsymbol{u})\big)\big|
\end{aligned} \tag{A.7}$$

$$+ \sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \right| \big|H_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}(\boldsymbol{u}) - C(\boldsymbol{u})\big| \tag{A.8}$$

$$+ \sup_{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d} \big|\mathbb{B}_n^{(m)}(s,t,\boldsymbol{H}_{\lfloor ns \rfloor + 1:\lfloor nt \rfloor}^{-1}(\boldsymbol{u})) - \mathbb{B}_n^{(m)}(s,t,\boldsymbol{u})\big|. \tag{A.9}$$

Each term requires a different treatment.

1- *The term* (A.7)*:* Fix $1 \le k \le l \le n$. Writing $\boldsymbol{U}_i = (U_{i1}, \ldots, U_{id})$, the d components of $\hat{\boldsymbol{U}}_i^{k:l}$ can be expressed as

$$\hat{U}_{ij}^{k:l} = H_{k:l,j}(U_{ij}), \qquad i \in \{k, \ldots, l\}.$$

The indicators $\mathbf{1}(\hat{U}_{ij}^{k:l} \le u_j)$ and $\mathbf{1}(U_{ij} \le H_{k:l,j}^{-1}(u_j))$ can be different only if $U_{ij} = H_{k:l,j}^{-1}(u_j)$. Because of the "no-ties" Condition 3.1 and the continuity of the marginal distributions, for every $j \in \{1, \ldots, d\}$, there is at most one index $i \in \{k, \ldots, l\}$ for which the indicators are different. It follows that the term (A.7) is bounded by

$$\frac{2d}{\sqrt{n}} \max_{1 \le i \le n} |\xi_{i,n}^{(m)}|.$$

Using the fact that, from (M1), for any $\nu \ge 1$, $\sup_{n \ge 1} \mathrm{E}[|\xi_{1,n}^{(m)}|^\nu] < \infty$, we have that, for every $\alpha > 0$ and $\nu \ge 1$ such that $\nu > 1/\alpha$,

$$\mathrm{P}\left(\max_{1 \le i \le n} |\xi_{i,n}^{(m)}| \ge n^\alpha \right) \le n \mathrm{P}\big(|\xi_{1,n}^{(m)}| \ge n^\alpha\big) \le n^{1-\nu\alpha}\ \mathrm{E}[|\xi_{1,n}^{(m)}|^\nu] \to 0. \tag{A.10}$$

Take $\alpha \in (0, 1/2)$ to conclude that (A.7) converges to zero in probability.

2- *The term* (A.8)*:* Let $\delta \in (0, 1/2)$, to be specified later. We split the supremum into two parts, according to whether $t - s$ is smaller or larger than $a_n = n^{-1/2-\delta}$:

$$A_{n,1} = \sup_{\substack{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d \\ t - s \le a_n}} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \right| \big| H_{\lfloor ns \rfloor + 1 : \lfloor nt \rfloor}(\boldsymbol{u}) - C(\boldsymbol{u}) \big|,$$

$$A_{n,2} = \sup_{\substack{(s,t,\boldsymbol{u}) \in \Delta \times [0,1]^d \\ t - s \ge a_n}} \left| \frac{1}{\sqrt{n}} \sum_{i=\lfloor ns \rfloor + 1}^{\lfloor nt \rfloor} \xi_{i,n}^{(m)} \right| \big| H_{\lfloor ns \rfloor + 1 : \lfloor nt \rfloor}(\boldsymbol{u}) - C(\boldsymbol{u}) \big|.$$

We will show that both $A_{n,1}$ and $A_{n,2}$ converge to zero in probability.

(a) Since both $H_{k:l}$ and C take values in $[0, 1]$, a crude bound for $A_{n,1}$ is

$$A_{n,1} \le \frac{1}{\sqrt{n}} (n\, a_n + 1) \max_{1 \le i \le n} |\xi_{i,n}^{(m)}| \le 2\, n^{-\delta} \max_{1 \le i \le n} |\xi_{i,n}|.$$

Apply (A.10) with $\alpha \in (0, \delta)$ to find that $A_{n,1}$ converges to zero in probability.

(b) Recall $\mathbb{B}_n$ in (3.4). Observe that

$$\mathbb{B}_n(s, t, \boldsymbol{u}) = \frac{\lfloor nt \rfloor - \lfloor ns \rfloor}{\sqrt{n}} \{ H_{\lfloor ns \rfloor + 1 : \lfloor nt \rfloor}(\boldsymbol{u}) - C(\boldsymbol{u}) \}.$$

We have

$$
\begin{aligned}
A_{n,2} &= \sup_{\substack{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d \\ t-s\geq a_n}} \left| \frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \right| \left|\mathbb{B}_n(s,t,\boldsymbol{u})\right| \\
&\leq \sup_{\substack{\lfloor ns\rfloor<\lfloor nt\rfloor \\ \lfloor nt\rfloor+1-\lfloor ns\rfloor\geq na_n}} \left| \frac{1}{\lfloor nt\rfloor - \lfloor ns\rfloor} \sum_{i=\lfloor ns\rfloor+1}^{\lfloor nt\rfloor} \xi_{i,n}^{(m)} \right| \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\mathbb{B}_n(s,t,\boldsymbol{u})\right| \\
&\leq \max_{\substack{1\leq k\leq l\leq n \\ l-k\geq na_n}} \left| \frac{1}{l-k+1} \sum_{i=k}^{l} \xi_{i,n}^{(m)} \right| \sup_{(s,t,\boldsymbol{u})\in\Delta\times[0,1]^d} \left|\mathbb{B}_n(s,t,\boldsymbol{u})\right|.
\end{aligned}
$$

By weak convergence $\mathbb{B}_n \rightsquigarrow \mathbb{B}_C$ in $\ell^\infty(\Delta\times[0,1]^d)$, the supremum at the end of the previous display is bounded in probability. Writing $h_n = na_n$, it is sufficient to show that

$$
\max_{\substack{1\leq k\leq l\leq n \\ l-k\geq h_n}} \left| \frac{1}{l-k+1} \sum_{i=k}^{l} \xi_{i,n}^{(m)} \right| \xrightarrow{\mathrm{P}} 0, \qquad n\to\infty.
$$

Fix $\eta > 0$. The probability that the previous maximum exceeds η is bounded by

$$
\sum_{\substack{1\leq k\leq l\leq n \\ l-k\geq h_n}} \mathrm{P}\left[\left| \frac{1}{l-k+1} \sum_{i=k}^{l} \xi_{i,n}^{(m)} \right| > \eta \right]. \tag{A.11}
$$

Fix $\nu \geq 2$, to be specified later. By stationarity and Markov's inequality, the previous expression is bounded by

$$
\sum_{\substack{1\leq k\leq l\leq n \\ l-k\geq h_n}} \eta^{-\nu}\,(l-k+1)^{-\nu}\,\mathrm{E}\left[\left| \sum_{i=1}^{l-k+1} \xi_{i,n}^{(m)} \right|^\nu \right] \leq \eta^{-\nu}\, n \sum_{r=h_n}^{n} r^{-\nu}\,\mathrm{E}\left[\left| \sum_{i=1}^{r} \xi_{i,n}^{(m)} \right|^\nu \right].
$$

Recall that the sequence $(\xi_{in}^{(m)})_{i\in\mathbb{Z}}$ is ℓ_n-dependent from (M2) and assume that n is sufficiently large so that $n \geq 2\ell_n + h_n$. Then, by Corollary A.1 in Romano and Wolf (2000), there exists a constant C_ν, depending only on ν, such that

$$
\mathrm{E}\left[\left| \sum_{i=1}^{r} \xi_{i,n}^{(m)} \right|^\nu \right] \leq C_\nu^\nu\,(4\ell_n r)^{\nu/2}\,\mathrm{E}[|\xi_{1,n}^{(m)}|^\nu].
$$

Using the fact that, from (M1), $\sup_{n\geq 1}\mathrm{E}[|\xi_{1,n}^{(m)}|^\nu] < \infty$, and up to a multiplicative constant, the expression in (A.11) is bounded by

$$
\begin{aligned}
n \sum_{r=h_n}^{n} r^{-\nu}(\ell_n r)^{\nu/2} &\leq n^2 h_n^{-\nu/2} \ell_n^{\nu/2} \\
&= O\big(n^{2-(1/2-\delta)\nu/2+(1/2-\gamma)\nu/2}\big) \\
&= O\big(n^{2+(\delta-\gamma)\nu/2}\big).
\end{aligned}
$$

The right-hand side converges to zero if we choose $\delta = \gamma/2$ and then $\nu > 8/\gamma$.

3- *The term* (A.9)*:* We claim that, for every $\varepsilon \in (0,1)$ and every $j \in \{1,\dots,d\}$, we have

$$\limsup_{n\to\infty} \max_{\substack{1\le k\le l\le n \\ l-k+1\ge \varepsilon n}} \sup_{u\in[0,1]} \big|H_{k:l,j}(u) - u\big| = 0 \qquad \text{a.s.} \tag{A.12}$$

By a well-known equality (Shorack and Wellner, 1986, page 86), equation (A.12) continues to hold if we replace $H_{k:l,j}$ by $H^{-1}_{k:l,j}$. Convergence in probability to zero of (A.9) then follows from weak convergence of the process $\mathbb{B}_n^{(m)}$ in $\ell^\infty(\Delta \times [0,1]^d)$ to the continuous process $\mathbb{B}_C$ which vanishes on the diagonal $s = t$ (split the supremum into two parts, according to whether $t - s$ is smaller than ε or not).

It remains to show (A.12). The almost sure invariance principle established in Berkes and Philipp (1977), whose assumptions were weakened by Yoshihara (1979), implies a functional law of the iterated logarithm for $u \mapsto H_{1:n,j}(u) - u$ as soon as $a > 3$, which in turn implies the Glivenko–Cantelli lemma under strong mixing, i.e.,

$$\limsup_{n\to\infty} \sup_{u\in[0,1]} |H_{1:n,j}(u) - u| = \lim_{m\to\infty} \sup_{k\ge m} \sup_{u\in[0,1]} \big|H_{1:k,j}(u) - u\big| = 0 \qquad \text{a.s.} \tag{A.13}$$

If $2 \le k \le l$ and for any $u \in [0,1]$, we have, by the triangle inequality,

$$\big|H_{k:l,j}(u) - u\big| \le \frac{l}{l-k+1}\big|H_{1:l,j}(u) - u\big| + \frac{k-1}{l-k+1}\big|H_{1:k-1,j}(u) - u\big|.$$

For $k = 1$, the previous inequality remains valid trivially, where $H_{1:0}$ can be defined arbitrarily. It follows that, for fixed $s \in (0,1)$,

$$\begin{aligned}\max_{\substack{1\le k\le l\le n \\ l-k+1\ge \varepsilon n}} \sup_{u\in[0,1]} \big|H_{k:l,j}(u) - u\big| &\le 2 \max_{1\le k\le n} \sup_{u\in[0,1]} \frac{k}{\varepsilon n}\big|H_{1:k,j}(u) - u\big| \\ &\le \frac{2s}{\varepsilon} + \frac{2}{\varepsilon}\max_{\substack{1\le k\le n \\ k>ns}} \frac{k}{n} \sup_{u\in[0,1]} \big|H_{1:k}(u) - u\big| \le \frac{2s}{\varepsilon} + \frac{2}{\varepsilon}\sup_{\substack{k\in\mathbb{N} \\ k>ns}} \sup_{u\in[0,1]} \big|H_{1:k,j}(u) - u\big|.\end{aligned}$$

Combine the previous display with (A.13) to find, for fixed $s \in (0,1)$,

$$\limsup_{n\to\infty} \max_{\substack{1\le k\le l\le n \\ l-k+1\ge \varepsilon n}} \sup_{u\in[0,1]} \big|H_{k:l,j}(u) - u\big| \le \frac{2s}{\varepsilon} \qquad \text{a.s.}$$

Since $s \in (0,1)$ was arbitrary, we find (A.12).

■

B Details for the experiments under serial dependence

For the numerical experiments involving serially dependent observations, we restricted ourselves to the bivariate case and only focused on the tests based on $\check{S}_n$ and $\hat{S}_n$. Given a bivariate copula C, two models were used to generate serially dependent observations under H_0 defined in (1.1).

- The first one is a simple autoregressive model of order one, AR(1). Let $\boldsymbol{U}_i$, $i \in \{-100, \ldots, 0, \ldots, n\}$, be a bivariate i.i.d. sample from a copula C. Then, set $\boldsymbol{\epsilon}_i = (\Phi^{-1}(U_{i1}), \Phi^{-1}(U_{i2}))$, where Φ is the c.d.f. of the standard normal distribution, and $\boldsymbol{X}_{-100} = \boldsymbol{\epsilon}_{-100}$. Finally, for any $j \in \{1, 2\}$ and $i \in \{-99, \ldots, 0, \ldots, n\}$, compute recursively
$$X_{ij} = 0.5 X_{i-1,j} + \epsilon_{ij}. \tag{AR1}$$
- The second model is a bivariate version of the exponential autoregressive (EXPAR) model considered in Auestad and Tjøstheim (1990) and Paparoditis and Politis (2001, Section 3.3) (see also Bücher and Kojadinovic, 2013). The sample $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ is generated as previously with (AR1) replaced by
$$X_{ij} = \{0.8 - 1.1 \exp(-50 X_{i-1,j}^2)\} X_{i-1,j} + 0.1 \epsilon_{ij}. \tag{EXPAR}$$

Data under $(\neg H_0) \cap H_{0,m}$, with $H_{0,m}$ as in (1.2), were generated using the procedures described above except that the bivariate random vectors $\boldsymbol{U}_i$, $i \in \{-100, \ldots, 0, \ldots, n\}$ are independent such that $\boldsymbol{U}_i$, $i \in \{-100, \ldots, 0, \ldots, k^\star\}$ are i.i.d. from a copula C_1 and $\boldsymbol{U}_i$, $i \in \{k^\star + 1, \ldots, n\}$ are i.i.d. from a copula C_2, where $C_1 \neq C_2$ and $k^\star = \lfloor nt \rfloor$ for some $t \in (0, 1)$. The resulting samples $\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n$ are therefore not samples under $H_{1,c} \cap H_{0,m}$ since the change in the dependence is gradual by (AR1) or (EXPAR). The copulas C_1 and C_2 were taken to be both either bivariate Clayton, Gumbel–Hougaard, Normal or Frank copulas such that C_1 has a Kendall's tau of 0.2 and C_2 a Kendall's tau of $\tau \in \{0.4, 0.6\}$. The parameter t defining $k^\star$ was chosen in $\{0.25, 0.5\}$.

The dependent multiplier sequences necessary to carry out the tests were generated using the "moving average approach" proposed initially in Bühlmann (1993, Section 6.2) and revisited in some detail in Bücher and Kojadinovic (2013, Section 6.1). A standard normal sequence was used for the required initial i.i.d. sequence. The kernel function κ in that procedure was chosen to be the Parzen kernel defined by $\kappa_P(x) = (1 - 6x^2 + 6|x|^3)\mathbf{1}(|x| \leq 1/2) + 2(1 - |x|^3)\mathbf{1}(1/2 < |x| \leq 1)$, $x \in \mathbb{R}$, which amounts to choosing the function φ in Condition (M3) as $x \mapsto (\kappa_P \star \kappa_P)(2x)/(\kappa_P \star \kappa_P)(0)$, where '$\star$' denotes the convolution operator. The value of the bandwidth parameter ℓ_n defined in Condition (M2) was chosen using the procedure described in Bücher and Kojadinovic (2013, Section 5). Two choices for the "combining" function ψ in that procedure were considered: the median and the maximum. Both choices led to similar rejection rates. The results reported in Tables 2 and 6 below are those obtained with $\psi =$ maximum.

C Selected results of the simulation study

Tables 1 up to 6 provide partial results of the large-scale Monte Carlo simulation experiment described in Section 5.

Table 1: Percentage of rejection of H_0 computed from 1000 random samples of size $n \in \{50, 100, 200\}$ generated under H_0, where C is either the d-dimensional Clayton (Cl), the Gumbel–Hougaard (GH) or the normal (N) copula whose bivariate margins have a Kendall's tau of τ.

			Cl			GH			N		
d	n	τ	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R
2	50	0.00	6.2	4.0	4.6	5.4	2.9	4.5	7.3	3.4	4.8
		0.25	6.7	6.2	5.6	5.5	3.3	5.4	4.4	3.0	6.3
		0.50	5.6	7.9	6.0	4.4	3.3	4.6	4.4	5.3	4.9
		0.75	6.0	16.6	5.5	3.2	6.7	4.3	3.6	9.1	4.9
	100	0.00	4.9	3.5	5.3	5.2	4.1	5.5	4.3	2.8	5.5
		0.25	6.1	6.6	5.0	5.0	3.3	6.2	5.3	4.0	5.5
		0.50	4.4	9.3	5.9	3.7	2.8	5.7	3.1	3.4	5.3
		0.75	2.7	10.0	4.6	2.5	4.7	4.4	2.1	6.0	5.6
	200	0.00	4.0	3.5	5.2	4.3	4.0	5.2	5.4	4.9	4.3
		0.25	4.7	5.2	6.3	3.3	3.0	3.8	4.0	3.9	5.2
		0.50	5.1	8.5	4.9	3.2	2.3	4.5	4.0	4.7	4.8
		0.75	2.6	9.3	5.9	1.5	3.1	5.2	1.9	4.8	5.7
3	50	0.00	4.3	1.5	3.0	4.2	2.1	3.6	5.5	2.8	3.4
		0.25	6.3	5.0	5.1	5.5	1.0	5.1	5.3	3.0	4.3
		0.50	8.2	9.1	5.9	2.7	0.9	5.7	3.0	2.2	4.6
		0.75	2.0	2.9	6.9	0.5	0.4	6.3	1.1	1.3	4.1
	100	0.00	4.5	3.4	4.5	4.5	2.8	4.6	4.5	2.7	3.9
		0.25	5.0	5.1	5.4	4.2	2.6	4.4	5.4	3.5	4.5
		0.50	5.7	7.6	6.3	3.3	1.3	5.0	3.2	3.1	3.9
		0.75	2.5	4.9	5.0	1.0	1.0	5.2	0.8	1.6	5.5
	200	0.00	3.3	2.5	4.3	3.5	3.2	4.3	4.8	4.0	4.7
		0.25	6.6	7.1	5.5	5.0	3.3	4.5	4.8	4.1	5.0
		0.50	6.0	9.2	4.5	3.0	2.4	5.9	4.8	4.3	4.8
		0.75	2.9	6.4	6.4	0.7	0.9	3.8	1.3	2.2	4.9

Table 2: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200\}$ generated under H_0 as explained in Appendix B, where C is either the bivariate Clayton (Cl), the Gumbel–Hougaard (GH), the normal (N) or the Frank (F) copula with a Kendall's tau of τ. The columns $\overline{\hat{\ell}_n^{opt}}$ and std give the mean and the standard deviation of the values of ℓ_n used for creating the dependent multiplier sequences.

			AR1				EXPAR			
C	n	τ	$\overline{\hat{\ell}_n^{opt}}$	std	$\hat{S}_n$	$\check{S}_n$	$\overline{\hat{\ell}_n^{opt}}$	std	$\hat{S}_n$	$\check{S}_n$
Cl	100	0.00	14.2	8.5	0.7	4.2	16.7	9.5	0.6	5.1
		0.25	14.1	8.6	1.7	5.9	16.7	10.2	2.3	5.5
		0.50	14.0	10.3	2.9	4.5	16.4	10.7	3.2	6.1
		0.75	13.3	10.0	3.3	2.7	15.5	10.7	4.7	5.1
	200	0.00	16.7	8.0	2.6	5.1	20.9	9.5	1.7	4.5
		0.25	16.0	7.3	3.0	5.1	20.5	9.7	2.0	4.1
		0.50	15.8	7.8	2.5	2.6	19.8	9.9	2.8	3.5
		0.75	15.5	9.0	3.6	1.6	19.0	9.0	4.7	4.3
GH	100	0.00	14.5	9.7	0.9	4.6	16.9	8.2	0.6	4.4
		0.25	14.1	8.7	1.5	4.9	17.1	10.2	0.6	5.0
		0.50	14.0	9.5	1.2	3.9	15.8	9.5	0.3	4.0
		0.75	13.7	10.0	0.5	2.6	15.2	9.2	0.4	1.6
	200	0.00	16.8	7.9	1.8	4.3	21.5	10.1	1.5	3.6
		0.25	16.6	8.8	2.0	5.5	20.9	11.5	1.1	5.1
		0.50	15.9	7.4	1.6	3.7	20.1	10.9	0.7	2.9
		0.75	15.5	8.7	0.9	1.3	18.8	9.1	0.2	1.6
N	100	0.00	14.1	7.9	1.1	5.0	17.3	9.2	1.4	5.4
		0.25	13.5	8.1	1.4	5.9	17.3	10.8	1.1	5.0
		0.50	13.5	9.1	1.4	3.3	16.4	9.7	1.0	4.5
		0.75	12.9	7.9	1.7	1.7	15.7	10.8	1.1	2.7
	200	0.00	16.3	6.2	1.9	5.4	20.7	8.7	1.5	3.7
		0.25	16.0	7.1	2.4	4.2	20.9	8.9	1.5	5.0
		0.50	16.1	7.8	3.2	4.2	19.8	10.4	1.8	2.9
		0.75	15.4	7.5	1.4	0.9	19.3	10.7	0.5	0.8
F	100	0.00	13.8	7.8	1.5	5.8	17.4	9.7	0.8	6.4
		0.25	14.2	9.3	1.3	5.5	16.6	9.7	0.5	4.8
		0.50	13.9	9.0	1.7	3.3	16.8	11.4	0.5	3.6
		0.75	13.5	9.7	0.6	1.5	15.8	10.3	1.2	3.1
	200	0.00	16.8	7.2	2.3	4.2	20.9	9.4	1.4	4.3
		0.25	16.0	7.0	2.9	6.0	20.6	8.7	1.1	3.6
		0.50	16.1	8.2	1.5	3.1	20.3	10.4	1.2	3.2
		0.75	15.8	8.8	0.5	0.9	19.6	9.0	0.8	1.2

Table 3: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{50, 100, 200\}$ generated under $H_{0,m} \cap H_{1,c}$, where $H_{1,c}$ is defined in (5.1), $k^\star = \lfloor nt \rfloor$, C_1 and C_2 are both either bivariate Clayton (Cl), Gumbel–Hougaard (GH) or normal (N) copulas such that C_1 has a Kendall's tau of 0.2 and C_2 a Kendall's tau of τ.

			Cl			GH			N		
n	τ	t	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R
50	0.4	0.10	7.5	8.1	5.7	6.1	3.7	4.3	6.1	4.8	4.3
		0.25	12.1	10.6	4.0	9.4	5.1	5.0	10.4	7.6	5.5
		0.50	18.0	16.1	6.3	12.1	7.8	4.8	12.3	8.4	5.8
	0.6	0.10	14.5	17.0	5.4	11.4	7.7	6.2	11.4	9.8	7.2
		0.25	35.5	34.4	7.4	29.9	21.4	6.4	31.3	21.4	7.1
		0.50	47.3	41.6	7.0	45.3	30.3	8.9	46.0	33.9	9.1
100	0.4	0.10	7.1	8.7	6.1	6.6	5.1	5.1	5.8	5.2	5.3
		0.25	18.8	19.9	5.2	16.9	13.2	5.8	14.9	12.5	6.1
		0.50	26.5	26.4	7.3	23.8	18.8	7.3	22.6	19.1	7.9
	0.6	0.10	21.5	25.1	5.8	16.7	12.0	5.1	17.5	16.9	6.1
		0.25	65.1	66.0	6.3	61.2	51.5	7.5	62.9	54.8	9.9
		0.50	82.1	81.6	14.7	78.8	69.7	14.9	79.1	73.7	11.8
200	0.4	0.10	11.1	13.8	5.8	8.3	8.1	5.5	9.5	9.8	5.0
		0.25	30.8	33.9	5.9	27.6	24.8	6.4	29.6	28.3	6.8
		0.50	47.1	48.6	9.0	45.8	41.4	8.7	47.1	46.1	9.4
	0.6	0.10	36.4	41.3	6.8	34.4	31.7	7.1	36.0	36.3	6.7
		0.25	92.6	93.2	12.3	91.4	88.9	16.7	91.3	90.2	12.0
		0.50	98.9	99.3	22.2	98.5	98.1	22.0	99.3	99.1	21.1

Table 4: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200\}$ generated under $H_{0,m} \cap H_{1,c}$, where $H_{1,c}$ is defined in (5.1), $k^\star = \lfloor nt \rfloor$, C_1 (resp. C_2) is a d-dimensional Clayton (resp. Gumbel–Hougaard) copula whose bivariate margins have a Kendall's tau of τ.

			$d=2$			$d=3$		
n	τ	t	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R
100	0.25	0.25	5.7	4.1	5.3	5.3	3.1	4.4
		0.50	6.2	5.7	5.6	9.1	6.0	5.8
		0.75	6.6	6.3	3.5	5.8	5.4	4.9
	0.50	0.25	5.5	5.9	5.8	4.6	2.9	5.0
		0.50	10.5	12.2	5.1	15.1	15.1	6.8
		0.75	8.3	11.9	4.4	7.7	9.9	5.3
	0.75	0.25	4.0	7.6	5.1	2.5	1.9	4.3
		0.50	12.5	19.9	6.0	9.8	13.2	5.2
		0.75	8.2	16.4	6.5	4.8	6.7	3.8
200	0.25	0.25	5.8	5.3	6.1	5.8	3.7	5.8
		0.50	8.5	8.7	5.4	9.8	9.7	6.6
		0.75	6.4	6.9	5.5	9.0	9.1	5.3
	0.50	0.25	10.4	11.5	6.8	12.6	10.1	6.7
		0.50	30.9	37.5	5.3	44.0	45.8	7.1
		0.75	16.3	23.2	6.1	20.1	27.0	5.1
	0.75	0.25	11.3	18.0	4.9	15.6	16.7	7.3
		0.50	43.4	54.4	6.2	58.9	63.1	5.2
		0.75	21.3	36.4	4.6	23.6	36.0	6.8

Table 5: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{50, 100, 200\}$ such that the $\lfloor nt_1 \rfloor$ first observations of each sample are from a d-dimensional c.d.f. with normal copula and $N(0,1)$ margins (that is, from a multivariate standard normal c.d.f.), and the $n - \lfloor nt_1 \rfloor$ last observations are from a d-dimensional c.d.f. with normal copula whose first margin is the $N(\mu, 1)$ and whose d-1 remaining margins are the $N(0,1)$. The bivariate margins of the normal copula have a Kendall's tau of τ.

	$(\mu, t_1) =$		$(0.5, 0.25)$			$(0.5, 0.5)$			$(2, 0.25)$			$(2, 0.5)$		
d	n	τ	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R	$\check{S}_n$	$\hat{S}_n$	S_n^R
2	50	0.00	6.7	3.8	9.0	6.5	3.1	17.7	4.6	2.4	70.1	5.1	2.6	98.5
		0.25	5.6	2.8	8.1	4.8	3.6	15.0	6.0	3.1	57.7	4.7	1.6	98.5
		0.50	4.4	4.1	9.6	3.4	2.9	13.0	18.4	5.1	39.6	5.7	0.9	99.3
	100	0.00	5.3	4.0	19.0	6.3	5.2	30.4	6.1	4.1	99.0	5.5	2.5	100.0
		0.25	5.0	4.4	13.9	3.7	2.3	24.9	8.6	4.7	97.5	4.4	2.2	100.0
		0.50	4.0	3.8	11.5	3.1	3.1	22.6	29.5	13.1	91.6	12.4	2.0	100.0
	200	0.00	5.5	5.2	34.1	3.9	3.4	61.9	4.3	3.3	100.0	5.6	4.1	100.0
		0.25	3.9	3.4	27.6	4.1	3.5	51.3	13.6	8.9	100.0	8.1	4.0	100.0
		0.50	3.4	3.9	18.9	2.8	2.9	43.1	57.8	39.4	100.0	34.8	8.3	100.0
3	50	0.00	4.9	1.6	5.0	4.5	1.7	10.0	4.8	1.9	36.5	6.0	2.3	79.6
		0.25	5.0	2.7	6.9	5.0	2.6	9.9	6.9	2.6	24.8	5.1	2.6	87.4
		0.50	3.7	2.6	5.5	3.3	1.3	8.9	9.4	3.5	17.4	3.0	0.7	94.2
	100	0.00	4.5	2.3	11.3	4.2	2.5	18.6	4.8	3.0	87.5	3.6	2.0	99.6
		0.25	4.9	3.1	10.7	5.1	3.5	14.8	6.7	3.8	67.9	5.9	4.2	99.9
		0.50	2.8	2.0	7.3	3.0	2.4	13.7	16.9	8.6	60.6	6.0	1.3	100.0
	200	0.00	3.0	2.4	20.1	4.5	4.0	37.3	5.3	3.3	100.0	4.3	3.2	100.0
		0.25	4.8	4.1	15.3	5.6	4.4	30.6	11.9	8.1	99.2	7.4	5.2	100.0
		0.50	4.9	3.7	11.8	3.8	3.2	24.9	41.0	30.9	99.2	23.9	7.8	100.0

Table 6: Percentage of rejection of H_0 computed from 1000 samples of size $n \in \{100, 200\}$ generated under $\neg H_0$ as explained in the second paragraph of Appendix B, where C_1 and C_2 are both bivariate Gumbel–Hougaard copulas such that C_1 has a Kendall's tau of 0.2 and C_2 a Kendall's tau of τ. The columns $\overline{\hat{\ell}_n^{opt}}$ and std give the mean and the standard deviation of the values of ℓ_n used for creating the dependent multiplier sequences.

			AR1				EXPAR			
n	t	τ	$\overline{\hat{\ell}_n^{opt}}$	std	$\hat{S}_n$	$\check{S}_n$	$\overline{\hat{\ell}_n^{opt}}$	std	$\hat{S}_n$	$\check{S}_n$
100	0.25	0.4	14.2	10.1	3.8	13.8	16.9	10.5	1.5	10.6
		0.6	14.1	9.1	10.6	39.9	15.8	9.5	8.5	32.6
	0.50	0.4	14.3	10.1	5.3	18.0	17.1	10.3	2.5	13.5
		0.6	13.9	8.6	25.2	57.2	16.5	9.3	17.1	54.8
200	0.25	0.4	16.5	8.6	8.6	16.4	20.6	9.8	4.5	14.6
		0.6	16.4	8.0	46.0	71.2	19.6	9.0	32.5	63.1
	0.50	0.4	16.6	7.4	19.0	31.9	20.7	8.8	12.4	25.8
		0.6	16.3	7.1	75.5	89.8	20.6	9.1	62.5	80.9