Riemann-Hilbert approach to gap probabilities for the Bessel process

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Abstract

We consider the gap probability for the Bessel process in the single-time and multi-time case. We prove that the scalar and matrix Fredholm determinants of such process can be expressed in terms of determinants of integrable kernels in the sense of Its-Izergin-Korepin-Slavnov and thus related to suitable Riemann-Hilbert problems. In the single-time case, we construct a Lax pair formalism and we derive a Painlevé III equation related to the Fredholm determinant.

1 Introduction

Many models in Mathematical Physics rely on the notion of a determinantal random point process (DPP). A few examples are offered by the statistical distribution of the eigenvalues of random matrix models pioneered by Dyson ([8]), certain models of random growth of crystals ([2], [10], [20]), and mutually avoiding random walkers (usually referred as Dyson's processes).

The present paper falls into the last category and establishes a connection between certain "gap" probabilities and a particular class of boundary value problems in the complex plane, generally referred to as "Riemann-Hilbert problems" (see e.g. [7]). We will show that such boundary value problem, suitably formulated, allows to express the "gap" probabilities in term of theory of equation of Painlevé type; this relationship is quite well-known originally in two dimensional statistical physics ([25]) and it was extensively studied in the eighties and nineties ([17], [18], [19], [29], [34], [35]).

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In order to frame our results in a narrower context, we mention the well-known "Tracy-Widom" distribution ([34]) which describes the fluctuations of the largest eigenvalue of a random Gaussian matrix (suitably scaled) in terms of the solution of a nonlinear ODE (Painlevé II). Similarly in [35] the authors connected the fluctuation of the smallest eigenvalue of the "Laguerre ensemble" to the third member of the Painlevé hierarchy. Our results are closely related to these and will extend this connection to the case of the "Bessel process" using a completely different method.

We also mention the recent paper [28] where a new Lax pair related to the Painlevé III equation is defined and the first component of their eigenvector is the smallest eigenvalue probability near the hard edge in the large n limit of the Laguerre ensemble.

Before getting into the details pertinent to the Bessel process we will briefly review for the sake of the reader the main concepts about DPP in timeless and dynamic regimes and explain the connection with our case. For surveys on DPP, we refer to Soshnikov [31] and Johansson [21].

Determinantal Point Process. Consider a random collection of points on the real line. A configuration $\mathcal{X}$ is a subset of $\mathbb{R}$ that locally contains a finite number of points, i.e. $\#(\mathcal{X} \cap [a, b]) < +\infty$ for every bounded interval $[a, b] \subset \mathbb{R}$. Then, a (locally finite) point process on $\mathbb{R}$ is a probability measure on the space of all configurations, so that, loosely speaking, it is possible to evaluate the probability of any given configuration.

Given a point process on $\mathbb{R}$, the mapping $A \mapsto \mathbb{E}[\#(\mathcal{X} \cap A)]$, which assigns to a Borel set A the expected value of the number of points in A, is a measure on $\mathbb{R}$. We assume there exists a density ρ_1 with respect to Lebesgue measure and we call it 1-point correlation function for the point process. Then, we have

$$\mathbb{E}[\#(\mathcal{X} \cap A)] = \int_A \rho_1(x) \mathrm{d}x. \tag{1.1}$$

and $\rho_1(x)dx$ represents the probability to have a point in the infinitesimal interval $[x, x + dx]$. In general, given disjoints sets $A_1, \ldots, A_k$, we have

$$\mathbb{E}\left[\prod_{j=1}^{k} \#(\mathcal{X} \cap A_j)\right] = \int_{A_1} \ldots \int_{A_k} \rho_k(x_1, \ldots, x_k) \mathrm{d}x_1 \ldots \mathrm{d}x_k \tag{1.2}$$

i.e. the expected number of k-tuples $(x_1, \ldots, x_k) \in \mathcal{X}^k$ such that $x_j \in A_j$ for every j. In case the A_j's are not disjoint it is still possible to define the quantity above, with little modifications.

A point process with correlation functions $\{\rho_k\}_{k\geq 1}$ is a DPP if there exists a kernel $K(x,y)$ such that for every k and every $x_1,\ldots,x_k$ we have

$$\rho_k(x_1,\ldots,x_k) = \det[K(x_i,x_j)]_{i,j=1}^k. \tag{1.3}$$

In a DPP all quantities of interest can be expressed in terms of K. In particular, given a Borel set A, we are interested in the gap probability, i.e. the probability to find no points in A. It is possible to show that such quantity is equal to the Fredholm determinant

$$\det(\mathrm{Id} - \chi_A K \chi_A) \tag{1.4}$$

of the (trace class) integral operator K defined by

$$K[f](x) = \int_{\mathbb{R}} K(x,y) f(y) \mathrm{d}y. \tag{1.5}$$

and restricted to the Borel set A (χ_A is the projection on such subset).

An example of a DPP is the set of eigenvalues of a random matrix ensemble as in the case of the aforementioned Gaussian and Laguerre ensembles. The theory of gap probabilities for DPP has been extensively studied (see [1], [32], [33] and [36] to mention a few).

Remark 1.1. *The same arguments are valid when performing a scaling limits of the process, as the number of points goes to ∞, at different sectors in the domain of the points (see for example [31]). As we will see, this is the case of the Bessel process, which appears as limiting kernel of certain matrix ensembles as the dimension of the matrix goes to infinity.*

A generalization of such theory is given by introducing multiple times $0 < \tau_1 < \ldots < \tau_n$ at which one can simultaneously study the behaviour of the points. We call a time-dependent process "multi-time process".

The idea behind the introduction of multi-time processes is to be able to consider not only static models in timeless equilibrium, but also dynamical systems which may be in an arbitrary non-equilibrium state changing with time.

The first implementation of this dynamic regime was proposed by Dyson ([8]) for the study of the random eigenvalues of a Gaussian Ensemble.

Given a collection of times $\{\tau_k\}_{k=1,\ldots,n}$, within a fixed time interval, say $(0,T)$, and subsets $A_k \subset \mathbb{R}$, $k = 1,\ldots,n$, the quantity of interest is the probability that for all k no points lie in A_k at time τ_k. We call again this quantity "gap probability".

Applying classical results from Karlin and McGregor ([22]) and Eynard and Mehta ([9]), it can be proved that the gap probability is equal to the

Fredholm determinant of a suitable integral operator $[K]$, with matrix kernel $[K_{ij}]_{i,j=1}^{n}$, restricted to the sets $\vec{A} = A_1 \sqcup \ldots \sqcup A_n$:

$$P\,(\text{no points in } A_k \text{ at time } \tau_k,\, \forall k) = \det(\mathrm{Id} - \chi_{\vec{A}}\,[K]\,\chi_{\vec{A}}). \tag{1.6}$$

The Bessel process. We introduce now the subject of the present paper. The Bessel process is a determinantal point process as detailed above ([31]) defined in terms of a trace-class integral operator acting on $L^2(\mathbb{R}_+)$, with kernel

$$K_B(x,y) = \frac{J_\nu(\sqrt{x})\sqrt{y}J_{\nu+1}(\sqrt{y}) - J_{\nu+1}(\sqrt{x})\sqrt{x}J_\nu(\sqrt{y})}{2(x-y)} \tag{1.7}$$

where J_ν are Bessel functions with parameter $\nu > -1$.

The Bessel kernel K_B arose originally as the correlation function in the scaling limit of the Laguerre and Jacobi Unitary Ensembles near the hard edge of their spectrum at zero ([12], [26], [27]) as well as of generalized LUEs and JUEs ([24], [37]).

In this article we focus on the gap probabilities of this process. In particular, we will be concerned with the Fredholm determinant of such operator on a collection of (finite) intervals $I := \bigcup_{i=1}^{N}[a_{2i-1}, a_{2i}]$, i.e. the Tracy-Widom distribution $\det(\mathrm{Id} - K_B\chi_I)$, and the emphasis is on the determinant thought of as function of the endpoint a_i, $i = 1, \ldots, 2N$.

The gap probabilities of the Bessel process were first studied in [35]; we refer to this paper for a comparison with the differential equations showed in the present paper (in particular, Theorem 2.15 and formula (2.47)). We point out that such equations are not the same as those shown in [35] and they are derived through a completely different method.

The second part of this paper will examine the Bessel process in a time-dependent regime. Consider n times $\tau_1, \ldots, \tau_n$ in a given time interval $(0,T)$; the so called multi-time or Extended Bessel process (see [23] and [36]) is a determinantal point process with matrix kernel $[K_B]$ with entries

$$K_{B;ij}(x,y) = \begin{cases} \int_0^1 e^{u\Delta_{ij}} J_\nu(\sqrt{xu})J_\nu(\sqrt{yu})\,\mathrm{d}u & i \geq j \\ -\int_1^\infty e^{u\Delta_{ij}} J_\nu(\sqrt{xu})J_\nu(\sqrt{yu})\,\mathrm{d}u & i < j \end{cases} \tag{1.8}$$

$i,j = 1,\ldots,n$; with $\Delta_{ij} = \tau_i - \tau_j$ the time gap between two times and $\nu > -1$.

Remark 1.2. *In the case $T = \tau_1 = \ldots = \tau_n = 0$, we can recover the time-less Bessel kernel (1.7).*

As shown by Forrester, Nagao and Honner in [13], the multi-time Bessel process (with its correspondent kernel) appears as scaling limit of the Extended (multi-time) Laguerre process at the hard edge of the spectrum.

Although the multi-time Bessel process has been known since a long time, the study of its gap probabilities has never been performed before and it is addressed in this paper. Again, we will focus on the Fredholm determinant of such process on a collection of intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$, I_j refers to the Borel set considered at time τ_j for all j. The result is a set of relations that describes the Fredholm determinant as a function of the endpoints of the intervals and of the n times.

Fredholm determinant of integrable kernels and τ-function. The Fredholm determinant of the time-less Bessel kernel and, as it will be clear in the paper, the Fredholm determinant of its multi-time counterpart are instances of determinants of operators with integrable kernel in the sense of Its-Izergin-Korepin-Slavnov [16] (IIKS). We point out that the multi-time Bessel kernel (1.8) is not of the integrable form, but we will show how to reduce it to such form in the course of the paper (Section 3, Theorem 3.4); this is also the main conceptual contribution of the paper.

We will recall here the main results that will be used later (for a concise and clear description we refer to [14]).

Let us consider a $p \times p$ matrix operator K acting on $L^2(\Sigma)$, with $\Sigma \subseteq \mathbb{C}$ a collection of piecewise smooth, oriented contours (possibly extending to infinity), defined as

$$K(\phi)(\lambda) = \int_\Sigma K(\lambda, \mu)\phi(\mu)\, \mathrm{d}\mu$$

with kernel

$$K(\lambda, \mu) = \frac{\mathbf{f}^T(\lambda)\mathbf{g}(\mu)}{\lambda - \mu}$$

where $\mathbf{f}, \mathbf{g}$ are rectangular $r \times p$ matrix valued functions ($p < r$), such that $\mathbf{f}^T(\lambda)\mathbf{g}(\lambda) = 0$. We assume that $\mathbf{f}$ and $\mathbf{g}$ are sufficiently smooth functions along the connected components of Σ.

We are interested in its Fredholm determinant $\det(\mathrm{Id} - K)$. The key observation is the Jacobi's formula for the variation of the determinant

$$\partial \ln \det(\mathrm{Id} - K) = -\mathrm{Tr}\left((\mathrm{Id} + R)\,\partial K\right) \tag{1.9}$$

where ∂ signifies the differential with respect to any auxiliary parameter (let's denote them globally by $\vec{s}$), on which K may depend, and R is the resolvent operator $R := (I - K)^{-1}K$, which belongs to the same class of integrable operators

$$R(\lambda, \mu) = \frac{\mathbf{F}^T(\lambda)\mathbf{G}(\mu)}{\lambda - \mu}$$

if K is an IIKS operator; moreover, $\mathbf{F}$ and $\mathbf{G}$ can be determined explicitly

$$\mathbf{F}(\lambda) = \Gamma(\lambda)\mathbf{f}(\lambda), \qquad \mathbf{G}(\lambda) = \Gamma(\lambda)\mathbf{g}(\lambda),$$

by solving the following Riemann-Hilbert problem (RHP)

$$\Gamma_+(\lambda) = \Gamma_-(\lambda)M(\lambda) \quad \lambda \in \Sigma \tag{1.10a}$$

$$\Gamma(\lambda) = \mathbf{I}_r + \mathcal{O}\left(\frac{1}{\lambda}\right) \quad \lambda \to \infty \tag{1.10b}$$

$$M(\lambda) = \mathbf{I}_r - 2\pi i \mathbf{f}(\lambda)\mathbf{g}^T(\lambda) \tag{1.10c}$$

(the dependence on the parameters $\vec{s}$ is implicit).

In several cases of interest, and in our case as well, the Fredholm determinant for such a kernel coincides with the notion of the isomonodromic τ-function introduced by Jimbo, Miwa and Ueno ([17], [18] and [19]) to study monodromy-preserving deformations of rational connections on $\mathbb{P}^1$.

We report here just the main results that will be used. We refer to [3] and [5] for a thorough exposition.

First of all we consider a slightly more general notion of τ-function associated to any Riemann-Hilbert problem depending on parameters.

Given a generic RHP (1.10a)-(1.10b), we introduce the following one form on the space of (deformation) parameters

$$\omega(\partial) := \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_\partial(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} \tag{1.11a}$$

$$\Xi_\partial(\lambda) := \partial M(\lambda) M^{-1}(\lambda); \tag{1.11b}$$

for the sake of simplicity, we are assuming that all the expressions involved are depending smoothly on the parameters (here and below we will denote with $'$ the derivative with respect to λ).

Then, the following equality holds

$$\omega(\partial) = \partial \ln \det(\mathrm{Id} - K) - H(M) \tag{1.12}$$

where

$$H(M) := \int_\Sigma \left(\partial \mathbf{f}'^{\,T}\mathbf{g} + \mathbf{f}'^{\,T}\partial\mathbf{g}\right) \mathrm{d}\lambda - 2\pi i \int_\Sigma \mathbf{g}^T\mathbf{f}'\partial\mathbf{g}^T\mathbf{f}\,\mathrm{d}\lambda.$$

On the other hand, it is possible to show that ω is the logarithmic total differential of the isomonodromic τ-function of Jimbo, Miwa and Ueno ([17], [18], [19]), in the case of RHPs corresponding to rational ODEs.

Thus, it is possible to define, up to normalization, the isomonodromic τ-function $\tau_{JMU} := \exp\left(\int \omega\right)$ which, thanks to (1.12), will coincide with the Fredholm determinant $\det(\mathrm{Id} - K)$, provided that $H(M) \equiv 0$.

The main steps in our study of the gap probabilities for the Bessel process are the following: we will first find an integrable operator in the sense specified above, acting on $L^2(\Sigma)$, with Σ a suitable collection of contours. Through an appropriate Fourier transform, we will prove that such operator has the same Fredholm determinant as the Bessel process. We will then set up a RHP for this integrable operator and connect it to the Jimbo-Miwa-Ueno τ function.

This strategy will be applied separately to both the single-time and the multi-time Bessel process. Our approach derives from the one used in [4] and [5] for the Airy and Pearcey processes in the dynamic and time-less regime respectively.

Whereas the part dedicated to the single-time process is mostly a review of known results (see [11], [17] and [35]), re-derived using an alternative approach, the results on the multi-time Bessel are new and never appeared in the literature before.

The present paper is organized as follows: in Section 2 we will deal with the single-time Bessel determinantal process in the general case of several intervals; in the Subsection 2.1 we will focus on the single-time process restricted to a single interval $[0, a]$: we will find a Lax pair and we will be able to make a connection between the Fredholm determinant and the third Painlevé transcendent. This provides a different and direct proof of this known connection ([17], [35]); in particular our approach directly specifies the monodromy data of the associated isomonodromic system and allows to use the steepest descent method to investigate asymptotic properties, if so desired. In Section 3 we will study the gap probabilities for the multi-time Bessel process. Although the results of Section 3 strictly include those of Section 2, we have decided to separate the two cases for the benefit of a clearer exposition.

2 Gap probabilities for the single-time Bessel process and Painlevé Transcendent

We recall the definition of the Bessel kernel

$$K_B(x, y) = \frac{J_\nu(\sqrt{x})\sqrt{y}J_{\nu+1}(\sqrt{y}) - J_{\nu+1}(\sqrt{x})\sqrt{x}J_\nu(\sqrt{y})}{2(x-y)}; \tag{2.1}$$

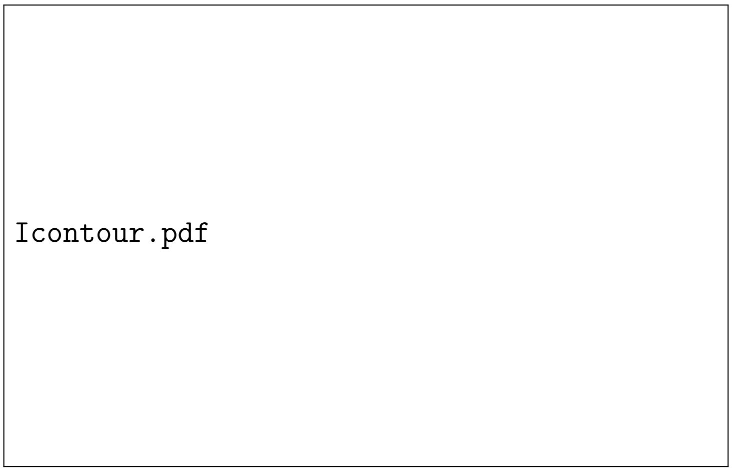

Figure 2.1: The contours appearing in the definition of the Bessel kernel (2.2).

writing the Bessel functions as explicit contour integrals, it is possible to show, through some suitable manipulations and integrations by parts, that the Bessel kernel can be written also in the following form

$$K_B(x,y) = \left(\frac{y}{x}\right)^{\nu/2} \iint_{\gamma\times\hat{\gamma}} \frac{e^{xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{t-s} \left(\frac{s}{t}\right)^{\nu} \frac{\mathrm{d}t}{2\pi i}\frac{\mathrm{d}s}{2\pi i} \tag{2.2}$$

with $\nu > -1$, $x, y > 0$ and γ a curve that extends to $-\infty$ and winds around the zero counterclockwise, while the curve $\hat{\gamma}$ is simply the transformed curve under the map $t \to 1/t$ (γ and $\hat{\gamma}$ are disjoint); the logarithmic cut is on $\mathbb{R}_-$. The contours are as in Figure 2.1.

We want to study the Fredholm determinant of the Bessel operator; in particular, we will focus on the following quantity

$$\det\left(\mathrm{Id} - K_B\chi_I\right) \tag{2.3}$$

where χ_I is the characteristic function of a collection of finite intervals $I := [a_1, a_2] \cup [a_3, a_4] \cup \ldots \cup [a_{2N-1}, a_{2N}]$ $(0 \leq a_1 < \ldots < a_{2N})$, which assumes values 0 or 1 depending on the variable being respectively outside or within I.

Remark 2.1. *The Bessel operator is not trace-class on an infinite interval. Thus, it is meaningless to consider the operator restricted to such interval.*

Remark 2.2. *Defining $K_a := K_B(x,y)\chi_{[0,a]}(y)$, then we have*

$$K_B(x,y)\chi_I(y) := \sum_{j=1}^{2N} (-1)^j K_{a_j}(x,y) \tag{2.4}$$

Our goal is to setup a Riemann-Hilbert problem associated to the Fredholm determinant of $K_B\chi_I$.

Theorem 2.3. *The following identity between Fredholm determinants holds*

$$\det\left(\mathrm{Id} - K_B\chi_I\right) = \det\left(\mathrm{Id} - \mathbb{B}\right) \tag{2.5}$$

where $\mathbb{B}$ *is a trace-class integrable operator acting on* $L^2\left(\gamma \sqcup \hat{\gamma}\right)$ *with kernel*

$$\mathbb{B}(s,t) := \frac{\vec{f}(s)^T \cdot \vec{g}(t)}{s-t} \tag{2.6a}$$

$$\vec{f}(s) = \frac{1}{2\pi i}\begin{bmatrix} e^{\frac{a_1 s}{2} - \frac{1}{4s}} s^{-\nu} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \chi_\gamma(s) + \frac{1}{2\pi i}\begin{bmatrix} 0 \\ e^{-a_1 s + \frac{1}{4s}} s^{\nu} \\ -e^{-a_2 s + \frac{1}{4s}} s^{\nu} \\ \vdots \\ e^{-a_{2N-1} s + \frac{1}{4s}} s^{\nu} \\ -e^{-a_{2N} s + \frac{1}{4s}} s^{\nu} \end{bmatrix} \chi_{\hat{\gamma}}(s) \tag{2.6b}$$

$$\vec{g}(t) = \begin{bmatrix} 0 \\ e^{\frac{a_1 t}{2}} \\ e^{t\left(a_2 - \frac{a_1}{2}\right)} \\ \vdots \\ e^{t\left(a_{2N} - \frac{a_1}{2}\right)} \end{bmatrix} \chi_\gamma(t) + \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \chi_{\hat{\gamma}}(t) \tag{2.6c}$$

Remark 2.4. *The space* $L^2(\gamma \sqcup \hat{\gamma})$ *is the space of square integrable functions in the arclength measure, defined on the curves* $\gamma \sqcup \hat{\gamma}$*.*

Proof. We work on a single kernel K_{a_j} and we will later sum them up (as in Remark 2.2), thanks to the linearity of the operations that we are going to perform.

First of all, we can notice that, if $x < 0$ or $y < 0$, $K_B(x,y) \equiv 0$; in particular, if $x < 0$, then a simple residue calculation shows that the kernel vanishes. Similar arguments lead to the same conclusion for $y < 0$.

Then, using Cauchy's theorem, we can write

$$\begin{aligned} K_{a_j}(x,y) &= K_B(x,y)\chi_{[0,a_j]}(y) = \\ &= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{\xi(a_j - y)} \iint_{\gamma\times\hat{\gamma}} \frac{e^{xt - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t-s)} \left(\frac{s}{t}\right)^\nu \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2} = \\ &= \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}\xi}{2\pi i} e^{-\xi y} \int_{i\mathbb{R}+\epsilon} \frac{\mathrm{d}t}{2\pi i} e^{xt} \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi a_j - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t-s)} \left(\frac{s}{t}\right)^\nu \end{aligned} \tag{2.7}$$

where $i\mathbb{R}+\epsilon$ ($\epsilon > 0$) is a translated imaginary axis; thanks to the analyticity of the kernel, we continuously deformed the curve γ into such translated imaginary axis, in order to make the Fourier operator defined below more explicit. We also discarded the conjugation term $\left(\frac{y}{x}\right)^{\nu/2}$, due to the invariance of the Fredholm determinant under conjugation by a positive function.

Defining the following Fourier transform operators:

$$\begin{array}{c|c} \mathcal{F} : L^2(\mathbb{R}) \to L^2(i\mathbb{R}+\epsilon) & \mathcal{F}^{-1} : L^2(i\mathbb{R}+\epsilon) \to L^2(\mathbb{R}) \\ f(x) \mapsto \frac{1}{\sqrt{2\pi i}} \int_{\mathbb{R}} f(x) e^{\xi x} \mathrm{d}x & h(\xi) \mapsto \frac{1}{\sqrt{2\pi i}} \int_{i\mathbb{R}+\epsilon} h(\xi) e^{-\xi x} \mathrm{d}\xi \end{array} \tag{2.8}$$

it is straightforward to deduce that

$$K_B \chi_I = \mathcal{F}^{-1} \circ \overline{\mathcal{K}_B} \circ \mathcal{F} \tag{2.9}$$

with $\overline{\mathcal{K}_B} = \sum_j (-1)^j \overline{\mathcal{K}_{a_j}}$ and $\forall\, j = 1, \ldots, N$ $\overline{\mathcal{K}_{a_j}}$ is an operator on $L^2(i\mathbb{R}+\epsilon)$ with kernel

$$\overline{\mathcal{K}_{a_j}}(\xi, t) = \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi a_j - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t - s)} \left(\frac{s}{t}\right)^{\nu}$$

In order to ensure convergence of the Fourier-transformed Bessel kernel, we conjugate $\overline{\mathcal{K}_B}$ by a suitable function

$$\begin{aligned} &\mathcal{K}_B(\xi, t) := e^{\frac{a_1 t}{2} - \frac{a_1 \xi}{2}} \overline{\mathcal{K}_B}(\xi, t) \\ = &\sum_{j=1}^{2N} (-1)^j \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi\left(a_j - \frac{a_1}{2}\right) + \frac{a_1 t}{2} - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t - s)} \left(\frac{s}{t}\right)^{\nu} =: \sum_{j=1}^{2N} (-1)^j \mathcal{K}_{a_j}(\xi, t) \end{aligned} \tag{2.10}$$

and we continuously deform the translated imaginary axis $i\mathbb{R}+\epsilon$ into its original shape γ; note that $a_j - \frac{a_1}{2} > 0$, $\forall\, j = 2, \ldots, 2N$ and $\frac{a_1}{2} \geq 0$.

Lemma 2.5. *For each $j = 1, \ldots, 2N$, the operator $\mathcal{K}_{a_j}$ with kernel*

$$\mathcal{K}_{a_j}(\xi, t) = \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\xi\left(a_j - \frac{a_1}{2}\right) + \frac{a_1 t}{2} - \frac{1}{4t} - a_j s + \frac{1}{4s}}}{(\xi - s)(t - s)} \left(\frac{s}{t}\right)^{\nu} \tag{2.11}$$

is trace-class. Moreover, the following decomposition holds $\mathcal{K}_{a_j} = \mathcal{A} \circ \mathcal{B}_{a_j}$, with

$$\begin{array}{c|c} \mathcal{A} : L^2(\hat{\gamma}) \to L^2(\gamma) & \mathcal{B}_{a_j} : L^2(\gamma) \to L^2(\hat{\gamma}) \\ h(s) \mapsto t^{-\nu} e^{\frac{a_1 t}{2} - \frac{1}{4t}} \int_{\hat{\gamma}} \frac{h(s)}{t-s} \frac{\mathrm{d}s}{2\pi i} & f(t) \mapsto s^{\nu} e^{-a_j s + \frac{1}{4s}} \int_{\gamma} \frac{e^{t\left(a_j - \frac{a_1}{2}\right)}}{t-s} f(t) \frac{\mathrm{d}t}{2\pi i} \end{array} \tag{2.12}$$

$\mathcal{A}$ and $\mathcal{B}_{a_j}$ are trace-class operators as well.

Proof. It is easy to verify that $\mathcal{A}$ and $\mathcal{B}_{a_j}$ are Hilbert-Schmidt and that their composition gives $\mathcal{K}_{a_j}$.

Moreover, we have the following decomposition of kernels. Introducing an additional contour $i\mathbb{R}+\delta$ not intersecting either of $\gamma, \hat{\gamma}$, we have $\mathcal{A} = \mathcal{P}_2 \circ \mathcal{P}_1$ with

$$
\begin{array}{r|l}
\mathcal{P}_1 : L^2(\hat{\gamma}) \to L^2(i\mathbb{R}+\delta) & \mathcal{P}_2 : L^2(i\mathbb{R}+\delta) \to L^2(\gamma) \\
\mathcal{P}_1[f](u) = \displaystyle\int_{\hat{\gamma}} \frac{f(s)}{u-s}\frac{\mathrm{d}s}{2\pi i} & \mathcal{P}_2[h](t) = \dfrac{e^{\frac{a_1 t}{2}-\frac{1}{4t}}}{t^{\nu}} \displaystyle\int_{i\mathbb{R}+\delta} \frac{h(u)}{t-u}\frac{\mathrm{d}u}{2\pi i}
\end{array}
$$

Analogously, $\mathcal{B}_{a_j} = \mathcal{O}_{2,j} \circ \mathcal{O}_{1,j}$ with

$$
\begin{array}{r|l}
\mathcal{O}_{1,j} : L^2(\gamma) \to L^2(i\mathbb{R}+\delta) & \mathcal{O}_{2,j} : L^2(i\mathbb{R}+\delta) \to L^2(\hat{\gamma}) \\
\mathcal{O}_{1,j}[f](w) = \displaystyle\int_{\gamma} \frac{e^{t\left(a_j-\frac{a_1}{2}\right)}f(t)}{t-w}\frac{\mathrm{d}t}{2\pi i} & \mathcal{O}_{2,j}[h](s) = s^{\nu}e^{-a_j s+\frac{1}{4s}} \displaystyle\int_{i\mathbb{R}+\delta} \frac{h(w)}{w-s}\frac{\mathrm{d}s}{2\pi i}
\end{array}
$$

It is straightforward to check that $\mathcal{P}_i$ and $\mathcal{O}_{i,j}$ are Hilbert-Schmidt operators, $i = 1,2$ and $j = 1, \ldots, 2N$. Therefore, $\mathcal{A}$ and $\mathcal{B}_{a_j}$ are trace-class. □

Remark 2.6. *The kernel $\mathcal{A}$ does not depend on the set of parameters $\{a_j\}_2^{2N}$, but only on the first endpoint a_1.*

Before proceeding further, we notice that any operator acting on the Hilbert space $H := L^2(\gamma \sqcup \hat{\gamma}) \simeq L^2(\gamma) \oplus L^2(\hat{\gamma}) = H_1 \oplus H_2$ can be written as a 2×2 matrix of operators with (i,j)-entry given by an operator $H_j \to H_i$.

According to such split and using matrix notation, we can thus write $\det(\mathrm{Id} - \mathcal{K}_B)$ as

$$
\begin{aligned}
&\det\left(\mathrm{Id}_{L^2(\gamma)} - \sum_{j=1}^{2N}(-1)^j \mathcal{A}\circ\mathcal{B}_{a_j}\right) \\
&= \det\left(\mathrm{Id}_{L^2(\gamma)} \otimes \mathrm{Id}_{L^2(\hat{\gamma})} - \left[\begin{array}{c|c} 0 & \mathcal{A} \\ \hline \sum_{j=1}^{2N}(-1)^j\mathcal{B}_{a_j} & 0 \end{array}\right]\right) = \det(\mathrm{Id}_{L^2(\gamma\cup\hat{\gamma})} - \mathbb{B})
\end{aligned}
\tag{2.13}
$$

The first identity comes from multiplying the right hand side on the left by the following matrix (with determinant equal 1)

$$
\mathrm{Id}_{L^2(\gamma)\oplus L^2(\hat{\gamma})} + \left[\begin{array}{c|c} 0 & -\mathcal{A} \\ \hline 0 & 0 \end{array}\right].
$$

□

The Riemann-Hilbert problem for the Bessel process. Thanks to Theorem 2.3 we can relate the computation of the Fredholm determinant of the Bessel operator to the theory of isomonodromic equations. We start by setting up a suitable Riemann-Hilbert problem which is naturally related to the Fredholm determinant of the operator $\mathbb{B}$.

Proposition 2.7. *Given the integrable kernel (2.6a)-(2.6c), the associated Riemann-Hilbert problem is the following:*

$$\begin{cases} \Gamma_+(\lambda) = \Gamma_-(\lambda)\,(I - J(\lambda)) & \lambda \in \Sigma := \gamma \sqcup \hat{\gamma} \\ \Gamma(\lambda) = I + \mathcal{O}\left(\frac{1}{\lambda}\right) & \lambda \to \infty \end{cases} \tag{2.14}$$

where Γ *is a* $(2N+1)\times(2N+1)$ *matrix such that it is analytic on* $\mathbb{C}\backslash\Sigma$, *bounded near* $\lambda = 0$ *and satisfies the jump conditions above with*

$$J(\lambda) := \begin{bmatrix} 0 & e^{\theta_1} & e^{\theta_2} & \dots & e^{\theta_{2N}} \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \chi_\gamma(\lambda) + \begin{bmatrix} 0 & 0 & \dots & 0 \\ e^{-\theta_1} & 0 & \dots & 0 \\ -e^{-\theta_2} & 0 & \dots & 0 \\ \vdots & & & \vdots \\ (-1)^{2N+1}e^{-\theta_{2N}} & 0 & \dots & 0 \end{bmatrix} \chi_{\hat{\gamma}}(\lambda) \tag{2.15}$$

$\theta_j := a_j\lambda - \frac{1}{4\lambda} - \nu\ln\lambda$, $\forall\, j = 1, \dots, 2N$.

Proof. It is straightforward to verify that $I - J(\lambda) = I - \vec{f}(\lambda)\cdot\vec{g}(\lambda)^T$. □

Theorem 2.8. *The Tracy-Widom distribution of the Bessel process, i.e. the Fredholm determinant* $\det(\mathrm{Id} - K_B\chi_I)$, *is equal to the isomonodromic* τ-*function related to the Riemann-Hilbert problem defined in Proposition 2.7. In particular,* $\forall\, j = 1, \dots, 2N$

$$\partial_{a_j} \ln\det\left(\mathrm{Id} - K_B\chi_I\right) = \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_{\partial_{a_j}}(\lambda)\right)\frac{\mathrm{d}\lambda}{2\pi i} \tag{2.16a}$$

$$\Xi_\partial(\lambda) := -\partial J(\lambda)\cdot(I - J(\lambda))^{-1} \tag{2.16b}$$

where $I = [a_1, a_2] \cup \dots [a_{2N-1}, a_{2N}]$ *is a collection of finite intervals and* $\Sigma = \gamma \sqcup \hat{\gamma}$; *we denote by* $'$ *the derivative with respect to the spectral parameter* λ.

Proof. Looking at the formula (1.12), we just need to verify that $H(M) \equiv 0$, with $M(\lambda) := I - J(\lambda)$. □

Starting from Theorem 2.8, it it possible to derive more explicit differential identities by using the Jimbo-Miwa-Ueno residue formula. First, we notice that the jump matrix $J(\lambda)$ can be written as

$$J(\lambda, \vec{a}) = e^{T(\lambda,\vec{a})} \cdot J_0 \cdot e^{-T(\lambda,\vec{a})} \tag{2.17}$$

where J_0 is a constant matrix, consisting only on 0 and ± 1, and

$$\begin{gathered} T(\lambda, \vec{a}) = \operatorname{diag}(T_0, T_1, \dots, T_{2N}) \\ T_0 = \frac{1}{2N+1} \sum_{j=1}^{2N} \theta_j \qquad T_j = T_0 - \theta_j \end{gathered} \tag{2.18}$$

Therefore, the matrix $\Psi(\lambda, \vec{a}) := \Gamma(\lambda, \vec{a}) e^{T(\lambda,\vec{a})}$ solves a Riemann-Hilbert problem with constant jumps and it is (sectionally) a solution to a polynomial ODE.

Moreover, recalling the Jimbo-Miwa-Ueno residue formula and adapting it to the case at hand (compare with [5, Formula 3.26]; the equality was shown in [3]), the following equality holds

$$\int_\Sigma \operatorname{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_{\partial_{a_j}}(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} = -\operatorname*{res}_{\lambda=\infty} \operatorname{Tr}\left(\Gamma^{-1}(\lambda)\Gamma'(\lambda)\partial_{a_j} T(\lambda)\right). \tag{2.19}$$

In conclusion,

Proposition 2.9. *For all $j = 1, \dots, 2N$, the Fredholm determinant satisfies the following differential relation*

$$\partial_{a_j} \ln\det(\mathrm{Id} - K_B \chi_I) = -\Gamma_{1;j+1,j+1} \tag{2.20}$$

with $\Gamma_{1;j+1,j+1}$ the $(j+1, j+1)$ component of the residue matrix $\Gamma_1 = \lim_{\lambda\to\infty} \lambda(I - \Gamma(\lambda))$ and Γ is the solution to the Riemann-Hilbert problem (2.14).

Proof. (the proof follows the same guidelines as in [5, Proposition 3.2])

Given the definition of $T(\lambda)$,

$$\partial_{a_j} T(\lambda, \vec{a}) = \lambda\left(\frac{1}{N+1} I - E_{j+1,j+1}\right) \tag{2.21}$$

and plugging into (2.19), we have

$$\partial_{a_j} \ln\det(\mathrm{Id} - K_B\chi_I) = \frac{\operatorname{Tr}\Gamma_1}{N+1} - \Gamma_{1;j+1,j+1} = -\Gamma_{1;j+1,j+1} \tag{2.22}$$

since $\det\Gamma(\lambda) \equiv 1$, thus $\operatorname{Tr}\Gamma_1 = 0$. $\square$

`bessel1time.pdf`

Figure 2.2: The jump matrices for the Bessel-RHP in the single-time case.

2.1 The single interval case and Painlevé III equation

We consider now the case where the Bessel kernel is restricted to a single finite interval $[0, a]$.

We will see that from the 2×2 Bessel Riemann-Hilbert problem we will derive a suitable Lax pair which matches with the Lax pair of the Painlevé III transcendent, as shown in [11] (the Lax pair described there is slightly different from the one found in our present work, but it can be shown that the two formulations are equivalent).

In order to make the connection with the Painlevé transcendent more explicit, we will work on a rescaled version of the Bessel kernel, which can be easily derived from our original definition (2.2) through suitable scalings.

By specializing the results of the previous section, we get a (Fourier transformed) Bessel operator on $L^2(\gamma)$ with the following kernel

$$\mathcal{K}_B(\xi, t) = \int_{\hat{\gamma}} \frac{\mathrm{d}s}{2\pi i} \frac{e^{\frac{\xi x}{4} + \frac{x}{2}\left(\frac{t}{2} - \frac{1}{t}\right) - \frac{x}{2}\left(s - \frac{1}{s}\right)}}{(\xi - s)(t - s)} \left(\frac{s}{t}\right)^{\nu} \tag{2.23}$$

where $x := \sqrt{a}$.

It can be easily shown that $\mathcal{K}_B$ is a trace-class operator, since product of two Hilbert-Schmidt operators $\mathcal{K}_B = \mathcal{A}_2 \circ \mathcal{A}_1$ with kernels

$$\mathcal{A}_1(t, s) = \frac{1}{2\pi i} \frac{\exp\left\{\frac{tx}{4} - \frac{x}{2}\left(s - \frac{1}{s}\right)\right\}}{t - s} s^{\nu} \cdot \chi_{\gamma}(t)\chi_{\hat{\gamma}}(s) \tag{2.24}$$

$$\mathcal{A}_2(s, t) = -\frac{1}{2\pi i} \frac{\exp\left\{\frac{x}{2}\left(\frac{s}{2} - \frac{1}{s}\right)\right\}}{t - s} s^{-\nu} \cdot \chi_{\gamma}(s)\chi_{\hat{\gamma}}(t). \tag{2.25}$$

Proposition 2.10. *The operators $\mathcal{A}_j$, $j = 1, 2$, are trace-class.*

Proof. The proof follows the same arguments as the proof of Lemma 2.5. $\square$

Theorem 2.11. *Consider the interval $[0, a]$, then the following identity holds*

$$\det\left(\mathrm{Id} - K_B \chi_{[0,a]}\right) = \det\left(\mathrm{Id} - \mathbb{B}\right) \tag{2.26}$$

with $\mathbb{B}$ a trace-class integrable operator with kernel defined as follows

$$\begin{aligned}\mathbb{B}(t,s) &= \frac{1}{2\pi i} \frac{e^{\frac{tx}{4} - \frac{x}{2}\left(s - \frac{1}{s}\right)} s^{\nu} \cdot \chi_{\gamma}(t) \chi_{\hat{\gamma}}(s) - e^{\frac{x}{2}\left(\frac{s}{2} - \frac{1}{s}\right)} s^{-\nu} \cdot \chi_{\gamma}(s) \chi_{\hat{\gamma}}(t)}{t - s} \\ &= \frac{\vec{f}(t)^T \cdot \vec{g}(s)}{t - s}\end{aligned} \tag{2.27a}$$

with

$$\vec{f}(t) = \frac{1}{2\pi i} \begin{bmatrix} e^{\frac{tx}{4}} \\ 0 \end{bmatrix} \chi_{\gamma}(t) + \frac{1}{2\pi i} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \chi_{\hat{\gamma}}(t) \tag{2.27b}$$

$$\vec{g}(s) = \begin{bmatrix} e^{\frac{x}{2}\left(-s + \frac{1}{s}\right)} s^{\nu} \\ 0 \end{bmatrix} \chi_{\hat{\gamma}}(s) + \begin{bmatrix} 0 \\ -e^{\frac{x}{2}\left(\frac{s}{2} - \frac{1}{s}\right)} s^{-\nu} \end{bmatrix} \chi_{\gamma}(s) \tag{2.27c}$$

The associated 2×2 Riemann-Hilbert problem has jump matrix $M(\lambda) := I - J(\lambda)$ on $\Sigma := \gamma \cup \hat{\gamma}$ with

$$J(\lambda) = \begin{bmatrix} 0 & -e^{\frac{x}{2}\left(\lambda - \frac{1}{\lambda}\right)} \lambda^{-\nu} \\ 0 & 0 \end{bmatrix} \chi_{\gamma}(\lambda) + \begin{bmatrix} 0 & 0 \\ e^{\frac{x}{2}\left(-\lambda + \frac{1}{\lambda}\right)} \lambda^{\nu} & 0 \end{bmatrix} \chi_{\hat{\gamma}}(\lambda) \tag{2.28}$$

and the solution Γ to the RHP is bounded near the origin when $x = 0$. See Figure 2.2 for a sketch of the jumps.

It is easy to see that

$$\begin{aligned} M(\lambda) &= e^{T(\lambda)} M_0 e^{-T(\lambda)} \\ \text{with } \ T(\lambda) &:= \frac{\theta_x}{2} \sigma_3, \quad \theta_x := \frac{x}{2}\left(\lambda - \frac{1}{\lambda}\right) - \nu \ln \lambda \end{aligned} \tag{2.29}$$

where M_0 is a constant matrix. Thus, the matrix $\Psi(\lambda) := \Gamma(\lambda) e^{T_x(\lambda)}$ solves a Riemann-Hilbert problem with constant jumps and it is (sectionally) a solution to a polynomial ODE.

Applying again Theorem 2.8 and Jimbo-Miwa-Ueno residue formula, we get

$$\begin{aligned} \partial_x \ln \det(\mathrm{Id} - \mathbb{B}) &= \int_{\Sigma} \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda) \Gamma_-'(\lambda) \Xi_x(\lambda)\right) \frac{\mathrm{d}\lambda}{2\pi i} \\ &= -\operatorname*{res}_{\lambda=\infty} \mathrm{Tr}\left(\Gamma^{-1} \Gamma' \partial_x T\right) + \operatorname*{res}_{\lambda=0} \mathrm{Tr}\left(\Gamma^{-1} \Gamma' \partial_x T\right). \end{aligned} \tag{2.30}$$

Proposition 2.12. *The Fredholm determinant of the single-interval Bessel operator satisfies the following identity*

$$\partial_x \ln\det(\mathrm{Id} - \mathbb{B}) = -\frac{1}{2}\Gamma_{1;22} + \frac{1}{2}\tilde{\Gamma}_{1;2,2} \tag{2.31}$$

where $\Gamma_{1;22}$ *is the* $(2,2)$*-entry of the residue matrix at infinity, while* $\tilde{\Gamma}_{1;2,2}$ *is the* $(2,2)$*-entry of residue matrix at zero.*

Proof. As in the proof of Proposition 2.9, we can easily get the result by calculating the derivative of the conjugation matrix

$$\partial_x T(\lambda) = \frac{1}{2}\left(\lambda - \frac{1}{\lambda}\right)\left(\frac{1}{2}I - E_{2,2}\right) \tag{2.32}$$

and by keeping into account that, since $\det\Gamma(\lambda) \equiv 1$, $\operatorname{Tr}\Gamma_1 = \operatorname{Tr}\tilde{\Gamma}_1 = 0$. $\square$

From the asymptotic behaviour at infinity of the matrix Ψ, we can calculate the Lax pair associated to our Riemann-Hilbert problem.

$$A := \partial_\lambda \Psi \cdot \Psi^{-1}(\lambda) = A_0 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} \tag{2.33}$$

$$B := \partial_x \Psi \cdot \Psi^{-1}(\lambda) = B_0 + \lambda B_1 + \frac{B_{-1}}{\lambda} \tag{2.34}$$

with coefficients

$$\begin{aligned}
A_0 &= \frac{x}{4}\sigma_3 \\
A_{-1} &= \frac{x}{4}[\Gamma_1, \sigma_3] - \frac{\nu}{2}\sigma_3 \\
A_{-2} &= \frac{x}{4}[\Gamma_2, \sigma_3] + \frac{x}{4}[\sigma_3\Gamma_1, \Gamma_1] - \frac{\nu}{2}[\Gamma_1, \sigma_3] + \frac{x}{4}\sigma_3 - \Gamma_1 \\
B_1 &= \frac{1}{4}\sigma_3 \\
B_0 &= \frac{1}{4}[\Gamma_1, \sigma_3] \\
B_{-1} &= \frac{1}{4}\left([\Gamma_2, \sigma_3] + [\sigma_3\Gamma_1, \Gamma_1] - \sigma_3 + 4\frac{\mathrm{d}\Gamma_1}{\mathrm{d}x}\right)
\end{aligned} \tag{2.35}$$

The form of the coefficients matches with the results in [11, Chapter 5, Section 3, Formulæ (5.3.32) and (5.3.34)]. In particular, using the same notation as in [11, Chapter 5, Section 3, Formulæ (5.3.7) and (5.3.8)], we have

$$A_0 = \begin{bmatrix} \frac{x}{4} & 0 \\ 0 & -\frac{x}{4} \end{bmatrix}, \quad A_{-1} = \begin{bmatrix} -\frac{\nu}{2} & Y(x) \\ V(x) & \frac{\nu}{2} \end{bmatrix},$$

$$B_1 = \begin{bmatrix} \frac{1}{4} & 0 \\ 0 & -\frac{1}{4} \end{bmatrix}, \quad B_0 = \begin{bmatrix} 0 & \frac{Y(x)}{x} \\ \frac{V(x)}{x} & 0 \end{bmatrix}, \tag{2.36}$$

$$A_{-2} = \begin{bmatrix} \frac{x}{4} - U(x) & -W(x)U(x) \\ \frac{U(x)-\frac{x}{2}}{W(x)} & -\frac{x}{4} + U(x) \end{bmatrix}, \qquad B_{-1} = -\frac{1}{x} A_{-2}. \tag{2.37}$$

Calculating the compatibility equation, we get the following system of ODEs

$$\frac{\mathrm{d}U}{\mathrm{d}x} = -2\frac{WUV}{x} - 2\frac{UY}{xW} + \frac{U}{x} + \frac{Y}{W} \tag{2.38a}$$

$$\frac{\mathrm{d}V}{\mathrm{d}x} = -\frac{\nu V}{x} - \frac{U}{W} + \frac{x}{2W} \tag{2.38b}$$

$$\frac{\mathrm{d}W}{\mathrm{d}x} = 2\frac{VW^2}{x} - \frac{\nu W}{x} - 2\frac{Y}{x} \tag{2.38c}$$

$$\frac{\mathrm{d}Y}{\mathrm{d}x} = -WU + \frac{\nu Y}{x} \tag{2.38d}$$

and the constant

$$\frac{\Theta_0}{4} := -\frac{\nu}{4} + \frac{UY}{xW} - \frac{WUV}{x} - \frac{Y}{2W} + \frac{\nu U}{x} \tag{2.38e}$$

which can be proven to be the monodromy exponent at 0 and equal to $-\nu$.

Setting now

$$F(x) := \frac{Y(x)}{W(x)U(x)} \tag{2.39}$$

and substituting in the equations above, we get that F and L satisfy

$$x\frac{\mathrm{d}F}{\mathrm{d}x} = (4U - x)F^2 + (2\nu - 1)F - x \tag{2.40}$$

$$x\frac{\mathrm{d}U}{\mathrm{d}x} = -4FU^2 + 2xUF - (2\nu - 1)U \tag{2.41}$$

Remark 2.13. *The latter equation for the function U is a Bernoulli 1st-order ODE with $n = 2$.*

Remark 2.14 (Behaviour as $x \to 0_+$). *To inspect the behaviour of the functions L, F as $x \to 0_+$, consider the matrix*

$$Y(\lambda, x) := x^{-\frac{\nu}{2}\sigma_3}\Gamma(\lambda, x)x^{\frac{\nu}{2}\sigma_3} \tag{2.42}$$

which solves a similar RHP with jumps on contours like in Fig. 2.2 but where the off diagonal terms are multiplied by a factor $x^{\pm\nu}$.

For the sake of simplicity we consider only the case $\nu > 0$. Given $\mathbb{D}$ a disk containing $\hat{\gamma}$ and entirely contained in γ, define the following matrix

$$\Phi(\lambda, x) := \begin{cases} \left(I + \frac{C_1}{\lambda} + \frac{C_2}{\lambda^2}\right)(I + A(\lambda, x)\sigma_+) =: \Phi_\infty & \lambda \in \mathbb{C}\backslash\mathbb{D} \\ \Phi_\infty \left[I + \left(B(\lambda, x) - \frac{x^\nu J_{\nu+1}(x)}{\lambda} - \frac{x^\nu J_{\nu+2}(x)}{\lambda^2}\right)\sigma_-\right] & \lambda \in \mathbb{D} \end{cases} \tag{2.43}$$

where

$$A(\lambda,x)=x^{-\nu}\int_\gamma \frac{s^{-\nu}e^{\frac{x}{2}\left(s-\frac{1}{s}\right)}\mathrm{d}s}{(s-\lambda)2\pi i},\qquad B(\lambda,x)=x^{\nu}\int_{\hat\gamma} \frac{s^{\nu}e^{-\frac{x}{2}\left(s-\frac{1}{s}\right)}\mathrm{d}s}{(s-\lambda)2\pi i},\tag{2.44}$$

$\sigma_\pm$ is a 2×2 matrix where the only non-zero entry is the upper (respectively, lower) diagonal entry equal to 1 and the matrices C_1, C_2 can be explicitly computed by inspecting the behaviour of Φ inside the disk. Such matrix has the same jumps on γ and $\hat\gamma$ as for Y and it displays an extra jump on the circle $\partial\mathbb{D}$ (counterclockwise oriented). The "error" matrix $\mathcal{E} := Y\Phi^{-1}$ has only a jump on $\partial\mathbb{D}$ by construction:

$$\mathcal{E}_+=\mathcal{E}_-\left(\Phi_-\Phi_+^{-1}\right)\quad \lambda\in\partial\mathbb{D},\qquad \mathcal{E}=I+\mathcal{O}\left(\frac{1}{\lambda}\right)\quad \lambda\to\infty.\tag{2.45}$$

It is possible to show that the surviving jump on $\partial\mathbb{D}$ is a perturbation of the identity of the order $|x|^{2\nu+3}$. Therefore, thanks to a small norm argument ([15]), the matrix Φ can be considered as an explicit approximant of the matrix Y (and of the original matrix Γ). Inspecting its behaviour at ∞, it is possible to recover the functions Y, V, U, W and $F := \frac{Y}{WU}$ appearing in the Lax pair; in particular,

$$U(x)=C_\nu x^{2\nu+1}+\mathcal{O}\left(x^{2\nu+2}\right),\qquad F(x)=\frac{-2\nu}{x}+\mathcal{O}\left(1\right),\tag{2.46}$$

where C_ν is a constant depending on the parameter ν.

For $-1<\nu\le 0$ the argument is similar, but one needs to be more careful with the asymptotic expansion and the rate of convergence.

Moreover, differentiating (2.40) and using (2.41), we get the following Painlevé III equation:

$$\frac{\mathrm{d}^2F}{\mathrm{d}x^2}=\frac{1}{F}\left(\frac{\mathrm{d}F}{\mathrm{d}x}\right)^2-\frac{1}{x}\frac{\mathrm{d}F}{\mathrm{d}x}+\frac{2}{x}\left(\Theta_0F^2+\nu-1\right)+F^3-\frac{1}{F}\tag{2.47}$$

Given the expression of the matrix A, we can find an expression for the residue matrix Γ_1. Moreover, focusing on the residue at 0, we can perform similar calculation with the already known Lax pair (2.33)-(2.34). We finally get

$$\begin{aligned}&\operatorname*{res}_{\lambda=0}\mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial_xT\right)=-\operatorname*{res}_{\lambda=\infty}\mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial_xT\right)\\&=\frac{1}{2x}\left[-2U^2F^2+\left(xF^2-2\nu F+x\right)U-\frac{x^2}{4}\right]\end{aligned}\tag{2.48}$$

In conclusion,

Theorem 2.15. *The Tracy-Widom distribution of the Bessel process restricted to a single interval satisfies the following identity*

$$\det(\mathrm{Id} - \mathbb{B}) = \exp\left\{\int_0^x H_{\mathrm{III}}(s)\,\mathrm{d}s\right\} \tag{2.49}$$

where H_{III} is the Hamiltonian associated to the Painlevé III equation (see [17, Appendix C])

$$H_{\mathrm{III}}(F,U;x) = \frac{1}{2x}\left[-2U^2F^2 + \left(xF^2 - 2\nu F + x\right)U - \frac{x^2}{4}\right]. \tag{2.50}$$

Remark 2.16. *The Hamiltonian H_{III} is singular at 0, but it is integrable in a (right) neighborhood of the origin: $H_{\mathrm{III}} \in L^1(0,\epsilon)$, $\epsilon > 0$. Given that $\partial \ln \tau = H_{\mathrm{III}}$ and τ is continuous, this yields $\tau(0) = \det(\mathrm{Id}) = 1$, as expected.*

Remark 2.17. *We recall that the above results found for the single-time Bessel operator are not completely new, but they appeared in earlier works. In particular, we refer to the article by Tracy and Widom [35] for the connection between the Fredholm determinant of the Bessel kernel and the Painlevé III equation, and to the article by Jimbo, Miwa and Ueno [17, Appendix C] for the expression of the Hamiltonian that characterizes the τ-function.*

3 Gap probabilities for the multi-time Bessel process

The multi-time Bessel process on $L^2(\mathbb{R}_+)$ with times $\tau_1 < \ldots < \tau_n$ is governed by the matrix operator $[K_B] := [\tilde{K}_B] + [H_B]$ with kernels $[K_B]$, $[\tilde{K}_B]$ and $[H_B]$ given as follows

$$K_{B;ij}(x,y) := \tilde{K}_{B;ij}(x,y) + H_{B;ij}(x,y) \tag{3.1a}$$

$$\tilde{K}_{B;ij}(x,y) := \frac{1}{(2\pi i)^2}\left(\frac{y}{x}\right)^{\frac{\nu}{2}} \iint_{\gamma\times\hat{\gamma}_j} \frac{\mathrm{d}t\,\mathrm{d}s}{ts}\,\frac{e^{\Delta_{ij}+xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{\frac{1}{4t}-\frac{1}{4s}-\Delta_{ij}}\left(\frac{s}{t}\right)^{\nu} \tag{3.1b}$$

$$H_{B;ij}(x,y) := \chi_{\tau_i<\tau_j}\frac{1}{\Delta_{ji}}\left(\frac{y}{x}\right)^{\frac{\nu}{2}}\int_{\gamma} e^{\frac{x}{4\Delta_{ji}}(t-1)+\frac{y}{4\Delta_{ji}}\left(\frac{1}{t}-1\right)} t^{-\nu-1}\frac{\mathrm{d}t}{2\pi i} \tag{3.1c}$$

with the same curve γ as in the single-time Bessel kernel (a contour that winds around zero counterclockwise and extends to $-\infty$), $\hat{\gamma}_j := \frac{1}{\gamma+4\tau_j}$, $\forall\, i,j = 1,\ldots,n$, $\Delta_{ij} := \tau_i - \tau_j$; also in this case the contours γ and $\{\hat{\gamma}_j\}_j$ are all disjoint.

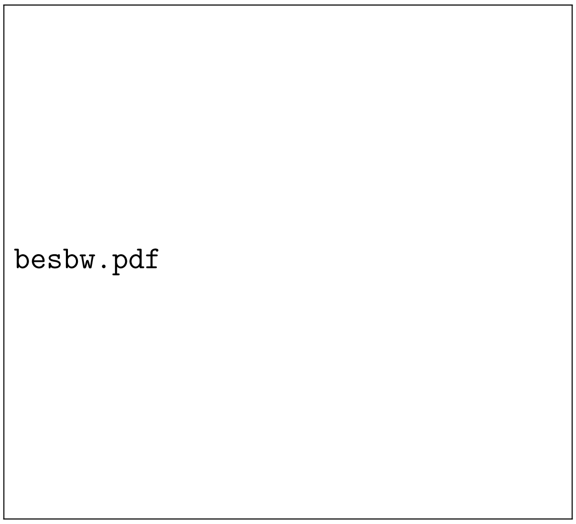

Figure 2.3: Numerical computation of the Fredholm determinant $\det(\mathrm{Id} - K_B \chi_{[0,a]})$ as function of a, in the case $\nu = -.5$ (solid line), $\nu = 0$ (dotted line) and $\nu = .5$ (dashed line). The outcome has been obtained by directly calculating the Fredholm determinant of the Bessel operator (following the ideas of [6]) and not by integrating the Hamiltonian H_{III} given in (2.50).

Remark 3.1. *The matrix with entries $H_{B;ij}$ $(i, j = 1, \ldots, n)$ is strictly upper triangular, by construction.*

Remark 3.2. *The integral expression (3.1a)-(3.1c) for the multi-time Bessel kernel is equivalent to the one given in the introduction (1.8) (see [23] and [36]). To prove the equivalency, one simply needs to write the Bessel functions as contour integrals and perform some suitable integrations by parts.*

As in the single-time case, we are interested in the following quantity

$$\det\left(\mathrm{Id} - [K_B]\, \chi_{\mathcal{I}}\right) \tag{3.2}$$

which is equal to the gap probability of the multi-time Bessel kernel restricted to a collection of multi-intervals $\mathcal{I} = \{I_1, \ldots, I_n\}$,

$$I_j := [a_1^{(j)}, a_2^{(j)}] \cup \ldots \cup [a_{2k_j-1}^{(j)}, a_{2k_j}^{(j)}], \qquad 0 \leq a_1^{(j)} < \ldots < a_{2k_j}^{(j)}.$$

Remark 3.3. *The multi-time Bessel operator fails to be trace-class on infinite intervals.*

For the sake of clarity, we will focus on the simple case $I_j = [0, a^{(j)}]$, $j = 1, \ldots, n$. The general case follows the same guidelines described below, but it involves calculations with complicated notation which we prefer to avoid.

Theorem 3.4. *The following identity between Fredholm determinants holds*

$$\det\left(\mathrm{Id} - [K_B]\,\chi_{\mathcal{I}}\right) = \det\left(\mathrm{Id} - \mathbb{K}_B\right) \tag{3.3}$$

with $\chi_{\mathcal{I}} = \mathrm{diag}\,(\chi_{I_1}, \ldots, \chi_{I_n})$ *the characteristic matrix of the collection of multi-intervals. The operator* $\mathbb{K}_B$ *is an integrable operator with a* $2n \times 2n$ *matrix kernel of the form*

$$\mathbb{K}_B(t,\xi) = \frac{\boldsymbol{f}(t)^T \cdot \boldsymbol{g}(\xi)}{t-\xi} \tag{3.4}$$

acting on the Hilbert space

$$H := L^2\left(\gamma \cup \bigcup_{k=1}^{n} \gamma_{-k}, \mathbb{C}^n\right) \sim L^2\left(\bigcup_{k=1}^{n} \gamma_{-k}, \mathbb{C}^n\right) \oplus L^2(\gamma, \mathbb{C}^n), \tag{3.5}$$

with $\gamma_{-k} := \frac{1}{\gamma} - 4\tau_k$.

The functions $\boldsymbol{f}$, $\boldsymbol{g}$ *are the following* $2n \times 2n$ *matrices*

$$\boldsymbol{f}(t) = \left[\begin{array}{c|c} \mathrm{diag}\,\mathcal{N}(t) & 0 \\ \hline 0 & A(\mathcal{M}(t)) \\ \hline 0 & B(\mathcal{H}(t)) \end{array}\right] \tag{3.6}$$

$$\boldsymbol{g}(\xi) = \left[\begin{array}{c|c} 0 & \mathrm{diag}\,\mathcal{N}(\xi) \\ \hline C(\mathcal{M}(\xi)) & 0 \\ \hline 0 & D(\mathcal{H}(\xi)) \end{array}\right] \tag{3.7}$$

where $\mathrm{diag}\,\mathcal{N}$ *is a* $n \times n$ *matrix,* A *and* C *are two rows with* n *entries and*

B and D are $(n-1)\times n$ matrices,

$$\operatorname{diag}\mathcal{N}(t) := \operatorname{diag}\left(-4e^{-\frac{a^{(1)}}{t_1}}\chi_\gamma(t),\ldots,-4e^{-\frac{a^{(n)}}{t_n}}\chi_\gamma(t)\right)$$

$$A(\mathcal{M}(t)) := \left[e^{-\frac{t}{4}}t_1^\nu\chi_{\gamma-1}(t),\ldots,e^{-\frac{t}{4}}t_n^\nu\chi_{\gamma-n}(t)\right]$$

$$B(\mathcal{H}(t)) := \begin{bmatrix} -4e^{-\frac{a^{(2)}}{t_2}}\frac{t_1^\nu}{t_2^\nu}\chi_{\gamma-1} & 0 & & & \\ -4e^{-\frac{a^{(3)}}{t_3}}\frac{t_1^\nu}{t_3^\nu}\chi_{\gamma-1} & -4e^{-\frac{a^{(3)}}{t_3}}\frac{t_2^\nu}{t_3^\nu}\chi_{\gamma-2} & & & \\ \vdots & \vdots & \ddots & & \\ -4e^{-\frac{a^{(n)}}{t_n}}\frac{t_1^\nu}{t_n^\nu}\chi_{\gamma-1} & -4e^{-\frac{a^{(n)}}{t_n}}\frac{t_2^\nu}{t_n^\nu}\chi_{\gamma-2} & & -4e^{-\frac{a^{(n)}}{t_n}}\frac{t_{n-1}^\nu}{t_n^\nu}\chi_{\gamma-(n-1)} & 0 \end{bmatrix}$$

$$\operatorname{diag}\mathcal{N}(\xi) := \operatorname{diag}\left(e^{\frac{a^{(1)}}{\xi_1}}\chi_{\gamma-1}(\xi),\ldots,e^{\frac{a^{(n)}}{\xi_n}}\chi_{\gamma-n}(\xi)\right)$$

$$C(\mathcal{M}(\xi) := \left[e^{\frac{\xi}{4}}\xi_1^{-\nu}\chi_\gamma(\xi),\ldots,e^{\frac{\xi}{4}}\xi_n^{-\nu}\chi_\gamma(\xi)\right]$$

$$D(\mathcal{H}(\xi)) = \begin{bmatrix} 0 & e^{\frac{a^{(2)}}{\xi_2}}\chi_{\gamma-2}(\xi) & & & & \\ & 0 & e^{\frac{a^{(3)}}{\xi_3}}\chi_{\gamma-3}(\xi) & & & \\ & & 0 & e^{\frac{a^{(4)}}{\xi_4}}\chi_{\gamma-4}(\xi) & & \\ & & & & \ddots & \\ & & & & 0 & e^{\frac{a^{(n)}}{\xi_n}}\chi_{\gamma-n}(\xi) \end{bmatrix} \blacksquare$$

with $\xi_k := \xi + 4\tau_k$, $t_k := t + 4\tau_k$, for $k = 1,\ldots,n$.

Remark 3.5. *The naming of Fredholm determinant in the theorem above needs some clarification: by "det" we denote the determinant defined through the Fredholm expansion*

$$\det(\mathrm{Id}-K) := 1 + \sum_{k=1}^{\infty}\frac{1}{k!}\int_{X^k}\det[K(x_i,x_j)]_{i,j=1}^k \mathrm{d}\mu(x_1)\ldots\mathrm{d}\mu(x_k) \qquad (3.8)$$

with K an integral operator acting on the Hilbert space $L^2(X,\mathrm{d}\mu(x))$, with kernel $K(x,y)$.

In our case, the operator $[K_B]\chi_{\mathcal{I}} = ([\tilde{K}_B]+[H_B])\chi_{\mathcal{I}}$ is actually the sum of a trace-class operator $[\tilde{K}_B]\chi_{\mathcal{I}}$ and a Hilbert-Schmidt operator $[H_B]\chi_{\mathcal{I}}$ whose kernel is diagonal-free, as it will be clear along the proof.

Thus, to be precise, we have the following chain of identities

$$\begin{aligned} \text{"}\det\text{"}(\mathrm{Id}-[K_B]\chi_{\mathcal{I}}) &= \text{"}\det\text{"}(\mathrm{Id}-[\tilde{K}_B]\chi_{\mathcal{I}} - [H_B]\chi_{\mathcal{I}}) \\ &= e^{\mathrm{Tr}[\tilde{K}_B]}\det{}_2(\mathrm{Id}-[\tilde{K}_B]\chi_{\mathcal{I}} - [H_B]\chi_{\mathcal{I}}) \qquad (3.9) \end{aligned}$$

where $\det_2$ *denotes the regularized Carleman determinant (see [30] for a detailed description of the theory).*

Proof. Thanks to the invariance of the Fredholm determinant under kernel conjugation, we can discard the term $\left(\frac{y}{x}\right)^{\nu/2}$ in our further calculations.

We will work on the entry (i,j) of the kernel. We can notice that for $x<0$ or $y<0$ the kernel is identically zero, $K_B(x,y)\equiv 0$, as in the single-time case. Then, applying Cauchy's theorem and after some suitable calculations, we have

$$
\begin{aligned}
&\chi_{[0,a^{(j)}]}(y)\cdot \tilde{K}_{B;ij}(x,y)\\
&=\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}e^{\xi(a^{(j)}-y)}\iint_{\gamma\times\hat{\gamma}_j}\frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2 ts}\frac{e^{\Delta_{ij}+xt-\frac{1}{4t}-a^{(j)}s+\frac{1}{4s}}}{(\xi-s)\left(\frac{1}{4t}-\frac{1}{4s}-\Delta_{ij}\right)}\left(\frac{s}{t}\right)^{\nu}\\
&=\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}e^{-\xi y}\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}t}{2\pi i}e^{xt}\int_{\hat{\gamma}_j}\frac{\mathrm{d}s}{2\pi i}\frac{e^{\Delta_{ij}+\xi a^{(j)}-\frac{1}{4t}-a^{(j)}s+\frac{1}{4s}}}{(\xi-s)\left(\frac{1}{4t}-\frac{1}{4s}-\Delta_{ij}\right)}\left(\frac{s}{t}\right)^{\nu}\frac{1}{ts}\\
&=\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}e^{-\xi y}\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}t}{2\pi i}e^{xt}\int_{\gamma}\frac{\mathrm{d}s}{2\pi i}\frac{4e^{\tau_i+\xi a^{(j)}-\frac{1}{4t}-\frac{a^{(j)}}{s+4\tau_j}+\frac{s}{4}}}{\left(\frac{1}{\xi}-4\tau_j-s\right)\left(\frac{1}{t}-4\tau_i-s\right)}\left(\frac{1}{(s+4\tau_j)t}\right)^{\nu}\frac{1}{t\xi}
\end{aligned}
\tag{3.10}
$$

where we deformed γ into a translated imaginary axis $i\mathbb{R}+\epsilon$ ($\epsilon>0$) in order to make Fourier transform operator more explicit; the last equality follows from the change of variable on s: $s\mapsto 1/(s+4\tau_j)$ (thus the contour $\hat{\gamma}_j$ becomes similar to γ and can be continuously deformed into that).

On the other hand

$$
\begin{aligned}
&\chi_{[0,a^{(j)}]}(y)\cdot H_{B;ij}(x,y)\\
&=\frac{-1}{\Delta_{ji}}\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}\frac{e^{\xi(a^{(j)}-y)}}{\xi-\frac{1}{4\Delta_{ji}}\left(1-\frac{1}{t}\right)}\int_{\gamma}e^{\frac{x}{4\Delta_{ji}}(t-1)-\frac{a^{(j)}}{4\Delta_{ji}}\left(1-\frac{1}{t}\right)}t^{-\nu-1}\frac{\mathrm{d}t}{2\pi i}\\
&=\frac{-1}{\Delta_{ji}}\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}e^{-\xi y}\int_{i\mathbb{R}+\epsilon}\frac{e^{\xi a^{(j)}+\frac{x}{4\Delta_{ji}}(t-1)-\frac{a^{(j)}}{4\Delta_{ji}}\left(1-\frac{1}{t}\right)}}{\xi-\frac{1}{4\Delta_{ji}}\left(1-\frac{1}{t}\right)}t^{-\nu-1}\frac{\mathrm{d}t}{2\pi i}\\
&=-4\int_{i\mathbb{R}+\epsilon}\frac{\mathrm{d}\xi}{2\pi i}e^{-\xi y}\int_{i\mathbb{R}+\epsilon}e^{xt}\frac{\mathrm{d}t}{2\pi i}\frac{e^{a^{(j)}\left(\xi-\frac{t}{4\Delta_{ji}t+1}\right)}}{t\xi\left(4\Delta_{ji}+\frac{1}{t}-\frac{1}{\xi}\right)}(4\Delta_{ji}t+1)^{-\nu}
\end{aligned}
\tag{3.11}
$$

It is easily recognizable the conjugation with a Fourier-like operator as in (2.8), so that

$$
([K_B]\chi_{\mathcal{I}})_{ij}=\mathcal{F}^{-1}\circ\left(\mathcal{B}_{ij}+\chi_{i<j}\mathcal{H}_{ij}\right)\circ\mathcal{F}
\tag{3.12}
$$

with

$$\mathcal{B}_{ij}(t,\xi) = \int_\gamma \frac{\mathrm{d}s}{2\pi i} \frac{4e^{\tau_i+\xi a^{(j)}-\frac{1}{4t}-\frac{a^{(j)}}{s+4\tau_j}+\frac{s}{4}}}{\left(\frac{1}{\xi}-4\tau_j-s\right)\left(\frac{1}{t}-4\tau_i-s\right)}\left(\frac{1}{(s+4\tau_j)t}\right)^\nu \frac{1}{t\xi} \tag{3.13}$$

$$\mathcal{H}_{ij}(t,\xi) := -4\frac{e^{a^{(j)}\left(\xi-\frac{t}{4\Delta_{ji}t+1}\right)}}{4\tau_j-4\tau_i+\frac{1}{t}-\frac{1}{\xi}}(4\Delta_{ji}t+1)^{-\nu}\frac{1}{\xi t} \tag{3.14}$$

Now we will perform a change of variables on the Fourier-transformed kernel $\mathcal{B}_{ij}+\chi_{i<j}\mathcal{H}_{ij}$: $\xi_j := \frac{1}{\xi}-4\tau_j$ and $\eta_i := \frac{1}{t}-4\tau_i$. This will lead to the following expression for the (Fourier-transformed) multi-time Bessel kernel

$$\mathcal{K}_{B;ij} = \mathcal{B}_{ij}(\eta,\xi)+\chi_{\tau_i<\tau_j}\mathcal{H}_{ij}(\eta,\xi) =$$

$$4\int_\gamma \frac{\mathrm{d}t}{2\pi i}\frac{e^{\frac{a^{(j)}}{\xi+4\tau_j}-\frac{\eta}{4}-\frac{a^{(j)}}{t+4\tau_j}+\frac{t}{4}}}{(\xi-t)(\eta-t)}\left(\frac{\eta+4\tau_i}{t+4\tau_j}\right)^\nu+\chi_{\tau_i<\tau_j}\cdot 4\frac{e^{\frac{a^{(j)}}{\xi+4\tau_j}-\frac{a^{(j)}}{\eta+4\tau_j}}}{\xi-\eta}\left(\frac{\eta+4\tau_j}{\eta+4\tau_i}\right)^{-\nu} \tag{3.15}$$

with $\xi\in\frac{1}{\gamma}-4\tau_j =: \gamma_{-j}$ and $\eta\in\frac{1}{\gamma}-4\tau_i =: \gamma_{-i}$, $\forall\, i,j=1,\dots,n$. Such operator is acting on the Hilbert space $L^2\left(\bigcup_{k=1}^n\gamma_{-k},\mathbb{C}^n\right)\sim\bigoplus_{k=1}^n L^2\left(\gamma_{-k},\mathbb{C}^n\right)$.

Lemma 3.6. *The operator $\mathcal{B}$ is trace-class and the operator $\mathcal{H}$ is Hilbert-Schmidt. Moreover, the following decomposition holds $\mathcal{K}_B = \mathcal{M}\circ\mathcal{N}+\mathcal{H}$, where*

$$\begin{aligned}
\mathcal{M}:\ & L^2(\gamma,\mathbb{C}^n)\to L^2\left(\textstyle\bigcup_{k=1}^n\gamma_{-k},\mathbb{C}^n\right)\\
\mathcal{N}:\ & L^2\left(\textstyle\bigcup_{k=1}^n\gamma_{-k},\mathbb{C}^n\right)\to L^2(\gamma,\mathbb{C}^n)\\
\mathcal{H}:\ & L^2\left(\textstyle\bigcup_{k=1}^n\gamma_{-k},\mathbb{C}^n\right)\to L^2\left(\textstyle\bigcup_{k=1}^n\gamma_{-k},\mathbb{C}^n\right)
\end{aligned} \tag{3.16}$$

with entries

$$\mathcal{M}_{ij}(t,\eta) := \frac{1}{2\pi i}\frac{e^{-\frac{\eta}{4}+\frac{t}{4}}}{\eta-t}\left(\frac{\eta_i}{t_j}\right)^\nu\cdot\chi_\gamma(t)\cdot\chi_{\gamma_{-i}}(\eta) \tag{3.17a}$$

$$\mathcal{N}_{ij}(\xi,t;a^{(j)}) = 4\delta_{ij}\cdot\frac{e^{a^{(j)}\left(\frac{1}{\xi_j}-\frac{1}{t_j}\right)}}{\xi-t}\cdot\chi_{\gamma_{-j}}(\xi)\cdot\chi_\gamma(t) \tag{3.17b}$$

$$\mathcal{H}_{ij}(\xi,\eta) = \chi_{\tau_i<\tau_j}\cdot 4\frac{e^{a^{(j)}\left(\frac{1}{\xi_j}-\frac{1}{\eta_j}\right)}}{\xi-\eta}\left(\frac{\eta_j}{\eta_i}\right)^{-\nu}\cdot\chi_{\gamma_{-i}}(\eta)\cdot\chi_{\gamma_{-j}}(\xi) \tag{3.17c}$$

$\zeta_k := \zeta+4\tau_k$ *($\zeta=\xi,t,\eta$) and* $\gamma_{-k} := \frac{1}{\gamma}-4\tau_k$, $\forall\, k=1,\dots,n$.

Proof. All the kernels are of the general form $H(z,w)$ with z and w on disjoint supports, that we indicate now temporarily by S_1, S_2. It is then simple to see that in each instance $\int_{S_1}\int_{S_2} |H(z,w)|^2 |\mathrm{d}z||\mathrm{d}w| < +\infty$ and hence each operator is Hilbert-Schmidt. Then $\mathcal{B}$ is trace class because it is the composition of two HS operators. $\square$

Now consider the Hilbert space

$$H := L^2\left(\gamma \cup \bigcup_{k=1}^{n} \frac{1}{\gamma} - 4\tau_k, \mathbb{C}^n\right) \sim L^2\left(\bigcup_{k=1}^{n} \frac{1}{\gamma} - 4\tau_k, \mathbb{C}^n\right) \otimes L^2(\gamma, \mathbb{C}^n), \tag{3.18}$$

and the matrix operator $\mathbb{K}_B : H \to H$ defined as

$$\mathbb{K}_B = \left[\begin{array}{c|c} 0 & \mathcal{N} \\ \hline \mathcal{M} & \mathcal{H} \end{array}\right] \tag{3.19}$$

due to the splitting of the space H into its two main addenda.

For now, we denote by "det" the determinant defined by the Fredholm expansion (3.8); then, "det"$(\mathrm{Id} - \mathbb{K}_B) = \det_2(\mathrm{Id} - \mathbb{K}_B)$, since its kernel is diagonal-free. Moreover, we introduce another Hilbert-Schmidt operator

$$\mathbb{K}'_B = \left[\begin{array}{c|c} 0 & -\mathcal{N} \\ \hline 0 & 0 \end{array}\right]$$

which is only Hilbert-Schmidt, but nevertheless its Carleman determinant ($\det_2$) is well defined and $\det_2(I - \mathbb{K}'_B) \equiv 1$.

Collecting all the results we have so far, we perform the following chain of equalities

$$\begin{aligned} \text{“}\det\text{”}(\mathrm{Id}_{L^2(\mathbb{R}_+)} - [K_B]\chi_{\mathcal{I}}) &= \det{}_2(\mathrm{Id} - [K_B]\chi_{\mathcal{I}})\, e^{-\operatorname{Tr}[\tilde{K}_B]} \\ = \det{}_2\left(\mathrm{Id}_{L^2\left(\bigcup_{k=1}^n \gamma_{-k}\right)} - \mathcal{K}_B\right) e^{-\operatorname{Tr}(\mathcal{B})} &= \det{}_2(\mathrm{Id}_H - \mathbb{K}_B)\det{}_2(\mathrm{Id}_H - \mathbb{K}'_B) \\ &= \det{}_2(\mathrm{Id}_H - \mathbb{K}_B) = \text{“}\det\text{”}(\mathrm{Id}_H - \mathbb{K}_B) \end{aligned} \tag{3.20}$$

The first equality follows from the fact that $[K_B] - [\tilde{K}_B]$ is diagonal-free; the second equality follows from invariance of the determinant under Fourier transform; the first identity on the last line is just an application of the following result: given $\mathbb{K}_B$, $\mathbb{K}'_B$ Hilbert-Schmidt operators, then

$$\det{}_2(\mathrm{Id} - \mathbb{K}_B)\det{}_2(\mathrm{Id} - \mathbb{K}'_B) = \det{}_2(\mathrm{Id} - \mathbb{K}_B - \mathbb{K}'_B + \mathbb{K}_B\mathbb{K}'_B)e^{\operatorname{Tr}(\mathbb{K}_B\mathbb{K}'_B)}.$$

It is finally just a matter of computation to show that $\mathbb{K}_B$ is an integrable operator of the form (3.4)-(3.4). $\square$

For the sake of clarity, let us consider a simple example of the multi-time Bessel process with two times τ_1, τ_2, restricted to the finite intervals $I_1 := [0, a]$ and $I_2 := [0, b]$:

$$
\begin{gathered}
[K_B](x,y)\ \mathrm{diag}\left[\chi_{[0,a]}(y), \chi_{[0,b]}(y)\right] = \left(\frac{y}{x}\right)^{\frac{\nu}{2}} \times \\
\left\{ \begin{bmatrix} 4\chi_{[0,a]}(y) \int_{\gamma\times\hat{\gamma}_1} \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2} \frac{e^{xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{t-s} \left(\frac{s}{t}\right)^{\nu} & \frac{\chi_{[0,b]}(y)}{(2\pi i)^2} \int_{\gamma\times\hat{\gamma}_2} \frac{\mathrm{d}t\,\mathrm{d}s}{ts} \frac{e^{\Delta_{12}+xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{\frac{1}{4t}-\frac{1}{4s}-\Delta_{12}} \left(\frac{s}{t}\right)^{\nu} \\ \frac{\chi_{[0,a]}(y)}{(2\pi i)^2} \int_{\gamma\times\hat{\gamma}_1} \frac{\mathrm{d}t\,\mathrm{d}s}{ts} \frac{e^{\Delta_{21}+xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{\frac{1}{4t}-\frac{1}{4s}-\Delta_{21}} \left(\frac{s}{t}\right)^{\nu} & 4\chi_{[0,b]}(y) \int_{\gamma\times\hat{\gamma}_2} \frac{\mathrm{d}t\,\mathrm{d}s}{(2\pi i)^2} \frac{e^{xt-\frac{1}{4t}-ys+\frac{1}{4s}}}{t-s} \left(\frac{s}{t}\right)^{\nu} \end{bmatrix} \right. \\
\left. + \begin{bmatrix} 0 & -\chi_{[0,b]}(y) \frac{1}{\Delta_{12}} \int_{\gamma} e^{\frac{x}{4\Delta_{12}}(1-t)+\frac{y}{4\Delta_{12}}\left(1-\frac{1}{t}\right)} t^{-\nu-1} \frac{\mathrm{d}t}{2\pi i} \\ 0 & 0 \end{bmatrix} \right\} \qquad (3.21)
\end{gathered}
$$

Then, the integral operator $\mathbb{K}_B : H \to H$ on the space $H := L^2\left(\gamma \cup \gamma_{-1} \cup \gamma_{-2}, \mathbb{C}^2\right)$ has the following expression

$$
\mathbb{K}_B = \left[\begin{array}{c|c} 0 & \mathcal{N} \\ \hline \mathcal{M} & \mathcal{H} \end{array}\right] =
$$

$$
\left[\begin{array}{cc|cc}
0 & 0 & -4e^{\frac{a}{\xi_1}}\chi_{\gamma_{-1}}e^{-\frac{a}{t_1}}\chi_{\gamma} & 0 \\
0 & 0 & 0 & -4e^{\frac{b}{\xi_2}}\chi_{\gamma_{-2}}e^{-\frac{b}{t_2}}\chi_{\gamma} \\ \hline
e^{\frac{\xi}{4}}\xi_1^{-\nu}\chi_{\gamma}e^{-\frac{t}{4}}t_1^{\nu}\chi_{\gamma_{-1}} & e^{\frac{\xi}{4}}\xi_2^{-\nu}\chi_{\gamma}e^{-\frac{t}{4}}t_1^{\nu}\chi_{\gamma_{-1}} & 0 & -4e^{\frac{b}{\xi_2}}\chi_{\gamma_{-2}}e^{-\frac{b}{t_2}}\frac{t_1^{\nu}}{t_2^{\nu}}\chi_{\gamma_{-1}} \\
e^{\frac{\xi}{4}}\xi_1^{-\nu}\chi_{\gamma}e^{-\frac{t}{4}}t_2^{\nu}\chi_{\gamma_{-2}} & e^{\frac{\xi}{4}}\xi_2^{-\nu}\chi_{\gamma}e^{-\frac{t}{4}}t_2^{\nu}\chi_{\gamma_{-2}} & 0 & 0
\end{array}\right] \qquad (3.22)
$$

and the equality between Fredholm determinants holds

$$
\det\left(\mathrm{Id}_{L^2(\mathbb{R}_+,\mathbb{C}^2)} - K_B \mathrm{diag}(\chi_{[0,a]}, \chi_{[0,b]})\right) = \det\left(\mathrm{Id}_H - \mathbb{K}_B\right). \qquad (3.23)
$$

The Riemann-Hilbert problem for the multi-time Bessel process. As explained in the introduction, we can relate the computation of the Fredholm determinant of the matrix Bessel operator to the theory of isomonodromic equations, through a suitable Riemann-Hilbert problem.

Proposition 3.7. *Given the integrable kernel (3.4)-(3.7), the associated Riemann-Hilbert problem is the following:*

$$
\Gamma_+(\lambda) = \Gamma_-(\lambda)\left(I - 2\pi i J_B(\lambda)\right) \qquad \lambda \in \Sigma \qquad (3.24a)
$$

$$
\Gamma(\lambda) = I + \mathcal{O}\left(\frac{1}{\lambda}\right) \qquad \lambda \to \infty \qquad (3.24b)
$$

where Γ *is a* $2n \times 2n$ *matrix* Γ *such that it is analytic on the complex plane except at* $\Sigma := \gamma \cup \bigcup_{k=1}^{n} \frac{1}{\gamma} - 4\tau_k$*; the jump matrix* $J_B(\lambda) := \boldsymbol{f}(\lambda) \cdot \boldsymbol{g}(\lambda)^T$ *has the expression*

$$J_B(\lambda) := \left[\begin{array}{c|c|c} 0 & \star_1 & 0 \\ \hline \star_2 & 0 & \star_3 \\ \hline \star_4 & 0 & \star_5 \end{array}\right]$$

$$\star_1 := \left[-4e^{\theta_1}\chi_\gamma, \ldots, -4e^{\theta_n}\chi_\gamma\right]^T$$

$$\star_2 := \left[e^{-\theta_1}\chi_{\gamma_{-1}}, \ldots, e^{-\theta_n}\chi_{\gamma_{-n}}\right]$$

$$\star_3 := \left[e^{-\theta_2}\chi_{\gamma_{-2}}, \ldots, e^{-\theta_n}\chi_{\gamma_{-n}}\right]$$

$$\star_4 := \begin{bmatrix} -4e^{-\theta_1+\theta_2}\chi_{\gamma_{-1}} & 0 & & & \\ -4e^{-\theta_1+\theta_3}\chi_{\gamma_{-1}} & -4e^{-\theta_2+\theta_3}\chi_{\gamma_{-2}} & 0 & & \\ -4e^{-\theta_1+\theta_4}\chi_{\gamma_{-1}} & -4e^{-\theta_2+\theta_4}\chi_{\gamma_{-2}} & -4e^{-\theta_3+\theta_4}\chi_{\gamma_{-3}} & 0 & \\ \vdots & & & \ddots & \\ -4e^{-\theta_1+\theta_n}\chi_{\gamma_{-1}} & \ldots & & -4e^{-\theta_{n-1}+\theta_n}\chi_{\gamma_{-(n-1)}} & 0 \end{bmatrix}$$

$$\star_5 := \begin{bmatrix} 0 & & & \\ -4e^{-\theta_2+\theta_3}\chi_{\gamma_{-2}} & & & \\ -4e^{-\theta_2+\theta_4}\chi_{\gamma_{-2}} & -4e^{-\theta_3+\theta_4}\chi_{\gamma_{-3}} & & \\ \vdots & \vdots & & \\ & & & \\ -4e^{-\theta_2+\theta_n}\chi_{\gamma_{-2}} & & -4e^{-\theta_{n-1}+\theta_n}\chi_{\gamma_{n-1}} & 0 \end{bmatrix}$$

$$\theta_i := \frac{\lambda}{4} - \frac{a_i}{\lambda_i} - \nu \ln \lambda_i, \qquad \lambda_i = \lambda + 4\tau_i \tag{3.25}$$

∎

We recall that we are considering the simple case $\mathcal{I} = \bigsqcup_j I_j$ with $I_j := [0, a^{(j)}]$, $\forall\, j = 1, \ldots, n$.

Applying again the results stated in [3] and [5], we can claim the following.

Theorem 3.8. *Given* n *times* $\tau_1 < \tau_2 < \ldots < \tau_n$ *and given the multi-interval matrix* $\chi_{\mathcal{I}} := \operatorname{diag}(\chi_{I_1}, \ldots, \chi_{I_n})$*, the Tracy-Widom distribution of the multi-time Bessel operator, i.e. the Fredholm determinant* $\det(\mathrm{Id} - [K_B]\chi_{\mathcal{I}})$*, is equal to the isomonodromic* τ*-function related to the above Riemann-Hilbert problem.*

In particular, we have

$$\partial \ln \det(\mathrm{Id} - [K_B]\chi_{\mathcal{I}}) = \int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_\partial(\lambda)\right)\frac{\mathrm{d}\lambda}{2\pi i} \tag{3.26a}$$

$$\Xi_\partial(\lambda) = -2\pi i\, \partial J_B\,(I + 2\pi i J_B) \tag{3.26b}$$

the $'$ notation means differentiation with respect to λ, while with ∂ we denote any of the derivatives with respect to times ∂_{τ_k} or endpoints $\partial_{a^{(k)}}$ $(k = 1, \ldots, n)$.

Proof. Keeping into account formula 1.12, it is enough to verify that $H(M) = 0$ with $M(\lambda) = I - J_B(\lambda)$. □

In the simple 2-times case, the jump matrix is

$$J_B(\lambda) = \mathbf{f}(\lambda) \cdot \mathbf{g}(\lambda)^T =$$
$$\left[\begin{array}{cc|c|c} 0 & 0 & -4e^{\frac{\lambda}{4}-\frac{a}{\lambda_1}}\lambda_1^{-\nu}\chi_\gamma & 0 \\ 0 & 0 & -4e^{\frac{\lambda}{4}-\frac{b}{\lambda_2}}\lambda_2^{-\nu}\chi_\gamma & 0 \\ \hline e^{-\frac{\lambda}{4}+\frac{a}{\lambda_1}}\lambda_1^{\nu}\chi_{\gamma_{-1}} & e^{-\frac{\lambda}{4}+\frac{b}{\lambda_2}}\lambda_2^{\nu}\chi_{\gamma_{-2}} & 0 & e^{-\frac{\lambda}{4}+\frac{b}{\lambda_2}}\lambda_2^{\nu}\chi_{\gamma_{-2}} \\ \hline -4e^{\frac{a}{\lambda_1}-\frac{b}{\lambda_2}}\left(\frac{\lambda_1}{\lambda_2}\right)^{\nu}\chi_{\gamma_{-1}} & 0 & 0 & 0 \end{array}\right] \tag{3.27}$$

Thanks to Theorem 3.8, it is possible to derive some more explicit differential identities by using the Jimbo-Miwa-Ueno residue formula (see [3]), as in the single-time case.

First we notice that the jump matrix is equivalent up to conjugation with a constant matrix J_0:

$$J_B(\lambda) = e^{T_B(\lambda)} J_B^0 e^{-T_B(\lambda)} \tag{3.28}$$

with

$$T_B(\lambda) := \mathrm{diag}\left[\theta_1 - \frac{\kappa}{2n}, \ldots, \theta_n - \frac{\kappa}{2n}, 1 - \frac{\kappa}{2n}, \theta_2 - \frac{\kappa}{2n}, \ldots, \theta_n - \frac{\kappa}{2n}\right]$$
$$\kappa := \theta_1 + 2\sum_{k=2}^{n} \theta_k \tag{3.29}$$

Therefore, the matrix $\Psi_B(\lambda) = \Gamma(\lambda)e^{T_B(\lambda)}$ solves a Riemann-Hilbert problem with constant jumps and it is (sectionally) a solution to a polynomial ODE.

Theorem 3.9. *The quantity (3.26a) can be computed explicitly as*

$$\int_\Sigma \mathrm{Tr}\left(\Gamma_-^{-1}(\lambda)\Gamma_-'(\lambda)\Xi_\partial(\lambda)\right)\frac{\mathrm{d}\lambda}{2\pi i} = -\operatorname*{res}_{\lambda=\infty}\mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial T_B\right) +$$
$$+\sum_{i=1}^{n}\operatorname*{res}_{\lambda=-4\tau_i}\mathrm{Tr}\left(\Gamma^{-1}\Gamma'\partial T_B\right) \tag{3.30}$$

More specifically, regarding the derivative with respect to the endpoints $a^{(i)}$ $(i = 1, \dots, n)$, we have

$$\operatorname*{res}_{\lambda=-4\tau_1} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{a^{(1)}}T_B\right) = \left(\frac{1}{2n} - 1\right)\left(\Gamma_0^{-1}\Gamma_1\right)_{(1,1)} \tag{3.31a}$$

$$\operatorname*{res}_{\lambda=-4\tau_i} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{a^{(i)}}T_B\right) = \left(\frac{1}{n} - 1\right)\left[\left(\Gamma_0^{-1}\Gamma_1\right)_{(i,i)} + \left(\Gamma_0^{-1}\Gamma_1\right)_{(i+n,i+n)}\right] \tag{3.31b}$$

and, regarding the derivative with respect to the times τ_i $(i = 1, \dots, n)$, we have

$$\begin{aligned}\operatorname*{res}_{\lambda=-4\tau_1} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{\tau_1}T_B\right) = 4\nu\left(\frac{1}{2n} - 1\right)\left(\Gamma_0^{-1}\Gamma_1\right)_{(1.1)} + \\ +4a^{(1)}\left(1 - \frac{1}{2n}\right)\left(-\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}\Gamma_1 + 2\Gamma_0^{-1}\Gamma_2\right)_{(1,1)}\end{aligned} \tag{3.32a}$$

$$\begin{aligned}\operatorname*{res}_{\lambda=-4\tau_i} \operatorname{Tr}\left(\Gamma^{-1}\Gamma'\partial_{\tau_i}T_B\right) = 4\nu\left(\frac{1}{n} - 1\right)\left[\left(\Gamma_0^{-1}\Gamma_1\right)_{(i.i)} + \left(\Gamma_0^{-1}\Gamma_1\right)_{(i+n,i+n)}\right] + \\ +4a^{(i)}\left(1 - \frac{1}{n}\right)\left[\left(-\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}\Gamma_1 + 2\Gamma_0^{-1}\Gamma_2\right)_{(i,i)} + \right. \\ \left. + \left(-\Gamma_0^{-1}\Gamma_1\Gamma_0^{-1}\Gamma_1 + 2\Gamma_0^{-1}\Gamma_2\right)_{(i+n,i+n)}\right]\end{aligned} \tag{3.32b}$$

where the Γ_i's are coefficients of the asymptotic expansion of the matrix Γ near ∞ and $-4\tau_j$. We recall that each asymptotic expansion (the Γ_i's) is different in a neighborhood of each point $-4\tau_j$ and it's different from the one near ∞.

The residue at infinity does not give any contribution in either case.

Proof. We calculate the derivatives of the conjugation factor

$$
\begin{aligned}
\partial_{a^{(1)}} T_B(\lambda) &= \operatorname{diag}\left[\partial_{a^{(1)}}\left(\theta_1 - \tfrac{\kappa}{2n}\right), 0, \dots, 0\right] \\
&= \operatorname{diag}\left[\tfrac{1}{\lambda_1}\left(\tfrac{1}{2n} - 1\right), 0, \dots, 0\right] = \tfrac{1}{\lambda_1}\left(\tfrac{1}{2n} - 1\right) \cdot E_{(1,1)} \\
\partial_{a^{(i)}} T_B(\lambda) &= \operatorname{diag}\left[0, \dots, \partial_{a^{(i)}}\left(\theta_i - \tfrac{\kappa}{2n}\right), \dots, \partial_{a^{(i)}}\left(\theta_i - \tfrac{\kappa}{2n}\right), \dots, 0\right] \\
&= \tfrac{1}{\lambda_i}\left(\tfrac{1}{n} - 1\right) \cdot E_{(i,i),\,(i+n,i+n)} \\
\partial_{\tau_1} T_B(\lambda) &= \operatorname{diag}\left[\partial_{\tau_1}\left(\theta_1 - \tfrac{\kappa}{2n}\right), 0, \dots, 0\right] \\
&= \operatorname{diag}\left[\left(\tfrac{4a^{(1)}}{\lambda_1^2} - \tfrac{4\nu}{\lambda_1}\right)\left(1 - \tfrac{1}{2n}\right), 0, \dots, 0\right] \\
&= \left(\tfrac{4a^{(1)}}{\lambda_1^2} - \tfrac{4\nu}{\lambda_1}\right)\left(1 - \tfrac{1}{2n}\right) \cdot E_{(1,1)} \\
\partial_{\tau_i} T_B(\lambda) &= \operatorname{diag}\left[0, \dots, \partial_{\tau_i}\left(\theta_i - \tfrac{\kappa}{2n}\right), \dots, \partial_{\tau_i}\left(\theta_i - \tfrac{\kappa}{2n}\right), \dots, 0\right] \\
&= \left(\tfrac{4a^{(i)}}{\lambda_i^2} - \tfrac{4\nu}{\lambda_i}\right)\left(1 - \tfrac{1}{n}\right) \cdot E_{(i,i),\,(i+n,i+n)}
\end{aligned} \tag{3.33}
$$

where $E_{(i,i)\,(i+n,i+n)}$ is the zero matrix with only two non-zero entries (which are 1's) in the (i,i) and $(i+n, i+n)$ positions.

Then, recalling the (formal) asymptotic expansion of the matrix Γ near ∞ and $-4\tau_i$ for all i (see [38] for a detailed discussion on the topic), the results follow from straightforward calculations.

□

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