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A SIMPLE MEASURE OF RISK AVERSION IN THE LARGE AND AN APPLICATION

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Abstract

We introduce a simple measure of risk aversion in the large. Besides satisfying properties which are conceptually analogous to the usual properties of the Arrow-Pratt measure, the index of risk aversion in the large leads to a stronger concept of decreasing risk aversion, which necessarily implies decreasing absolute risk aversion but not necessarily non-increasing relative risk aversion.

The index also leads to a recursive procedure for refining the set of $vN - M$ utility functions. We show that the majority of refinements considered in the theory of multiple risk bearing including that of mixed risk aversion can be obtained from this procedure. Finally, as an illustration, we apply the measure to characterize individual behaviour under uncertainty in the principal-agent model of optimal income tax enforcement in which the risks involved are indeed large.

Keywords: expected utility, risk aversion, certainty equivalent, multiple risk bearing, principal-agent, audit probability.

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1 Introduction

Pratt (1964) and Arrow (1971) introduce a simple measure of risk aversion in the small, which has come to be known as the Arrow-Pratt measure. However, they both leave open the question of a similar measure of risk aversion in the large. We propose below such a measure. We introduce the index of risk aversion in the large and show that it approximates risk premia for risks which are not *necessarily* small. The index not only satisfies properties which are conceptually analogous to those of the Arrow-Pratt measure, but also implies an ordering in terms of “concavifying” transforms which is similar to the Arrow-Pratt ordering over vN-M utility functions.

The assumption of decreasing absolute risk aversion (DARA) in the sense of Arrow-Pratt is known to yield many economically reasonable results concerning individual choice under uncertainty. However, in applications it is often too weak to obtain unambiguous comparative static results. For this reason, sometimes the stronger assumption of nonincreasing relative risk aversion is used. We show that the index of risk aversion in the large implies a stronger concept of decreasing risk aversion, to be called strong decreasing risk aversion (SDRA) ¹. This concept is stronger than that of DARA, but it does not imply nonincreasing relative risk aversion and the two are not generally comparable. Furthermore, convexity of risk aversion in the large implies convexity of absolute risk aversion (i.e. risk aversion in the small) but the converse is not true.

¹This is not to be confused with the concept of strong risk aversion introduced by Ross (1981). The concept of SDRA is about how an individual’s aversion to risk decreases with wealth when complete insurance is possible as in the Arrow-Pratt theory. Whereas Ross’s measure is about how averse an individual is to risk when only some risks can be insured against and others must be retained.

It is well accepted by now that DARA is also not sufficient for obtaining plausible behavior in models, which unlike the Arrow-Pratt theory involve more than one risk. Additional restrictions need to be imposed so as to refine the set of vN-M utility functions. Thus, Pratt and Zeckhauser (1987) require the utility function to satisfy properness. Similarly, Kimball (1993) introduces a stronger restriction, namely that of standard risk aversion ². More recently, Caballé and Pomansky (1996) propose that the utility function must satisfy mixed risk aversion, which is even stronger than standard (and thus proper) risk aversion and equivalent to the complete monotonicity of its first derivative over the interval $(0, \infty)$ ³. We show that SDRA can also be seen as a step further in this refinement strategy. By using the basic idea underlying SDRA, we propose a recursive procedure for systematically refining the set of vN-M utility functions and obtain mixed risk aversion as a particular case ⁴. The concept of SDRA thus enables us to narrow the gap between the standard Arrow-Pratt model concerning a single risk and the theory of multiple risk bearing.

Finally, in order to illustrate, we apply the concept of SDRA to characterize individual behavior under uncertainty in the principal-agent model of optimal income tax enforcement in which the risks involved are indeed large.

²A vN-M utility function is proper if individually undesirable, independent risks are always jointly undesirable, that is, the two risks must aggravate each other. A standard utility function is the one for which every undesirable risk is aggravated by every independent, loss-aggravating risk.

³A real valued function $f(w)$ defined on $(0, \infty)$ is *completely monotone* iff its derivatives $f^n(w)$ of all orders exist and $(-1)^n f^n(w) \geq 0$, for all $w > 0$ and $n = 0, 1, 2, \dots$

⁴Caballé and Pomansky also propose in the concluding section of their paper the refinement that the absolute risk aversion of the utility function be completely monotone. It is seen below that this too obtains from our procedure.

The paper is organized as follows. The next two sections introduce the index of risk aversion in the large and the concept of SDRA and establish their properties including their relationship with the theory of multiple risk bearing. Section 4 introduces the principal-agent model of optimal income tax enforcement and Section 5 characterizes individual behavior under uncertainty in this model. Section 6 draws the conclusion.

2 Risk Aversion in The Large

Consider a risk averse individual with wealth w and vN-M utility function $u(w)$ and facing a lottery which offers 0 and $y \geq 0$ with probabilities p and $1 - p$, respectively, i.e. $(0, y; p, 1 - p)$, $y \geq 0$ ⁵. Let x be the *certainty equivalent* of this lottery, that is,

$$u(w + x) = pu(w) + (1 - p)u(w + y). \quad (1)$$

Note that any lottery $(y_1, y_2; p, 1 - p)$ with $y_1 \leq y_2$ can be expressed as the combination of a lottery of the form $(0, y; p, 1 - p)$, $y \geq 0$, and an appropriate deterministic downward or upward shift in wealth. Our restriction to lotteries of the form $(0, y; p, 1 - p)$, $y \geq 0$, may be thus viewed as equivalent to assuming that the individual has been compensated for the wealth effect arising from the risk. For this reason it seems appropriate to refer to the lottery $(0, y; p, 1 - p)$, $y \geq 0$, as the *compensated risk*⁶. In order to fix our arguments we assume henceforth that $0 < p < 1$ and $y > 0$.

⁵Our analysis below does not depend on the assumption that the lottery has only two possible outcomes.

⁶Drèze and Modigliani (1972) and Kimball (1990) consider risks that are expected marginal utility increasing. In contrast, compensated risks belong to the class of expected marginal utility decreasing risks.

We assume that u is smooth and thus its derivatives of all orders exist with $u'(w) > 0$ and $u''(w) < 0$ for all $w \geq 0$, i.e., u is strictly increasing and strictly concave or strictly risk averse. In order to keep the notation simple, we also assume that $u(0) = 0$ ⁷.

Equation (1) above which defines the certainty equivalent of the compensated risk $(0, y; p, 1 - p)$ can be rewritten as

$$u_w(x) = (1 - p)u_w(y), \quad (2)$$

where $u_w(z) \equiv u(w + z) - u(w)$ for all $w, z \geq 0$.

Clearly, like $u(z)$, the utility function $u_w(z)$ satisfies for each $w \geq 0$, $u_w(0) = 0$, $u'_w(z) > 0$ and $u''_w(z) < 0$ for all $z \geq 0$. Define

$$z = (1 - p)y - x. \quad (3)$$

Then, z is the *risk premium* corresponding to the compensated risk $(0, y; p, 1 - p)$. Under our assumptions, $x < y$ and, by the implicit function theorem, x is a differentiable function of y . By differentiation, it follows from (2) that

$$u'_w(x) \frac{dx}{dy} = (1 - p)u'_w(y). \quad (4)$$

By differentiating (3) and substituting from (4), we get

$$\frac{dz}{dy} = (1 - p) \left(1 - \frac{u'_w(y)}{u'_w(x)} \right). \quad (5)$$

Since $x < y$ and u is strictly concave, $0 < dz/dy < 1$. Similarly, (4) implies $0 < dx/dy < 1$. By differentiating the expression on the right in (5), we obtain

⁷Such a normalization (as we know from the expected utility theory) is legitimate, but not always possible. For example, $u(w) = \alpha w^\alpha$, with $\alpha < 0$. Nevertheless, it can be approximated by the utility function $\alpha(w + a)^\alpha - \alpha a^\alpha$, where $a > 0$ is arbitrarily small.

$$\frac{d^2z}{dy^2} = (1-p) \frac{u'_w(y)}{u'_w(x)} \left[\frac{-u''_w(y)}{u'_w(y)} - \left(\frac{-u''_w(x)}{u'_w(x)} \right) \frac{dx}{dy} \right].$$

Since, as noted earlier, $0 < dx/dy < 1$ and $x < y$, a *necessary condition* for d^2z/dy^2 to be nonpositive is that $-u''_w(y)/u'_w(y)$ is decreasing in y , that is, absolute risk aversion of u_w is decreasing. By substituting from (2) and (4) in the equality above and rearranging, we obtain

$$\frac{d^2z}{dy^2} = (1-p) \frac{(u'_w(y))^2}{u_w(y)u'_w(x)} \left[\frac{-u''_w(y)u_w(y)}{(u'_w(y))^2} - \frac{-u''_w(x)u_w(x)}{(u'_w(x))^2} \right].$$

A sufficient condition for d^2z/dy^2 to be nonpositive for every compensated risk $(0, y; p, 1-p)$ is thus : $-u''_w(y)u_w(y)/(u'_w(y))^2$ is nonincreasing in y . This condition is necessary as well, since as seen from (2) the certainty equivalent x can be arbitrarily close to y for p sufficiently close to zero. This leads to the following definition :

A vN-M utility function u satisfies *strong decreasing risk aversion* (SDRA) at wealth level $w \geq 0$, if

$$\left(\frac{-u''_w(y)}{u'_w(y)} \right) \left(\frac{u_w(y)}{u'_w(y)} \right) \text{ is nonincreasing in } y.$$

Let

$$R(y, u_w) \equiv \left(\frac{-u''(w+y)}{u'(w+y)} \right) \left(\frac{u(w+y) - u(w)}{u'(w+y)} \right). \quad (6)$$

Then, SDRA at wealth level w is equivalent to $R'(y, u_w) \leq 0$, i.e. the derivative with respect to y is nonpositive.

Observe that the co-efficient $R(y, u_w)$ is invariant with respect to positive affine transformations of u and thus the concept of SDRA is consistent with the expected utility theory. More importantly, however, unlike the Arrow-Pratt co-efficient of absolute risk aversion, $R(y, u_w)$ is not entirely a local concept

because of the involvement of the term $u(w + y) - u(w)$ for y not small ⁸. We claim below that $R(y, u_w)$ is fact a simple measure of risk aversion in the large.

Using the familiar Taylor series expansion, it is easily seen that the risk premium corresponding to the gamble $(0, y; p, 1 - p)$ for *infinitesimal* y is given by

$$\pi_p(w, y) \equiv (1 - p)y - x \approx (1 - p)\frac{y^2}{2} \left(\frac{-u''(w)}{u'(w)} \right), \quad (7)$$

where x is the certainty equivalent and $(1 - p)y$ is the expected monetary value of the gamble.

We can rewrite the defining equation (6) above as

$$\frac{(1 - p)y}{2} R(y, u_w) = (1 - p)\frac{y^2}{2} \left[\frac{-u''(w + y)}{u'(w + y)} \times \frac{\frac{u(w + y) - u(w)}{y}}{u'(w + y)} \right]. \quad (8)$$

Since

$$\lim_{y \rightarrow 0} \frac{-u''(w + y)}{u'(w + y)} \times \frac{\frac{u(w + y) - u(w)}{y}}{u'(w + y)} = \frac{-u''(w)}{u'(w)}, \quad (9)$$

it follows from the comparison of expressions on the right in (7), (8) and (9) that

$$\frac{(1 - p)y}{2} R(y, u_w) \approx \pi_p(w, y). \quad (10)$$

In words, $R(y, u_w)$ is approximately equal to twice the risk premium per unit of the expected monetary value of the gamble ⁹. Clearly, smaller the y , more accurate the approximation. For large y , as one would expect, the simple measure $R(y, u_w)$ contains some but not all information about the utility function u . This information can be, however, refined to any degree of precision by taking increasingly smaller values of y .

⁸In addition, $R(y, u_w)$ is independent of the units in which w and y are measured. Thus, like an elasticity, it is a pure number.

⁹This should clarify, as we observed earlier, why $R(y, u_w)$ is independent of the units in which wealth is measured.

In view of (10) and the preceding interpretation, it seems appropriate to term $((1-p)y/2)R(y, u_w)$ or simply $R(y, u_w)$ as the *index of risk aversion in the large*.

Note that the Arrow-Pratt theory relies, perhaps for the sake of a simpler presentation, on the assumption that the risk is actuarially neutral (besides being small). The Arrow-Pratt theory thereby rules out the possibility of any measure of risk premium per unit of the expected monetary value of the risk because to begin with this expected value is assumed to be zero. Our reliance on compensated risks rather than on actuarially neutral risks has thus paid off in terms of the index of risk aversion in the large.

The Arrow-Pratt co-efficient of absolute risk aversion $r(w, u) \equiv -u''(w)/u'(w)$ is often seen as an intuitive measure of the concavity of the utility function $u(w)$ at the point w . We claim that $R(y, u_w)$ can also be interpreted as a measure of concavity of the utility function $u(x)$ not just at the point $w + y$ but to an extent also over the interval $[w, w + y]$.

In Figure 1, let $h(y, u_w)$ denote the length of the segment $[a, w + y]$. More precisely,

$$h(y, u_w) = \frac{u_w(y)}{u'_w(y)} = \frac{u(w + y) - u(w)}{u'(w + y)}. \quad (11)$$

If the utility function $u(x)$ is linear over the entire interval $[w, w + y]$, $h(y, u_w) = y$ and $h'_w(y) = 1$, a constant. The length of the segment $[a, w + y]$ and the rate at which it increases with y clearly depend on the concavity of the utility function $u(x)$ not only around the point $w + y$ but also to an extent over the entire interval $[w, w + y]$. The dotted curve $v(\cdot)$ is identical to the curve $u(\cdot)$ in the neighborhood of the point $w + y$ but flatter or “less concave” over the rest of the interval $[w, w + y]$. Accordingly, as seen from the graph, both $h(y, v_w)$ - the length of the corresponding segment $[b, w + y]$ - and its derivative $h'(y, v_w)$ are

smaller, i.e., $h(y, v_w) \leq h(y, u_w)$ and $h'(y, v_w) \leq h'(y, u_w)$. By differentiating the expression on the right in (11), we obtain

$$\begin{aligned} h'(y, u_w) &= 1 + \left(\frac{-u_w''(y)}{u_w'(y)} \right) \left(\frac{u_w(y)}{u_w'(y)} \right) \\ &= 1 + \left(\frac{-u_w''(y)}{u_w'(y)} \right) h(y, u_w) \\ &= 1 + R(y, u_w). \end{aligned}$$

Since by assumption $-u_w''(y)/u_w'(y) = -v_w''(y)/v_w'(y)$, $R(y, u_w) \geq R(y, v_w)$ iff both $h(y, u_w) \geq h(y, v_w)$ and $h'(y, u_w) \geq h'(y, v_w)$, i.e., the curve $v(\cdot)$ is above the curve $u(\cdot)$ and flatter over the interval $[w, w + y]$ as shown in Figure 1.

The interpretations above suggest additional properties of the measure $R(x, u_w)$. These are verified in the following.

Proposition 1 : Given two vN-M utility functions $u(x)$ and $v(x)$, there exists an increasing concave function g such that $u(x) = g(v(x))$ for all $x \geq 0$ iff $R(y, u_w) \geq R(y, v_w)$ for all $w, y \geq 0$.

Proof of Proposition 1 : We first prove the sufficiency part. By definition, for each $w, y \geq 0$

$$\begin{aligned} u_w(y) &= u(w + y) - u(w), \\ v_w(y) &= v(w + y) - v(w), \end{aligned}$$

and by hypothesis

$$u(w + y) = g(v(w + y)),$$

and

$$u_w(y) = g(v(w + y)) - g(v(w))$$

with g increasing and concave. Differentiating with respect to y

$$u'_w(y) = u'(w + y) = g'(v(w + y))v'(w + y)$$

$$= g'(v(w+y))v'_w(y).$$

$$u''_w(y) = u''(w+y) = g''(v(w+y))(v'_w(y))^2 + g'(v(w+y))v''_w(y).$$

Using these equalities, we get

$$\begin{aligned} \frac{-u''_w(y)}{u'_w(y)} \frac{u_w(y)}{u'_w(y)} &= \frac{-g''(v(w+y))u_w(y)}{(g'(v(w+y)))^2} + \frac{-v''_w(y)(g(v(w+y)) - g(v(w)))}{v'_w(y)g'(v(w+y))v'_w(y)} \\ &\geq \frac{-v''_w(y)g'(v(w+y))(v(w+y) - v(w))}{v'_w(y)g'(v(w+y))v'_w(y)}, \end{aligned}$$

by using the concavity of the function g , i.e., $g'(v(w+y))(v(w+y) - v(w)) \leq g(v(w+y)) - g(v(w))$.

This proves

$$\frac{-u''_w(y)}{u'_w(y)} \frac{u_w(y)}{u'_w(y)} \geq \frac{-v''_w(y)v_w(y)}{v'_w(y)v'_w(y)}.$$

Next, we prove the necessity part. By assumption, for each $w \geq 0$ and $y > 0$,

$$\frac{-u''(w+y)}{u'(w+y)} \frac{\frac{u(w+y)-u(w)}{y}}{u'(w+y)} \leq \frac{-v''(w+y)}{v'(w+y)} \frac{\frac{v(w+y)-v(w)}{y}}{v'(w+y)}.$$

Taking the limit as $y \rightarrow 0$, we get

$$\frac{-u''(w)}{u'(w)} \leq \frac{-v''(w)}{v'(w)} \text{ for each } w \geq 0.$$

From the well known theorem of Pratt (1964), there exists an increasing concave function g such that $u(x) = g(v(x))$ for all $x \geq 0$. ■

Note that the proof of the sufficiency part does not require y to be small. This again clarifies that even for large gambles the measure $R(y, u_w)$ does not differ essentially from the actual risk premium z as defined by equations (2) and (3) as it never reverses the comparison between u and v . For the necessity part however a point by point comparison between u and v is necessary and this can be done only by taking the gambles to be arbitrarily small.

3 SDR A and The Theory of Multiple Risk Bearing

Having justified it conceptually, we now examine the implications of SDR A for the theory of multiple risk bearing. Since we consider comparative statics of risk aversion across the entire interval $[0, \infty]$, we fix the wealth level at the minimum, i.e., $w = 0$ but allow y to take values over $[0, \infty]$.

We denote the index of risk aversion in the large $R(y, u_w)$ for $w = 0$ simply by $R(y, u)$. That is ¹⁰

$$R(y, u) = \frac{-u''(y)}{u'(y)} \frac{u(y)}{u'(y)}. \quad (12)$$

And we refer to SDR A at wealth level $w = 0$, i.e. $R'(y, u) \leq 0$, simply as SDR A.

DARA, as defined by Arrow and Pratt, means that risk premium declines in wealth. Since we fix the wealth level, it may seem that a comparison of risk aversion across wealth levels is not to be considered. But this is not so. On the contrary, since y is allowed to be large, SDR A (or for that matter SDR A at any fixed wealth level) has some strong implications for risk aversion at all wealth levels ¹¹. To begin with, SDR A, that is

$$R'(y, u) = \frac{u''(y)}{u'(y)} \left[\left(\frac{u(y)}{u'(y)} \right) \left(\frac{-u'''(y)}{u''(y)} - 2 \frac{-u''(y)}{u'(y)} \right) - 1 \right] \leq 0. \quad (13)$$

implies DARA, i.e., $-u'''(y)/u''(y) \geq -u''(y)/u'(y)$. The ratio $-u'''(y)/u''(y)$ is

¹⁰More evidently, $R(y, u) = -u''(y) (u(y) - u(0)) / (u'(y))^2$. But since, by assumption, $u(0) = 0$, $R(y, u)$ is simply equal to the expression on the right in (12).

¹¹On the other hand, an alternative interpretation in terms of a model of random (i.e. not fixed) wealth as given in Section 5 is also possible.

the index of absolute prudence (Kimball (1990))¹². The relative magnitudes of $-u''(y)/u'(y)$ and $-u'''(y)/u''(y)$ are known to play an important part in many applications¹³. Inequality (13) clarifies why this might be so. Next we compare SDRA with nonincreasing relative risk aversion, that is,

$$\frac{u''(y)}{u'(y)} \left[y \left(\frac{-u'''(y)}{u''(y)} - \frac{-u''(y)}{u'(y)} \right) - 1 \right] \leq 0. \quad (14)$$

Since u is concave, i.e., $u'(x)$ is nonincreasing in x , $u(y)/u'(y) = (1/u'(y)) \int_0^y u'(x)dx \geq y$. The inequalities (13) and (14) are thus not comparable except when (13) holds with equality, i.e., $R'(y, u) = 0$ or risk aversion in the large is constant. This is so because in that case by twice integrating the expression on the right in (13) we obtain $u(x) = ax^\alpha$ with $0 < \alpha < 1$ and $a > 0$, that is, relative risk aversion is also constant¹⁴. Our argument is completed

¹²It captures the strength of agent's adjustment to risk or precautionary motive. It is defined by the equation $Eu'(w+y) = u'(w-x)$, where E is the expectation operator and x is the *precautionary premium*.

¹³Sinclair-Desgagné and Gabel (1997) study the problem of environmental auditing and show that the condition $-u'''(y)/u''(y) \geq -2u''(y)/u'(y)$ (which is necessary but not sufficient for SDRA) is sufficient for the condition under which an audit has to be performed. Drèze and Modigliani (1972) implicitly use the same condition to sign a precautionary saving effect. Carroll and Kimball (1996) consider the class of utility functions which satisfy $-u'''(y)/u''(y) \geq -k u''(y)/u'(y)$, where k can take different values, in order to establish the concavity of the consumption function.

¹⁴Intuitively, this is to be expected. Since constant risk aversion in the large implies that the risk premium is a constant proportion of the expected monetary value of the compensated risk which is mathematically equivalent to the same thing as that risk premium is a constant proportion of wealth. This equivalence between constant risk aversion in the large and constant relative risk aversion in the small clarifies why the utility function $u(x) = ax^\alpha$ with $0 < \alpha < 1$ and $a > 0$ has been so successful in applications. It seems to have been yet another case of practice running ahead of theory.

if we can exhibit a utility function which satisfies (13) with strict inequality, i.e., $R'(y, u) < 0$. It is easily seen that one such utility function is $u(y) = y + y^\alpha$ with $0 < \alpha < 1$.

Let $u_1(y) \equiv u(y)/u'(y)$. Then, $u_1(0) = 0$; $u'_1 = 1 + R(y, u) > 0$, since u is concave; and $u''_1 = R'(y, u)$. This means that u satisfies SDRA if and only if u_1 is increasing and concave.

In geometric terms, as seen from Figure 2, $u_1(y)$ is equal to the length of the subtangent at y , i.e., the length of the segment $[a, y]$ ¹⁵. Thus, risk aversion (i.e., concavity of u) is equivalent to requiring that the length of the subtangent be increasing and SDRA is equivalent to requiring that it be increasing at a decreasing rate.

Let $\mathbf{u}$ denote the set of all smooth utility functions u , which satisfy $u(0) = 0$, $u'(x) > 0$, and $u''(x) \leq 0$ for all $x \in [0, \infty]$. And let T denote the operator $T(u) = u/u'$. Then SDRA of $u \in \mathbf{u}$ is equivalent to $T(u) \in \mathbf{u}$.

Let $T^{n+1}(u) = T(T^n(u))$, and $u_n = T^n(u)$, $n = 1, 2, \dots$. We introduce the following definitions:

A vN-M utility function $u \in \mathbf{u}$ satisfies $S^{(n)}\text{DRA}$ if $T^k(u) \in \mathbf{u}$ for all $k = 1, 2, \dots, n$, where $S^{(n)}\text{DRA}$ means *strong strong ... (n times) decreasing risk aversion*¹⁶.

Table 1 below summarizes the relationship between these increasingly stronger concepts of decreasing risk aversion. The arrows indicate what implies what.

¹⁵For definition and more details concerning the early and classic concept of subtangent see, for example, Blakey (1962).

¹⁶Without stating it, we have been following the convention that decreasing means non-increasing unless specifically stated to be strictly decreasing.

The proofs can be seen from the proof of Theorem 1 below.

u	u_1	u_2	$\cdot$	$\cdot$	$\cdot$
$u' > 0$	$u'_1 > 0$	$u'_2 > 0$	$\cdot$	$\cdot$	$\cdot$
$u'' \leq 0$	$u''_1 \leq 0$	$u''_2 \leq 0$	$\cdot$	$\cdot$	$\cdot$
$u''' \geq 0$	$u'''_1 \geq 0$				
$u'''' \leq 0$	$\cdot$				
$\cdot$	$\cdot$				
$\cdot$	$\cdot$				
$\cdot$					

Table 1

In order to assure the reader that these definitions are not vacuous, we note that the class of functions $u(x) = ax^\alpha$, $a > 0$ and $0 < \alpha \leq 1$, satisfy $S^{(n)}DRA$ for all $n \geq 1$, i.e. *strongest decreasing risk aversion*.

Proposition 2 : If a vN-M utility function $u \in \mathbf{u}$ satisfies $S^{(2)}DRA$, then risk aversion in the large is decreasing and convex. Furthermore, if risk aversion in the large is decreasing and convex, then so must be the absolute risk aversion.

Proof of Proposition 2 : Since u satisfies $S^{(2)}DRA$, $u''_2 \leq 0$. Since $u_2 = u_1/u'_1$, $u'_2 = 1 + (-u''_1/u'_1)(u_1/u'_1)$. From (13), it follows that

$$\frac{-u'''_1(x)}{u''_1(x)} \geq 2 \frac{-u''_1(x)}{u'_1(x)} \text{ for all } x \geq 0.$$

Since by definition $u_1(x) = u(x)/u'(x)$, it means that

$$\frac{d^3}{dx^3} \frac{u(x)}{u'(x)} = \frac{d^2}{dx^2} \left(\frac{-u''(x)}{u'(x)} \frac{u(x)}{u'(x)} \right) = R''(x, u) > 0,$$

that is, risk aversion in the large is convex. From definition of $S^{(2)}DRA$, it is clearly decreasing. It is easily seen that the above inequality is equivalent to

$$\left(\frac{d^2}{dx^2} \frac{-u''(x)}{u'(x)} \right) \frac{u(x)}{u'(x)} + 2 \left(\frac{d}{dx} \frac{-u''(x)}{u'(x)} \right) \frac{d}{dx} \frac{u(x)}{u'(x)} + \frac{-u''(x)}{u'(x)} \frac{d^2}{dx^2} \frac{u(x)}{u'(x)} > 0.$$

Since

$$\frac{d^2}{dx^2} \frac{u(x)}{u'(x)} \leq 0 \text{ and } \frac{d}{dx} \frac{-u''(x)}{u'(x)} \leq 0$$

by SDRA, and $d/dx(u(x)/u'(x)) > 0$, it follows from the above inequality that

$$\frac{d^2}{dx^2} \frac{-u''(x)}{u'(x)} \geq 0.$$

This proves that absolute risk aversion must be nonincreasing and convex.

Proposition 2 confirms the intuition that if risk aversion in the large is decreasing and convex then so must be the risk aversion in the small. The converse of this, of course, cannot be true. ■

Convexity of absolute risk aversion means that higher the wealth, smaller the reduction in risk premium of a small risk for a given increase in wealth. It is a natural condition and known to be sufficient for risk vulnerability (Gollier and Pratt (1996))¹⁷. Proposition 2 thus shows that $S^{(2)}DRA$ implies risk vulnerability. Notice that the proof uses a condition which is barely necessary for $S^{(2)}DRA$. Actually, $S^{(2)}DRA$ is strong enough such that even the stronger concept of standardness (i.e. both prudence $(-u'''/u'')$ and absolute risk aversion $(-u''/u')$ are decreasing) and thus properness can be shown to follow similarly from it. Convexity of risk aversion in the large alone is not sufficient for standardness and standardness is not sufficient for convexity of absolute risk aversion. However, decreasing prudence and SDRA together are sufficient for both standardness and convexity of absolute risk aversion.

¹⁷Risk vulnerability means that adding an unfair background risk to wealth makes risk averse individuals more risk averse.

As noted earlier, mixed risk aversion is the most stringent refinement of the set of vN-M utility functions so far. We now show that strongest decreasing risk aversion implies mixed risk aversion.

Theorem 1: A vN-M utility function $u \in \mathbf{u}$ satisfies strongest decreasing risk aversion only if its first derivative u' is completely monotone on $(0, \infty)$.

Corollary 1: A vN-M utility function $u \in \mathbf{u}$ satisfies strongest decreasing risk aversion only if its absolute risk aversion $(-u''/u')$ is completely monotone on $(0, \infty)$.

Besides the class of utility functions $u(x) = ax^\alpha$, $0 < \alpha \leq 1$, another one with completely monotone absolute risk aversion is the class of functions which have a first derivative $u'(x) = e^{-x^{1+\alpha}}$, $0 < \alpha \leq 1$ ¹⁸.

A function f is *operator monotone* on $(0, \infty)$ if $(-1)^{n-1}f^n(x) \geq 0$ for $n = 1, 2, \dots$. Theorem 1 and Corollary 1 can be thus rephrased as follows : if u satisfies strongest decreasing risk aversion then u and $-\log u'$ are both operator monotone on $(0, \infty)$ ¹⁹.

Proof of Theorem 1: Let $D^n f$ denote the n -th derivative of f , i.e. $D^n f(x) = \frac{d^n}{dx^n} f(x)$.

Table 1 and a little reflection along with the fact that $u_1 = u/u'$ show that

¹⁸The integral of $u'(x) = e^{-x^{1+\alpha}}$ exists because it is bounded above by the integrable function e^{-x} over the interval $[1, \infty)$.

¹⁹Operator monotone functions have been widely used in matrix analysis (see, e.g. Bhatia (1996)). Very few functions have been identified to be operator monotone, a canonical example is the class of functions $f(x) = x^\alpha$, $0 < \alpha \leq 1$. It is known that if f is operator monotone on $(0, \infty)$ then f has a Taylor expansion $f(x) = \sum_{n=0}^{\infty} a_n(x-1)^n$ in which the coefficients a_n are positive for all odd n and negative for all even n . Clearly, the first derivative of an operator monotone function is completely monotone.

we need to prove only the following :

Given any $u \in \mathbf{u}$, let $m \geq 2$ be some integer. Then, $(-1)^n D^n(u/u') \leq 0$ for each n with $2 \leq n \leq m$ implies $(-1)^n D^n(u') \geq 0$ for each n with $2 \leq n \leq m$.

The proof for this assertion has two parts.

Claim 1: If $(-1)^n D^n(u/u') \leq 0$ for each n with $2 \leq n \leq m$, then $(-1)^n D^{n-1}(-u''/u') \leq 0$ for each n with $2 \leq n \leq m$.

The claim is clearly true for $n = 2$. From induction in n and the identity

$$D^n((u/u')(-u''/u')) = \sum_{r=0}^n \frac{n!}{(n-r)!r!} D^r(u/u') D^{n-r}(-u''/u'),$$

it is seen that it also holds for all n with $2 \leq n \leq m$.

Next, define $f(x) = -\log u'(x)$. Then $D^n f(x) = D^{n-1}(-u''/u')$ and $u'(x) = e^{-f(x)}$.

Claim 2 : If $(-1)^n D^n f(x) \leq 0$ for $2 \leq n \leq m$, then $(-1)^n D^n u'(x) \geq 0$ for $2 \leq n \leq m$. This claim follows from the following two identities

$$\begin{aligned} D^n u'(x) &= D^{n-1}(e^{-f(x)}(-Df(x))), \\ D^{n-1}(e^{-f(x)}(-Df(x))) &= \sum_{r=0}^{n-1} \frac{n!}{(n-r-1)!r!} D^r e^{-f(x)} D^{n-r-1}(-Df(x)) \end{aligned}$$

and induction in n after noting that the claim is true for $n = 2$.

Claims 1 and 2 together complete the required proof. ■

Proof of Corollary 1: If u satisfies strongest decreasing risk aversion, then as seen from the proof of Theorem 1 and Table 1, $(-1)^n D^n u_1 \leq 0$ for each $n \geq 2$, i.e. u'_1 is completely monotone. Claim 1 in the proof of Theorem 1 proves that if $(-1)^n D^n u_1 \leq 0$ for $n \geq 2$, then $(-1)^n D^{n-1}(-u''/u') \leq 0$ for $n \geq 2$ which proves that $-u''/u'$ is completely monotone. ■

Note that complete monotonicity of the first derivative of the utility function and / or of its absolute risk aversion are necessary but not sufficient for strongest decreasing risk aversion ²⁰.

How successful is our refinement strategy? As noted earlier, strongest decreasing risk aversion is satisfied by the class of functions $u(x) = ax^\alpha$, $a > 0$ and $0 < \alpha \leq 1$. The question is therefore whether this is the only class with this property? This is an interesting, but a difficult question. I do not know the complete answer, but I provide below some insights that might be of help.

Note, first that the operator $T : u \rightarrow u/u'$ has one and only one fixed point, namely $u(x) = x$ ²¹.

Theorem 2: If a vN-M utility function $u \in \mathbf{u}$ satisfies strongest decreasing risk aversion, then the successive iterates $u_{n+1} = Tu_n$, $n = 0, 1, \dots$ and $u_0 = u$, converge pointwise to the fixed point of T , i.e., $\lim_{n \rightarrow \infty} u_n(x) = x$ for all $x \geq 0$.

Proposition 3 : The successive iterates $u_{n+1} = Tu_n$, $n = 0, 1, \dots$ and $u_0 \in \mathbf{u}$, converge finitely iff u_0 belongs to the family $u(x) = ax^\alpha$, $a > 0$ and $0 < \alpha \leq 1$.

Proof of Theorem 2 : Since $u \in \mathbf{u}$ satisfies strongest decreasing risk aversion, $u_n \in \mathbf{u}$ for each $n \geq 0$, i.e., $u'_n > 0$ and $u''_n \leq 0$ for all $n \geq 0$.

(a) For each $n \geq 1$, since $u'_{n-1}(x)$ is nonincreasing in x , $u_n(x) = u_{n-1}(x)/u'_{n-1}(x) = \int_0^x u'_{n-1}(y)dy/u'_{n-1}(x) \geq x$ (using $u(0) = 0$).

(b) Since $u_n = u_{n-1}/u'_{n-1}$ and $u''_{n-1} \leq 0$, it is seen from differentiation

²⁰The utility function $u(x) = x/1 + x$, for example, does not satisfy even SDRA, but its first derivative and absolute risk aversion are both completely monotone.

²¹Clearly, if u is a fixed point of T , then $u = Tu = u/u'$ implies $u' = 1$.

that $u'_n(x) \geq 1$ for each $x \geq 0$ and $n \geq 1$.

(c) Since $u_{n+1} = u_n/u'_n$ and as shown in (b) $u'_n \geq 1$, for each x , $u_n(x)$ is nonincreasing in n for $n \geq 1$.

(d) In view of (c), let $v(x) = \lim_{n \rightarrow \infty} u_n(x)$, And in view of (a) $v(x) \geq x$.

Since for each x , $u_n(x)$ and $u_{n+1}(x)$ converge to the same limit and $u_{n+1}(x) = u_n(x)/u'_n(x)$, $\lim_{n \rightarrow \infty} u'_n(x) = 1$ if $v(x) > 0$, i.e., if $x > 0$. This means that if $x > 0$, then $u'_n(x)$ is bounded, i.e., there exist m and n_0 such that $u'_n(x) \leq m$ for $n \geq n_0$. Hence the set $\{u'_n(y) : y \geq x, n \geq 1\}$ is bounded, since $u'_n(x)$ is nonincreasing in x . Therefore, by the dominated convergence theorem

$$u_n(y) - u_n(x) = \int_x^y u'_n(z) dz$$

converges to $\int_x^y 1 dz = y - x$. Thus, $v(y) - v(x) = y - x$ for $y \geq x > 0$. However, since $u_n(0) = 0$ for each n , $v(0) = \lim_{n \rightarrow \infty} u_n(0) = 0$. This means that $\lim_{n \rightarrow \infty} u_n(y) = v(y) = y$ for each $y \geq 0$. This completes the proof. ■

Proof of Proposition 3 : The proof of “if” part is obvious. We prove the “only if” part. It is easily seen that

$$\frac{u''_n}{u'_n} = \frac{u'_n}{u_n} - \frac{u'_{n+1}}{u_{n+1}} \text{ for } n \geq 0.$$

If convergence is finite, then $u''_n = 0$ for some finite n . If $n = 0$, then by integration $u(x) = ax$; $a > 0$. If $n = 1$, then again by integration, and from the fact that $u_n = u_{n-1}/u'_{n-1}$, we obtain $u(x) = ax^\alpha$, $a > 0$ and $0 < \alpha < 1$. Similarly, if $n \geq 2$, then

$$u_{n-2} = ae^{(1/\alpha)x^{\frac{1}{\alpha}}} - a, \quad a > 0 \text{ and } 0 < \alpha < 1.$$

But this means that u_{n-2} is not concave, which contradicts that $u_{n-2} \in \mathbf{u}$. Hence $n \leq 1$, and $u(x) = ax^\alpha$, $a > 0$ and $0 < \alpha \leq 1$. ■

Theorem 2 and Proposition 3, reduce our question to the following one : does strongest decreasing risk aversion imply *finite* convergence of the successive iterates $u_{n+1} = Tu_n$, $n \geq 0$ and $u_0 \in \mathbf{u}$?

4 Tax Evasion and Risk Aversion

In the literature on optimal income tax enforcement (Reinganum and Wilde (1984)), Border and Sobel (1987), and Chander and Wilde (1998)) the tax evasion decision of the agent is modeled as a problem of choice under uncertainty involving large risks. Chander and Wilde (1998) show that if the agent is risk neutral (i.e. risk aversion is identically zero), the optimal income tax function must be increasing and concave ²². The same result however need not obtain when the agent is risk averse. First, the incentive to underreport income, which is equivalent to taking a large risk, is weaker if the agent is risk averse, thus making it easier or less costly for the principal to enforce any given tax function. Second, the incentive to underreport income is even weaker if the true income of the agent is higher and risk aversion is not sufficiently decreasing with income.

We show that if the vN-M utility function of the agent satisfies SDRA, then indeed the optimal income tax function, as in the risk neutral case, is increasing and concave.

²²In other words, the inability of the tax authority to costlessly observe true income severely restricts his ability to redistribute.

Consider an agent with random income y , which can take values over an interval $[0, \infty]$, $\bar{y} > 0$, according to a probability density function g with $g(y) > 0$ for all $y \in (0, \bar{y})$. The principal does not know the true income of the agent, but knows the density function g and can choose a *tax function* $t : [0, \bar{y}] \rightarrow R_+$, an *audit function* $p : [0, \bar{y}] \rightarrow [0, 1]$, and a *penalty function* $f : [0, \bar{y}] \times [0, \bar{y}] \rightarrow R_+$ so as to maximize his objective. The agent treating the policy of the principal parametrically submits a report so as to maximize his own expected utility, i.e.,

$$\max_x [p(x)u(y - t(x)) + (1 - p(x))u(y - f(y, x))],$$

where x is the reported income, y is the true income, and $u \in \mathbf{u}$ is the utility function of the agent. We assume that the agent can never pay more than his true income, accordingly, the tax and penalty functions must satisfy the following restriction for each y ,

$$0 \leq t(y) \leq y,$$

and

$$t(y) \leq f(y, x) \leq y \text{ for all } x.$$

Standard arguments (Mookherjee and Png (1989)) show that the revelation principle applies in this setting. Consequently, to solve for an optimal scheme, the principal can restrict his attention to schemes whereby the agent is induced to report his income truthfully. Thus, (t, p, f) must be such that for each y ,

$$0 \leq t(y) \leq f(y, y) \leq y,$$

$$0 \leq p(y) \leq 1,$$

and

$$(1 - p(y))u(y - t(y)) + p(y)u(y - f(y, y)) \geq (1 - p(x))u(y - t(x)) + p(x)u(y - f(x, y))$$

for all x with $t(x) \leq y$.

The latter inequalities say that agent's expected utility is maximized if he reports his income truthfully.

Observe that if $f(y, y) > t(y)$ then we can increase $t(y)$ and lower $f(y, y)$ in such a way that the incentive constraints for income level y are not reversed and they are weaker for income levels other than y . This means that $f(y, y)$ should be as small as possible. Thus $f(y, y) = t(y)$. Similarly $f(x, y)$, $x \neq y$ with $t(x) \leq y$ should be as large as possible. Thus, $f(x, y) = y$ for $x \neq y$ with $t(x) \leq y$. In view of these observations we only need to consider those schemes that satisfy for each y :

$$0 \leq p(y) \leq 1, \tag{15}$$

$$0 \leq t(y) \leq y, \tag{16}$$

and the incentive constraints

$$u(y - t(y)) \geq (1 - p(x))u(y - t(x)) \text{ for all } x \text{ with } t(x) \leq y, \tag{17}$$

since by assumption $u(0) = 0$.

As in most analysis of the principal-agent problem we assume henceforth that an agent reports truthfully whenever reporting truthfully is optimal.

Let Q denote the set of all schemes (t, p) that satisfy (15), (16) and (17). Then Q is the *set of all feasible schemes*.

Let S be some subset of Q , i.e., $S \in Q$.

A scheme $(t, p) \in S$ is *efficient* in S if there is no other scheme $(t', p') \in S$ such that $p' \leq p$, $t' \geq t$, and $p' \neq p$ or $t' \neq t$. That is, if it is not possible to not lower the taxes and decrease an audit probability without increasing any other audit probability, and it is not possible not to increase the audit probabilities and raise the tax of some income level without lowering it at any other level.

An optimal scheme must clearly be efficient if the objective of the principal is to maximize revenue net of audit cost, that is $\max_{t,p} \left[\int_0^{\bar{y}} t(y)g(y)dy - c \int_0^{\bar{y}} p(y)g(y)dy \right]$, where $c > 0$ is the cost per audit. It is possible to show that for a variety of other objectives also an optimal scheme must be efficient including when the objective of the principal is purely redistributive. Banerjee, Chakraborty and Chander (2000) show that for reasonable values of the parameters g , c and α the optimal scheme in the following case is also efficient : $\max_{t,p} \left[\int_0^{\bar{y}} (y + r - t(y))^\alpha g(y)dy \right]$ subject to $\int_0^{\bar{y}} t(y)g(y)dy - c \int_0^{\bar{y}} p(y)g(y)dy \geq r$, where the parameter α , $0 < \alpha \leq 1$, indicates principal's aversion to inequality. We thus only need to characterize efficient schemes.

5 A General Characterization

We do the characterization in two steps. In the first step we show that in an efficient scheme the tax and audit functions must be monotonic. In the second step we show that if the agent's utility function satisfies SDRA then the tax function must be concave.

Lemma 1. A scheme $(t, p) \in Q$ is efficient in Q only if t is nondecreasing and p is nonincreasing.

The lemma distinguishes itself from other results on this subject (Border

and Sobel (1987)) and Chander and Wilde (1998)) by allowing the agent to be risk averse ²³. Though the lemma is of interest in itself ²⁴, we need it for the proof of Theorem 3 below which is the main application of SDRA. The proof is thus included in the appendix to the paper.

We now consider an alternative interpretation of SDRA. Consider an agent with vN-M utility function u who is indifferent between a certain wealth w and a random wealth $(0, z; p, 1 - p)$ ²⁵. That is, w and z satisfy

$$u(w) = pu(0) + (1 - p)u(z) = (1 - p)u(z), \quad (18)$$

since by assumption $u(0) = 0$. Let $z(w)$ denote the solution of (18) for a given w . Given that u is strictly increasing, it follows that for p fixed $z(w)$ must be strictly increasing in w , i.e., $dz(w)/dw > 0$. In fact, since u is concave, $dz(w)/dw > 1$. By differentiating (18) twice it is seen that $d^2z(w)/dw^2 \leq 0$ for all $p \geq 0$ if and only if u satisfies SDRA. The relationship between z and w as implied by SDRA of u is illustrated in Figure 3 ²⁶. The “level curve” is parameterized by the probability p of the unfavorable outcome. Lower the p higher the level curve.

Theorem 3 : If the vN-M utility function u satisfies SDRA, then a scheme (t, p) is efficient in Q only if t is concave.

The proof involves showing that for each $w \in [0, \bar{y}]$ there exists a probability

²³Mookherjee and Png (1989) on the other hand allow the agent to be risk averse but do not obtain similar monotonicity results.

²⁴This is so because the results reported in the lemma are consistent with certain stylized facts in insurance markets.

²⁵This interpretation might explain why SDRA is related to the concepts in the theory of multiple risk bearing in which also initial wealth is considered to be random.

²⁶In words, the value of the lottery, w , increases more than proportionately with its prize, z .

$p \in [0, 1]$ and a level curve passing through $(w, z(w))$ such that the function $y - t(y)$ lies entirely above it.

Proof of Theorem 3 : Since (t, p) is efficient, given any $\hat{y} \in [0, \bar{y}]$ (as shown in the proof of Lemma 1) there exists a $\hat{x}$ such that $\hat{y} \geq t(\hat{x})$ and $u(\hat{y} - t(\hat{y})) = (1 - p(\hat{x}))u(\hat{y} - t(\hat{x}))$.

Three cases arise : $p(\hat{x}) = 0$, $p(\hat{x}) = 1$, and $0 < p(\hat{x}) < 1$. We consider these in the given order.

If $p(\hat{x}) = 0$, $u(\hat{y} - t(\hat{y})) = u(\hat{y} - t(\hat{x}))$ and thus $t(\hat{y}) = t(\hat{x})$. Since $u(y - t(y)) \geq u(y - t(\hat{x}))$ for all $y \geq t(\hat{x})$ (from (17) and that $p(\hat{x}) = 0$) and t is nondecreasing by Lemma 1, it follows that $t(y) = t(\hat{x})$ for all $y \geq \hat{x} \geq t(\hat{x})$. On the other hand since t is nondecreasing, $t(y) \leq t(\hat{x})$ for $y \leq \hat{x}$. Thus all of t lies below the line $\ell(y) = t(\hat{x}) = t(\hat{y})$ for all $y \in [0, \bar{y}]$.

If $p(\hat{x}) = 1$, $u(\hat{y} - t(\hat{y})) = 0$ i.e., $t(\hat{y}) = \hat{y}$. Since $y \geq t(y)$ for all $y \geq 0$, it follows that all of $t(y)$ lies below the 45° line $y = \ell(y)$ and $\ell(\hat{y}) = t(\hat{y})$.

For the remaining case $0 < p(\hat{x}) < 1$, define $z = y - t(\hat{x})$, $\hat{w} = \hat{y} - t(\hat{y})$, and $\hat{z} = \hat{y} - t(\hat{x})$. Consider the level curve corresponding to $p(\hat{x})$, that is the set of w and z satisfying $u(w) = (1 - p(\hat{x}))u(z)$ for all $z \geq 0$. As seen from Figure 3, there exists a line $k(z) = a(z - \tilde{z})$, $a > 0$, $\tilde{z} \geq 0$, such that $\hat{w} = k(\hat{z})$ and $w \geq k(z)$ for all w satisfying $u(w) \geq (1 - p(\hat{x}))u(z)$, $z \geq \tilde{z} \geq 0$.

Since from (17) $u(y - t(y)) \geq (1 - p(\hat{x}))u(y - t(\hat{x}))$ for all $y \geq t(\hat{x})$, $z = y - t(\hat{x})$ and $\tilde{z} = \tilde{y} - t(\hat{x})$ for some $\tilde{y} \geq t(\hat{x})$,

$$\begin{aligned} y - t(y) &\geq a[(y - t(\hat{x})) - (\tilde{y} - t(\hat{x}))] \text{ for all } y \geq \tilde{y} \geq t(\hat{x}) \\ &= a(y - \tilde{y}) \text{ for all } y \geq \tilde{y} \geq t(\hat{x}), \end{aligned}$$

and

$$\hat{y} - t(\hat{y}) = a(\hat{y} - \tilde{y}).$$

Since $y - t(y) \geq 0$ for all $y \geq 0$, the above inequality is true for $y \leq \tilde{y}$ as well.

This means that there exists α and β such that

$$t(y) \leq \alpha y + \beta \text{ for all } y \geq 0$$

and

$$t(\hat{y}) = \alpha \hat{y} + \beta.$$

We have thus shown that in all three cases given any $\hat{y} \in [0, \bar{y}]$, there exists an affine function $\ell(y)$ such that $t(\hat{y}) = \ell(\hat{y})$ and $t(y) \leq \ell(y)$ for all $y \geq 0$. This proves that t is concave.

6 Conclusion

By restricting ourselves to compensated risks, we have been able to discover a simple measure of risk aversion in the large. Besides satisfying the usual properties of the Arrow-Pratt measure, the index of risk aversion in the large leads to a stronger concept of decreasing risk aversion. We have been able to show that SDRA is sufficient to characterize individual behaviour under uncertainty in a model concerning large risks. We have been also able to relate SDRA to some other applications. Though our treatment has been illustrative rather than exhaustive, the concept of SDRA is clearly of more general interest and applicability. An obvious direction for future research is to find additional applications.

The concept of $S^{(2)}DRA$ too can be justified behaviorally as it implies convexity of risk aversion in the large, which can be similarly interpreted as

the natural property of convex absolute risk aversion, which too is implied by $S^{(2)}DRA$. In addition, $S^{(2)}DRA$ enables us to relate the central concepts of risk vulnerability, properness, and standardness, which are rooted in the theory of multiple risk bearing, to the standard Arrow-Pratt model concerning a single risk.

Though the concept of strongest decreasing risk aversion is difficult to justify behaviorally, it implies mixed risk aversion, which has been often studied in the literature (Pratt and Zeckhauser (1987) and Caballé and Pomansky (1996)) but which is equally difficult to justify in behavioral terms. It is seen that for the class of utility functions $u(x) = ax^\alpha$, $a > 0$ and $0 < \alpha \leq 1$, the subtangent is linear, that is, $u(x)/u'(x) = bx$. In terms of our refinement procedure, the canonical class of utility functions which mirrors the strongest decreasing risk aversion is “nearly” the same as the class of linear utility functions. In a sense, therefore, the strongest decreasing risk aversion is close to Machina’s (1982) characterization of declining risk aversion in a model with multiple risks in which neither initial wealth nor wealth increments are necessarily nonstochastic. He finds that in such a general setting risk aversion is nonincreasing only if the utility function is linear.

Appendix :

Proof of Lemma 1. The proof has two parts.

Claim 1 : If $(t, p) \in Q$ is efficient in Q , then the incentive constraints for each income level y must be binding at some x , that is, for each y there exists an x such that $u(y - t(y)) = (1 - p(x))u(y - t(x))$.

Suppose not, i.e., for some y , $u(y - t(y)) > (1 - p(x))u(y - t(x))$ for all x with $t(x) \leq y$. Then, $t(y) < y$ and we can find $t'(y)$ satisfying $y > t'(y) > t(y)$ such that $u(y - t'(y)) > (1 - p(x))u(y - t(x))$ for all x with $t(x) \leq y$. Since raising $t(y)$ to $t'(y)$ will not reverse the incentive constraints for income level other than y , we have been able to find a scheme $(t', p') \in Q$ such that $t' \geq t$, $t' \neq t$, and $p' = p$, which contradicts that (t, p, f) is efficient in Q . This proves Claim 1.

Claim 2 : If the incentive constraints for income levels y and y' are binding at x and x' respectively, and $y' > y$, then $t(x') \geq t(x)$. That is, if $y' > y$,

$$u(y - t(y)) = (1 - p(x))u(y - t(x)) \text{ for some } x \text{ with } t(x) \leq y$$

and

$$u(y' - t(y')) = (1 - p(x'))u(y' - t(x')) \text{ for some } x' \text{ with } t(x') \leq y', \text{ then } t(x') \geq t(x).$$

Suppose not, that is suppose $y' > y$, but $t(x') < t(x)$. Then from (17) $u(y' - t(y')) = (1 - p(x'))u(y' - t(x')) \geq (1 - p(z))u(y' - t(z))$ for all z with $t(z) \leq y'$ and

$$u(y - t(y)) = (1 - p(x))u(y - t(x)) \geq (1 - p(z))u(y - t(z)) \text{ for all } z \text{ with } t(z) \leq y.$$

In particular, since $y \geq t(x) > t(x')$,

$$u(y' - t(y')) = (1 - p(x'))u(y' - t(x')) \geq (1 - p(x))u(y' - t(x))$$

and

$$u(y - t(y)) = (1 - p(x))u(y - t(x)) \geq (1 - p(x'))u(y - t(x')).$$

Thus,

$$\frac{u(y' - t(x'))}{u(y' - t(x))} \geq \frac{1 - p(x)}{1 - p(x')} \geq \frac{u(y - t(x'))}{u(y - t(x))}.$$

which means

$$\frac{u(y' - t(x'))}{u(y - t(x'))} \geq \frac{u(y' - t(x))}{u(y - t(x))}$$

Which can be rewritten as

$$\frac{\int_0^{y-t(x')} u'(s)ds + \int_{y-t(x')}^{y'-t(x')} u'(s)ds}{\int_0^{y-t(x')} u'(s)ds} \geq \frac{\int_0^{y-t(x)} u'(s)ds + \int_{y-t(x)}^{y'-t(x)} u'(s)ds}{\int_0^{y-t(x)} u'(s)ds}.$$

Thus,

$$\frac{\int_{y-t(x')}^{y'-t(x')} u'(s)ds}{\int_0^{y-t(x')} u'(s)ds} \geq \frac{\int_{y-t(x)}^{y'-t(x)} u'(s)ds}{\int_0^{y-t(x)} u'(s)ds}.$$

Since u is concave, that is u' is decreasing, the above inequality cannot be true if $t(x') < t(x)$. Hence our supposition is wrong. This proves Claim 2.

We are now in a position to show that t must be nondecreasing. Suppose not, that is $y' > y$ and $t(y') < t(y)$. Then, $y \geq t(y) > t(y') \geq t(x')$ where x' is the point at which the incentive constraints are binding for taxpayer y' . It follows from (6) that $u(y - t(y)) \geq (1 - p(x'))u(y - t(x'))$. Since u is concave and $t(y) > t(x')$ it follows that $u(y' - t(y)) > (1 - p(x'))u(y' - t(x'))$. Since $t(y) > t(y')$ by supposition, $u(y' - t(y')) > (1 - p(x'))u(y' - t(x'))$. But this contradicts that the incentive constraints for taxpayer y' are binding at x' . Hence our supposition is wrong and $t(y') \geq t(y)$ for $y' > y$.

Finally, we prove that p is nonincreasing. The inequalities (17) imply that for each x , $p(x)$ must satisfy

$$p(x) \geq 1 - \frac{u(y - t(y))}{u(y - t(x))} \text{ for all } y \geq t(x).$$

Given that (t, p, f) is efficient, we must have

$$p(x) = 1 - \inf_{y > t(x)} \frac{u(y - t(y))}{u(y - t(x))}.$$

Since $t(x)$ is nondecreasing with x (as shown), it follows that $p(x)$ is nonincreasing.

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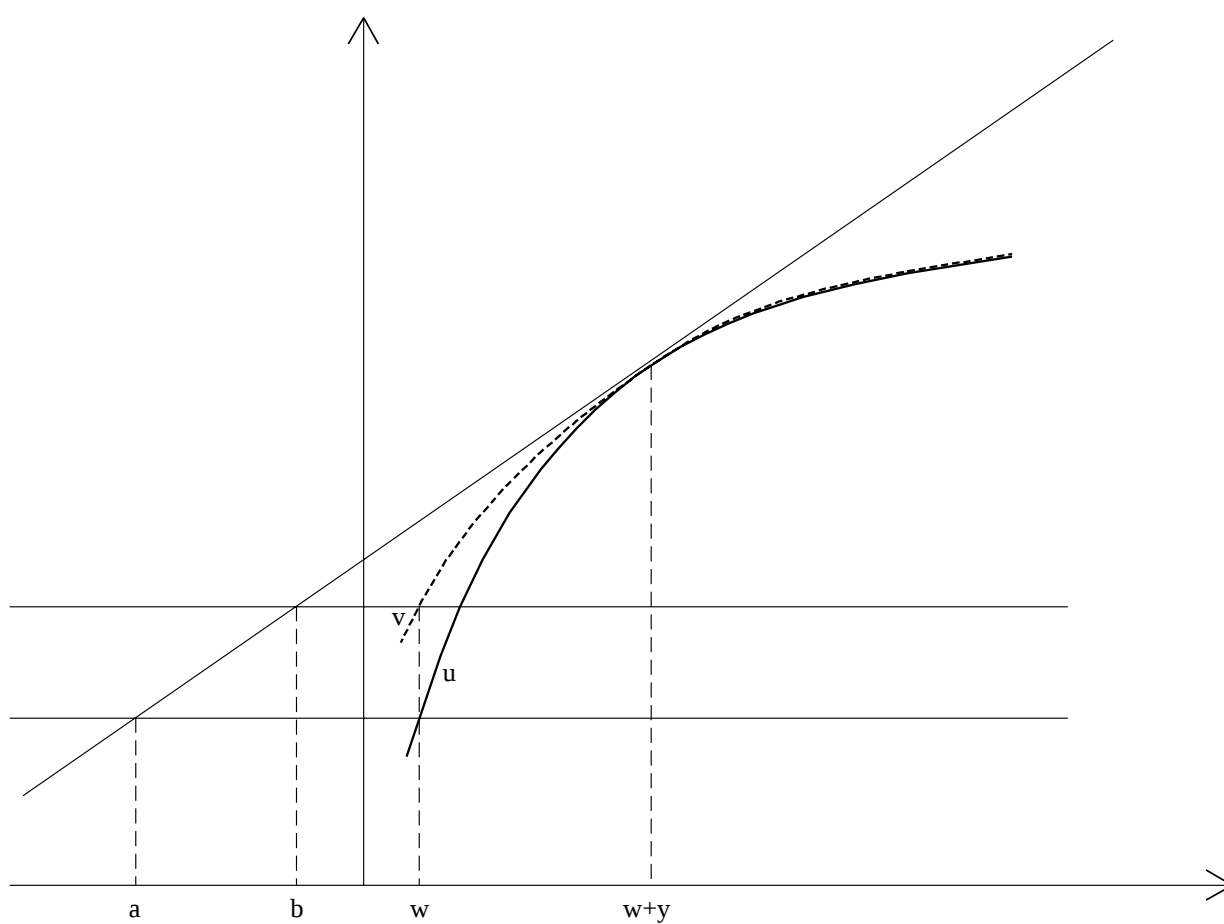


Figure 1

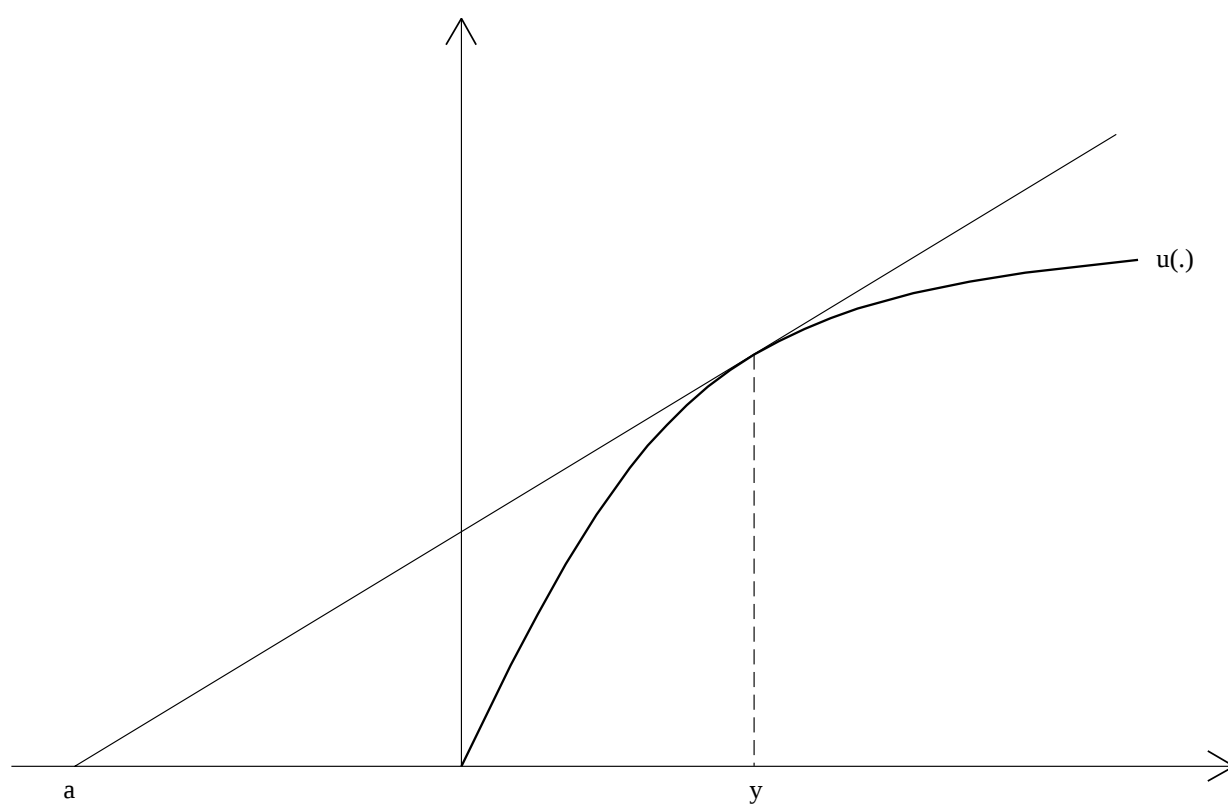


Figure 2

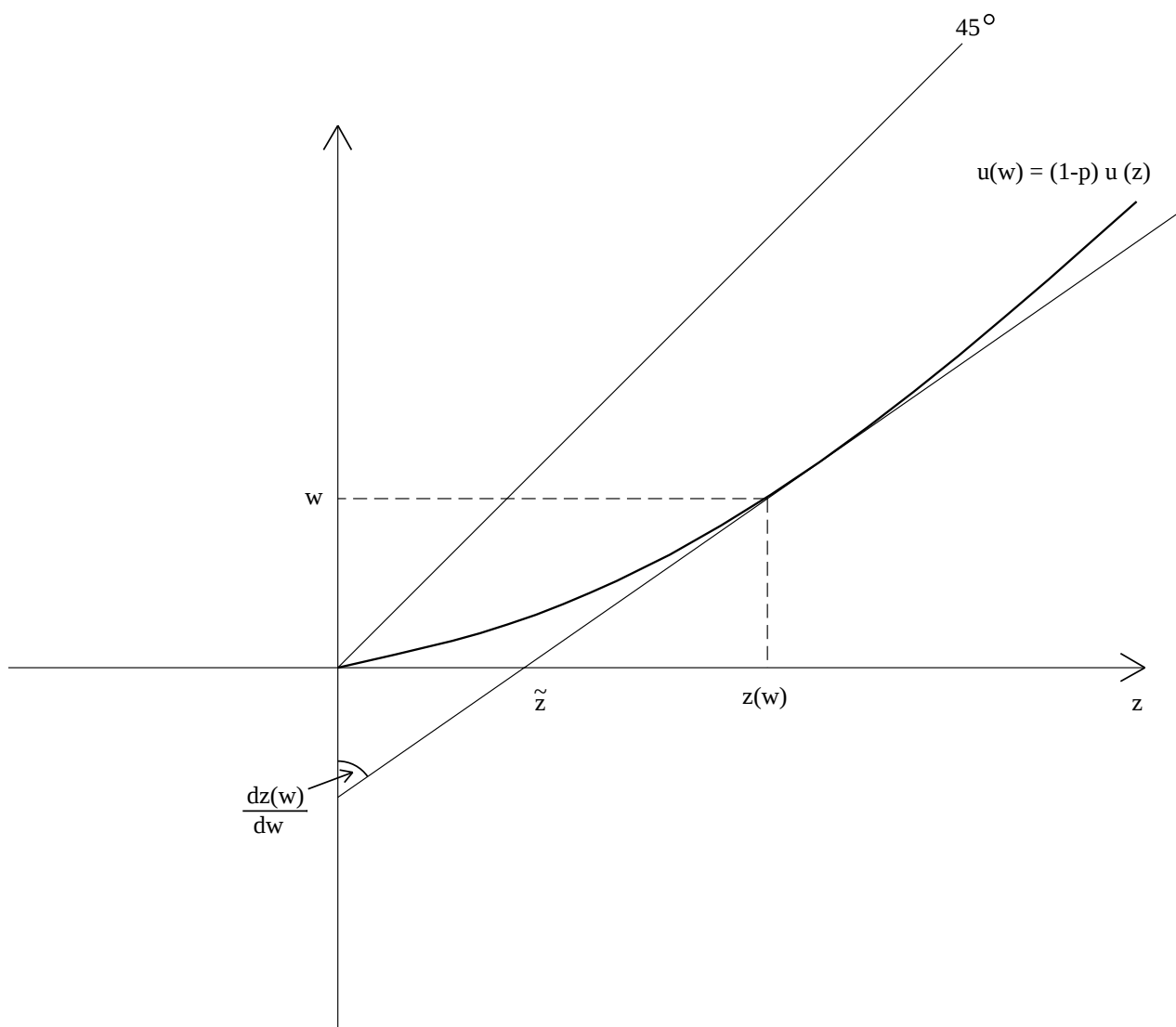


Figure 3