A modified 4D ROOSTER method using the Chambolle-Pock algorithm

Cyril Mory[1] and Laurent Jacques[2]

Abstract—**The 4D RecOnstructiOn using Spatial and TEmporal Regularization method is a recent 4D cone beam computed tomography algorithm. 4D ROOSTER has not been rigorously proved to converge. This paper aims to reformulate it using the Chambolle & Pock primal-dual optimization scheme. The convergence of this reformulated 4D ROOSTER is therefore guaranteed.**

I. Introduction

Four dimensional cone beam computed tomography (4D CBCT) of the free breathing thorax is important for image-guided radiation therapy (IGRT). Mainly two families of methods have been proposed to handle the problem: respiratory motion compensation, in which the motion of organs during breathing is estimated and used to deform the volume during reconstruction, and respiration-correlated reconstruction, in which several volumes are computed, each one from a subset of the projections that has been acquired during the correct respiratory phase. Advances in compressed sensing have been used to improve respiration-correlated reconstruction by enforcing spatial regularity constraints [1]–[4], but only a few recent methods exploit the strong correlation between successive respiratory phases [5]–[8]. 4D ROOSTER [8], which was recently proposed for 4D cardiac CBCT, is one such method. However, it came without a rigorous proof of convergence. In this paper, we propose a reformulation of 4D ROOSTER using the Chambolle & Pock primal-dual optimization scheme [9]. The convergence of this reformulated 4D ROOSTER is therefore guaranteed.

II. The original 4D ROOSTER method

This algorithm assumes that a rough segmentation of the patient's heart is available, and that movement is expected to occur only inside this segmented region. The method consists in iteratively enforcing five different constraints in an alternating manner. It starts by minimizing a quadratic data-attachment term $\sum_\alpha \|R_\alpha S_\alpha x - p_\alpha\|_2^2$, with α the projection angle, x a 4D sequence of volumes, R_α the forward projection operator at angle α, S_α a linear interpolator, and p_α the measured projection at angle α. This data-attachment term is minimized by conjugate gradient. Then the following regularization steps are applied sequentially: positivity enforcement, averaging along time outside the heart region, spatial total-variation denoising, and temporal total-variation denoising. This constitutes one iteration of the main loop, the output of which is fed back to the conjugate gradient minimizer for the next iteration. This algorithm offers no convergence guarantees. In [10], [11], using the theory of non-expansive mappings, it is only proved that if the main iteration has at least one fixed point, the algorithm converges to one of them. However, we show in this paper that each step of this method can be interpreted as a proximal operator [8], thus little effort is required to make it fit into the Chambolle & Pock framework. We remind that for $f : \mathbb{R}^N \to \mathbb{R}$ a closed convex function, the proximal operator of f is defined as $\mathrm{prox}_f(v) = \underset{x\in\mathbb{R}^N}{\arg\min}\ \frac{1}{2}\|x - v\|_2^2 + f(x)$.

III. Chambolle & Pock 4D ROOSTER

Setting $R = \begin{pmatrix} R_{\alpha_1} S_{\alpha_1} \\ \vdots \\ R_{\alpha_m} S_{\alpha_m} \end{pmatrix}$ and $p = \begin{pmatrix} p_{\alpha_1} \\ \vdots \\ p_{\alpha_m} \end{pmatrix}$, with m the number of projections, the data-attachment term becomes a single L^2 norm $\sum_\alpha \|R_\alpha S_\alpha x - p_\alpha\|_2^2 = \|Rx - p\|_2^2$. The 4D ROOSTER optimization problem can then be expressed as the search for

$$\begin{aligned} \underset{x\in\mathbb{R}^N}{\arg\min}\ & \frac{1}{2}\|Rx - p\|_2^2 + \lambda_2\|x\|_{TV_{space}} \\ & + \lambda_3\|x\|_{TV_{time}} + i_{\mathbb{R}^{N+}}(x) + i_{ROI}(x) \end{aligned} \tag{1}$$

with x the 3D+t sequence of volumes, with in total N voxels, p the set of measured projections, with in total P pixels, $R : \mathbb{R}^N \to \mathbb{R}^P$ the forward projection operator, $\|.\|_{TV_{space}}$ the spatial total-variation norm, $\|.\|_{TV_{time}}$ the temporal total-variation norm, $\mathbb{R}^{N+}$ the set of sequences of volumes in which all voxels have non-negative values, ROI the set of sequences of volumes in which all voxels outside the heart have equal values, $i_{\mathbb{R}^{N+}}(x)$ and $i_{ROI}(x)$ their respective convex indicator functions, and $\lambda_2, \lambda_3 > 0$ two parameters weighing the relative importance of the terms. We adopt the same formalism as in [12], and therefore use the following notations:

$$\begin{aligned} F_1 &: \mathbb{R}^P \to \mathbb{R}, t \to \frac{1}{2}\|t - p\|_2^2 \\ F_2 &: \mathbb{R}^N \to \mathbb{R}, t \to \lambda_2\|t\|_{TV_{space}} \\ F_3 &: \mathbb{R}^N \to \mathbb{R}, t \to \lambda_3\|t\|_{TV_{time}} \\ H &: \mathbb{R}^N \to \mathbb{R} \cup \{+\infty\}, t \to i_{\mathbb{R}^{N+}}(t) + i_{ROI}(t) \end{aligned}$$

Because the cost function has more than two terms, we have to reformulate the problem into a search in $\mathbb{R}^{3N}$ by

[1]iMagX project, ICTEAM Institute, Université catholique de Louvain, Louvain-la-Neuve, Belgium and Université de Lyon, CREATIS ; CNRS UMR5220 ; Inserm U1044 ; INSA-Lyon ; Université Lyon 1 ; Centre Léon Bérard, France

[2]ISP GROUP, ICTEAM/ELEN, Université catholique de Louvain, Louvain-la-Neuve, Belgium

defining $x' = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$, with x_1, x_2 and $x_3 \in \mathbb{R}^N$, and $\prod_{1,i} = \{x' \in \mathbb{R}^{3N} \mid x_1 = x_i\}$ for $i = 2, 3$. The 4D ROOSTER optimization problem becomes

$$\begin{aligned} \underset{x \in \mathbb{R}^{3N}}{\arg\min}\, & F_1(Rx_1) + F_2(x_2) + F_3(x_3) \\ & + i_{\mathbb{R}^{N+}}(x_1) + i_{ROI}(x_1) + i_{\prod_{1,2}}(x') + i_{\prod_{1,3}}(x') \end{aligned} \tag{2}$$

We define the total dimension $W = P + 2N$, the convex functions F and G and the linear operator K:

$$\begin{aligned} & F : \mathbb{R}^W \to \mathbb{R}, s = \begin{pmatrix} s_1 \\ s_2 \\ s_3 \end{pmatrix} \to \sum_{j=1..3} F_j(s_j) \\ & \quad \text{with } s_1 \in \mathbb{R}^P, s_2 \in \mathbb{R}^N, s_3 \in \mathbb{R}^N \\ & K : \mathbb{R}^{3N} \to \mathbb{R}^W, K = \begin{pmatrix} R & 0 & 0 \\ 0 & I_N & 0 \\ 0 & 0 & I_N \end{pmatrix} \\ & G : \mathbb{R}^{3N} \to \mathbb{R}, x' \to i_{\prod_{1,2}}(x') + i_{\prod_{1,3}}(x') + H(x_1) \end{aligned}$$

With these new notations, we can write the cost function as a sum of two functions $\underset{x \in \mathbb{R}^{3N}}{\arg\min}\, F(Kx') + G(x')$ on which the Chambolle-Pock algorithm applies. In compact form, the Chambolle-Pock algorithm is as follows: let $v'^{(0)} \in \mathbb{R}^W, x'^{(0)} \in \mathbb{R}^{3N}, y'^{(0)} \in \mathbb{R}^{3N}$, the update step is

$$\begin{cases} v'^{(k+1)} = \operatorname{prox}_{\gamma F^*}\left(v'^{(k)} + \gamma K y'^{(k)}\right) \\ x'^{(k+1)} = \operatorname{prox}_{\mu G}\left(x'^{(k)} - \mu K^* v'^{(k+1)}\right) \\ y'^{(k+1)} = 2x'^{(k+1)} - x'^{(k)} \end{cases}$$

If we expand this for our problem and simplify it since $x_1 = x_2 = x_3$ and $y_1 = y_2 = y_3$, we obtain:

$$\begin{cases} v_1^{(k+1)} = \operatorname{prox}_{\gamma F_1^*}\left(v_1^{(k)} + \gamma R y_1^{(k)}\right) \to \text{Data fidelity} \\ v_2^{(k+1)} = \operatorname{prox}_{\gamma F_2^*}\left(v_2^{(k)} + \gamma y_2^{(k)}\right) \to \text{TV in space} \\ v_3^{(k+1)} = \operatorname{prox}_{\gamma F_3^*}\left(v_3^{(k)} + \gamma y_3^{(k)}\right) \to \text{TV in time} \\ x^{(k+1)} = \operatorname{prox}_{\frac{\mu}{3} H}\left(3x_1^{(k)} - \mu\left(R^* v_1^{(k+1)}\right.\right. \\ \quad \left.\left. + v_2^{(k+1)} + v_3^{(k+1)}\right)\right) \to \text{Positivity and ROI} \\ y^{(k+1)} = 2x^{(k+1)} - x^{(k)} \to \text{Update step} \end{cases}$$

Four new operators appear in this formulation. $R^* : \mathbb{R}^P \to \mathbb{R}^N$ is the back projection operator. To express the other ones, we use the relation between the proximal operator of a function and the proximal operator of its adjoint:

$$\operatorname{prox}_{\gamma F^*}(x) = x - \gamma \operatorname{prox}_{\frac{1}{\gamma} F}\left(\frac{1}{\gamma} x\right)$$

Therefore

$$\operatorname{prox}_{\gamma F_1^*}(x) = x - \gamma\left(\frac{x + p}{\gamma + 1}\right) = \frac{x - \gamma p}{\gamma + 1}$$

The proximal operator for the spatial and temporal TV norms can be computed iteratively using the Chambolle algorithm [13], simply by using either the spatial or the temporal gradient and divergence operators

$$\begin{aligned} & \operatorname{prox}_{\frac{1}{\gamma} TV}(x) = x - \tfrac{1}{\gamma} \operatorname{div} \overline{p}, \text{where } \overline{p} = \lim_{n \to +\infty} p^{(n)} \\ & p^{(0)} = 0 \text{ and for every voxel } i, \\ & p_i^{(n+1)} = \frac{p_i^n + h\left(\nabla\left(\operatorname{div} p^{(n)} - \gamma x\right)\right)_i}{1 + h\left(\nabla\left(\operatorname{div} p^{(n)} - \gamma x\right)\right)_i}, \text{with } h > 0 \end{aligned}$$

And the proximal operator of γF_2^*(and similarly γF_3^*) can be obtained from the same relationship as previously

$$\begin{aligned} & \operatorname{prox}_{\gamma F_2^*}\left(v_2^{(k)} + \gamma y_2^{(k)}\right) \\ & \quad = v_2^{(k)} + \gamma y_2^{(k)} - \gamma \operatorname{prox}_{\frac{1}{\gamma} F_2}\left(\frac{1}{\gamma}\left(v_2^{(k)} + \gamma y_2^{(k)}\right)\right) \\ & \quad = v_2^{(k)} + \gamma\left(y_2^{(k)} - \operatorname{prox}_{\frac{1}{\gamma} F_2}\left(\frac{1}{\gamma} v_2^{(k)} + y_2^{(k)}\right)\right) \end{aligned}$$

With this new formulation, the modified 4D ROOSTER is guaranteed to converge. The theory of proximal algorithms also helps find the suitable parameters based on the norm of R, while they can only be determined empirically in the original method.

IV. Perspectives

4D ROOSTER can be applied to respiration-correlated 4D CBCT in order to reduce the level of streak artifacts in the reconstructions. Preliminary results are provided in Figure 1 and 2. They have been obtained using the original 4D ROOSTER method, and are compared with a standard ECG-gated FDK reconstruction. The modified 4D ROOSTER method presented in this paper will be implemented in the near future using the RTK framework [14].

References

[1] F. Bergner, T. Berkus, M. Oelhafen, P. Kunz, T. Pan, R. Grimmer, L. Ritschl, and M. Kachelrieß, "An investigation of 4D cone-beam CT algorithms for slowly rotating scanners," *Medical Physics*, vol. 37, no. 9, p. 5044, 2010.

[2] Z. Qi and G.-H. Chen, "Extraction of tumor motion trajectories using PICCS-4DCBCT: a validation study," *Medical physics*, vol. 38, no. 10, pp. 5530–5538, Oct. 2011, PMID: 21992371 PMCID: PMC3195374.

[3] S. Leng, J. Tang, J. Zambelli, B. Nett, R. Tolakanahalli, and G.-H. Chen, "High temporal resolution and streak-free four-dimensional cone-beam computed tomography," *Physics in medicine and biology*, vol. 53, no. 20, pp. 5653–5673, Oct. 2008, PMID: 18812650.

[4] E. Y. Sidky and X. Pan, "Image reconstruction in circular cone-beam computed tomography by constrained, total-variation minimization," *Physics in Medicine and Biology*, vol. 53, no. 17, pp. 4777–4807, Sep. 2008.

[5] H. Gao, R. Li, Y. Lin, and L. Xing, "4D cone beam CT via spatiotemporal tensor framelet," *Medical Physics*, vol. 39, no. 11, pp. 6943–6946, Nov. 2012, PMID: 23127087 PMCID: PMC3494730.

[6] H. Wu, A. Maier, R. Fahrig, and J. Hornegger, "Spatial-temporal total variation regularization (STTVR) for 4D-CT reconstruction," in *Proceedings of SPIE Medical Imaging 2012*, N. J. Pelc, R. M. Nishikawa, and B. R. Whiting, Eds., San Diego, CA, USA, Feb. 2012, p. 83133J.

[7] Z. Tian, X. Jia, B. Dong, Y. Lou, and S. B. Jiang, "Low-dose 4DCT reconstruction via temporal nonlocal means," *Medical physics*, vol. 38, no. 3, pp. 1359–1365, Mar. 2011, PMID: 21520846.

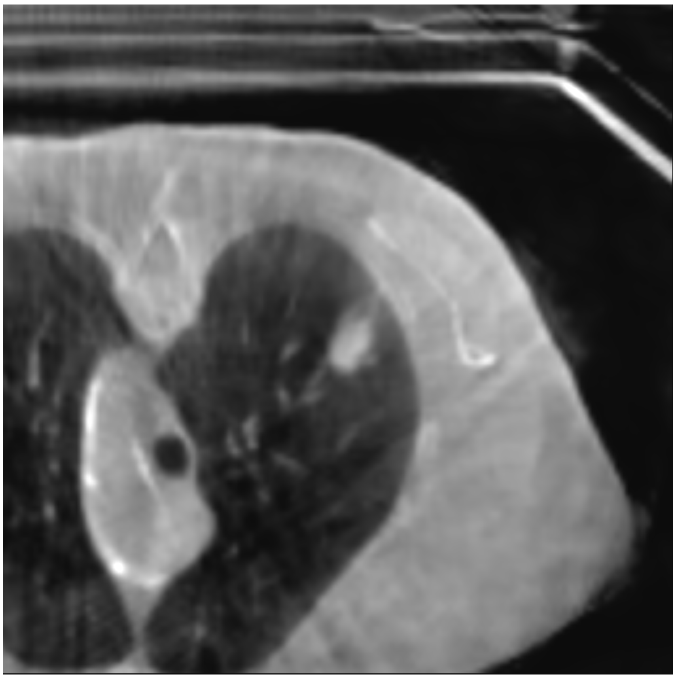

Figure 1. Respiratory-gated reconstruction of a lung tumor using 4D ROOSTER

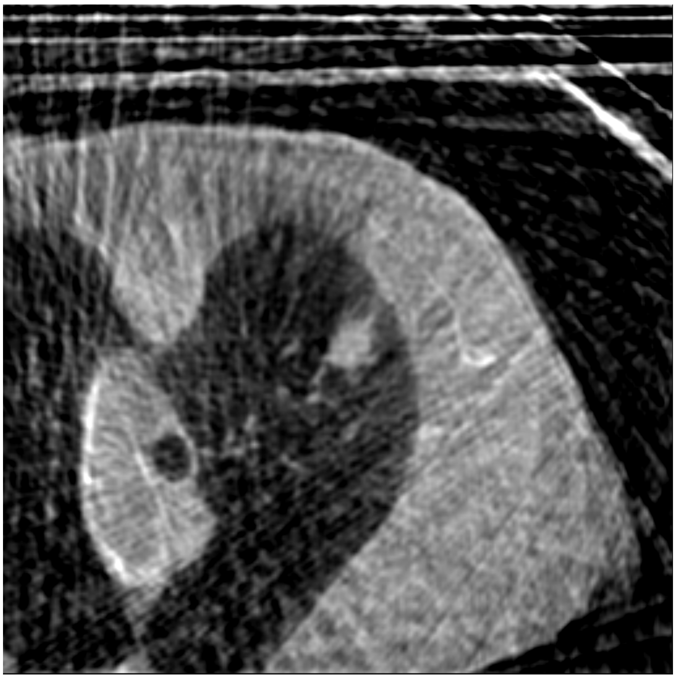

Figure 2. Respiratory-gated reconstruction of a lung tumor using the Feldkamp, Davis and Kress method

[8] C. Mory, "Cardiac c-arm computed tomography," Ph.D. dissertation, Université Lyon 1, Feb. 2014.

[9] A. Chambolle and T. Pock, "A first-order primal-dual algorithm for convex problems with applications to imaging," *Journal of Mathematical Imaging and Vision*, vol. 40, no. 1, pp. 120–145, May 2011.

[10] J. J. Moreau, "Fonctions convexes duales et points proximaux dans un espace hilbertien," *Comptes Rendus de l'Académie des Sciences (Paris)*, vol. 255, pp. 2897–2899, 1962.

[11] P. L. Combettes and J.-C. Pesquet, "Proximal splitting methods in signal processing," in *Fixed-Point Algorithms for Inverse Problems in Science and Engineering*, ser. Springer Optimization and Its Applications, H. H. Bauschke, R. S. Burachik, P. L. Combettes, V. Elser, D. R. Luke, and H. Wolkowicz, Eds. Springer New York, Jan. 2011, pp. 185–212.

[12] A. Gonzalez, L. Jacques, C. De Vleeschouwer, and P. Antoine, "Compressive optical deflectometric tomography: A constrained total-variation minimization approach," *Journal of Inverse Problems and Imaging*, In press.

[13] A. Chambolle, "An algorithm for total variation minimization and applications," *J. Math. Imaging Vis.*, vol. 20, no. 1-2, p. 89–97, Jan. 2004.

[14] S. Rit, M. Vila Oliva, S. Broumische, R. Labarbe, D. Sarrut, and G. C. Sharp, "The reconstruction toolkit (RTK), an open-source cone-beam CT reconstruction toolkit based on the insight toolkit (ITK)," in *Proceedings of the International Conference on the Use of Computers in Radiation Therapy (ICCR)*, 2013.