

MICHELL TRUSSES IN TWO DIMENSIONS AS A Γ -LIMIT OF OPTIMAL DESIGN PROBLEMS IN LINEAR ELASTICITY

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ABSTRACT. We reconsider the minimization of the compliance of a two dimensional elastic body with traction boundary conditions for a given weight. It is well known how to rewrite this optimal design problem as a nonlinear variational problem. We take the limit of vanishing weight by sending a suitable Lagrange multiplier to infinity in the variational formulation. We show that the limit, in the sense of Γ -convergence, is a certain Michell truss problem. This proves a conjecture by Kohn and Allaire.

1. INTRODUCTION

The aim of the present article is to derive a certain form of the Michell truss problem from an optimal design problem in linear elasticity in two dimensions. The optimal design problem we consider is the following classical question: Consider an elastic body of given weight loaded in plane stress. Which shape of the body minimizes the compliance (work done by the load)? There exist several different approaches to this problem; here we are going to be concerned with the “homogenization method” that has been developed by Lurie et al. [22], Gibiansky and Cherkasov [17], Murat and Tartar [26], Kohn and Strang [20], and others. The homogenization method rephrases the compliance optimization problem as a two-phase design problem, and then enlarges the set of permissible designs via relaxation. We refer the interested reader to Allaire’s book [1] for a more thorough account of the method.

On a formal level, it has been noted by Allaire and Kohn [2] that the relaxed formulation of the problem leads to a different variational problem in the limit of vanishing weight, namely a certain variant of the Michell truss problem (see also [4, 28]). This problem was first stated by Michell in 1904 [24]. Michell trusses are elastic structures that consist of linear truss elements, each of which can withstand a certain tensile or compressive stress. The variational problem consists in finding the Michell truss of least weight that is admissible in the sense that it resists a given load. Michell himself already knew that this problem has no solution in general, and relaxation is required to assure existence of solutions. Since this day, the theory of Michell trusses has been very popular in the engineering and mathematics community. We refrain from attempting to give a comprehensive list of the relevant literature, and refer the reader to [19, 27].

In the present article, we are going to cast the formal observations by Kohn and Allaire into a rigorous statement. More precisely, we are going to prove that the Michell truss problem is the limit of the compliance minimization problem in linear elasticity for vanishing weight in the sense of Γ -convergence [9].

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The plan of the paper is as follows: Section 2 below is supposed to give the reader a quick overview over the setting and main result. It consists of four Subsections: In Sections 2.1 and 2.2, we are going to state the compliance minimization problem for positive weight and the Michell problem respectively. In Section 2.3, our aim is to give the reader a good idea of our main result as quickly as possible, without too many preparatory definitions. This is why we first state a special case of our main theorem, Theorem 2.2. In Section 2.4, we give a short explanation of the form of the variational formulation of the compliance minimization problem that we had presented in Section 2.1. In Section 3, we are going to collect some results from the literature. In Section 4, we explain the manner in which we use Airy potentials for the solution of elasticity problems, and we state our main Γ -convergence result, Theorem 4.7. Section 5 contains the proof of the compactness and upper bound part of Theorem 4.7, and Section 6 contains the proof of the lower bound. The appendix consists of two parts: In Section A, we prove some facts on the relaxation of integral functionals whose integrands depend on second gradients, which we were unable to find in the literature. In Section B, we derive the 2-quasiconvexification of the integrand in the compliance minimization problem.

Notation. The symbol “ C ” is used as follows: A statement such as “ $f \leq Cg$ ” is shorthand for “there exists a constant $C > 0$ such that $f \leq Cg$ ”. The value of C may change within the same line. For $f \leq Cg$, we also write $f \lesssim g$.

2. SETTING AND (A SPECIAL CASE OF THE) MAIN RESULT

In the present section, our aim is to present first, the optimal design problems in linear elasticity, second, the Michell truss problem in its variational form, and third, a special case of our main theorem that links these problems via Γ -convergence. On the one hand, this special case does not require a lot of preparatory definitions, and on the other hand, it is not much weaker than the full result.

For a bounded open set $U \subset \mathbb{R}^n$, $k \in \mathbb{N}$ and $1 \leq p \leq \infty$, the Sobolev space $W^{k,p}(U)$ is defined by its norm

$$\|u\|_{W^{k,p}(U)} = \sum_{|\alpha| \leq k} \|\partial^\alpha u\|_{L^p(U)},$$

where the sum runs over multiindices $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n$ with $|\alpha| = \sum_i \alpha_i \leq k$ and $\partial^\alpha = \prod_i \partial_i^{\alpha_i}$. For $p = 2$, we use the notation

$$H^k(U) = W^{k,2}(U).$$

For the spaces with homogeneous boundary conditions, we use the notation

$$W_0^{k,p}(U) = \{u \in W^{k,p}(U) : \nabla^\alpha u = 0 \text{ on } \partial\Omega \text{ for } |\alpha| \leq k-1\}.$$

The fractional Sobolev spaces $W^{s,p}(U)$ with $k \in \mathbb{N}$, $k < s < k+1$ and $p \in [1, \infty)$ are defined by the Gagliardo norm,

$$\|u\|_{W^{s,p}(U)} = \|u\|_{W^{k,p}(U)} + \int_U dx \int_U dy \frac{|\nabla^k u(x) - \nabla^k u(y)|^p}{|x - y|^{n+sp}}.$$

For compact, $n-1$ -rectifiable subsets $S \subset \mathbb{R}^n$, we define the norms $\|u\|_{W^{k,p}(S)}$, $\|u\|_{W^{s,p}(S)}$ by a suitable cover of S by the ranges of Bilipschitz maps, and we write $W^{s,2}(S) = H^s(S)$. The dual of $H^s(S)$ is denoted by $H^{-s}(S)$, and the dual of $W^{k,\infty}(S)$ is denoted by $W^{-k,1}(S)$.

We write $\mathbb{R}_{\text{sym}}^{n \times n} := \{M \in \mathbb{R}^{n \times n} : M^T = M\}$. On $\mathbb{R}^{n \times n}$, we introduce a scalar product by

$$\xi : \xi' = \sum_{i,j} \xi_{ij} \xi'_{ij}.$$

We will also use the notation $|\xi|^2 = \xi : \xi$ for $\xi \in \mathbb{R}^{n \times n}$.

In the present paper, we are going to derive a result for bounded open sets $\Omega \subset \mathbb{R}^2$. More precisely, the symbol Ω will be reserved for sets satisfying the following definition:

Definition 2.1. *From now on, we assume that $\Omega \subset \mathbb{R}^2$ has the following properties:*

- (i) Ω is open, bounded, connected and simply connected
- (ii) There exists a finite number of points $x_i \in \partial\Omega$, $i = 1, \dots, N$, with pairwise disjoint neighborhoods U_i of x_i and C^2 -diffeomorphisms $\varphi_i : U_i \rightarrow \varphi(U_i)$ such that $\varphi(x_i) = 0$, $\varphi(\Omega \cap U_i) \subset (0, 1)^2$ and $\varphi(\partial\Omega \cap U_i) \subset [0, 1] \times \{0\} \cup \{0\} \times [0, 1]$.
- (iii) $\partial\Omega$ is C^2 away from $\{x_i : i = 1, \dots, N\}$.

The purpose of (ii) and (iii) above is that the operator $W^{2,1}(\Omega) \rightarrow L^1(\partial\Omega)$, $u \mapsto \nabla u \cdot n$ is surjective, where n denotes the outer unit normal to $\partial\Omega$, see Theorem 3.10 below. We will denote the outer unit normal of $\partial\Omega$ by $n = (n_1, n_2)$, and define a tangent vector $\tau = n^\perp = (-n_2, n_1)$. We denote the tangential derivative by ∂_τ , and the normal derivative by ∂_n .

2.1. The compliance minimization problem. For $\lambda > 0$, let $F_\lambda : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}$ be defined by

$$F_\lambda(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ \lambda + |\xi|^2 & \text{else.} \end{cases}$$

In the following, the parameter λ is a Lagrange multiplier for the weight of the two-dimensional elastic structure. Taking the limit of vanishing weight corresponds to the limit $\lambda \rightarrow \infty$. We define the functionals for finite λ with boundary conditions fixed by the choice of some $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^2)$. The space of allowed stresses is given by

$$S_g(\Omega) = \{\sigma \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) : \operatorname{div} \sigma = 0 \text{ in } \Omega, \sigma \cdot n = g \text{ on } \partial\Omega\},$$

where n denotes the unit outer normal of Ω . The integral functional for finite λ is given by

$$\mathcal{G}_{g,\lambda}(\sigma) = \begin{cases} \lambda^{-1/2} \int_\Omega F_\lambda(\sigma) dx & \text{if } \sigma \in S_g \\ +\infty & \text{else.} \end{cases}$$

For explanation of the fact that the traditional form of compliance minimization is equivalent to the minimization of $\mathcal{G}_{g,\lambda}$, see Section 2.4.

2.2. The Michell truss problem. For the definition of the limit functional, we need to collect some more notation.

For any Borel set $U \subset \mathbb{R}^n$, let $\mathcal{M}(U)$ denote the set of signed Radon measures on U . We denote by $\mathcal{M}(U; \mathbb{R}^p)$ the $\mathbb{R}^p$ valued Radon measures on U . Furthermore, let $\mathcal{M}(U; \mathbb{R}_{\text{sym}}^{n \times n})$ denote the space $\{\mu \in \mathcal{M}(U; \mathbb{R}^{n \times n}) : \mu_{ij} = \mu_{ji} \text{ for } i \neq j\}$. For $\mu \in \mathcal{M}(U; \mathbb{R}^p)$, let $|\mu|$ denote the total variation measure (see Section 3.1). For $\mu \in \mathcal{M}(U; \mathbb{R}^p)$, we have by the Radon-Nikodym differentiation Theorem (see Theorem 3.1 below) that for $|\mu|$ -almost

every $x \in U$, the derivative $d\mu/d|\mu|$ exists. For any one-homogeneous function $h : \mathbb{R}^p \rightarrow \mathbb{R}$ and any $\mu \in \mathcal{M}(U; \mathbb{R}^p)$, we may hence define

$$h(\mu) = h\left(\frac{d\mu}{d|\mu|}\right) d|\mu|.$$

This is a well defined Borel measure.

Now let $U \subset \mathbb{R}^n$ be open. Let $\mathcal{E}'(U; \mathbb{R}^p)$ denote the dual of $C^1(U; \mathbb{R}^p)$, i.e., the space of $\mathbb{R}^p$ valued distributions whose support is compactly contained in U . Let $\mu \in \mathcal{M}(\bar{U}; \mathbb{R}^n)$, $f \in \mathcal{E}'(\bar{U})$. We say that $-\operatorname{div} \mu = f$ if and only if

$$\int_{\bar{U}} \nabla \varphi \cdot \frac{d\mu}{d|\mu|} d|\mu| = \langle f, \varphi \rangle \quad (1)$$

for every $\varphi \in C^1(\mathbb{R}^n)$. Here μ and f are viewed, respectively, as a measure and a distribution on $\mathbb{R}^n$ supported on $\bar{U}$. If f has support in the boundary ∂U , then this induces a boundary condition for μ . Just as in the equation above, the notation $\langle \cdot, \cdot \rangle$ will denote the pairing of topological vector spaces with their dual in the sequel. It will always be clear from the context which pairing is meant. For $\mu \in \mathcal{M}(U; \mathbb{R}_{\text{sym}}^{n \times n})$ and $f \in \mathcal{E}'(U; \mathbb{R}^n)$, we say that $-\operatorname{div} \mu = f$ if the equation holds for every row, $-\operatorname{div} \mu_i = f_i$ for $i = 1, \dots, n$.

For $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$, let $\lambda_1(\xi), \lambda_2(\xi)$ denote the eigenvalues of ξ . We set

$$\rho^0(\xi) := \sum_{i=1}^2 |\lambda_i(\xi)|.$$

We will repeatedly use the following estimates:

$$|\xi| \leq \rho^0(\xi) \leq 2|\xi|. \quad (2)$$

Note that $\rho^0 : \mathbb{R}_{\text{sym}}^{2 \times 2} \rightarrow \mathbb{R}$ is sublinear and positively one-homogeneous.

For $U \subset \mathbb{R}^2$, $\mu = (\mu_1, \mu_2) \in \mathcal{M}(U; \mathbb{R}^2)$, we write $\mu^\perp := (-\mu_2, \mu_1)$ and $\operatorname{curl} \mu = \operatorname{div} \mu^\perp$. Again, for $\mu \in \mathcal{M}(U; \mathbb{R}_{\text{sym}}^{2 \times 2})$ and $f \in \mathcal{E}'(U; \mathbb{R}^2)$, we say that $\operatorname{curl} \mu = f$ if the equation holds for every row.

Now let Ω be as in Definition 2.1. For $g \in W^{-1,1}(\partial\Omega; \mathbb{R}^2)$, let the space of permissible stresses be given by

$$\Sigma_g(\Omega) = \{\sigma \in \mathcal{M}(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{2 \times 2}) : -\operatorname{div} \sigma = g\mathcal{H}^1 \llcorner \partial\Omega\}.$$

With these preparations, we are ready to define the Michell problem for for traction boundary values $g \in W^{-1,1}(\partial\Omega; \mathbb{R}^2)$,

$$\mathcal{G}_g(\sigma) = \begin{cases} 2\rho^0(\sigma)(\bar{\Omega}) & \text{if } \sigma \in \Sigma_g(\bar{\Omega}) \\ +\infty & \text{else.} \end{cases}$$

For a motivation of this functional in the context of structural optimization, we refer to [1, 2].

2.3. Gamma convergence. We want to approximate the functional $\mathcal{G}_g$ by the functionals $\mathcal{G}_{g,\lambda}$ in the sense of Γ -convergence. We assume that Ω satisfies Definition 2.1. We introduce the trace operators

$$\begin{aligned}\gamma_0 : u &\mapsto u|_{\partial\Omega} \\ \gamma_1 : u &\mapsto \nabla u|_{\partial\Omega} \cdot n.\end{aligned}$$

For the properties of the trace operators, see Section 3.4 below.

As a special case of our main theorem, we have that for $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^2)$,

$$\mathcal{G}_{g,\lambda} \xrightarrow{\Gamma} \mathcal{G}_g.$$

More precisely:

Theorem 2.2. *Let $\Omega \subset \mathbb{R}^2$ satisfy Definition 2.1 and let $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^2)$.*

- (i) *Compactness: Let $\{\sigma_\lambda\}_\lambda \subset L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ be such that $\mathcal{G}_{g,\lambda}(\sigma_\lambda) < C$. Then there exists a subsequence (no relabeling) and $\sigma \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ such that $\sigma_\lambda \rightarrow \sigma$ weakly $*$ in the sense of measures.*
- (ii) *Lower bound: If $\sigma_\lambda \rightarrow \sigma$ weakly $*$ in the sense of measures, then*

$$\liminf_{\lambda \rightarrow \infty} \mathcal{G}_{g,\lambda}(\sigma_\lambda) \geq \mathcal{G}_g(\sigma). \quad (3)$$

- (iii) *Upper bound: For every $\sigma \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ there exists a sequence $\{\sigma_\lambda\}_\lambda \subset L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ such that $\sigma_\lambda \rightarrow \sigma$ weakly $*$ in the sense of measures and $\lim_{\lambda \rightarrow \infty} \mathcal{G}_{g,\lambda}(\sigma_\lambda) = \mathcal{G}_g(\sigma)$.*

Remark 2.3. (i) For the sake of simplicity, we have here set the same boundary conditions $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^2)$ for the approximating and the limit problem. Actually, one would like to obtain a larger class of allowed “boundary conditions” for the limit problem. For example, one would like to consider $g = \sum_{i=1}^M v_i \delta_{x_i}$, where $x_i \in \partial\Omega$, $v_i \in \mathbb{R}^2$ for $i = 1, \dots, M$ and δ_x denotes the distribution defined by $\delta_x(f) = f(x)$. These boundary values are the ones that one considers in the Michell problem, see [5]. Distributions g of this type are not in $H^{-1/2}(\partial\Omega; \mathbb{R}^2)$, but they do belong to $W^{-1,1}(\partial\Omega; \mathbb{R}^2)$, which at first glance might look like the “natural” space for the boundary conditions of the limit problem. In our main theorem, we will allow boundary values from a certain subset of $W^{-1,1}(\partial\Omega; \mathbb{R}^2)$ in the limit problem. In particular, this subset contains the aforementioned “delta-type” distributions, provided that the applied forces do not act tangentially, see Lemma 4.4. The functions in this space will be approximated by boundary values $g_\lambda \in H^{-1/2}(\partial\Omega; \mathbb{R}^2)$.

- (ii) The main idea of our proof can be summarized as follows: By the well-known representation of divergence free stresses via Airy potentials, we can formulate the variational problems for finite and vanishing weight as the minimization of integral functionals in the spaces H^2 and BH respectively, where the latter denotes the space of functions of bounded Hessian, i.e., the space of functions $u \in W^{1,1}$ such that the distributional derivative D^2u defines a vector-valued Radon measure. We may then use the blow-up technique developed by Fonseca and Müller [3, 14, 15] and the results by Kohn and Strang [20] and Allaire and Kohn [2] to prove the lower bound part of the Γ -convergence result. For the construction of the upper bound, we use approximation and relaxation results that are well known to specialists. Nevertheless, for some of them, we could not find a proof in the literature and provide them here.

- (iii) In their formal derivations of the Michell truss problem in [2], Allaire and Kohn also discussed the three-dimensional case. We are not able to say anything new about this case: The representation of divergence free stresses via Airy potentials is limited to two dimensions.
- (iv) We restrict ourselves to the case of vanishing Poisson ratio for the sake of simplicity and readability. The interested reader will be able to generalize our results without difficulty to a general “soft” isotropic phase, defined by the elasticity tensor $A_0 \in \text{Lin}(\mathbb{R}_{\text{sym}}^{2 \times 2}; \mathbb{R}_{\text{sym}}^{2 \times 2})$ with

$$A_0 \xi = 2\mu \left(\xi - \frac{1}{2}(\text{Tr } \xi) \text{Id}_{2 \times 2} \right) + \kappa \text{Tr } \xi \text{Id}_{2 \times 2} \quad \text{for } \xi \in \mathbb{R}_{\text{sym}}^{2 \times 2},$$

where μ, κ are the shear and bulk modulus respectively, and $\text{Lin}(V; W)$ denotes the set of linear operators $V \rightarrow W$. In that case, the functionals for finite λ are given by

$$\mathcal{G}_{g, \lambda}^{A_0}(\sigma) = \begin{cases} \lambda^{-1/2} \int F_{\lambda}^{A_0}(\sigma) dx & \text{if } \sigma \in S_g \\ +\infty & \text{else,} \end{cases}$$

where

$$F_{\lambda}^{A_0}(\xi) = \begin{cases} 0 & \text{if } \xi = 0 \\ (A_0^{-1} \xi) : \xi + \lambda & \text{else,} \end{cases} \quad (4)$$

and the limit functional is given by $\frac{\kappa + \mu}{\sqrt{4\kappa\mu}} \mathcal{G}_g$. It suffices to take the formulas for the quasiconvex envelope for $F_{\lambda}^{A_0}$ from [2], and adapt our proof accordingly.

2.4. Derivation of the variational form of the compliance minimization problem.

We give a brief derivation of the compliance minimization problem in its variational form,

$$\inf_{\sigma \in S_g(\Omega)} \mathcal{G}_{g, \lambda}(\sigma),$$

starting from the standard formulation in linear elasticity. What we present here is a subset of the derivation by Allaire and Kohn, see [1] for more details.

As before, let $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^2)$. Consider an elastic body $\Omega \subset \mathbb{R}^2$, characterized by its elasticity tensor $A_0 \in \text{Lin}(\mathbb{R}_{\text{sym}}^{2 \times 2}; \mathbb{R}_{\text{sym}}^{2 \times 2})$, where we assume that A_0 is invertible. We remove a subset $H \subset \Omega$ from the elastic body and the new boundaries from that process shall be traction-free. The resulting linear elasticity problem is to find $u : \Omega \setminus H \rightarrow \mathbb{R}^2$ such that

$$\begin{aligned} \sigma &= A_0 e(u) \\ \text{div } \sigma &= 0 \quad \text{in } \Omega \\ \sigma \cdot n &= g \quad \text{on } \partial\Omega \\ \sigma \cdot n &= 0 \quad \text{on } \partial H, \end{aligned}$$

where $e(u) = \frac{1}{2}(\nabla u + \nabla u)^T$. The compliance (work done by the load) is given by

$$c(H) = \int_{\partial\Omega} g \cdot u d\mathcal{H}^1 = \int_{\Omega \setminus H} (A_0 e(u)) : e(u) dx,$$

where $u : \Omega \setminus H \rightarrow \mathbb{R}^2$ is the unique solution to the linear elasticity system above. We want to minimize the compliance under a constraint on the “weight” $\mathcal{L}^2(\Omega \setminus H)$. We do

so by the introduction of a Lagrange multiplier λ , and are interested in the minimization problem

$$\min_H (c(H) + \lambda \mathcal{L}^2(\Omega \setminus H)) .$$

Taking the limit of vanishing weight corresponds to the limit $\lambda \rightarrow \infty$. We rephrase the problem by considering space-dependent elasticity tensors of the form $A(x) = \chi(x)A_0$, where $\chi \in L^\infty(\Omega; \{0, 1\})$. The elasticity system from above becomes

$$\begin{aligned} \sigma &= A(x)e(u) \\ \operatorname{div} \sigma &= 0 \quad \text{in } \Omega \\ \sigma \cdot n &= g \quad \text{on } \partial\Omega . \end{aligned} \tag{5}$$

Now the compliance is a functional on the set of permissible elasticity tensors, and is given by

$$c(A) = \int_{\Omega} (A(x)e(u)) : e(u) dx ,$$

where u is the solution of (5). By the principle of minimum complementary energy, we have that the compliance can be written as

$$c(A) = \int_{\Omega} G(A(x), \sigma(x)) dx ,$$

where

$$G(\bar{A}, \xi) = \begin{cases} +\infty & \text{if } \xi \neq 0 \text{ and } \bar{A} = 0 \\ 0 & \text{if } \xi = 0 \text{ and } \bar{A} = 0 \\ (\bar{A}^{-1}\xi) : \xi & \text{else,} \end{cases}$$

and $\sigma \in L^\infty(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ is a solution of the PDE

$$\begin{aligned} \operatorname{div} \sigma &= 0 \quad \text{in } \Omega \\ \sigma \cdot n &= g \quad \text{on } \partial\Omega , \end{aligned}$$

i.e., $\sigma \in S_g(\Omega)$. We see that the compliance minimization problem can be understood as the variational problem of finding the infimum

$$\inf \left\{ \int_{\Omega} (G(\chi(x)A_0, \sigma(x)) + \lambda \chi(x)) dx : \chi \in L^\infty(\Omega; \{0, 1\}), \sigma \in S_g(\Omega) \right\} .$$

Of course, the compliance of a pair (χ, σ) is infinite if there exists a set of positive measure U such that $\chi = 0$ and $\sigma \neq 0$ on U . Hence the above variational problem is equivalent with

$$\inf \left\{ \int_{\Omega} F_{\lambda}^{A_0}(\sigma) dx : \sigma \in S_g(\Omega) \right\} , \tag{6}$$

where $F_{\lambda}^{A_0}$ has been defined in (4). As is well known, the variational problem (6) does not possess a solution in general and requires relaxation. For simplicity, we assume here that A_0 is the identity on $\mathbb{R}_{\text{sym}}^{2 \times 2}$, see Remark 2.3. Then (6) is just the variational problem $\mathcal{G}_{g, \lambda}(\sigma) \rightarrow \inf$.

3. PRELIMINARIES

For the proof of our main result, we are going to rely heavily on two sets of results from the literature: On the one hand, on the results on optimal design in the “relaxed formulation” [1, 2, 20], and on the other hand, on the blow-up technique for the derivation of relaxed functionals and Γ -limits developed by Fonseca and Müller [3, 13, 14]. In order to present them, we need to review some basic facts about measures, BV functions and quasiconvexity.

In the following, let $U \subset \mathbb{R}^n$ be open and bounded.

3.1. Measures. At the basis of the blow-up argument that we use in the present work is a refinement of the well known Radon-Nikodym differentiation theorem:

Theorem 3.1 (Proposition 2.2 in [3]). *Let λ, μ be Radon measures in U with $\mu \geq 0$. Then there exists a Borel set $E \subset U$ with $\mu(E) = 0$ such that for any $x_0 \in \text{supp } \mu \setminus E$ we have*

$$\lim_{\rho \downarrow 0} \frac{\lambda(x_0 + \rho K)}{\mu(x_0 + \rho K)} = \frac{d\lambda}{d\mu}(x_0)$$

for any bounded convex set K containing the origin. Here, the set E is independent of K .

Let $\mu_j \in \mathcal{M}(U; \mathbb{R}^p)$ for $j = 1, 2, \dots$. We say that $\mu_j \rightarrow \mu \in \mathcal{M}(U; \mathbb{R}^p)$ weakly $*$ in the sense of measures if

$$\int_U \varphi \cdot \frac{d\mu_j}{d|\mu_j|} d|\mu_j| \rightarrow \int_U \varphi \cdot \frac{d\mu}{d|\mu|} d|\mu| \quad \text{for every } \varphi \in C_c^0(U; \mathbb{R}^p).$$

It is a well known fact that if $\sup_{j \in \mathbb{N}} |\mu_j|(U) < \infty$, then there exists a weakly $*$ convergent subsequence.

The next result concerns the convergence of positively one-homogeneous functions of measures. The statement below is contained in Theorem 1.15 in [12].

Theorem 3.2. *Let $h : \mathbb{R}^p \rightarrow \mathbb{R}$ be positively one-homogeneous and continuous, and let $\mu_j \in \mathcal{M}(U; \mathbb{R}^p)$, $j \in \mathbb{N}$, such that $\mu_j \rightarrow \mu$, $|\mu_j| \rightarrow |\mu|$ weakly $*$ in the sense of measures. Then*

$$h(\mu_j) \rightarrow h(\mu) \quad \text{weakly } * \text{ in the sense of measures.}$$

3.2. BV and BH functions. The space of functions of bounded variation $BV(U)$ is defined as the set of functions $f \in L^1(U)$ that satisfy

$$\sup \left\{ - \int_U f \operatorname{div} \varphi \, dx : \varphi \in C_c^1(U; \mathbb{R}^n), \|\varphi\|_{C^0} \leq 1 \right\} < \infty.$$

In this case, the map $\varphi \mapsto - \int_U f \operatorname{div} \varphi \, dx$ defines a vector valued Radon measure, which is denoted by Dv .

According to the Theorem 3.1, we have the following decomposition of the measure Dv for $v \in BV(U)$,

$$Dv = \nabla v \mathcal{L}^n + D^s v,$$

where $\nabla v = \frac{d(Dv)}{d\mathcal{L}^n}$ and $D^s v$ is the so-called singular part of Dv .

The following theorem determines the structure of blow-ups of BV functions:

Theorem 3.3 (Theorem 2.3 in [3]). *Let $u \in BV(U; \mathbb{R}^m)$ and for a bounded convex open set K containing the origin, and let ξ be the density of Du with respect to $|Du|$, $\xi = \frac{d(Du)}{d(|Du|)}$.*

For $x_0 \in \text{supp}(|Du|)$, assume that $\xi(x_0) = \eta \otimes \nu$ with $\eta \in \mathbb{R}^m$, $\nu \in \mathbb{R}^n$, $|\eta| = |\nu| = 1$, and for $\rho > 0$ let

$$v_\rho(y) = \frac{\rho^{n-1}}{|Du|(x_0 + \rho K))} \left(u(x_0 + \rho y) - \int_{x_0 + \rho K} u(x') dx' \right).$$

Then for every $\sigma \in (0, 1)$ there exists a sequence ρ_j converging to 0 such that v_{ρ_j} converges in $L^1(K; \mathbb{R}^m)$ to a function $v \in BV(K; \mathbb{R}^m)$ which satisfies $|Dv|(\sigma \bar{K}) \geq \sigma^n$ and can be represented as

$$v(y) = \psi(y \cdot \nu) \eta$$

for a suitable non-decreasing function $\psi : (a, b) \rightarrow \mathbb{R}$, where $a = \inf\{y \cdot \nu : y \in K\}$ and $b = \sup\{y \cdot \nu : y \in K\}$.

We will also need Alberti's rank-one theorem:

Theorem 3.4. *Let $v \in BV(U; \mathbb{R}^m)$. Then $D^s v$ is rank-one.*

For later reference, we also mention that for $u \in W^{1,1}(U; \mathbb{R}^m)$, we have as a consequence of the classical Sobolev embeddings, that for almost every $x_0 \in U$, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\int_{Q(x_0, \varepsilon)} |u(x) - u(x_0) - \nabla u(x_0) \cdot (x - x_0)|^{n/(n-1)} dx \right)^{(n-1)/n} = 0. \quad (7)$$

Here, we have used the notation $Q(x_0, \varepsilon) = x_0 + [-\varepsilon/2, \varepsilon/2]^n$.

The space of functions of bounded Hessian is defined as

$$BH(U) = \{u \in W^{1,1}(U), \nabla u \in BV(U; \mathbb{R}^n)\}.$$

It can be made into a normed space by setting

$$\|u\|_{BH(U)} = \|u\|_{W^{1,1}(U)} + |D^2 u|(U).$$

We say that a sequence $u_j \in BH(U)$ converges weakly $*$ to $u \in BH(U)$ if $u_j \rightarrow u$ in $W^{1,1}(U)$ and $D^2 u_j \rightarrow D^2 u$ weakly $*$ in $\mathcal{M}(U; \mathbb{R}^{n \times n})$.

The space $BH(U)$ has been investigated first in [10, 11]. In particular, these papers contain theorems about compactness and extension properties of this space. The first theorem that we cite is a weakened form of Theorem 1.3 in [11]:

Theorem 3.5. *Let u_j be a bounded sequence in $BH(U)$. Then there exists a subsequence (no relabeling) and $u \in BH(U)$ such that*

$$u_j \rightarrow u \quad \text{weakly } * \text{ in } BH(U).$$

In two dimensions, functions in BH are continuous:

Theorem 3.6 ([10], Theorem 3.3). *Let $U \subset \mathbb{R}^2$ with C^2 boundary. Then*

$$BH(U) \subset C^0(U).$$

3.3. Relaxation of integral functionals that depend on higher derivatives. In the proof of the upper bound of Proposition 4.6, we will need a relaxation result for integral functionals that depend on second gradients. This is the case $k = p = 2$ in Theorem 3.7 below.

A function $f : \mathbb{R}^{m \times n^k} \rightarrow \mathbb{R}$ is called k -quasiconvex if

$$f(\xi) = \inf \left\{ \int_{[-1/2, 1/2]^n} f(\xi + \nabla^k \varphi) dx : \varphi \in W_0^{k, \infty}([-1/2, 1/2]^n; \mathbb{R}^m) \right\}, \quad (8)$$

see [23]. The so-called k -quasiconvexification of $f : \mathbb{R}^{m \times n^k} \rightarrow \mathbb{R}$ is given by the right hand side above,

$$Q_k f(\xi) = \inf \left\{ \int_{[-1/2, 1/2]^n} f(\xi + \nabla^k \varphi) dx : \varphi \in W_0^{k, \infty}([-1/2, 1/2]^n; \mathbb{R}^m) \right\}.$$

As is well known, in the case $k = 1$, one obtains the relaxation of integral functionals $u \mapsto \int f(\nabla u) dx$ by replacing f by its 1-quasiconvexification $Q_1 f$. Concerning higher k , there exist some relaxation results in the literature, but we could not find any theorems that fit our situation. In particular, Theorem 1.3 in [6] deals with the case of the relaxation of Caratheodory functions. In our case, where the integrand only depends on the second gradient, this means that continuity of the integrand $f(\xi)$ with respect to ξ is required. We are interested in the non-continuous case $f = F_\lambda$, so we cannot use this theorem. The following theorem suits our purpose, and we prove it in the appendix:

Theorem 3.7. *Let $1 \leq p < \infty$, and let $f : \mathbb{R}^{m \times n^2} \rightarrow \mathbb{R}$ such that*

$$0 \leq f(A) \leq C(1 + |A|^p) \quad \text{for all } A \in \mathbb{R}^{m \times n^2}.$$

Let $\Omega \subset \mathbb{R}^2$ satisfy Definition 2.1, and let $u_0 \in W^{2,p}(\Omega)$. Let $u \in u_0 + W_0^{2,p}(\Omega; \mathbb{R}^m)$ and $\varepsilon > 0$. Then there exists $v \in u_0 + W_0^{2,p}(\Omega; \mathbb{R}^m)$ with

$$\begin{aligned} \|u - v\|_{W^{1,p}(\Omega; \mathbb{R}^m)} &< \varepsilon \\ \int_{\Omega} f(\nabla^2 v) dx &< \int_{\Omega} Q_2 f(\nabla^2 u) dx + \varepsilon. \end{aligned}$$

Remark 3.8. The theorem can be straightforwardly generalized to the case of higher derivatives, if in Definition 2.1 one replaces C^2 -regularity with C^k -regularity of $\partial\Omega$ in the appropriate sense. Another straightforward generalization is the case of general dimension n of the domain Ω . Moreover, (simple) connectedness and boundedness of Ω are not necessary here.

We will need to determine the 2-quasiconvexification of $F_\lambda : \mathbb{R}^{2 \times 2} \rightarrow \mathbb{R}$. In principle this is contained in [2, 20]. However we could not find a clear statement in the literature, so we give a proof of the following theorem in the appendix.

Theorem 3.9. *We have*

$$Q_2 F_\lambda(\sigma) = \begin{cases} 2\sqrt{\lambda} \rho^0(\sigma) - 2|\det \sigma| & \text{if } \rho^0(\sigma) \leq \sqrt{\lambda} \\ |\sigma|^2 + \lambda & \text{else.} \end{cases}$$

3.4. Trace and extension operators. Recall that we assume that $\Omega \subset \mathbb{R}^2$ satisfies Definition 2.1. The trace operator

$$\gamma_0 : u \mapsto u|_{\partial\Omega}$$

is linear surjective as a map $W^{1,1}(\Omega) \rightarrow L^1(\Omega)$ (see [16]) and also as a map $BV(\Omega) \rightarrow L^1(\Omega)$ (see [25]). For the spaces $W^{2,1}(\Omega)$ and $BH(\Omega)$, it also makes sense to consider the operator

$$\gamma_1 : u \mapsto \nabla u|_{\partial\Omega} \cdot n.$$

The following theorem combines statements from Chapter 1.8 of [21] and Chapter 2 as well as the appendix of [10].

Theorem 3.10. (i) *The operator (γ_0, γ_1) is linear surjective both as a map*

$$H^2(\Omega) \rightarrow H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$$

and as a map

$$BH(\Omega) \rightarrow \gamma_0(W^{2,1}(\Omega)) \times L^1(\partial\Omega).$$

(ii) *There exist continuous right inverses $(\gamma_0, \gamma_1)^{-1}$, defined as maps*

$$H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \rightarrow H^2(\Omega)$$

and

$$\gamma_0(W^{2,1}(\Omega)) \times L^1(\partial\Omega) \rightarrow W^{2,1}(\Omega).$$

Remark 3.11. (i) The norm on $\gamma_0(W^{2,1}(\Omega))$ is the one induced by γ_0 ,

$$\|u\|_{\gamma_0(W^{2,1}(\Omega))} := \inf\{\|v\|_{W^{2,1}(\Omega)} : \gamma_0(v) = u\}.$$

Note that $\gamma_0(W^{2,1}(\Omega)) \subsetneq W^{1,1}(\partial\Omega)$, see the appendix of [10]. This fact together with the theorem explains our choice of assumptions on the boundary conditions in our main theorem; see equations (15) and (17) below.

(ii) In [10], the statement on the surjectivity of the trace operator $BH(\Omega) \rightarrow \gamma_0(W^{2,1}(\Omega)) \times L^1(\partial\Omega)$ is only made for C^2 -regular boundary. For the sake of brevity, we only sketch the changes of that proof that have to be made to show that the claim also holds true for $\Omega \subset \mathbb{R}^2$ satisfying Definition 2.1. In Proposition 1 of the appendix of [10], it is shown that for $g \in L^1([0,1] \times \{0\})$, there exists $u \in W^{2,1}([0,1]^2)$ such that $u|_{[0,1] \times \{0\}} = 0$, $\partial_2 u|_{[0,1] \times \{0\}} = g$. The proof works by an explicit construction, modifying an idea by Gagliardo. In fact, using the explicit formulas for u and its partial derivatives that are given in the proof, one easily deduces that $u_{\{0\} \times [0,1]} = \partial_1 u|_{\{0\} \times [0,1]} = 0$. Hence, one may use the Proposition twice to obtain $\tilde{u} \in W^{2,1}([0,1]^2)$ that vanishes on $\Gamma := [0,1] \times \{0\} \cup \{0\} \times [0,1]$, and whose normal derivative agrees with a given $\tilde{g} \in L^1(\Gamma)$ on Γ . With this slightly more general version of the Proposition, the proof of Theorem 1 in the appendix of [10], which states the surjectivity of the trace operator, can be extended to the case where Ω satisfies Definition 2.1 without additional changes: One uses a suitable cover of the boundary $\partial\Omega$ by open sets, an associated partition of unity and C^2 -regular diffeomorphisms that reduce the problem to the situation of the proposition.

We have the following Poincaré inequality for BH :

Lemma 3.12. *Let $u \in BH(\Omega)$, $\gamma_0(u) = f$, $\gamma_1(u) = g$, then we have*

$$\|u\|_{W^{1,1}(\Omega)} \lesssim \|f\|_{W^{1,1}(\partial\Omega)} + \|g\|_{L^1(\partial\Omega)} + |D^2 u|(\Omega).$$

Proof. By the Poincaré inequality for BV functions (see [30] Chapter 5) we have

$$\|\nabla u\|_{L^1(\Omega)} \lesssim \|\nabla u\|_{L^1(\partial\Omega)} + |D^2u|(\Omega)$$

and

$$\|u\|_{L^1(\Omega)} \lesssim \|u\|_{L^1(\partial\Omega)} + \|\nabla u\|_{L^1(\Omega)}.$$

This proves the claim. $\square$

Next, we quote a general extension result from [29]. In the following, we slightly change a definition from [29]:

Definition 3.13. *Let $U \subset \mathbb{R}^n$ be open and bounded. We say that the boundary ∂U is said to satisfy minimal conditions if*

- (i) *There exists a cover of ∂U by a finite number of open sets $U_1, U_2, \dots, U_m$*
- (ii) *For every $i = 1, \dots, m$, $\partial U \cap U_i$ can be represented as the graph of a Lipschitz function $\tilde{U}_i \rightarrow \mathbb{R}$ with $\tilde{U}_i \subset \mathbb{R}^{n-1}$.*

Note that if Ω satisfies Definition 2.1, then $\partial\Omega$ also satisfies the minimal conditions.

Theorem 3.14 (Theorem 5 and 5' in Chapter 6 of [29]). *Let $U \subset \mathbb{R}^n$ such that ∂U satisfies the minimal conditions. Then there exists an extension operator*

$$E : L^1(U) \rightarrow L^1(\mathbb{R}^n)$$

that is continuous as a map $W^{k,p}(U) \rightarrow W^{k,p}(\mathbb{R}^n)$ for every $k \in \mathbb{N}$ and every $1 \leq p \leq \infty$. Moreover, the norm of this operator only depends on n and on the maximum of the Lipschitz constants of the functions that appear in Definition 3.13 (ii).

4. AIRY POTENTIALS AND BOUNDARY CONDITIONS

In the present section, we assume that $\Omega \subset \mathbb{R}^2$ satisfies Definition 2.1.

4.1. Airy potentials. Here we are going to rephrase the compliance minimization problem and the Michell problem. We use the representation of divergence free stresses in two dimensions by Airy potentials. We recall that for $A \in \mathbb{R}_{\text{sym}}^{2 \times 2}$, the cofactor matrix $\text{cof } A$ is defined by

$$\text{cof } A = \begin{pmatrix} A_{22} & -A_{12} \\ -A_{12} & A_{11} \end{pmatrix}.$$

Note that in two dimensions, we have $\text{cof } \text{cof } A = A$.

In the compliance minimization problem, we say that $u \in H^2(\Omega)$ is an Airy potential for $\sigma \in S_g(\Omega)$ if

$$\nabla^2 u = \text{cof } \sigma \quad \text{in } \Omega. \tag{9}$$

Note that in such a situation, we have $\text{div } \sigma = \text{curl } \text{cof } \sigma = 0$. Since $A \mapsto \text{cof } A$ is linear on two by two matrices, the object $\text{cof } \mu$ is well defined for $\mu \in \mathcal{M}(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{2 \times 2})$. We say that the function $u \in W^{1,1}(U)$ is an Airy potential for $\sigma \in \Sigma_g(\Omega)$ if U is a neighborhood of $\bar{\Omega}$, and

$$D^2 u \llcorner \bar{\Omega} = \text{cof } \sigma \tag{10}$$

as elements of $\mathcal{M}(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{2 \times 2})$. Our definitions of Airy potentials make sense by the Poincaré Lemma; this statement is made precise in the following lemma.

Lemma 4.1. *We have*

$$\{\sigma \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) : \text{curl } \sigma = 0\} = \{D^2 u : u \in BH(\Omega)\}$$

and

$$\{\sigma \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) : \text{curl } \sigma = 0\} = \{\nabla^2 u : u \in H^2(\Omega)\}.$$

Proof. The inclusion $\{D^2 u : u \in BH(\Omega)\} \subseteq \{\sigma \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2}) : \text{curl } \sigma = 0\}$ is obvious. For the opposite inclusion, let $\sigma \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ with $\text{curl } \sigma = 0$. Let φ be a standard mollifier, i.e., $\varphi \in C_c^\infty(\mathbb{R}^2; \mathbb{R})$, $\text{supp } \varphi \subset \{x \in \mathbb{R}^2 : |x| < 1\}$, $\int_{\mathbb{R}^2} \varphi dx = 1$, and $\varphi_\varepsilon := \varepsilon^{-2} \varphi(\cdot/\varepsilon)$. On $\bar{\Omega}_\varepsilon := \{x \in \Omega : \text{dist}(\partial\Omega, x) > \varepsilon\}$, set $\sigma_\varepsilon := \sigma * \varphi_\varepsilon$. Note that we have $\sigma_\varepsilon \in C^\infty(\bar{\Omega}_\varepsilon; \mathbb{R}_{\text{sym}}^{2 \times 2})$ with $\text{curl } \sigma_\varepsilon = 0$ on $\bar{\Omega}_\varepsilon$. For ε small enough, $\bar{\Omega}_\varepsilon$ is simply connected, and by the Poincaré Lemma there exists $v_\varepsilon \in C^\infty(\bar{\Omega}_\varepsilon; \mathbb{R}^2)$ such that $\nabla v_\varepsilon = \sigma_\varepsilon$. For every $(\bar{x}, \bar{y}) \in \bar{\Omega}_\varepsilon$, there exists an open square $Q \subset \bar{\Omega}_\varepsilon$ with center $(\bar{x}, \bar{y})$. On Q , we have

$$v_\varepsilon(x, y) = v(\bar{x}, \bar{y}) + \begin{pmatrix} \int_{\bar{x}}^x (\sigma_\varepsilon)_{11}(t, 0) dt + \int_{\bar{y}}^y (\sigma_\varepsilon)_{12}(x, t) dt \\ \int_{\bar{x}}^x (\sigma_\varepsilon)_{21}(t, y) dt + \int_{\bar{y}}^y (\sigma_\varepsilon)_{22}(0, t) dt \end{pmatrix}.$$

Using $(\sigma_\varepsilon)_{12} = (\sigma_\varepsilon)_{21}$, one easily obtains $\text{curl } v_\varepsilon = 0$ on Q , and hence on all of $\bar{\Omega}_\varepsilon$. Again by the Poincaré Lemma there exists $\tilde{u}_\varepsilon \in C^\infty(\bar{\Omega}_\varepsilon)$ such that $\sigma_\varepsilon = \nabla^2 \tilde{u}_\varepsilon$ on $\bar{\Omega}_\varepsilon$. Of course, the sets $\bar{\Omega}_\varepsilon$ have Lipschitz boundary. Moreover, there exist open sets $U_1, \dots, U_M$ such that for ε small enough, $U_i \cap \partial\bar{\Omega}_\varepsilon$ can be represented as the graph of some Lipschitz function $w_{i,\varepsilon}$, and the Lipschitz constants of $w_{i,\varepsilon}$ are uniformly bounded. By Theorem 3.14, we may extend $\tilde{u}_\varepsilon \in W^{2,1}(\bar{\Omega}_\varepsilon)$ to $u_\varepsilon \in W^{2,1}(\Omega)$ such that

$$\|u_\varepsilon\|_{W^{2,1}(\Omega)} \leq C \|\tilde{u}_\varepsilon\|_{W^{2,1}(\bar{\Omega}_\varepsilon)}, \quad (11)$$

where C does not depend on ε . After subtracting suitable affine functions, we may assume

$$\int_{\Omega} u_\varepsilon dx = 0, \quad \int_{\Omega} \nabla u_\varepsilon dx = 0.$$

From (11) and the Poincaré inequality in BH (see [10]) it follows

$$\|u_\varepsilon\|_{BH(\Omega)} \lesssim \|\nabla^2 u_\varepsilon\|_{L^1(\Omega)} \lesssim |\sigma|(\Omega).$$

By Theorem 3.5, we obtain that there exists $u \in BH(\Omega)$ such that $u_\varepsilon \rightarrow u$ in $BH(\Omega)$, with $D^2 u = \sigma$. This proves the first statement. The second statement is proved in the same way, using Theorem 3.14 for the extension $H^2(\bar{\Omega}_\varepsilon) \rightarrow H^2(\Omega)$, and weak compactness of the resulting bounded sequence u_ε in $H^2(\Omega)$. $\square$

4.2. Boundary values. We say that $g \in W^{-1,1}(\partial\Omega; \mathbb{R}^2)$ is balanced if

$$\int_{\partial\Omega} (Mx + b) \cdot g(x) d\mathcal{H}^1 = 0 \quad \text{for all } M \in \mathbb{R}_{\text{skew}}^{2 \times 2} \text{ and } b \in \mathbb{R}^2.$$

It only makes sense to consider balanced traction boundary values, as can be seen from the following well known lemma (see e.g. [5]):

Lemma 4.2. *If $\Sigma_g(\Omega) \neq \emptyset$, then g is balanced.*

Proof. Assume $\sigma \in \Sigma_g(\Omega)$. Taking $\varphi(x) = (1, 0)$ or $\varphi(x) = (0, 1)$ and testing these functions against the identity $-\text{div } \sigma = g \mathcal{H}^1 \llcorner \partial\Omega$ (see (1)), we obtain

$$\int_{\partial\Omega} g d\mathcal{H}^1 = 0.$$

Secondly, taking $\varphi(x) = x^\perp = (-x_2, x_1)$ as a test function, we get

$$\int_{\partial\Omega} x^\perp \cdot g \, d\mathcal{H}^1 = \int_{\Omega} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} : \frac{d\sigma}{d|\sigma|} d|\sigma| = 0.$$

The latter holds since σ has values in the symmetric matrices. This proves the lemma, since for every $M \in \mathbb{R}_{\text{skew}}^{2 \times 2}$, there exists $c \in \mathbb{R}$ such that $Mx = cx^\perp$. $\square$

For certain $h \in W^{-1,1}(\partial\Omega; \mathbb{R}^2)$, we now define two integrals $h^{(1)}, h^{(2)}$. Let $x_0 \in \partial\Omega$ be fixed, $L := \mathcal{H}^1(\partial\Omega)$, and let $\vartheta_{x_0} : [0, L] \rightarrow \partial\Omega$ denote the positively oriented simple Lipschitz curve that satisfies

$$|\vartheta'_{x_0}| = 1, \quad \vartheta_{x_0}(0) = \vartheta_{x_0}(L) = x_0, \quad \vartheta_{x_0}([0, L]) = \partial\Omega.$$

Obviously, $\vartheta_{x_0}|_{(0,L)}$ is a Bilipschitz homeomorphism.

For $\varphi \in L^\infty(\partial\Omega)$ with $\int_{\partial\Omega} \varphi d\mathcal{H}^1 = 0$, we may define its first integral $\Phi(\varphi) \in W^{1,\infty}(\partial\Omega)$ by

$$\Phi(\varphi)(\vartheta_{x_0}(x)) = \int_0^x \varphi \circ \vartheta_{x_0}(t) dt - c_\varphi,$$

where c_φ is chosen such that $\int_{\partial\Omega} \Phi(\varphi) d\mathcal{H}^1 = 0$. We may extend this definition to $h \in W^{-1,1}(\partial\Omega) = (W^{1,\infty}(\partial\Omega))'$ with $\langle h, \chi_{\partial\Omega} \rangle = 0$, where $\chi_{\partial\Omega}$ is the function defined by $\chi_{\partial\Omega}(x) = 1$ for all $x \in \partial\Omega$: We let $\Phi(h) \in (L^\infty(\partial\Omega))'$ (to be thought of as the first integral of h) be defined by

$$\langle \Phi(h), \varphi \rangle = - \left\langle h, \Phi \left(\varphi - \int_{\partial\Omega} \varphi d\mathcal{H}^1 \right) \right\rangle \quad \text{for all } \varphi \in L^\infty(\partial\Omega).$$

For vector valued arguments, we may define $\Phi : W^{-1,1}(\partial\Omega; \mathbb{R}^2) \rightarrow (L^\infty(\partial\Omega; \mathbb{R}^2))'$ by its action on the components of its argument.

We recall that n denotes the unit outer normal of $\partial\Omega$, and $\tau = (-n_2, n_1)$. Let these objects be understood as functions in $L^\infty(\partial\Omega; \mathbb{R}^2)$. If $h \in W^{-1,1}(\partial\Omega; \mathbb{R}^2)$ with $\langle h_i, \chi_{\partial\Omega} \rangle = 0$ for $i = 1, 2$, then $\tau \cdot \Phi(h)$ can be understood as an element of $(L^\infty(\partial\Omega))'$, by

$$\langle \tau \cdot \Phi(h), \varphi \rangle = \langle \Phi(h), \tau \varphi \rangle \quad \text{for all } \varphi \in L^\infty(\partial\Omega).$$

If we assume furthermore $\langle \tau \cdot \Phi(h), \chi_{\partial\Omega} \rangle = 0$, we can define the first and second integrals $h^{(1)} \in (L^\infty(\partial\Omega; \mathbb{R}^2))'$ and $h^{(2)} \in \Phi((L^\infty(\partial\Omega))')$ by

$$\begin{aligned} h^{(1)} &= \Phi(h) \\ h^{(2)} &= \Phi(\tau \cdot \Phi(h)). \end{aligned} \tag{12}$$

In order to make the transition between stresses and their Airy potentials, the following definition will be convenient: Let

$$X := \left\{ g \in W^{-1,1}(\partial\Omega; \mathbb{R}^2) : n \cdot (g^\perp)^{(1)} \in L^1(\partial\Omega), (g^\perp)^{(2)} \in \gamma_0(W^{2,1}(\Omega)) \right\}.$$

We make X into a topological vector space by letting the topology on X be the strongest one that makes the following map continuous:

$$\begin{aligned} X &\rightarrow L^1(\partial\Omega) \times \gamma_0(W^{2,1}(\Omega)) \\ g &\mapsto (n \cdot (g^\perp)^{(1)}, (g^\perp)^{(2)}). \end{aligned}$$

Remark 4.3. (i) The requirements for the existence of $(g^\perp)^{(1)}, (g^\perp)^{(2)}$, namely that

$$\langle g_i, \chi_{\partial\Omega} \rangle = 0 \quad \text{for } i = 1, 2, \quad \left\langle \tau \cdot \Phi(g^\perp), \chi_{\partial\Omega} \right\rangle = 0, \quad (13)$$

precisely express that g has to be balanced, $\int_{\partial\Omega} (Mx + b) \cdot g d\mathcal{H}^1(x) = 0$ for all $M \in \mathbb{R}_{\text{skew}}^{2 \times 2}$, $b \in \mathbb{R}^2$. To see that $\int_{\partial\Omega} Mx \cdot g d\mathcal{H}^1 = 0$ for all $M \in \mathbb{R}_{\text{skew}}^{2 \times 2}$ is equivalent with the second equation in (13), we observe that

$$\begin{aligned} \int_{\partial\Omega} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} x \cdot g d\mathcal{H}^1 &= - \int_{\partial\Omega} x \cdot g^\perp d\mathcal{H}^1 \\ &= - \int_{\partial\Omega} \left(x_0 + \int_0^{\gamma_{x_0}^{-1}(x)} \tau d\mathcal{H}^1 \right) \cdot g^\perp(x) d\mathcal{H}^1(x) \\ &= - \left\langle \Phi(g^\perp), \tau \right\rangle = - \left\langle \tau \cdot \Phi(g^\perp), \chi_{\partial\Omega} \right\rangle. \end{aligned}$$

- (ii) The space X is a replacement for $W^{-1,1}(\partial\Omega; \mathbb{R}^2)$; the latter is slightly too large for our purposes. In the formulation of the compliance minimization problem via Airy potentials, we need to translate the integrals $(g^\perp)^{(2)}, (g^\perp)^{(1)} \cdot n$ of the boundary values $g \in X$ into boundary values of a function in $BH(\Omega)$ and its normal derivative. This is not possible for $W^{-1,1}(\partial\Omega; \mathbb{R}^2)$, firstly because $L^1(\partial\Omega) \subsetneq \Phi(W^{-1,1}(\partial\Omega))$, and secondly because $\gamma_0(BH(\Omega)) = \gamma_0(W^{2,1}(\Omega)) \subsetneq W^{1,1}(\partial\Omega) = \Phi(L^1(\partial\Omega))$. Nevertheless, with our choice of X we have $H^{-1/2}(\partial\Omega; \mathbb{R}^2) \subset X$ and furthermore X contains balanced finite sums of delta distributions that do not contain applied forces tangential to $\partial\Omega$. For the precise statement, see Lemma 4.4 below.

The upcoming lemma only serves to prove the claim made in the previous remark and can be skipped by the reader who is only interested in the statement and proof of the main theorem.

For $v \in \mathbb{R}^2$ and $x \in \partial\Omega$, we write $v \in \mathcal{T}_x(\partial\Omega)$ if there exists $\varepsilon > 0$ such that either $\{x + tv : t \in (-\varepsilon, \varepsilon)\} \cap \Omega = \emptyset$ or $\{x + tv : t \in (-\varepsilon, \varepsilon)\} \subset \overline{\Omega}$. This is the case, for example, if v is the tangent vector to $\partial\Omega$ in a point x where the curvature does not change sign. (If the curvature changes sign at x , then $\mathcal{T}_x(\partial\Omega) = \emptyset$.) If $\partial\Omega$ is not C^2 near x , then the set $\mathcal{T}_x(\partial\Omega)$ is larger, see Figure 1.

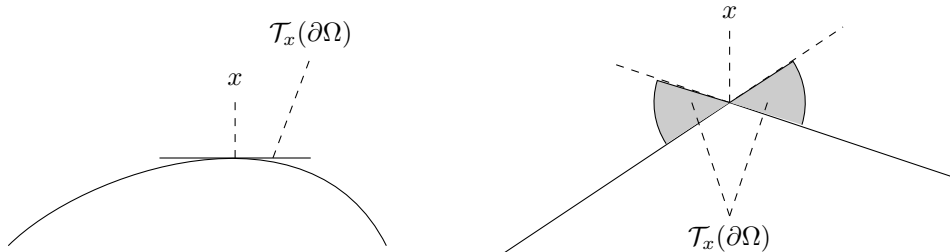


FIGURE 1. In the left panel, the space $\mathcal{T}_x(\partial\Omega)$ is just the tangent space at $x \in \partial\Omega$. In the right panel, the space $\mathcal{T}_x(\partial\Omega)$ is a cone.

Lemma 4.4. For $i = 1, \dots, N$ let $x_i \in \partial\Omega$ and $v_i \in \mathbb{R}^2$ such that $v_i \notin \mathcal{T}_{x_i}(\partial\Omega)$, and additionally

$$\sum_i v_i = \sum_i v_i \cdot x_i^\perp = 0.$$

Then $g = \sum_i \delta_{x_i} v_i$ is balanced, and $g \in X$.

Proof. Using the definitions, the fact that g is balanced is obvious. With the notation we have introduced above, and the assumption $x_0 \neq x_i$ for $i = 1, \dots, N$, we have

$$(g^\perp)^{(1)}(\vartheta_{x_0}(x)) = \sum_{i: \vartheta_{x_0}^{-1}(x_i) < x} v_i^\perp.$$

Hence $(g^\perp)^{(1)}$ is a piecewise constant function $\partial\Omega \rightarrow \mathbb{R}^2$. Let $F_i \in \mathbb{R}^2$ denote the value of $(g^\perp)^{(1)}$ on the arc connecting x_i and x_{i+1} in $\partial\Omega$ (counterclockwise). Furthermore we note that $(g^\perp)^{(2)} \in W^{1,\infty}(\partial\Omega)$ with $\partial_\tau(g^\perp)^{(2)} = (g^\perp)^{(1)} \cdot \tau$. For every $i = 1, \dots, N$, choose $\varepsilon_i > 0$ at least so small that

$$\begin{aligned} [x_i - \varepsilon_i v_i, x_i + \varepsilon_i v_i] \cap \partial\Omega &= \{x_i\} \\ [x_i - \varepsilon_i v_i, x_i + \varepsilon_i v_i] \cap [x_j - \varepsilon_j v_j, x_j + \varepsilon_j v_j] &= \emptyset \quad \text{for } i \neq j. \end{aligned}$$

For $i = 1, \dots, N$, there is exactly one out of the two points $x_i \pm \varepsilon_i v_i$ that is contained in Ω . Denote this point by $\bar{x}_i$. Let $\tilde{\Omega} \Subset \Omega$ be a simply connected polygonal domain with $\bar{x}_i \in \partial\tilde{\Omega}$ for $i = 1, \dots, N$. Let Q_i denote the open subset of $\Omega \setminus \tilde{\Omega}$ that is bounded by $\partial\Omega$, $[x_i, \bar{x}_i]$, $\partial\tilde{\Omega}$ and $[x_{i+1}, \bar{x}_{i+1}]$, see Figure 2.

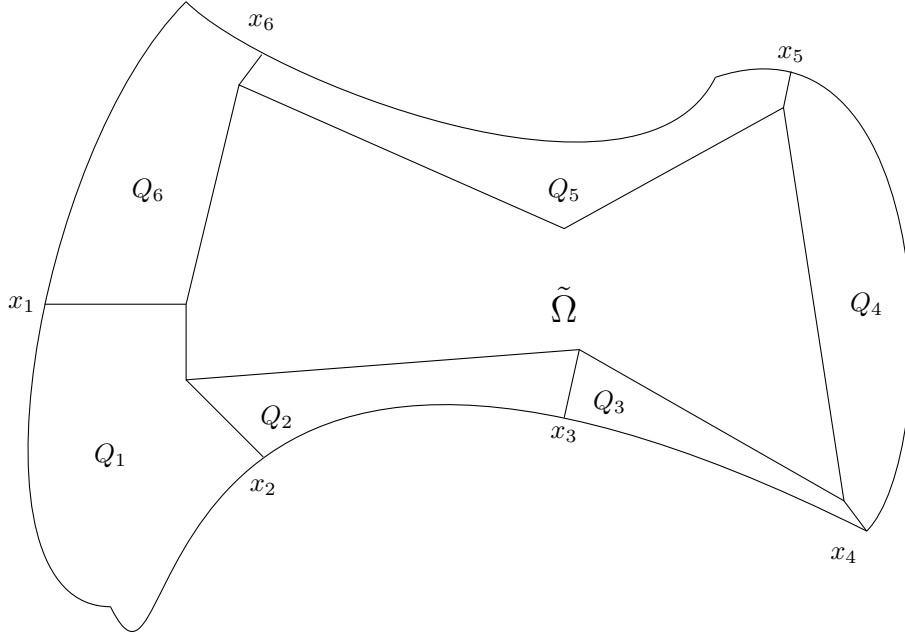


FIGURE 2. The partition of Ω into subsets Q_i and a polygonal set $\tilde{\Omega}$.

On $\Omega \setminus \tilde{\Omega}$, we may define $u \in BH(Q)$ almost everywhere by setting

$$u(x) := (g^\perp)^{(2)}(x_i) + F_i \cdot (x - x_i) \quad \text{if } x \in Q_i.$$

This makes u affine on each subset Q_i , and we claim that there exists a continuous extension to $\Omega \setminus \tilde{\Omega}$. Indeed, we need to check continuity only at the boundaries $\partial Q_i \cap \partial Q_{i+1} = [x_{i+1}, \bar{x}_{i+1}] = [x_{i+1}, x_{i+1} \pm \varepsilon_{i+1} v_{i+1}]$. For $x \in [x_{i+1}, \bar{x}_{i+1}]$, we have

$$\begin{aligned} \lim_{\substack{z \rightarrow x \\ z \in Q_i}} u(z) - \lim_{\substack{z \rightarrow x \\ z \in Q_{i+1}}} u(z) &= (u(x_{i+1}) + F_i \cdot (x - x_{i+1})) - (u(x_{i+1}) + F_{i+1} \cdot (x - x_{i+1})) \\ &= (F_i - F_{i+1}) \cdot (x - x_{i+1}) \\ &= -v_{i+1}^\perp \cdot \tilde{\varepsilon} v_{i+1} \quad \text{for some } \tilde{\varepsilon} \in [-\varepsilon_{i+1}, \varepsilon_{i+1}] \\ &= 0. \end{aligned}$$

This proves the existence of the continuous extension of u to $\Omega \setminus \tilde{\Omega}$. Let T be a triangulation of $\tilde{\Omega}$, and extend u to a function that is affine on each triangle of T and continuous on Ω . Since piecewise affine continuous functions are of bounded Hessian, we have $u \in BH(\Omega)$, and

$$\begin{aligned} \gamma_0(u) &= (g^\perp)^{(2)} \in \gamma_0(BH(\Omega)) = \gamma_0(W^{2,1}(\Omega)) \\ \gamma_1(u) &= (g^\perp)^{(1)} \cdot n \in L^1(\partial\Omega) \end{aligned}$$

Hence $g \in X$, and the lemma is proved. $\square$

For $\sigma \in C^0(\bar{\Omega}; \mathbb{R}_{\text{sym}}^{2 \times 2})$, we have that $\sigma \in \Sigma_g(\Omega)$ implies $\sigma \cdot n = g$ on $\partial\Omega$. If additionally $\sigma = \text{cof } \nabla^2 u$ for some $u \in C^2(\bar{\Omega})$, then $g^\perp = (\text{cof } \nabla^2 u \cdot n)^\perp = -\nabla^2 u \cdot \tau = -\partial_\tau \nabla u$. This implies that the integral $(g^\perp)^{(1)}$ is equal to $-\nabla u$ up to a constant, and $(g^\perp)^{(2)}$ is equal to $-u$ up to an affine function. The following lemma restates these observations for the non-smooth case.

Lemma 4.5. *Let $g \in W^{-1,1}(\partial\Omega; \mathbb{R}^2)$, $\sigma \in \Sigma_g(\Omega)$, and let U be a neighborhood of $\bar{\Omega}$, such that $u \in W^{1,1}(U)$, and $D^2 u \llcorner \bar{\Omega} = \text{cof } \sigma$. Then there exists $\zeta \in L^1(\partial\Omega)$ and an affine function F such that*

$$\begin{aligned} (g^\perp)^{(1)} \cdot n &= -\gamma_1(u) - \zeta + \nabla F \cdot n \\ (g^\perp)^{(2)} &= -\gamma_0(u) + F. \end{aligned}$$

The same conclusion holds true if $g \in H^{-1/2}(\partial\Omega; \mathbb{R}^2)$, $\sigma \in L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$, and $u \in H^2(\Omega)$ with $\sigma = \text{cof } \nabla^2 u$.

Proof. To prove the first claim, we show that there exists $\zeta \in L^1(\partial\Omega)$ and some vector $c \in \mathbb{R}^2$ such that

$$(g^\perp)^{(1)} = -\nabla u|_{\partial\Omega} - \zeta n + c, \tag{14}$$

where the right hand side is understood in the sense of traces of BV functions. To prove this claim, let $\varphi \in C^1(U)$. Then we have

$$\begin{aligned} \langle \partial_\tau \varphi, (g^\perp)^{(1)} \rangle &= -\langle \varphi, g^\perp \rangle \\ &= -\int_{\bar{\Omega}} (\nabla \varphi \cdot d\sigma)^\perp \\ &= -\int_{\bar{\Omega}} \nabla \varphi \cdot d(\text{cof } D^2 u) \\ &= \int_{\bar{\Omega}} (\nabla \varphi)^\perp \cdot d(D^2 u). \end{aligned}$$

Let $\mu = \frac{d(D^2u)}{d\mathcal{H}^1} \mathcal{H}^1 \llcorner \partial\Omega$ denote the restriction of the jump part of D^2u to $\partial\Omega$. Then we have

$$D^2u \llcorner \bar{\Omega} = D^2u \llcorner \Omega + \mu.$$

Using Theorem 3.4 and the symmetry of D^2u , we have that μ is $\mathcal{H}^1$ -almost everywhere parallel to $n \otimes n$, and we may write $\mu = \zeta n \otimes n \mathcal{H}^1$. Hence,

$$\int_{\bar{\Omega}} (\nabla \varphi)^\perp \cdot d\mu = - \int_{\partial\Omega} \zeta n \partial_\tau \varphi d\mathcal{H}^1.$$

By the Gauss' Theorem for BV functions (see e.g. [18]), we have that

$$\begin{aligned} \int_{\bar{\Omega}} (\nabla \varphi)^\perp \cdot d(D^2u \llcorner \Omega) &= \int_{\partial\Omega} (\nabla u \otimes \nabla \varphi^\perp) \cdot n d\mathcal{H}^1 \\ &= \int_{\partial\Omega} -\nabla u \partial_\tau \varphi d\mathcal{H}^1. \end{aligned}$$

Hence, we have

$$- \int_{\partial\Omega} (\nabla u + \zeta n) \partial_\tau \varphi d\mathcal{H}^1 = \int_{\partial\Omega} (g^\perp)^{(1)} \partial_\tau \varphi d\mathcal{H}^1 \quad \text{for all } \varphi \in C^1(U).$$

This proves (14) and hence the first claim. Next we observe that

$$\begin{aligned} \partial_\tau \left((g^\perp)^{(2)} + \gamma_0(u) \right) &= \left((g^\perp)^{(1)} + \nabla u|_{\partial\Omega} + \zeta n \right) \cdot \tau \\ &= c \cdot \tau, \end{aligned}$$

which proves the second claim. Finally, the situation $u \in H^2(\Omega)$ is just a special case of what we have just proved, by extending u to some $\tilde{u} \in H^2(U)$, where U is some neighborhood of $\bar{\Omega}$, and $\tilde{u}|_\Omega = u$, which is possible by Theorem 3.14. $\square$

4.3. Statement of the main theorem. First we will state a proposition that is basically equivalent to our main theorem, using Airy potentials and the most general boundary values that are allowed within this framework.

For $\lambda > 0$, $\bar{f} = (\bar{f}_1, \bar{f}_2) \in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$ let the functional $\mathcal{F}_{\bar{f}, \lambda} : H^2(\Omega) \rightarrow \mathbb{R}$ be defined by

$$\mathcal{F}_{\bar{f}, \lambda}(u) = \begin{cases} \lambda^{-1/2} \int_{\Omega} F_\lambda(\nabla^2 u) dx & \text{if } \gamma_0(u) = \bar{f}_1 \text{ and } \gamma_1(u) = \bar{f}_2 \\ +\infty & \text{else.} \end{cases}$$

Furthermore, for $\bar{f} = (\bar{f}_1, \bar{f}_2) \in \gamma_0(W^{2,1}(\Omega)) \times L^1(\partial\Omega)$, let the functional $\mathcal{F}_{\bar{f}} : BH(\Omega) \rightarrow \mathbb{R}$ be defined by

$$\mathcal{F}_{\bar{f}}(u) = \begin{cases} 2\rho^0(D^2u)(\Omega) + 2 \int_{\partial\Omega} |\gamma_1(u) - \bar{f}_2| d\mathcal{H}^1 & \text{if } \gamma_0(u) = \bar{f}_1 \\ +\infty & \text{else.} \end{cases}$$

In the statement of the following theorem, we use the standard norm on the Cartesian product $H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)$, i.e.

$$\|(\bar{f}_1, \bar{f}_2)\|_{H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} := \|\bar{f}_1\|_{H^{3/2}(\partial\Omega)} + \|\bar{f}_2\|_{H^{1/2}(\partial\Omega)}.$$

Proposition 4.6. *Let $\Omega \subset \mathbb{R}^2$ satisfy Definition 2.1, and assume that*

$$\begin{aligned} \bar{f}_\lambda &\in H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega) \\ \bar{f} &\in \gamma_0(W^{2,1}(\Omega)) \times L^1(\partial\Omega) \\ \bar{f}_\lambda &\rightarrow \bar{f} \text{ weakly in } \gamma_0(W^{2,1}(\Omega)) \times L^1(\partial\Omega) \\ \frac{\|\bar{f}_\lambda\|_{H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)}}{\lambda^{1/4}} &\rightarrow 0 \text{ as } \lambda \rightarrow \infty. \end{aligned} \tag{15}$$

- (i) Compactness: *Let $\{u_\lambda\}_\lambda \subset H^2(\Omega)$ be such that $\mathcal{F}_{\bar{f}_\lambda, \lambda}(u_\lambda) < C$. Then there exists a subsequence (no relabeling) and $u \in BH(\Omega)$ such that $u_\lambda \rightarrow u$ weakly $*$ in $BH(\Omega)$.*
- (ii) Lower bound: *If $u_\lambda \rightarrow u$ weakly $*$ in $BH(\Omega)$, then*

$$\liminf_{\lambda \rightarrow \infty} \mathcal{F}_{\bar{f}_\lambda, \lambda}(u_\lambda) \geq \mathcal{F}_{\bar{f}}(u). \tag{16}$$

- (iii) Upper bound: *For every $u \in BH(\Omega)$ there exists a sequence $\{u_\lambda\}_\lambda \subset H^2(\Omega)$ such that $u_\lambda \rightarrow u$ weakly $*$ in $BH(\Omega)$ and $\lim_{\lambda \rightarrow \infty} \mathcal{F}_{\bar{f}_\lambda, \lambda}(u_\lambda) = \mathcal{F}_{\bar{f}}(u)$.*

With the proposition at hand, we can prove our main theorem, which contains Theorem 2.2 as a special case.

Theorem 4.7. *Let $\Omega \subset \mathbb{R}^2$ satisfy Definition 2.1, and assume that*

$$\begin{aligned} g_\lambda &\in H^{-1/2}(\partial\Omega; \mathbb{R}^2) \\ g &\in W^{-1,1}(\partial\Omega; \mathbb{R}^2), \quad (g^\perp)^{(2)} \in \gamma_0(W^{2,1}(\Omega)) \\ g_\lambda^\perp &\rightarrow g^\perp \text{ weakly in } X \\ \lambda^{-1/2} \|g\|_{H^{-1/2}(\partial\Omega; \mathbb{R}^2)}^2 &\rightarrow 0 \text{ as } \lambda \rightarrow \infty. \end{aligned} \tag{17}$$

- (i) Compactness: *Let $\{\sigma_\lambda\}_\lambda \subset L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ be such that $\mathcal{G}_{g_\lambda, \lambda}(\sigma_\lambda) < C$. Then there exists a subsequence (no relabeling) and $\sigma \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ such that $\sigma_\lambda \rightarrow \sigma$ weakly $*$ in the sense of measures.*
- (ii) Lower bound: *If $\sigma_\lambda \rightarrow \sigma$ weakly $*$ in the sense of measures, then*

$$\liminf_{\lambda \rightarrow \infty} \mathcal{G}_{g_\lambda, \lambda}(\sigma_\lambda) \geq \mathcal{G}_g(\sigma). \tag{18}$$

- (iii) Upper bound: *Assume that Ω contracts nicely. For every $\sigma \in \mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ there exists a sequence $\{\sigma_\lambda\}_\lambda \subset L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ such that $\sigma_\lambda \rightarrow \sigma$ weakly $*$ in the sense of measures and $\lim_{\lambda \rightarrow \infty} \mathcal{G}_{g_\lambda, \lambda}(\sigma_\lambda) = \mathcal{G}_g(\sigma)$.*

Remark 4.8. The reason for the assumption $\lambda^{-1/2} \|g_\lambda\|_{H^{-1/2}(\partial\Omega; \mathbb{R}^2)}^2 \rightarrow 0$ here, and for the analogous assumption $\lambda^{-1/4} \|\bar{f}_\lambda\|_{H^{3/2}(\partial\Omega) \times H^{1/2}(\partial\Omega)} \rightarrow 0$ in the Proposition, is technical. This allows us to control the behavior of boundary layers in the upper bound construction, and gives a convenient estimate for error terms appearing in the proof of the lower bound, see the proof of Proposition 4.6. However, these assumptions are not a restriction, in the sense that every $g \in W^{-1,1}(\partial\Omega; \mathbb{R}^2)$ with $(g^\perp)^{(2)} \in \gamma_0(W^{2,1}(\Omega))$ possesses an approximating sequence g_λ with these properties.

Proof. Recall the definition of the “integrals” $(g^\perp)^{(1)}, (g^\perp)^{(2)}$ from (12). We may assume that g_λ, g are balanced, since otherwise $S_{g_\lambda} = \emptyset$ or $\Sigma_g = \emptyset$ by Lemma 4.2. By Remark 4.3

(i), the requirements for the existence of $(g^\perp)^{(1)}, (g^\perp)^{(2)}, (g_\lambda^\perp)^{(1)}, (g_\lambda^\perp)^{(2)}$ are met. Now we set

$$\begin{aligned}\bar{f}_\lambda &:= -((g_\lambda^\perp)^{(2)}, (g_\lambda^\perp)^{(1)} \cdot n) \\ \bar{f} &:= -((g^\perp)^{(2)}, (g^\perp)^{(1)} \cdot n).\end{aligned}\tag{19}$$

With these definitions, (17) implies (15).

For the compactness part, assume that $\mathcal{G}_{g_\lambda, \lambda}(\sigma_\lambda) < C$. Using Lemma 4.1, we obtain a sequence u_λ in $H^2(\Omega)$ with $\mathcal{G}_{g_\lambda, \lambda}(\sigma_\lambda) = \mathcal{F}_{\bar{f}_\lambda, \lambda}(u_\lambda)$. By Proposition 4.6, we obtain weak * convergence of $\sigma_\lambda = \text{cof } D^2 u_\lambda$ to $\text{cof } D^2 u$.

For the lower bound part, we assume $\{\sigma_\lambda\}_\lambda \subset L^2(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ with $\sigma_\lambda \rightarrow \sigma$ weakly * in $\mathcal{M}(\Omega; \mathbb{R}_{\text{sym}}^{2 \times 2})$ and $\lim_{\lambda \rightarrow \infty} \mathcal{G}_{g_\lambda, \lambda}(\sigma_\lambda) = M < \infty$. We may assume $-\text{div } \sigma_\lambda = g_\lambda \mathcal{H}^1 \llcorner \partial\Omega$, since otherwise $\mathcal{G}_{g_\lambda, \lambda}(\sigma_\lambda) = +\infty$. Using Lemma 4.1, let $u_\lambda \in H^2(\Omega)$ and $u \in BH(\Omega)$ such that $\text{cof } \nabla^2 u_\lambda = \sigma_\lambda$ and $\text{cof } D^2 u = \sigma$. By Lemma 4.5, we may assume that by the addition of suitable affine functions, we have

$$\begin{aligned}\gamma_0(u_\lambda) &= -(g_\lambda^\perp)^{(2)}, \quad \gamma_1(u_\lambda) = -(g_\lambda^\perp)^{(1)} \cdot n, \\ \gamma_0(u) &= -(g^\perp)^{(2)}, \quad \gamma_1(u) = -(g^\perp)^{(1)} \cdot n.\end{aligned}$$

This implies $\mathcal{G}_{g_\lambda, \lambda}(\sigma_\lambda) = \mathcal{F}_{\bar{f}_\lambda, \lambda}(u_\lambda)$ and $\mathcal{G}_g(\sigma) = \mathcal{F}_{\bar{f}}(u)$, and hence the lower bound follows from the lower bound part of Proposition 4.6.

For the upper bound part, let $\sigma \in \Sigma_g(\Omega)$. Applying Lemma 4.1 and Lemma 4.5 we have that after the addition of an affine function, $u \in BH(\Omega)$ with $(\gamma_0(u), \gamma_1(u)) = \bar{f}$. The upper bound part of Proposition 4.6 and (19) immediately yield the desired recovery sequence. $\square$

The rest of this paper is concerned with the proof of Proposition 4.6.

5. PROOF OF COMPACTNESS AND UPPER BOUND

In this section and the next, we use the notation

$$G_\lambda(\xi) = \lambda^{-1/2} Q_2 F_\lambda(\xi).$$

By Theorem 3.9, we have

$$G_\lambda(\xi) = \begin{cases} 2(\rho^0(\xi) - \lambda^{-1/2} |\det \xi|) & \text{if } \rho^0(\xi) \leq \sqrt{\lambda} \\ \lambda^{1/2} + \lambda^{-1/2} |\xi|^2 & \text{else.} \end{cases}$$

Proof of compactness in Proposition 4.6. We claim that for $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$,

$$\frac{1}{2} \rho^0(\xi) \leq G_\lambda(\xi).\tag{20}$$

By $|\xi| \leq \rho^0(\xi)$, this implies

$$\frac{1}{2} |D^2 u_\lambda|(\Omega) = \frac{1}{2} \|\nabla^2 u_\lambda\|_{L^1(\Omega)} \leq \mathcal{G}_\lambda(u_\lambda).\tag{21}$$

By Lemma 3.12, we obtain that u_λ is a bounded sequence in BH . By Theorem 3.5 we obtain that a subsequence converges weakly * in $BH(\Omega)$. It remains to prove (20).

Let $x \in \Omega$. Let a_1, a_2 denote the absolute values of the eigenvalues of ξ . For $\rho^0(\xi) = a_1 + a_2 \leq \sqrt{\lambda}$ we have

$$a_1 + a_2 - 2\lambda^{-1/2} a_1 a_2 \geq \lambda^{-1/2} ((a_1 + a_2)^2 - 2a_1 a_2) \geq 0.$$

Hence we have

$$G_\lambda(\xi) = 2(a_1 + a_2 - \lambda^{-1/2}a_1a_2) \geq a_1 + a_2 = \rho^0(\xi). \quad (22)$$

If $a_1 + a_2 \geq \sqrt{\lambda}$, then

$$G_\lambda(\xi) \geq \frac{a_1^2 + a_2^2}{\sqrt{\lambda}} \geq \frac{1}{2} \frac{(a_1 + a_2)^2}{\sqrt{\lambda}} \geq \frac{1}{2}(a_1 + a_2) = \frac{1}{2}\rho^0(\xi)$$

Combining these two cases, we obtain (20) which proves the compactness part of the theorem. $\square$

Proof of the upper bound in Proposition 4.6. By Theorem 3.10, the application of the right inverse of the trace operator (γ_0, γ_1) to $\bar{f}_\lambda$ and $\bar{f}$ yields a sequence $f_\lambda := (\gamma_0, \gamma_1)^{-1}\bar{f}_\lambda$ in $H^2(\Omega)$ and $f := (\gamma_0, \gamma_1)^{-1}\bar{f} \in W^{2,1}(\Omega)$ such that

$$\begin{aligned} f_\lambda &\rightarrow f \quad \text{weakly in } W^{2,1}(\Omega) \\ \lambda^{-1/2}\|f_\lambda\|_{H^2(\Omega)}^2 &\rightarrow 0. \end{aligned} \quad (23)$$

By Theorem 3.14, we may extend f_λ and f to all of $\mathbb{R}^2$ such that

$$\begin{aligned} f_\lambda &\rightarrow f \quad \text{weakly in } W^{2,1}(\mathbb{R}^2) \\ \lambda^{-1/2}\|f_\lambda\|_{H^2(\mathbb{R}^2)}^2 &\rightarrow 0. \end{aligned}$$

Let $v \in BH(\mathbb{R}^2)$ be defined by

$$v(x) = \begin{cases} u(x) - f(x) & \text{for } x \in \Omega \\ 0 & \text{else.} \end{cases}$$

Let $V \subset \mathbb{R}^2$ be some neighborhood of $\bar{\Omega}$, and let $\phi : V \rightarrow \phi(V) \subset \mathbb{R}^2$ be a C^2 -diffeomorphism, such that $\phi(\Omega) = K$, where K is a convex polygon that contains the origin. Such a map ϕ exists by our assumptions on Ω , see Definition 2.1. Choose $C_1 > 0$ such that

$$\text{dist}\left(\phi^{-1}\left(\frac{K}{\sqrt{1+C_1\varepsilon}}\right), \partial\Omega\right) > \varepsilon$$

for all $\varepsilon > 0$ small enough. For such ε , we set

$$\Theta_\varepsilon(x) = \phi^{-1}\left(\frac{\phi(x)}{\sqrt{1+C_1\varepsilon}}\right),$$

and

$$v_\varepsilon(x) := v(\Theta_\varepsilon(x)).$$

Note that $v_\varepsilon \in BH(\mathbb{R}^2)$ and $v_\varepsilon = 0$ on $\{x : \text{dist}(x, \partial\Omega) \leq \varepsilon\}$. Additionally, we claim that in the limit $\varepsilon \rightarrow 0$, we have

$$\begin{aligned} v_\varepsilon &\rightarrow v \quad \text{in } W^{1,1}(\mathbb{R}^2), \\ |D^2v_\varepsilon|(\mathbb{R}^2) &\rightarrow |D^2v|(\mathbb{R}^2). \end{aligned} \quad (24)$$

To prove this claim, we observe that $\nabla\Theta_\varepsilon(x) \rightarrow \text{Id}_{2 \times 2}$ and $\nabla^2\Theta_\varepsilon(x) \rightarrow 0$ uniformly as $\varepsilon \rightarrow 0$. Now we have

$$\begin{aligned} \nabla v_\varepsilon(x) &= \nabla v(\Theta_\varepsilon(x))\nabla\Theta_\varepsilon(x) \\ D^2v_\varepsilon &= (D^2v \circ \Theta_\varepsilon) : (\nabla\Theta_\varepsilon \otimes \nabla\Theta_\varepsilon) \\ &\quad + \nabla v \circ \Theta_\varepsilon \cdot \nabla^2\Theta_\varepsilon \mathcal{L}^2, \end{aligned}$$

where by $(D^2v \circ \Theta_\varepsilon)$, we mean the Radon measure that is defined by

$$\int_{\Omega} \varphi : d(D^2v \circ \Theta_\varepsilon) = \int_{\Omega} \varphi(\Theta_\varepsilon^{-1}(z)) \det \nabla \Theta_\varepsilon^{-1}(z) : d(D^2v)(z) \quad \text{for } \varphi \in C_c^0(\Omega; \mathbb{R}^{2 \times 2}).$$

From the uniform convergences $(\Theta_\varepsilon(x) - x) \rightarrow 0$, $\nabla \Theta_\varepsilon(x) \rightarrow \text{Id}_{2 \times 2}$ and $\nabla^2 \Theta_\varepsilon(x) \rightarrow 0$, the claim (24) follows.

We let $\varphi \in C_c^\infty(\mathbb{R}^2)$ be such that $\int \varphi(x) dx = 1$ and $\text{supp } \varphi \subset \{x \in \mathbb{R}^2 : |x| < 1\}$, and $\varphi_\varepsilon = \varepsilon^{-2} \varphi(\cdot/\varepsilon)$. Next we set

$$\tilde{v}_\varepsilon := \varphi_\varepsilon * v_\varepsilon,$$

and by the properties of $\varphi_\varepsilon, v_\varepsilon$, we have

$$\tilde{v}_\varepsilon = 0, \quad \nabla \tilde{v}_\varepsilon = 0 \quad \text{on } \partial\Omega,$$

$$\tilde{v}_\varepsilon \rightarrow v \quad \text{in } W^{1,1}(\mathbb{R}^2),$$

$$\|\nabla^2 v_\varepsilon\|_{L^1(\mathbb{R}^2)} \rightarrow |D^2v|(\mathbb{R}^2).$$

Furthermore, there exists a constant $C_2 > 0$ such that

$$\|\nabla^2 v_\varepsilon\|_{L^\infty(\mathbb{R}^2)} \leq C_2 \varepsilon^{-1} |D^2v|(\mathbb{R}^2).$$

We set

$$\varepsilon(\lambda) := 4\lambda^{-1/2} C_2 |D^2v|(\mathbb{R}^2),$$

and define the recovery sequence by

$$u_\lambda = f_\lambda + \tilde{v}_{\varepsilon(\lambda)}.$$

Our choice of $\varepsilon(\lambda)$ implies that

$$\|\nabla^2 \tilde{v}_{\varepsilon(\lambda)}\|_{L^\infty} < \frac{\sqrt{\lambda}}{4}. \quad (25)$$

From (23) and Theorem 3.5, we see that u_λ converges weakly $*$ in $BH(\mathbb{R}^2)$ to the function

$$\tilde{u}(x) = \begin{cases} u(x) & \text{if } x \in \Omega \\ f(x) & \text{else.} \end{cases}$$

Next let

$$\tilde{E}_\lambda := \{x \in \Omega : \rho^0(\nabla^2 u_\lambda) > \sqrt{\lambda}\}.$$

By (2) and (25), we have

$$\tilde{E}_\lambda \subset \{x \in \Omega : |\nabla^2 f_\lambda| > \sqrt{\lambda}/4\} =: E_\lambda.$$

Let

$$\Omega_\varepsilon := \{x \in \mathbb{R}^2 : \text{dist}(x, \Omega) < \varepsilon\}.$$

For every $\varepsilon > 0$, we have that

$$\begin{aligned} \int_{\Omega_\varepsilon \setminus E_\lambda} G_\lambda(\nabla^2 u_\lambda) dx &\leq 2 \int_{\Omega_\varepsilon \setminus E_\lambda} \rho^0(\nabla^2 u_\lambda) dx \\ &\int_{E_\lambda} G_\lambda(\nabla^2 u_\lambda) dx \lesssim \lambda^{1/2} \mathcal{L}^2(E_\lambda) + \int_{E_\lambda} \frac{|\nabla^2 u_\lambda|^2}{\sqrt{\lambda}} dx \\ &\lesssim \frac{\|\nabla^2 f_\lambda\|_{L^2}^2}{22\sqrt{\lambda}} \rightarrow 0 \quad \text{as } \lambda \rightarrow \infty, \end{aligned}$$

where we have used the assumption (15). This implies, using the non-negativity of G_λ ,

$$\begin{aligned} \limsup_{\lambda \rightarrow \infty} \int_{\Omega} G_\lambda(\nabla^2 u_\lambda) dx &\leq \limsup_{\lambda \rightarrow \infty} \int_{\Omega_\varepsilon} G_\lambda(\nabla^2 u_\lambda) dx \\ &\leq \limsup_{\lambda \rightarrow \infty} 2 \int_{\Omega_\varepsilon} \rho^0(\nabla^2 u_\lambda) dx \\ &= 2 \int_{\Omega_\varepsilon} \rho^0(D^2 \tilde{u}), \end{aligned} \quad (26)$$

where we have used Theorem 3.2 in the last step, and the fact that

$$\rho^0(D^2 \tilde{u})(\partial\Omega_\varepsilon) = 0.$$

Taking the limit $\varepsilon \rightarrow 0$ in the estimate (26), we obtain

$$\begin{aligned} \limsup_{\lambda \rightarrow \infty} \int_{\Omega} G_\lambda(\nabla^2 u_\lambda) dx &\leq 2 \int_{\Omega} \rho^0(D^2 \tilde{u}) \\ &= 2 \int_{\Omega} \rho^0(D^2 u) + 2 \int_{\partial\Omega} |\nabla u - \nabla f| d\mathcal{H}^1. \end{aligned} \quad (27)$$

On the right hand side above, $\nabla u|_{\partial\Omega}$ has to be understood as the trace of $\nabla u \in BV(\Omega)$. By Theorem 3.6, we have that $\tilde{u}$ is continuous. In particular, we must have $f|_{\partial\Omega} = \gamma_0(u)$, and hence

$$\begin{aligned} \int_{\partial\Omega} |\nabla u - \nabla f| d\mathcal{H}^1 &= \int_{\partial\Omega} |\gamma_1(u) - \partial_n f| d\mathcal{H}^1 \\ &= \int_{\partial\Omega} |\gamma_1(u) - \bar{f}_2| d\mathcal{H}^1, \end{aligned} \quad (28)$$

which implies

$$\limsup_{\lambda \rightarrow \infty} \int_{\Omega} G_\lambda(\nabla^2 u_\lambda) dx \leq 2 \int_{\Omega} \rho^0(D^2 u) + 2 \int_{\partial\Omega} |\gamma_1(u) - \bar{f}_2| d\mathcal{H}^1.$$

By Theorem 3.7, we may find $U_\lambda \in H^2(\Omega)$ with $U_\lambda = f_\lambda$ on $\partial\Omega$ such that it satisfies

$$\begin{aligned} \lambda^{-1/2} \int_{\Omega} F_\lambda(\nabla^2 U_\lambda) dx &\leq \int_{\Omega} G_\lambda(\nabla^2 u_\lambda) dx + \frac{1}{\lambda} \\ \|U_\lambda - u_\lambda\|_{W^{1,2}(\Omega)} &\leq \frac{1}{\lambda}. \end{aligned}$$

This implies that $U_\lambda \rightarrow u$ in $W^{1,1}(\Omega)$. Also, we have

$$\limsup_{\lambda \rightarrow \infty} \mathcal{F}_{\bar{f}_\lambda, \lambda}(U_\lambda) \leq 2 \int_{\Omega} \rho^0(D^2 u) + 2 \int_{\partial\Omega} |\gamma_1(u) - \bar{f}_2| d\mathcal{H}^1.$$

By the compactness part, it follows that $U_\lambda \rightarrow u$ weakly $*$ in $BH(\Omega)$. This proves that U_λ is the required recovery sequence. $\square$

6. PROOF OF THE LOWER BOUND

Lemma 6.1. *Let $\Omega \subset \mathbb{R}^2$ be open and bounded, $\varphi \in C^0(\Omega; \mathbb{R}^2)$ and $w_\lambda \rightarrow 0$ in $L^1(\Omega)$ with $\|\nabla w_\lambda\|_{L^1} \leq C$. Then $G_\lambda(\varphi \otimes w_\lambda) \rightarrow 0$ in L^1 .*

Proof. By the Poincaré inequality,

$$\left\| w_\lambda - \left(\int_\Omega w_\lambda(x) dx \right) \right\|_{L^2} \leq C \|\nabla w_\lambda\|_{L^1} \leq C.$$

For λ large enough we may assume $|\int_\Omega w_\lambda(x) dx| < 1$, and hence $\|w_\lambda\|_{L^2} < C$. In particular, we have

$$\mathcal{L}^2 \left(\{x : |w_\lambda| > C^{-1}\sqrt{\lambda}\} \right) < \frac{C}{\lambda}.$$

Now by $G_\lambda(\xi) = 2\rho^0(\xi) - 2\lambda^{-1/2}|\det \xi| \lesssim |\xi|$ for $\rho^0(\xi) \leq \sqrt{\lambda}$ and $|\xi| \leq \rho^0(\xi)$, we have

$$\begin{aligned} \int_\Omega G_\lambda(\varphi \otimes w_\lambda) dx &\leq C \left(\int_\Omega |w_\lambda| + \frac{|w_\lambda|^2}{\sqrt{\lambda}} dx \right) + \lambda^{1/2} \mathcal{L}^2 \left(\{x : |w_\lambda| > C^{-1}\sqrt{\lambda}\} \right) \\ &\rightarrow 0 \quad \text{as } \lambda \rightarrow \infty. \end{aligned}$$

This proves the lemma. $\square$

Lemma 6.2. (i) Let $\Omega \subset \mathbb{R}^2$ be open and bounded, $\xi_0 \in \mathbb{R}^{2 \times 2}$, and $w_\lambda \rightarrow 0$ in $L^1(\Omega)$ as $\lambda \rightarrow \infty$. Then

$$\liminf_{\lambda \rightarrow \infty} \int_\Omega G_\lambda(\xi_0 + \nabla w_\lambda) dx \geq 2\mathcal{L}^2(\Omega)\rho^0(\xi_0).$$

(ii) Let $\nu \in \mathbb{R}^2$ with $|\nu| = 1$ and let $\tilde{Q} \subset \mathbb{R}^2$ be a cube such that one of its sides is parallel to ν , and let $v \in L^1(\tilde{Q}; \mathbb{R}^2)$ such that

$$v(y) = \psi(y \cdot \nu)\eta \quad \text{for all } y \in \tilde{Q}$$

for some $\eta \in \mathbb{R}^2$ and $\psi \in BV((0, 1))$. Furthermore let $v_j \rightarrow v \in L^1(\tilde{Q})$, and $\lambda_j \rightarrow \infty$. Then

$$\liminf_{j \rightarrow \infty} \int_{\tilde{Q}} f_{\lambda_j}(\nabla v_j) dx \geq 2\rho^0(Dv)(\tilde{Q}).$$

Proof of (i). We may assume $\lim_{\lambda \rightarrow \infty} \int_\Omega G_\lambda(\xi_0 + \nabla w_\lambda) = C < \infty$. We claim that for $A, B \in \mathbb{R}^{2 \times 2}$, we have

$$G_\lambda(A + B) \leq C(G_\lambda(A) + G_\lambda(B)). \quad (29)$$

Indeed, by the sublinearity of ρ^0 , we may assume $\rho^0(A) \geq \rho^0(A+B)/2$ and $\rho^0(A) \geq \rho^0(B)$. Using the fact that $|\xi|/2 \leq \rho^0(\xi) \leq 2|\xi|$, we can make the following case distinction:

Case 1: If $\rho^0(A+B), \rho^0(A) > \sqrt{\lambda}$, then we have

$$G_\lambda(A) = \lambda^{1/2} + \lambda^{-1/2}|A|^2 \geq \lambda^{1/2} + \lambda^{-1/2} \left(\frac{\rho^0(A+B)}{4} \right)^2 \geq CG_\lambda(A+B)$$

Case 2: If $\rho^0(A) \leq \sqrt{\lambda}, \rho^0(A+B) \geq \lambda$, then $\sqrt{\lambda}/2 \leq \rho^0(A) \leq \rho^0(A+B) \leq 2\sqrt{\lambda}$. Additionally, by (22), we have $\rho^0(A) \leq G_\lambda(A)$. This allows us to estimate

$$G_\lambda(A) \geq \rho^0(A) \geq \sqrt{\lambda}/2 \geq C \left(\sqrt{\lambda} + \lambda^{-1/2}(2\lambda^{1/2})^2 \right) \geq CG_\lambda(A+B).$$

Case 3: If $\rho^0(A), \rho^0(A+B) \leq \sqrt{\lambda}$, then again by (22), we have $\rho^0(A) \leq G_\lambda(A)$, and hence

$$G_\lambda(A) \geq \rho^0(A) \geq \frac{1}{2} \left(\rho^0(A+B) - \lambda^{-1/2}|\det(A+B)| \right) = \frac{1}{4}G_\lambda(A+B).$$

This proves (29). Hence we have

$$\int_{\Omega} G_{\lambda}(\nabla w_{\lambda}) dx < C,$$

and there exists a non-negative Radon measure μ such that

$$G_{\lambda}(\nabla w_{\lambda}) \mathcal{L}^2 \rightarrow \mu \text{ weakly } ^* \text{ in the sense of measures.}$$

Let Ω_k be an increasing sequence of subdomains s.t.

$$\Omega_k \Subset \Omega, \quad \Omega = \cup_{k \in \mathbb{N}} \Omega_k$$

and let φ^k be smooth cutoff functions with $0 \leq \varphi^k \leq 1$, $\varphi^k|_{\Omega_k} = 1$, $\varphi^k|_{\Omega \setminus \Omega_{k+1}} = 0$. We set

$$w_{\lambda}^k = \varphi^k w_{\lambda}.$$

By the quasiconvexity of G_{λ} , we have

$$\begin{aligned} \mathcal{L}^2(\Omega) G_{\lambda}(\xi_0) &\leq \int_{\Omega} G_{\lambda}(\xi_0 + \nabla w_{\lambda}^k) dx \\ &\leq \mathcal{L}^2(\Omega \setminus \Omega_{k+1}) G_{\lambda}(\xi_0) dx + \int_{\Omega_{k+1} \setminus \Omega_k} G_{\lambda}(\xi_0 + \nabla w_{\lambda}^k) dx \\ &\quad + \int_{\Omega_k} G_{\lambda}(\xi_0 + \nabla w_{\lambda}) dx. \end{aligned}$$

This implies

$$\mathcal{L}^2(\Omega_{k+1}) G_{\lambda}(\xi_0) \leq \int_{\Omega_{k+1} \setminus \Omega_k} G_{\lambda}(\xi_0 + \nabla w_{\lambda}^k) dx + \int_{\Omega_k} G_{\lambda}(\xi_0 + \nabla w_{\lambda}) dx. \quad (30)$$

We estimate

$$\begin{aligned} \int_{\Omega_{k+1} \setminus \Omega_k} G_{\lambda}(\xi_0 + \nabla w_{\lambda}^k) dx &\lesssim \int_{\Omega_{k+1} \setminus \Omega_k} \left(G_{\lambda}(\xi_0) + G_{\lambda}(\nabla \varphi^k \otimes w_{\lambda}) + G_{\lambda}(\varphi^k \nabla w_{\lambda}) \right) dx \\ &\lesssim \mathcal{L}^2(\Omega_{k+1} \setminus \Omega_k) G_{\lambda}(\xi_0) + \int_{\Omega_{k+1} \setminus \Omega_k} G_{\lambda}(\nabla \varphi^k \otimes w_{\lambda}) \\ &\quad + \int_{\Omega} G_{\lambda}(\nabla w_{\lambda})(\varphi^{k+1} - \varphi^{k-1}) dx, \end{aligned} \quad (31)$$

where we have used (29) in the first inequality. Subtracting the inequality (30) from $\int_{\Omega} G_{\lambda}(\xi_0 + \nabla w_{\lambda}) dx$, and additionally using (31), we obtain

$$\begin{aligned} &\int_{\Omega} G_{\lambda}(\xi_0 + \nabla w_{\lambda}) dx - \mathcal{L}^2(\Omega_{k+1}) G_{\lambda}(\xi_0) \\ &\quad \gtrsim \int_{\Omega \setminus \Omega_k} G_{\lambda}(\xi_0 + \nabla w_{\lambda}) dx - \mathcal{L}^2(\Omega_{k+1} \setminus \Omega_k) G_{\lambda}(\xi_0) \\ &\quad \quad - \int_{\Omega_{k+1} \setminus \Omega_k} G_{\lambda}(\nabla \varphi^k \otimes w_{\lambda}) dx - \int_{\Omega} G_{\lambda}(\nabla w_{\lambda})(\varphi^{k+1} - \varphi^{k-1}) dx. \end{aligned} \quad (32)$$

We have $\|\nabla w_{\lambda}\|_{L^1} \lesssim \|G_{\lambda}(\nabla w_{\lambda})\|_{L^1} \leq C$, and hence by Lemma 6.1, we have

$$G_{\lambda}(\nabla \varphi^k \otimes w_{\lambda}) \rightarrow 0 \quad \text{in } L^1. \quad (33)$$

Sending $\lambda \rightarrow \infty$ in (32) and using (33) and the non-negativity of G_λ , we obtain

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \int_{\Omega} G_\lambda(\xi_0 + \nabla w_\lambda) dx - \mathcal{L}^2(\Omega_{k+1}) 2\rho^0(\xi_0) \\ \gtrsim -\mathcal{L}^2(\Omega_{k+1} \setminus \Omega_k) \rho^0(\xi_0) - \int_{\Omega} (\varphi^{k+1} - \varphi^{k-1}) d\mu. \end{aligned}$$

Summing up from $k = 2$ to $k = l$ and dividing by $l - 1$, we get

$$\begin{aligned} \lim_{\lambda \rightarrow \infty} \int_{\Omega} G_\lambda(\xi_0 + \nabla w_\lambda) dx - \left(\frac{1}{l-1} \sum_{k=2}^l \mathcal{L}^2(\Omega_{k+1}) \right) 2\rho^0(\xi_0) \\ \gtrsim -\frac{1}{l-1} \mathcal{L}^2(\Omega_l \setminus \Omega_2) \rho^0(\xi_0) - \frac{1}{l-1} \int_{\Omega} (\varphi^{l+1} + \varphi^l - \varphi^2 - \varphi^1) d\mu \\ \gtrsim -\frac{1}{l-1} (\mathcal{L}^2(\Omega) \rho^0(\xi_0) + |\mu|(\Omega)). \end{aligned}$$

Sending $l \rightarrow \infty$, we obtain the desired result. $\square$

Proof of (ii). The proof of Lemma 4.3 (ii) in [3] can be copied word by word; except for the last step, where instead of Lemma 4.3 (i) in that reference we use part (i) of the present lemma. $\square$

Proof of the lower bound in Proposition 4.6. We may assume $\mathcal{F}_{\bar{f}}(u) < \infty$ and after choosing an appropriate subsequence, we may also assume $\lim_{\lambda \rightarrow \infty} \mathcal{F}_{\bar{f}_\lambda, \lambda}(u_\lambda) = \mathcal{F}_{\bar{f}}(u)$.

Let $(\gamma_0, \gamma_1)^{-1}$ be the right inverse of the trace operator from Theorem 3.10, and let E be the extension operator $L^1(\Omega) \rightarrow L^1(\mathbb{R}^2)$ from Theorem 3.14. Letting $f_\lambda := E \circ (\gamma_0, \gamma_1)^{-1} \bar{f}_\lambda$, we have that $\tilde{u}_\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by

$$\tilde{u}_\lambda(x) := \begin{cases} u_\lambda(x) & \text{if } x \in \Omega \\ f_\lambda(x) & \text{else} \end{cases}$$

satisfies $\tilde{u}_\lambda \in W^{2,2}(\mathbb{R}^2)$. Setting $A(f_\lambda) := \{x \in \mathbb{R}^2 : \rho^0(\nabla^2 f_\lambda(x)) > \lambda^{1/2}\}$ and $\tilde{\Omega}_\varepsilon := \{x \in \mathbb{R}^2 \setminus \Omega : \text{dist}(x, \Omega) < \varepsilon\}$, we have

$$\begin{aligned} \int_{\tilde{\Omega}_\varepsilon} G_\lambda(\nabla^2 \tilde{u}_\lambda) dx &\lesssim \int_{\tilde{\Omega}_\varepsilon \setminus A(f_\lambda)} |\nabla^2 f_\lambda| dx \\ &\quad + \int_{\tilde{\Omega}_\varepsilon \cap A(f_\lambda)} \frac{|\nabla^2 f_\lambda|^2}{\lambda^{1/2}} dx + \mathcal{L}^2(\Omega_\varepsilon \cap A(f_\lambda)) \lambda^{1/2} \\ &\lesssim \|\nabla^2 f_\lambda\|_{L^1(\tilde{\Omega}_\varepsilon)} + \int_{\mathbb{R}^2} \frac{|\nabla^2 f_\lambda|^2}{\lambda^{1/2}} dx \end{aligned} \tag{34}$$

By assumption (15) and the continuity of $E \circ (\gamma_0, \gamma_1)^{-1}$ we have

$$\int_{\mathbb{R}^2} \frac{|\nabla^2 f_\lambda|^2}{\lambda^{1/2}} dx \rightarrow 0 \quad \text{and} \quad f_\lambda \rightharpoonup E \circ (\gamma_0, \gamma_1)^{-1} \bar{f} \quad \text{in } W^{2,1}(\mathbb{R}^2). \tag{35}$$

The latter implies in particular that $|\nabla^2 f_\lambda|$ is equiintegrable, and hence we obtain from (34) and (35) that

$$\lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \int_{\tilde{\Omega}_\varepsilon} G_\lambda(\nabla^2 \tilde{u}_\lambda) dx = 0. \tag{36}$$

From now on we write $v_\lambda = \nabla \tilde{u}_\lambda$. By assumption, the sequence $\mathcal{F}_{\tilde{f}_\lambda, \lambda}(u_\lambda)$ is bounded, and hence

$$\int_{\mathbb{R}^2} G_\lambda(\nabla v_\lambda) dx = \mathcal{F}_{\tilde{f}_\lambda, \lambda}(u_\lambda) + \int_{\Omega \setminus \mathbb{R}^2} G_\lambda(\nabla^2 \tilde{u}_\lambda) dx \leq C \quad (37)$$

by (34) (with $\varepsilon = \infty$). Also, by $|\xi| \lesssim G_\lambda(\xi)$ for all $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$, we have

$$\int_{\mathbb{R}^2} |\nabla v_\lambda| dx \leq C. \quad (38)$$

By (37) and (38) and the compactness theorems for BV functions and Radon measures respectively, there exists a subsequence of v_λ (no relabeling), a measure $\mu \in \mathcal{M}(\mathbb{R}^2)$ and $v \in BV(\mathbb{R}^2; \mathbb{R}^2)$ with $v = \nabla u$ on Ω , such that

$$\begin{aligned} v_\lambda &\rightarrow v \quad \text{weakly } * \text{ in } BV(\mathbb{R}^2; \mathbb{R}^2) \\ G(\nabla v_\lambda) \mathcal{L}^2 &\rightarrow \mu \quad \text{weakly } * \text{ in } \mathcal{M}(\mathbb{R}^2). \end{aligned}$$

By Theorem 3.1 there exists an $\mathcal{L}^2$ measurable function ξ and a $|D^s v|$ measurable function ζ such that

$$\mu = \xi \mathcal{L}^2 + \zeta |D^s v|.$$

We are going to show

$$\xi(x_0) \geq 2\rho^0(\nabla v(x_0)) \quad \text{for } \mathcal{L}^2 - \text{a.e. } x_0 \in \Omega \quad (39)$$

$$\zeta(x_0) \geq 2 \quad \text{for } |D^s v| - \text{a.e. } x_0 \in \overline{\Omega}. \quad (40)$$

We claim that this implies (16). Indeed, recall that the right hand side in (16) reads

$$2\rho^0(Dv)(\Omega) + 2 \int_{\partial\Omega} |v \cdot n - \tilde{f}_2| d\mathcal{H}^1 = 2\rho^0(Dv)(\overline{\Omega}),$$

see equations (27) and (28) in the proof of the upper bound. Let $\Omega_\varepsilon := \{x \in \mathbb{R}^2 : \text{dist}(x, \Omega) < \varepsilon\}$. By (36), the left hand side of (16) is equal to

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \int_{\Omega_\varepsilon} G_\lambda(\nabla v_\lambda) dx &= \lim_{\varepsilon \rightarrow 0} \mu(\Omega_\varepsilon) \\ &= \mu(\overline{\Omega}). \end{aligned}$$

Now we see that (39) and (40) imply $\mu(\overline{\Omega}) \geq 2\rho^0(Dv)(\overline{\Omega})$, and hence prove the lower bound part. It remains to show (39) and (40).

First we prove (39). Let $Q(x_0, \varepsilon) := x_0 + [-\varepsilon/2, \varepsilon/2]^2$. For $\mathcal{L}^2$ -almost every x_0 , we may choose a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ converging to zero, such that $\mu(\partial Q(x_0, \varepsilon_j)) = 0$ for every $j \in \mathbb{N}$. When we write $\varepsilon \rightarrow 0$ in the sequel, we actually mean the limit $j \rightarrow \infty$ for such a sequence. For every j , we have

$$\lim_{\lambda \rightarrow \infty} \int_{Q(x_0, \varepsilon_j)} G_\lambda(\nabla v_\lambda) dx \rightarrow \mu(Q(x_0, \varepsilon_j)).$$

Note that by Theorem 3.1 we have

$$\begin{aligned} \xi(x_0) &= \lim_{\varepsilon \rightarrow 0} \frac{\mu(Q(x_0, \varepsilon))}{\mathcal{L}^2(Q(x_0, \varepsilon))} \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \int_{Q(x_0, \varepsilon)} G_\lambda(\nabla v_\lambda) dx. \end{aligned} \quad (41)$$

For ε small enough, define $w_{\lambda,\varepsilon} : Q \rightarrow \mathbb{R}^2$ by

$$w_{\lambda,\varepsilon}(x) = \varepsilon^{-1} (v_\lambda(x_0 + \varepsilon x) - v(x_0)) .$$

Furthermore let $w_0(x) = \nabla v(x_0) \cdot x$. Using a change of variables, the Cauchy-Schwarz inequality and (7), we have

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \|w_{\lambda,\varepsilon} - w_0\|_{L^1(Q)} &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_Q |v(x_0 + \varepsilon x) - v(x_0) - \nabla v(x_0) \cdot \varepsilon x| dx \\ &= \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^3} \int_{Q(x_0, \varepsilon)} |v(x) - v(x_0) - \nabla v(x_0) \cdot (x - x_0)| dx \\ &\leq \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon^2} \left(\int_{Q(x_0, \varepsilon)} |v(x) - v(x_0) - \nabla v(x_0) \cdot (x - x_0)|^2 dx \right)^{1/2} \\ &= 0 . \end{aligned} \tag{42}$$

By (41) and (42), it is possible to choose a sequence $\lambda_j \rightarrow \infty$ and a subsequence $\varepsilon_j \rightarrow 0$ (no relabeling) such that with $w_j := w_{\lambda_j, \varepsilon_j}$

$$\begin{aligned} \lim_{j \rightarrow \infty} \|w_j - w_0\| &= 0 \\ \lim_{j \rightarrow \infty} \int_{Q(x_0, \varepsilon_j)} G_{\lambda_j}(\nabla v_{\lambda_j}) dx &= \xi(x_0) . \end{aligned}$$

Noting $\int_{Q(x_0, \varepsilon_j)} G_{\lambda_j}(\nabla v_{\lambda_j}) dx = \int_Q G_{\lambda_j}(\nabla w_j) dx$, we obtain from Lemma 6.2 that

$$\xi(x_0) = \lim_{j \rightarrow \infty} \int_Q G_{\lambda_j}(\nabla w_j) dx \geq 2\rho^0(\nabla w_0) = 2\rho^0(\nabla v(x_0)) .$$

This proves (39).

We turn to the proof of (40). By Theorem 3.1 we have that for $|D^s v|$ -almost every $x_0 \in \overline{\Omega}$, there exist $\eta, \nu \in \mathbb{R}^2$ with $|\eta| = |\nu| = 1$ such that for any open bounded convex set K containing the origin, we have

$$\lim_{\varepsilon \rightarrow 0} \frac{Dv(x_0 + \varepsilon K)}{|Dv|(x_0 + \varepsilon K)} = \eta \otimes \nu .$$

Let $Q^\nu \subset \mathbb{R}^2$ denote the cube of sidelength one, with one axis parallel to ν :

$$Q^\nu := \left\{ x \in \mathbb{R}^2 : |x \cdot \nu| < \frac{1}{2}, |x \cdot \nu^\perp| < \frac{1}{2} \right\} .$$

Furthermore, let $Q^\nu(x_0, \varepsilon) = x_0 + \varepsilon Q^\nu$. From now on, let the limit $\varepsilon \rightarrow 0$ be understood only to involve a sequence $(\varepsilon_j)_{j \in \mathbb{N}}$ such that $\mu(\partial Q^\nu(x_0, \varepsilon_j)) = 0$ for all $j \in \mathbb{N}$. By the definition of ζ , we have

$$\begin{aligned} \zeta(x_0) &= \lim_{\varepsilon \rightarrow 0} \frac{\mu(Q^\nu(x_0, \varepsilon))}{|Dv|(Q^\nu(x_0, \varepsilon))} \\ &= \lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \frac{1}{|Dv|(Q^\nu(x_0, \varepsilon))} \int_{Q^\nu(x_0, \varepsilon)} G_\lambda(\nabla v_\lambda) dx . \end{aligned} \tag{43}$$

We define

$$w_{\lambda,\varepsilon}(x) = \frac{\varepsilon}{|Dv|(Q^\nu(x_0, \varepsilon))} \left(v_\lambda(x_0 + \varepsilon x) - \int_{Q^\nu(x_0, \varepsilon)} v_\lambda(x') dx' \right)$$

$$w_\varepsilon(x) = \frac{\varepsilon}{|Dv|(Q^\nu(x_0, \varepsilon))} \left(v(x_0 + \varepsilon x) - \int_{Q^\nu(x_0, \varepsilon)} v(x') dx' \right).$$

Let $\sigma \in (0, 1)$. By Theorem 3.3 there exists $\tilde{w} \in BV_{\text{loc}}(\mathbb{R})$ such that with

$$w_0(x) := \tilde{w}(x \cdot \nu)\eta$$

we have $|Dw_0|(Q_\nu) \geq \sigma^2$ and

$$\lim_{\varepsilon \rightarrow 0} \|w_\varepsilon - w_0\|_{L^1(Q^\nu)} = 0.$$

By the convergence $v_\lambda \rightarrow v$ in $L^1(\Omega)$, we have $\lim_{\lambda \rightarrow \infty} \|w_{\lambda,\varepsilon} - w_\varepsilon\|_{L^1(Q^\nu)} = 0$, and hence

$$\lim_{\varepsilon \rightarrow 0} \lim_{\lambda \rightarrow \infty} \|w_{\lambda,\varepsilon} - w_0\|_{L^1(Q^\nu)} = 0. \quad (44)$$

By (43) and (44), we can choose a subsequence λ_j and a sequence ε_j such that

$$\lambda_j \frac{|Dv|(Q_{x_0, \varepsilon_j})^2}{\varepsilon_j^4} \rightarrow \infty$$

$$\lim_{j \rightarrow \infty} \frac{1}{|Dv|(Q_{x_0, \varepsilon_j})} \int_{Q_{x_0, \varepsilon_j}} G_{\lambda_j}(\nabla v_{\lambda_j}) dx = \zeta(x_0) \quad (45)$$

$$\lim_{j \rightarrow \infty} \|w_{\lambda_j, \varepsilon_j} - w_0\|_{L^1(Q)} = 0.$$

Now let $\tilde{\varepsilon}_j := \frac{|Dv|(Q_{x_0, \varepsilon_j})}{\varepsilon_j^2}$. Then we have $\lambda_j \tilde{\varepsilon}_j^2 \rightarrow \infty$ and

$$\frac{1}{|Dv|(Q_{x_0, \varepsilon_j})} \int_{Q_{x_0, \varepsilon_j}} G_{\lambda_j}(\nabla v_{\lambda_j}) dx = \int G_{\lambda_j \tilde{\varepsilon}_j^2}(\nabla w_{\lambda_j, \varepsilon_j}) dx.$$

Hence using Lemma 6.2 (ii) it follows

$$\zeta(x_0) = \lim_{j \rightarrow \infty} \int G_{\lambda_j \tilde{\varepsilon}_j^2}(\nabla w_{\lambda_j, \varepsilon_j}) dx \geq 2|Dw_0|(Q^\nu) \geq 2\sigma^2.$$

Sending $\sigma \uparrow 1$ proves (40) and completes the proof of the lower bound. $\square$

APPENDIX A. PROOF OF THEOREM 3.7

In analogy to the proof of the relaxation leading to 1-quasiconvex integrands, we first need to prove an approximation lemma. This is slightly more complicated here than in the case of 1-quasiconvexity, since we cannot use the approximation by finite elements here, which is possible there (see [7]). Instead, we are going to need Whitney's extension theorem, that we quote here in a version that can be found in Stein's book [29]. Let $\Omega \subset \mathbb{R}^n$. Let the Greek letters $\alpha, \beta, \gamma \in \mathbb{N}_0^n$ denote multiindices. We will write $|\alpha| = \sum_i \alpha_i$, $\alpha! = \prod_i \alpha_i!$, and $\nabla^\alpha = \partial_1^{\alpha_1} \dots \partial_n^{\alpha_n}$. Furthermore, for $x \in \mathbb{R}^n$, we write $x^\alpha = x_1^{\alpha_1} \dots x_n^{\alpha_n}$. We shall say

that a function $f : \Omega \rightarrow \mathbb{R}$ belongs to $\text{Lip}^{(k,1)}(\Omega)$ if there exists a collection of real-valued functions $\{f^{(\alpha)} : |\alpha| \leq k\}$ and a constant $M > 0$ such that

$$f^{(\alpha)}(y) = \sum_{\beta: |\alpha+\beta| \leq k} \frac{f^{(\alpha+\beta)}(x)}{\beta!} (x-y)^\beta + R_\alpha(x, y)$$

and

$$|f^{(\alpha)}(y)| \leq M \text{ and } |R_\alpha(x, y)| \leq M|x-y|^{k+1-|\alpha|}$$

for all $x, y \in \Omega$ and all multiindices α with $|\alpha| \leq k$. The set $\text{Lip}^{(k,1)}(\Omega)$ is a normed space, where the norm of f is given by the smallest constant M such that the above relations hold true.

Theorem A.1 (Theorem 4 in Chapter 6 of [29]). *Let k be a non-negative integer and let $\Omega \subset \mathbb{R}^n$ be closed. Then there exists a continuous extension operator $\text{Lip}^{(k,1)}(\Omega) \rightarrow \text{Lip}^{(k,1)}(\mathbb{R}^n)$. The norm of this mapping has a bound that is independent from Ω .*

For a closed set $\Gamma \subset \mathbb{R}^n$, let $d_\Gamma \in C^\infty(\mathbb{R}^n; [0, \infty))$ denote a regularized distance function, that is, a function with the property that there exists a constant $C > 0$ such that

$$C^{-1} \text{dist}(x, \Gamma) \leq d_\Gamma(x) \leq C \text{dist}(x, \Gamma) \quad \text{for all } x \in \mathbb{R}^n.$$

$$|\nabla^\alpha d_\Gamma(x)| \leq C|d_\Gamma(x)|^{1-|\alpha|} \quad \text{for all multiindices } \alpha.$$

Such a regularized distance function exists by Theorem 2 of Chapter 6 in [29]. We will use it to construct suitable cutoff functions in Lemma A.2 below. This lemma is a preparation for the approximation lemma, Lemma A.3 below.

Let $\phi \in C^\infty([0, \infty))$ such that $\phi(t) = 0$ for $t \leq \frac{1}{2}$ and $\phi(t) = 1$ for $t \geq 1$.

Lemma A.2. *Let $p \geq 1$, $U \subset [0, 1]^2$, $(u_\varepsilon)_{\varepsilon>0}$ a sequence in $W^{2,p}(U)$ that converges strongly to $u \in W^{2,p}(U)$ for $\varepsilon \rightarrow 0$, and let*

$$\Gamma \subset ([0, 1] \times \{0\} \cup \{0\} \times [0, 1]) \cap \partial U$$

be a closed set of positive $\mathcal{H}^1$ measure, and $u_0 \in W^{2,p}(U)$ such that $u = u_0$ and $\nabla u = \nabla u_0$ on Γ . Furthermore let

$$\tilde{u}_\varepsilon(x) = \phi\left(\frac{d_\Gamma(x)}{\varepsilon}\right) u_\varepsilon(x) + \left(1 - \phi\left(\frac{d_\Gamma(x)}{\varepsilon}\right)\right) u_0(x).$$

Then $\tilde{u}_\varepsilon = u_0$ and $\nabla \tilde{u}_\varepsilon = \nabla u_0$ on Γ , and

$$\|\tilde{u}_\varepsilon - u\|_{W^{2,p}(U)} \rightarrow 0$$

$$\int_{U \cap \{x: d_\Gamma(x) \leq \varepsilon\}} (1 + |\nabla^2 u_\varepsilon|^p + |\nabla u|^p) dx \rightarrow 0.$$

Proof. The first claim is obvious. To show the second claim, it suffices to show that $\|\nabla^2(\tilde{u}_\varepsilon - u)\|_{L^p} \rightarrow 0$, since $\tilde{u}_\varepsilon - u$ and its gradient vanish on Γ . This is a straightforward computation, that estimates the integral over $|\nabla^2(\tilde{u}_\varepsilon - u)|^p$ in the “bulk” $U \cap \{x : d_\Gamma(x) > \varepsilon\}$ and in boundary layer

$$U \cap \{x : d_\Gamma(x) \leq \varepsilon\} \subset (U \cap \{x : x_1 < C\varepsilon\}) \cup (U \cap \{x : x_2 < C\varepsilon\}),$$

where the constant $C > 0$ is chosen appropriately. The contribution of the bulk vanishes in the limit $\varepsilon \rightarrow 0$ due to the assumption. Writing $\tilde{u}_\varepsilon - u$ and its gradient as integrals in x_i -direction in the set $U \cap \{x : x_i < C\varepsilon\}$ for $i = 1, 2$ and using Fubini’s theorem, the

proof that the contribution of the boundary layer vanishes in the limit is a straightforward but lengthy computation that we omit here for the sake of brevity. The third claim is an immediate consequence of the L^p integrability of $\nabla^2 u$ and the strong convergence $\nabla^2 u_\varepsilon \rightarrow \nabla^2 u$ in L^p . $\square$

Lemma A.3. *Let $\Omega \subset \mathbb{R}^2$ satisfy Definition 2.1, and let $p \in [1, \infty)$. Furthermore let $u \in \{u_0\} + W_0^{2,p}(\Omega)$ and $\delta > 0$. Then there exists $w \in \{u_0\} + W_0^{2,p}(\Omega)$ and $\Omega_w \subset \Omega$ such that Ω_w is the union of mutually disjoint closed cubes, w is piecewise a polynomial of degree k on Ω_w , and furthermore*

$$\|u - w\|_{W^{2,p}(\Omega)} < \delta, \\ \int_{\Omega \setminus \Omega_w} (1 + |\nabla^2 u|^p + |\nabla^2 w|^p) dx < \delta.$$

Proof. Let $v := u - u_0 \in W_0^{2,p}(\Omega)$. We may extend u_0 to $\mathbb{R}^2$ such that

$$\|u_0\|_{W^{2,p}(\mathbb{R}^2)} \lesssim \|u_0\|_{W^{2,p}(\Omega)}.$$

Also, v may be understood as an element of $W^{2,p}(\mathbb{R}^2)$ by a trivial extension. Let $\{\Omega_i : i = 1, \dots, P\}$ be an open cover of $\bar{\Omega}$, let $\{\psi_i : i = 1, \dots, P\}$ be a subordinate partition of unity, and let $\xi_i : \Omega_i \rightarrow \xi_i(\Omega_i) =: \tilde{\Omega}_i$, $i = 1, \dots, P$, be a set of C^2 diffeomorphisms such that $\xi_i(\Omega \cap \Omega_i) \subset (0, 1)^2$ and $\Gamma_i := \xi_i(\Omega_i \cap \partial\Omega) \subset [0, 1] \times \{0\} \cup \{0\} \times [0, 1]$. (Such ξ_i exist by the assumption on Ω .)

Let $\varphi \in C_c^\infty(\mathbb{R}^n)$ with $\int \varphi dx = 1$, and $\varphi_\varepsilon = \varepsilon^{-n} \varphi(\cdot/\varepsilon)$. For every i , we have that

$$\|(u - (\varphi_\varepsilon * u)) \circ \xi_i^{-1}\|_{W^{2,p}(\tilde{\Omega}_i)} \rightarrow 0.$$

By the previous lemma, there exists $\hat{u}_\varepsilon^{(i)} \in W^{2,p}(\tilde{\Omega}_i)$ such that $\hat{u}_\varepsilon^{(i)} = (\varphi_\varepsilon * u) \circ \xi_i^{-1}$ on $\tilde{\Omega}_i \cap \{x : d_{\Gamma_i(x)} \geq \varepsilon\}$,

$$\hat{u}_\varepsilon^{(i)} = u \circ \xi_i^{-1}, \quad \nabla \hat{u}_\varepsilon^{(i)} = \nabla u \circ \xi_i^{-1} \quad \text{on } [0, 1] \times \{0\} \cap \tilde{\Omega}_i,$$

and additionally,

$$\|\hat{u}_\varepsilon^{(i)} - u \circ \xi_i^{-1}\|_{W^{2,p}(U)} \rightarrow 0 \\ \int_{\tilde{\Omega}_i \cap \{x : d_{\Gamma_i(x)} < \varepsilon\}} 1 + |\nabla^2 \hat{u}_\varepsilon^{(i)}|^p + |\nabla u \circ \xi_i^{-1}|^p dx \rightarrow 0.$$

Setting $u_\varepsilon^{(i)} := \hat{u}_\varepsilon^{(i)} \circ \xi_i$, we obviously have that $u_\varepsilon^{(i)} = \varphi_\varepsilon \circ u$ on $\Omega_i \setminus \xi_i^{-1}(\tilde{\Omega}_i \cap \{x : d_{\Gamma_i(x)} < \varepsilon\})$, and

$$u_\varepsilon^{(i)} = u, \quad \nabla u_\varepsilon^{(i)} = \nabla u \quad \text{on } \Omega_i \cap \partial\Omega, \\ \|u_\varepsilon^{(i)} - u\|_{W^{2,p}(\Omega_i)} \rightarrow 0 \\ \int_{\xi_i^{-1}(\tilde{\Omega}_i \cap \{x : d_{\Gamma_i(x)} < \varepsilon\})} 1 + |\nabla^2 u_\varepsilon^{(i)}|^p + |\nabla u|^p dx \rightarrow 0.$$

Setting

$$\Omega_\varepsilon := \Omega \setminus \bigcup_{i=1}^N \xi_i^{-1}(\tilde{\Omega}_i \cap \{x : d_{\Gamma_i}(x) < \varepsilon\})$$

$$u_\varepsilon := \sum_i \psi_i u_\varepsilon^{(i)},$$

we have $u_\varepsilon = \varphi_\varepsilon * u$ on Ω_ε and

$$u_\varepsilon = u, \quad \nabla u_\varepsilon = \nabla u \quad \text{on } \partial\Omega,$$

$$\|u_\varepsilon - u\|_{W^{2,p}(\Omega)} \rightarrow 0$$

$$\int_{\Omega \setminus \Omega_\varepsilon} 1 + |\nabla^2 u_\varepsilon|^p + |\nabla u|^p dx \rightarrow 0.$$

and hence we may fix ε such that

$$\|u_\varepsilon - u\|_{W^{k,p}(\Omega)} < \delta/2$$

$$\int_{\Omega \setminus \Omega_\varepsilon} (1 + |\nabla^k u|^p + |\nabla^k u_\varepsilon|^p) dx < \delta/2.$$

Note that Ω_ε is closed and $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$.

Let $0 < h \ll 1$ to be chosen later, and let Q_i , $i = 1, \dots, N$ be mutually disjoint closed cubes of sidelength h contained in Ω_ε such that

$$\text{dist}(Q_i, Q_j) \geq h^{4/3} \quad \text{for all } i, j = 1, \dots, N, i \neq j$$

$$\text{dist}(\partial\Omega_\varepsilon, Q_i) \geq h^{4/3} \quad \text{for all } i = 1, \dots, N$$

$$\mathcal{L}^n \left(\Omega_\varepsilon \setminus \bigcup_i Q_i \right) \leq Ch^{1/3}.$$

This is always possible assuming that h is small enough. Let x_i denote the midpoint of Q_i . Then we define $\tilde{u}_\varepsilon$ to be the Taylor polynomial of degree two at x_i on each Q_i ,

$$\tilde{u}_\varepsilon(y) = \sum_{|\alpha| \leq 2} \nabla^\alpha u_\varepsilon(x_i) \frac{(y - x_i)^\alpha}{\alpha!} \quad \text{for } y \in Q_i.$$

Let $V = (\cup_i Q_i) \cup \partial\Omega_\varepsilon$. We claim that there exists an extension of $\tilde{u}_\varepsilon$ from V to Ω_ε such that $\|\tilde{u}_\varepsilon\|_{W^{2,\infty}(\Omega_\varepsilon)} \lesssim \|u_\varepsilon\|_{W^{2,\infty}(\Omega_\varepsilon)}$. In order to prove our claim, we invoke Theorem A.1 with $k = 1$. We verify that $\tilde{u}_\varepsilon \in \text{Lip}^{(1,1)}(V)$: Firstly, we have for all $y \in V$ and all multiindices α with $|\alpha| \leq 1$,

$$|\nabla^\alpha \tilde{u}_\varepsilon(y) - \nabla^\alpha u_\varepsilon(y)| \leq Ch^{2-|\alpha|}, \quad (46)$$

where the constant C depends on $\|u_\varepsilon\|_{C^3(\Omega_\varepsilon)}$. Furthermore, we have $u_\varepsilon \in \text{Lip}^{(1,1)}(\Omega_\varepsilon) = C^{1,1}(\Omega_\varepsilon)$, namely there exists a constant $0 < M_1 \lesssim \|u_\varepsilon\|_{C^{1,1}(\Omega_\varepsilon)}$ such that for all $x, y \in \Omega_\varepsilon$ and all multiindices α with $|\alpha| \leq 1$ we have

$$\nabla^\alpha u_\varepsilon(y) = \sum_{|\alpha+\beta| \leq 1} \nabla^{\alpha+\beta} u_\varepsilon(x) \frac{(y-x)^\beta}{\beta!} + R_\alpha(x, y) \quad (47)$$

with

$$|\nabla^\alpha u_\varepsilon(x)| < M_1, \quad |R_\alpha(x, y)| < M_1 |x - y|^{2-|\alpha|}.$$

Now let $x, y \in Q_i$. Then we have for all multiindices α with $|\alpha| \leq 1$,

$$\nabla^\alpha \tilde{u}_\varepsilon(y) = \sum_{|\alpha+\beta| \leq 2} \nabla^\alpha u_\varepsilon(x) \frac{(y-x)^\beta}{\beta!}. \quad (48)$$

Next let $x \in Q_i, y \in Q_j$ with $i \neq j$, or $x \in \partial\Omega, y \in Q_i$. In this case, we have $|x-y| \geq h^{4/3}$. By inserting (46) in (47), we obtain

$$\begin{aligned} \nabla^\alpha \tilde{u}_\varepsilon(y) &= \sum_{|\alpha+\beta| \leq 1} \nabla^{\alpha+\beta} \tilde{u}_\varepsilon(x) \frac{(y-x)^\beta}{\beta!} + O(h^{3-|\alpha|}) + R_\alpha(x, y) \\ &= \sum_{|\alpha+\beta| \leq 1} \nabla^{\alpha+\beta} \tilde{u}_\varepsilon(x) \frac{(y-x)^\beta}{\beta!} + \tilde{R}_\alpha(x, y), \end{aligned} \quad (49)$$

where we have introduced $\tilde{R}_\alpha = R_\alpha(x, y) + O(h^{3-|\alpha|})$, which by $|x-y| \geq h^{4/3}$ implies

$$|\tilde{R}_\alpha(x, y) - R_\alpha(x, y)| = O(h^{3-|\alpha|}) = o(1)|x-y|^{2-|\alpha|},$$

where the last estimate holds since for all multiindices α with $|\alpha| \leq 1$, we have

$$\frac{4}{3} < \frac{3-|\alpha|}{2-|\alpha|}.$$

Summarizing (48) and (49), we have proved that $\tilde{u}_\varepsilon \in \text{Lip}^{(1,1)}(V)$, with

$$|\nabla^\alpha \tilde{u}_\varepsilon(x)| < M_2, \quad |\tilde{R}_\alpha(x, y)| < M_2|x-y|^{2-|\alpha|},$$

where

$$M_2 \leq M_1 + o(1) \lesssim \|u_\varepsilon\|_{C^{1,1}(\Omega_\varepsilon)} \lesssim \|u_\varepsilon\|_{W^{2,\infty}(\Omega_\varepsilon)}.$$

By Theorem A.1, there exists an extension of $\tilde{u}_\varepsilon$ to $\text{Lip}^{(1,1)}(\Omega_\varepsilon)$, with

$$\|\tilde{u}_\varepsilon\|_{W^{2,\infty}(\Omega_\varepsilon)} \lesssim \|u_\varepsilon\|_{W^{2,\infty}(\Omega_\varepsilon)}.$$

Comparing the extension with u_ε , we have the estimates

$$\begin{aligned} \int_{\Omega_\varepsilon \setminus V} |\nabla^2 \tilde{u}_\varepsilon - \nabla^2 u_\varepsilon|^p dx &\lesssim h^{1/3} \|u_\varepsilon\|_{W^{2,\infty}(\Omega_\varepsilon)}^p, \\ \int_V |\nabla^2 \tilde{u}_\varepsilon - \nabla^2 u_\varepsilon|^p dx &\lesssim \mathcal{L}^2(V) h^p. \end{aligned} \quad (50)$$

Choosing h small enough, we have

$$\int_{\Omega_\varepsilon \setminus V} (1 + |\nabla^k u|^p) dx < \delta/3,$$

and $\|\tilde{u}_\varepsilon - u_\varepsilon\|_{W^{2,p}(\Omega_\varepsilon)} < \delta/2$. We claim that the function $w \in W^{2,p}(\Omega)$, defined by

$$w(x) = \begin{cases} \tilde{u}_\varepsilon & \text{if } x \in \Omega_\varepsilon \\ u_\varepsilon & \text{else} \end{cases}$$

satisfies all the properties that are stated in the lemma. Indeed, we have $w = u_0$ and $\nabla w = \nabla u_0$ on $\partial\Omega$, w is a polynomial of degree 2 on $\Omega_w := \cup_i Q_i$, and

$$\|w - u\|_{W^{2,p}(\Omega)} \leq \|u - u_\varepsilon\|_{W^{2,p}(\Omega)} + \|u_\varepsilon - \tilde{u}_\varepsilon\|_{W^{2,p}(\Omega)} < \delta,$$

$$\int_{\Omega \setminus \Omega_w} (1 + |\nabla^2 u|^p + |\nabla^2 w|^p) dx \leq 2\delta/3 + Ch^{1/3} \|u_\varepsilon\|_{W^{2,\infty}(\Omega_\varepsilon)}^p < \delta.$$

This proves the lemma. $\square$

Proof of Theorem 3.7. Let $u \in u_0 + W_0^{2,p}(\Omega)$. By Lemma A.3, there exists $w \in u_0 + W_0^{2,p}(\Omega)$ and $\Omega_w \subset \Omega$ such that Ω_w is a union of mutually disjoint closed cubes, $\Omega_w = \cup_{i=1}^N Q_i$, w is piecewise a polynomial of degree 2 on Ω_w , and furthermore

$$\begin{aligned} \|u - w\|_{W^{2,p}(\Omega)} &< \varepsilon/2, \\ \int_{\Omega \setminus \Omega_w} 1 + |\nabla^2 w|^p + |\nabla^2 u|^p dx &< \varepsilon/2. \end{aligned}$$

For $i \in \{1, \dots, N\}$, choose $\tilde{\xi}_i \in W_0^{2,\infty}([-1/2, 1/2]^2; \mathbb{R}^m)$ such that

$$\int_{[-1/2, 1/2]^2} f(\nabla^2 w(x_i) + \nabla^2 \tilde{\xi}_i(x)) dx < Q_2 f(\nabla^2 w(x_i)) + \frac{\varepsilon}{N \mathcal{L}^2(Q_i)},$$

where x_i denotes the center of the cube Q_i . We identify $\tilde{\xi}_i$ with its periodic extension to $\mathbb{R}^2$. Let d_i denote the sidelength of the cube Q_i , and let $M_i \in \mathbb{N}$ to be chosen later. We define $\xi_i \in W_0^{2,p}(Q_i; \mathbb{R}^m)$ by

$$\xi_i(x) = \left(\frac{d_i}{M_i} \right)^2 \tilde{\xi}_i \left(\frac{M_i(x - x_i)}{d_i} \right).$$

Then we have

$$\begin{aligned} \int_{Q_i} f(\nabla^2 w(x_i) + \nabla^2 \xi_i) dx &= \mathcal{L}^2(Q_i) \int_{[0,1]^2} f(\nabla^2 w(x_i) + \nabla^2 \tilde{\xi}_i(x)) dx \\ &< \mathcal{L}^2(Q_i) Q_2 f(\nabla^2 w(x_i)) + \frac{\varepsilon}{N} \\ &= \int_{Q_i} Q_2 f(\nabla^2 w(x)) dx + \frac{\varepsilon}{N}. \end{aligned}$$

Choosing M_i large enough, we may assume

$$\|\xi_i\|_{W^{1,p}(Q_i; \mathbb{R}^m)}^p < \frac{(\varepsilon/2)^p}{N}.$$

Now the function

$$v(x) = \begin{cases} w + \xi_i & \text{on } Q_i \\ w & \text{on } \Omega \setminus \Omega_w \end{cases}$$

has all the required properties. $\square$

APPENDIX B. PROOF OF THEOREM 3.9

For the convenience of the reader, we repeat the statement. We set

$$\bar{G}_\lambda(\sigma) := \begin{cases} 2\sqrt{\lambda}\rho^0(\sigma) - 2|\det \sigma| & \text{if } \rho^0(\sigma) \leq \sqrt{\lambda} \\ |\sigma|^2 + \lambda & \text{else,} \end{cases}$$

and the theorem we want to prove is

Theorem B.1. *We have*

$$Q_2 F_\lambda(\sigma) = \bar{G}_\lambda(\sigma).$$

For the proof, we will need to carry out proofs of statements whose analogues for first gradients are well known. We closely follow the proofs in [8], adapting them to the current situation.

Definition B.2. Let $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$. We say that f is symmetric rank one convex if

$$f(t\xi_1 + (1-t)\xi_2) \leq tf(\xi_1) + (1-t)f(\xi_2)$$

for all $t \in [0, 1]$, and for all $\xi_1, \xi_2 \in \mathbb{R}_{\text{sym}}^{n \times n}$ such that $\xi_1 - \xi_2 = \alpha\eta \otimes \eta$ for some $\alpha \in \mathbb{R}$, $\eta \in \mathbb{R}^n$.

Furthermore, for $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$, we set

$$R^{\text{sym}}f(\xi) := \sup\{g(\xi) : g \leq f \text{ and } g \text{ is symmetric rank one convex}\}.$$

Lemma B.3. Let $\alpha, \beta \in \mathbb{R}$, $t \in [0, 1]$, $\varepsilon > 0$ and

$$u_t : [a, b] \rightarrow \mathbb{R}, \quad x \mapsto \frac{1}{2}(t\alpha + (1-t)\beta)x^2.$$

Then there exist $I, J \subset [0, 1]$ and $u : [0, 1] \rightarrow \mathbb{R}$ such that $\bar{I} \cup \bar{J} = [0, 1]$, $I \cap J = \emptyset$, $|I| = t$, $|J| = (1-t)$ and

$$\begin{aligned} u(0) &= u_t(0), & u'(0) &= u'_t(0), \\ u(1) &= u_t(1), & u'(1) &= u'_t(1), \\ \|u - u_t\|_{L^\infty} + \|u' - u'_t\|_{L^\infty} &< \varepsilon \\ u''(x) &= \alpha \quad \text{for } x \in I \\ u''(x) &= \beta \quad \text{for } x \in J. \end{aligned}$$

Proof. For $q \in [0, 1-t]$, let $\varphi_{t,q} : [0, 1] \rightarrow \mathbb{R}$ be defined by

$$\varphi_{t,q}(x) = \begin{cases} (1-t)(\alpha - \beta) & \text{if } q \leq x < q+t \\ -t(\alpha - \beta) & \text{else.} \end{cases}$$

Note that $\int_0^1 ds \varphi_{t,q}(s) = 0$ independently of q . In fact, we may choose q such that we also have

$$\int_0^1 ds \int_0^s d\tilde{s} \varphi_{t,q}(\tilde{s}) = 0.$$

This choice of q shall be fixed from now on. We extend $\varphi_{t,q}$ periodically on $\mathbb{R}$. For $k \in \mathbb{N}$, we set $\Phi_k(x) := \varphi_{t,q}(kx)$. Choosing $k \in \mathbb{N}$ large enough, we set

$$u(x) := u_t(x) + \int_0^x ds \int_0^s d\tilde{s} \Phi_k(\tilde{s}).$$

It is obvious that this function has all the desired properties (for large enough k). $\square$

Lemma B.4. Let $\varepsilon > 0$, $t \in [0, 1]$ and let $\xi_1, \xi_2 \in \mathbb{R}_{\text{sym}}^{n \times n}$ and $\alpha \in \mathbb{R}$, $\eta \in \mathbb{R}^n$ such that $\xi_1 - \xi_2 = \alpha\eta \otimes \eta$. Let $l : [0, 1]^n \rightarrow \mathbb{R}$ be affine, and $u_t(x) = l(x) + \frac{1}{2}x^T(t\xi_1 + (1-t)\xi_2)x$.

Then there exists a function $u : [0, 1]^n \rightarrow \mathbb{R}$ and open sets $\Omega_1, \Omega_2 \subset [0, 1]^n$ such that

$$\begin{aligned} \|u - u_t\|_{L^\infty} + \|u - u_t\|_{L^\infty} &< \varepsilon \\ \nabla^2 u &= \xi_1 \text{ on } \Omega_1 \\ \nabla^2 u &= \xi_2 \text{ on } \Omega_2 \\ \|\nabla^2 u\|_{L^\infty} &\leq C \\ |\mathcal{L}^n(\Omega_1) - t| &< \varepsilon \\ |\mathcal{L}^n(\Omega_2) - (1 - t)| &< \varepsilon. \end{aligned}$$

Proof. We may fill the cube $[0, 1]^n$ by smaller cubes with one of the axes parallel to η , and set $u = u_t$ on the (small) remainder. In this way, we reduce the problem to the case where $\eta = e_1$. Now let $\Omega_\varepsilon := [\varepsilon, 1 - \varepsilon]^n$, and $\eta \in C_0^\infty((0, 1)^n)$ such that

$$\begin{aligned} \eta &= 1 \text{ on } \Omega_\varepsilon \\ \|\nabla \eta\|_{L^\infty} &\leq \frac{L}{\varepsilon} \\ \|\nabla^2 \eta\|_{L^\infty} &\leq \frac{L}{\varepsilon^2}, \end{aligned}$$

where L is some numerical constant that does not depend on ε . For $x \in [0, 1]$, we write $\tilde{u}_t(s) = \frac{1}{2}(t(\xi_1)_{11} + (1 - t)(\xi_2)_{11})s^2$. Let $\delta > 0$ to be chosen later. According to Lemma B.3, we may choose $\tilde{u} : [0, 1] \rightarrow \mathbb{R}$ and $I, J \subset [0, 1]$ such that $\bar{I} \cup \bar{J} = [0, 1]$, $I \cap J = \emptyset$, $|I| = t$, $|J| = (1 - t)$, and

$$\begin{aligned} \tilde{u}(0) &= \tilde{u}_t(0), \quad \tilde{u}'(0) = \tilde{u}'_t(0), \\ \tilde{u}(1) &= \tilde{u}_t(1), \quad \tilde{u}'(1) = \tilde{u}'_t(1), \\ \|\tilde{u} - \tilde{u}_t\|_{L^\infty} + \|\tilde{u}' - \tilde{u}'_t\|_{L^\infty} &< \delta \\ \tilde{u}''(x) &= \xi_1 \quad \text{for } x \in I \\ \tilde{u}''(x) &= \xi_2 \quad \text{for } x \in J. \end{aligned}$$

We set $\psi(x_1, \dots, x_n) = \tilde{u}(x_1)$ and

$$u := \eta(\psi + l) + (1 - \eta)u_t.$$

Choosing δ small enough (e.g. $\delta < \min(\varepsilon^3, \varepsilon^3/L)$), this choice of u satisfies all the requirements. We leave it to the reader to carry out the straightforward computations that lead to this statement. $\square$

Lemma B.5. Assume that $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ is bounded from above by a continuous function $\tilde{f} \in C^0(\mathbb{R}_{\text{sym}}^{n \times n})$. Then we have

$$Q_2 f \leq R^{\text{sym}} f.$$

Proof. Since $Q_2 f$ is the largest 2-quasiconvex function that is less or equal to f , and $R^{\text{sym}} f$ is the largest symmetric rank one convex function that is less or equal to f , it suffices to show that if $g : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is 2-quasiconvex, and $g \leq f$, then it is symmetric-rank-one convex. So let us suppose $g : \mathbb{R}^{n \times n} \rightarrow \mathbb{R}$ is 2-quasiconvex, and let $\xi_1, \xi_2 \in \mathbb{R}^{n \times n}$ and $\alpha \in \mathbb{R}$, $\eta \in \mathbb{R}^n$ such that $\xi_1 - \xi_2 = \alpha \eta \otimes \eta$. We need to show

$$g(t\xi_1 + (1 - t)\xi_2) \leq tg(\xi_1) + (1 - t)g(\xi_2) \tag{51}$$

for all $t \in [0, 1]$. Let $u_t(x) := \frac{1}{2}x^T(t\xi_1 + (1-t)\xi_2)x$, and $\varepsilon > 0$ to be chosen later. Let u be the approximating function of Lemma B.4, with the sets $\Omega_1, \Omega_2 \subset [0, 1]^n$ as in the statement of that lemma. Then we have $u - u_t \in W_0^{2,\infty}([0, 1]^n)$ and

$$\nabla^2 u = t\xi_1 + (1-t)\xi_2 + \nabla^2(u - u_t).$$

Hence

$$\begin{aligned} g(t\xi_1 + (1-t)\xi_2) &= \int_{[0,1]^n} g(\nabla^2 u_t) dx \\ &\leq \int_{[0,1]^n} g(\nabla^2 u) dx \\ &= \mathcal{L}^n(\Omega_1)g(\xi_1) + \mathcal{L}^n(\Omega_2)g(\xi_2) + \int_{[0,1]^n \setminus (\Omega_1 \cup \Omega_2)} g(\nabla^2 u) dx \\ &\leq \mathcal{L}^n(\Omega_1)g(\xi_1) + \mathcal{L}^n(\Omega_2)g(\xi_2) + \int_{[0,1]^n \setminus (\Omega_1 \cup \Omega_2)} \tilde{f}(\nabla^2 u) dx \end{aligned}$$

Choosing ε small enough and using the properties of u, Ω_1, Ω_2 from the statement of Lemma B.4, we see that the right hand side is smaller than $tg(\xi_1) + (1-t)g(\xi_2) + \delta$ for any given $\delta > 0$; here we also used the assumption that $\tilde{f}$ is continuous. This proves (51) and hence the lemma. $\square$

Definition B.6. Let $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$. We set $R_0^{\text{sym}} f = f$ and

$$\begin{aligned} R_{k+1}^{\text{sym}} f(\xi) &:= \inf \{ tR_k^{\text{sym}}(\xi_1) + (1-t)R_k^{\text{sym}}(\xi_2) : \\ &\quad t\xi_1 + (1-t)\xi_2 = \xi, \xi_1 - \xi_2 = \alpha\eta \otimes \eta \text{ for some } \alpha \in \mathbb{R}, \eta \in \mathbb{R}^n \}. \end{aligned}$$

In complete analogy to Theorem 6.10, part 2 in [8], we show

Lemma B.7. Let $f : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$, and let $g : \mathbb{R}_{\text{sym}}^{n \times n} \rightarrow \mathbb{R}$ be symmetric rank one convex with $g \leq f$. Then we have

$$R^{\text{sym}} f = \inf_{k \in \mathbb{N}} R_k^{\text{sym}} f.$$

Proof. First we observe that for any $k \in \mathbb{N}$, we have

$$g \leq R_{k+1}^{\text{sym}} f \leq R_k^{\text{sym}} f,$$

and hence obtain that

$$R' f := \inf_{k \in \mathbb{N}} R_k^{\text{sym}} f = \lim_{k \rightarrow \infty} R_k^{\text{sym}} f$$

is well defined. For any symmetric rank one convex function g we have $R' g = g$, and hence $R'(R^{\text{sym}} f) = R^{\text{sym}} f$. Furthermore, if $g \leq g'$, then $R' g \leq R' g'$. Combining these observations with the fact $R^{\text{sym}} f \leq f$, we obtain

$$R^{\text{sym}} f \leq R' f \leq f.$$

It remains to show that $R' f$ is symmetric rank one convex. Assume that $\xi_1, \xi_2 \in \mathbb{R}_{\text{sym}}^{n \times n}$, and $\alpha \in \mathbb{R}, \eta \in \mathbb{R}^n$ such that $\xi_1 - \xi_2 = \alpha\eta \otimes \eta$. Let $\varepsilon > 0$. By definition of $R' f$, there exist $i, j \in \mathbb{N}$ such that

$$R_i^{\text{sym}} f(\xi_1) \leq R' f(\xi_1) + \varepsilon, \quad R_j^{\text{sym}} f(\xi_2) \leq R' f(\xi_2) + \varepsilon.$$

Without loss of generality, we may assume $i \leq j$, which yields $R_j^{\text{sym}} f(\xi_1) \leq R_i^{\text{sym}} f(\xi_1)$. Thus we obtain for every $t \in [0, 1]$,

$$\begin{aligned} R'f(t\xi_1 + (1-t)\xi_2) &\leq R_{j+1}^{\text{sym}}(t\xi_1 + (1-t)\xi_2) \\ &\leq tR_j^{\text{sym}}(\xi_1) + (1-t)R_j^{\text{sym}}(\xi_2) \\ &\leq tR'f(\xi_1) + (1-t)R'f(\xi_2) + \varepsilon. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we obtain that $R'f$ is symmetric rank one convex, which proves the lemma. $\square$

Lemma B.8. *We have*

$$R^{\text{sym}}F_\lambda \leq \bar{G}_\lambda.$$

Proof. From the definition of $R^{\text{sym}}F_\lambda$, we see that for $\xi \in \mathbb{R}_{\text{sym}}^{2 \times 2}$, $R \in SO(2)$, we have

$$R^{\text{sym}}F_\lambda(R^T \xi R) = R^{\text{sym}}F_\lambda(\xi).$$

Hence it suffices to consider ξ of diagonal form,

$$\xi = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}.$$

We may assume $|x| + |y| < \lambda$, since otherwise we know $F_\lambda(\xi) = \bar{G}_\lambda(\xi) = \lambda + x^2 + y^2$. Similarly, we may assume $0 < |x| + |y|$, since otherwise $F_\lambda(\xi) = \bar{G}_\lambda(\xi) = 0$. Let $\alpha, \beta \in (0, 1)$ to be chosen later, and set

$$\xi_1 = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_2 = \begin{pmatrix} x/\alpha & 0 \\ 0 & 0 \end{pmatrix}, \quad \xi_3 = \begin{pmatrix} x & 0 \\ 0 & y/\beta \end{pmatrix}.$$

Note that $\beta\xi_3 + (1-\beta)(\alpha\xi_2 + (1-\alpha)\xi_1) = \xi$, and $\xi_3 - (\alpha\xi_2 + (1-\alpha)\xi_1)$, $\xi_2 - \xi_1$ are both symmetric-rank-one. By Lemma B.7, we have

$$\begin{aligned} R^{\text{sym}}F_\lambda(\xi) &\leq \beta F_\lambda(\xi_3) + (1-\beta)(\alpha F_\lambda(\xi_2) + (1-\alpha)F_\lambda(\xi_1)) \\ &= \beta \left(\lambda + x^2 + \frac{y^2}{\beta^2} \right) + (1-\beta)\alpha \left(\lambda + \frac{x^2}{\alpha^2} \right). \end{aligned}$$

Now we assume $|x| > 0$. The right hand side in the last estimate is convex in α ; it attains its minimum at $\alpha = \frac{|x|}{\sqrt{\lambda}}$. Hence,

$$\begin{aligned} R^{\text{sym}}F_\lambda(\xi) &\leq \beta \left(\lambda + x^2 + \frac{y^2}{\beta^2} \right) + (1-\beta)2|x|\sqrt{\lambda} \\ &= 2|x|\sqrt{\lambda} + \beta(\sqrt{\lambda} - |x|)^2 + \frac{y^2}{\beta} \end{aligned}$$

Choosing $\beta = |y|/(\sqrt{\lambda} - |x|)$, we obtain

$$R^{\text{sym}}F_\lambda(\xi) \leq 2\sqrt{\lambda}(|x| + |y| - |xy|) = \bar{G}_\lambda(\xi).$$

It remains to prove the claim for the case $|x| = 0$. Then we have

$$\begin{aligned} R^{\text{sym}}F_\lambda(\xi) &\leq \beta F_\lambda(\xi_3) + (1-\beta)F_\lambda(\xi_1) \\ &= \beta \left(\lambda + x^2 + \frac{y^2}{\beta^2} \right). \end{aligned}$$

Again setting $\beta = |y|/(\sqrt{\lambda} - |x|)$, we obtain the same conclusion as before. This proves the lemma. $\square$

Proof of Theorem B.1. By Theorem 6.28 in [8], we have $\bar{G}_1 = Q_1 F_1$. We have $F_\lambda = \lambda F(\cdot/\sqrt{\lambda})$, and hence by the definition of the quasiconvex envelope (47) it is easily seen that $Q_1 F_\lambda = \lambda Q_1 F_1(\cdot/\sqrt{\lambda})$. It is also easily verified that $\bar{G}_\lambda = \lambda \bar{G}(\cdot/\sqrt{\lambda})$, and since $Q_1 F_\lambda \leq Q_2 F_\lambda$, we obtain $\bar{G}_\lambda \leq Q_2 F_\lambda$. By Lemma B.5 and Lemma B.8, we also have the opposite inequality $\bar{G}_\lambda \geq Q_2 F_\lambda$. This proves the theorem. $\square$

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