Auctions with Discrete Increments: a Structural Econometric Approach Based on Dominated Strategies *

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Abstract

We propose a structural econometric evaluation for auctions with discrete increments. Although very common in practice, this kind of mechanism raises many theoretical difficulties. First, there are no closed form equilibrium strategies. Thus the econometrician cannot rely on a single formula to build a model. Second, the discrete nature of the outcome complicates the inference.

Our econometric model is built on the weak assumption that bidders do not play some given dominated strategies. This implies that our results remain valid for many different kinds of equilibrium concepts. The specification of our model allows the computation of a Maximum Likelihood Estimator. Usual asymptotic tests and confidence regions derived from the likelihood function are shown to be unreliable. A new exact inference method proposed by Dufour [1998] is implemented to circumvent this difficulty. The procedure is applied to an auction in which travel agents bid to discount airline tickets.

From an economic viewpoint, we find that although the profit of the travel agents may be small, it is very unlikely that competition is tight. We also compare the rent kept by the winner of a discrete increment auction with that of a continuous auction. Our data show that the number of players is crucial. If the number of player is small, then the winning price is smaller when the tick

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is zero. In this case, the Vickrey auction is dominated (from the risk-neutral principal viewpoint) by the discrete increment mechanism. This may explain why we observe substantial ticks in many actual auction mechanisms.

Keywords: Auction with discrete increments. Finite sample inference. Simulated inference.

1 Introduction

This paper proposes a structural estimation of an auction model when the bids are multiples of a tick. Our estimation techniques rely on the hypothesis that some strategies are never played because they are dominated. A precise specification of the equilibrium correspondence may thus be avoided. This modeling choice induces particular inference problems which are handled using a new valid estimation procedure based on some recent results obtained by Dufour [1998].

The literature on empirical game theory owes much to "*the case of auction data*" (Laffont [1997]). Starting with Paarsch [1992], many papers have adopted a structural approach to the study of empirical auction data. The recent empirical literature is characterized by four modeling choices. First, it often relies on a sophisticated specification of the equilibrium, *e.g.*, a symmetric, Nash-Bayes equilibrium as in Laffont, Ossard and Vuong [1995]. Second, these papers are almost exclusively devoted to continuous auction mechanisms. Third, the econometric inference relies on asymptotic approximations. Fourth, recent empirical literature moves towards "weaker" distributional assumptions by choosing non- or semi-nonparametric settings (see in particular, Laffont and Vuong [1993] and Laffont, Ossard and Vuong [1995]). Guerre, Perrigne and Vuong [2000] provide justifications for this approach.

These choices are not innocuous and raise many questions. First, the empirical literature on game theory owes also much to experiments. Now, it is a well documented fact that auction participants often face difficulties when computing optimal strategies; see for instance Bazerman and Samuelson [1983], Kagel [1995] and Rubinstein [1999]. Consequently, the approach relying on an elaborate player's behavior may lead to misspecifications regardless how "weak" the distributional assumptions.

Second, many actual auction mechanisms display substantial ticks. For instance, in the auction game we consider in the paper, the tick of \$5 is a non negligible part of the winning bids. This again may lead to misspecification if this feature is ignored. It is also puzzling to observe so many discrete auctions since some theoretical results (*e.g.*, Chwe [1989]) indicate that the principal should be better off with a continuous mechanism.

Finally, statistical asymptotic inference may be difficult to justify from a strict structural viewpoint as noted in Jouneau [2000]. Moreover, there exists a vast literature documenting the lack of reliability of asymptotic approximations in nonlinear models (see for instance Dagenais and Dufour [1991]).

The first problem — namely that actual economic behaviors may deviate from sophisticated equilibrium concepts — calls for a relaxation of the assumptions made on player's admissible strategies. A weak requirement on actual player's behavior

is that nobody choose dominated strategies. Now, this imposes constraints on the distribution of the outcomes of a game, which turn out to be useful for inference purposes. The model is still structural since the econometric modeling directly follows from economic constraints. Also notice that exhibiting some dominated strategies is far less demanding than computing the set of equilibria of actual games.

This last statement is linked to the second problem, *i.e.*, the presence of a tick. Indeed, the theoretical literature concerning discrete auctions, and more generally discontinuous mechanisms, is meager. Thus the econometrician cannot rely on a single equilibrium formula to build a model. The approach we follow, *i.e.*, a modeling based on the elimination of some dominated strategies, accounts easily for the presence of a tick and avoids the computation of an equilibrium.

But this approach also leads to nonlinear models with complex latent structure, for which asymptotic procedures are known to be unreliable. This, together with the argument already presented by Jouneau [2000] motivates our choice for a *valid* approach.

Our finite sample inference procedure builds on recent work by Dufour [1998]. It requires more specific distributional assumptions than in recent semi- and nonparametric setups. As a matter of fact, our modeling choice can be interpreted as a move of the robustness requirement from the statistical distributional assumptions to the economic equilibrium concepts.

The empirical auction game considered here concerns airline tickets. The internet site Travel Bids organizes auctions in which travel agencies bid to discount tickets already booked by future travelers. We collected 618 recordings of such auctions. We exhibit two dominated strategies for the travel agencies. These dominated strategies are then used to derive the likelihood of the model. We perform a maximum likelihood estimation for two competing formulations of the number of players. We show that this number is very difficult to forecast. Moreover, we also find that the asymptotic estimates are not reliable. In particular, different choices for the estimators of the asymptotic variance/covariance matrix lead to completely different confidence regions. Using simulations, we show that this problem does not result from eventual distributional misspecifications. Together with the theoretical arguments mentioned above, this motivates our choice for a valid inference technique.

A recent paper by Dufour [1998] extends greatly the scope of finite sample inference procedures. We show how to apply these techniques to our context. In particular, they can cope with a number of nuisance parameters as large as the sample itself because their estimation is not needed. Such a large heterogeneity comes from possible variations of the number of players and of the link between the fundamentals, *i.e.*,

the signals, and the outcome of the game, *i.e.*, the winning bid. We extend Dufour's approach to compute confidence regions. The validity of the test procedure allows us to control the coverage probability of the confidence regions. Finally, the latter are used to derive a point estimator and to forecasting.

When applied to our dataset, the finite technique provides much more useful results than the asymptotic procedure. As a last contribution, we examine the economic consequences of our estimation. In particular, we conclude that the discrete auction mechanism most probably selects the best travel agency. On the other hand, it is unlikely that the winning auction equals the largest possible discount. Thus, the empirical results in terms of competition are mixed. Finally, we use our estimations to compare the discrete and Vickrey [1961] mechanisms. It appears that the principal of the game may well be better off with a tick when the number of bidders is small. This is not contradictory with Chwe [1989]'s result (as the contexts differ) and may explain why so many actual auctions display substantial ticks.

The rest of the paper is organized as follows. Section 2 describes the auction mechanism. Some features are very specific to our dataset, but the main aspects are shared by many actual auction mechanisms. In Section 3, we present the econometric model on which we perform Maximum Likelihood Estimation. Sections 4 and 5 respectively deal with finite sample test procedures and confidence regions. In Section 6, we investigate the behavior of the asymptotic and exact techniques. We also perform a cross-validation experiment. Section 7 is devoted to the economic interpretations of the results. In particular, we compare the discrete auction with the Vickrey [1961] second price mechanism. Section 8 concludes.

2 Description of the Travel Bids auction mechanism

Travel Bids is "An on-line auction that allows travelers to save money by getting bids from many travel agents" (Travel Bids [1999]). The detailed description of the auctions is available at `http://www.travelbids.com`. It is possible to download some results of closed auctions at `http://www.travelbids.com/gg00002.htm` (select the "`Single Trip`" option, then enter an identification code). For instance, the code `L001833` yields the following result:

```
Here is the single listing you requested:

Regular Listing:L001833 Quoted Price:$303 Bidding Deadline:02/18/98 02:00 PM
Description:(Air)
Air on UA, PHL-SFO RT 4/25-4/27
Bid #B003794 Disc Bid Amt:$30 9.9% Bid Time:02/18/98 12:24 PM (MST) A
```

```
Bid #B003786 Disc Bid Amt:$25 8.3% Bid Time:02/17/98 12:34 PM (MST)
Bid #B003777 Disc Bid Amt:$20 6.6% Bid Time:02/17/98 10:00 AM (MST)

This list was generated on 2/5/99 at 8:19:44 AM (Mountain Time) (v 3.53)

Traveler's Price = Quoted Price LESS Discount Bid Amount
```

The "`Regular Listing`" refers to a genuine auction mechanism. There are two other listings, namely "`Maximum Price`" and "`Full Service`". The latter is not an auction: future travelers ask for information about their trips (time schedule, accommodation, ...). This information is not publicly available and is directly sent by the travel agents to the future traveler. The "`Maximum Price`" listing refers to an auction mechanism in which the future traveler sets a maximum price to pay for a given flight. Contrarily to the "`Regular Listing`" mechanism, the deal needs not be feasible. More precisely, the "`Regular Listing`" is designed for travelers who have already booked a ticket. The airline company and flight schedule are already known. As a consequence, the nature of the object to be auctioned is very precise. Note however that some travelers using the "`Regular Listing`" do not provide the full description of the ticket. This feature is captured by the *AIRLINE* exogenous variable of our model (see Section **??** for details). The main reason why we excluded the "`Maximum Price`" listing from the dataset is that very little exogenous information (see below) is observed since the ticket has never been designed before the auction starts.

In the "`Regular Listing`", the "`Quoted Price`" refers (in principle) to a first proposal. The travel agents may then overbid. In the above example, three bids have been submitted, each one referring to a given discount ($20, $25 and $30, respectively). As indicated by the letter "A", the ticket has finally been awarded at a price of $273= $303 − $30. Both the winning travel agent and the traveler pay an intermediary fee $f = \$5$ to Travel Bids. This fee is not a listing fee. If the traveler gets no proposal, he pays nothing. Finally the bids must be multiples of $5 (*i.e.*, the tick is $t = \$5$). Notice that this tick is not a negligible fraction of the final bid. As a consequence, an econometric model which neglects the tick is likely to be misspecified.

For a given auction l, the data we collected consist in a set of variables:

- q_l: the quoted price,
- w_l: the winning bid,
- o_l: other bids,
- z_l: exogenous variables (company, quantity, time...).

We collected this information for $L = 618$ flights. Table **??** provides some descriptive statistics.

Table 1 Descriptive Statistics

Statistic[1]	Quoted ($)	Winning bid ($)	Quantity	$\frac{\text{Winning bid}}{\text{Quoted}}$
Average	890.65	71.48	1.86	0.0806
Std. err.	914.34	77.85	1.32	0.0202
Maximum	9808	665	23	0.1980
q95	2362	197	4	0.1155
q75	1032	80	2	0.0877
Median	599	45	2	0.0748
q25	405	30	1	0.0676
q05	296	25	1	0.0622
Minimum	250	20	1	0.0602

[1] qx stands for the empirical quantile of order x%.

Table **??** does not report statistics about the variable "other bids". This is clearly an endogenous variable. However, we will not try to model it. Several reasons may be given for this. First, more than a quarter of the auctions stops with only one bid. So the variable o_l is sometimes missing. Second, and more importantly, we could not come up with a satisfactory way to deal with these bids (see Section **??**).

3 Econometric modeling of the private values

For the ease of exposition, we will first assume that the number of bidders is known. A first caveat is at hand: this is clearly not the case for the econometrician and needs not be the case for the bidders either. We will handle this problem more precisely in Section **??**. In what follows, an uppercase letter denotes a random variable, whereas its realization is denoted by the same lowercase letter.

3.1 Model when the number of players is known

Consider the l-th auction and let n_l be the number of players, $n_l \geq 2$. Second caveat: n_l refers to the total number of potential bidders all along the auction. The number of potential bidders may change since the auction is continuously open (the web site is open 24 hours/day). Some travel agencies may be closed when the auction takes place. As we will see, the fact that the number of players may change from one auction to the other creates no difficulty in our setup.

3.1.1 Specification of the private values

Let $C_{i,l}$, $i = 1, \ldots, n_l$ denote player i's private value in auction l. $C_{i,l}$ may be interpreted as travel agent i's cost for providing the ticket designed in auction l. It is defined as the value such that bidder i has zero profit in auction l by offering a discount equal to $q_l - C_{i,l} - f$. This last quantity may be interpreted as a threat point. The cost $C_{i,l}$ is not observed. From an economic viewpoint, it is a private information. We thus model costs as latent variables. Consider the following density for these latent variable, jointly with quoted price and exogenous variables:

$$\mathcal{L}_{C_{1,l},\ldots,C_{n_l,l},Q_l,Z_l} = \mathcal{L}_{C_{1,l},\ldots,C_{n_l,l}|Q_l,Z_l} \times \mathcal{L}_{Q_l,Z_l}.$$

We are mainly interested in modeling the conditional density $\mathcal{L}_{C_{1,l},\ldots,C_{n_l,l}|Q_l,Z_l}$. The quoted price is given when the auction starts, so it plays the same role as the announced price in an auction book. We do not model this quoted price as a result from a strategic behavior. This point may be questioned since the information given by Travel Bids, according to which the "quoted price refers to a real price" (Travel Bids [1999]), is not completely reliable. Indeed, the future traveler may strategically propose a price below (or above) what he really booked.

We assume that the $C_{i,l}$'s are independent, conditionally on Q_l, Z_l.[1] Therefore, the relevant density may be written as

$$\mathcal{L}_{C_{1,l},\ldots,C_{n_l,l}|Q_l,Z_l}(c_{1,l}, \ldots, c_{n_l,l}, q_l, z_l) = \prod_{i=1}^{n_l} \mathcal{L}_{C_{i,l}|Q_l,Z_l}(c_{i,l}, q_l, z_l).$$

Moreover, we assume that the $C_{i,l}$'s are conditionally identically distributed, so that $\mathcal{L}_{C_{i,l}|Q_l,Z_l} = \mathcal{L}_{C_{j,l}|Q_l,Z_l}$ for all $i, j = 1, \ldots, n_l$. Finally, we restrict our model to a log-normal parametric specification:

$$\mathcal{L}_{\log(C)|Q,Z}(\log(c)|q,z) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left[-\frac{1}{2}\left(\frac{\log(c) - h_\beta(q,z)}{\sigma}\right)^2\right].$$

[1] We will show later that the independence assumption can be relaxed.

The latent model for the cost is then

$$\log(C_{i,l}) = h_\beta(q_l, z_l) + \sigma\tilde{\epsilon}_{i,l}, \quad i = 1, \ldots, n_l, \quad l = 1, \ldots, L,$$

where the $\tilde{\epsilon}_{i,l}$'s are (conditionally) *i.i.d.* standard Gaussian random variables. The normality assumption is not crucial and is made only to simplify the exposition. Other distributions are possible.

The conditional mean is specified as

$$\begin{aligned} h_\beta(q_l, z_l) = \beta_0 + \beta_1 \log(q_l) + \beta_2 \log^2(q_l) + \beta_3 \log(QUANTITY_l) + \\ \beta_4 \log^2(QUANTITY_l) + \beta_5 \big[\log(q_l) \times \log(QUANTITY_l)\big] + \beta_6 AIRLINE_l, \end{aligned} \quad (1)$$

with

$$AIRLINE_l = \begin{cases} 0, & \text{if the airline company is not mentioned in auction } l \\ 1, & \text{otherwise} \end{cases}$$

$QUANTITY_l$ = the number of tickets to be supplied in auction l.

This specification is a straightforward adaptation of the translog cost function. The presence of q_l as an explanatory variable is clear: the quoted price is a first estimation of the ticket price. However, q_l is not an equilibrium price since discounts have been observed. At best, it could be interpreted as the price which would have arisen, had the competition generated by Travel Bids not existed. The explanation for *AIRLINE* is somewhat different. On the one hand, it could detect auctions in which the future traveler has not booked a ticket yet (since the company is not reported). On the other hand, if no company is reported, the nature of the competition changes since the object to be auctioned is not fully designed. In particular, if $AIRLINE = 0$, the tickets supplied by two travel agents could differ in "quality". Note however that the outcome for this part of the strategic behavior is not observed (only the winning bid is reported). Moreover, we stress that the winning bid is always the highest discount, whatever the (unobserved) proposed qualities. Thus, if some "quality competition" took place, it is not supplanted by the price competition. The dominated strategies discussed below are not affected by the eventual presence of a quality competition.

Finally, the variable $QUANTITY$ is meant to capture a "quantity" effect. Consider the following specification for the conditional mean of the logarithm of the average cost:

$$\begin{aligned} E\big[\log(C_{i,l}/QUANTITY_l)\big] = \beta_0 + \beta_1 \log(q_l/QUANTITY_l) + \beta_2 \log^2(q_l/QUANTITY_l) \\ + (\beta_3 + \beta_1 - 1)\log(QUANTITY_l) + (\beta_4 - \beta_2)\log^2(QUANTITY_l) \\ + (\beta_5 + 2\beta_2)\big[\log(q_l) \times \log(QUANTITY_l)\big] + \beta_6 AIRLINE_l. \end{aligned} \quad (2)$$

The reason why we consider this particular formulation is that the quoted price is itself a first cost of the tickets. So, if we want to understand the quantity effect "others things being equal", we have to consider the quoted price *per* ticket. Now, suppose $\beta_3 + \beta_1 - 1$ is negative and large, whereas $\beta_4 = \beta_2$ and $\beta_5 + 2\beta_2 = 0$. Then it is profitable for a traveler who wishes to buy twice the same ticket to split his demand into two different auctions. The effect has to be strong enough to compensate for doubling of the fee. We suspect that some travelers did try to capture this effect. More precisely, our dataset contains some (rare) couples of observations such that $q_l = q_{l+1}$ and $z_l = z_{l+1}$. Considering the cost function, if $\beta_3 + \beta_1 - 1$ is negative (resp. null) (resp. positive) then the tickets are produced by a decreasing (resp. constant) (resp. increasing) average cost technology. Note that this terminology applies to the costs function of the travel agencies. As for the traveler, he always faces a decreasing average cost function since Travel Bids collects a fee.

Clearly, other effects could be introduced. For instance, one could distinguish between long/short distance flights, among the airline companies, the timing, *etc.* However, the information we use in our estimation is quite weak and we cannot hope to measure significative effects with a fairly small number of observations. Notice also that the log-normal specification for the costs entails observed heteroskedasticity. In particular, if $\beta_6 \neq 0$, the variance of the costs varies with $AIRLINE$.

3.1.2 Two dominated strategies and their statistical consequences

The difficult point is to relate this model to the observed outcomes, namely the bids. We are not aware of a closed form solution for the bids in the present context. Hughart [1975] proposed an approach criticized by Wilson [1975]. Chwe [1989] tackles the problem of first price, independent private values auctions with discrete increments. He derives a Nash equilibrium for this game in the special case where the private values are uniformly distributed on $[0, 1]$. He also proved in this case that the associated continuous auction yields a larger expected payoff. We shall come back to this result at the end of this paper.

Without further knowledge of the set of equilibria, our inference relies on some dominated strategies. More precisely, we only require that the outcome of the game does not result from some given dominated strategies. Clearly, the piece of information we use is weak — from an economic viewpoint — and we hope to obtain very structural estimators robust to modifications of the equilibrium concept.

We now describe the dominated strategies on which our inference is based. There are clearly an infinite number of dominated strategies. The point is to select some "valuable" ones (from an inferential viewpoint). Let $c_{(1:n_l)}$ be the smallest element

of $\{c_{1,l}, c_{2,l}, \ldots, c_{n_l,l}\}$. Let f be the fee paid by the winning agent (recall $f = \$5$ in our dataset). We must have

$$q_l - w_l - f - c_{(1:n_l)} \geq 0. \tag{3}$$

Indeed if this is not the case then, the winner has a strictly negative profit when offering a discount w_l or larger. Such an outcome is impossible if the players do not play dominated strategies.

Let t be the tick (recall that $t = \$5$ in our dataset). Consider now $c_{(2:n_l)}$, the second smallest element in $\{c_{1,l}, c_{2,l}, \ldots, c_{n_l,l}\}$. We claim that

$$q_l - w_l - f - t - c_{(2:n_l)} \leq 0. \tag{4}$$

Suppose this is not the case. Then at least two bidders could overbid w_l by offering $w_l + t$. It would have been profitable for at least one of them to do so, since at most one of them has bid w_l. Inference about $\theta = (\beta', \sigma)'$ is based on equations (**??**) and (**??**).

One should remark that the dominated strategies described above are closely related to the argument first stated by Vickrey [1961]. If the tick were 0, our last point induces $q_l - w_l - f = c_{(2:n_l)}$.

Since the (joint) conditional distribution function of the $C_{i,l}$'s has been specified, the conditional probability of the event

$$\{C_{(1:n_l)} \leq q_l - w_l - f\} \cap \{q_l - w_l - f - t \leq C_{(2:n_l)}\} \tag{5}$$

is known. This probability is given by the following proposition.

Proposition 1
Let a and b be two real numbers such that $b < a$. Then

$$\mathbf{P}(\{C_{(1:n_l)} \leq a\} \cap \{b \leq C_{(2:n_l)}\}) = n_l F_C(b)(1 - F_C(b))^{n_l - 1} + (1 - F_C(b))^{n_l} - (1 - F_C(a))^{n_l},$$

where F_C denotes the conditional c.d.f. of $C_{1,l}$.

Proof.
First we distinguish two events:

$$\{C_{(1:n_l)} \leq a\} \cap \{C_{(2:n_l)} \geq b\} = \Big(\{C_{(1:n_l)} \leq b\} \cap \{C_{(2:n_l)} \geq b\}\Big) \cup \Big(\{b \leq C_{(1:n_l)} \leq a\} \cap \{C_{(2:n_l)} \geq b\}\Big).$$

Since the $C_{i,l}$'s are *i.i.d.* random variables, the conditional probability of the first term is the probability that exactly one $C_{i,l} \leq b$, which is given by the binomial distribution $\mathcal{B}(n_l, F_C(b))$:

$$\mathbf{P}\big(\{C_{(1:n_l)} \leq b\} \cap \{C_{(2:n_l)} \geq b\}\big) = n_l F_C(b)(1 - F_C(b))^{n_l - 1}.$$

For the remaining term, notice that by definition $C_{(1:n_l)} \leq C_{(2:n_l)}$. Hence:

$$\begin{aligned}\mathbf{P}\big(\{b \leq C_{(1:n_l)} \leq a\} \cap \{C_{(2:n_l)} \geq b\}\big) &= \mathbf{P}\big(\{b \leq C_{(1:n_l)} \leq a\}\big) \\ &= (1 - F_C(b))^{n_l} - (1 - F_C(a))^{n_l}.\end{aligned}$$

■

From now on, we assume that for all auctions the event (**??**) is realized. We may now make three remarks.

First, notice that the decomposition of the event used in the proof has some economic content. The second event $\{b < C_{(1:n_l)} \leq a\} \cap \{C_{(2:n_l)} \geq b\}$, with $b = b_l \equiv q_l - w_l - f - t$ and $a = a_l \equiv q_l - w_l - f$ means that nobody (including the player with the smallest cost) could have bid strictly higher than w_l. In Section **??**, this event will be referred to as "maximum degree of competition" for auction l.

Second, given the specification of the conditional mean, the event $\{b_l \leq C_{(1:n_l)} \leq a_l\}$ suffices to identify the parameter β/σ (provided the matrix of the exogenous variables has full column rank). Now the event $\{C_{(1:n_l)} \leq b_l\} \cap \{C_{(2:n_l)} \geq b_l\}$ makes σ identifiable since $\tilde{\epsilon}_{(1:n_l)}$ and $\tilde{\epsilon}_{(2:n_l)}$ do not have the same expectation. Therefore the parameter $\theta = (\beta', \sigma)'$ is identifiable.

Finally, we should also mention that our model may be directly extended to the case where the tick is variable (this is the case for instance in wine auctions, see Ginsburgh [1997]). The only point is that the amounts by which the price increases must be known (at least to the bidders).

3.2 The unobservability of the number of players

A problem with auction datasets is that the number of players is often not observed. This is for instance the case in first price auctions. It could even be unknown to the bidders themselves. This creates difficulties from a game-theoretic viewpoint, as well as for econometric inference.

From the game-theoretic viewpoint, the strategies we consider are such that: (i) the bidders will not prefer negative profits, (ii) it is always valuable to bid over an opponent if this does not lead to a negative payoff. Clearly, this does not require any assumption on the knowledge of the number of players. It could be totally unknown, common or private information.

We now consider the econometric difficulties introduced by the unobservability of n_l. In practice, only sealed bid mechanisms allow a precise knowledge of the number of bidders. At the other extreme, descending auction mechanisms totally preclude a direct observation of this number.

To the best of our knowledge, only three papers deal with the estimation of the number of players. Laffont, Ossard and Vuong [1995] assume that this number is the same for all auctions and lies between 11 and 23 (see Sareen [1998] for a Bayesian equivalent). They justify this choice by the fact that 11 individuals attend all of the auctions. For many auctions, this kind of justification cannot be invoked. This point is discussed in Ginsburgh [1997] for wine auctions (see also Albano and Jouneau [1998]). The main objection is that the observed size of the audience may be a bad proxy for the number of *active* players.

One could also approximate the number of active players by the number of submitted bids. However, as overbidding is allowed, the number of bids needs not equal the number of bidders (see Section **??** for evidence on this point).

Laffont, Ossard and Vuong [1995] consider the number of players as a constant parameter. In our model, this amounts to setting $n_l = n$, $\forall l$. Then n is simply an extra integer-valued parameter. From an asymptotic point of view, this creates no major difficulties since the estimator of n converges much faster than the estimators of the continuous parameters (Laffont, Ossard and Vuong [1995] also raised this point). Thus the asymptotic behavior of $\hat{\theta}_L$ remains the same whether n is known or not (provided it is constant).

However considering the number of players constant may seem too stringent a constraint. For instance, our model may be used to forecast the rent left by the auction to the travel agency (see Section **??**). If the number of players is constant from one auction to another, then the average rent should be constant (other things being equal). On the contrary, if we let n change, we may capture some unobserved heterogeneity due to variations of the competition. Notice that this is a major source of heteroskedaticity. Indeed, the distribution of $(C_{(1:n)}, C_{(2:n)})$ varies with n.

To cope with possible variations of the number of players, we introduce the following notation. We define the variable $D_{k,l}$ in the following way: $D_{k,l} = 1$ if the number of players in auction l is k and 0 otherwise, $l = 1, \ldots, L$, $k \geq 2$. The number of players in auction l is then defined as:

$$N_l = \sum_{k \geq 2} k \; D_{k,l}.$$

With this notation, two models can be considered.

3.2.1 A mixture model

Given specified latent density, the likelihood function of auction l is directly obtained from Proposition **??**:

$$\mathcal{L}^m_{W_l|Q_l,Z_l,D_{2,l},D_{3,l},\ldots}(w_l|q_l,z_l,d_{2,l},d_{3,l},\ldots) = \sum_{k\geq 2} d_{k,l}\big[k\Phi(y_l)\Phi(-y_l)^{k-1} + \Phi(-y_l)^k - \Phi(-x_l)^k\big], \quad (6)$$

where

$$x_l = \frac{\log(q_l - w_l - f) - h_\beta(q_l, z_l)}{\sigma},$$
$$y_l = \frac{\log(q_l - w_l - f - t) - h_\beta(q_l, z_l)}{\sigma},$$

and Φ stands for the c.d.f. of the $\mathcal{N}(0,1)$ distribution.

This likelihood is conditional on the number of players. This number is not observed however but may be thought of as resulting from a draw. In order to obtain the marginal likelihood which corresponds with what we do observe, we need to integrate out the number of players.

A simple solution is to assume an upper bound $N_{\max}$ on the number of players. The marginal distribution for the number of players may then be written:

$$\mathbf{P}\left(\{N_l = k\}\right) = \pi_k\,,\ k = 2,\ldots,N_{\max}. \quad (7)$$

The sample likelihood is obtained by averaging over all possible $D_l = (D_{2,l},\ldots,D_{N_{\max},l})$ for $l = 1,\ldots,L$:

$$\prod_{l=1}^{L}\sum_{k=2}^{N_{\max}} \pi_k\left(k\Phi(y_l)\Phi(-y_l)^{k-1} + \Phi(-y_l)^k - \Phi(-x_l)^k\right).$$

Other specifications for the distribution of the number of players may be considered. In particular Poisson or geometrical distributions may be used to allow for a large number of players while keeping the number of parameters reasonably small.[2] One may also choose a distribution which depends on the exogenous variables of the model.

The previous model is a mixture of distributions, each one corresponding to a specific number of players. This kind of distribution may be estimated by one of the techniques described in Titterington, Makov and Smith [1983]. Notice that since

[2]This strategy has been suggested by one of the referee to Laffont, Ossard and Vuong [1995].

we constrain the parameter θ to be the same whatever the number of players, the consistency of the Maximum Likelihood Estimator (MLE) is easily established. More precisely our model does not suffer from a lack of identifiability and the likelihood is everywhere bounded even if one component of the mixture is never observed.

Straightforward maximization of the likelihood yields the results presented in Tables **??** and **??**. (All the computer programs used in this paper are available upon request.)[3]

Table 2 ML estimates of θ (mixture model)

Parameters	Estimate	Std.err. BHHH	Std.err. GN	Std.err. EW
β_0	-0.6102	666.1760	6.8694	0.0765
β_1	1.1384	196.0660	2.0456	0.0232
β_2	-0.0091	14.2549	0.1511	0.0017
β_3	0.0250	198.0200	1.7536	0.0017
β_4	0.0102	40.3257	0.3418	0.0039
β_5	-0.0051	31.4864	0.2929	0.0030
β_6	0.0079	59.1502	0.4056	0.0029
σ	0.0922	14.1623	0.3304	0.0125

Table 3 ML estimates of the marginal distribution of N_l (mixture model)

Parameters	Estimate
π_2	1.0000
π_3	2.8×10^{-8}
π_4	4.5×10^{-9}
π_5	1.3×10^{-10}

The sample log-likelihood (divided by the number of observations) at its maximum is -0.651778. The LR-statistic associated with the global significance hypothesis is 25.5345 (the asymptotic p-value is 2.5×10^{-4}).

[3]We also implemented an EM estimation procedure and obtain the very same results.

The estimated occurence of the event $\{N_l = 2\}$ is extremely high. Thus the other regimes (namely when $\{N_l > 2\}$) have no influence on the likelihood. Not surprisingly, the hypothesis $\pi_2 = 1$ is not rejected (by the usual 5%-level likelihood ratio test). This is a common finding when estimating mixture models and it usually precludes the estimation of the poorly represented regimes. Notice however that this is harmless for the parameter θ as it is the same in each regime. Thus even if only one regime is represented, the parameter θ should be correctly estimated.

The standard errors reported in the last three columns of Table **??** refer to different asymptotic approximations. The results in the second column rely on the first derivatives of the likelihood function only (BHHH procedure). The third column reports the standard errors obtained using the Gauss-Newton (GN) approach, and the last column contains the Eicker-White (EW) approximation proposed by White [1982].

Note that the three asymptotic approximations are not in accordance. In particular, the estimations obtained from the first two procedures give very large standard errors whereas the last one provides much smaller ones. The asymptotic confidence regions as well as the significance tests give very different results from one procedure to another. In particular, according to the usual 5% Wald test procedure, all of the parameters, except β_5, are found significantly different from 0 by the Eicker-White method, whereas none is significant according to the two other procedures (at the same level).

The discrepancies in the estimation of the standard errors will be further investigated in Section **??**.

3.2.2 A cluster model

One may be surprised by the small forcasted number of bidders in such an auction. The impossibility to account for many players in this kind of models seems to be the rule ; *e.g.*, see Albano and Jouneau [1998]. Sareen [1998] obtains similar results: her *a posteriori* distribution of the number of bidders displays a spike at the lower bound (namely 11).

One way to account for possibly many players is to consider the $d_{k,l}$'s as extra *parameters* of the model. In this second model, the number of players in each auction does not result from a draw. The only requirement we maintain is that n_l is below $N_{\max}$.

The likelihood of the sample may now be written as:

$$\prod_{l=1}^{L}\sum_{k=2}^{N_{\max}} d_{k,l}\left(k\Phi(y_l)\Phi(-y_l)^{k-1}+\Phi(-y_l)^k-\Phi(-x_l)^k\right). \tag{8}$$

This kind of cluster models has been used for quite a long time (see, for instance Titterington, Makov and Smith [1983]). They raise the famous "incidental parameter" problem (see Lancaster [2000]). In any case, they do not enjoy a good reputation. The main reason is that a consistent estimator is difficult to exhibit. To see that, first assume the true value of θ is known and consider the estimation of the parameter $d_{k,l}$. The MLE of $d_{k,l}$ is $\hat{d}_{k,l}=1$ provided that

$$\begin{aligned}\mathbf{P}\big(\{C_{(1:k)}<q_l-w_l-f\}\cap\{C_{(2:k)}>q_l-w_l-f-t\}\big)\\ >\mathbf{P}\big(\{C_{(1:j)}<q_l-w_l-f\}\cap\{C_{(2:j)}>q_l-w_l-f-t\}\big),\end{aligned}$$

for all $j\neq k$. First notice $\hat{d}_{k,l}$ is computed from a single observation. So for $\hat{d}_{k,l}$ to be consistent, we must have $\hat{d}_{k,l}=\overline{d}_{k,l}$. Second, as the joint distributions of $(C_{(1:k)},C_{(2:k)})$ share the same support (namely $^+\times{}^+$), there is a strictly positive probability of making an error when estimating the number of players. As this probability does not vanish when increasing the sample size, the MLE cannot be consistent.

This inconsistency of the MLE of $d_{k,l}$ generally prevents the MLE of θ from being consistent. Indeed, consider a given sequence of estimators $\hat{d}_{k,l}$. As the sample size grows, the frequency of errors in the sequence $\hat{d}_{k,l}$ does not vanish. Consequently, the limit criterion one maximizes with respect to θ differs from the limit likelihood computed with the $\overline{d}_{k,l}$. At best, the MLE may converge towards a pseudo-true value (see White [1982]). But may also fail to exhibit a LLN-type behavior.

To the best of our knowledge, valid inference for cluster models has not been considered yet. Nevertheless, Sections **??** and **??** show that such models may be very useful for finite sample inference purposes. In this Section consider usual asymptotic estimation.

A strict MLE of the cluster model requires the computation of the likelihood for any admissible path of the process n_l, $l=1,\ldots,L$. As n_l may range from 2 to $N_{\max}$, and $L=618$, such an estimator is clearly not computable. But the infeasible global MLE may be approximated by an iterative algorithm.

The estimation technique relies on a first step estimator denoted $\hat{\theta}(0)$. The likelihood of each observation l computed at $\hat{\theta}(0)$ is then maximized with respect to $d_{k,l}$. This gives an estimation for $d_{k,l}$ denoted $\hat{d}_{k,l}(0)$, $k=2,\ldots,N_{\max}$, $l=1,\ldots,L$. The likelihood function is maximized w.r.t. to θ as if $\hat{d}_{k,1}(0),\ldots,\hat{d}_{k,L}(0)$ were the true numbers of players in auction l. This gives $\hat{\theta}(1)$ which is used to compute a new

process $\hat{d}_{k,l}(1)$, $k = 2, \ldots, N_{\max}$, $l = 1, \ldots, L$ and so on. This iterative procedure stops when a fixed point has been found.

This algorithm provides an increasing sequence of likelihoods. Indeed, let $\mathcal{L}^c\big(\hat{\theta}(h), \hat{d}(h)\big)$ be the likelihood one obtains after iteration h, where d denotes the vector of the $d_{k,l}$'s. We have

$$\mathcal{L}^c\big(\hat{\theta}(h+1), \hat{d}(h+1)\big) - \mathcal{L}^c\big(\hat{\theta}(h), \hat{d}(h)\big) = \Big[\mathcal{L}^c\big(\hat{\theta}(h+1), \hat{d}(h+1)\big) - \mathcal{L}^c\big(\hat{\theta}(h), \hat{d}(h+1)\big)\Big] + \Big[\mathcal{L}^c\big(\hat{\theta}(h), \hat{d}(h+1)\big) - \mathcal{L}^c\big(\hat{\theta}(h), \hat{d}(h)\big)\Big].$$

The two bracketed terms of the RHS above are positive. As a consequence, the iterative estimator achieves a local maximum of the likelihood.

The Vickrey mechanism yields a first step estimator. If we consider the approximation $t \simeq 0$, then following Vickrey [1961] we have $c_{(2:n_l)} \simeq q_l - w_l - f$, *i.e.*, the winning bid w_l is such that the player with the second smallest cost has a zero profit by offering w_l. Moreover, in this fisrt step, we shall set $\hat{d}_{2,l}(0) = 1$ for all of the auctions. Now, a draw in the distribution of the private values is observed and a simple least squares estimation may be performed yielding $\hat{\theta}(0)$. This proxy model will also be used in Sections **??** to **??**.

The iterative estimator will be denoted $\hat{\theta} = (\hat{\beta}', \hat{\sigma})'$. Accordingly, $\hat{n}_l$ is defined as $\sum_{k=2}^{N_{\max}} k\, \hat{d}_{k,l}$, $l = 1, \ldots, L$. Our results are presented in Tables **??** and **??**.

To check whether our results were sensitive to $N_{\max}$ we perform an estimation with $N_{\max} = 99$. The results for θ are much alike. Setting $N_{\max} = 99$ slightly decreases the estimated frequency of two players. From now on, unless stated otherwise, we keep the specification $N_{\max} = 5$.

With $N_{\max} = 5$, the sample log-likelihood is -0.632000, whereas when $N_{\max} = 99$, the sample log-likelihood is -0.589551. The LR-statistics associated with the global significance hypothesis are respectively 37.75 (p-value 8.0×10^{-5}) and 64.00 (p-value 1.7×10^{-5}). Recall however that the standard asymptotic distribution does not apply for the cluster model (see above). The LR-statistics and their p-values are given as benchmarks.

Significance tests as well as confidence regions will be discussed in the next two sections. For the time being, one may simply notice that the quantity and "quality" effects (namely β_3 to β_6) seem to be very small for both models. Also, as we already mentioned the estimated occurence of 2 players is quite high. Finally the standard errors approximations display the same patterns as in the mixture model. We will come back to the problem of the point estimation later on.

Table 4 Iterative estimates of θ (cluster model)

	$N_{\max} = 5$			
Parameters	Estimate	Std. err. BHHH	Std. err. GN	Std. err. EW
β_0	−0.5691	451.2750	4.7067	0.0542
β_1	1.1269	133.9430	1.4003	0.0164
β_2	−0.0083	9.8627	0.1033	0.0012
β_3	0.0319	144.1800	1.1969	0.0113
β_4	0.0101	29.2096	0.2331	0.0028
β_5	−0.0061	23.4870	0.1995	0.0020
β_6	0.0088	55.7770	0.2762	0.0015
σ	0.0645	5.6972	0.1431	0.0057

	$N_{\max} = 99$			
Parameters	Estimate	Std. err. BHHH	Std. err. GN	Std. err. EW
β_0	−0.5131	265.0740	2.9080	0.0384
β_1	1.1128	78.4137	0.8655	0.0117
β_2	−0.0073	5.7716	0.0639	0.0009
β_3	0.0268	94.5608	0.7362	0.0070
β_4	0.0085	18.1771	0.1435	0.0019
β_5	−0.0051	16.1281	0.1221	0.0012
β_6	0.0082	32.9085	0.1704	0.0010
σ	0.0419	2.0773	0.0491	0.0021

4 Finite sample tests

The standard asymptotic techniques give quite disappointing results. Although the models we estimated seem globally significant (at least for the mixture specification), the discrepancies between the different asymptotic approximations of the standard errors are rather surprising.

The inconclusiveness of standard asymptotic techniques clearly calls for the application of other and more reliable inference procedures. We now turn to finite sample inference techniques.

We first show how to perform tests on the parameter θ with a correct level at

Table 5 Sample distributions of the number of players (cluster model)

$N_{\max} = 5$	
Number of players	Frequency
2	0.88
3	0.06
4	0.02
5	0.04

$N_{\max} = 99$	
Number of players	Frequency
2	0.73
3	0.11
4	0.05
$\hat{n}_l \geq 5$	0.07

any sample size (exact tests). Second, these tests allow us to derive exact confidence regions with a correct level. Finally, these are used to compute a point estimator. We recall that a test (confidence region) is said "exact" when its type I risk (coverage probability) is smaller (larger) than a pre-specified level (see Lehmann [1986]). More precisely, for a given hypothesis H_0, a test associated with the critical region A has level α whenever $\sup_{H_0} \mathbf{P}(A) \leq \alpha$ holds, where $\sup_{H_0}$ indicates that the supremum is over the set of all probability distributions defining H_0. Similarly, a confidence region R has level $1 - \alpha$ if $\mathbf{P}(R) \geq 1 - \alpha$ is true for any admissible probability distribution.

In this section we describe how to perform exact tests. We follow a procedure proposed by Dufour [1998] called Maximized Monte Carlo (MMC) test.

4.1 The MMC test procedure: a guideline

4.1.1 Simple null hypotheses

In what follows, we consider a given statistical parametric model. The distributions of the model are indexed by a parameter γ whose true value is denoted $\overline{\gamma}$. The MMC test is an extension of the Monte Carlo test procedure proposed by Dwass [1953]. The Monte Carlo test procedure is concerned with a simple null hypothesis:

$$H_0 : \overline{\gamma} = \gamma_H,$$

where γ_H is a given value of the parameter.

We describe the main aspects of the Monte Carlo test procedure. For a complete description, see Dufour [1998]. The technique of Monte Carlo tests relies on simulations of a statistic under the null hypothesis. One must compute the same statistic using the simulated trials and the data. Let $\Lambda^{(0)}$ be the statistic when the real data are used and $\Lambda^{(s)}$ its s-th simulated counterpart, $s = 1, \ldots, \mathcal{S}$.

Consider the case where a single simulation is made. Then *under* H_0 the two random variables $\Lambda^{(0)}$ and $\Lambda^{(1)}$ are *i.i.d.* and therefore exchangeable. It follows that, under the null, the event $\left\{\Lambda^{(0)} < \Lambda^{(1)}\right\}$ has probability 0.5. Then the test defined by the decision rule "reject the null whenever $\Lambda^{(0)} < \Lambda^{(1)}$ and otherwise accept it" has level 0.5.[4]

By extension of the same argument, if we simulate 19 statistics $\Lambda^{(1)}, \ldots, \Lambda^{(19)}$, the test associated with the decision rule "reject the null whenever $\Lambda^{(0)} < \Lambda^{(s)}$, $\forall s = 1, \ldots, 19$ and otherwise accept it" has level 0.05. More generally tests at any (rational) level may be performed along the same lines.

4.1.2 Composite null hypotheses

When H_0 is composite, nuisance parameters are present under the null. In this case, H_0 corresponds to several probability distributions. Therefore the problem of simulating under the null is much more complicated. Indeed, under H_0, $\lambda^{(0)}$ is considered a draw from a null distribution associated with one particular *unknown* value of the nuisance parameters. To preserve the exchangeability between the statistic $\Lambda^{(0)}$ and its simulated versions, one should know the value of the nuisance parameters.

A solution proposed by Dufour [1998] is based on the intuition that the null hypothesis with nuisance parameters may be viewed as the *union* of simple hypotheses where the nuisance parameters are fixed. More precisely, if the null is

$$H_0 : \overline{\eta} = \eta_H,$$

where η refers to some of the components of γ, it may be written

$$H_0 : \{\gamma | \overline{\gamma} = (\eta_H, \nu) \text{ for some } \nu\} = \bigcup_{\nu} \{\gamma | \overline{\gamma} = (\eta_H, \nu)\},$$

ν being the vector of nuisance parameters.

The decision rule "reject the null whenever for all ν the statistic simulated under (η_H, ν) is larger than $\Lambda^{(0)}$ and accept it otherwise" has level 0.5 (again see Dufour [1998, Propositions 3.1 and 3.2] for a detailed formulation and proofs).

In practice, this procedure involves a search over the set of nuisance parameters (see, for instance Dufour and Khalaf [2000]). Indeed we seek a value of the nuisance parameter such that the corresponding simple null hypothesis is accepted. If no such value is found, then the null is rejected at level 0.5. This decision rule is consistent with the interpretation of the composite null as the union of several simple hypotheses.

[4] Dufour [1998]'s procedure slightly differs from ours in the case of ties between the simulated statistic and its counterpart. Our decision rule may have a smaller power but is much easier to implement.

Such a criterion can be reformulated as "reject the composite null hypothesis if and only if no value of the nuisance parameter leads to the acceptance of the corresponding simple null hypothesis."

Clearly, this solution leads to conservative critical regions. Indeed $\Lambda^{(1)}$ may be smaller than $\Lambda^{(0)}$ for some value of the nuisance parameter but not for the true value. Then, it may be that the MMC test procedure leads to the acceptance of the composite null hypothesis while it would have been rejected for the true value of the nuisance parameter. This conservative feature would typically induce a loss of power. However, the next Section establishes that this loss is likely to be small in practice. More importantly, the power of a MMC test should be compared with that of other *valid* and *feasible* procedures.

The extension to any rational level follows along the same lines as in the case of a simple null hypothesis. Consider the case of a test at level 0.05 where we need to simulate $\mathcal{S} = 19$ versions of the test statistic. The decision rule "reject the null whenever for each value of the nuisance parameter, all the simulated statistics are larger than $\Lambda^{(0)}$, and accept it otherwise", has level 0.05 (see Dufour [1998]).

4.2 Application to progressive auctions

We now explain how to apply the MMC test technique to our context. We have to distinguish the real data from their simulated counterparts. In what follows, the former have a $^{(0)}$ superscript while the latter have $^{(s)}$, $s = 1, \ldots, \mathcal{S}$ superscripts. We first consider the cluster model. We then discuss the case of the mixture model.

4.2.1 Simulation of the model and nuisance parameters

Consider the null hypothesis H_0:

$$H_0 : \overline{\theta} = \theta_H = (\beta_H', \sigma_H)'.$$

It is clear that the latent model

$$\log(C_{i,l}) = h_{\beta_H}(q_l, z_l) + \sigma_H \tilde{\epsilon}_{i,l}, \quad i = 1, \ldots, n_l, \quad l = 1, \ldots, L,$$

where $\tilde{\epsilon}_{i,l}$ are *i.i.d.* Gaussian, is easy to simulate. The problem is that $\Lambda^{(0)}$ is computed from the observed variables. Recall however that the private values $C_{i,l}^{(0)}$ are not observed. Therefore as the exchangeability is required, we have to simulate the endogenous observed variables, namely the winning bids.

Now, since our model does not require the specification of a closed-form equilibrium, the precise nature of the link between the private values and the winning bid is unspecified. In any case, when $\bar{\theta} = \theta$, we denote this link $W_l = g_l(C_{1,l}, \ldots, C_{n_l,l}, \theta)$.

Clearly, the g_l's are unknown functions. However, the auction design and the maintained assumption that some dominated strategies are never observed impose some restrictions on the class of admissible functions for g_l. In particular, this function must be such that for any θ, any auction l, any number of players n_l and possible configuration for the private values $c_{1,l} \ldots, c_{n_l,l}$, we have

$$q_l - f - c_{(1:n_l)} \geq g_l(c_{1,l}, \ldots, c_{n_l,l}, \theta)\,, \tag{9}$$

$$q_l - f - t - c_{(2:n_l)} \leq g_l(c_{1,l}, \ldots, c_{n_l,l}, \theta)\,, \tag{10}$$

$$g_l(c_{1,l}, \ldots, c_{n_l,l}, \theta) \bmod t \; = 0\,, \tag{11}$$

$$g_l(c_{1,l}, \ldots, c_{n_l,l}, \theta) \geq 5. \tag{12}$$

Equality (**??**) simply states that the winning bid is a multiple of the tick t. The last inequality (**??**) deserves some comments. Our actual dataset only contains auctions from which an exchange has arisen. This implies that the winning bid must be larger than \$5. One explanation is that the fee paid by the traveler to Travel Bids is \$5. Thus if we want to simulate a sample of winning bids exchangeable with our actual dataset, we must impose this last constraint.

Let $\mathcal{G}_l$ be the set of functions satisfying (**??**) to (**??**). In our context, the hypothesis H_0:

$$H_0 : \overline{\theta} = \theta_H$$

is composite and $g_1, \ldots, g_L, d_{k,1}, \ldots, d_{k,L}$, $k \in \{2, \ldots, N_{\max}\}$ act as nuisance parameters.[5] More formally, the complete formulation of the null hypothesis is associated with the following set of parameters:

$$H_0 : \{(\theta_H, g_1, \ldots, g_L, d_{2,1}, \ldots, d_{N_{\max},L}) \,|d_{k,l} \in \{0,1\}, \forall\, k \in \{2, \ldots, N_{\max}\}, \text{ and } g_l \in \mathcal{G}_l,\ \forall l = 1, \ldots, L\}.$$

The MMC test procedure requires a search over the set of nuisance parameters. As $\mathcal{G}_l$ is a set of functions, this looks infeasible at first glance.

Notice however that the constraints on the functions g_l induced by the underlying economic theory are very stringent. Consider a given draw of the private values

[5]The parameters $g_1, \ldots, g_L$, do not enter the likelihood functions $\mathcal{L}^m$ and $\mathcal{L}^c$. This is why they do not previously appear.

$c^{(1)}_{1,l}\ \dots\ , c^{(1)}_{n_l,l}$. The constraints (**??**) to (**??**) under $H_0 : \overline{\theta} = \theta_H$ become

$$\begin{aligned}
&q_l - f - c^{(1)}_{(1:n_l)} \geq g_l(c^{(1)}_{1,l}\ \dots\ , c^{(1)}_{n_l,l}, \theta_H)\,,\\
&q_l - f - t - c^{(1)}_{(2:n_l)} \leq g_l(c^{(1)}_{1,l}\ \dots\ , c^{(1)}_{n_l,l}, \theta_H)\,,\\
&g_l(c^{(1)}_{1,l}\ \dots\ , c^{(1)}_{n_l,l}, \theta_H) \bmod t\ = 0\,,\\
&g_l(c^{(1)}_{1,l}\ \dots\ , c^{(1)}_{n_l,l}, \theta_H) \geq 5.
\end{aligned} \tag{13}$$

This set of inequalities appears as restrictions on the possible *values* of the function g_l, that is the winning bids, when the private values are given. With the notation $w^{(1)}_l = g_l(c^{(1)}_{1,l}\ \dots\ , c^{(1)}_{n_l,l}, \theta_H)$, (**??**) is simply

$$\begin{aligned}
&q_l - f - t - c^{(1)}_{(2:n_l)} \leq w^{(1)}_l \leq q_l - f - c^{(1)}_{(1:n_l)}\,,\\
&w^{(1)}_l \bmod t\ = 0\,,\\
&w^{(1)}_l \geq 5.
\end{aligned} \tag{14}$$

This rewriting together with an appropriate choice of the test statistic leads to a feasible number of simulations.

4.2.2 A convenient test statistic

Consider the following statistic:

$$\sum_{l=1}^{L} \left(\log(q_l) - \log(q_l - w_l - f)\right)^2. \tag{15}$$

To interpret it, we introduce a constrained version of our model defined by:

$$h_\beta(q_l, z_l) = \log(q_l)\,,\ \forall l.$$

Using an approximation suggested by Vickrey's mechanism, $t \simeq 0$ implies $c_{(2:n_l)} \simeq q_l - w_l - f$. The statistic may then be interpreted as the sum of the constrained squared residuals in our proxy model. Note however that this interpretation is slightly missleading since the "error terms" are not centered (indeed $E[\tilde{\epsilon}_{(2:n)}] \neq 0$).

The statistic given in (**??**) depends upon the parameters through the endogenous variables $w_1, \dots, w_L$ only. When computed with private values simulated under the null, the statistic is denoted

$$\lambda(w^{(1)}_1, \dots, w^{(1)}_L, n_1, \dots, n_L, \theta_H)$$

This statistic has several good properties. First, the statistic computed with the observed winning bids $w_1^{(0)}, \ldots, w_L^{(0)}$ — hereafter $\lambda^{(0)}$ — is a constant. Notice $\lambda^{(0)}$ depends on the true value of the parameter through $w_1^{(0)}, \ldots, w_L^{(0)}$ only. Second, the simulated statistic may be written as a sum of terms

$$\left(\log(q_l) - \log(q_l - w_l^{(1)} - f)\right)^2.$$

These are the two key points which allow for a quick algorithm test. Before implementing such a procedure we shall make a remark.

Suppose we want to perform a test of $H_0 : \overline{\theta} = \theta_H$ at level 0.5.[6] The procedure then consists in rejecting the null whenever the simulated statistic is larger than its sample counterpart, $\Lambda^{(0)}$. Obviously choosing to reject H_0 when the simulated statistic is *larger* than $\Lambda^{(0)}$ is somewhat arbitrary when we deal with a 0.5 level test. We could indeed decide to reject the null at the same level when the simulated statistic is *smaller* than $\Lambda^{(0)}$. However some justifications may be given for this choice. Suppose we consider the null hypothesis

$$H_0 : \overline{\theta} = (\overline{\beta}', \overline{\sigma})' = (0, \beta_1, 0, 0, 0, 0, 0, 1),$$

where β_1 is very large (more precisely far from 1). The constraints:

$$q_l - f - c_{(1:n_l)} \geq w_l \geq q_l - f - t - c_{(2:n_l)}$$

are equivalent to:

$$c_{(2:n_l)} + t \geq q_l - f - w_l \geq c_{(1:n_l)}.$$

For β_1 sufficiently large, we have:

$$\frac{c_{(2:n_l)} + t}{q_l^{\beta_1}} \simeq \frac{c_{(1:n_l)}}{q_l^{\beta_1}} \simeq 1.$$

Thus $q_l - f - w_l \simeq q_l^{\beta_1}$ and the simulated statistic is very large.

If we want to avoid erroneously accepting hypotheses with arbitrarily large values of β_1, we have to reject the null when the simulated statistic is too large. Notice we choose the decision rule "reject when the simulated statistic is *large*" rather than "reject when the simulated statistic is *small*", because of the latter's lack of power. The comparison cannot rest on something else, since both decision rules have the same level. In other words, as they have the same behavior under the null, the decision rules must be compared under the alternative.

[6] Needless to say, the tests we will perform on our dataset have levels smaller than 0.5, see Section ??.

4.2.3 Implementation of the test

With our choice for the statistic, and according to the interpretation of the decision rule, the MMC test at level 0.5 requires solving the following problem:

$$\begin{aligned}
\min \quad & \lambda(w_1^{(1)}, \ldots, w_L^{(1)}, n_1, \ldots, n_L, \theta_H) \\
\text{w.r.t} \quad & w_1^{(1)}, \ldots, w_L^{(1)}, n_1, \ \ldots, n_L \\
\text{s.t} \quad & n_l \in \{2, \ldots, N_{\max}\}, \ \forall\, l \\
& q_l - f - c^{(1)}_{(1:n_l)} \geq w_l^{(1)} \geq q_l - f - t - c^{(1)}_{(2:n_l)}, \ \forall\, l \\
& \log(c^{(1)}_{i,n_l}) = h_{\beta_H}(q_l, z_l) + \sigma_H \tilde{\epsilon}^{(1)}_{i,n_l}, \forall\, i = 1, \ldots, n_l, \ \forall\, l \\
& w_l^{(1)} \bmod t \ = 0, \quad w_l^{(1)} \geq 5, \ \forall\, l.
\end{aligned}$$

Indeed, if there exist a value of the nuisance parameters such that $\lambda(w_1^{(1)}, \ldots, w_L^{(1)}, n_1, \ldots, n_L, \theta_H) \leq \lambda^{(0)}$, then we must accept the null hypothesis. This minimization problem is easily solved following steps 1 and 2 below.

`Step 1.` Assume for a moment that the number of players always equals two. In our context, the MMC test procedure requires two steps. First we compute a draw of the private values:

$$\log(c^{(1)}_{i,l}) = h_{\beta_H}(q_l, z_l) + \sigma_H \epsilon^{(1)}_{i,l}, \quad i = 1, 2 \quad l = 1, \ldots, L.$$

Second, the MMC test procedure requires the following minimization:

$$\begin{aligned}
\min \quad & \lambda(w_1^{(1)}, \ldots, w_L^{(1)}, 2, \ldots, 2, \theta_H) \\
\text{w.r.t} \quad & w_1^{(1)}, \ldots, w_L^{(1)} \\
\text{s.t} \quad & q_l - f - c^{(1)}_{(1:2)} \geq w_l^{(1)} \geq q_l - f - t - c^{(1)}_{(2:2)}, \ \forall\, l \\
& \log(c^{(1)}_{i,2}) = h_{\beta_H}(q_l, z_l) + \sigma_H \tilde{\epsilon}^{(1)}_{i,2}, \forall\, i = 1, 2, \ \forall\, l \\
& w_l^{(1)} \bmod t \ = 0, \quad w_l^{(1)} \geq 5, \ \forall\, l.
\end{aligned}$$

As already mentioned, the test statistic is separable so that the problem boils

down to L minimizations of the following type:

$$\begin{aligned}
\min \quad & \Big(\log(q_l) - \log(q_l - w_l^{(1)} - f)\Big)^2 \\
\text{w.r.t} \quad & w_l^{(1)} \\
\text{s.t} \quad & q_l - f - c_{(1:2)}^{(1)} \geq w_l^{(1)} \geq q_l - f - t - c_{(2:2)}^{(1)}, \\
& \log(c_{i,2}^{(1)}) = h_{\beta_H}(q_l, z_l) + \sigma_H \tilde{\epsilon}_{i,2}^{(1)}, \forall\, i = 1, 2, \\
& w_l^{(1)} \bmod t = 0, \quad w_l^{(1)} \geq 5.
\end{aligned}$$

Now notice that this problem is trivially solved since $(\log(q_l) - \log(q_l - x - f))^2$ is a convex function of x that reaches its minimum when $x = -f$. So this function is increasing for $x \geq 5$. Thus, the minimum of the simulated statistic is reached for the smallest admissible values of the simulated bids.

`Step 2.` Suppose now that the numbers of players are no longer constrained to be equal to two. The same arguments apply. Since $q_l - f - c_{(1:n_l)}^{(1)}$ and $q_l - f - t - c_{(2:n_l)}^{(1)}$, are decreasing functions of n_l we simply have to examine the smallest multiple of the tick larger than $q_l - f - c_{(1:2)}^{(1)}$. We compute $\Big(\log(q_l) - \log(q_l - w_l^{(1)} - f)\Big)^2$ for this value and the minimum of the simulated statistic is found.

To extend the test procedure to other levels, consider now $\mathcal{S}$ simulations of the test statistic under the null. Then the decision rule "reject H_0 whenever all of the simulated statistics are larger than $\Lambda^{(0)}$, and otherwise accept it" has level $1/(\mathcal{S}+1)$ (see Dufour [1998]).

4.2.4 Application to the constant average cost hypothesis and significance tests

Up to now, we are only able to test for given values of θ. For instance, we test whether we may accept $\overline{\theta} = (0, 1, 0, \ldots, 0, 0.03)'$. We reject this hypothesis at level 0.05.

A more interesting hypothesis is:

$$H_0 : \overline{\theta} = (0, 0.983, 0, 0.017, 0, 0, 0, 0.03)'.$$

It turns out that we cannot rejected it at level 0.05. Two conclusions may be drawn from this decision. First, under the null, we have $\overline{\beta}_1 + \overline{\beta}_3 = 1$, $\overline{\beta}_2 = \overline{\beta}_4$ and $\overline{\beta}_5 = -2\overline{\beta}_2$. Thus, the constant average cost hypothesis cannot be rejected (see equation (**??**)). This is not very surprising as the market we study mainly consists of mere dealers.

Second, from the acceptance of H_0, we may automatically conclude that no slope coefficient is significatively different from 0, except perhaps β_1 and β_3. Indeed, suppose we want to test whether $\overline{\beta}_6 = 0$ (say). According to the MMC procedure, for accepting $\overline{\beta}_6 = 0$, it is sufficient to prove that there exist values $\beta_0, \ldots, \beta_5$ and σ such that the hypothesis $\overline{\theta} = (\beta_0, \beta_1, \beta_2, \ldots, 0, \sigma)'$ is accepted.

4.2.5 MMC test for the mixture model and the robustness of MMC tests

We explained above how to perform MMC tests for our cluster model. What about the mixture model? We notice that the $d_{k,l}$ are no longer parameters but realizations of unobserved random variables $D_{k,l}$. However the mixture model involves the π's, *i.e.*, the parameters of the marginal distribution of the $D_{k,l}$'s (see (**??**)). According to the MMC technique the $D_{k,l}$'s should now be drawn in the marginal distribution for given values of the π's.

Suppose we focus on hypotheses involving the parameter θ only (and not the π's). The π's then act as nuisance parameters. According to the general MMC test procedure, we should try all possible values of the π's and simulate the $D_{k,l}$ for all of these choices. Notice however that once the $D_{k,l}$ are simulated, we can apply the same argument as in `step 1` above. Moreover recall the test statistic we developed for the cluster model involved a minimization of λ over the set of *all possible paths* of $d_{k,l}$. Thus, as far as the null hypothesis involves the parameter θ only, the test derived for the cluster specification has the correct level for the mixture model! It is remarkable that the finite sample test we propose is robust to *any* specification of the number of players.

Of course, when the $\pi's$ are present under the null, it is in general better to simulated the $D_{k,l}$ according to their marginal distribution under the null. Then, the test is performed with known $d_{k,l}$ as in the cluster specification. Considering all possible paths of $d_{k,l}$'s typically induce a loss of power when the π's are known.

More generally, suppose we want to "enlarge" the model for instance, to avoid a possible misspecification. Any null hypothesis which is accepted at a given level in the "small" nested model is automatically accepted at the same level in the "enlarged" one. Thus the argument according to which a given hypothesis is accepted because the model is misspecified is not well founded as long as we may enlarge the current model so that the misspecification disappears.

Along the same lines, suppose we suspect the omission of an effect (observed or not) which would appear in a larger model. We may never conclude that this effect is significative. For instance, we may introduce some individual correlation of the form:

$$\tilde{\epsilon}_{i,l} = \tilde{\zeta}_{i,l} + \rho\, \tilde{\xi}_l$$

to account for some unobserved common value feature. The hypothesis

$$H_0 : \overline{\theta} = (0, 0.983, 0, 0.017, 0, 0, 0, 0.03)' \text{ and } \overline{\rho} = 0$$

is accepted in this "larger" model. Thus we cannot reject the absence of unobserved common value feature.

5 Confidence regions

We now consider the problem of deriving confidence sets.

5.1 The problem and the general methodology

Suppose we first want to build a confidence interval for β_6 (say). It is well known that, if $\Theta_H(\alpha)$ is the set of all values θ_H such that $H_0 : \overline{\theta} = \theta_H$ cannot be rejected at level α, then $\Theta_H(\alpha)$ is a confidence region at level $1-\alpha$ for θ. Similarly, the set of all values of β_6 that cannot be rejected by our test procedure is a confidence region for β_6. It may well be that this region is not connected. The convex hull of this region is an interval. Its coverage probability cannot be less than the level $1-\alpha$. In practice, building a confidence interval at level $1-\alpha$ for β_6 amounts to finding the largest and the smallest values of β_6 that cannot be rejected according to our test procedure at level α.

Again, consider first a confidence interval at level 0.5. For a given draw of the error terms, the bounds of this interval are obtained by maximizing and minimizing β_6, under the constraint that the simulated test statistic computed at the point $(\beta_0, \beta_1, \beta_2, \ldots, \beta_6, \sigma)$ is smaller than $\lambda^{(0)}$. More precisely, we face the following optimisation problems

$$\begin{aligned}
\min \quad & \pm\ \beta_6 \\
\text{w.r.t} \quad & w_1^{(1)}, \ldots, w_L^{(1)}, n_1, \ \ldots, n_L, \theta \\
\text{s.t} \quad & \lambda(w_1^{(1)}, \ldots, w_L^{(1)}, n_1, \ldots, n_L, \theta) \leq \lambda^{(0)} \\
& q_l - f - c^{(1)}_{(1:n_l)} \geq w_l^{(1)} \geq q_l - f - t - c^{(1)}_{(2:n_l)}, \ \forall\, l \\
& \log(c^{(1)}_{i,n_l}) = h_\beta(q_l, z_l) + \sigma \tilde{\epsilon}^{(1)}_{i,n_l}, \forall\, i = 1, \ldots, n_l, \ \forall\, l \\
& w_l^{(1)} \bmod t = 0, \quad w_l^{(1)} \geq 5, \ \forall\, l.
\end{aligned}$$

Such a constrained maximization may be performed using the Constrained Optimisation package provided by Gauss. The following proposition is useful for generalizing

the approach to other levels.

Proposition 2 *Consider $\mathcal{S}$ independent simulations and define $R^{(s)}$ a 0.5-level confidence region obtained from simulation s, $s = 1, \ldots, \mathcal{S}$, and $R_{\mathcal{S}} = \bigcup_{s=1}^{\mathcal{S}} R^{(s)}$.*

(*i*) *$R_{\mathcal{S}}$ has level $\mathcal{S}/(\mathcal{S}+1)$.*

(*ii*) *$R_{\mathcal{S}}$ is empty with probability less than $1/(\mathcal{S}+1)$.*

(*iii*) *This probability is a decreasing function of $\mathcal{S}$.*

(*iv*) *For any pre-specified $0 < \tau < 1$ there exists almost surely finite $\mathcal{S}$ such that the probability of $R_{\mathcal{S}'}$ being empty is less than τ, $\forall \mathcal{S} > \mathcal{S}'$.*

Proof

(*i*) Consider $\mathcal{S} = 2$ simulated statistics. The decision rule "accept the null whenever *either* $\Lambda^{(1)} < \Lambda^{(0)}$ *or* $\Lambda^{(2)} < \Lambda^{(0)}$ and otherwise accept it" has level 1/3. The corresponding confidence region then has a coverage probability of 2/3 (at least). This confidence region can be seen as the *union* of the two confidence regions at level 0.5 derived respectively from simulation $s = 1$ and $s = 2$. This argument generalizes to any value of $\mathcal{S}$.

(*ii*) If $R_{\mathcal{S}}$ is empty, then it does not contain $\overline{\theta}$. Therefore from (*i*), the probability of $R_{\mathcal{S}}$ being empty is less than $1/(\mathcal{S}+1)$.

(*iii*) By construction $R_{\mathcal{S}} \subseteq R_{\mathcal{S}+1}$. Therefore if $R_{\mathcal{S}+1}$ is empty, then so is $R_{\mathcal{S}}$.

(*iv*) From (*iii*) by choosing $\mathcal{S}$ sufficiently large, one can make the probability of $R_{\mathcal{S}'}$ being empty arbitrarily small, $\forall \mathcal{S} > \mathcal{S}'$.

■

We use the result (*i*) to build the 0.9 level confidence intervals provided in the next section. The results (*ii*) to (*iv*) show that, provided that the level is large enough, the confidence intervals are not empty.

5.2 Results

The optimisation procedure takes approximately 20 minutes for a 0.5-level confidence interval (the program has been tested on a Pentium II PC 166 Mhz). So the construction of the eight 0.9 level confidence intervals takes approximately 42 hours. In Table **??**, we report the confidence intervals for each component of θ, obtained as the

union of nine 0.5-level confidence regions. We also provide the mid-interval estimators whose statistical properties will be discussed in the next section.[7]

Table 6 Exact 0.9-level confidence intervals

Parameters	Smallest	Largest	Mid-interval
β_0	−0.00481	0.02748	0.01134
β_1	0.98226	0.98784	0.98505
β_2	−0.00011	0.00084	0.00037
β_3	−0.00925	0.07178	0.03127
β_4	−0.00875	0.01904	0.00515
β_5	−0.00136	0.00628	0.00246
β_6	−0.07755	0.05662	−0.01047
σ	0.01877	0.04955	0.03416

The first striking result is that the exact confidence intervals are very narrow. In particular, they are much smaller than the asymptotic confidence regions derived by the BHHH and the Newton procedures. This result should be emphasized since exact techniques are often thought to be too conservative compared to their asymptotic counterparts. Recall how the MMC technique circumvents the difficulty raised by the nuisance parameters. However, this statement is difficult to assess as asymptotic confidence regions do not necessarily fulfill the level constraint in finite sample. But our results show that some asymptotic confidence regions may be even "too large". In this case, the MMC procedure may provide much more useful results.

These narrow intervals are due to the fact that the model is highly constrained. Consider a given 0.5-level confidence region. First, the simulated winning bids must be such that the simulated statistic is smaller than its sample counterpart. Second, for given values of the error terms and for a given path of the w_l's, the constraints induced by the absence of some dominated strategies are linear with respect to θ. Indeed we have:

$$q_l - f - c^{(1)}_{(1:n_l)} \geq w^{(1)}_l \geq q_l - f - t - c^{(1)}_{(2:n_l)}$$

[7]For our dataset, all of the nine 0.5 level confidence regions are non-empty. They are also close to the resulting 0.9 level confidence interval.

if and only if

$$\Big[h_\beta(q_l,z)+\sigma\epsilon^{(1)}_{(2:n_l)} \geq \log(q_l-f-t-w_l^{(1)}) \\ \text{and}\quad h_\beta(q_l,z)+\sigma\epsilon^{(1)}_{(1:n_l)} \leq \log(q_l-f-w_l^{(1)})\Big]$$

holds. There are L couples of such constraints and the number of parameters is only 8. So we optimize the parameters on a narrow domain.

It is the economic theory which implies such narrow bands. Notice indeed that our maintained hypothesis (namely that the players do not play some dominated strategies) is used differently in the asymptotic ML approach and in our finite sample approach. Indeed, the ML approach seeks for values of the parameters that maximize the *probability* of the events $q_l - f - C_{(1:N_l)} \geq w_l \geq q_l - f - t - C_{(2:N_l)}$ for all auction l. Clearly, the probability of this event cannot be 1 (if only because the variance of the error term is strictly positive). On the contrary, our finite sample approach *imposes* the constraint $q_l - f - c^{(s)}_{(1:n_l)} \geq w_l^{(s)} \geq q_l - f - t - c^{(s)}_{(2:n_l)}$ on each simulated auction. Of course, if the economic theory imposes constraints which are violated by the data, then our approach fails. Recall however that we impose some minimal assumptions on the rationality of the players. Thus, our emphasis on the robustness of the economic hypotheses and the econometric treatment we choose are linked.

The second conclusion one could draw is that the asymptotic point estimators are probably not very reliable (see Section **??**). Only two components of the MLE belong to the exact 0.9-level confidence intervals (namely β_3 and β_4). For the cluster model, three components belong to these intervals (β_3, β_4 and β_6) for the specification $N_{\max} = 5$ and four for the specification $N_{\max} = 99$ (namely σ and the same components as in the other specification). It should be stressed that the upper bound for σ is about twice as small as the MLE.

Finally, we note that $\overline{\beta}_1$ is significantly different from zero at level 0.1. Incidentally, this also shows that the model is globally significant (at the same level). We also remark that $\overline{\beta}_1$ is strictly smaller than one. This means that we cannot accept $\overline{\beta}_j = 0$, $j = 0, 2, \ldots, 6$ and $\overline{\beta}_1 = 1$. Recall that these are the values of the slopes defining the proxy model in the interpretation of the test statistic (**??**). From an economic viewpoint, $\overline{\beta}_1 < 1$ implies that the tarification of the travel agencies participating to the auction differs from q_l.

5.3 The Mid-Interval Estimator

The confidence intervals may also be used to compute an interesting point estimator. Clearly any point in the confidence set may be chosen as an estimator. We shall

consider the Mid-Interval Estimators (MIEs) defined as the mid-intervals points (see Table **??** again). We have the following result:

Proposition 3 *Let $R_{j,\mathcal{S}}$ be the confidence interval for β_j at level $\mathcal{S}/(\mathcal{S}+1)$, $\ell(R_{j,\mathcal{S}})$ the length of $R_{j,\mathcal{S}}$ and $\tilde{\beta}_{j,\mathcal{S}}$ its midpoint. Then we have:*

$$\mathbf{P}\left(\{|\overline{\beta}_j - \tilde{\beta}_{j,\mathcal{S}}| \leq \frac{\ell(R_{j,\mathcal{S}})}{2}\}\right) \geq \frac{\mathcal{S}}{\mathcal{S}+1},\ \forall \overline{\beta}_j.$$

Proof

By construction, $\tilde{\beta}_{j,\mathcal{S}} \in R_{j,\mathcal{S}}$ always. Therefore, the events $\{\overline{\beta}_j \in R_{j,\mathcal{S}}\} \cap \{\tilde{\beta}_{j,\mathcal{S}} \in R_{j,\mathcal{S}}\}$ and $\{\overline{\beta}_j \in R_{j,\mathcal{S}}\}$ have the same probability, bounded from below by $\mathcal{S}/(\mathcal{S}+1)$. ■

A similar property is typically difficult to establish for usual asymptotic estimators.

6 Simulations and cross validation experiments

In this section we investigate the reliability of the econometric techniques used in the paper. First, one may be concerned with the accuracy of the asymptotic approximations. Second, one may also question the performance of the finite sample approaches to estimation and inference.[8] We handle these problems by means of simulations and cross validation experiments.

6.1 The reliability of asymptotic procedures

As noted in Section **??**, the three estimators (BHHH, GN and EW) of the parameter standard errors provide very different results. The Eicker-White estimator is asymptotically robust to some misspecification. As a consequence, our results may be due to some misspecification of the latent model. However, in finite sample, this may also come from bad asymptotic approximations. To investigate this point, we choose to simulate and estimate a version of our auction model. In such an experiment, we fully control the specification of the model.

These simulations entail a large number of cumbersome computations. So we restrict the sample size to 200. The true value of the parameter θ is set to $(0, 0.98, 0, 0, 0, 0, 0, 0.06)'$ and we draw the numbers of participants according to the empirical distribution reported in Table 5 with $N_{\max} = 5$. For each of the 200 auctions, theses values, together with the actual quoted prices, provide a set of admissible winning bids. The simulated winning bid used for estimation is drawn uniformly in this set.

[8] These concerns roughly cover the comments raised by participants at various seminars.

Note that the model is compatible with the specification of the previous Sections. We then estimate the model under the constraint $\beta_2 = \beta_3 = \beta_4 = \beta_5 = \beta_6 = 0$. This experiment is repeated 50 times. Tables **??** and **??** provide some descriptive statistics on the estimation of the parameters.

Table 7 Estimation of the simulated model (cluster specification)

	β_0	β_1	σ
Mean	0.01353	0.98199	0.35732
Standard error	0.01545	0.00262	0.01026
Maximum	0.07961	0.99204	0.44536
q75	0.04651	0.98534	0.37839
Median	0.00819	0.98222	0.35791
q25	−0.00988	0.97703	0.33473
Minimum	−0.05301	0.97182	0.25019

Table 8 Estimation of the simulated model (mixture specification)

	β_0	β_1	σ
Mean	0.00775	0.98268	0.36550
Standard error	0.04510	0.00715	0.00845
Maximum	0.07785	0.99418	0.44141
q75	0.03281	0.98567	0.38392
Median	0.00952	0.98317	0.36703
q25	−0.01095	0.97898	0.34655
Minimum	−0.06659	0.97188	0.29610

From these Tables, it is clear that the estimation of β_0 and β_1 are close to the true values. However, the results for σ are not quite as good. This may come from the difficulty to capture the unobserved heteroskedasticity due to the numbers of players. To investigate this point, we simulate the model under the constraint that the number of players always equals 2. We then compute the constrained MLEs of this model. The results are presented in Table **??**.

Table 9 Estimation of the simulated model (constrained specification $n_l = 2,\ \forall l$)

	β_0	β_1	σ
Mean	0.02295	0.98127	0.33954
Standard error	0.03562	0.00527	0.02148
Maximum	0.11660	0.99318	0.41113
q75	0.04948	0.98542	0.36060
Median	0.01890	0.98214	0.33558
q25	−0.00060	0.97754	0.31775
Minimum	−0.05148	0.96662	0.28399

It now appears that the problem with σ is more serious. It may be due to the fact that the likelihood we use is written as a difference between the c.d.f. computed at two different values. This kind of likelihood does not penalize high values of σ as a density would.

We also compute for all these simulations the three different standard errors and the associated confidence regions. The results are reported in Table **??**. In order to make a comparison with our finite sample approach, these confidence regions are computed for an asymptotic level equal to 0.9.

The findings of these experiments are several. First, the discrepancies between the asymptotic standard errors estimations remain. Of course, these cannot be due to misspecifications since the model is well specified by construction. Notice that the differences are of the same magnitude as what we obtained from the real dataset. This is more probably due to poor first and second order approximations. Such findings are well documented. For instance Dagenais and Dufour [1991, 1992] and Dufour and Dagenais [1992] pointed out that usual asymptotic test procedures are not invariant with respect to the estimator of the information matrix. This may explain the discrepancies between BHHH- and GN-type estimations. As for the Eicker-White method, see Dagenais and Dufour [1992] and Dufour and Trognon [1997].

Second, although the confidence regions have an asymptotic coverage probability of 0.9, notice that the confidence intervals for σ *never* contain the true value when we use the EW standard error estimation. The EW-based confidence regions are likely to be far from valid. The two other techniques (namely GN and BHHH) provide very wide confidence regions. Of course they always contain the true value, but they are empirically "useless".

Constrained model									
Mean	989.38318	10.25107	0.10960	149.77056	1.57247	0.01701	518.86077	7.40478	0.10
Maximum	1330.75798	12.40572	0.14210	200.28885	1.90308	0.02229	360.71940	5.31563	0.14
q75	1068.09673	10.88544	0.11872	162.28024	1.66985	0.01843	288.65571	4.40646	0.11
Median	998.01511	10.13039	0.10759	151.34136	1.55392	0.01660	259.08108	3.96924	0.10
q25	903.58963	9.59859	0.10096	138.10373	1.47238	0.01541	222.79802	3.66460	0.09
Minimum	716.25032	8.65590	0.09052	106.95574	1.31600	0.01378	167.37449	3.11960	0.08
Estim. cov. proba.	**1.00000**	**1.00000**	**0.78000**	**1.00000**	**1.00000**	**0.84000**	**1.00000**	**1.00000**	**0.00**

Third, the estimated coverage probabilities for β_0 and β_1 turn out to be correct even for the EW estimations.

Finally, the problems pertain when the number of players is known. The only new finding is that the quality of the EW confidence region seems to deteriorate slightly (this may be due to the relatively small number of simulations, however).

In short, these results are in accordance with our belief that the failures of the asymptotic techniques is due to poor approximations and not to misspecifications issues. The above mentioned problems definitely remain even when the model is well-specified.

6.2 The performance of finite sample procedures

We now investigate the behavior of finite sample techniques using the very same series of checks. First we consider the MIE, then we turn to the study of the MMC-based confidence regions.

Table 11 Descriptive statistics of the MIE

	β_0	β_1	σ
Mean	0.75668	0.99921	0.17074
Standard error	0.02755	0.00002	0.00459
Maximum	0.79616	0.99925	0.18082
q75	0.76820	0.99923	0.17401
Median	0.76340	0.99921	0.17065
q25	0.75118	0.99920	0.16842
Minimum	0.58938	0.99918	0.16018

Compared to their asymptotic counterparts, the performances of the MIE are mixed. For the parameter σ, the MIE is closer to the true value and for the other parameters the asymptotic estimations perform better. This suggests that our MMC-based confidence regions are not centered around $\overline{\theta}$. This comes from the fact that the underlying test statistic is not symmetrical around the true value.

The confidence intervals undoubtly outperform the asymptotic ones in two crucial aspects. First, they are all valid. Second, they provide much narrower bounds than the BHHH and GN procedures. Notice that they shall not be compared to the EW procedure for the latter has a zero empirical coverage probability.

Table 12

	β_0	β_1	σ
Mean	1.63422	0.04898	0.34148
Maximum	1.88743	0.05135	0.36165
q75	1.64965	0.04951	0.34801
Median	1.63838	0.04906	0.34130
q25	1.61481	0.04846	0.33684
Minimum	1.54236	0.04657	0.32036
Estim. cov. proba.	**1.00000**	**1.00000**	**1.00000**

We also notice that because the MMC-based confidence regions involve a maximization and then a projection step they are conservative. This is clear from the reported coverage probabilities (see Table **??**).

6.3 Cross validation experiment

The Maximized Monte Carlo Tests approach we used definitely calls for a precise specification of the DGP. Indeed the model must be simulated. This is clearly impossible if the DGP is not completely specified. One may see this requirement as a weakness of the approach. Indeed, it is clear that if the model is misspecified the confidence intervals may not have the correct level. This raises the question of possible misspecifications of the model.

However, the misspecification tests proposed in the literature are based on asymptotic approximations. We already showed that the asymptotic approximations perform badly in our model. As we focus on finite sample evaluation, such misspecification tests are not consistent with our main approach. We propose to address the question by a cross-validation experiment.

We randomly draw 500 auctions from the entire sample. The out-of-sample experiment is formed by the remaining 118 auctions. Usually, the cross-validation exercise amounts to compare the forecast endogenous variable with their observed counterparts. The difficulty with our model is that we cannot forecast the endogenous observed variable (*i.e.* the winning bid) directly. Indeed, the cost may be forecast but, as we already stressed the link between the costs and the winning bid is not explicitly stated. The model does provide a set of admissible winning bids, however. We thus choose to report the errors in terms of distance from this forecast set to the realized

Table 13 Forecasts with MIE (average number of winning bids forcasted = 7.12)

	Structural	Naive
Exact	51.7%	86.4%
Error less than $5	72.0%	94.9%
Error less than $10	75.4%	96.6%
Error less than $15	82.2%	96.6%
Error less than $20	88.1%	98.3%
Error larger than $20	11.9%	1.7%

bids. We also compare our structural forecasting with a "naive" one. The naive forecasting simply derived from the sample distribution of the winning bid/quoted price ratio. We compute the average and the standard error for this series. The naive forecast set of admissible bids is then:

$$\left\{ t \times k | k \in \ , \ \hat{m} - \alpha \hat{s} \leq \frac{t \times k}{q_r} \leq \hat{m} + \alpha \hat{s} \right\}$$

where:

$$\hat{m} = \tfrac{1}{500} \sum_{r=1}^{500} \frac{w_r}{q_r}$$
$$\hat{s} = \left(\tfrac{1}{500} \sum_{r=1}^{500} (\hat{m} - \frac{w_r}{q_r})^2 \right)^{1/2} .$$

The quantity α in the naive forecast set is chosen so that the average out-of-sample number of forecast bids are the same for the structural and the naive approaches.

In the previous table the term 'exact' means that the observed bid belongs to the forecast sets of admissible winning bids. This happens for about half the out-of-sample auctions for the structural estimator and more than 80% for the naive forecasting procedure. Clearly, the latter dominates the former. However, a closer look at the results reveals that the forecast errors are roughly balanced between over- and under-estimation of the winning bid. This motivates our second contest in which the estimator for σ has been set to a larger value in a 0.9 level confidence region.[9]

The results are now much closer. Of course, the uncertainty of the forecast is larger (the average number of possible winning bids is now 11 instead of 7). This

[9]Of course, this confidence region has been recomputed from the 500 auctions.

Table 14

	Structural	Naive
Exact	96.6%	95.7%
Error less than \$5	98.3%	96.6%
Error less than \$10	98.3%	96.6%
Error less than \$15	98.3%	98.3%
Error less than \$20	98.3%	99.1%
Error larger than \$20	1.7%	0.8%

result shows that the performance of the model in terms of out-of-sample prediction is reasonably good. However, the MIE for σ seems to be a bit optimistic.

7 Interpretation of the discrete auction model

We are now able to study the economic implications of the model. As we already said, the theoretical results concerning discrete mechanisms are relatively few compared to the large body of literature devoted to continuous auctions schemes. Some "naive" questions may thus be raised. Does the current mechanism select the highest signal? Is the competition tight? How is surplus shared? We answer these questions using our model and the estimations already discussed.

In the following, the $\check{}$ notation refers to the ML estimation of the mixture model, the $\hat{}$ to the estimation of the cluster model with $N_{\max} = 5$ and $\tilde{}$ to the MIE.

7.1 The degree of competition

For each point estimator, one may check the occurence of "maximum degree of competition". Recall that we identified this event as

$$\{b_l \leq C_{(1:N_l)} \leq a_l\} \cap \{C_{(2:N_l)} \geq b_l\}.$$

As the costs are not observed, we may only compare the probabilities of the two events appearing in the proof of Proposition 1.

More formally, for the estimation of the cluster model, we forecast that the maximum degree of competition has been reached in auction l whenever:

$$\hat{\mathbf{P}}_l^1 = \hat{\mathbf{P}}\big(\{C_{(1:N_l)} < b_l\} \cap \{C_{(2:N_l)} > b_l\}\big) < \hat{\mathbf{P}}_l^2 = \hat{\mathbf{P}}\big(\{b_l < C_{(1:N_l)} < a_l\} \cap \{C_{(2:N_l)} > b_l\}\big).$$

For the cluster model this occurs only once in the data set for a one way flight from Detroit to Pittsburgh. The quoted price is \$332 and the winning bid is \$65. This auction also corresponds to the second largest winning bid/quoted price ratio. The largest ratio is achieved for two tickets "San Fransisco to Rome, *etc*" (*sic*) for which the quoted price is \$3358 and the winning bid \$665. For this last ticket the two probabilities are also very close to each other. On the whole dataset, $\hat{\mathbf{P}}_l^1$ is on average 13 times larger than $\hat{\mathbf{P}}_l^2$. Moreover $\hat{\mathbf{P}}_l^1$ is always at least twice as large as $\hat{\mathbf{P}}_l^2$ except for 5 flights only (these flights achieve respectively the 2nd, 4th, 5th, 7th and 10th largest values of the winning bid/quoted price ratio).

Interestingly, the likelihood of the "maximum degree of competition" does not seem to be related to the number of *submitted* bids. More precisely, the correlation between the number of submitted bids and $\hat{\mathbf{P}}_l^2$ is only -0.02. Thus a large number of submitted bids shall not be interpreted as a high degree of competition. Not surprisingly however, the number of forecast players $\hat{n}_l$ is correlated with $\hat{\mathbf{P}}_l^2$: this correlation is 0.45.

The same conclusions arise under the specification $N_{\max} = 99$. In particular $\hat{\mathbf{P}}_l^{\mathbf{2}}$ is larger than $\hat{\mathbf{P}}_l^{\mathbf{1}}$ for 6 flights only, which are exactly the 5 flights mentioned above, and the flight achieving the largest winning bid/quoted price ratio.

This analysis may also be performed on the mixture model. One should first take into account that the number of players is now a random variable. The definitions for $\check{\mathbf{P}}_l^{\mathbf{1}}$ and $\check{\mathbf{P}}_l^{\mathbf{2}}$ are:

$$\check{\mathbf{P}}_l^1 = \check{\mathbf{P}}\big(\{C_{(1:N_l)} < b_l\} \cap \{C_{(2:N_l)} > b_l\}\big),$$
$$\check{\mathbf{P}}_l^2 = \check{\mathbf{P}}\big(\{b_l < C_{(1:N_l)} < a_l\} \cap \{C_{(2:N_l)} > b_l\}\big).$$

The event $\check{\mathbf{P}}_l^1 < \check{\mathbf{P}}_l^2$ is much more frequent in the mixture specification. It arises 69 times in the sample. The results are less clearcut than for the cluster specification. In particular, the occurence of maximum degree of competition does not seem to be clearly related to the winning bid/quoted price ratio. This result may be due a larger estimate for σ in the mixture specification than in the cluster model. For any given value of n_l, a larger value of σ induces larger probabilities $\check{\mathbf{P}}\big(\{b_l < C_{(1:n_l)} < a_l\} \cap \{C_{(2:n_l)} > b_l\}\big)$, other things being equal. The average contribution of the $\check{\mathbf{P}}_l^{\mathbf{2}}$'s to the likelihood of observation l equals 0.25 in the mixture specification and 0.11 in the cluster specification (recall that the log-likelihood of a single observation is $\log(\check{\mathbf{P}}_l^1 + \check{\mathbf{P}}_l^2)$.)

For the MIE, the problem is to find a reasonable estimate for the number of players. To increase the probability of the event "maximum degree of competition" one could then choose $n_l = 5$ for all auction. It turns out that even with this choice

the maximum degree of competition is never forecast by our finite sample approach. The reason is that $\tilde{\sigma}$ is much smaller than its asymptotic counterparts. Thus this event $\{b_l < C_{(1:n_l)} < a_l\}$ is less frequent. This also explain why the occurence of maximum degree of competition is more frequently forecast in the mixture model.

The answer to the question "could the auction have gone higher?" crucially depends on the value $\overline{\sigma}$. Unfortunately, this parameter seems difficult to estimate (see the previous Section). In any case, the event "maximum of competition" does not seem to be frequent. More probably, this type of auction leaves a reasonable surplus to the bidder with the smallest cost.

7.2 The rates of profit and comparison with the Vickrey mechanism

The previous statement is relevant only if the mechanism is able to select the best travel agency. But this is not automatically the case. To investigate this point further, we also examine the rates of profit. Suppose the player with the ith lowest cost in auction l wins the auction by offering the discount w_l. Then its rate of profit may be forecast as:

$$1 - \frac{E[C_{(i:N_l)}]}{q_l - w_l - f}, \tag{16}$$

where the expectation is taken conditionally on the events described in (**??**) and (**??**).

In Tables **??** to **??** we reproduce some descriptive statistics on the simulated profit rates, depending on N_l the number of players, and on i the rank of the winner of the auction. These statistics have been obtained by numerical approximation of the expectation in (**??**), and assuming all the auctions have the same number of participants: $N_l = N \in \{2, 3, 4, 5\}$, $\forall l$.

These Tables show that the auctions are most probably won by the travel agent with the least cost. Indeed, suppose there are two players, then the forecast rate of profit — had the second best player won the auction — is always negative (indeed its maximum is -0.0513, -0.0300, -0.0200, according to both of the MLEs and the MIE, respectively).

For the player with the second lowest cost (hereafter second best player) to win the auction and make a positive profit, at least 5 players are necessary, according to the cluster specification (see Table **??**). This is obviously not sufficient, since the profits of the second best player are negative for more than 95% of the auctions. Moreover, even with 5 players, the two other estimators (MLE and MIE) still always predict a negative profit rate for the second best player.

41mm4	1	0.0734	0.0061	0.0937	0.0831	0.0776	0.0738	0.0699	
	2	−0.0225	0.0059	**−0.0082**	−0.0132	−0.0182	−0.0225	−0.0264	−
	3	−0.0613	0.0081	−0.0418	−0.0491	−0.0565	−0.0605	−0.0659	−
	4	−0.1223	0.0111	−0.0971	−0.1073	−0.1151	−0.1203	−0.1278	−
51mm5	1	0.0734	0.0067	0.0912	0.0845	0.0777	0.0736	0.0689	
	2	−0.0155	0.0057	**−0.0002**	−0.0065	−0.0114	−0.0160	−0.0192	−
	3	−0.0437	0.0073	−0.0264	−0.0330	−0.0387	−0.0432	−0.0479	−
	4	−0.0791	0.0091	−0.0583	−0.0653	−0.0734	−0.0780	−0.0837	−
	5	−0.1368	0.0119	−0.1107	−0.1200	−0.1287	−0.1351	−0.1436	−

41mm4	1	0.0524	0.0057	0.0696	0.0613	0.0562	0.0526	0.0494
	2	−0.0139	0.0059	**−0.0006**	−0.0050	−0.0098	−0.0138	−0.0176
	3	−0.0414	0.0080	−0.0232	−0.0299	−0.0367	−0.0404	−0.0453
	4	−0.0833	0.0107	−0.0605	−0.0696	−0.0768	−0.0812	−0.0878
51mm5	1	0.0527	0.0061	0.0697	0.0623	0.0565	0.0530	0.0489
	2	−0.0088	0.0057	**0.0054**	−0.0003	−0.0049	−0.0092	−0.0125
	3	−0.0289	0.0073	−0.0127	−0.0185	−0.0241	−0.0283	−0.0328
	4	−0.0537	0.0090	−0.0361	−0.0410	−0.0481	−0.0522	−0.0575
	5	−0.0931	0.0113	−0.0703	−0.0783	−0.0860	−0.0910	−0.0980

41mm4	1	0.0173	0.0051	0.0366	0.0278	0.0201	0.0164	0.0136
	2	−0.0278	0.0170	**−0.0026**	−0.0068	−0.0145	−0.0238	−0.0384
	3	−0.0519	0.0214	−0.0167	−0.0218	−0.0355	−0.0478	−0.0661
	4	−0.0802	0.0241	−0.0345	−0.0443	−0.0626	−0.0769	−0.0964
51mm5	1	0.0174	0.0052	0.0347	0.0274	0.0204	0.0166	0.0136
	2	−0.0229	0.0161	**−0.0000**	−0.0038	−0.0104	−0.0183	−0.0326
	3	−0.0424	0.0202	−0.0109	−0.0153	−0.0269	−0.0384	−0.0562
	4	−0.0612	0.0227	−0.0238	−0.0283	−0.0443	−0.0578	−0.0771
	5	−0.0865	0.0247	−0.0415	−0.0488	−0.0680	−0.0833	−0.1035

These results may be compared to the usual Vickrey auction mechanism where the tick is zero. In this case, the profit of the second best player is exactly zero (Vickrey [1961]). From the viewpoint of the traveler, a discrete increment auction mechanism thus seems advantageous (see Figure **??**.a). But this cannot be the case when the number of players is very large. Indeed, in this case, $C_{(1:N_l)}$ is likely to be very close to $C_{(2:N_l)}$. So the probability that there exists a multiple of t lying between $q - C_{(2:N_l)} - f$ and $q - C_{(1:N_l)} - f$ is small. The winning bid when the tick is zero is $q - C_{(2:N_l)} - f$. But in a discrete increment mechanism with a large number of players, it is very unlikely that the first best player could bid higher than $q - C_{(2:N_l)} - f$ (see Figure **??**.b).

Figures **??**.a and **??**.b display the typical result obtained from our data for a small number of players and the general result one should have for a large number of players. In the first case, the traveler is better off with a strictly positive tick. When the number of players is large, the traveler is better off with a continuous mechanism. So if the tick is chosen by the traveler (the principal of the game), he should choose a Vickrey auction, *i.e.*, $t = 0$, only when the number of players is sufficiently large. Recall however that the MLE estimate of π_2 is close to 1. This suggest that the number of bidders may actually be small.

This empirical findings contrast with the result obtained by Chwe [1989] for uniformly distributed private signals. Note that the differences between Chwe's approach and our are many. First, Chwe [1989] deals with a particular Nash equilibrium whereas we rely on some dominated strategies. Second, the distribution we specified for the private signal is not uniform.[10] Finally, our result is empirical, not theoretical. As the number of players seems small, we forecast that the future traveler is better off with a discrete mechanism. This is by no means a theoretical proof. Note however that discrete mechanisms are very common and that the auction mechanism is often chosen by the principal. These facts imply that some discrete mechanisms should make the principal better off.

It is also interesting to compare the magnitude of the rate of profit with the winning bid/quoted price ratio. Indeed the traveler has booked the ticket and the maximum amount he is willing to pay is q_l. The quoted price q_l may thus be viewed as a surplus which will be shared by the traveler, the travel agent and Travel Bids. The rate of profit may be viewed as the travel agency's share. Travel Bids's share is

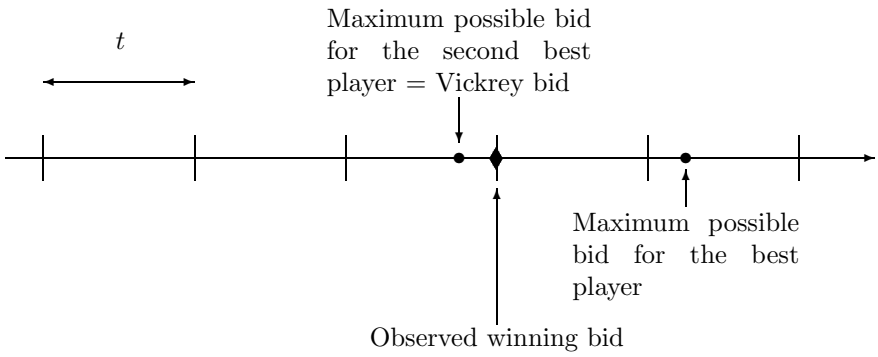
$$\frac{f+5}{q_l}$$

On average this share is 0.0177. Using Table **??** and including the fee, we find that

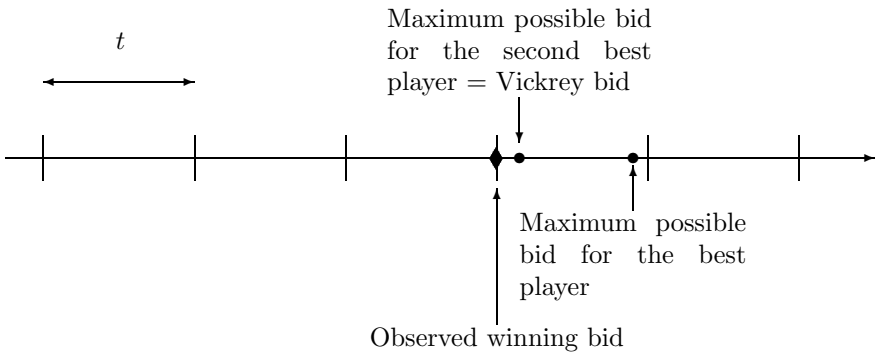
[10] Note we could straightforwardly handle the case of uniformly distributed costs.

Figure 1: Typical situations for the number of players.

a. *Typical situation with a small number of players.*



b. *Typical situation with a large number of players.*



the traveler's share is, on average 0.0718.

The profit rates derived from the asymptotic MLE are comparable to that of the traveler's share. On the contrary, our MIE makes the travel agency's share a bit less than that of Travel Bids, whereas the cluster estimate lies somewhere in between. In any case, the competition looks tighter when we use finite sample estimation techniques.

8 Conclusion

This paper provides a structural estimation of a progressive auction mechanism with discrete increments. The estimation approach rests on the consideration of some dominated strategies. This implies that the results remain valid for many kinds of equilibrium concepts.

Finite sample test procedures are given as well as valid conservative confidence regions. These procedures build on previous work by Dufour [1998]. They are applied to auctions on air travel tickets data. Point estimation is also examined.

From an economic viewpoint, we find that although the profit of the travel agents may be small, competition rarely reaches its maximum degree according to our estimates. We also compare the rent kept by the winner of a discrete increment auction with that of a continuous mechanism. Our analysis shows that the number of players plays an important role. If the number of players is small, then the winning discount is larger when the tick is zero. In this case, the Vickrey auction is dominated (from the viewpoint of a risk-neutral principal) by the discrete increment mechanism. This may explain why we observe substantial ticks in many actual auction mechanisms.

On econometric grounds, we find that some usual asymptotic confidence regions are dominated by a finite sample approach. Moreover, we show that dominated strategies may be helpfull to provide useful estimation of game-theoretic models.

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