Convergence, Consensus, and Dissensus in the Weighted-Median Opinion Dynamics

Wenjun Mei, *Member, IEEE*, Julien M. Hendrickx, *Member, IEEE*, Ge Chen, *Member, IEEE*, Francesco Bullo, *Fellow, IEEE*, and Florian Dörfler, *Member, IEEE*

***Abstract*—Mechanistic and tractable mathematical models play a key role in understanding how social influence shapes public opinions. Recently, a weighted-median mechanism has been proposed as a new microfoundation of opinion dynamics and validated via experimental data. Numerical studies indicate that this new mechanism recreates some nontrivial real-world features of opinion evolution. In this article, we conduct a thorough analysis of the weighted-median opinion dynamics. We fully characterize the equilibria set, and establish the almost-sure convergence for any initial condition. Moreover, we prove a necessary and sufficient condition for the almost-sure convergence to consensus, as well as a sufficient condition for almost-sure dissensus. We related the rich dynamical behavior of the weighted-median opinion dynamics to two delicate network structures: the cohesive sets and the decisive links. To complement our sufficient conditions for almost-sure dissensus, we further prove that, given the influence network, determining whether the system almost surely achieves persistent dissensus is NP-hard, which reflects that the complexity of the network topology contributes to opinion evolution.**

***Index Terms*—Consensus, opinion dynamics, social networks, weighted median.**

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I. Introduction

A. Background and Motivation

OPINION dynamics study how individuals exchange their opinions via interpersonal influence as well as the role of social network structures in shaping public opinions. Due to the complexity of social influence, a key to understanding opinion dynamics relies on the construction of mechanistic and tractable mathematical models. A large class of opinion dynamics models have been proposed based on the classic French–DeGroot model [1], [2], which assumes that individuals update their opinions by taking some weighted averages of their social neighbors' opinions. However, as pointed out in [3], the widely adopted weighted-averaging mechanism features a nonnegligibly unrealistic implication. That is, the attractiveness between any two individuals' opinions is proportional to their opinion distance, which leads to consensus under mild network connectivity assumptions.

To resolve the aforementioned unrealistic implication, in [3], the authors derive a new microfoundation of opinion dynamics: the weighted-median mechanism. That is, each individual updates their opinion by taking a weighted median, instead of a weighted average, of the current opinions held by their social neighbors. This new mechanism is derived from the psychological theory of cognitive dissonance and exhibits various desirable features: Empirical validation via a set of online experiment data indicates that, compared with the weighted-averaging mechanism, the new mechanism enjoys significantly lower errors in predicting individuals' opinion shifts driven by social influence. Numerical comparisons show that the weighted-median mechanism recreates some nontrivial features of real-world opinion evolution, which some widely studied extensions of the French–DeGroot model do not fully capture. Those features include the decaying likelihood of reaching consensus with increasing group size or clustering coefficient; the emergence of various empirically observed public opinion distributions; and the pattern of how extreme opinions are located in social networks.

Numerical studies indicate that opinion evolution via the weighted-median mechanism exhibits rich patterns dependent on some delicate network structures [3]. However, its dynamical behavior remains to be rigorously analyzed. In terms of dynamical behavior, researchers on opinion dynamics primarily focus on the equilibrium set, the convergence of the systems, and

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graph-theoretic conditions for reaching consensus and persistent dissensus (the equilibrium at which not all the individuals hold the same opinion), respectively. Typically, these properties are studied in the framework of consensus algorithms by leveraging certain network connectivity conditions. Nevertheless, this framework does not apply to the analysis of the weighted-median opinion dynamics since more sophisticated network-structure properties rather than connectivity are involved.

In this article, we identify two important network structures that shape the asymptotic behavior of the weighted-median opinion dynamics: *cohesive sets* and *decisive links*. By establishing some important properties of these two structures, we conduct a thorough theoretical analysis of the weighted-median opinion dynamics, including fully characterizing the equilibrium set and establishing its almost-sure convergence in finite time. A necessary and sufficient graph-theoretic condition for asymptotic consensus is provided. Last but not least, by building connections between the weighted-median model and the monotone non-all-equal 3 satisfiability (NAE3SAT) problem, we prove that determining whether weighted-median opinion dynamics almost surely reach persistent dissensus is an NP-hard problem.

B. Literature Review

The French–DeGroot model [1], [2] is one of the earliest models of opinion dynamics. It assumes that individuals' opinions are denoted by real numbers and are updated by taking some weighted averages of their social neighbors' opinions. The interpersonal weights constitute an influence matrix, which in turn induces a directed and weighted graph called the influence network. The weighted-averaging mechanism implies overly large attractions between distant opinions, which drive the system to consensus under mild conditions. Essentially, the French–DeGroot model is a linear consensus algorithm. Spectral analysis of the interpersonal influence matrix indicates that the French–DeGroot model achieves asymptotic opinion consensus as long as the associated influence network has a globally reachable and aperiodic strongly connected component [4]. This is an overly simplified prediction since many real-world social systems, with their influence networks being connected, do not always achieve consensus.

To explain the prevalence of persistent dissensus, various important extensions have been proposed by introducing additional mechanisms. To name a few widely studied representatives, the Friedkin–Johnsen (F–J) model [5] assumes that individuals have persistent attachments to their initial opinions. The F–J model is a discrete-time linear system. Its convergence is established via matrix spectral analysis. In this model, a group reaches persistent dissensus if and only if at least two individuals start with distinct opinions.

The Altafini model [6] assumes the presence of negative weights in the influence network. This model is still a discrete-time linear system but the presence of negative weights adds some difficulty to spectral analysis. One way of analyzing such a system is to introduce an auxiliary "lifted graph" [7], which transforms the original Altafini model into a higher dimensional linear system without negative coefficients. Spectral analysis of this augmented system indicates that the Altafini model achieves bipartite consensus if and only if the influence network is strongly connected and structurally balanced. For any strongly connected but structurally unbalanced network, all the individuals' opinions converge to the trivial state 0.

The bounded-confidence models, including the synchronous Hegselmann–Krause (H–K) model [8] and the gossip-like Deffuant–Weisbuch (D–W) model [9], assume that individuals only assign weights to opinions within certain distances from their own opinions. Such discontinuous truncation of social influence makes bounded-confidence models very challenging to analyze. In the case of homogeneous individuals, convergence and convergence rate are thoroughly studied in the framework of linear consensus algorithms with time-varying topologies [10], [11], [12]. The behavior of the H–K model with heterogeneous individuals in general "remain a mystery" [13]. Only the convergence under some specific conditions or with noises has been established [14], [15], [16], [17]. For the heterogeneous D–W model, Chen et al. [18] establish its almost-sure exponential convergence for certain range of model parameters by leveraging the technique of "transforming randomness into control inputs" [19], which is also used in the analysis of the weighted-median opinion dynamics in this article.

The biased-assimilation model [20] introduces a highly nonlinear modification of the French–DeGroot model to characterize the effect that individuals weigh confirming information more than disconfirming information. Conditions for the convergence to polarization, persistent dissensus, or consensus are analyzed in [20] for specific types of networks. Some local stability and attractivity properties are established in [21].

Bizyaeva et al. [22] propose a continuous-time multioption nonlinear generalization of classical linear weighted-average opinion dynamics. By introducing tunable sensitivity, this model manages to capture bifurcation and clustering behavior in opinion formation processes. We refer to [23] and [24] for an insightful survey of recent progress in the modeling and analysis of opinion dynamics.

A prominent difference between the aforementioned averaging-based models and the weighted-median model studied in this article is that the former imply pairwise independent interactions between each individual and their neighbors. On the other hand, the weighted-median mechanism requires the ordering of all the opinions of one's neighbors. Namely, neighbors' influences are not independent. In [25], a similar idea of neighbor-dependent interactions is incorporated into a DeGroot-like model and an *observer effect*, i.e., bounded expressed opinions, is also studied.

Outside the context of opinion dynamics, median-based mechanisms have been used in resilient consensus protocols, e.g., [26] and [27], or distributed optimization algorithms, e.g., [28]. The setups and motivations of these models are quite different from the model studied in this article. In [26], the agents update their states according to a mechanism that mixes averaging with median. In [27], the nodal states are vectors following local linear dynamics. Medians of neighbors' encoded states are used as control inputs. These two models focus on the resilience of median-based consensus protocols to attacks from

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malicious agents. The model proposed in [28] is constructed on unweighted graphs and a median-based mechanism is used as the information aggregation rule, mixed with a gradient-based iteration. None of the aforementioned literature analyzes the scenarios where each node updates its state by merely taking a weighted-median of their neighbors' states, which is thoroughly studied in this article.

C. Contribution

As indicated by simulations in [3], the weighted-median opinion dynamics often lead to opinion clustering and exhibit rich dynamical behavior regarding the conditions for reaching consensus or dissensus. In this article, we conduct a thorough analysis of the weighted-median opinion dynamics [3] and identify two important network structures that shape the opinion evolution. The contributions of this article include the following aspects.

First, we fully characterize the set of all the equilibria. It turns out that the equilibria of the weighted-median opinion dynamics exhibit a clear pattern that depends on an important delicate structure of the influence network: the cohesive sets. The notion of cohesive set is first proposed by Morris [29] and is widely adopted in the analysis of linear threshold models of network diffusion, e.g., see [30]. In this article, we adopt a special case in [29] as the definition of cohesive sets, and provide some useful properties of it.

Second, we establish the almost-sure finite-time convergence of the weighted-median opinion dynamics, with respect to initial conditions and individual update sequences. Then, we provide a necessary and sufficient graph-theoretic condition for the almost-sure convergence to consensus, and give a sufficient graph-theoretic condition for the convergence to almost-sure persistent dissensus. Moreover, different from the H–K and D–W models, which explain dissensus by having the population becoming disconnected, our weighted-median model proposes an alternative reasons where dissensus happens even if the network remains connected. Note that almost-sure persistent dissensus is not automatic in the absence of almost-sure consensus, as there is a middle ground where different outcomes have positive probabilities. Theoretical analysis results indicate that, under various network structure conditions, the weighted-median model either almost surely converges to consensus, or almost surely converges to persistent dissensus, or has a nonzero probability of reaching dissensus, depending on the initial conditions. In addition, network connectivity conditions are not sufficient to guarantee consensus. Therefore, the weighted-median opinion dynamics are less likely to reach consensus and exhibit richer dynamical behavior than the French–DeGroot model.

Third, we show that, given an influence network, determining whether the weighted-median opinion dynamics almost surely achieve persistent dissensus is equivalent to determining that for a finite set of initial conditions. We then further prove that the latter problem is NP-hard by relating it to the monotone NAE3SAT problem, which is known to be NP-hard. This result reflects the complexity that network topology contributes to the dynamical behavior of the weighted-median model, which highlights another important difference with averaging-based consensus problems and shows an arguably more benign behavior.

D. Organization

The rest of this article is organized as follows. Section II introduces some basic definitions and notions, as well as the model setup of the weighted-median opinion dynamics. Section III presents all the theoretical analysis and proofs of the main results. Finally, Section IV concludes this article. Proofs of lemmas are provided in the appendixes.

II. Basic Definitions and Model Setup

Let $\subseteq$ and $\subset$ be the symbols for subset and proper subset, respectively. Denote by $\mathbb{N}$ the set of natural numbers, i.e., $\mathbb{N}=\{0,1,2,\dots\}$. Let $\mathbb{Z}$ (resp., $\mathbb{Z}_+$) be the set of (resp., positive) integers. Let $\mathbb{1}_n$ and $\mathbb{0}_n$ be the n-dimensional vector whose entries are all ones and all zeros, respectively.

Consider a group of n individuals, whose interpersonal influences are characterized by an *influence matrix* $W=(w_{ij})_{n\times n}$. Denote by $\mathcal{G}(W)$ the directed and weighted graph associated with the adjacency matrix W, referred to as an *influence network*. In this article, we use the terms "graph" and "network" interchangeably. Let $\mathcal{V}=\{1,\dots,n\}$ be the index set of the nodes, and $w_{ij}>0$ means that there is a link from node i to node j with weight w_{ij}, or, equivalently, the individual i assigns $w_{ij}>0$ weight to individual j. Denote by $\mathcal{N}_i$ the set of node i's out-neighbors, i.e., $\mathcal{N}_i=\{j\in\mathcal{V}|w_{ij}\neq 0\}$, which includes node i itself if $w_{ii}\neq 0$. Conventionally, an influence matrix W is assumed to be row stochastic.

The weighted-median opinion dynamics is proposed to resolve a nonnegligibly unrealistic implication of the widely adopted weighted-averaging mechanism. Namely, opinion attractiveness increases with opinion distance. This implication is regardless of whether the opinion updates are synchronous or asynchronous. Here, we briefly review the argument articulated in [3, Sec. B.1]: Given the influence network $\mathcal{G}(W)$ and the individuals' opinions $x(t)$ at time t, the opinion update of an individual i at time $t+1$ can be interpreted as the best response that minimizes i's cognitive dissonance caused by disagreeing with others, characterized as

$$x(t+1)\in \operatorname{argmin}_{z\in\mathbb{R}} \sum_{j=1}^{n} w_{ij}|z-x_j(t)|^{\alpha}.$$

Here, $\alpha>1$ implies that moving toward distant opinions reduces cognitive dissonance more efficiently, i.e., distant opinions are more attractive, e.g., $\alpha=2$ for the classic DeGroot model. On the other hand, $\alpha<1$ implies that nearby opinions are more attractive. What we would like to propose is a mechanism without any intrinsic assumption on how opinion attractiveness depends on opinion distance. Therefore, we set $\alpha=1$ and obtained the weighted-median opinion dynamics, formulated as follows.

Definition 1 (Weighted median [3]): Given any n-tuple of real values $x=(x_1,\dots,x_n)$ and any n-tuple of nonnegative weights $w=(w_1,\dots,w_n)$ with $\sum_{i=1}^n w_i=1$, $x^*\in\{x_1,\dots,x_n\}$ is a

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weighted median of x associated with the weights w if x^* satisfies

$$\sum_{i:\, x_i<x^*} w_i \le 1/2, \quad \text{and} \quad \sum_{i:\, x_i>x^*} w_i \le 1/2.$$

For simplicity, we also say that x^* is a weighted median of x associated with w.

The following lemma rephrases [3, Appendix A]. It presents some immediate results regarding the uniqueness of weighted median.

Lemma 1 (Properties of weighted median): Given any n-tuple of real values $x = (x_1, \dots, x_n)$ and any n-tuple of the associated nonnegative weights $w = (w_1, \dots, w_n)$ with $\sum_{i=1}^n w_i = 1$, let $x_{(1)}, x_{(2)}, \dots, x_{(n)}$ be a reordering of $x_1, \dots, x_n$ such that $x_{(1)} \le x_{(2)} \le \cdots \le x_{(n)}$. The weighted median of x associated with w is unique if and only if there exists $1 < i^* < n$ such that

$$\sum_{i=1}^{i^*-1} w_{(i)} < \frac{1}{2}, \quad w_{(i^*)} > 0, \quad \sum_{i=i^*+1}^{n} w_{(i)} < \frac{1}{2}.$$

In this case, $x_{(i^*)}$ is the unique weighted median of x associated with w. When such i^* does not exists, there exist $1 < \underline{i} < \bar{i} < n$ such that

$$\sum_{i=1}^{\underline{i}-1} w_{(i)} < \frac{1}{2}, \ \sum_{i=1}^{\underline{i}} w_{(i)} = \sum_{i=1}^{\bar{i}} w_{(i)} = \frac{1}{2}, \ \sum_{i=\bar{i}+1}^{n} w_{(i)} < \frac{1}{2}$$

which also implies that $w_{(\underline{i}+1)} = \cdots = w_{(\bar{i}-1)} = 0$. In this case, $x_{\underline{i}}, x_{\underline{i}+1}, \dots, x_{\bar{i}}$ are all weighted medians of x associated with w.

The weighted-median opinion dynamics have been proposed in [3] as a discrete-time stochastic process, in which one individual updates their opinion at each time step.

Definition 2 (Weighted-median opinion dynamics [3]): Consider a group of n individuals in an influence network associated with a row-stochastic influence matrix W. For any $i \in \mathcal{V}$ and any $t \in \mathbb{N}$, denote by $x_i(t)$ individual i's opinion at time t. The weighted-median opinion dynamics is defined as the following stochastic process: At each time $t+1$, one individual i is uniformly randomly picked and updates their opinion according to the following equation:

$$x_i(t+1) = \text{Med}_i\left(x(t); W\right)$$

where $\text{Med}_i(x(t); W)$ is the weighted median of $x(t)$ associated with the weights given by the ith row of W, i.e., $(w_{i1}, w_{i2}, \dots, w_{in})$. If the weighted-median is not unique, then let $\text{Med}_i(x(t); W)$ be the weighted median that is the closest to $x_i(t)$.

Remark 1: The model setup in Definition 2 guarantees that $\text{Med}_i(x(t); W)$ is always unique for any i and t; see [3] for a detailed discussion.

Remark 2: As pointed out in [3], the weighted-averaging mechanism, as the microfoundation of a large class of opinion dynamics models, features an unrealistic implication that the attractiveness between two opinions is proportional to their distance. The weighted-median mechanism and the D–W bounded-confidence model [9] share a similar idea that such overly large attractions should be mitigated. The D–W model achieves this by introducing an additional assumption. That is, the opinion attractiveness first increases with opinion distance and becomes zero once the distance exceeds a prescribed bound. Such sudden truncation of an increasing trend is not empirically supported and makes it too difficult to rigorously establish convergence of the model and the conditions for reaching consensus. On the other hand, the weighted-median mechanism mitigates but does not completely truncate the attractions by distant opinions. Distant opinions are less likely to be the weighted-median opinions. But the weights assign to distant opinions could still affect which opinions are in the weighted-median positions. Moreover, as shown in Section III, the dynamic behavior of the weighted-median model is mathematically tractable.

Remark 3: Another major difference between the weighted-median and the weighted-averaging mechanisms is that the latter depends heavily on numerical representation of opinions, which could be nonunique and artificial, especially for issues where the opinions are not intrinsically quantitative. On the other hand, the weighted-median operation does not require any numerical representation of opinions, but instead is well-defined as long as the opinions are mapped into an ordered set. Therefore, the weighted-median mechanism is naturally applicable to multiple-choice issues with ordered options. Opinion dynamics over such issues have not been extensively studied by quantitative models, but they are prevalent in real life. For example, in modern societies, many political issues are evaluated along 1-D ideology spectra and political solutions often do not lend themselves to a continuum of viable choices.

Remark 4 (Connections with other models): According to Lemma 1, the weighted median is a self-map. Hence, any individual i in the weighted-median opinion dynamics can only adopts opinions $x_i(t)$ that initially exist in the system, i.e., $x(t) \in \{x_1(0), \dots, x_n(0)\}^n$ for any $t \in \mathbb{N}$. Here, the notation $\{x_1(0), \dots, x_n(0)\}^n$ means the set of all the n-dimensional vectors of which each entry's value is taken from the discrete set $\{x_1(0), \dots, x_n(0)\}$. Therefore, for any fixed initial condition, the weighted-median model given by Definition 2 is essentially a discrete-time Markov chain. In the case when there are only two distinct initial opinions, the weighted-median model coincides with several well-studied network dynamics, e.g., the deterministic voter model based on the local majority rule [31], [32] and the linear threshold model with homogeneous threshold $1/2$ [30], [33]. If the underlying influence network is a lattice, then it becomes the Glauber model of social segregation [34]. However, the two-opinion scenario does not reflect a key feature of the weighted-median opinion dynamics: the discrete but ordered opinions set.

We end this section by a proposition stating that the weighted-median update mechanism is a continuous function of the current opinion vector.

Proposition 2 (Continuity of weighted median): Given any row-stochastic matrix $W = (w_{ij})_{n\times n}$, for any $i \in \mathcal{V}$, the function $\text{Med}_i(x; W)$ is continuous at any $x \in \mathbb{R}^n$.

The proof is provided in Appendix A. This proposition implies that the weighted-median opinion update is robust to additive noises of opinions. That is, slight perturbations of individuals' current opinions only cause slight changes of individuals'

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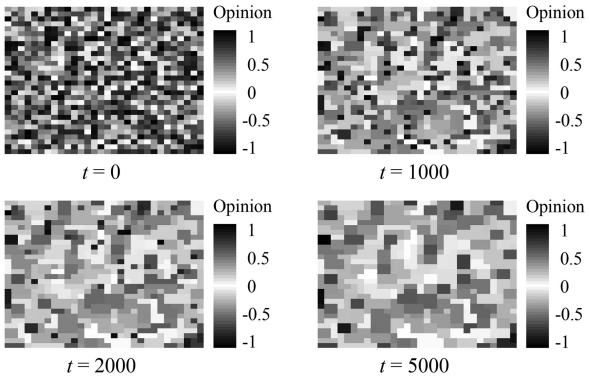


Fig. 1. One simulation of the weighted-median model on a 30 × 30 lattice graph. This figure is taken from [3, Fig. 3]. In this figure, each block is an individual bilaterally connected with all their adjacent blocks (not including the diagonally adjacent blocks). Each individual has a self-loop and uniformly assigns weights to all their neighbors including themselves. Initial opinions are independently randomly generated according to the uniform distribution on $[-1, 1]$. The gray scale of each block is proportional to the absolute value of the individual's final opinion. After 5000 time steps, the evolution reaches an equilibrium.

opinion updates. At the end of Section III-B, we have a remark on the weighted-median opinion dynamics' robustness with respect to perturbations of the influence network as well.

III. Dynamical Behavior of the Weighted-Median Opinion Dynamics

It is widely believed that influence network structure plays an important role in shaping opinion evolution in social groups. However, in some widely studied averaging-based models, e.g., the French–DeGroot model and the bounded-confidence models, whether a system reaches consensus is only determined by the connectivity of the (potentially time-varying) influence network. The effects of other finer network structures are not fully unveiled. The intuition behind this problem is that the weighted-averaging mechanism implies too large "attractive forces" between distant opinions. As a result, no structural property other than the lack of connectivity is capable of harnessing the system from converging to consensus.

On the other hand, the weighted-median mechanism resolves the aforementioned issue by assuming independence between opinion attraction and opinion distance. With this fundamental change, the weighted-median opinion dynamics exhibit some new features in terms of how influence network structure shapes opinion evolutions. Fig. 1, taken from [3], shows a typical example of a single simulation of the weighted-median model on a lattice graph. One could observe that the system does not reach a consensus at the equilibrium. Moreover, opinions might form local clusters, which indicates that some local structures of the influence network might play roles in shaping the system's asymptotic behavior.

In this section, we identify two such structures and analyze how they affect the weighted-median model's dynamical behavior. These two structures are cohesive sets and decisive links. We provide some important properties of them, based on which we characterize the set of equilibria, establish its convergence, and propose conditions for reaching asymptotic consensus and dissensus, respectively.

A. Important Concepts and Lemmas

The concept of cohesive sets was first proposed in [29] and used in the study of linear-threshold models of network diffusion [35], with a generalized form. In this article, we adopt a specific version of it.

Definition 3 (Cohesive set and maximal cohesive set): Given an influence network $\mathcal{G}(W)$ with node set $\mathcal{V}$, a cohesive set $\mathcal{M} \subseteq \mathcal{V}$ is a subset of nodes that satisfies $\sum_{j\in\mathcal{M}} w_{ij} \geq 1/2$ for any $i \in \mathcal{M}$. A cohesive set $\mathcal{M}$ is a maximal cohesive set if there does not exists $i \in \mathcal{V}\setminus\mathcal{M}$ such that $\sum_{j\in\mathcal{M}} w_{ij} > 1/2$.

Note that, since the influence matrix W is assumed to be row-stochastic, according to Definition 3, the whole set of nodes $\mathcal{V}$ is always a maximal cohesive set. Moreover, in the weighted-median opinion dynamics, if all the nodes in a cohesive set adopt the same opinion, then that opinion will also be the weighted-median opinion for each individual in the cohesive set. Namely, these individuals' opinions will not be updated via the weighted-median mechanism. In this sense, the concept of cohesive set can be interpreted as a characterization of "echo chamber" (a metaphorical description of a situation in which beliefs are amplified by communication and repetition inside a closed system).

Given the concept of cohesive set, we further define a process called *cohesive expansion*.

Definition 4 (Cohesive expansion): Given an influence network $\mathcal{G}(W)$ with node set $\mathcal{V}$ and a subset of nodes $\mathcal{M} \subseteq \mathcal{V}$, a subset of $\mathcal{V}$ is a cohesive expansion of $\mathcal{M}$ if it can be constructed via the following iteration algorithm.

1) Let $\mathcal{M}_0 = \mathcal{M}$.
2) For $k = 0, 1, 2, \ldots$, if there exists $i \in \mathcal{V}\setminus\mathcal{M}_k$ such that $\sum_{j\in\mathcal{M}_k} w_{ij} > 1/2$, then let $\mathcal{M}_{k+1} = \mathcal{M}_k \cup \{i\}$.
3) Terminate the iteration at step k as long as there does not exists any $i \in \mathcal{V}\setminus\mathcal{M}_k$ such that $\sum_{j\in\mathcal{M}_k} w_{ij} > 1/2$, and let $\text{Expansion}(\mathcal{M}) = \mathcal{M}_k$.

In what follows, we present and prove some novel results on the fundamental properties of cohesive sets, cohesive expansion, and maximal cohesive sets.

The following lemma states that cohesive expansion is unique. As a result, we can denote by $\text{Expansion}(\mathcal{M})$ the cohesive expansion of a node subset $\mathcal{M}$ without ambiguity.

Lemma 3 (Uniqueness of cohesive expansion): Given an influence network $\mathcal{G}(W)$ with the node set $\mathcal{V}$, for any $\mathcal{M} \subseteq \mathcal{V}$, the cohesive expansion of $\mathcal{M}$ is unique, i.e., independent of the order of node additions.

The proof of Lemma 3 is given in Appendix B.

The following lemma presents some fundamental properties of cohesive sets and cohesive expansions.

Lemma 4 (Properties of cohesive sets/expansions): Given an influence network $\mathcal{G}(W)$ with the node set $\mathcal{V}$, the following statements hold.

1) If $\mathcal{M}_1, \mathcal{M}_2 \subseteq \mathcal{V}$ are both cohesive sets, then $\mathcal{M}_1 \cup \mathcal{M}_2$ is also a cohesive set.
2) For any $\mathcal{M}, \tilde{\mathcal{M}} \subseteq \mathcal{V}$, if $\mathcal{M} \subseteq \tilde{\mathcal{M}}$, then $\text{Expansion}(\mathcal{M}) \subseteq \text{Expansion}(\tilde{\mathcal{M}})$.

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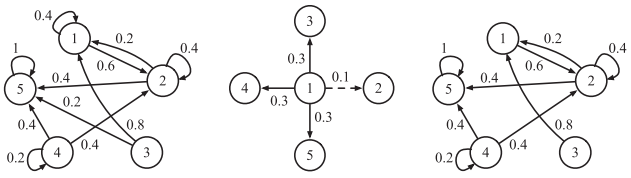


Fig. 2. Visualized examples illustrating the concepts of cohesive set, maximal cohesive set, cohesive expansion, decisive link, and indecisive link. In Panel (a), the node set $\mathcal{M}=\{1\}$ is not cohesive since node 1 assigns less than 1/2 weights to itself. Its cohesive expansion Expansion$(\mathcal{M})=\{1,3\}$ is not cohesive either. The node set $\tilde{\mathcal{M}}=\{1,2\}$ is a cohesive set. Consistent with statement 4) of Lemma 4, its cohesive expansion Expansion$(\tilde{\mathcal{M}})=\{1,2,3\}$ is a maximal cohesive set. All the maximal cohesive sets containing $\tilde{\mathcal{M}}$ are $\{1,2,3\}$, $\{1,2,3,4\}$, and $\{1,2,3,4,5\}$, among which Expansion$(\tilde{\mathcal{M}})$ is the "smallest," i.e., Expansion$(\tilde{\mathcal{M}})$ is a subset of any other maximal cohesive set containing $\tilde{\mathcal{M}}$. In Panel (b), the link (1,2) is indecisive, while the other links are all decisive. Even though individual 1 assigns nonzero weight to individual 2, according to the weighted-median mechanism, this weight is so small that individual 2's opinion plays no role in determining individual 1's neighbors' weighted-median opinion. That is, individual 2 has no actual influence on individual 1's opinion updates. Panel (c) shows the graph in Panel (a) with all the indecisive links removed. This graph has a maximal cohesive set that is not the whole node set $\mathcal{V}$, e.g., the node set $\{1,2,3\}$, and has a globally reachable node, i.e., node 5.

3) For any $\mathcal{M}$, $\tilde{\mathcal{M}}\subseteq\mathcal{V}$, Expansion$(\mathcal{M})\cup$ Expansion$(\tilde{\mathcal{M}})\subseteq$ Expansion$(\mathcal{M}\cup\tilde{\mathcal{M}})$.
4) If $\mathcal{M}$ is a cohesive set, then Expansion$(\mathcal{M})$ is also cohesive and is the smallest maximal cohesive set that includes $\mathcal{M}$, that is, for any maximal cohesive set $\hat{\mathcal{M}}$ such that $\mathcal{M}\subseteq\hat{\mathcal{M}}$, we have Expansion$(\mathcal{M})\subseteq\hat{\mathcal{M}}$.

The proof is given in Appendix C.

Remark 5: Note that statement 4) of Lemma 4 holds if $\mathcal{M}$ is a cohesive set. If a node set $\mathcal{M}$ is not cohesive, then the cohesive expansion of $\mathcal{M}$ is not necessarily cohesive. The concepts of cohesive set, maximal cohesive sets, and cohesive expansion, as well as their relations, are illustrated via examples in Fig. 2(a).

In the following, we present another useful lemma on cohesive sets, which is a straightforward consequence of Definition 3 and statement 4) of Lemma 4.

Lemma 5 (Cohesive partition): Given an influence network $\mathcal{G}(W)$ with node set $\mathcal{V}$ and a cohesive set $\mathcal{M}\subseteq\mathcal{V}$, if $\mathcal{M}$ is maximally cohesive, then $\mathcal{V}\setminus\mathcal{M}$ is also maximally cohesive; If $\mathcal{M}$ is not maximally cohesive, then either of the following two statements holds.

1) Expansion$(\mathcal{M})=\mathcal{V}$.
2) Expansion$(\mathcal{M})$ and $\mathcal{V}\setminus$ Expansion$(\mathcal{M})$ are both nonempty and maximally cohesive.

Another important concept involved is called *decisive link*; see Fig. 2(b) for visualized illustrations. Loosely speaking, a link (i,j) is decisive if node j could potentially be a "tie-breaker" when i computes their weighted-median opinion.

Definition 5 (Decisive and indecisive out links): Given an influence network $\mathcal{G}(W)$ with the node set $\mathcal{V}$, define the out-neighbor set of each node i as $\mathcal{N}_i=\{j\in\mathcal{V}|\,w_{ij}\neq 0\}$. A link (i,j) is a decisive out-link of the node i, if there exists a subset $\theta\subseteq\mathcal{N}_i$ such that the following three conditions hold: 1) $j\in\theta$; 2) $\sum_{k\in\theta}w_{ik}>1/2$; and 3) $\sum_{k\in\theta\setminus\{j\}}w_{ik}<1/2$. Otherwise, the link (i,j) is an indecisive out-link of the node i.

Remark 6: According to the definition of weighted median, if a link (i,j) with weight w_{ij} is indecisive, then, whatever opinion individual j holds, it has no influence on the value of i's neighbors' weighted-median opinion. That is, indecisive links play no role in the weighted-median opinion dynamics.

B. Equilibria, Convergence, Consensus, and Persistent Dissensus

In this subsection, we present and prove our main results on the dynamical behavior of the weighted-median opinion dynamics. First of all, the following theorem characterizes the set of all the equilibria.

Theorem 6 (Set of equilibria): Consider an influence network $\mathcal{G}(W)$ with n individuals. An opinion vector $x^*\in\mathbb{R}^n$ is an equilibrium of the weighted-median opinion dynamics given by Definition 2, i.e., $x_i^*=\text{Med}_i(x^*;W)$ for any $i\in\mathcal{V}$, if and only if x^* is either a consensus vector, i.e., $x_i^*=x_j^*$ for any $i,\,j\in\mathcal{V}$, or satisfies the following condition: for any y satisfying $\min_i x_i^*<y\leq\max_i x_i^*$, both $\{i\in\mathcal{V}|\,x_i^*<y\}$ and $\{i\in\mathcal{V}|\,x_i^*\geq y\}$ are maximal cohesive sets on $\mathcal{G}(W)$.

Proof: We first prove the sufficient part. If x^* is a consensus vector, then we have $\text{Med}_i(x^*;W)=x_i^*$ for any i, since the weighted-median of only one opinion is always the opinion itself. Now suppose that x^* is NOT a consensus vector and satisfies that, for any $y\in(\min_k x_k^*,\max_k x_k^*)$, $\{j\in\mathcal{V}|x_j^*<y\}$ and $\{j\in\mathcal{V}|x_j^*\geq y\}$ are both maximal cohesive sets. For any given i, let $y=x_i^*$. Since $i\in\{j:\,x_j^*\geq x_i^*\}$, and $\{j:\,x_j^*\geq x_i^*\}$ is a maximal cohesive set, we have

$$\sum_{j:\,x_j^*\geq x_i^*}w_{ij}\geq\frac{1}{2}\quad\Rightarrow\quad\text{Med}_i(x^*;W)\geq x_i^*.$$

Let $\tilde{y}=\min_k\{x_k^*\,|\,x_k^*>x_i^*\}$. Since $i\in\{j:\,x_i^*<\tilde{y}\}$, and $\{j:\,x_i^*<\tilde{y}\}$ is a maximal cohesive set, we have

$$\sum_{j:\,x_j^*<\tilde{y}}w_{ij}\geq\frac{1}{2}\quad\Rightarrow\quad\text{Med}_i(x^*;W)<\tilde{y}.$$

The inequality $x_i^*\leq\text{Med}_i(x^*;W)<\tilde{y}$ together with $\text{Med}_i(x^*;W)\geq x_i^*$ leads to $x_i^*=\text{Med}_i(x^*;W)$. This concludes the proof for the sufficient part.

Now we proceed to prove the necessary part. Suppose x^* is not a consensus vector and there exists $y\in(\min_k x_k^*,\max_k x_k^*)$ such that $\{j:\,x_j^*<y\}$ and $\{j:\,x_j^*\geq y\}$ are not both maximal cohesive sets. Since these two sets form a disjoint partition of the node set $\{1,\ldots,n\}$, one of them must not be cohesive. Otherwise, by definition, we must have

$$\sum_{j:\,x_j^*<y}w_{pj}=1-\sum_{j:\,x_j^*\geq y}w_{pj}\leq\frac{1}{2},\ \text{for any}\ p\in\{j:\,x_j^*\geq y\}$$

$$\sum_{j:\,x_j^*\geq y}w_{pj}=1-\sum_{j:\,x_j^*<y}w_{pj}\leq\frac{1}{2},\ \text{for any}\ p\in\{j:\,x_j^*<y\}$$

which contradicts the preassumption that $\{j:\,x_j^*<y\}$ and $\{j:\,x_j^*\geq y\}$ are not both maximal cohesive sets.

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Suppose $\{j : x_j^* \geq y\}$ is not cohesive. As a direct consequence, there exists i with $x_i^* \geq y$ but

$$\sum_{j:\, x_j^* < y} w_{ij} > \frac{1}{2}$$

which in turn implies that $\text{Med}_i(x^*; W) < y \leq x_i^*$. Therefore, such x^* is not an equilibrium of the weighted-median opinion dynamics. Similarly, if $\{j : x_j^* < y\}$ is not cohesive, then there exists i with $x_i^* < y$ such that $\text{Med}_i(x^*; W) \geq y > x_i^*$. That is, x^* is not an equilibrium. Therefore, in order for x^* to be an equilibrium, it must be either a consensus vector or $\{j : x_j^* < y\}$ and $\{j : x_j^* \geq y\}$ are both maximal cohesive sets for any $y \in (\min_k x_k^*, \max_k x_k^*)$. This concludes the proof for the necessary part. ■

Definition 6 (Consensus and dissensus equilibrium): Consider the weighted-median opinion dynamics given by Definition 2 for a group of n individuals. If $x^* \in \mathbb{R}^n$ is an equilibrium and $x_i^* = x_j^*$ for any $i, j \in \{1, 2, \ldots, n\}$, then x^* is referred to as a consensus equilibrium. If the opinion trajectory $x(t) \in \mathbb{R}^n$ converges to a consensus equilibrium, we say that the system achieves consensus. If $x^* \in \mathbb{R}^n$ is an equilibrium and there exists $i, j \in \{1, 2, \ldots, n\}$ such that $x_i^* \neq x_j^*$, then x^* is referred to as a dissensus equilibrium. If the opinion trajectory $x(t) \in \mathbb{R}^n$ converges to a dissensus equilibrium, we say that the system achieves persistent dissensus.

In what follows, we prove that the weighted-median opinion dynamics, with any initial condition, almost surely converge to an equilibrium in finite time. (Here, "almost surely" is in terms of the random initial condition and the random individual update sequence given by Definition 2.) Moreover, we propose and prove necessary and sufficient graph-theoretic conditions for almost-sure convergence to consensus equilibria, as well as a sufficient graph-theoretic condition for almost-sure convergence to dissensus equilibria.

The following lemma is a foundation of the proof for convergence and is akin to results about absorbing states in Markov chains.

Lemma 7 (Transforming randomness to sequence design): Consider the weighted-median opinion dynamics given by Definition 2. If, starting from any x, there exists an update sequence $i_1, \ldots, i_{T_x}$, along which the opinion trajectory reaches an equilibrium at time step T_x, then the weighted-median opinion dynamics almost surely converges to an equilibrium in finite time, for any initial condition $x(0)$.

The proof is provided in Appendix D.

With Lemma 7, we can establish the convergence of the weighted-median opinion dynamics by manually inputting to the system the ordering of individuals' opinion updates. That is, Lemma 7 provides a way of transforming randomness into control inputs. Similar approaches have been used in previous literature on synchronization and consensus analysis; e.g., see [18], [19], and [36].

Now we are ready to present the main theorem of this subsection. The core of its proof is to establish the convergence of the weighted-median model. We prove the convergence by first considering a two-opinion scenario, and then, extending the argument to general cases where there are more than two distinct opinions initially.

Theorem 8 (Convergence, consensus, and dissensus): Consider the weighted-median opinion dynamics given by Definition 2 on an influence network $\mathcal{G}(W)$ with node set $\mathcal{V}$. Denote by $\mathcal{G}_{\text{decisive}}(W)$ the subgraph of $\mathcal{G}(W)$ with all the indecisive out links removed. The following statements hold

1) For any initial condition $x(0) \in \mathbb{R}^n$, the solution $x(t)$ almost surely converges to an equilibrium x^* in finite time.
2) For any initial condition $x(0) \in \mathbb{R}^n$, the solution $x(t)$ almost surely converges to a consensus equilibrium if and only if the only maximal cohesive set of $\mathcal{G}(W)$ is $\mathcal{V}$. Moreover, if $\mathcal{G}(W)$ has a maximal cohesive set $\mathcal{M} \neq \mathcal{V}$, then there exists a subset of initial conditions $X_0 \subseteq \mathbb{R}^n$ with nonzero Lebesgue measure [37] in $\mathbb{R}^n$ such that, for any $x(0) \in X_0$, there is no update sequence along which the system achieves consensus.
3) If $\mathcal{G}_{\text{decisive}}(W)$ does not have a globally reachable node, then, for any initial condition $x(0) \in \mathbb{R}^n \setminus \hat{X}$, where $\hat{X} = \{x \in \mathbb{R}^n \,|\, \exists\, i \neq j \text{ s.t. } x_i = x_j\}$ has zero Lebesgue measure in $\mathbb{R}^n$, the solution $x(t)$ almost surely reaches a dissensus equilibrium in finite time.

Proof: Regarding statement 1), we first point out that it is equivalent to the following claim.

Statement 1):* For any initial state $x(0)$, there exists an update sequence $\{i_1, \ldots, i_T\}$ along which the solution $x(t)$ reaches an equilibrium at time step T.

"Statement 1) ⇒ Statement 1*)" is straightforward and "1*) ⇒ 1)" is an immediate result of Lemma 7. Therefore, in order to prove statement 1), we only need to prove 1*).

Now we prove that statement 1*) is true. We first consider the case in which there are initially only two different opinions. Without loss of generality, let the two opinions be y_1 and y_2. Let

$$\mathcal{V}_1(t) = \{i \in \mathcal{V} |\, x_i(t) = y_1\}, \quad \mathcal{V}_2(t) = \{i \in \mathcal{V} |\, x_i(t) = y_2\}$$

for any $t \in \mathbb{N}$. Due to Definition 2, no new opinion will be created during the weighted-median opinion dynamics. Therefore, for any initial state $x(0) \in \{y_1, y_2\}^n$, the solution $x(t)$ satisfies $x(t) \in \{y_1, y_2\}^n$ for any $t \geq 0$. As a consequence, at any t, the sets $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$ form a partition of the node set $\mathcal{V}$. We neglect the trivial cases when $\mathcal{V}_1(0) = \mathcal{V}$ or $\mathcal{V}_2(0) = \mathcal{V}$, otherwise the system would be already at an equilibrium.

For any given $x(0)$, we now manually design an individual update sequence, along which $x(t)$ reaches an equilibrium in finite time. The sketch of this process is clear: We first let the individuals holding opinion y_1 convert to y_2 via the weighted-median mechanism, until no individual holding opinion y_1 can change their opinion. Then, we let the individuals holding opinion y_2 convert to y_1 via the weighted-median mechanism, until no individual holding opinion y_2 can do that. Such a process must end in finite time, and at the end, $x(t)$ reaches an equilibrium. The aforementioned sketch is specified as follows.

We first construct an update sequence as follows.

1) For any time step $t+1$, $t = 0, 1, 2, \ldots$, if there exists some $i_{t+1} \in \mathcal{V}_1(t)$ such that $\sum_{j \in \mathcal{V}_2(t)} w_{i_{t+1} j} > 1/2$, then update node i_{t+1} at time step $t+1$, and thereby,

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we get $\mathcal{V}_1(t+1) = \mathcal{V}_1(t) \setminus \{i_{t+1}\}$ and $\mathcal{V}_2(t+1) = \mathcal{V}_2(t) \cup \{i_{t+1}\}$.

2) The update stops at time step T if there does not exists any $i \in \mathcal{V}_1(T)$ such that $\sum_{j\in\mathcal{V}_2(T)} w_{ij} > 1/2$.

At the end of the update sequence $\{i_1, \ldots, i_T\}$, we obtain a partition $\mathcal{V}_1(T)$ and $\mathcal{V}_2(T)$, and all the individuals in $\mathcal{V}_1(T)$ ($\mathcal{V}_2(T)$, resp.) hold the opinion y_1 (y_2, resp.). Note that $\mathcal{V}_2(T)$ is the cohesive expansion of $\mathcal{V}_2(0)$. However, since $\mathcal{V}_2(0)$ is not necessarily cohesive, $\mathcal{V}_2(T)$ is not necessarily cohesive either.

If $\mathcal{V}_1(T)$ is empty, then the system is already at an equilibrium where all the nodes hold opinion y_2. If $\mathcal{V}_1(T)$ is not empty, then, for any $i \in \mathcal{V}_1(T) = \mathcal{V}\setminus \mathcal{V}_2(T)$, since $\mathcal{V}_2(T)$ is already the cohesive expansion of $\mathcal{V}_2(0)$, we have $\sum_{j\in\mathcal{V}_2(T)} w_{ij} \le 1/2$, which implies that

$$\sum_{j\in\mathcal{V}_1(T)} w_{ij} = \sum_{j\in\mathcal{V}\setminus\mathcal{V}_2(T)} w_{ij} = 1 - \sum_{j\in\mathcal{V}_2(T)} w_{ij} \ge 1/2.$$

Therefore, $\mathcal{V}_1(T)$ is a cohesive set. Denote by $\mathcal{E}_1 = \mathcal{V}_1(T) \cup (j_1, \ldots, j_k)$ the cohesive expansion of $\mathcal{V}_1(T)$, and the nodes are added to $\mathcal{V}_1(T)$ along the sequence $j_1, \ldots, j_k$. That is, the entire update sequence now becomes $i_1, \ldots, i_T, j_1, \ldots, j_k$. If $\mathcal{E}_1 = \mathcal{V}$, then the system along this update sequence reaches the equilibrium where all the nodes adopt opinion y_1. If $\mathcal{E}_1 \neq \mathcal{V}$, then the system along such an update sequence reaches the state in which all the nodes in $\mathcal{E}_1$ adopt opinion y_1 while all the nodes in $\mathcal{V}\setminus \mathcal{E}_1$ adopt opinion y_2. According to Lemma 5, $\mathcal{E}_1$ and $\mathcal{V}\setminus \mathcal{E}_1$ are both maximally cohesive sets. Therefore, the system reaches an equilibrium along the update sequence $i_1, \ldots, i_T, j_1, \ldots, j_k$. Till now we have proved statement 1*) in the case when there are only two distinct opinions initially.

Now we extend the aforementioned argument to the case of any arbitrary initial condition $x(0) \in \mathbb{R}^n$ by induction. Suppose statement 1*) holds whenever there are less than or equal to $R-1$ distinct opinions initially, for some $2 < R \le n$. Suppose that there are R distinct values among the initial opinions $\{x_1(0), \ldots, x_n(0)\}$ and denote by these R values $y_1, y_2 \ldots, y_R$, with $y_1 < y_2 < \ldots < y_R$.

Let $A_1 = \{y_1\}$ and $B_1 = \{y_2, \ldots, y_R\}$, and consider A_1 and B_1 as two possible states of the nodes. Due to the weighted-median mechanism, whether a node converts its state from A_1 to B_1 depends only on which of its neighbors are in state B_1. Therefore, when discussing the nodes' state transitions from A_1 to B_1, we can assume that the nodes in state B_1 all have an identical opinion $\tilde{y}_1 > y_1$. That is, the process of the state transitions from A_1 to B_1 is reduced to the two-opinion scenario, which we have previously discussed. Repeating the argument for the two-opinion case, we can construct an update sequence $i_{1,1}, \ldots, i_{1,k_1}$ such that, at time k_1, either no individual hold the opinion y_1, or the node set $\mathcal{V}$ at time k_1 is divided into two sets $\mathcal{E}_1$ and $\mathcal{V}\setminus \mathcal{E}_1$, such that

1) all the nodes in $\mathcal{E}_1$ hold the opinion y_1;
2) $\mathcal{E}_1$ is a maximal cohesive set.

If no individual holds the opinion y_1 at time k_1, then this situation is reduced to the case when there are $R-1$ distinct opinions initially, and the convergence is established by the preassumption. If the node set $\mathcal{V}$ at time k_1 is divided into two sets $\mathcal{E}_1$ and $\mathcal{V}\setminus \mathcal{E}_1$, then, after the update sequence $i_{1,1}, \ldots, i_{1,k_1}$, nodes in $\mathcal{E}_1$ never switch their opinion from y_1 to any other opinion, while nodes in $\mathcal{V}\setminus \mathcal{E}_1$ never switch their opinions to y_1.

Let $A_2 = \{y_1, y_2\}$ and $B_2 = \{y_3, \ldots, y_R\}$. We have shown that, after the update sequence $i_{1,1}, \ldots, i_{1,k_1}$, the nodes holding opinion y_1 will never change their opinions. Therefore, due to the weighted-median mechanism, for all the nodes in $\mathcal{V}\setminus \mathcal{E}_1$, it makes no difference to their opinion updates whether the nodes in $\mathcal{E}_1$ hold opinion y_1 or y_2. As the result, in terms of determining the behavior of the nodes in $\mathcal{V}\setminus \mathcal{E}_1$, we can take y_1 and y_2 as the same opinion. Following the same line of argument in the previous paragraph, there exists another update sequence $i_{2,1}, \ldots, i_{2,k_2}$, right after the sequence $i_{1,1}, \ldots, i_{1,k_1}$, such that, after these two sequences of updates, all the nodes are partitioned into two sets $\mathcal{E}_2$ and $\mathcal{V}\setminus \mathcal{E}_2$, where $\mathcal{E}_2$ is the set of all the nodes holding either opinion y_1 or opinion y_2, and $\mathcal{E}_2$ is a maximal cohesive set.

Repeating the argument in the previous paragraph, we obtain the sets $\mathcal{E}_1, \ldots, \mathcal{E}_{n-1}$, which are all maximal cohesive sets, and the entire update sequence $i_{1,1}, \ldots, i_{1,k_1}, \ldots, i_{n-1,1}, \ldots, i_{n-1,k_{n-1}}$. Define

$$\mathcal{V}_1 = \mathcal{E}_1$$
$$\mathcal{V}_r = \mathcal{E}_r \setminus \cup_{s=1}^{r-1} \mathcal{E}_s, \text{ for any } r = 2, \ldots, n-1$$
$$\mathcal{V}_n = \mathcal{V}\setminus \cup_{s=1}^{n-1} \mathcal{E}_s.$$

The way we construct $\mathcal{E}_1 \ldots, \mathcal{E}_{n-1}$ implies that, after the update sequence $i_{1,1}, \ldots, i_{1,k_1}, \ldots, i_{n-1,1}, \ldots, i_{n-1,k_{n-1}}$, the system reaches a state in which, for any $r \in \{1, \ldots, n\}$, all the nodes in $\mathcal{V}_r$ hold the opinion y_r and will not switch to any other opinion. Therefore, for any initial condition $x(0) \in \mathbb{R}^n$, we have constructed an update sequence, along which the system reaches an equilibrium in finite time. This concludes the proof for the case of R distinct initial opinions and thereby concludes the proof of statement 1*) by induction, which is equivalent to statement 1).

Now we proceed to prove the sufficient part of statement 2). If the only maximal cohesive set in $\mathcal{G}(W)$ is $\mathcal{V}$ itself, then, according to statement 4) of Lemma 4, the cohesive expansion of any cohesive set is $\mathcal{V}$ itself. Repeating the construction of the update sequence in the proof of statement 1*), we have that $\mathcal{E}_1$ in that proof is either an empty set or the cohesive expansion of a cohesive set, which is $\mathcal{V}$ itself under the condition of statement 2). If $\mathcal{E}_1$ is empty, then the situation is reduced to the case of $R-1$ distinct initial opinions. Following the same argument, $\mathcal{E}_2$ is either empty or $\mathcal{V}$ itself. As this argument goes on, will find some $r \in \{1, \ldots, R\}$ such that $\mathcal{E}_r = \mathcal{V}$ while $\mathcal{E}_1, \ldots, \mathcal{E}_{r-1}$ are all empty. That is, for any initial condition, there exists an update sequence, along which the system reaches an equilibrium where all the individuals hold the same opinion.

Regarding the necessary part of statement 2), suppose there exists a maximal cohesive set $\mathcal{M} \subseteq \mathcal{V}$. We construct the set X_0 of initial conditions as

$$X_0 = \Big\{x(0) \in \mathbb{R}^n \;\Big|\; \max_{j\in\mathcal{M}} x_j(0) < \min_{k\in\mathcal{V}\setminus\mathcal{M}} x_k(0), \text{ or } \min_{j\in\mathcal{M}} x_j(0) > \max_{k\in\mathcal{V}\setminus\mathcal{M}} x_k(0)\Big\}.$$

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By definition, the set X_0 has nonzero Lebesgue measure in $\mathbb{R}^n$. Moreover, according to the weighted-median mechanism, starting from any $x(0) \in X_0$, the opinions of the nodes in $\mathcal{M}$will always be lower (higher resp.) than the opinion of any node in $\mathcal{V}\setminus\mathcal{M}$, if $\max_{j\in\mathcal{M}} x_j(0) < \min_{k\in\mathcal{V}\setminus\mathcal{M}} x_k(0)$ ($\min_{j\in\mathcal{M}} x_j(0) > \max_{k\in\mathcal{V}\setminus\mathcal{M}} x_k(0)$ resp.). This concludes the proof of statement 2).

Now we proceed to prove statement 3). By definition, $\hat{X}$ has zero Lebesgue measure in $\mathbb{R}^n$. Consider any $x(0) \in \mathbb{R}^n \setminus \hat{X}$. Since no new opinion is created along the weighted-median opinion dynamics, the system achieves consensus if and only if there exists a node i and sometime $T \in \mathbb{N}$ such that $x_j(T) = x_i(0)$ for any $j \in \mathcal{V}$. Since no individual other than i initially holds the opinion $x_i(0)$, as indicated by Remark 6, in order for a node j to adopt the opinion $x_i(0)$ at time T, there must exist at least one path on the influence network $\mathcal{G}(W)$ from j to i via only decisive links. According to the conditions in statement 3), $\mathcal{G}_{\text{decisive}}(W)$ does not have a globally reachable node. Therefore, such node i does not exist. This concludes the proof of statement 3). ■

Note that, as implied by Theorem 8, if $\mathcal{G}_{\text{decisive}}(W)$ has no globally reachable node, then $\mathcal{V}$ is the not the only maximal cohesive set in $\mathcal{G}(W)$. However, the converse of this statement is not true; see Fig. 2(c) for an example. Fig. 2(c) presents the graph in Fig. 2(a) with all the indecisive links removed. In Fig. 2(c), the graph has a maximal cohesive set that is not $\mathcal{V}$, e.g., the node set $\{1, 2, 3\}$, but this graph has a globally reachable node, i.e., node 5.

At the end of this subsection, we briefly compare the dynamical behavior of the weighted-median opinion dynamics with some widely studied averaging-based models. Regardless of whether the opinion updates are synchronous or asynchronous, the French–DeGroot model and the F–J model lead to somewhat oversimplified outcomes in terms of the conditions for achieving consensus or persistent dissensus: The system either almost surely achieves consensus or almost surely achieves persistent dissensus (with respect to initial conditions), dependent on whether a single graph-theoretic condition is satisfied. These conditions are usually about network connectivity [23], [24], [38], and thus, highly sensitive to perturbations of networks. For example, adding an edge with an infinitesimal weight can make a disconnected network strongly connected. Some other extensions of the French–DeGroot model, e.g., the biased assimilation model and the bounded-confidence model, are highly nonlinear such that the conditions for consensus/dissensus are too difficult to analyze.

On the other hand, as indicated by Theorem 8, the weighted-median opinion dynamics exhibit three different behaviors: almost-sure consensus, almost-sure dissensus, or having a nonzero probability of reaching dissensus. Which behavior the model exhibits depends on two important network structure properties: the presence of nontrivial maximal cohesive sets and the connectivity of the influence network with all the indecisive links removed. Generically, given a graph where the link weights are independently randomly generated from continuous distributions, the conditions in Definitions 1, 3, and 5 hold with strict inequalities. As a consequence, the notions of weighted-median opinion, cohesive set, and decisive link are to some extent robust to network uncertainty. That is, under sufficiently small perturbations of link weights, the individuals' weighted-median opinions, the cohesive sets and the decisive links remain unchanged. Therefore, the weighted-median opinion dynamics, despite its simplicity in form, leads to somehow richer, more robust and yet mathematically tractable dynamical behavior.

C. Further Analysis: Determining Almost-Sure Dissensus is NP-Hard

In the previous section, we establish the almost-sure finite-time convergence of the weighted-median opinion dynamics and provide a necessary and sufficient graph-theoretic condition for almost-sure consensus. A sufficient condition for almost-sure dissensus is also provided. In this subsection, we provide necessary and sufficient conditions for almost-sure convergence to dissensus equilibria, and prove that, given an arbitrary directed and weighted graph, determining whether these conditions hold is NP-hard. This result highlights the fact that median consensus yields significantly richer and more complex phenomena than average consensus, where most convergence questions can be solved by linear algebraic tools.

The basic strategy is to relate our problem mentioned previously to the monotone NAE3SAT problem, which is known to be NP-hard.

Definition 7 (Monotone NAE3SAT): Given m clauses $c_1, \ldots, c_m$, where each clause c_j is a set of three indices $\{k_j^1, k_j^2, k_j^3\}$ (a same index can appear in multiple clauses or multiple times in the same clause), find n binary variables $x_1, \ldots, x_n \in \{-1, 1\}$ such that, for each clause c_j, $x_{k_j^1}$, $x_{k_j^2}$, and $x_{k_j^3}$ are not all equal.

The following result is well-known in previous literature [39].

Lemma 9 (Monotone NAE3SAT is NP-hard): The monotone NAE3SAT problem as described in Definition 7 is NP-hard unless P=NP.

As indicated by Theorem 8, the weighted-median opinion dynamics given by Definition 2 almost surely achieves either a consensus equilibrium or a dissensus equilibrium in finite time, for any initial condition. The following proposition presents some equivalent statements for not achieving almost-sure dissensus.

Proposition 10 (Almost-sure persistent dissensus): Given an influence network $\mathcal{G}(W)$ and the weighted-median opinion dynamics as in Definition 2, the following statements are equivalent.

1) There exists a set $X_0 \subseteq \mathbb{R}^n$ with positive measure in $\mathbb{R}^n$ such that, for any $X(0) \in X_0$, there exists an update sequence, along which consensus is achieved in finite time.
2) There exists $x_0 \in \mathbb{R}^n$ with n distinct entries and an update sequence $\{i_1, \ldots, i_T\}$ such that, along this update sequence, the trajectory $x(t)$ starting from $x(0) = x_0$ achieves consensus at time T.
3) There exists a ternary vector $y \in \{-1, 0, 1\}^n$, with $y_i = 0$ for some $i \in \{1, \ldots, n\}$ and $y_j \in \{-1, 1\}$ for any $j \in$

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$\{1,\dots,n\}\setminus\{i\}$, and an update sequence $\{i_1,\dots,i_T\}$ such that the trajectory $y(t)$ starting from $y(0)=y$ achieves consensus on 0 at time T, i.e., $y(T)=\mathbb{0}_n$.

The weighted-median opinion dynamics achieve persistent dissensus if and only if any of 1)–3) does not hold.

The proof is provided in Appendix E.

From statement 3) of Proposition 10 we know that, in order to check whether trajectories of the weighted-median opinion dynamics almost surely achieve persistent dissensus, we need to check only a finite set of initial conditions, whose entries take values from $\{-1,0,1\}$. However, in what follows, we show that, given any influence network $\mathcal{G}(W)$, determining whether statement 3) in Proposition 10 holds is an NP-hard problem. We tackle this problem in the following steps.

1) We show that, for any instance of the monotone NAE3SAT problem, we can build an associated influence network $\mathcal{G}(W)$ such that the NAE3SAT instance admits a solution if and only if the weighted-median opinion dynamics on $\mathcal{G}(W)$ do not almost surely achieve dissensus equilibria. Moreover, this construction can be done in polynomial time with respect to the size of the initial monotone NAE3SAET problem.
2) The aforementioned Step (1) implies that, if we had an algorithm to determine in polynomial time whether the weighted-median opinion dynamics on any influence network $\mathcal{G}(W)$ achieves almost-sure dissensus, we could then combine it with the construction in Step (1) to create an algorithm solving monotone NAE3SAT in polynomial time. That is, our problem is solvable in polynomial time only if NAE3SAT is.
3) The NP-hardness of the monotone NAE3SAT, unless N=NP, implies then the NP-hardness of determining whether the weighted-median opinion dynamics achieve almost-sure dissensus.

The key in the aforementioned steps is to construct a class of influence networks and relate the testing of statement 3) in Proposition 10 on these networks to an instance of the monotone NAE3SAT problem.

Given a monotone NAE3SAT problem, we construct the following specific class of influence networks and refer to them as "sink-variable-clause" (SVC) graphs.

Definition 8 (SVC graphs): Given a monotone NAE3SAT problem with n binary variables and m clauses as in Definition 7, an associated SVC graph is constructed as a weighted and directed graph with $2n+m+1$ nodes classified into the following three categories:

1) the sink node with the index s;
2) the variable nodes that come in pairs (n pairs in total). The indices for each pair are denoted by $(v_i,\tilde{v}_i)$, $i\in\{1,\dots,n\}$;
3) the clause nodes, with the indices c_j, $j\in\{1,\dots,m\}$.

Nodes in the SVC graph are connected with each other obeying the following rules.

1) The sink node has a self-loop with weight 1 on itself.
2) For each $i\in\{1,\dots,n\}$, we have a pair of variable nodes $(v_i,\tilde{v}_i)$. Node $\tilde{v}_i$ assigns 100% of its weight to node v_i. Node v_i assigns a weight 1/3 to itself, a weight 1/3 to node $\tilde{v}_i$, and a weight 1/3 to the last clause node c_m; see Fig. 3(a).
3) Any clause node c_j is associated with three variable nodes $k_j^1,k_j^2,k_j^3\in\{v_1,\dots,v_n\}$, at most two of which might be the same. The clause node c_j assigns a weight 1/5 to each of these 3 associated variable nodes (or 2/5 if an index appears twice), and a weight 2/5 to the previous clause node c_{j-1} if $j>1$, or a weight 2/5 to the sink node s if $j=1$; see Fig. 3(b).
4) Each variable node is linked to by at least one clause node. That is, $\{k_1^1,k_1^2,k_1^3,\dots,k_m^1,k_m^2,k_m^3\}=\{v_1,\dots,v_n\}$.

The idea of constructing the aforementioned SVC graph is as follows. First, if the sink node starts with opinion 0, it remains 0 all the time and this 0 shall propagate to the entire graph if the other nodes' initial opinions and update sequence are chosen properly. Second, each pair of variable nodes start with opposing opinions -1 and 1, and we keep them from changing their opinions until all the clause nodes have their opinions updated to 0. Third, if the clause nodes start with opinions from $\{-1,\,1\}$ and update their opinions sequentially according to their indices $c_1,\,c_2,\,\dots,\,c_m$, their opinions will all become 0, if and only if the variable nodes' opinions satisfy the rules of monotone NAE3SAT. Then, the variable nodes' opinions can all be updated to 0 following some update sequence. The following lemma will show that certain of the properties mentioned previously do indeed hold. This lemma is proved by directly applying Definition 2.

Lemma 11 (Properties of SVC graphs): For any SVC graph constructed as in Definition 8, the following statements hold.

1) For any $i\in\{1,\dots,n\}$, if $x_{c_m}(t)=0$ and $x_{v_i}(t)=-x_{\tilde{v}_i}(t)\neq 0$ for some t, then an update of node v_i followed by an update of node $\tilde{v}_i$ leads to $x_{v_i}(t+2)=x_{\tilde{v}_i}(t+2)=0$.
2) As long as $x_{v_i}(t)=x_{v_i}(0)\neq 0$ for any $i\in\{1,\dots,n\}$, node c_j ($j\in\{1,\dots,m\}$) can update to 0 from time step t to $t+1$ if and only if the following conditions are simultaneously satisfied.
   a) $x_{c_{j-1}}(t)=0$ for $j>1$ and $x_s(t)=0$ for $j=1$. The latter holds as long as $x_s(0)=0$.
   b) $x_{k_j^1}(t)$, $x_{k_j^2}(t)$, and $x_{k_j^3}(t)$ are not all equal.

With all the aforementioned preparation work, now we state and prove the main theorem.

Theorem 12 (Testing almost-sure dissensus is NP-hard): Given an influence graph $\mathcal{G}(W)$, the problem of determining whether statement 3) in Proposition 10 holds is NP-hard. Hence, unless P=NP, no polynomial time algorithm can solve it.

Proof: First of all, we point out that, according to Definition 8, an SVC graph with any given n and m can be constructed in polynomial time.

The main part of this proof is to show that, for any SVC graph with $2n+m+1$ nodes constructed according to Definition 8, statement 3) in Proposition 10 hold if and only if there exists a solution to the associated monotone NAE3SAT with m clauses and n variables.

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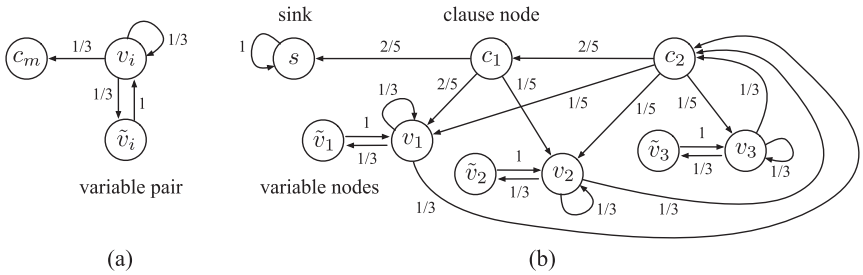


Fig. 3. Illustrations of the structure of the SVC graphs. Panel (a) shows how a pair of variable nodes are connected to each other and to the last clause node. Panel (b) shows a simple example of an SVC graph with $m = 2$ and $n = 3$. In this graph, variable pairs $(v_1, \tilde{v}_1)$, $(v_2, \tilde{v}_2)$ are associated with clause node c_1, and variable pairs $(v_1, \tilde{v}_1)$, $(v_2, \tilde{v}_2)$, and $(v_3, \tilde{v}_3)$ are associated with clause node c_2.

Regarding the sufficient part, given any variable vector $x^* \in \mathbb{R}^n$ satisfying a monotone NAE3SAT problem with m clauses, let $\mathcal{G}(W)$ be the associated SVC graph with $2n + m + 1$ nodes. We now construct an initial condition $y(0) \in \mathbb{R}^{2n+m+1}$ and an update sequence, along which the weighted-median opinion dynamics over $\mathcal{G}(W)$ achieves consensus on 0 in finite time. The initial condition $y(0)$ is constructed as follows:

$$\begin{aligned} &y_s(0) = 0 \\ &y_{v_i}(0) = x^*_{v_i}, \quad y_{\tilde{v}_i}(0) = -x^*_{v_i} \quad \forall\, i = 1, \ldots, n \\ &y_{c_j}(0) \in \{-1, 1\} \quad \forall j = 1, \ldots, m. \end{aligned}$$

The update sequence is updated as follows: We first update the opinion of the clause node c_1. Note that $y_{k_1^1}(0) = x^*_{k_1^1}$, $y_{k_1^2}(0) = x^*_{k_1^2}$, and $y_{k_1^3}(0) = x^*_{k_1^3}$ are in the set $\{-1, 1\}$ and are not all equal. Therefore, according to how the SVC graph $\mathcal{G}(W)$ is constructed, we have

$$\sum_{j:\, y_j(0)=-1} w_{c_1 j} \le 2/5 < 1/2$$

$$\sum_{j:\, y_j(0)=1} w_{c_1 j} \le 2/5 < 1/2, \quad \text{and}$$

$$\sum_{j:\, y_j(0)=0} w_{c_1 j} = w_{c_1 \tilde{} s} = 2/5 > 0.$$

That is, the opinion of c_1 will be updated to 0 at time 1. Then, we update the opinion of node c_2. Since $y_{k_2^1}(1) = y_{k_2^1}(0) = x^*_{k_2^1}$, $y_{k_2^2}(1) = y_{k_2^2}(0) = x^*_{k_2^2}$ and $y_{k_2^3}(1) = y_{k_2^3}(0) = x^*_{k_2^3}$ are in the set $\{-1, 1\}$ and are not all equal, and since $y_{c_1}(1) = 0$, according to how $\mathcal{G}(W)$ is constructed, we immediately have that the opinion of c_2 will be updated to 0 at time 2. According to the same argument and following the update sequence $\{c_1, c_2, \ldots, c_m\}$, we have

$$y_{c_1}(m) = \cdots = y_{c_m}(m) = 0.$$

Now we update the opinions of the variable pairs in the following way. For each variable pair $\{v_i, \tilde{v}_i\}$, note that at time m, one of these two nodes' opinions is 1 and the others' is -1. Moreover, we have $y_{c_m}(m) = 0$. Therefore, according to how $\mathcal{G}(W)$ is constructed, an update of node v_i's opinion followed by an update of node $\tilde{v}_i$'s opinion will make their opinions both become 0. As the result, after $2n$ time steps, all the variable nodes' opinions will also be updated to 0. In summary, for the SVC graph, we have constructed an initial condition $y(0)$ and an update sequence $\{c_1, \ldots, c_m, v_1, \tilde{v}_1, \ldots, v_n, \tilde{v}_n\}$, along which the weighted-median opinion dynamics achieve consensus on the opinion 0 at time $2n + m$. This concludes the proof for the sufficient part.

Regarding the necessary part, suppose that there exist $y(0) \in \mathbb{R}^{2n+m+1}$ and an update sequence that together satisfy statement 3) in Proposition 10 on an SVC graph $\mathcal{G}(W)$. We now prove that the vector x^*, with $x^*_{k_i^j} = y_{k_i^j}(0)$ for any $i \in \{1, \ldots, m\}$ and $j \in \{1, 2, 3\}$, must satisfy the rules for the associated monotone NAE3SAT.

For $y(0)$ and the update sequence given previously that lead to consensus on opinion 0, we first conclude that $y_s(0)$ must be 0, since the sink node s can never change its opinion. Then, we have $y_i(0) \in \{1, -1\}$ for any $i \in \{1, \ldots, 2n + m + 1\} \setminus \{s\}$. Denote by T_m the first time step at which the opinion of c_m is updated to 0. For any $i \in \{1, \ldots, n\}$, according to how the out links of the nodes v_i and $\tilde{v}_i$ are constructed, we know that, up to time T_m, none of the opinions of v_i and $\tilde{v}_i$'s out neighbors in $\mathcal{G}(W)$ has ever been updated to 0. That is, we have $y_{v_i}(t) \in \{-1, 1\}$ and $y_{\tilde{v}_i}(t) \in \{-1, 1\}$ for any $t \le T_m$. Moreover, since $\{v_i, \tilde{v}_i\}$ forms a cohesive set in $G(W)$, we have that $y_{v_i}(t) = -y_{\tilde{v}_i}(t) \neq 0$ for any $t \le T_m$. Otherwise, there exists some $t_i \le T_m$ such that $y_{v_i}(t) = y_{\tilde{v}_i}(t) \neq 0$ for any $t \ge t_i$, which will contradict the pre-assumption that $y(t)$ eventually achieves consensus on opinion 0. Furthermore, since $y_{v_i}(t) = -y_{\tilde{v}_i}(t) \neq 0$ remains true for all $t \le T_m$ and only one node can update its opinion at each time, we have that neither node v_i nor node $\tilde{v}_i$ has ever updated their opinion before T_m. Therefore, for any $i \in \{1, \ldots, n\}$ and any $t \le T_m$

$$y_{v_i}(t) = y_{v_i}(0) = -y_{\tilde{v}_i}(0) = -y_{\tilde{v}_i}(t).$$

Now we look at the node c_m. By assumption, node c_m's opinion is updated to 0 at time T_m, i.e., the weighted-median opinion for node c_m at time $T_m - 1$ is 0. In addition, since none of $y_{k_m^1}(T_m - 1)$, $y_{k_m^2}(T_m - 1)$ and $y_{k_m^3}(T_m - 1)$ is 0, the following two statements must simultaneously hold.

1) $y_{k_m^1}(T_m - 1)$, $y_{k_m^2}(T_m - 1)$, and $y_{k_m^3}(T_m - 1)$ are not all equal, which implies that $y_{k_m^1}(0)$, $y_{k_m^2}(0)$, and $y_{k_m^3}(0)$ are not all equal.
2) $y_{c_{m-1}}(T_m - 1) = 0$. Denote by T_{m-1} the first time that $y_{c_{m-1}}(t)$ is updated to 0. Then, we immediately have $T_{m-1} < T_m$.

Apply the same aforementioned argument to the clause nodes $c_{m-1}, \ldots, c_1$ sequentially, we have $T_1 < T_2 < \ldots < T_m$, and

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moreover, $y_{k_j^1}(0)$, $y_{k_j^2}(0)$, $y_{k_j^3}(0)$ are not all equal for any $j \in \{1,\dots,m\}$. Let x^* be such that

$$x^*_{k_i^j} = y_{k_i^j}(0)$$

for any $i \in \{1,\dots,m\}$ and any $j \in \{1,2,3\}$. Then, x^* satisfies the rules for the associated monotone NAE3SAT. This concludes the proof for the necessary part.

Now we have proved that for the specific class of SVC graphs, testing statement 3) in Proposition 10 is equivalent to finding a solution to the monotone NAE3SAT problem, which is known to be NP-hard. Suppose for any graph with arbitrary topology, testing statement 3) in Proposition 10 is not NP-hard, then the testing method can also be applied to the specific class of SVC graphs constructed, and thus, the latter is not NP-hard, which leads to a contradiction. Therefore, in general, testing statement 3) in Proposition 10 for any arbitrary graph is NP-hard. ■

Note that Theorem 12 does not exclude the possibility that, for some specific class of graphs, testing whether statement 3) in Theorem 12 holds is not an NP-hard problem.

IV. Conclusion

In this article, we conduct a thorough theoretical analysis of the rich dynamical behavior of the weighted-median opinion dynamics, previously exhibited via simulations in [3]. We characterize the set of all the equilibria and establish the almost-sure finite-time convergence of the dynamics. A necessary and sufficient condition for the almost-sure convergence and a sufficient condition for persistent dissensus are given. All these important behavior of the weighted-median opinion dynamics are related to some delicate structures of the influence networks, such as the cohesive sets and decisive links. The rich dynamical behavior revealed by the theoretical analysis in this article, together with other desirable properties discussed via experimental validation and numerical comparisons in [3], supports the weighted-median mechanism as a well-founded microfoundation of opinion dynamics. In terms of future research directions, various meaningful extensions could be made by incorporating the weighted-median mechanism with previous important extensions to the French–DeGroot model, e.g., the continuous-time weighted-median model, the persistent attachment to initial opinions or the presence of negative weights characterizing antagonistic relations.

Appendix A
Proof of Proposition 2

We prove the continuity of $\mathrm{Med}_i(x;W)$ by definition. For any $x \in \mathbb{R}^n$ and any $i \in \mathcal{V}$, suppose i has m out-neighbors (i itself is among these neighbors if i has a self-loop). Rank the opinions of i's neighbors in the ascending order as $x_{i_1} \le x_{i_2} \le \cdots \le x_{i_m}$. Suppose $\mathrm{Med}_i(x;W) = x_{i_r}$ for some $r \in \{1,\dots,m\}$. For any $\epsilon > 0$, let $\delta = \epsilon/2$. According to Lemma 1 and Definition 2, the following three cases are exhaustive.

Case 1: $w_{ii_1} + \cdots w_{ii_{r-1}} < 1/2$ and $w_{ii_1} + \cdots w_{ii_r} > 1/2$. According to Lemma 1, in this case, x_{i_r} is the unique weighted median of $(x_{i_1},\dots,x_{i_m})$ with respect to the weights $(w_{ii_1},\dots,w_{ii_m})$. For any $\Delta x \in \mathbb{R}^n$ with $\|\Delta x\|_\infty < \delta$, $x_{i_k} + \Delta x_{i_k} < x_{i_r} - \epsilon/2$ implies $k < r$, while $x_{i_k} + \Delta x_{i_k} > x_{i_r} + \epsilon/2$ implies $k > r$. Therefore

$$\sum_{k:\, x_k + \Delta x_k < x_{i_r} - \epsilon} w_{ik} \le \sum_{k<r} w_{ii_k} < 1/2$$

$$\sum_{k:\, x_k + \Delta x_k > x_{i_r} + \epsilon} w_{ik} \le \sum_{k>r} w_{ii_k} < 1/2.$$

According to Lemma 1, the aforementioned two inequalities implies that $\mathrm{Med}_i(x+\Delta x;W) \in (\mathrm{Med}_i(x;W) - \epsilon/2, \mathrm{Med}_i(x;W) + \epsilon/2)$.

Case 2: $w_{ii_1} + \cdots w_{ii_r} > 1/2$ and $w_{ii_1} + \cdots + w_{ii_{r-1}} = 1/2$. According to Definition 2, we have $x_i \ge x_{i_r}$ in this case. For any $\Delta x \in \mathbb{R}^n$ with $\|\Delta x\|_\infty < \delta$, similar to the argument for Case 1, we have

$$\sum_{k:\, x_k + \Delta x_k < x_{i_r} - \epsilon/2} w_{ik} \le \sum_{k<r} w_{ii_k} \le 1/2$$

$$\sum_{k:\, x_k + \Delta x_k > x_{i_r} + \epsilon/2} w_{ik} \le \sum_{k>r} w_{ii_k} < 1/2$$

and $x_i + \Delta x_i > x_{i_r} - \epsilon/2$. According to Lemma 1 and Definition 2, we have $\mathrm{Med}_i(x+\Delta x;W) \in (\mathrm{Med}_i(x;W) - \epsilon/2, \mathrm{Med}_i(x;W) + \epsilon/2)$.

Case 3: $w_{ii_1} + \cdots + w_{ii_r} = 1/2$. According to Definition 2, this implies that $x_i \le x_{i_r}$. For any $\Delta x \in \mathbb{R}^n$ with $\|\Delta x\|_\infty < \delta$, similar to the argument for Case 1, we have

$$\sum_{k:\, x_k + \Delta x_k < x_{i_r} - \epsilon/2} w_{ik} \le \sum_{k<r} w_{ii_k} < 1/2$$

$$\sum_{k:\, x_k + \Delta x_k > x_{i_r} + \epsilon/2} w_{ik} \le \sum_{k>r} w_{ii_k} \le 1/2$$

and $x_i + \Delta x_i < x_{i_r} + \epsilon/2$, which, according to Lemma 1 and Definition 2, implies that $\mathrm{Med}_i(x+\Delta x;W) \in (\mathrm{Med}_i(x;W) - \epsilon/2, \mathrm{Med}_i(x;W) + \epsilon/2)$.

The aforementioned three cases together imply that, for any $\Delta x \in \mathbb{R}^n$ with $\|\Delta x\|_\infty < \delta$, $|\mathrm{Med}_i(x+\Delta x;W) - \mathrm{Med}_i(x;W)| < \epsilon$. This concludes the proof. ■

Appendix B
Proof of Lemma 3

We prove the uniqueness of cohesive expansion by contradiction. For any cohesive set $\mathcal{M} \subseteq \mathcal{V}$, suppose that $\mathcal{E}_1 = \mathcal{M} \cup (i_1,\dots,i_k)$ and $\mathcal{E}_2 = \mathcal{M} \cup (j_1,\dots,j_\ell)$ are two distinct cohesive expansions of $\mathcal{M}$. Here, the ordered sets $(i_1,\dots,i_k)$ and $(j_1,\dots,j_l)$ indicate the orderings of node additions along the corresponding cohesive expansions. Since $\mathcal{E}_1 \neq \mathcal{E}_2$ means that one of these two sets must contain at least one element that is not in the other, without loss of generality, suppose that there exists some $s_0 \in \{1,\dots,l\}$ such that $j_{s_0} \notin (i_1,\dots,i_k)$. Now we prove that this is impossible.

First of all, s_0 cannot be 1, otherwise

$$\sum_{r \in \mathcal{E}_1} w_{j_1 r} \ge \sum_{r \in \mathcal{M}} w_{j_1 r} > 1/2$$

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implies that $\mathcal{E}_1$ can be further expanded to $\mathcal{E}_1 \cup (j_1)$. Second, there must exist $s_1 \in \{1, \ldots, s_0 - 1\}$ such that $j_{s_1} \notin (i_1, \ldots, i_k)$, otherwise $\mathcal{M} \cup (j_1, \ldots, j_{s_0-1}) \subseteq \mathcal{E}_1$ and

$$\sum_{r \in \mathcal{E}_1} w_{j_{s_0} r} \geq \sum_{r \in \mathcal{M} \cup (j_1, \ldots, j_{s_0-1})} w_{j_{s_0} r} > 1/2$$

which implies that $\mathcal{E}_1$ can be further expanded to $\mathcal{E}_1 \cup (j_{s_0})$. As the same argument goes on, we will obtain that $j_1 \notin (i_1, \ldots, i_k)$. But we have already shown in this paragraph that $j_1 \notin (i_1, \ldots, i_k)$ cannot be true. Therefore, $\mathcal{E}_1 \neq \mathcal{E}_2$ cannot be true. This concludes the proof. ■

Appendix C
Proof of Lemma 4

The proof of statement 1) is straightforward: For any $i \in \mathcal{M}_1 \cup \mathcal{M}_2$, since either $i \in \mathcal{M}_1$ or $i \in \mathcal{M}_2$, we have

$$\sum_{j \in \mathcal{M}_1 \cup \mathcal{M}_2} w_{ij} \geq \max \left\{ \sum_{j \in \mathcal{M}_1} w_{ij}, \sum_{j \in \mathcal{M}_2} w_{ij} \right\} \geq \frac{1}{2}.$$

Therefore, $\mathcal{M}_1 \cup \mathcal{M}_2$ is cohesive.

Let Expansion$(\mathcal{M}) = \mathcal{M} \cup (i_1, \ldots, i_k)$, where the indices $1, \ldots, k$ indicate the ordering of node additions along the cohesive expansion of $\mathcal{M}$. For any given $p \in \{1, \ldots, k\}$, according to Definition 4 and since $\mathcal{M} \subseteq \tilde{\mathcal{M}}$, we have

$$\sum_{j \in \tilde{M} \cup \{i_s : s < p\}} w_{i_p j} \geq \sum_{j \in \mathcal{M} \cup \{i_s : s < p\}} w_{i_p j} > \frac{1}{2}.$$

Therefore, all of the nods $i_1, \ldots, i_k$ are included in the cohesive expansion of $\tilde{M}$, which in turn implies Expansion$(\mathcal{M}) \subseteq$ Expansion$(\tilde{\mathcal{M}})$. This concludes the proof of Statement 2).

According to Lemma 3, since $\mathcal{M} \subseteq \mathcal{M} \cup \tilde{\mathcal{M}}$ and $\tilde{\mathcal{M}} \subseteq \mathcal{M} \cup \tilde{\mathcal{M}}$, we have Expansion$(\mathcal{M}) \subseteq$ Expansion$(\mathcal{M} \cup \tilde{\mathcal{M}})$ and Expansion$(\tilde{\mathcal{M}}) \subseteq$ Expansion$(\mathcal{M} \cup \tilde{\mathcal{M}})$. Therefore, Expansion$(\tilde{\mathcal{M}}) \cup$ Expansion$(\mathcal{M}) \subseteq$ Expansion$(\mathcal{M} \cup \tilde{\mathcal{M}})$. This concludes the proof of statement 3).

Now we prove statement 4). Let $\mathcal{M}$ be a cohesive set. Since each newly added node assigns more than 1/2 weights to the existing nodes in the cohesive expansion process, Expansion$(\mathcal{M})$ is also a cohesive set. Let $\hat{\mathcal{M}}$ be a maximal cohesive set containing $\mathcal{M}$, i.e., $\mathcal{M} \subseteq \hat{\mathcal{M}}$. By applying statement 3) of Lemma 4, we have

$$\text{Expansion}(\mathcal{M}) \subseteq \text{Expansion}(\hat{\mathcal{M}}).$$

In addition, since $\hat{\mathcal{M}}$ is already a maximal cohesive set, according to Definition 4, the cohesive expansion of $\hat{\mathcal{M}}$ is $\hat{\mathcal{M}}$ itself. Therefore, we have Expansion$(\mathcal{M}) \subseteq \hat{\mathcal{M}}$. Since $\hat{\mathcal{M}}$ is any arbitrary maximal cohesive set containing $\mathcal{M}$, in this sense, Expansion$(\mathcal{M})$ is the smallest maximal cohesive set containing $\mathcal{M}$. This concludes the proof. ■

Appendix D
Proof of Lemma 7

For any given $x(0) \in \mathbb{R}^n$, due to the definition of weighted median, we have $x(t) \in \Omega = \{x_1(0), \ldots, x_n(0)\}^n$ along any update sequence. Here, Ω is a finite set of at most n^n distinct elements. According to Definition 2, at any time $t+1$, one individual is uniformly randomly picked and update their opinion via the weighted-median mechanism. To put in mathematically, for any $x \in \Omega$

$$\text{Prob}[x(t+1) = x^{(i)} \,|\, x(t) = x] = 1/n \tag{1}$$

for any $x^{(i)} \in \Omega$ satisfying $x_i^{(i)} = \text{Med}_i(x; W)$ and $x_j^{(i)} = x_j$ for any $j \neq i$. Therefore, the weighted-median opinion dynamics is a Markov chain over the finite state space Ω. This Markov chain has absorbing states, i.e., all the equilibria characterized in Theorem 6. Moreover, according to the assumption of this lemma, for any $x \in \Omega$, there exists at least one update sequence along which the trajectory $x(t)$ starting from x reaches an equilibrium. Therefore, the weighted-median opinion dynamics is an absorbing Markov chain. According to [40, Th. 11.3], $x(t)$ starting from $x(0)$ almost surely converges to an equilibrium. Since the stochastic process $x(t)$ is a finite-state Markov chain, $x(t)$ reaches an equilibrium almost surely in finite time. ■

Appendix E
Proof of Proposition 10

"Statement 2) ⇒ Statement 1)": Suppose 2) is true, i.e., there exists $x_0 \in \mathbb{R}^n$ with n distinct entries and an update sequence $\{i_1, \ldots, i_T\}$ such that, along this sequence, $x(t)$ starting from $x(t) = x_0$ achieves consensus at time T. Let $r = \min_{i \neq j} |x_{0,i} - x_{0,j}|$. Since the entries of x_0 are all distinct, we have $r > 0$. Let $X_0 = \left\{x \in \mathbb{R}^n \,\middle|\, \|x - x_0\|_\infty < r/2\right\}$. For any $\tilde{x}_0 \in X_0$, the entries of $\tilde{x}_0$ have the same ordering as the entries of x_0 in terms of their values. That is, if x_0 satisfies $x_{0,j_1} \leq x_{0,j_2} \leq \cdots \leq x_{0,j_n}$, where $\{j_1, \ldots, j_n\}$ is a permutation of $\{1, \ldots, n\}$, then $\tilde{x}_0$ also satisfies $\tilde{x}_{0,j_1} \leq \tilde{x}_{0,j_2} \leq \cdots \leq \tilde{x}_{0,j_n}$. According to the definition of weighted median, for any $x \in \mathbb{R}^n$, if $\text{Med}_i(x; W) = x_j$ for some j, then $\text{Med}_i(\tilde{x}; W) = \tilde{x}_j$ as long as the entries of $\tilde{x}$ have the same ordering as those of x. That is, in the weighted-median opinion dynamics, only the ordering of the individual opinions matters. Therefore, for any $\tilde{x}_0 \in X_0$, the trajectory $x(t)$ starting with $x(0) = \tilde{x}_0$ and along the update sequence $\{i_1, \ldots, i_T\}$ also achieves consensus at time T. Therefore, statement 2) implies statement 1).

"Statement 1) ⇒ Statement 2)": Suppose statement 1) is true, i.e., there exists $X_0 \subseteq \mathbb{R}^n$ with positive measure in $\mathbb{R}^n$ such that, for any $x_0 \in X_0$, there exists an update sequence, along which the trajectory $x(t)$ with $x(0) = x_0$ achieves consensus in finite time. Since X_0 has positive measure in $\mathbb{R}^n$, there must exists $x_0 \in X_0$ such that the entries of x_0 are all distinct. For this x_0, there exists an update sequence $\{i_1, \ldots, i_T\}$ such that $x(t)$ starting with $x(0) = x_0$ achieves consensus at time T. That is, statement 1) implies 2).

"Statement 2) ⇒ Statement 3)": Given a vector x satisfying the conditions in statement 2) and the corresponding update sequence $\{i_1, \ldots, i_T\}$, since $x(t) \in \{x_1(0), \ldots, x_n(0)\}^n$ for any $t \in \mathbb{N}$, there exists $i \in \mathcal{V}$ such that $x_j(T) = x_i$ for any j, i.e., $x(T)$ achieves consensus at the value x_i. For any $t \in \mathbb{N}$, define

$$\mathcal{V}_1(t) = \left\{j \in \mathcal{V} \middle| \, x_j(t) < x_i\right\}$$

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$$\mathcal{V}_2(t) = \left\{ j \in \mathcal{V} \middle| \, x_j(t) > x_i \right\}.$$

Now define $y \in \{-1, 0, 1\}^n$ as follows: For the index i specified previously, let $y_i = 0$. Let $y_j = -1$ for any $j \in \mathcal{V}_1(0)$ and $y_j = 1$ for any $j \in \mathcal{V}_2(0)$. Consider the solution $y(t)$ to the weighted-median opinion dynamics starting with $y(0) = y$ and along the update sequence $\{i_1, \ldots, i_T\}$. Define

$$\mathcal{U}_1(t) = \left\{ j \in \mathcal{V} \middle| \, y_j(t) = -1 \right\}$$
$$\mathcal{U}_2(t) = \left\{ j \in \mathcal{V} \middle| \, y_j(t) = 1 \right\}.$$

By definition, $\mathcal{U}_1(0) = \mathcal{V}_1(0)$ and $\mathcal{U}_2(0) = \mathcal{V}_2(0)$. Now we prove $\mathcal{V}_1(t) = \mathcal{U}_1(t)$ and $\mathcal{V}_2(t) = \mathcal{U}_2(t)$ for any $t \in \mathbb{N}$. Without loss of generality, suppose $i_1 \in \mathcal{V}_1(0)$ (otherwise we can switch the definitions of $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$). At time step 1, only node i_1 updates her opinion. Since

$$i_1 \in \mathcal{V}_2(1) \;\Leftrightarrow\; \sum_{j \in \mathcal{V}_2(0)} w_{i_1,j} > \frac{1}{2} \;\Leftrightarrow\; i_1 \in \mathcal{U}_2(1)$$

$$i_1 \in \mathcal{V}_1(1) \;\Leftrightarrow\; \sum_{j \in \mathcal{V}_1(0)} w_{i_1,j} \leq \frac{1}{2} \;\Leftrightarrow\; i_1 \in \mathcal{U}_1(1)$$

$$i_1 \in \mathcal{V} \backslash \left(\mathcal{V}_1(1) \cup \mathcal{V}_2(1) \right) \;\Leftrightarrow\; x_{i_1}(1) = x_i$$

$$\Leftrightarrow \sum_{j \in \mathcal{V} \backslash \mathcal{V}_1(0)} w_{i_1,j} > \frac{1}{2} \sum_{j \in \mathcal{V}_2(0)} w_{i_1,j} < \frac{1}{2}$$

$$\Leftrightarrow\; y_{i_1}(1) = 0 \;\Leftrightarrow\; i_1 \in \mathcal{V} \backslash \left(\mathcal{U}_1(1) \cup \mathcal{U}_2(1) \right)$$

we have $\mathcal{V}_1(1) = \mathcal{U}_1(1)$ and $\mathcal{V}_2(1) = \mathcal{U}_2(1)$. Following the same argument for $t = 2, 3, \ldots$, we conclude that $\mathcal{V}_1(t) = \mathcal{U}_1(t)$ and $\mathcal{V}_2(t) = \mathcal{U}_2(t)$ for any t. Therefore

$$\mathcal{V}_1(T) = \mathcal{V}_2(T) \text{ is empty } \;\Leftrightarrow\; \mathcal{U}_1(T) = \mathcal{U}_2(T) \text{ is empty}$$

that is, $y_j(T) = 0$ for any $j \in \mathcal{V}$. This concludes the proof for "statement 2) $\Rightarrow$ statement 3)."

"Statement 3) $\Rightarrow$ Statement 2)": Given $y \in \{-1, 0, 1\}$ satisfying the conditions in statement 3) and given the corresponding update sequence $\{i_1, \ldots, i_T\}$, define $\mathcal{U}_1(t)$ and $\mathcal{U}_2(t)$ in the same way as aforementioned. For any $x \in \mathbb{R}^n$ that satisfies the following conditions:

1) $x_i = 0$;
2) $x_j < 0$ for any $j \in \mathcal{U}_1(0)$;
3) $x_j > 0$ for any $j \in \mathcal{U}_2(0)$;
4) all the entries of x are distinct.

Define $\mathcal{V}_1(t)$ and $\mathcal{V}_2(t)$ in the same way as in last paragraph. Following the same argument in the proof for "2) $\Rightarrow$ 3)," we obtain that $\mathcal{V}_1(T) = \mathcal{U}_1(T)$ and $\mathcal{V}_2(T) = \mathcal{U}_2(T)$ are all empty. That is, the vector x we construct satisfies the conditions for x_0 in statement 2). This concludes the proof for "3) $\Rightarrow$ 2)." ■

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