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LETTERS

A new measure of mortality differentials based on precedence probability

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Abstract
This short note aims to propose a new comparison measure for life tables: the age shift needed to restore equality of survival chance as measured by the precedence probability. Mortality by household income in France and by generation in Belgium is used as an illustration.

Keywords Stochastic precedence · Life tables · Mortality projection · Mortality differentials

1 Precedence probability

Comparing the respective performance of competing systems is of particular interest in reliability analysis. Stochastic precedence has been proposed as an effective tool in that respect. This concept is based on the relevant probability $\mathrm{P}[T_1 \leq T_2]$, where T_1 and T_2 denote the respective system lifetimes. Specifically, T_1 is said to stochastically precede T_2 when $\mathrm{P}[T_1 \leq T_2] \geq 0.5$. This definition can be traced back (at least) to Brown and Rutemiller [4]. For continuous, mutually independent or conditionally independent random variables T_1 and T_2, this holds true if, and only if, $\mathrm{P}[T_1 \leq T_2] \geq \mathrm{P}[T_1 \geq T_2]$ so that it is more likely that the first system has broken down while the second one was still working. If T_1 and T_2 are independent and identically distributed, we obviously have $\mathrm{P}[T_1 \leq T_2] = 0.5$.

The precedence probability $\mathrm{P}[T_1 \leq T_2]$ allows the analyst to classify the second system as better than, equivalent to or worse than the first system according to whether this probability is greater than, equal to or less than 0.5. It is rooted in the history of nonparametric testing, corresponding to the expected value of the

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celebrated Mann-Whitney statistic used to test the equality of two distributions. This concept is relevant in numerous engineering applications, such as stress-strength. It also naturally appears in the target-based approach to decision under risk, as pointed out by De Santis et al. [8]. Other references about stochastic precedence include Arcones et al. [1], Boland et al. [3], Navarro and Rubio [12] and Finkelstein [9].

The precedence probability can be used to compare life tables, considering independent remaining lifetimes $T_x^{[1]}$ and $T_x^{[2]}$ at attained age x obeying each of them. If $\mathrm{P}\big[T_x^{[1]} \leq T_x^{[2]}\big] \geq 0.5$ then the second life table is associated with better survival. Being based on outliving probabilities, stochastic precedence conforms with intuition. In this letter, we propose a related concept to compare life tables.

2 Mortality differentials

2.1 Age shift restoring equality of survival chance

Age shifting has a long tradition in life insurance. Often called Rueff's adjustment (see [13], Section 4.4.3), it consists in carrying out calculations not with the actual age x but with an adjusted age $x-\delta$. Here, δ is the age correction accounting for the switch from the reference life table to the mortality of the population of interest. The shift δ can be constant, depend on age x or on the year of birth, for instance. Age shifting correction is applied in regulatory life tables applied in France and Italy.

As regards the determination of the age-shift function, various criteria have been adopted. We just mention that most criteria are based on the analysis of the actuarial or demographic values (life annuities or life expectancies for instance) calculated using the appropriate probabilities and, respectively, using the probabilities obtained by age shifting, with the aim of minimizing the "distance" (conveniently defined) between the sets of actuarial values. We refer the reader to Section 3.4.5 in [13] where optimal age shifts are determined by minimizing the sum of squared differences between life expectancies at age x computed from insurance market data and reference one at shifted age $x-\delta$ or by Poisson likelihood maximization (looking for the age shift δ making market mortality statistics the most probable). The new measure of mortality differentials proposed in this paper follows a similar approach, defining the age shift as the one restoring fairness among the two groups under consideration, that is, ensuring equal probability of last survival.

Equality of chance (or opportunity) has become an important concept in public policy. Reaching a precedence probability $\mathrm{P}\big[T_x^{[1]} \leq T_x^{[2]}\big] = 0.5$ for two individuals aged x in a population of interest is thus appealing in that respect since it ensures that none of them is more likely to survive to the other. When the two individuals are taken from different socio-economic groups, the precedence probability might well differ from 0.5. Therefore, we suggest to use the age shift needed to restore equality of chance as mortality differential indicator. Precisely, for two remaining lifetimes such that $\mathrm{P}\big[T_x^{[1]} \leq T_x^{[2]}\big] \geq 0.5$, we define the age shift τ_x as the root of the equation

$$\mathrm{P}\left[T_x^{[1]} \le T_x^{[2]} - \tau | T_x^{[2]} > \tau\right] = 0.5 \quad \Leftrightarrow \quad \mathrm{P}\left[T_x^{[1]} \le T_{x+\tau}^{[2]}\right] = 0.5. \tag{1}$$

In words, considering an individual aged x taken from population 1, with lifetime $T_x^{[1]}$, the inequality $\mathrm{P}\left[T_x^{[1]} \le T_x^{[2]}\right] \ge 0.5$ means that he or she is less likely to live longer than an individual of the same age taken from population 2, with lifetime $T_x^{[2]}$. In order to ensure that both individuals have the same chance of being the last survivor (i.e. to restore equality of survival chances, in brief), we must consider an individual from population 2 not aged x but aged $x + \tau_x$. Within this pair of individuals, $\mathrm{P}\left[T_x^{[1]} \le T_{x+\tau}^{[2]}\right] = 0.5$ ensures that none of them is more likely to be the last survivor. The larger τ_x, the more population 2 outperforms population 1 in terms of longevity. Considering the evolution of τ_x as a function of attained age x is particularly instructive to detect whether mortality gaps tend to accentuate with age, or not.

Henceforth, we use a superscript "[k]" to indicate quantities relating to $T_x^{[k]}$, $k \in \{1, 2\}$, such as

$${}_tq_x^{[k]} = \mathrm{P}\left[T_x^{[k]} \le t\right] = 1 - {}_tp_x^{[k]} \text{ and } \mu_{x+t}^{[k]} = -\frac{\partial}{\partial t} \ln {}_tp_x^{[k]}.$$

Recall that life expectancy at attained age x, denoted as $\overline{e}_x^{[k]}$, is defined as

$$\overline{e}_x^{[k]} = \mathrm{E}\left[T_x^{[k]}\right] = \int_0^{\omega - x} t \quad {}_tp_x^{[k]} \quad \mu_{x+t}^{[k]} \quad \mathrm{d}t = \int_0^{\omega - x} {}_tp_x^{[k]} \quad \mathrm{d}t$$

where ω denotes the ultimate age for both lifetimes under consideration, possibly infinite. Comparing life expectancies is a convenient way to compare the mortality experienced by two groups of individuals. As every mathematical expectation, the use of life expectancies is legitimated by the law of large numbers. This concept differs from the median in that we do not control the percentage of individuals still alive at age $x + \overline{e}_x^{[k]}$, among those attaining age x. Also, even if $\overline{e}_x^{[1]} = \overline{e}_x^{[2]}$ holds true, the probability $\mathrm{P}\left[T_x^{[1]} \le T_x^{[2]}\right]$ is not necessarily equal to 0.5.

Let us now provide the reader with an example helping to figure out the relative merits of difference in life expectancies and age shifts: heterogeneity in survival and fairness of retirement age in public pension regimes. Assume that the population is divided into two homogeneous groups in terms of survival, white collars (group 1, with lifetime $T_x^{[1]}$) and blue collars (group 2, with lifetime $T_x^{[2]}$), say. White collars benefit from lower mortality rates at all ages. We do not discuss pension amounts here, only the time spent after retirement. Setting the same retirement age x_r for both groups may be considered as being unfair since blue collars tend to have shorter remaining lifetimes. To restore fairness, we could increase the retirement age for white collars by δ such that

$$\overline{e}_{x_r}^{[1]} = \overline{e}_{x_r+\delta}^{[2]}.$$

This ensures that within a sufficiently large population, the total time lived after retirement by all blue collars is equal to the total time lived after retirement by all white collars. This is a collective viewpoint, based on the law of large numbers classically adopted by actuaries. Another way to restore fairness, in line with (1), is by defining δ as the root of

$$\mathrm{P}\left[T^{[1]}_{x_r} \leq T^{[2]}_{x_r+\delta}\right] = 0.5.$$

Now, the shift δ guarantees that for every pair of retiring individuals, one blue collar and one white collar, none of them has an advantage over the other in terms of time spent after retirement. This alternative viewpoint, less common among actuaries is thus more individual. We refer the reader e.g. to [6] as well as to the French-written paper by [7] for a detailed analysis of the use of this kind of concept in the law.

2.2 Independent case

In this section, we assume that the remaining lifetimes under consideration are mutually independent. This hypothesis may be regarded as being reasonable for two individuals in different social classes in a period-based approach, for instance. The probability involved in (1) can then be written as

$$\mathrm{P}\left[T^{[1]}_{x} \leq T^{[2]}_{x+\tau}\right] = 1 - \int_0^{\omega-x-\tau} {}_tp^{[2]}_{x+\tau} \quad \mu^{[2]}_{x+\tau+t} \quad {}_tp^{[1]}_{x} \quad \mathrm{d}t.$$

A grid search can be performed to identify the integer part of τ_x. Its fractional part can then be obtained by interpolation. Different fractional age assumptions can be used to this end, depending on age x under consideration.

2.3 Conditionally independent case

In this section, we assume that $T^{[1]}_x$ and $T^{[2]}_x$ are only conditionally independent, given one or several time factors. This corresponds to the case where $T^{[1]}_x$ and $T^{[2]}_x$ obey some mortality projection model depending on future trajectory of time factors $\boldsymbol{\kappa}^{[1]}$ and $\boldsymbol{\kappa}^{[2]}$, respectively. For instance, $T^{[1]}_x$ and $T^{[2]}_x$ may refer to male and female lifetimes in a given country and $\boldsymbol{\kappa}^{[1]}$ and $\boldsymbol{\kappa}^{[2]}$ represent the gender-specific time factors in a bivariate Lee-Carter model. Or $T^{[1]}_x$ and $T^{[2]}_x$ may relate to lifetimes in two different countries in a multi-population model, or to lifetimes in two different generations.

The formula in the preceding section can then be applied conditional to $\left(\boldsymbol{\kappa}^{[1]}, \boldsymbol{\kappa}^{[2]}\right)$. In a period-based approach, a limited number of time factors enter the calculation (just two in the Lee-Carter case, for instance) and the integrals can easily be evaluated numerically. Calculations are more difficult when cohorts are followed over time since whole trajectories of $\left(\boldsymbol{\kappa}^{[1]}, \boldsymbol{\kappa}^{[2]}\right)$ enter the calculation, resulting in a large number of conditioning variables in the probability (1). Simulation can then be conducted to obtain the age shift τ_x. A first approximation of τ_x, say $\tau^{[0]}_x$, can be obtained by replacing the time factors $\left(\boldsymbol{\kappa}^{[1]}, \boldsymbol{\kappa}^{[2]}\right)$ with their expected values. We can then simulate a large number of pairs $\left(T^{[1]}_x, T^{[2]}_x\right)$ by first generating a trajectory of future $\left(\boldsymbol{\kappa}^{[1]}, \boldsymbol{\kappa}^{[2]}\right)$ and then simulate remaining lifetimes according to the life tables corresponding to these time factors values. The probability in (1) can be approached by the corresponding proportion and the value of τ_x can be obtained by a grid search for values close to $\tau_x^{[0]}$.

As another example of conditional independence, let us mention shared or correlated frailty models. These models account for heterogeneity in survival by including random effects multiplying baseline forces of mortality. Calculations are carried out in a similar way, conditioning on the random effects instead of time factors.

3 Numerical illustrations

3.1 Mortality according to household total income in France

The French National Institute of Statistics INSEE has published several studies of mortality by social class, educational levels and living standards. In this section, we consider the life tables produced by [2] based on the French permanent demographic sample (EDP), which incorporates tax data. Households are divided in 20 classes according to the quantiles of their total income over 2012–2016.

Consider for $T_x^{[1]}$ a remaining lifetime subject to the mortality for the poorest 5% of the population and for $T_x^{[2]}$ a remaining lifetime subject to the mortality for the richest 5% of the population. Here, we assume that $T_x^{[1]}$ and $T_x^{[2]}$ are mutually independent. We then get $\mathrm{P}\big[T_x^{[1]} \leq T_x^{[2]}\big] > 0.5$ for all ages x and both males and females. The left panel in Fig. 1 displays the age shifts τ_x restoring equal survival chances between these two groups for attained ages x from birth (entire lifetime) to 90. We can see there that the age shifts are pretty large and decrease with age. Life expectancies at birth in these two groups are 71.7 and 84.4 for males and 80.0 and 88.3 for females. The age shifts at birth equal to 11.7 years for males and 6.3 years for

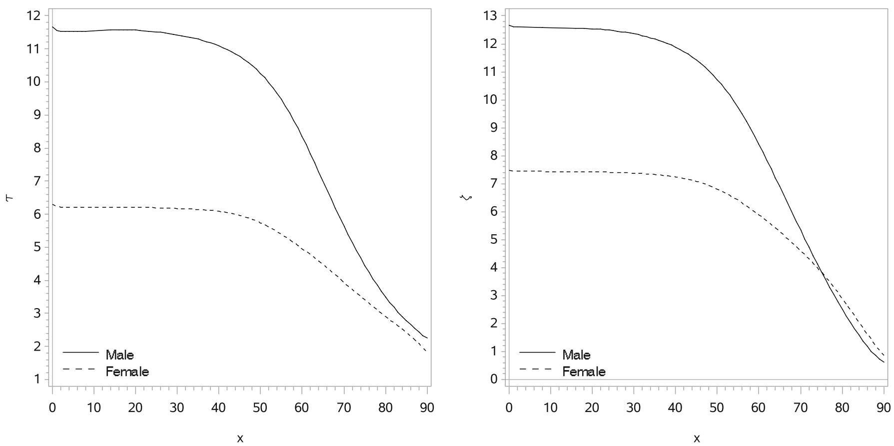

Fig. 1 Age shifts τ_x for $x \in \{0, 1, \dots, 90\}$ solving equation (1) in the left panel and medians ζ_x solving (2) in the right panel when $T_x^{[1]}$ corresponds to a lifetime subject to the mortality for the poorest 5% and $T_x^{[2]}$ to a lifetime subject to the mortality for the richest 5% of the French population. Values obtained from the life tables given in [2] by assuming mutual independence

females thus compare with the differences of these life expectancies. In words, this means that we have to pair a male newborn in the poorest 5% of the population with a boy aged 11.7 in the richest 5% of the population to ensure that both of them have equal chances of being the last survivor. We see that age shifts remain stable until age 30, because of the low mortality risk at these ages, but start to drop at older ages. Expected remaining lifetimes at age 65 in these two groups are 15.8 and 21.8 for males and 20.4 and 25.0 for females, of the same order of age shifts equal to 6.8 years and 4.4 years, respectively. Contrarily to differences in life expectancies, the meaning of age shifts is clearer as they represent the comparative advantage of the higher socio-economic group if equality of survival chance is regarded as a meaningful objective. Notice that the fact that age shifts tend to disappear at high ages is an illustration of the convergence principle.

For the sake of comparison, we also display in the right panel in Fig. 1 the values of ζ_x solving

$$\mathrm{P}\big[T_x^{[2]} - T_x^{[1]} > \zeta_x\big] = 0.5, \tag{2}$$

that is, the median of the difference in lifetimes $T_x^{[2]} - T_x^{[1]}$. In words, individuals with lifetime $T_x^{[2]}$ live a period ζ_x longer than individuals with lifetimes $T_x^{[1]}$ with probability 0.5 (whereas τ_x is the age increase to be added to the individual benefiting from better longevity to restore equality of survival chances). We can see that the values of ζ_x are slightly larger than the corresponding values of τ_x but that the age patterns are very similar for both ζ_x and τ_x. Both indicators have their respective merits, depending on the problem under consideration.

3.2 Mortality projection for Belgium

In this section, we consider mortality statistics for Belgium, general population. Observed mortality rates and corresponding exposures are available from https://www.plan.be/databases/. Here, we consider calendar years 1991–2018 and ages 30–109. Future mortality is projected using the models of [5], henceforth referred to as "CD" and of [11] , henceforth referred to as "LC". Package `StMoMo` contributed by [14] is used for applying the LC method and the function `auto.arima` in package `forecast` developed by [10] to forecast time series. Both packages are available in the `R` programming environment.

As an illustration, we consider calendar years $\{1992, 2005, 2018\}$ spanning the observation period under consideration and we follow generations over time so that the lifetimes under consideration are only conditionally independent, given the time factor involved in the CD or LC model. Precisely, we consider individuals aged $x \in \{30, 31, \ldots, 90\}$ in calendar year r, $r \in \{1992, 2005\}$, and we compare their survival with individuals reaching the same age x in calendar year $r + 13 \in \{2005, 2018\}$. The results are displayed in Fig. 2. The upper panels display the approximations $\tau_x^{[0]}$ obtained by replacing the time factors with their expected values whereas the age shifts appearing in the lower panels have been obtained by simulation.

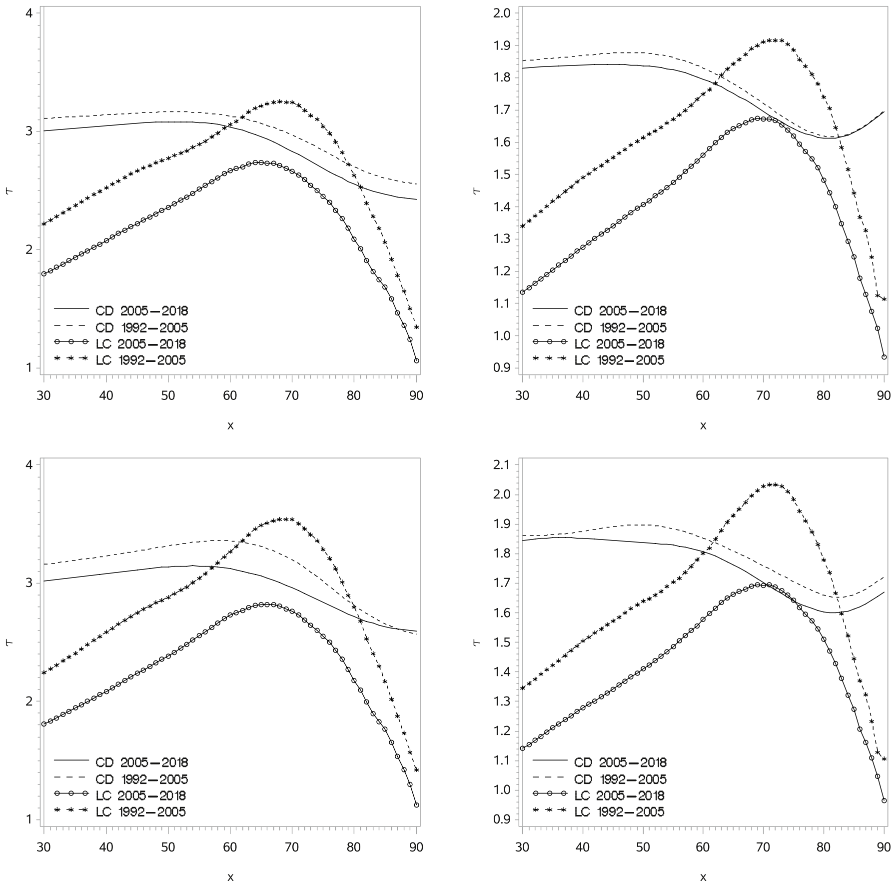

Fig. 2 Age shifts τ_x for $x \in \{30, 31, \ldots, 90\}$ solving equation (1) when $T_x^{[1]}$ corresponds to a lifetime subject to the mortality for generation aged x in calendar year r, $r \in \{1992, 2005\}$, and $T_x^{[2]}$ to a lifetime subject to the mortality for generation aged x in calendar year $r + 13$, Belgian general population. Initial approximations $\tau_x^{[0]}$ are displayed in the top panels, solutions τ_x to (1) obtained by simulation are displayed in the bottom panels. Males appear on the left, females on the right

We see there that the approximations $\tau_x^{[0]}$ are very close to the true values τ_x solving equation (1). It is also visible that the choice of the mortality projection model impacts on the age shifts restoring equality of survival chance, except for the middle ages 60–80 where both models produce similar τ_x values. Considering the magnitude of age shifts as a measure of mortality improvement, we see that CD forecasts more progress at younger ages (before 50) and at older ages, compared to LC. The graphs in Fig. 2 thus help to assess the longevity dynamics across ages. Notice that the differences in age shifts associated to models CD and LC are substantial. This may be attributed to the differences in mortality projections produced by the two models, as documented in [5]. Considering Figure 3.12

in that paper, we can see that mortality projections greatly differ between CD and LC, which materializes here in different patterns for the corresponding age shifts.

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