Isotopisms of nilpotent Leibniz algebras

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Part 1: Introduction to Leibniz algebras

- Let $\mathbb{F}$ be a field with $char(\mathbb{F}) \neq 2$. A *left Leibniz algebra* over $\mathbb{F}$ is a vector space $\mathfrak{g}$ over $\mathbb{F}$ endowed of a bilinear map (called *commutator* or *bracket*) $[-,-] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ which satisfies the *left Leibniz identity*

$$[x,[y,z]] = [[x,y],z] + [y,[x,z]] \;\; \forall x,y,z \in \mathfrak{g}.$$

- We can define a *right Leibniz algebra*, using the right Leibniz identity

$$[[x,y],z] = [[x,z],y] + [x,[y,z]] \;\; \forall x,y,z \in \mathfrak{g}.$$

- A Leibniz algebra that is both left and right is called *symmetric Leibniz algebra*.
- We assume that $\dim_{\mathbb{F}} \mathfrak{g} < \infty$.

- Every Lie algebra is a Leibniz algebra and every Leibniz algebra with skew-symmetric commutator is a Lie algebra.
- There is an adjunction between the category $\mathbf{LieAlg}_{\mathbb{F}}$ and the category $\mathbf{LeibAlg}_{\mathbb{F}}$.
- The left adjoint of the inclusion functor

$$\mathrm{i} : \mathbf{LieAlg}_{\mathbb{F}} \to \mathbf{LeibAlg}_{\mathbb{F}}$$

is

$$\pi : \mathbf{LeibAlg}_{\mathbb{F}} \to \mathbf{LieAlg}_{\mathbb{F}}$$

defined by

$$\mathfrak{g} \to \frac{\mathfrak{g}}{\mathrm{Leib}(\mathfrak{g})}$$

- The *Leibniz kernel*

$$\mathrm{Leib}(\mathfrak{g}) = \mathrm{Span}_{\mathbb{F}}\left\{[x,x] \,|\, x \in \mathfrak{g}\right\}$$

  is is the smallest bilateral ideal of $\mathfrak{g}$ such that $\mathfrak{g}/\,\mathrm{Leib}(\mathfrak{g})$ is a Lie algebra.

- The *left* and the *right center* of $\mathfrak{g}$ are

$$\mathrm{Z}_l(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [x,\mathfrak{g}] = 0\}$$

$$\mathrm{Z}_r(\mathfrak{g}) = \{x \in \mathfrak{g} \,|\, [\mathfrak{g},x] = 0\}$$

- The *center of* $\mathfrak{g}$ is $\mathrm{Z}(\mathfrak{g}) = \mathrm{Z}_l(\mathfrak{g}) \cap \mathrm{Z}_r(\mathfrak{g})$.
- $\mathrm{Leib}(\mathfrak{g}) \subseteq \mathrm{Z}_l(\mathfrak{g})$.

Solvable and Nilpotent Leibniz algebras

Definition

Let $\mathfrak{g}$ be a left Leibniz algebra and let

$$\mathfrak{g}^0 = \mathfrak{g},\ \mathfrak{g}^{k+1} = [\mathfrak{g}^k, \mathfrak{g}^k],\ \forall k \geq 0,$$

be the *derived series of* $\mathfrak{g}$. $\mathfrak{g}$ is $n-$*step solvable* if $\mathfrak{g}^{n-1} \neq 0$ and $\mathfrak{g}^n = 0$.

Definition

Let $\mathfrak{g}$ be a left Leibniz algebra and let

$$\mathfrak{g}^{(0)} = \mathfrak{g},\ \mathfrak{g}^{(k+1)} = [\mathfrak{g}, \mathfrak{g}^{(k)}],\ \forall k \geq 0,$$

be the *lower central series of* $\mathfrak{g}$. $\mathfrak{g}$ is $n-$*step nilpotent* if $\mathfrak{g}^{(n-1)} \neq 0$ and $\mathfrak{g}^{(n)} = 0$.

Levi decomposition

Theorem

Let $\mathbb{F}$ *be a field with* $\mathrm{char}(\mathbb{F}) = 0$ *and let* $\mathfrak{g}$ *be a Leibniz algebra over* $\mathbb{F}$*. Then there exists a Leibniz subalgebra* $\mathfrak{s}$ *(which is a semisimple Lie algebra) of* $\mathfrak{g}$ *such that*

$$\mathfrak{g} = \mathfrak{s} \oplus_s \mathfrak{r}$$

where $\mathfrak{r} = \mathrm{rad}(\mathfrak{g})$ *is the radical of* $\mathfrak{g}$*.*

Part 2: Two-step nilpotent Leibniz algebras

Proposition

If $\mathfrak{g}$ *is a left two-step nilpotent Leibniz algebra, then* $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathsf{Z}(\mathfrak{g})$ *and* $\mathfrak{g}$ *is symmetric.*

Proposition

If $\mathfrak{g}$ *is a left nilpotent Leibniz algebra with* $\dim_{\mathbb{F}} \mathfrak{g}^{(1)} = 1$*, then* $\mathfrak{g}^{(1)} \subseteq \mathsf{Z}(\mathfrak{g})$ *and* $\mathfrak{g}$ *is symmetric.*

- Let $\mathfrak{g}$ be a two-step nilpotent Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = Span_{\mathbb{F}}\{z_1, \ldots, z_t\}$, then

$$[x, y] = \phi_1(x, y)z_1 + \cdots + \phi_t(x, y)z_t, \ \ \forall x, y \in \mathfrak{g}$$

  where $\phi_i : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ is a bilinear form, $\forall i = 1, \ldots, t$.

- Our purpose is to reduce such a given bilinear form to a suitable canonical representation.

- Let $\mathfrak{g}$ be a nilpotent Leibniz algebra with $[\mathfrak{g}, \mathfrak{g}] = \mathbb{F}z$ and

$$[x, y] = \phi(x, y)z, \ \ \forall x, y \in \mathfrak{g}$$

- If $\mathfrak{g}$ is not a Lie algebra, then $0 \neq \mathrm{Leib}(\mathfrak{g}) = [\mathfrak{g}, \mathfrak{g}] \subseteq \mathrm{Z}(\mathfrak{g})$.
- $\phi = \phi^- + \phi^+$, where

$$\phi^- = \frac{\phi - \phi^t}{2}, \ \ \phi^+ = \frac{\phi + \phi^t}{2}.$$

- We use the simultaneous reduction to canonical form of a pair of bilinear forms.

Definition

A Kronecker module is a quadruple (V_1, V_2, f_1, f_2), where V_1, V_2 are vector spaces over $\mathbb{F}$ and $f_1, f_2 : V_1 \to V_2$ are linear maps.

- Given a pair of bilinear forms $\phi^-, \phi^+ : \mathfrak{g} \times \mathfrak{g} \to \mathbb{F}$ as above, we define the Kronecker module $\left(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+}\right)$, where

$$\overline{\phi^-}(x) : z \mapsto \phi^-(x, z), \quad \overline{\phi^+}(y) : z \mapsto \phi^+(y, z), \ \forall z \in \mathfrak{g}.$$

- $\mathfrak{g}$ is *decomposable* if $\mathfrak{g} = U \oplus U^\perp$.
- We can assume that $\mathfrak{g}$ is indecomposable.

The indecomposable modules $\left(\mathfrak{g}, \mathfrak{g}^*, \overline{\phi^-}, \overline{\phi^+}\right)$ turn out to be one of the following three pairs

$$\begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}, \begin{pmatrix} 0 & C_{f(x)^m} \\ C^t_{f(x)^m} & 0 \end{pmatrix} \tag{1}$$

$$\begin{pmatrix} 0 & J_0 \\ -J_0^t & 0 \end{pmatrix}, \begin{pmatrix} 0 & I_n \\ I_n & 0 \end{pmatrix} \tag{2}$$

$$\begin{pmatrix} 0 & J_1 \\ -J_1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & J_2 \\ J_2^t & 0 \end{pmatrix} \tag{3}$$

where $f(x) \in \mathbb{F}[x]$ is a monic irreducible polyonomial and J_1, J_2 are the matrices associated with

$$F_1, F_2 : \mathbb{F}^n \to \mathbb{F}^{n+1}$$

defined by

$$F_1(x_1, \ldots, x_n) = (x_1, \ldots, x_n, 0),$$
$$F_2(x_1, \ldots, x_n) = (0, x_1, \ldots, x_n).$$

Definition

Let $f(x) \in \mathbb{F}[x]$ be a monic irreducible polynomial. Let $n \in \mathbb{N}$ and let $A = C_{f(x)^n}$. The *Heisenberg* Leibniz algebra $\mathfrak{l}_{2n+1}^A$ is the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type (**??**).

- If $A = (a_{ij}) \in \mathrm{M}_n(\mathbb{F})$ and $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^A$, then the non-trivial commutators are

$$[e_i, f_j] = (\delta_{ij} + a_{ij})\,h, \ [f_j, e_i] = (-\delta_{ij} + a_{ij})\,h, \ \forall i, j = 1, \ldots, n$$

- If $(a_{ij}) = 0$, then we obtain the classical Heisenberg algebra $\mathfrak{h}_{2n+1}$.

Definition

Let $n \in \mathbb{N}$ and let $A = C_{x^n}$. The *Kronecker* Leibniz algebra $\mathfrak{k}_n$ is the $(2n+1)$-dimensional indecomposable Leibniz algebra with associated Kronecker module of type (**??**).

Definition

The *Dieudonné* Leibniz algebra $\mathfrak{d}_n$ is the $(2n+2)-$dimensional Leibniz algebra with associated Kronecker module of type (**??**).

The case $\mathbb{F} = \mathbb{C}$

Let $n \in \mathbb{N}$ and let $a \in \mathbb{C}$. Then

$$C_{(x-a)^n} \sim J_a = \begin{pmatrix} a & 0 & 0 & \cdots & 0 \\ 1 & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & \cdots & 1 & a \end{pmatrix}$$

Thus $\mathfrak{l}_{2n+1}^{A} \cong \mathfrak{l}_{2n+1}^{J_a}$ and the non-trivial Leibniz brackets are

$$[e_i, f_i] = (1+a)h,\ [f_i, e_i] = (-1+a)h\ \forall i = 1, \ldots, n$$

$$[e_i, f_{i-1}] = h,\ [f_{i-1}, e_i] = h,\ \forall i = 2, \ldots, n$$

where $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ is a basis of $\mathfrak{l}_{2n+1}^{J_a}$.

Proposition (*G. La Rosa, M. M.*, 2022)

Let $a \in \mathbb{C}$. The Heinseberg Leibniz algebras $\mathfrak{l}_{2n+1}^{J_a}$ and $\mathfrak{l}_{2n+1}^{J_{-a}}$ are isomorphic.

Proof.

- $\mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{-J_a^t}$ via the linear map φ defined by

$$\varphi(e_i) = f_i', \;\; \varphi(f_i) = e_i', \;\; \varphi(h) = -h', \;\; \forall i = 1, \cdots, n$$

- $-J_a^t \sim J_{-a} \Rightarrow \mathfrak{l}_{2n+1}^{J_a} \cong \mathfrak{l}_{2n+1}^{J_{-a}}$.

$\square$

Proposition (*G. La Rosa, M. M.*, 2022)

Let $a, a' \in \mathbb{C}$. Then $\mathfrak{l}_3^a \cong \mathfrak{l}_3^{a'} \iff a' = \pm a$.

Part 3: Lie Racks and the *Coquegigrue Problem*

- It is the problem, formulated by J.-L. Loday, of finding a generalization to Leibniz algebras of Lie's third theorem, which associates a Lie local group with any real or complex Lie algebra.
- Given any Leibniz algebra $\mathfrak{g}$, one wants to find a manifold endowed with a smooth map which plays the role of Ad for Lie groups, and such that the tangent space at a distinguished point, endowed with the differential ad of Ad, is a Leibniz algebra isomorphic to $\mathfrak{g}$.
- In the special case of Lie algebras $\mathfrak{g}$, one wants to obtain the usual simply connected Lie group associated with $\mathfrak{g}$.
- We assume now that $\mathbb{F} = \mathbb{R}$.

Definition

A (left) *rack* is a set X with a binary operation $\triangleright : X \times X \to X$ which is (left) autodistributive

$$x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)\,, \quad \forall x, y, z \in X$$

and such that $x \triangleright - : X \to X$ is a bijection $\forall x \in X$.
A rack is *pointed* if $\exists\, 1 \in X$ such that $1 \triangleright x = x$ and $x \triangleright 1 = 1$, $\forall x \in X$.
A rack $(X, \triangleright)$ is a *quandle* if $x \triangleright x = x$, $\forall x \in X$.

Example

Every group endowed with the conjugation is a pointed quandle.

Definition

A *Lie rack* is a pointed rack $(X, \triangleright, 1)$ such that X is a smooth manifold, $\triangleright$ is a smooth map and $x \triangleright -$ is a diffeomorphism $\forall x \in X$.

The tangent space of a Lie rack

Proposition (*M.-K. Kinyon*, 2007)

If X is a Lie rack, then the bracket

$$[x, y] = \mathrm{ad}_x(y),$$

where

- $\mathrm{ad} : \mathrm{T}_1 X \to \mathfrak{gl}(\mathrm{T}_1 X)$ *is the differential of* $\Phi : X \mapsto \mathrm{GL}(\mathrm{T}_1 X)$*;*
- $\Phi(x) = \mathrm{T}_1 \phi(x)$, $\forall x \in X$*;*
- $\phi(x) = x \rhd -$, $\forall x \in X$

defines on $\mathrm{T}_1 X$ *a Leibniz algebra structure.*

- M.-K. Kinyon solves the *coquecigrue problem* for the class of *split Leibniz algebras*, which are Leibniz algebras $\mathfrak{g}$ with a bilateral ideal $\mathrm{Leib}(\mathfrak{g}) \subseteq \mathfrak{a} \subseteq \mathrm{Z}_l(\mathfrak{g})$ and with a Lie subalgebra $\mathfrak{h} \subseteq \mathfrak{g}$ such that

$$\mathfrak{g} = \mathfrak{h} \oplus_s \mathfrak{a}$$

  and

$$[(x,a),(y,b)] = ([x,y], \rho_x(b)), \ \ \forall (x,a),(y,b) \in \mathfrak{h} \oplus \mathfrak{a}$$

  where $\rho : \mathfrak{h} \times \mathfrak{a} \to \mathfrak{a}$ is the action on the $\mathfrak{h}$-module $\mathfrak{a}$.

Proposition (*M.-K. Kinyon*, 2007)

Let $\mathfrak{g}$ be a split Leibniz algebra. Then there exists a Lie rack X such that $\mathrm{T}_1 X$ is isomorphic to $\mathfrak{g}$.

The local integration of a Leibniz algebra

- S. Covez gives a local solution to the coquegigrue problem: he shows how to integrate every Leibniz algebra into a Lie local rack.
- The central point of his result is to see every Leibniz algebra $\mathfrak{g}$ as an abelian extension with kernel $Z_l(\mathfrak{g})$ and to integrate explicitly the corresponding Leibniz algebra 2-cocycle into a Lie local rack 2-cocycle.

- We associate with $\mathfrak{g}$ the short exact sequence

$$0 \to \mathrm{Z}_l(\mathfrak{g}) \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0 \to 0$$

where $\mathfrak{g}_0 = \mathfrak{g}/\,\mathrm{Z}_l(\mathfrak{g})$.
- $\mathrm{Z}_l(\mathfrak{g})$ is a $\mathfrak{g}_0-$module $\Rightarrow \exists\, \omega \in \mathrm{ZL}^2\,(\mathfrak{g}_0, \mathrm{Z}_l(\mathfrak{g}))$ such that

$$\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathrm{Z}_l(\mathfrak{g})$$

with

$$[(x, a)\,, (y, b)] = \Big([x, y]_{\mathfrak{g}_0}\,, \rho_x(b) + \omega(x, y)\Big)\,,$$

where $\rho : \mathfrak{g}_0 \times \mathrm{Z}_l(\mathfrak{g}) \to \mathrm{Z}_l(\mathfrak{g})$ is the action induced by the $\mathfrak{g}_0$-module structure of $\mathrm{Z}_l(\mathfrak{g})$.

Theorem (*S. Covez*, 2013)

Every Leibniz algebra $\mathfrak{g} = \mathfrak{g}_0 \oplus_\omega \mathfrak{a}$ *can be integrated into a Lie local rack of the form*

$$G_0 \times_f \mathfrak{a}$$

with operation defined by

$$(g, a) \rhd (h, b) = \left(ghg^{-1}, \phi_g(b) + f(g, h)\right), \;\; \forall (g, a), (h, b) \in G_0 \times \mathfrak{a}$$

and unit $(1, 0)$*, where* G_0 *is a Lie group such that* $\mathrm{Lie}(G_0) = \mathfrak{g}_0$*,* ϕ *is the exponentiation of the action* ρ*,* $f : G_0 \times G_0 \to \mathfrak{a}$ *is the Lie local racks* $2-$ *cocycle defined by*

$$f(g, h) = \int_{\gamma_h} \left(\int_{\gamma_g} \tau^2(\omega)^{eq} \right)^{eq}, \;\; \forall g, h \in G_0$$

and $\tau^2(\omega) \in \mathrm{ZL}^1(\mathfrak{g}_0, Hom(\mathfrak{g}_0, \mathfrak{a}))$ *is defined by* $\tau^2(\omega)(x)(y) = \omega(x, y)$*,* $\forall x, y \in \mathfrak{g}_0$.

Theorem (*G. La Rosa, M. M.*, 2022)

Let R be a Lie rack integrating a Leibniz algebra $\mathfrak{g}$. R is a quandle if and only if $\mathfrak{g}$ is a Lie algebra. In particular $R = \mathrm{Conj}(G)$, where $\mathrm{Lie}(G) = \mathfrak{g}$.

Theorem (*G. La Rosa, M. M.*, 2022)

Every nilpotent real Leibniz algebra $\mathfrak{g}$ has a global integration into a Lie rack.

Integration of two-step nilpotent Leibniz algebras

Theorem (*G. La Rosa, M. M.*, 2022)

Let $(\mathfrak{g}, [-,-])$ *be a two-step nilpotent Leibniz algebra and let* $\omega : \mathfrak{g}_0 \times \mathfrak{g}_0 \to [\mathfrak{g}, \mathfrak{g}]$, *where* $\mathfrak{g}_0 = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, *be the Leibniz algebras* $2-$*cocycle associated with the short exact sequence*

$$0 \to [\mathfrak{g}, \mathfrak{g}] \hookrightarrow \mathfrak{g} \twoheadrightarrow \mathfrak{g}_0 \to 0.$$

Then the multiplication

$$(x, a) \rhd (y, b) = (y, b + \omega(x, y)) = (y, b) + [(x, a), (y, b)]$$

defines a Lie global rack structure on $\mathfrak{g}_0 \times [\mathfrak{g}, \mathfrak{g}]$, *such that* $\mathsf{T}_{(0,0)}(\mathfrak{g}_0 \times_\omega [\mathfrak{g}, \mathfrak{g}], \rhd)$ *is a Leibniz algebra isomorphic to* $\mathfrak{g}$.

Example

Let $A = (a_{ij})_{i,j} \in \mathrm{M}_n(\mathbb{R})$ and let $\mathfrak{g} = \mathfrak{l}^A_{2n+1}$ be the $(2n+1)-$dimensional Heisenberg Leibniz algebra with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ and bracket

$$\left[(x_1, \ldots, x_n, y_1, \ldots, y_n, z), (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z')\right] =$$

$$= \left(0, \ldots, 0, \sum_{i,j=1}^{n} (\delta_{ij} + a_{ij})\, x_i y'_j + \sum_{i,j=1}^{n} (-\delta_{ij} + a_{ij})\, x'_i y_j \right)$$

We have that $[\mathfrak{g}, \mathfrak{g}] = \mathbb{R}h \subseteq \mathrm{Z}(\mathfrak{g})$, thus the Lie global rack integrating $\mathfrak{g}$ is $R^A_{2n+1} = (\mathfrak{g}_0 \times_f \mathbb{R}h, \rhd)$ with multiplication

$$(x_1, \ldots, x_n, y_1, \ldots, y_n, z) \rhd (x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z') =$$

$$= \left(x'_1, \ldots, x'_n, y'_1, \ldots, y'_n, z' + \sum_{i,j=1}^{n} \left[(\delta_{ij} + a_{ij})\, x_i y'_j + (-\delta_{ij} + a_{ij})\, x'_i y_j\right]\right)$$

Definition

We define $(R^A_{2n+1}, \triangleright)$ as the *Heisenberg rack*.

- The multiplication $\triangleright$ in R^A_{2n+1} is a *perturbation* of the conjugation of the Heisenberg Lie group H_{2n+1}.
- We use the canonical matrix representation

$$H_{2n+1} = \left\{ \begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} \middle| \, x, y \in \mathbb{R}^n, z \in \mathbb{R} \right\} \leq \mathrm{GL}_{n+2}(\mathbb{R})$$

The conjugation formula for two matrices in H_{2n+1} is given by

$$\begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x' & z' \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix}^{-1} =$$

$$= \begin{pmatrix} 1 & x' & z' + \sum_{i=1}^{n} (x_i y'_i - y_i x'_i) \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix}$$

With the same representation, the multiplication $\triangleright$ of R^A_{2n+1} turns into

$$\begin{pmatrix} 1 & x & z \\ 0 & I_n & y^t \\ 0 & 0 & 1 \end{pmatrix} \triangleright \begin{pmatrix} 1 & x' & z' \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & x' & z' + \sum_{i,j=1}^{n} \left[(\delta_{ij} + a_{ij})\, x_i y'_j + (\delta_{ij} - a_{ij})\, x'_i y_j \right] \\ 0 & I_n & y'^t \\ 0 & 0 & 1 \end{pmatrix},$$

hence for $A = 0_{n\times n}$, it holds $R^{0_{n\times n}}_{2n+1} = \mathrm{Conj}(H_{2n+1})$.

Definition

Let $\mathfrak{g} = \mathfrak{k}_n$ and let $\{e_1, \ldots, e_n, f_1, \ldots, f_n, h\}$ be a basis of $\mathfrak{g}$. The *Kronecker* rack $K_n = (\mathfrak{g}_0 \times_f \mathbb{R}h, \rhd)$ is the Lie rack with multiplication

$$u \rhd v = v + [u, v]_{\mathfrak{k}_n}, \quad \forall u, v \in \mathbb{F}^{2n+1}$$

Definition

Let $\mathfrak{g} = \mathfrak{d}_n$ and let $\{e_1, \ldots, e_{2n+1}, h\}$ be a basis of $\mathfrak{g}$. Then *Dieudonnée* rack $D_n = (\mathfrak{g}_0 \times_f \mathbb{R}h, \rhd)$ is the Lie rack with multiplication

$$u \rhd v = v + [u, v]_{\mathfrak{d}_n}, \quad \forall u, v \in \mathbb{F}^{2n+2}$$

Remark

We have that $\mathrm{T}_{(0,0)}\, K_n \cong \mathfrak{k}_n$ and $\mathrm{T}_{(0,0)}\, D_n \cong \mathfrak{d}_n$.

Part 4: Isotopisms

- The concept of isotopism between two algebraic structures was introduced in 1942 by Abraham Adrian Albert in order to classify non-associative algebras.
- Every isomorphism between two algebraic structures is a particular isotopism.
- Our goal is to find isotopism classes of indecomposable two-step nilpotent Leibniz algebras and Lie racks integrating them.

Definition

Let $\mathfrak{g}$ and $\mathfrak{h}$ be left Leibniz algebras over $\mathbb{F}$. An *isotopism* between $\mathfrak{g}$ and $\mathfrak{h}$ is a triple of linear isomorphisms

$$(f, g, h) : \mathfrak{g} \leftrightarrows \mathfrak{h}$$

such that

$$[f(x), g(y)]_{\mathfrak{h}} = h([x, y]), \ \ \forall x, y \in \mathfrak{g}$$

Definition

Let R and S be left racks. An *isotopism* between R and S is a triple of bijections

$$(f, g, h) : R \leftrightarrows S$$

such that

$$f(x) \triangleright_S g(y) = h(x \triangleright y), \ \ \forall x, y \in R$$

Principal Isotopism

Remark

Let $(A, \cdot)$ and $(B, \circ)$ be two algebraic structures and let $(f, g, h) : A \leftrightarrows B$ be an isotopism. We define a new operation $* : B \times B \to B$ by

$$h(x) * h(y) = h(xy), \quad \forall x, y \in A.$$

In this way $(A, \cdot)$ and $(B, *)$ are isomorphic and $h(x) * h(y) = f(x) \circ g(y)$. Thus $(hf^{-1})(u) * (hg^{-1})(v) = u \circ v$, where $u = f(x)$, $v = g(y)$ and we have an isotopism, called **principal isotopism**

$$(fh^{-1}, gh^{-1}, \mathrm{id}_B) : (B, \circ) \leftrightarrows (B, *).$$

$$\begin{tikzcd}
(B, \circ) \arrow[r, "princ.isot."] & (B, *) \\
(A, \cdot) \arrow[u, "isotopism"] \arrow[ur, dashed, "iso."] &
\end{tikzcd}$$

Example

Let $\mathfrak{h}_3$ be the three-dimensional Heisenberg Lie algebra, let $A \in GL_2(\mathbb{F})$, $B = A \cdot \mathrm{diag}\{\lambda, \mu\}$ with $\lambda, \mu \in \mathbb{F}^*$ and let $\mathfrak{g}_3^{A,\lambda,\mu}$ be the nilpotent Leibniz algebra with one-dimensional commutator ideal and structure matrix

$$\begin{pmatrix} 0 & a_{11}b_{22} - a_{21}b_{12} & 0 \\ a_{12}b_{21} - a_{22}b_{11} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Then the triple of linear isomoprhisms $(f, g, \mathrm{id}_{\mathbb{F}_3}) : \mathfrak{g}_3^{A,\lambda,\mu} \leftrightarrows \mathfrak{h}_3$ given by

$$f \equiv A \oplus \alpha, \;\; g \equiv B \oplus \beta, \;\; \alpha, \beta \in \mathbb{F}^*$$

is an isotopism.

Remark

Given $\alpha \in \mathbb{F} \setminus \{\pm 1\}$, it is always possible to find A,λ and μ such that $\mathfrak{g}_3^{A,\lambda,\mu} = \mathfrak{l}_3^{\alpha}$. Thus $\mathfrak{h}_3$ and $\mathfrak{l}_3^{\alpha}$ are isotopic for every $\alpha \in \mathbb{F}$, $\alpha \neq \pm 1$.

Proposition

Let $\mathfrak{g}, \mathfrak{h}$ be Leibniz algebras and let $(f, g, h) : \mathfrak{g} \leftrightarrows \mathfrak{h}$ be an isotopism. Then

$$f(\mathrm{Z}_l(\mathfrak{g})) \subseteq \mathrm{Z}_l(\mathfrak{h}), \;\; g(\mathrm{Z}_r(\mathfrak{g})) \subseteq \mathrm{Z}_r(\mathfrak{h}).$$

Remark

We have that $\dim_{\mathbb{F}} \mathrm{Z}_l(\mathfrak{l}_3^{\alpha}) = 1$, for every $\alpha \in \mathbb{F} \setminus \{\pm 1\}$, and $\dim_{\mathbb{F}} \mathrm{Z}_l(\mathfrak{l}_3^{\pm 1}) = 2$. Thus $\mathfrak{l}_3^{\alpha}$ and $\mathfrak{l}_3^{\pm 1}$ are not isotopic.

Proposition

Let R, S be left pointed racks and let $(f, g, h) : R \leftrightarrows S$ be an isotopism. Then

$$f(\mathrm{Ann}(R)) \subseteq \mathrm{Ann}(S),$$

where $\mathrm{Ann}(R) = \{x \in R \mid x \triangleright y = 1_R, \ \forall y \in R\}$.

Isotopism classes of nilpotent Leibniz algebras with one-dimensional commutator ideal

Theorem (*G. La Rosa, M. M., G.P. Nagy*, 2022)

Let $\mathbb{F}$ *be a field with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let* $\mathfrak{g}$ *be a nilpotent Leibniz algebra with* $\dim_{\mathbb{F}} \mathfrak{g} = t \geq 3$ *and* $\dim_{\mathbb{F}}[\mathfrak{g}, \mathfrak{g}] = 1$.

- *If* $t = 2n + 2$, *then* $\mathfrak{g}$ *is isomorphic to the Dieudonnée Leibniz algebra* $\mathfrak{d}_n$;
- *If* $t = 2n + 1$, *then either* $\mathfrak{g}$ *is isomorphic to the Heisenberg Leibniz algebra* $\mathfrak{l}_{2n+1}^{J_1}$, *where* J_1 *is the* $n \times n$ *Jordan block of eigenvalue* 1, *or* $\mathfrak{g}$ *is isotopic to the Heisenberg Lie algebra* $\mathfrak{h}_{2n+1}$.

Theorem (*G. La Rosa, M. M., G.P. Nagy*, 2022)

Let $\mathfrak{g}, \mathfrak{h}$ be nilpotent Leibniz algebras with one-dimensional commutator ideal.

- *If $\mathfrak{g}$ and $\mathfrak{h}$ are isotopic, it is always possible to find a isotopism of the form $(f, \mathrm{id}, \mathrm{id})$ between them.*
- *Let $\mathbb{F} = \mathbb{R}$ and let R, S be the Lie global racks integrating resp. $\mathfrak{g}$ and $\mathfrak{h}$. Then R is isotopic to S if and only if $\mathfrak{g}$ is isotopic to $\mathfrak{h}$. Moreover an isotopism of the form $(f, \mathrm{id}, \mathrm{id})$ between $\mathfrak{g}$ and $\mathfrak{h}$ becomes also an isotopism between R and S.*

Sketch of the Proof.

- The multiplication of R and S can be written as

  $$u \rhd_R v = v + [u, v]_{\mathfrak{g}}, \;\; u' \rhd_S v' = v' + [u', v'], \;\; \forall u, v \in R, \; \forall u', v' \in S$$

  Thus, if $(f, \mathrm{id}, \mathrm{id}) : \mathfrak{g} \leftrightarrows \mathfrak{h}$ is an isotopism, then

  $$f(u) \rhd_S v = v + [f(u), v]_{\mathfrak{h}} = v + [u, v]_{\mathfrak{g}} = u \rhd_R v,$$

  for every $u, v \in R$.

Corollary (*G. La Rosa, M. M., G.P. Nagy*, 2022)

Let $\mathbb{F} = \mathbb{R}$ *and let* R *be a Lie rack integrating a nilpotent Leibniz algebra* $\mathfrak{g}$ *with* $\dim_{\mathbb{F}} \mathfrak{g} = t$ *and* $\dim_{\mathbb{F}}[\mathfrak{g}, \mathfrak{g}] = 1$.

- *If* $t = 2n + 2$, *then* R *is isomorphic to the Dieudonnée rack* D_n;
- *If* $t = 2n + 1$, *then either* R *is isomorphic to the Heisenberg rack* $R^{J_1}_{2n+1}$, *where* J_1 *is the* $n \times n$ *Jordan block of eigenvalue* 1, *or* R *is isotopic to the conjugation of the Heisenberg Lie group* $\mathrm{Conj}(H_{2n+1})$.

Final Goal

Let $\mathfrak{g}$ and $\mathfrak{h}$ be real Leibniz algebras and let suppose that there exist Lie global racks R and S integrating resp. $\mathfrak{g}$ and $\mathfrak{h}$. If there exists an isotopism between $\mathfrak{g}$ and $\mathfrak{h}$, then is it true that R and S are isotopic? How can we define an isotopism between them?

$$\begin{tikzcd} \mathfrak{g} \arrow[r, "int."] \arrow[d, "isotopism"'] & R \arrow[d, dashed, "isotopism(?)"] \\ \mathfrak{h} \arrow[r, "int."] & S \end{tikzcd}$$

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Thank you for your attention!