

LIE MODELS OF SIMPLICIAL SETS AND REPRESENTABILITY OF THE QUILLEN FUNCTOR

BY

URTZI BUIJS

*Departamento de Algebra, Geometría y Topología, Universidad de Málaga
Málaga, Ap. 59 29080, Spain
e-mail: ubuijs@uma.es*

AND

YVES FÉLIX

*Institut de Mathématiques et Physique
Université Catholique de Louvain-la-Neuve
Louvain-la-Neuve, Belgium
e-mail: Yves.felix@uclouvain.be*

AND

ANICETO MURILLO

*Departamento de Algebra, Geometría y Topología, Universidad de Málaga
Málaga, Ap. 59 29080, Spain
e-mail: aniceto@uma.es*

AND

DANIEL TANRÉ

*Département de Mathématiques, UMR-CNRS 8524, Université de Lille 1
Villeneuve d'Ascq Cedex 59655, France
e-mail: Daniel.Tanre@univ-lille1.fr*

Received May 29, 2017 and in revised form July 1, 2019

ABSTRACT

Extending the model of the interval, we explicitly define for each $n \geq 0$ a free complete differential graded Lie algebra $\mathfrak{L}_n$ generated by the simplices of Δ^n , with desuspended degrees, in which the vertices are Maurer–Cartan elements and the differential extends the simplicial chain complex of the standard n -simplex. The family $\{\mathfrak{L}_\bullet\}_{n \geq 0}$ is endowed with a cosimplicial differential graded Lie algebra structure which we use to construct two adjoint functors

$$\mathbf{SimpSet} \begin{array}{c} \xrightarrow{\mathfrak{L}} \\ \xleftarrow{\langle \cdot \rangle} \end{array} \mathbf{DGL}$$

given by $\langle L \rangle_\bullet = \mathbf{DGL}(\mathfrak{L}_\bullet, L)$ and $\mathfrak{L}(K) = \varinjlim_K \mathfrak{L}_\bullet$. This new tool lets us extend the Quillen rational homotopy theory approach to any simplicial set K whose path components are not necessarily simply connected.

We prove that $\mathfrak{L}(K)$ contains a model of each component of K . When K is a 1-connected finite simplicial complex, the Quillen model of K can be extracted from $\mathfrak{L}(K)$. When K is connected then, for a perturbed differential ∂_a , $H_0(\mathfrak{L}(K), \partial_a)$ is the Malcev Lie completion of $\pi_1(K)$. Analogous results are obtained for the realization $\langle L \rangle$ of any complete DGL.

CONTENTS

Introduction	315
1. The BCH product and the Lawrence–Sullivan model for the interval	319
2. Sequences of compatible models of Δ	324
3. Symmetric models of Δ and the cosimplicial structure . .	333
4. The realization functor and its adjoint	335
5. The $\mathfrak{L}$ -model of K	341
6. Differential graded Lie coalgebras and the Transfer Theorem	344
7. The cosimplicial structure via a transfer	349
8. Representability of the Quillen realization functor	352
9. The Malcev completion of the fundamental group	354
References	357

Introduction

In [18], R. Lawrence and D. Sullivan raise the following observation and subsequent general questions: the rational singular chains on a cellular complex are naturally endowed with a structure of cocommutative, coassociative infinity coalgebra and hence, taking the commutators of a “generalized bar construction”, it should give rise to a complete differential graded Lie algebra (DGL henceforth). What is the topological and geometrical meaning of this DGL? Are there closed formulae for its differential? Allowing 1-cells, what is the relation of this DGL with the fundamental group of the given complex? In the same reference they carefully construct such a DGL for the interval.

Here we attack these problems for any finite simplicial complex K . In fact, we construct in a functorial way a complete differential graded Lie algebra $\mathfrak{L}(K) = (\widehat{\mathbb{L}}(V_K), \partial)$ which is the only (up to isomorphism) free complete differential graded Lie algebra for which V_K together with the linear part ∂_1 of ∂ is the desuspension of the simplicial chain complex of K , and any generator of degree -1 (i.e., any 0-simplex) is a Maurer–Cartan element. The departure point is the construction of a family $\mathfrak{L}_n := \mathfrak{L}(\Delta^n)$ of such DGL’s for the standard simplices, together with coface and codegeneracy operators, for which we prove:

THEOREM A: *The family $\mathfrak{L}_\bullet := \{\mathfrak{L}(\Delta^n), \partial\}_{n \geq 0}$ forms a cosimplicial differential graded Lie algebra.*

We also prove uniqueness of such a family as Theorem 2.8, and give as examples explicit models $\mathfrak{L}_2$ and $\mathfrak{L}_3$ for the triangle and the tetrahedron in Proposition 2.7 and Example 2.9 respectively.

On the one hand, this cosimplicial structure lets us finally define $\mathfrak{L}(K)$ as a left Kan extension of $\mathfrak{L}_\bullet$ (see Definition 4.2),

$$\mathfrak{L}(K) = \varinjlim_K \mathfrak{L}_\bullet,$$

for any simplicial set K . Whenever K is a finite simplicial complex, $\mathfrak{L}(K)$ is trivially isomorphic to a sub DGL of a certain $\mathfrak{L}_n$ as $K \subset \Delta^n$ for some n .

On the other hand, the cosimplicial DGL $\mathfrak{L}_\bullet$ defines a functor $\langle \cdot \rangle$ from the category **DGL** of complete differential graded Lie algebras to the category **SimpSet** of simplicial sets via

$$\langle L \rangle = \mathbf{DGL}(\mathfrak{L}_\bullet, L).$$

Then, we prove:

THEOREM B: *The following are adjoint functors:*

$$\mathbf{SimpSet} \begin{matrix} \xrightarrow{\mathfrak{L}} \\ \xleftarrow{\langle \cdot \rangle} \end{matrix} \mathbf{DGL}.$$

These functors not only shed light on our original questions but also let us develop an Eckmann–Hilton dual of the Sullivan geometrical approach to rational homotopy theory, even for non-simply connected complexes. Here, $\mathfrak{L}_\bullet$ plays the dual role of the simplicial differential graded algebra $\mathcal{A}_\bullet = A_{PL}(\Delta^\bullet)$ of PL-forms on the standard simplices [12, 29, 30]. Recall that the Quillen approach [26] is based on the construction of several equivalences between homotopy categories joining the homotopy category of rational 2-reduced simplicial sets and that of 1-reduced differential graded Lie algebras.

Our work, although using a completely different method, extends Quillen theory in several ways.

First of all, if X is a simply connected simplicial set, we denote by Y the associated 2-reduced simplicial set. In the text, we abuse notation by denoting $\lambda(X)$ the image of Y by the Quillen construction. We prove:

THEOREM C: *Let K be a 1-connected finite simplicial complex. Then, for any vertex $a \in K$, there is a quasi-isomorphism of differential graded Lie algebras,*

$$(\mathfrak{L}(K), \partial_a) \xrightarrow{\simeq} \lambda(K).$$

In particular, $H_n(\mathfrak{L}(K), \partial_a) \cong \pi_{n+1}(K) \otimes \mathbb{Q}$, $n \geq 1$.

Here $\partial_a = \partial + \text{ad}_a$ is the twisting of the original differential ∂ of $\mathfrak{L}(K)$ via the Maurer–Cartan element a .

On the other hand, concerning the realization functor, we show that, for a non-negatively graded DGL $L = L_{\geq 0}$, there are explicit isomorphisms,

$$\pi_n \langle L \rangle \cong H_{n-1}(L), \quad n \geq 1,$$

where the group structure on $H_0(L)$ is given by the Baker–Campbell–Hausdorff product (see Proposition 4.6).

Moreover, recall the dual Sullivan geometric realization [12, 29] of a commutative differential graded algebra A given by

$$\langle A \rangle_S = \mathbf{CDGA}(A, \mathcal{A}_\bullet).$$

Then:

THEOREM D: *If L is a non-negatively graded finite type DGL, then there is a homotopy equivalence $\langle L \rangle \simeq \langle \mathcal{C}^*(L) \rangle_S$.*

Here, $\mathcal{C}^*$ denotes the cochain functor. In particular, our realization also equals Quillen realization for finite type 1-reduced DGL's in view of the main result in [21]. Moreover, in [15] Getzler has constructed another functor $\langle \cdot \rangle_G$ from **DGL** to **SimpSet**,

$$\langle L \rangle_G = \text{MC}(L \hat{\otimes} \mathcal{A}_\bullet),$$

in which MC denotes the set of Maurer–Cartan elements. By [15, Proposition 1.1], $\langle L \rangle_G = \langle \mathcal{C}^*(L) \rangle_S$ when L is a non-negatively graded finite type DGL, and thus, for these DGL's, all realizations coincide (see also [3, Cor. 1.3]).

Finally, we answer the last question which we begin with and explicitly describe the relation between the fundamental group of a complex and its associated DGL:

THEOREM E: *If K is a connected finite simplicial complex, then $H_0(\mathfrak{L}(K), \partial_a)$ is the Malcev Lie completion of the fundamental group of K .*

This result implies that, unlike the classical Quillen approach and their known interesting extensions [7, 19, 24], our model functor reflects geometrical properties of non-nilpotent spaces.

We also point out that our constructions apply to not necessarily connected complexes. Indeed, for any finite complex K , the connected components of K_+ , the disjoint union of K and a point, correspond to the classes of Maurer–Cartan elements of $\mathfrak{L}(K)$ modulo the gauge action [4]. Moreover, if a is a non-zero Maurer–Cartan element, then $(\mathfrak{L}(K), \partial_a)$ is a “model” of the corresponding path component of K .

On the other hand, in Theorem 4.7 we prove that, for a general DGL L , the components of the realization $\langle L \rangle$ correspond also to the classes of Maurer–Cartan elements of L modulo the gauge action. Moreover, if a is a Maurer–Cartan element, the corresponding component of $\langle L \rangle$ is the realization $\langle L^{(a)} \rangle$ of the “localization of L at a ”. In other words,

$$\langle L \rangle \simeq \bigcup_{a \in \widehat{\text{MC}}(L)} \langle L^{(a)} \rangle.$$

This extends [8, Theorem 5.5] to any (complete) DGL.

We remark that the definition and main properties of the model and realization functor purposely stay in the Lie side and do not rely upon any connection to commutative algebras via cochain functors. The main reason is the following: in [5], we endow the category of complete DGL's with a model structure which makes the model and realization functors a Quillen pair. This structure geometrically shapes complete Lie algebras as it is the result of “transferring” the classical model structure on simplicial sets. With this, weak equivalences and fibrations properly contain the weak equivalences and fibrations of the model structure defined in [19] via the cochain functor.

Since the original work of Quillen, several authors [16, 15, 3, 19, 1] have constructed geometrical realization functors or have studied the relations between them. Recently, in [27] and [6] it has been shown that the realization functor $\langle \cdot \rangle$ turns out to be isomorphic to the Getzler nerve $\gamma_\bullet$ defined in [15].

This paper is organized as follows:

Section 1 is devoted to recalling the Baker–Campbell–Hausdorff product and its connection with the basic properties of Lawrence–Sullivan model $\mathfrak{L}_1$ for the interval [18]. In Section 2 we extend this construction and build the sequence $\{\mathfrak{L}_n\}_{n \geq 0}$ of compatible models for Δ^n . We establish some technical properties and also prove uniqueness.

In Section 3 we prove the existence of compatible symmetric models, i.e., models $(\mathfrak{L}_n, \partial)$ that are Σ_{n+1} -DGL with an action that permutes the inclusions of the different copies of $\mathfrak{L}_{n-1}$. This leads to the existence of the cosimplicial DGL structure on $\mathfrak{L}_\bullet$. All of this proves Theorem A.

In Sections 4 and 5 we define the model $\mathfrak{L}$ and realization $\langle \cdot \rangle$ functors, we prove Theorem B, and give some properties and examples.

Section 6 is devoted to introducing the material on differential graded Lie coalgebras and the transfer theorem that we use in Section 7 to build a sequence of compatible models from a transfer process applied to a classical diagram of the form

$$\phi \circlearrowleft A_{PL}(K) \begin{matrix} \xrightarrow{p} \\ \xleftarrow{i} \end{matrix} C^*(K).$$

Using the uniqueness Theorem 2.8, we establish relations between the classical rational models of Sullivan and Quillen and the models constructed in the previous sections. In particular, we prove Theorem C as Theorem 7.4.

Section 8 contains the correspondence between our realization and Sullivan's realization. Theorem D corresponds to Theorem 8.1. Finally, Theorem E is proved in Section 9.

ACKNOWLEDGEMENTS. The authors would like to thank Benoit Fresse for helpful conversations.

The authors have been partially supported by the MINECO grant MTM2013-41768-P. The first author has also been partially supported by a U-Mobility Programme grant of the European Union's Seventh Framework Programme, G. A. no. 246550 and by the Ramón y Cajal MINECO program. The first and third authors have also been partially supported by the Junta de Andalucía grant FQM-213.

1. The BCH product and the Lawrence–Sullivan model for the interval

Throughout this paper we assume that $\mathbb{Q}$ is the base field. A **graded Lie algebra** consists of a $\mathbb{Z}$ -graded vector space

$$L = \bigoplus_{p \in \mathbb{Z}} L_p$$

together with a bilinear product called the **Lie bracket** denoted by $[-, -]$ such that

$$[x, y] = -(-1)^{|x||y|}[y, x]$$

and

$$(-1)^{|x||z|}[x, [y, z]] + (-1)^{|y||x|}[y, [z, x]] + (-1)^{|z||y|}[z, [x, y]] = 0.$$

Here $|x|$ denotes the degree of x .

A **differential graded Lie algebra** is a graded Lie algebra L endowed with a linear derivation ∂ of degree -1 such that $\partial^2 = 0$. It is called **free** if L is free as a Lie algebra, $L = \mathbb{L}(V)$ for some graded vector space V .

The **completion** $\widehat{L}$ of a graded Lie algebra L is

$$\widehat{L} = \varprojlim_n L/L^n$$

where $L^1 = L$, $L^n = [L, L^{n-1}]$ for $n \geq 2$, and the limit is taken on the topology arising from this filtration. A Lie algebra L is called **complete** if L is isomorphic to its completion. From now on, and unless explicitly stated otherwise, by a DGL we mean a complete differential graded Lie algebra.

A Maurer–Cartan element is an element $a \in L_{-1}$ such that $\partial a + \frac{1}{2}[a, a] = 0$. Denote by $\text{MC}(L)$ the set of Maurer–Cartan elements. These are preserved by DGL morphisms. In particular, if $L = (\widehat{\mathbb{L}}(V), \partial)$ is a complete free DGL, and θ is a derivation satisfying $\theta(V) \subset \widehat{\mathbb{L}}^{\geq 2}(V)$ and $[\theta, \partial] = 0$, then

$$e^\theta = \sum_{n \geq 0} \frac{\theta^n}{n!}$$

is an automorphism of L and so, if $a \in \text{MC}(L)$, then $e^\theta(a)$ is also a Maurer–Cartan element.

Given $(\widehat{\mathbb{L}}(V), \partial)$ a complete free DGL and $v \in V$, we will often write

$$\partial v = \sum_{n \geq 1} \partial_n v$$

where $\partial_n v \in \widehat{\mathbb{L}}^n(V)$.

Let (L, ∂) be a DGL and $a \in \text{MC}(L)$. Then, the derivation $\partial_a = \partial + \text{ad}_a$ is again a differential on L .

Given L a complete DGL, the **gauge action** $\mathcal{G}$ of L_0 on $\text{MC}(L)$ determines an equivalence relation among Maurer–Cartan elements defined as follows (see for instance [22, §4]): given $x \in L_0$ and $a \in \text{MC}(L)$,

$$x \mathcal{G} a = e^{\text{ad}_x}(a) - \frac{e^{\text{ad}_x} - 1}{\text{ad}_x}(\partial x).$$

Here and from now on, 1 inside an operator will denote the identity. Explicitly,

$$x \mathcal{G} a = \sum_{i \geq 0} \frac{\text{ad}_x^i(a)}{i!} - \sum_{i \geq 0} \frac{\text{ad}_x^i(\partial x)}{(i+1)!}.$$

We denote the quotient set by $\widehat{\text{MC}}(L) = \text{MC}(L)/\mathcal{G}$. Geometrically [17, 18], interpreting Maurer–Cartan elements as points in a space, one thinks of x as a flow taking $x \mathcal{G} a$ to a in unit time. In topological terms [8], the points a and $x \mathcal{G} a$ are in the same path component.

Let L be a complete Lie algebra concentrated in degree 0. We denote by UL its enveloping algebra, by I_L its augmentation ideal and by $\widehat{UL}$ and $\widehat{I_L}$ the completions of UL and I_L with respect to the powers of I_L ,

$$\widehat{UL} = \varprojlim_n UL/I_L^n, \quad \widehat{I_L} = \varprojlim_n I_L/I_L^n.$$

Denote finally by

$$G_L = \{x \in \widehat{UL} \mid \Delta(x) = x \widehat{\otimes} x\}$$

the group of grouplike elements in $\widehat{UL}$. Moreover, the injection of L in the set of primitive elements in $\widehat{UL}$ is an isomorphism and the functions $\exp$ and $\log$ give inverse bijections between L and G_L . This induces a product on L , called the **Baker–Campbell–Hausdorff product**, BCH product henceforth, defined by

$$a * b = \log(\exp(a) \cdot \exp(b)).$$

Note that $a * (-a) = 0$. Therefore, $-a$ is the inverse of a for the BCH product and we also use the notation $-a = a^{-1}$.

As the law in G_L is associative, the BCH product is also associative. An explicit form of the product is given by the Baker–Campbell–Hausdorff formula

$$a * b = a + b + \frac{1}{2}[a, b] + \frac{1}{12}[a, [a, b]] - \frac{1}{12}[b, [a, b]] + \cdots.$$

It follows from the Jacobi identity that in the Lie algebra of derivations of L we have $\text{ad}_{a*b} = \text{ad}_a * \text{ad}_b$. Hence $e^{\text{ad}_{a*b}} = e^{\text{ad}_a} \circ e^{\text{ad}_b}$.

Note that the BCH product is compatible with the gauge action on $\text{MC}(L)$.

LEMMA 1.1: *If $x, y \in L_0$ and $a \in \text{MC}(L)$, we have*

$$(x * y)\mathcal{G}a = x\mathcal{G}(y\mathcal{G}a).$$

Proof. Consider $L \oplus \mathbb{Q}c$ the cdgl where c is a cycle of degree -1 and the bracket extends the one on L defining $[x, c] = -(-1)^{|x|}\partial x$ for any $x \in L$ and $[c, c] = 0$. Then, for $x \in L_0$, a simple computation shows that

$$e^{\text{ad}_x}(b + c) = x\mathcal{G}b + c.$$

We thus have

$$\begin{aligned} (x * y)\mathcal{G}b + c &= e^{\text{ad}_{x*y}}(b + c) = e^{\text{ad}_x}e^{\text{ad}_y}(b + c) \\ &= e^{\text{ad}_x}(y\mathcal{G}b + c) = x\mathcal{G}(y\mathcal{G}b) + c, \end{aligned}$$

and therefore $(x * y)\mathcal{G}a = x\mathcal{G}(y\mathcal{G}a)$. ■

We also need the following property.

PROPOSITION 1.2: *Let L be a complete DGL and let $x, y \in L_0$. Then,*

$$x * y * (-x) = e^{\text{ad}_x}(y).$$

With the previous convention, the formula also reads

$$x * y * x^{-1} = e^{\text{ad}_x}(y).$$

Proof. First, we note that, in $\widehat{UL}$,

$$e^{\mathrm{ad}_x}(y) = e^x y e^{-x}.$$

Indeed,

$$\begin{aligned} e^{\mathrm{ad}_x}(y) &= \sum_{n=0}^{\infty} \frac{\mathrm{ad}_x^n(y)}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{i=0}^n (-1)^i \binom{n}{i} x^{n-i} y x^i \\ &= \sum_{i=0}^{\infty} \sum_{n=i}^{\infty} \frac{x^{n-i}}{(n-i)!} y \frac{(-x)^i}{i!} = e^x y e^{-x}. \end{aligned}$$

Replacing y by e^y , we deduce

$$(1) \quad e^{\mathrm{ad}_x}(e^y) = e^x e^y e^{-x}.$$

In a second step, we prove the equality

$$(2) \quad (e^{\mathrm{ad}_x})(y^n) = (e^{\mathrm{ad}_x}(y))^n.$$

On the left-hand side, the term of length k , $k \geq 0$, in x equals

$$\frac{\mathrm{ad}_x^k(y^n)}{k!}.$$

In the right-hand side, this term is

$$\sum_{k_1 + \dots + k_n = k} \frac{\mathrm{ad}_x^{k_1}(y)}{k_1!} \dots \frac{\mathrm{ad}_x^{k_n}(y)}{k_n!}.$$

As ad_x is a derivation, both terms coincide and the equality (2) is proved.

Therefore,

$$(3) \quad e^{\mathrm{ad}_x}(e^y) = \sum_{n \geq 0} \frac{(e^{\mathrm{ad}_x})(y^n)}{n!} = \sum_{n \geq 0} \frac{(e^{\mathrm{ad}_x}(y))^n}{n!} = e^{e^{\mathrm{ad}_x}(y)}.$$

Finally, the proposition follows from

$$x * y * (-x) = \log(e^x e^y e^{-x}) \stackrel{(1)}{=} \log(e^{\mathrm{ad}_x}(e^y)) \stackrel{(3)}{=} \log(e^{e^{\mathrm{ad}_x}(y)}) = e^{\mathrm{ad}_x}(y). \quad \blacksquare$$

The construction of a model for the interval was first introduced in [18].

Definition 1.3: The **Lawrence–Sullivan model for the interval**, LS-interval henceforth, is the complete DGL

$$(\mathfrak{L}_1, \partial) = (\widehat{\mathbb{L}}(a, b, x), \partial),$$

in which a and b are Maurer–Cartan elements, x is of degree 0 and

$$(4) \quad \partial x = \operatorname{ad}_x b + \sum_{n=0}^{\infty} \frac{B_n}{n!} \operatorname{ad}_x^n (b - a) = \operatorname{ad}_x b + \frac{\operatorname{ad}_x}{e^{\operatorname{ad}_x} - 1} (b - a),$$

where the B_n 's are the Bernoulli numbers.

Using the identity

$$\left(\frac{-x}{e^{-x} - 1} \right) = x + \left(\frac{x}{e^x - 1} \right),$$

we may also write

$$(5) \quad \partial x = \operatorname{ad}_x a + \frac{\operatorname{ad}_{-x}}{e^{-\operatorname{ad}_x} - 1} (b - a).$$

Let $(\widehat{\mathbb{L}}(a_0, a_1, a_2, x_1, x_2), \partial)$ be two glued LS-models of the interval. That is, a_0, a_1 and a_2 are Maurer–Cartan elements,

$$\partial x_1 = \operatorname{ad}_{x_1}(a_1) + \frac{\operatorname{ad}_{x_1}}{e^{\operatorname{ad}_{x_1}} - 1} (a_1 - a_0)$$

and

$$\partial x_2 = \operatorname{ad}_{x_2}(a_2) + \frac{\operatorname{ad}_{x_2}}{e^{\operatorname{ad}_{x_2}} - 1} (a_2 - a_1).$$

These two models give a model for the subdivision of an interval, as follows.

THEOREM 1.4 ([18, Theorem 2]): *The map*

$$\gamma: (\widehat{\mathbb{L}}(a, b, x), \partial) \longrightarrow (\widehat{\mathbb{L}}(a_0, a_1, a_2, x_1, x_2), \partial),$$

defined by $\gamma(a) = a_0$, $\gamma(b) = a_2$ and $\gamma(x) = x_1 * x_2$, is a DGL morphism.

Finally, we will make use of the following results (for the first cf. also [25]).

THEOREM 1.5: *Let $(\widehat{\mathbb{L}}(a, b, x), \partial')$ be a complete Lie algebra in which a, b are Maurer–Cartan elements and the linear part of the differential satisfies $\partial'_1 x = b - a$. Then*

$$(\widehat{\mathbb{L}}(a, b, x), \partial') = (\mathfrak{L}_1, \partial).$$

Proof. Let $\partial = \sum_{n \geq 1} \partial_n$, with $\operatorname{Im} \partial_n \subset \widehat{\mathbb{L}}^n(a, b, x)$, be the differential in $\mathfrak{L}_1$. We show that $\partial_n x = \partial'_n x$ for any $n \geq 1$. For $n = 1$ this is trivially true. For $n = 2$, by using $\partial_1 \partial'_2 + \partial'_2 \partial_1 = 0$, a short computation shows that

$$\partial_2 x = \partial'_2 x = \frac{1}{2} [x, a + b].$$

Assume $\partial_m x = \partial'_m x$, for $1 \leq m < n$. From $\partial^2 = \partial'^2 = 0$ and the induction hypothesis we deduce that $\partial_n x - \partial'_n x$ is a decomposable ∂_1 -cycle of degree -1 . Since $H_*(\widehat{\mathbb{L}}(a, b, x), \partial_1) = \mathbb{L}(a)$ and $\widehat{\mathbb{L}}(a, b, x)_0 = \mathbb{Q}x$, we conclude that

$$\partial_n x - \partial'_n x = 0. \quad \blacksquare$$

PROPOSITION 1.6: *Let L be a complete DGL which contains an LS-interval $(\widehat{\mathbb{L}}(a, b, x), \partial)$. Then, for any $v \in L$,*

$$\partial_a e^{\text{ad}_x}(v) = e^{\text{ad}_x}(\partial_b v).$$

In other words, the map

$$e^{\text{ad}_x} : (L, \partial_b) \rightarrow (L, \partial_a)$$

is an isomorphism of DGL's.

Proof. In fact,

$$\begin{aligned} \partial_b e^{-\text{ad}_x}(v) &= e^{-\text{ad}_x}(\partial_b v) + (-1)^{|v|} e^{-\text{ad}_x} \text{ad}_v \frac{e^{\text{ad}_x} - 1}{\text{ad}_x}(\partial_b x) \\ &= e^{-\text{ad}_x}(\partial_b v + (-1)^{|v|} \text{ad}_v(b - a)) \\ &= e^{-\text{ad}_x}(\partial_a v), \end{aligned}$$

where the first equality comes from [18, Lemma 1] and the second is a direct computation using (4). $\blacksquare$

2. Sequences of compatible models of Δ

For each $n \geq 0$, we consider the standard n -simplicial complex Δ^n , whose p -simplices are

$$\Delta_p^n = \{(i_0, \dots, i_p) \mid 0 \leq i_0 < \dots < i_p \leq n\}, \quad \text{if } p \leq n,$$

and $\Delta_p^n = \emptyset$ if $p > n$. Let $(\widehat{\mathbb{L}}(s^{-1}\Delta^n), d)$ be the complete free DGL on the desuspended rational simplicial chain complex on Δ^n ,

$$(6) \quad da_{i_0 \dots i_p} = \sum_{j=0}^p (-1)^j a_{i_0 \dots \widehat{i}_j \dots i_p}.$$

Here, $a_{i_0 \dots i_p}$ denotes the generator of degree $p - 1$ represented by the p -simplex $(i_0, \dots, i_p) \in \Delta_p^n$. Henceforth, unless explicitly needed, we drop the desuspension sign to avoid unnecessary notation and write simply $(\widehat{\mathbb{L}}(\Delta^n), d)$.

For each $0 \leq i \leq n$ consider the i -th coface map between simplicial complexes $\delta_i: \Delta^{n-1} \rightarrow \Delta^n$, defined on the vertices by

$$\delta_i(j) = \begin{cases} j, & \text{if } j < i, \\ j+1, & \text{if } j \geq i. \end{cases}$$

We use the same notation for the induced DGL morphism,

$$(7) \quad \delta_i: (\widehat{\mathbb{L}}(\Delta^{n-1}), d) \longrightarrow (\widehat{\mathbb{L}}(\Delta^n), d),$$

defined by

$$\delta_i(a_{j_0 \dots j_p}) = a_{\ell_0 \dots \ell_p} \quad \text{with } \ell_k = \begin{cases} j_k, & \text{if } j_k < i, \\ j_k + 1, & \text{if } j_k \geq i. \end{cases}$$

Finally, we denote by $\dot{\Delta}^n$ and Λ_i^n the boundary of Δ^n and the i -horn obtained by removing the i -th coface from $\dot{\Delta}^n$.

Definition 2.1: A **sequence of compatible models of Δ** is a family

$$\{(\mathfrak{L}_n, \partial) = (\widehat{\mathbb{L}}(\Delta^n), \partial)\}_{n \geq 0}$$

of DGL's satisfying the following properties:

- (1) For each $i = 0, \dots, n$, the generator $a_i \in \Delta_0^n$ is a Maurer–Cartan element, $\partial a_i = -\frac{1}{2}[a_i, a_i]$.
- (2) The linear part ∂_1 of ∂ is precisely d as in (6).
- (3) For each $i = 0, \dots, n$, the coface maps, $\delta_i: (\widehat{\mathbb{L}}(\Delta^{n-1}), \partial) \rightarrow (\widehat{\mathbb{L}}(\Delta^n), \partial)$, are DGL morphisms.

Each element $(\widehat{\mathbb{L}}(\Delta^n), \partial)$ of this sequence is called a **model** of Δ^n , which is thus implicitly endowed with models of Δ^q , $q < n$, satisfying the compatibility condition (3) above.

Definition 2.2: A sequence of models $\{(\widehat{\mathbb{L}}(\Delta^n), \partial)\}_{n \geq 0}$ is called **inductive** if, for $n \geq 2$, we have

$$(8) \quad \partial_{a_0} a_0 \dots a_n \in \widehat{\mathbb{L}}(\dot{\Delta}^n).$$

THEOREM 2.3: *There exist sequences of compatible inductive models of Δ .*

Its proof uses the following key proposition: Let $(\widehat{\mathbb{L}}(V \oplus W), \partial)$ be a complete free DGL in which V and W are of finite dimension. We denote by $\mathcal{I}$ the ideal generated by W , and by ∂_1 the linear part of the differential.

PROPOSITION 2.4: *With the above notations, if $\partial(\mathcal{J}) \subset \mathcal{J}$ and $H(W, \partial_1) = 0$, then the projection*

$$(\widehat{\mathbb{L}}(V \oplus W), \partial) \rightarrow (\widehat{\mathbb{L}}(V), \overline{\partial}) = (\widehat{\mathbb{L}}(V \oplus W)/\mathcal{J}, \overline{\partial})$$

is a quasi-isomorphism.

Moreover, every cycle in $\mathcal{J} \cap \widehat{\mathbb{L}}^{\geq n}(V \oplus W)$ is a boundary $\alpha = \partial\beta$ with

$$\beta \in \mathcal{J} \cap \widehat{\mathbb{L}}^{\geq n}(V \oplus W).$$

Proof. Let K be the kernel of the projection $(\widehat{\mathbb{L}}(V \oplus W), \partial_1) \rightarrow (\widehat{\mathbb{L}}(V), \partial_1)$. Since $H(W, \partial_1) = 0$, K is acyclic. Now let $\alpha \in \mathcal{J}$ be a ∂ -cycle. There exists n such that $\alpha = \alpha_n + \beta_{n+1}$, with $\alpha_n \in \mathcal{J} \cap \widehat{\mathbb{L}}^n(W \oplus V)$ and $\beta_{n+1} \in \mathcal{J} \cap \widehat{\mathbb{L}}^{>n}(W \oplus V)$. From $\partial\alpha = 0$, we deduce $\partial_1\alpha_n = 0$ and since $\alpha_n \in K \cap \widehat{\mathbb{L}}^n(W \oplus V)$ and $H(K, \partial_1) = 0$, there is an element

$$\gamma_n \in K \cap \widehat{\mathbb{L}}^n(W \oplus V) = \mathcal{J} \cap \widehat{\mathbb{L}}^n(W \oplus V)$$

such that $\alpha_n = \partial_1\gamma_n$. Thus, we have $\alpha - \partial\gamma_n \in \mathcal{J} \cap \widehat{\mathbb{L}}^{>n}(W \oplus V)$. By iterating this process, we obtain a sequence $(\gamma_j)_{j \geq n}$, such that $\gamma_j \in \mathcal{J} \cap \widehat{\mathbb{L}}^j(W \oplus V)$ and $\alpha = \partial(\sum_{j \geq n} \gamma_j)$. ■

COROLLARY 2.5: *Let $(\widehat{\mathbb{L}}(\Delta^n), \partial)$ be a model of Δ^n , and $n \geq 2$. Then:*

- (i) $H(\widehat{\mathbb{L}}(\Delta^n), \partial_{a_0}) = H(\widehat{\mathbb{L}}(\Lambda_i^n), \partial_{a_0}) = 0$ for any $i = 0, \dots, n$.
- (ii) *If the model is inductive and $n \geq 2$, then $H(\widehat{\mathbb{L}}(\dot{\Delta}^n), \partial_{a_0}) = \mathbb{L}(\Omega)$, with $\Omega = \partial_{a_0} a_{0\dots n}$.*

Proof. (i) This follows from Proposition 2.4. Let W be the vector space generated by the elements $a_i - a_0$ and the elements of higher degree. We verify directly that $H(W, d_1) = 0$. Moreover, the ideal generated by W is a differential ideal. This is clear by construction for $\mathfrak{L}_1$ and for degree reasons when $n \geq 2$.

(ii) By construction, $(\widehat{\mathbb{L}}(\dot{\Delta}^n), \partial) = (\widehat{\mathbb{L}}(\Lambda_n^n) \widehat{\mathbb{H}}\mathbb{L}(u), \partial)$, with $u = a_{0\dots n-1n}$. Let W be as above for Λ_n^n and $\mathcal{J}$ the ideal generated by W in $\widehat{\mathbb{L}}(\dot{\Delta}^n)$. Since $H(W, \partial_1) = 0$, by Proposition 2.4, the projection

$$(\widehat{\mathbb{L}}(\dot{\Delta}^n), \partial_{a_0}) \rightarrow (\widehat{\mathbb{L}}(\dot{\Delta}^n)/\mathcal{J}, \overline{\partial}_{a_0}) = (\widehat{\mathbb{L}}(a_0, u), \overline{\partial}_{a_0})$$

is a quasi-isomorphism. Now by the inductive hypothesis, $\overline{\partial}_{a_0}(u) = 0$. Then by Lemma 2.6, the injection $(\mathbb{L}(u), 0) \rightarrow (\widehat{\mathbb{L}}(a_0, u), \overline{\partial}_{a_0})$ is a quasi-isomorphism. Therefore, $H(\widehat{\mathbb{L}}(\dot{\Delta}^n), \partial_{a_0}) = \mathbb{L}(u)$. ■

LEMMA 2.6: Let $(\widehat{\mathbb{L}}(\mathbb{Q}a \oplus V), \partial)$ be a DGL in which a is a Maurer–Cartan element, $\dim V < \infty$ and $\partial_a(V) \subset \widehat{\mathbb{L}}(V)$. Then, the injection

$$(\widehat{\mathbb{L}}(V), \partial_a) \hookrightarrow (\widehat{\mathbb{L}}(\mathbb{Q}a \oplus V), \partial_a)$$

is a quasi-isomorphism.

Proof. Denote by J the ideal generated by V . Since V is finite dimensional,

$$J = \widehat{\mathbb{L}}(\mathrm{ad}_a^q v, \ v \text{ basis of } V \text{ and } q \geq 0).$$

We then denote by d_1 the linear part of the differential d in J .

We claim that for $v \in V$, and q odd,

$$d_1 \mathrm{ad}_a^q(v) = \mathrm{ad}_a^{q+1}(v) + (-1)^q \mathrm{ad}_a^q(\partial_1 v).$$

This holds for $q = 1$ because $d_1[a, v] = \frac{1}{2}[[a, a], v] - [a, \partial_1 v] = -\mathrm{ad}_a^2(v) - [a, \partial_1 v]$. Suppose this is true for $q - 2$. Then, since $\mathrm{ad}_a^q(v) = \frac{1}{2}[[a, a], \mathrm{ad}_a^{q-2}(v)]$, we have

$$\begin{aligned} d_1 \mathrm{ad}_a^q(v) &= \frac{1}{2} d_1 [[a, a], \mathrm{ad}_a^{q-2}(v)] \\ &= \frac{1}{2} [[a, a], \mathrm{ad}_a^{q-1}(v)] + (-1)^q \mathrm{ad}_a^q(\partial_1 v) \\ &= \mathrm{ad}_a^{q+1}(v) + (-1)^q \mathrm{ad}_a^q(\partial_1 v). \end{aligned}$$

The result is then a consequence of Proposition 2.4. ■

PROPOSITION 2.7: The model of the triangle is given by

$$(9) \quad \mathfrak{L}_2 = (\widehat{\mathbb{L}}(\Delta^2), \partial) = (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{12}, a_{02}, a_{012}), \partial),$$

in which $(\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial)$, $(\widehat{\mathbb{L}}(a_1, a_2, a_{12}), \partial)$ and $(\widehat{\mathbb{L}}(a_0, a_2, a_{02}), \partial)$ are LS-intervals and where their inclusions in $\mathfrak{L}_2$ give the coface maps. Moreover, the differential of a_{012} is defined by

$$\partial_{a_0} a_{012} = a_{01} * a_{12} * a_{02}^{-1}.$$

Proof. Let $(\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}), \partial)$ be the DGL in which $\widehat{\mathbb{L}}(a_0, a_1, a_{01})$, $\widehat{\mathbb{L}}(a_0, a_2, a_{02})$ and $\widehat{\mathbb{L}}(a_1, a_2, a_{12})$ are LS-intervals. Consider the DGL morphism

$$\varphi: (\widehat{\mathbb{L}}(a, b, x), \partial) \rightarrow (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}), \partial)$$

given by the composition

$$(\widehat{\mathbb{L}}(a, b, x), \partial) \xrightarrow{\psi} (\widehat{\mathbb{L}}(a_0, a_1, a_2, x_1, x_2), \partial) \xrightarrow{\psi'} (\widehat{\mathbb{L}}(a_0, a_1, a_2, a_{01}, a_{02}, a_{12}), \partial),$$

where ψ is the morphism of Theorem 1.4 and $\psi'(a_0) = a_0$; $\psi'(a_1) = a_1$; $\psi'(a_2) = a_0$; $\psi'(x_1) = a_{01}$; $\psi'(x_2) = a_{12} * a_{02}^{-1}$. The same argument as that in Theorem 1.4 shows that ψ' is a DGL morphism. Then, $\varphi(a) = \varphi(b) = a_0$ and $\varphi(x) = a_{01} * a_{12} * a_{02}^{-1}$. Hence,

$$\begin{aligned} \partial(a_{01} * a_{12} * a_{02}^{-1}) &= \partial\varphi(x) = \varphi(\partial x) \\ &= \varphi\left(\mathrm{ad}_x(b) + \sum_{k \geq 0} \frac{B_k}{k!} \mathrm{ad}_x^k(b - a)\right) \\ &= \mathrm{ad}_{a_{01} * a_{12} * a_{02}^{-1}}(a_0). \end{aligned}$$

In other terms, $a_{01} * a_{12} * a_{02}^{-1}$ is a ∂_{a_0} -cycle, and we can add a generator a_{012} such that $\partial_{a_0} a_{012} = a_{01} * a_{12} * a_{02}^{-1}$.

Finally, observe that the linear part of $\partial_{a_0}(a_{012})$ is $a_{12} - a_{02} + a_{01}$. In geometrical terms, this expression draws the border of Δ^2 starting from the base point a_0 . ■

Proof of Theorem 2.3. We proceed by induction on n . Let $(\mathfrak{L}_0, \partial) = (\widehat{\mathbb{L}}(a_0), \partial)$ in which a_0 is a Maurer–Cartan element. On the other hand, let

$$\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial)$$

be the LS-interval and let $\mathfrak{L}_2$, the first one satisfying the inductive statement, be the model of the triangle as in Proposition 2.7.

Assume that $\mathfrak{L}_q = (\widehat{\mathbb{L}}(\Delta^q), \partial)$ is defined for $q < n$, as in the statement, with $n \geq 3$. The condition (3) of Definition 2.1 defines the differential ∂ on each face of Δ^n . In particular, by induction, $\partial_{a_0} a_{0\dots n-1} \in \widehat{\mathbb{L}}(\Lambda_n^n)$. As $H(\widehat{\mathbb{L}}(\Lambda_n^n), \partial_{a_0}) = 0$ by Corollary 2.5(i), there exists $\Gamma \in \widehat{\mathbb{L}}(\Lambda_n^n)$ such that $|\Gamma| = n - 2$ and $\partial_{a_0} a_{0\dots n-1} = \partial_{a_0} \Gamma$. We set

$$\partial_{a_0} a_{0\dots n} = (-1)^n (a_{0\dots n-1} - \Gamma).$$

By construction, this model satisfies the conditions (1) and (3) of Definition 2.1. Now denote by $\partial_1 a_{0\dots n}$ the linear part of $\partial_{a_0} a_{0\dots n}$ and by Γ_1 the linear part of Γ . Since $\partial_1^2 = 0$, $\partial_1 \Gamma_1 = \partial_1(a_{0\dots n-1})$. Let ω be the difference

$$\omega = (-1)^{n-1} \Gamma_1 - \sum_{i=0}^{n-1} (-1)^i a_{0\dots \widehat{i} \dots n}.$$

Since $\partial_1(\sum_{i=0}^n (-1)^i a_{0\dots \widehat{i} \dots n}) = 0$,

$$\partial_1 \omega = (-1)^{n-1} \partial_1(a_{0\dots n-1}) + (-1)^n \partial_1(a_{0\dots n-1}) = 0.$$

Thus, ω is a ∂_1 -cycle of degree $n - 2$ in $\widehat{\mathbb{L}}(\Lambda_n^n)$. Hence, there is a linear element, γ , of degree $n - 1$ in $\widehat{\mathbb{L}}(\Lambda_n^n)$ such that $\partial_1 \gamma = \omega$. As $\widehat{\mathbb{L}}(\Lambda_n^n)$ is generated by elements of degree $\leq n - 2$, we have $\gamma = 0$ and $\omega = 0$. Therefore, we get the expected linear differential,

$$\partial_1 a_{0\dots n} = (-1)^n a_{0\dots n-1} + \sum_{i=0}^{n-1} (-1)^i a_{0\dots \widehat{i} \dots n-1}. \quad \blacksquare$$

We now prove the uniqueness up to isomorphism of the sequences of models.

THEOREM 2.8: *Two sequences $\{(\widehat{\mathbb{L}}(\Delta^n), \partial)\}_{n \geq 0}$ and $\{(\widehat{\mathbb{L}}(\Delta^n), \partial')\}_{n \geq 0}$ of compatible models of Δ are isomorphic: for $n \geq 0$, there are DGL isomorphisms,*

$$\varphi_n: (\widehat{\mathbb{L}}(\Delta^n), \partial) \xrightarrow{\cong} (\widehat{\mathbb{L}}(\Delta^n), \partial'),$$

which commute with the coface maps δ_i , for $i = 0, \dots, n$,

$$(10) \quad \begin{array}{ccc} (\widehat{\mathbb{L}}(\Delta^n), \partial) & \xrightarrow[\cong]{\varphi_n} & (\widehat{\mathbb{L}}(\Delta^n), \partial') \\ \delta_i \uparrow & & \uparrow \delta_i \\ (\widehat{\mathbb{L}}(\Delta^{n-1}), \partial) & \xrightarrow[\cong]{\varphi_{n-1}} & (\widehat{\mathbb{L}}(\Delta^{n-1}), \partial'), \end{array}$$

and such that $\text{Im}(\varphi_n - \text{id}) \subset \widehat{\mathbb{L}}^{\geq 2}(\Delta^n)$.

Proof. Obviously there is only one choice for $(\widehat{\mathbb{L}}(\Delta^0), \partial)$ while $(\widehat{\mathbb{L}}(\Delta^1), \partial)$ is also uniquely determined by Theorem 1.5. Suppose $n \geq 2$ and φ_m defined for all $m < n$. Then, the condition (3) of Definition 2.1 determines φ_n on every generator of $\widehat{\mathbb{L}}(\Delta^n)$ except on $x = a_{0\dots n}$.

As an inductive model exists, if we prove the existence of the isomorphism of the statement when the sequence

$$\{(\widehat{\mathbb{L}}(\Delta^n), \partial)\}_{n \geq 0}$$

is inductive, we will get the general result by transitivity. Hence, as $\partial_{a_0} x \in \widehat{\mathbb{L}}(\Delta^n)$, the hypothesis $\text{Im}(\varphi_i - \text{id}) \subset \widehat{\mathbb{L}}^{\geq 2}(\Delta^i)$ for $i < n$, implies that $\varphi_n \partial_{a_0} x - \partial_{a_0} x$ is a decomposable element. Thus, as $\partial_{a_0} x$ and $\partial'_{a_0} x$ have the same linear part, we deduce that $\varphi_n \partial_{a_0} x - \partial'_{a_0} x$ is a decomposable element. Since this is a ∂'_{a_0} -cycle, there exists $\omega \in \widehat{\mathbb{L}}(\Delta^n)$, $|\omega| = n - 1$, such that $\partial'_{a_0} \omega = \varphi_n \partial_{a_0} x - \partial'_{a_0} x$. The linear part, ω_1 , of ω is a ∂_1 -cycle and there exists

a linear element $\gamma \in \widehat{\mathbb{L}}(\Delta^n)$, $|\gamma| = n$, such that $\partial'_1 \gamma = \omega_1$. By degree reasons, we have $\gamma = 0$ and $\omega_1 = 0$. Thus, we set

$$\varphi_n(x) = x + \omega.$$

As $\omega \in \widehat{\mathbb{L}}^{\geq 2}(\Delta^n)$, the condition $\text{Im}(\varphi_n - \text{id}) \subset \widehat{\mathbb{L}}^{\geq 2}(\Delta^n)$ is fulfilled. Moreover, the coface maps being linear, the square (10) is commutative. ■

Example 2.9 (The model of the tetrahedron): Let (L, ∂) be a complete DGL, and $e_1, \dots, e_n \in L_1$. We form the DGL $(L', \partial') = (\widehat{\mathbb{L}}(e_i, u_i), \partial')$ where $\partial'(u_i) = 0$ and $\partial'(e_i) = u_i$ and the DGL morphism $\gamma: (L', \partial') \rightarrow (L, \partial)$ defined by

$$\gamma(e_i) = e_i \quad \text{and} \quad \gamma(u_i) = \partial e_i.$$

Now the product $u_1 * \dots * u_n$ is a linear combination of Lie brackets and in each of them we replace one and only one u_i by the corresponding e_i . This defines an element $A_{e_1 \dots e_n} \in L'$, and we define $B_{e_1 \dots e_n} = \gamma(A_{e_1 \dots e_n})$. By construction we have

$$\partial B_{e_1 \dots e_n} = \partial e_1 * \dots * \partial e_n,$$

and its linear part is precisely $\sum_{i=1}^n e_i$.

For the model of the tetrahedron, we observe that via Proposition 2.7,

$$\begin{aligned} \partial_{a_0} B_{a_{012}, a_{023}, -a_{013}} &= \partial_{a_0} a_{012} * \partial_{a_0} a_{023} * (-\partial_{a_0} a_{013}) \\ &= a_{01} * a_{12} * a_{23} * a_{13}^{-1} * a_{01}^{-1}. \end{aligned}$$

On the other hand, a direct computation, using successively Proposition 1.6, the model of the triangle described in Proposition 2.7 and Proposition 1.2, gives

$$\begin{aligned} \partial_{a_0}(e^{\text{ad}_{a_{01}}} a_{123}) &= e^{\text{ad}_{a_{01}}} \partial_{a_1} a_{123} \\ &= e^{\text{ad}_{a_{01}}} (a_{12} * a_{23} * a_{13}^{-1}) \\ &= a_{01} * a_{12} * a_{23} * a_{13}^{-1} * a_{01}^{-1}. \end{aligned}$$

Finally, define

$$\partial_{a_0} a_{0123} = e^{\text{ad}_{a_{01}}} a_{123} - B_{a_{012}, a_{023}, -a_{013}}.$$

We finish by finding base points in the barycentre of the simplex.

PROPOSITION 2.10: *The simplicial complex Δ^n admits a model $(\widehat{\mathbb{L}}(\Delta^n), \partial)$ where $\text{Im } \partial_b \subset \widehat{\mathbb{L}}(\Delta^n)$, and b is a Maurer–Cartan element whose linear part is the barycentre of Δ^n .*

Proof. Let $(\mathfrak{L}_n, \partial)$ be an inductive model for Δ^n , and let

$$x = -\frac{1}{n+1} \sum_{i=1}^n a_{0i}.$$

Then $b = x\mathfrak{G}a_0$ is a Maurer–Cartan element whose linear part is $\frac{1}{n+1} \sum a_i$. Consider the element $u = e^{\text{ad}_x} a_{0\dots n}$. By Proposition 1.6,

$$\partial_b u = e^{\text{ad}_x} \partial_{a_0} a_{0\dots n} \in \widehat{\mathbb{L}}(\dot{\Delta}^n).$$

Since

$$(\widehat{\mathbb{L}}(\{a_I\}_{I \subsetneq \{0, \dots, n\}}, u), \partial)$$

is isomorphic to $(\widehat{\mathbb{L}}(\Delta^n), \partial)$, the proposition is proved. $\blacksquare$

PROPOSITION 2.11: *For $n \geq 2$ we can suppose that $\partial_{a_0}(a_{0\dots n})$ belongs to the complete free Lie subalgebra generated by the elements $a_{i_0\dots i_r}$ with $0 < r < n$.*

Proof. Consider the complete sub-Lie algebra $\widehat{\mathbb{L}}(E_n) \subset \mathfrak{L}_n$ defined as follows: Write $c_0 = 0$ and, for $i > 0$, denote by

$$c_i = a_{01} * a_{12} * \cdots * a_{i-1,i}$$

the path joining a_0 and a_i by means of elements of the form $a_{r,r+1}$. Then, for $i < j$ consider the element

$$b_{ij} = c_i * a_{ij} * c_j^{-1}.$$

Then, b_{ij} is a loop at a_0 . By construction when $j = i + 1$, we have $b_{ij} = 0$.

Now for $r > 1$, let

$$b_{i_0\dots i_r} = e^{\text{ad}_{c_{i_0}}} (a_{i_0\dots i_r}).$$

Denote by E_n the vector space generated by elements b_{ij} , $i < j - 1$, and $b_{i_0\dots i_r}$ with $2 \leq r \leq n$. Since the natural morphism

$$\widehat{\mathbb{L}}(E_n) \hat{\Pi} \widehat{\mathbb{L}}(\{a_i\}, \{a_{i,i+1}\}) \rightarrow \mathfrak{L}_n$$

is an isomorphism on the indecomposable elements, it follows that it is an isomorphism.

We prove by induction on $r \geq 2$ that we can choose $\partial_{a_0}(a_{0\dots r}) \in \widehat{\mathbb{L}}(E_{r-1})$. The model of the triangle (Proposition 2.7) shows that this holds for the case $n = 2$, because $\partial_{a_0}(a_{012}) = -b_{02}$. We suppose now that this is true for $2 \leq r < n$.

Denote by $F_n \subset E_n$ the subspace generated by elements b_{ij} , $i < j - 1$ and $b_{i_0 \dots i_r}$, $2 \leq r < n$, with $\{i_0, \dots, i_r\} \neq \{0, 1, \dots, n-1\}$. Then the natural map

$$\varphi : \widehat{\mathbb{L}}(F_n) \widehat{\Pi} \widehat{\mathbb{L}}(\{a_i\}, \{a_{j,j+1}\}) \rightarrow \widehat{\mathbb{L}}(s^{-1}\Lambda_n^n)$$

is an isomorphism of graded Lie algebras. We show now that $\widehat{\mathbb{L}}(F_n)$ is stable for the differential ∂_{a_0} , i.e., $\partial_{a_0}(b_{i_0 \dots i_r}) \in \widehat{\mathbb{L}}(F_n)$ for any $b_{i_0 \dots i_r} \in F_n$. For $r = 2$ this is true by the following identities:

$$\begin{aligned} \partial_{a_0} b_{ijk} &= e^{\text{ad}_{c_i}} \partial_{a_i}(a_{ijk}) = e^{\text{ad}_{c_i}}(a_{ij} * a_{jk} * a_{ik}^{-1}) \\ &= c_i * a_{ij} * a_{jk} * a_{ik}^{-1} * c_i^{-1} \\ &= (c_i * a_{ij} * c_j^{-1}) * (c_j * a_{jk} * c_k^{-1}) * (c_k * a_{ik}^{-1} * c_i^{-1}) \\ &= b_{ij} * b_{jk} * b_{ik}^{-1}, \end{aligned}$$

where we have used Proposition 1.6 and Proposition 1.2.

Remark that every loop based at a_0 in the 1-skeleton of Λ_n^n is a Baker–Campbell–Hausdorff product of loops of the form b_{ij} . Moreover, if c is a path connecting a_0 to some point a_{i_0} , then $c = c * c_{i_0}^{-1} * c_{i_0}$ and

$$e^{\text{ad}_c}(a_{i_0 \dots i_r}) = e^{\text{ad}_{c * c_{i_0}^{-1}}} e^{\text{ad}_{c_{i_0}}}(a_{i_0 \dots i_r}) = e^{\text{ad}_{c * c_{i_0}^{-1}}}(b_{i_0 \dots i_r}).$$

Since $c * c_{i_0}^{-1}$ is a loop at a_0 and is a product of b_{ij} , we have that $e^{\text{ad}_c}(a_{i_0 \dots i_r})$ is in $\widehat{\mathbb{L}}(F_n)$.

Next, let $a_{i_0 \dots i_r}$ with $2 < r < n$. By the inductive hypothesis $\partial_{a_0}(a_{0 \dots r}) \in F_r$. The element $a_{i_0 \dots i_r}$ is the image of $a_{0 \dots r}$ along a composition of coface operators δ_i . Since the δ_i are cdgl morphisms, $\partial_{a_{i_0}}(a_{i_0 \dots i_r})$ belongs to the complete free Lie algebra generated by loops in the 1-skeleton based in a_{i_0} in degree 0 and elements of the form $e^{\text{ad}_c}(a_{j_0 \dots j_s})$ where c is a path joining a_{i_0} to a_{j_0} . Since by Proposition 1.6 we have that

$$\partial_{a_0}(b_{i_0 \dots i_r}) = e^{\text{ad}_{c_{i_0}}} \partial_{a_{i_0}}(a_{i_0 \dots i_r}),$$

it follows that $\partial_{a_0}(b_{i_0 \dots i_r})$ is in the complete free Lie algebra generated by loops at a_0 and elements of the form $e^{\text{ad}_t}(a_{j_0 \dots j_s})$ where t is a path joining a_0 to a_{j_0} . Therefore

$$\partial_{a_0}(b_{i_0 \dots i_r}) \in \widehat{\mathbb{L}}(F_n).$$

Since $H_*(\widehat{\mathbb{L}}(s^{-1}\Lambda_n^n), \partial_1) = \mathbb{L}(a_0)$, we have that

$$H(\widehat{\mathbb{L}}(F_n), \partial_1) = 0.$$

We continue with the construction of $\partial_{a_0}(a_{0\dots n})$. There is an element

$$\Gamma \in \widehat{\mathbb{L}}(s^{-1}\Lambda_n^n)$$

with $\partial_{a_0}\Gamma = \partial_{a_0}(a_{0\dots n-1})$, and $\partial_{a_0}(a_{0\dots n}) = (-1)^n(a_{0\dots n-1} - \Gamma)$. We will construct $\Gamma \in \widehat{\mathbb{L}}(F_n)$. By induction $\alpha = \partial_{a_0}(a_{0\dots n-1}) \in \widehat{\mathbb{L}}(F_n)$, and we decompose α as $\alpha = \alpha_1 + \alpha_2 + \dots$ with $\alpha_r \in \widehat{\mathbb{L}}^r(F_n)$. The element α_1 is a ∂_1 -cycle and so there is $u_1 \in F_n$ with $\partial_1(u_1) = \alpha_1$. We suppose by induction to have constructed $u_1, \dots, u_r$ such that $v = \alpha - \partial(u_1 + \dots + u_r) \in \widehat{\mathbb{L}}^{>r}(F_n)$. Then, its component v_{r+1} in bracket-length $r+1$ is a ∂_1 -cycle and so a ∂_1 -boundary, $v_{r+1} = \partial_1(u_{r+1})$. Then $\alpha - \partial(u_1 + \dots + u_{r+1})$ belongs to $\widehat{\mathbb{L}}^{>r+1}(F_n)$. By iteration we construct a sequence u_r and we define

$$\Gamma = \sum_r u_r \in \widehat{\mathbb{L}}(F_n).$$

Therefore $\partial_{a_0}(a_{0\dots n}) \in \widehat{\mathbb{L}}(F_n)$.

Writing each element of F_n in terms of the $a_{i_0\dots i_p}$, $p \geq 1$, implies that $\partial_{a_0}(a_{0\dots n})$ belongs to the complete free Lie algebra generated by the $a_{i_0\dots i_r}$ with $0 < r < n$. ■

3. Symmetric models of Δ and the cosimplicial structure

Let $a_{i_0\dots i_p}$ be a generator of $\widehat{\mathbb{L}}(\Delta^n)$ and $\sigma \in \Sigma_{p+1}$ of signature ε_σ . We set

$$a_{i_{\sigma(0)}\dots i_{\sigma(p)}} = \varepsilon_\sigma a_{i_0\dots i_p}.$$

With this notation, we define an action of the symmetric group Σ_{n+1} on the generators of $\widehat{\mathbb{L}}(\Delta^n)$ by the rule

$$\sigma a_{i_0\dots i_p} = a_{\sigma(i_0)\dots\sigma(i_p)}.$$

We extend it to brackets by $\sigma \cdot [a, b] = [\sigma \cdot a, \sigma \cdot b]$ and get an action on $\widehat{\mathbb{L}}(\Delta^n)$. In particular, we have

$$\sigma a_{0\dots n} = \varepsilon(\sigma) a_{0\dots n},$$

where $\varepsilon(\sigma)$ is the index of the permutation σ . We remark that the differential ∂_1 is compatible with this action, making $(\widehat{\mathbb{L}}(\Delta^n), \partial_1)$ a Σ_{n+1} -equivariant model.

Definition 3.1: A sequence $\{(\widehat{\mathbb{L}}(\Delta^n), \partial)\}_{n \geq 0}$ of compatible models of Δ is called **symmetric** if for the above action each $(\widehat{\mathbb{L}}(\Delta^n), \partial)$ is a Σ_{n+1} -DGL, that is, a DGL whose bracket and differential are compatible with the Σ_{n+1} action.

Since $(\mathfrak{L}_n, \partial_1)$ is Σ_{n+1} -DGL, we denote by L_ε the set of elements $a \in \widehat{\mathbb{L}}(\Delta^n)$ such that $\sigma(a) = \varepsilon(\sigma)a$.

LEMMA 3.2: $H_q(L_\varepsilon, \partial_1) = 0$ for $q > 0$.

Proof. Note that $(L_\varepsilon, \partial_1)$ is a retract of $(\widehat{\mathbb{L}}(\Delta^n), \partial_1)$ with a retraction ρ given by

$$\rho(a) = \frac{1}{(n+1)!} \sum_{\sigma} \varepsilon(\sigma) \sigma(a).$$

Therefore, $H(L_\varepsilon, \partial_1)$ injects into

$$H(\widehat{\mathbb{L}}(\Delta^n), \partial_1) = \mathbb{L}(a_0).$$

This implies the result. ■

THEOREM 3.3: *There exists a sequence of symmetric models of Δ .*

Proof. The model of Δ^0 is trivially symmetric. Let σ be the generator of Σ_2 and observe directly that in the LS-interval, $\mathfrak{L}_1 = (\widehat{\mathbb{L}}(a_0, a_1, a_{01}), \partial)$, the morphism defined by $\sigma(a_0) = a_1$, $\sigma(a_1) = a_0$ and $\sigma(a_{01}) = -a_{01}$ commutes with ∂ . We deduce then directly that $\mathfrak{L}_1$ is symmetric.

Assume $(\widehat{\mathbb{L}}(\Delta^{n-1}), \partial)$ to be symmetric with $n \geq 2$ and observe that, by construction, $(\mathfrak{L}(\dot{\Delta}^n), \partial)$ is a Σ_{n+1} -DGL. It remains only to define $\partial(a_{0\dots n})$ in order that $\sigma\partial(a_{0\dots n}) = \partial\sigma(a_{0\dots n})$ for every permutation σ . It means

$$\sigma\partial(a_{0\dots n}) = \varepsilon(\sigma)\partial(a_{0\dots n}).$$

Write $x = a_{0\dots n}$ and $\partial = \partial_1 + \partial_2 + \dots$ where ∂_q increases the length of the brackets by $q-1$. By definition $\partial_1 x = \sum_{i=0}^n (-1)^i a_{0\dots \widehat{i} \dots n}$ satisfies

$$\sigma\partial_1 x = \varepsilon(\sigma)\partial_1 x,$$

and we suppose by induction that, for all $r < q$, the elements $\partial_r x$ have been defined and satisfy

$$\sum_{i+j=r+1} \partial_i \partial_j x = 0, \quad \text{and} \quad \sigma(\partial_r x) = \varepsilon(\sigma) \partial_r x.$$

Then $\sum_{i=2}^q \partial_i \partial_{q+1-i} x$ is a ∂_1 -cycle in L_ε . By Lemma 3.2 there exists an element $\omega_q \in L_\varepsilon$ such that

$$\sum_{i=2}^q \partial_i \partial_{q+1-i} x = \partial_1 \omega_q.$$

We set $\partial_q x = \omega_q$ and check easily that the induction step is attained. ■

THEOREM 3.4: *Any sequence $\{\mathfrak{L}_n\}_{n \geq 0}$ of models of Δ admits a cosimplicial DGL structure for which the cofaces are the usual ones.*

Proof. Assume first that the models $\mathfrak{L}_n$ are symmetric. For $0 \leq i \leq n$, denote as usual by $\sigma_i: \{0, \dots, n+1\} \rightarrow \{0, \dots, n\}$ the morphism defined by

$$\sigma_i(j) = \begin{cases} j & \text{if } j \leq i, \\ j-1 & \text{if } j > i. \end{cases}$$

Then, we define a DGL morphism, denoted in the same way,

$$\sigma_i(a_{\ell_0 \dots \ell_q}) = \begin{cases} a_{\sigma_i(\ell_0) \dots \sigma_i(\ell_q)} & \text{if } \sigma_i(\ell_0) < \dots < \sigma_i(\ell_q), \\ 0 & \text{otherwise.} \end{cases}$$

When the elements $\sigma_i(\ell_j)$ are different, then σ_i extends to an element of Σ_{n+2} and $\partial \sigma_i(a_{\ell_0 \dots \ell_q}) = \sigma_i \partial(a_{\ell_0 \dots \ell_q})$.

In the case the sequence $\ell_0, \dots, \ell_q$ contains the elements i and $i+1$, then denote by τ the permutation $\tau = (i, i+1)$. We have $\sigma_i \circ \tau = \sigma_i$. Thus,

$$\sigma_i \partial(a_{\ell_0 \dots \ell_q}) = \sigma_i \tau \partial(a_{\ell_0 \dots \ell_q}) = -\sigma_i \partial(a_{\ell_0 \dots \ell_q}).$$

Therefore $\sigma_i \partial(a_{\ell_0 \dots \ell_q}) = 0 = \partial \sigma_i(a_{\ell_0 \dots \ell_q})$, and thus the σ_i 's are DGL morphisms. The cosimplicial identities are trivially satisfied.

Now let $\{\mathfrak{L}'_n\}_{n \geq 0}$ be another sequence of compatible models. By Theorem 2.8 we have compatible isomorphisms

$$\varphi_n: \mathfrak{L}'_n \xrightarrow{\cong} \mathfrak{L}_n.$$

We define then the codegeneracies as $\varphi_n^{-1} \sigma_i \varphi_n$. Since the φ_n 's commute with the cofaces, the cosimplicial identities are also satisfied in this case. ■

4. The realization functor and its adjoint

Based on a sequence $\mathfrak{L}_\bullet$ of compatible models of Δ with the cosimplicial structure given by Theorem 3.4, we define a pair of adjoint functors

$$\text{SimpSet} \begin{array}{c} \xrightarrow{\mathfrak{L}} \\ \xleftarrow{\langle \cdot \rangle} \end{array} \text{DGL}$$

between the categories of simplicial sets and complete DGL's.

Definition 4.1: Let $L \in \mathbf{DGL}$. The **realization** of L is the simplicial set

$$\langle L \rangle_{\bullet} = \mathbf{DGL}(\mathfrak{L}_{\bullet}, L).$$

On the other hand, let $\underline{\Delta}$ be the category whose objects are the sets $[n] = \{0, \dots, n\}$, $n \geq 0$, and whose morphisms are monotone maps. Now, let $I: \underline{\Delta} \rightarrow \mathbf{SimpSet}$ be the functor that associates to $[n]$ the simplicial set $\underline{\Delta}^n$ whose p -simplices are the sequences $0 \leq i_0 \leq \dots \leq i_p \leq n$. Observe that, by construction, $\mathfrak{L}_{\bullet}$ is a functor from $\underline{\Delta}$ to $\mathbf{DGL}$.

Definition 4.2: The functor **model** $\mathfrak{L}: \mathbf{SimpSet} \rightarrow \mathbf{DGL}$ is defined as the left Kan extension of $\mathfrak{L}_{\bullet}$ along I ,

$$\begin{array}{ccc} \underline{\Delta} & \xrightarrow{I} & \mathbf{SimpSet} \\ \mathfrak{L}_{\bullet} \downarrow & \swarrow \mathfrak{L} = \text{Lan}_I \mathfrak{L}_{\bullet} & \\ \mathbf{DGL} & & \end{array}$$

The DGL $\mathfrak{L}(K)$ is thus the colimit of $\mathfrak{L}_{\bullet}$ over the comma category $I \downarrow K$,

$$\mathfrak{L}(K) = \text{Lan}_I \mathfrak{L}_{\bullet}(K) = \varinjlim_{f: \underline{\Delta}^n \rightarrow K} \mathfrak{L}_n.$$

For simplicity, we write

$$\mathfrak{L}(K) = \varinjlim_K \mathfrak{L}_{\bullet}$$

and refer to it as the **$\mathfrak{L}$ -model** of the simplicial set K .

In the case K is a finite simplicial complex, then $K \subset \Delta^n$ for some n , and $\mathfrak{L}(K)$ is trivially isomorphic to the complete sub-DGL $(\widehat{\mathbb{L}}(V), \partial) \subset \mathfrak{L}_n$ where (V, ∂_1) is the desuspension of the chain complex of K .

THEOREM 4.3: *The functors $\mathfrak{L}$ and $\langle \cdot \rangle$ are adjoint. More precisely, for any simplicial set K and any complete differential graded Lie algebra L , there is a bijection*

$$\mathbf{SimpSet}(K, \langle L \rangle) \cong \mathbf{DGL}(\mathfrak{L}(K), L).$$

Proof. The result follows from classical properties of commutation of limits with hom functors, i.e.,

$$\begin{aligned} \mathbf{DGL}(\mathfrak{L}(K), L) &= \mathbf{DGL}(\varinjlim_K \mathfrak{L}_n, L) = \varprojlim_K \mathbf{DGL}(\mathfrak{L}_n, L) \\ &= \varprojlim_K \langle L \rangle_n = \varprojlim_K \mathbf{SimpSet}(\underline{\Delta}^n, \langle L \rangle) \\ &= \mathbf{SimpSet}(\varinjlim_K \underline{\Delta}^n, \langle L \rangle) = \mathbf{SimpSet}(K, \langle L \rangle). \quad \blacksquare \end{aligned}$$

PROPOSITION 4.4: *The realization $\langle L \rangle$ of a cdgl L is a Kan complex.*

Proof. Recall that $\langle L \rangle$ is a Kan complex if for any diagram in solid arrows

$$\begin{array}{ccc} \Lambda_i^n & \xrightarrow{\quad} & \langle L \rangle \\ \downarrow & \nearrow \text{dashed} & \\ \Delta^n & & \end{array}$$

there exists a dotted lifting making the diagram commutative. By adjunction this is equivalent to the existence of the dotted lifting in the corresponding cdgl diagram

$$\begin{array}{ccc} \mathfrak{L}(\Lambda_i^n) & \xrightarrow{\quad} & L \\ \downarrow & \nearrow \text{dashed} & \\ \mathfrak{L}_n & & \end{array}$$

Now, the inclusion $(\widehat{\mathbb{L}}(\Lambda_i^n), \partial_{a_0}) \hookrightarrow \mathfrak{L}_n$ extends to a morphism

$$f: (\widehat{\mathbb{L}}(W), d) = \mathfrak{L}(\Lambda_i^n) \widehat{\Pi} \widehat{\mathbb{L}}(u, v) \rightarrow \mathfrak{L}_n,$$

where u is a degree $n-1$ generator with $du=v$, and $f(u)=a_{0\dots n}$, $f(v)=\partial_{a_0}a_{0\dots n}$. Since the linear part f_1 is an isomorphism, f is an isomorphism: We define the inverse $g: \mathfrak{L}_n \rightarrow \widehat{\mathbb{L}}(W)$ to be the identity on $\mathfrak{L}(\Lambda_i^n)$. We put $g(a_{0\dots n}) = u$ and we define g on $x = a_{0\dots \widehat{i}\dots n}$ as follows. By definition of f_1 ,

$$f(f_1^{-1}(x)) = x + \omega_2 + \omega_3 + \dots$$

with $\omega_p \in \mathfrak{L}_n^p$. Since f_1 is an isomorphism, there is $\gamma_2 \in \mathbb{L}^2(W)$ with $\omega_2 = f_1(\gamma_2)$. Then

$$f(f_1^{-1}(x) - \gamma_2) = x + \omega'_3 + \omega'_4 + \dots$$

with $\omega'_q \in \mathfrak{L}_n^q$. We define $\gamma_3 \in \mathbb{L}^3(W)$ with $f_1(\gamma_3) = \omega'_3$, and we continue inductively. At the end, we obtain a sequence of elements $\gamma_r \in \mathbb{L}^r(W)$ and we define

$$g(x) = f_1^{-1}(x) - \sum_{k \geq 2} \gamma_k \in \widehat{\mathbb{L}}(W).$$

Clearly g is an inverse of f .

As

$$\mathfrak{L}_n \cong \mathfrak{L}(\Lambda_i^n) \widehat{\Pi} \widehat{\mathbb{L}}(u, du),$$

the dotted arrow in the cdgl diagram always exists. $\blacksquare$

We now interpret the homotopy groups of the realization of a DGL and its path component.

PROPOSITION 4.5: *For any DGL, (L, ∂) , there is a natural bijection*

$$\pi_0 \langle L \rangle \cong \widehat{\text{MC}}(L).$$

Proof. By [7, Proposition 3.1], two Maurer–Cartan elements $z_0, z_1 \in \text{MC}(L)$ are gauge equivalent if there is a map $\varphi: \mathfrak{L}_1 = (\widehat{\mathbb{L}}(a, b, x), \partial) \rightarrow L$ with $\varphi(a) = z_0$ and $\varphi(b) = z_1$. By Definition 4.1, $\langle L \rangle_0$ is the set of Maurer–Cartan elements of L , and $\langle L \rangle_1$ is the set of DGL morphisms from the LS-interval $\mathfrak{L}_1$ to L . This implies the result. ■

PROPOSITION 4.6: *Let (L, ∂) be a non-negatively graded DGL. Then, $\langle L \rangle$ is a connected simplicial set and there are natural group isomorphisms*

$$\pi_n \langle L \rangle \cong H_{n-1}(L, d), \quad n \geq 1,$$

in which $H_0(L, d)$ is considered with the group structure given by the Baker–Campbell–Hausdorff product.

Proof. By Proposition 4.5, $\langle L \rangle$ is connected. The coface maps, $\delta_j: \mathfrak{L}_{n-1} \rightarrow \mathfrak{L}_n$, induce the face maps

$$d_i = \mathbf{DGL}(\delta_j, L): \langle L \rangle_n \rightarrow \langle L \rangle_{n-1}.$$

We denote $\ker d_j = \{f: (\mathfrak{L}_n, \partial) \rightarrow (L, \partial) \mid d_j f = 0\}$. Since $\langle L \rangle$ is a Kan complex by Proposition 4.4,

$$\pi_n \langle L \rangle = \cap_{i=0}^n \ker d_i / \sim$$

where $f \sim g$ if there is $h \in \langle L \rangle_{n+1}$ such that $d_n h = f$, $d_{n+1} h = g$ and $d_i h = 0$ for $i < n$.

The space $K_n = \cap_{i=0}^n \ker d_i$ is the subspace of maps f such that $f(a_{i_0 \dots i_q}) = 0$ for $(i_0, \dots, i_q) \neq (0, \dots, n)$. In particular, $f(a_{0 \dots n})$ is a cycle. This defines a surjective map

$$\rho_n: K_n \rightarrow H_{n-1}(L), \quad \rho_n(f) = [f(a_{0 \dots n})].$$

Suppose first $n > 1$. If $h \in \langle L \rangle_{n+1}$ is a homotopy between f and g , then $h: \mathfrak{L}_{n+1} \rightarrow L$ vanishes on all the generators except $a_{0 \dots n}$, $a_{0 \dots \widehat{n} n+1}$ and $a_{0 \dots n+1}$. By definition of homotopy, $h(a_{0 \dots \widehat{n} n+1}) = f(a_{0 \dots n})$ and $h(a_{0 \dots n}) = g(a_{0 \dots n})$. Now, for degree reasons,

$$\partial h(a_{0 \dots n+1}) = (-1)^n (f(a_{0 \dots n}) - g(a_{0 \dots n})),$$

and therefore ρ_n induces a quotient map $\rho_n: \pi_n \langle L \rangle \rightarrow H_{n-1}(L)$ which is clearly bijective: the surjectivity is already done. For the injectivity suppose $\rho_n(f) = \rho_n(g)$. There is an element u with $\partial u = f(a_{0\dots n}) - g(a_{0\dots n})$ and we define a homotopy $h: \mathfrak{L}_{n+1} \rightarrow L$ by

$$h(a_{0\dots n+1}) = (-1)^n u, \quad h(a_{0\dots n}) = g(a_{0\dots n}) \quad \text{and} \quad h(a_{0\dots \widehat{n}n+1}) = f(a_{0\dots n}).$$

Now ρ_n is easily seen to be a morphism of groups: given $f, g \in K_n$, define $h: \mathfrak{L}_{n+1} \rightarrow L$ by

$$\begin{aligned} h(a_{0\dots n+1}) &= 0, & h(a_{0\dots \widehat{n}n+1}) &= f(a_{0\dots n}) + g(a_{0\dots n}), \\ d_{n+1}h &= f & \text{and} & \quad d_{n-1}h = g. \end{aligned}$$

In $\pi_n \langle L \rangle$, $[f] + [g] = [d_n(h)]$. Therefore $\rho_n([f] + [g]) = \rho_n[f] + \rho_n[g]$.

Next, we consider the case $n = 1$. Since $L = L_{\geq 0}$, a map $f: \mathfrak{L}_1 \rightarrow L$ corresponds to the cycle $f(a_{01}) \in L$. Suppose $f \sim g$. Then there is a map $h: \mathfrak{L}_2 \rightarrow L$ with $d_2h = g$, $d_1h = f$ and $d_0h = 0$. In $\mathfrak{L}_2$, $\partial_{a_0}(a_{012}) = a_{01} * a_{12} * a_{02}^{-1}$. Applying h , this gives $f(a_{01}) * g(a_{01})^{-1} = 0$ in homology. Therefore $\rho_1(f) = \rho_1(g)$ and thus ρ_1 also induces a map $\rho_1: \pi_1 \langle L \rangle \rightarrow H_0(L)$ which is also bijective.

Let $f, g: \mathfrak{L}_1 \rightarrow L$. The product $[f] \cdot [g]$ is the class of a map $k: \mathfrak{L}_1 \rightarrow L$ for which there is another map $h: \mathfrak{L}_2 \rightarrow L$ with $d_1h = f$, $d_0h = g$ and $d_2h = k$. A solution is given by the map $k: \mathfrak{L}_1 \rightarrow L$ defined by $k(a_{01}) = f(a_{01}) * g(a_{01})$, and $h(a_{012}) = 0$. This shows that $\rho_1([f] \cdot [g]) = \rho_1[f] * \rho_1[g]$. ■

For any differential graded Lie algebra L and any Maurer–Cartan element $z \in \text{MC}(L)$ consider the *localization of L at z* which is the DGL,

$$L^{(z)} = (L, \partial_z) / (L_{<0} \oplus M) \cong L_{>0} \oplus (L \cap \ker \partial_z),$$

where M is a complement of $\ker \partial_z$ in L_0 . Observe that the injection of $(L^{(z)}, d_z) \hookrightarrow (L, d_z)$ induces an isomorphism in homology in degrees ≥ 0 .

THEOREM 4.7: $\langle L \rangle \simeq \bigcup_{z \in \widehat{\text{MC}}(L)} \langle L^{(z)} \rangle$.

Proof. As we know, the components of $\langle L \rangle$ are identified with $\widehat{\text{MC}}(L)$. Via this identification, since $\langle L \rangle$ is a Kan complex (Proposition 4.4), the component of a given $z \in \widehat{\text{MC}}(L)$ is of the same homotopy type as the reduced simplicial set which we denote by $\langle L \rangle_z$ whose n -simplices are the DGL morphisms $f: \mathfrak{L}_n \rightarrow L$ such that $f(a_i) = z$ for any 0-simplex a_i , $i = 0, \dots, n$.

The simplicial set $\langle L \rangle_z$ has only one 0-simplex $\bar{z}: \mathfrak{L}_0 \rightarrow L$ and its degeneracies are the maps $\bar{z}: \mathfrak{L}_n \rightarrow L$ such that $\bar{z}(a_i) = z$ for all i and which vanish on all generators of non-negative degrees.

Recall that $\mathfrak{L}_n$ verifies the properties of Proposition 2.11. We now define an isomorphism of simplicial sets,

$$\varphi: \langle L, d \rangle_z \longrightarrow \langle L^z, \partial_z \rangle,$$

as follows. Let $f: \mathfrak{L}_n \rightarrow (L, \partial)$ be an element of $\langle L, \partial \rangle_z$. We set

$$\varphi(f)(a_i) = 0 \quad \text{and} \quad \varphi(f)(a_{i_0 \dots i_r}) = f(a_{i_0 \dots i_r}), \quad \text{if } r > 0.$$

Recall that

$$(L^z)_n = \begin{cases} 0, & \text{if } n < 0, \\ \ker \partial_z, & \text{if } n = 0, \\ L, & \text{if } n > 0. \end{cases}$$

We easily see that the association $f \mapsto \varphi(f)$ is a bijection of morphisms of Lie algebras. We are reduced to proving that $\varphi(f)$ is compatible with the differentials, that is,

$$\varphi(f) \circ \partial = \partial_z \circ \varphi(f).$$

We first observe that

$$\varphi(f) \circ \partial(a_i) = -\frac{1}{2}\varphi(f)[a_i, a_i] = 0 = \partial_z \varphi(f)(a_i).$$

First, since $f(a_i) = z$ for all i , $\partial f(a_{ij}) = f\partial(a_{ij}) = [-z, f(a_{ij})]$, so that

$$\partial_z \varphi(f)(a_{ij}) = \partial_z f(a_{ij}) = \partial f(a_{ij}) + [z, f(a_{ij})] = 0 = \varphi(f)\partial(a_{ij}).$$

Let $a_{i_0 \dots i_r}$ with $r > 1$ be a generator of $\mathfrak{L}_n$. Since $d_{a_{i_0}}(a_{i_0 \dots i_r})$ does not contain any generator a_i , the elements $\partial_{a_{i_0}} f$ and $\varphi(\partial_{a_{i_0}} f)$ are equal. We then have the following sequence of equalities that give the result:

$$\begin{aligned} d_z \varphi(f)(a_{i_0 \dots i_r}) &= \partial_z f(a_{i_0 \dots i_r}) = \partial f(a_{i_0 \dots i_r}) + [z, f(a_{i_0 \dots i_r})] \\ &= f\partial(a_{i_0 \dots i_r}) + [f(a_{i_0}), f(a_{i_0 \dots i_r})] = f\partial_{a_{i_0}}(a_{i_0 \dots i_r}) \\ &= \varphi(f)\partial_{a_{i_0}}(a_{i_0 \dots i_r}) = \varphi(f)(\partial(a_{i_0 \dots i_r}) + [a_0, a_{i_0 \dots i_r}]) \\ &= \varphi(f)\partial(a_{i_0 \dots i_r}). \end{aligned}$$

Finally, the compatibility of φ with the faces and degeneracies follows from the cosimplicial structure of $\mathfrak{L}_n$. ■

5. The $\mathfrak{L}$ -model of K

We first prove that the homology of $\mathfrak{L}(K)$ for the differential ∂_a depends only on the component of a .

PROPOSITION 5.1: *Let K be a finite simplicial complex and let a be a 0-simplex. Denote by K_a the component of a in K and denote also by a the corresponding generator of degree -1 in $\mathfrak{L}(K)$. Then, the injection $(\mathfrak{L}(K_a), \partial_a) \xrightarrow{\sim} (\mathfrak{L}(K), \partial_a)$ is a quasi-isomorphism.*

Proof. Write $K = K_a \amalg K'$, $\mathfrak{L}(K_a) = \widehat{\mathbb{L}}(V \oplus \mathbb{Q}a)$ and $\mathfrak{L}(K') = \widehat{\mathbb{L}}(W)$. Recall (see for instance [30, Proposition 6.2.(7)]) that the ideal I in $\mathbb{L}(V \oplus W \oplus \mathbb{Q}a)$ generated by $V \oplus W$ is the free DGL $I = \mathbb{L}(\overline{V} \oplus \overline{W})$, where

$$\begin{aligned}\overline{V} &= \{\mathrm{ad}_a^n(v_i), n \geq 0, \text{ with } (v_i) \text{ a basis of } V\}, \\ \overline{W} &= \{\mathrm{ad}_a^n(w_j), n \geq 0, \text{ with } (w_j) \text{ a basis of } W\}.\end{aligned}$$

Since $V \oplus W$ is finite dimensional and a has degree -1 , the kernel J of the projection $(\widehat{\mathbb{L}}(V \oplus W \oplus \mathbb{Q}a), \partial_a) \rightarrow (\mathbb{L}(a), \partial_a)$ is the DGL $(\widehat{\mathbb{L}}(\overline{V} \oplus \overline{W}), \partial)$. Denote by d_1 the linear part of the differential ∂ in J . We claim that for $w \in W$ and q even, $d_1 \mathrm{ad}_a^q(w) = \mathrm{ad}_a^{q+1}(w) + (-1)^q \mathrm{ad}_a^q(d_1 w)$. Suppose this is true for $q-2$. Then, since $\mathrm{ad}_a^q(v) = \frac{1}{2}[[a, a], \mathrm{ad}_a^{q-2}(v)]$ we have

$$\begin{aligned}d_1 \mathrm{ad}_a^q(v) &= \frac{1}{2} d_1 [[a, a], \mathrm{ad}_a^{q-2}(v)] \\ &= \frac{1}{2} [[a, a], \mathrm{ad}_a^{q-1}(v) + (-1)^q \mathrm{ad}_a^{q-2}(d_1 v)] \\ &= \mathrm{ad}_a^{q+1}(v) + (-1)^q \mathrm{ad}_a^q(d_1 v).\end{aligned}$$

Therefore the homology $H(\overline{W}, \partial_1) = 0$ and the injection

$$(\widehat{\mathbb{L}}(\overline{V}), \partial_a) \xrightarrow{\sim} (\mathfrak{L}(K), \partial_a)$$

is a quasi-isomorphism. This implies the result. $\blacksquare$

Definition 5.2: Let K be a simplicial set. A **model** of K is a free complete differential Lie algebra $(\widehat{\mathbb{L}}(Z), \partial)$ connected to $\mathfrak{L}(K)$ by a sequence of quasi-isomorphisms. Such a model is called **minimal** if the linear part of the differential is null, i.e., $\partial_1 = 0$.

We make precise the behaviour of some models in the connected case.

Let K be a finite connected simplicial complex. Then $(\mathfrak{L}(K), \partial) = (\widehat{\mathbb{L}}(V), \partial)$, where (V, ∂_1) is the desuspension of the chain complex of K . Let $a_0, \dots, a_n$ be the 0-simplices of K viewed as generators of V_{-1} . We write

$$W = \bigoplus_{q \geq -1} W_q$$

with $W_q = V_q$ for $q \geq 0$ and W_{-1} the vector space generated by the elements $a_i - a_0$.

PROPOSITION 5.3: *Let K be a finite connected simplicial complex. With the previous notation, the differential ∂ in $\mathfrak{L}(K)$ can be chosen so that*

$$\partial_{a_0}(W) \subset \widehat{\mathbb{L}}(W)$$

and there is a quasi-isomorphism

$$(\mathfrak{L}(K), \partial_{a_0}) \xrightarrow{\simeq} (\widehat{\mathbb{L}}(W), \partial_{a_0}).$$

Moreover, if we denote by $(\mathfrak{L}(K)/(a_0), \overline{\partial})$ the quotient of $(\mathfrak{L}(K), \partial)$ by the ideal generated by a_0 , then there is an isomorphism

$$(\mathfrak{L}(K)/(a_0), \overline{\partial}) \xrightarrow{\cong} (\widehat{\mathbb{L}}(W), \partial_{a_0}).$$

Proof. We prove by induction on n that $\partial_{a_0}(W_n) \subset \widehat{\mathbb{L}}(W)$. By the naturality of the differential in $\mathfrak{L}(K)$ we have only to prove it when $K = \Delta^{n+1}$. It is obvious for $n = -1$ and directly deduced for $n = 0$ from the definition of the differential of the LS-interval. Suppose that $\partial_{a_0}(W_q) \subset \widehat{\mathbb{L}}(W)$ for $q < n - 1$ and $n \geq 2$. From the induction hypothesis, we get

$$(\widehat{\mathbb{L}}(\dot{\Delta}^n), \partial_{a_0}) = (\mathbb{L}(a_0), \partial_{a_0}) \hat{\Pi} (\widehat{\mathbb{L}}(Z), \partial_{a_0}),$$

where Z is the graded vector space generated by the elements $\{a_i - a_0 \mid i > 0\}$, together with the generators of degree ≥ 0 . Here, $\hat{\Pi}$ denotes the coproduct in the category of DGL's which is obtained by completing the non-complete coproduct. Since $H(\mathbb{L}(a_0), \partial_{a_0}) = 0$, the inclusion of the second factor is a quasi-isomorphism

$$(11) \quad (\widehat{\mathbb{L}}(Z), \partial_{a_0}) \xrightarrow{\simeq} (\mathbb{L}(a_0), \partial_{a_0}) \hat{\Pi} (\widehat{\mathbb{L}}(Z), \partial_{a_0}).$$

On the other hand, in view of Corollary 2.5,

$$H(\widehat{\mathbb{L}}(\dot{\Delta}^n), \partial_{a_0}) = \mathbb{L}(u),$$

where $|u| = n - 2$. With (11), we can choose $u \in \widehat{\mathbb{L}}(Z)$, and set $\partial_{a_0}(a_0 \dots n) = u$. Since $W = Z \oplus \mathbb{Q}a_0 \dots n$, the inclusion $\partial_{a_0}(W) \subset \widehat{\mathbb{L}}(W)$ is established. Then, it follows that

$$(\mathfrak{L}(K), \partial_{a_0}) \xrightarrow{\cong} (\mathbb{L}(a_0), \partial_{a_0}) \hat{\amalg} (\widehat{\mathbb{L}}(W), \partial_{a_0}).$$

By Lemma 2.6 this implies that $(\mathfrak{L}(K), \partial_{a_0})$ is quasi-isomorphic to $(\widehat{\mathbb{L}}(W), \partial_{a_0})$. Moreover, on the quotient $\mathfrak{L}(K)/(a_0)$ the induced differentials $\overline{\partial}$ and $\overline{\partial_{a_0}}$ coincide. This implies the isomorphism of the statement. ■

COROLLARY 5.4: *Let K be a connected finite simplicial complex. Then, $(\mathfrak{L}(K), \partial_{a_0})$ has a minimal model of the form $(\widehat{\mathbb{L}}(R), \partial)$, with*

$$R = R_{\geq 0} \cong s^{-1} \widetilde{H}(K; \mathbb{Q}).$$

Proof. Denote by T a maximal tree contained in the graph formed by the 0-simplices and the 1-simplices of K . Since K is connected, this tree contains all the vertices. Moreover, in $\mathfrak{L}(K)$, the ideal $\mathcal{J}$, generated by the $\{a_i - a_0 \mid i > 0\}$ together with the generators of dimension 0 corresponding to the edges of T , is acyclic. Therefore, the quotient $\mathfrak{L}(K)/\mathcal{J}$ is quasi-isomorphic to a DGL of the form

$$(\mathbb{L}(a_0), \partial_{a_0}) \hat{\amalg} (\widehat{\mathbb{L}}(R), \partial),$$

with $R = R_{\geq 0}$.

Now, suppose that for some $x \in R$ we have $\partial_1 x = y \neq 0$. Then, replace y by ∂x , and by Proposition 2.4, taking the quotient by the ideal generated by x and y gives a quasi-isomorphism $(\widehat{\mathbb{L}}(R), \partial) \xrightarrow{\cong} (\widehat{\mathbb{L}}(R)/(x, y), \overline{\partial})$. We continue in the same way and finally obtain a minimal complete free DGL quasi-isomorphic to $(\mathfrak{L}(K), \partial_{a_0})$. ■

Example 5.5: If K is a connected finite simplicial complex of dimension 1, then $(\mathfrak{L}(K), \partial_a) \simeq (\widehat{\mathbb{L}}(V), 0)$ with

$$V = s^{-1} H_1(K).$$

Example 5.6: The minimal model of a finite simplicial set K of the homotopy type of $S^1 \vee S^2$ is of the form $(\widehat{\mathbb{L}}(a, b), 0)$, with $|a| = 0$ and $|b| = 1$. In particular, $H(\mathfrak{L}_K)$ is the completion of $\mathbb{L}(a, b)$ for the adjoint action of a on b .

6. Differential graded Lie coalgebras and the Transfer Theorem

In this and in the next section, we show how the homotopy transfer theorem and the structure of free Lie coalgebras let us also associate a DGL to a finite simplicial complex K . The uniqueness theorem (Theorem 2.8) will then identify this DGL with our construction $\mathfrak{L}(K)$. Following this approach we may understand the relation between the combinatorial properties of $\mathfrak{L}(K)$ with the algebra of PL-forms on K .

Recall that a graded Lie coalgebra is a graded vector space, V , with a comultiplication $\Delta: V \rightarrow V \otimes V$ that satisfies two properties,

$$(1 + \tau) \circ \Delta = 0 \quad \text{and} \quad (1 + \sigma + \sigma^2) \circ (1 \otimes \Delta) \circ \Delta = 0.$$

Here, $\tau: V \otimes V \rightarrow V \otimes V$ is the graded permutation and $\sigma: V \otimes V \otimes V \rightarrow V \otimes V \otimes V$ is the graded cyclic permutation [14, 23].

Any graded coalgebra (C, Δ) admits a Lie coalgebra structure defined by $\Delta_L = \Delta - \tau \circ \Delta$. If $T^c(V)$ denotes the tensor coalgebra on a graded vector space V , then Δ_L induces a Lie coalgebra structure on the indecomposables for the shuffle product, i.e., on the quotient of $T^c(V)$ by the shuffle products. This quotient is called the **free Lie coalgebra on V** and is denoted by $\mathbb{L}^c(V)$. We denote by **DGLC** the category of differential graded Lie coalgebras and by **CDGC** the category of differential connected graded cocommutative coalgebras. If **Vect** denotes the category of graded vector spaces, there is a natural isomorphism

$$\mathbf{Vect}(\Gamma, V) = \mathbf{DGLC}(\Gamma, \mathbb{L}^c(V))$$

when V is a graded vector space and Γ a graded Lie coalgebra [14, §4.2.1]. The graded universal coenveloping coalgebra of a graded Lie coalgebra L is a coalgebra, denoted $U^c L$ with the property that for any graded coalgebra C , we have

$$\mathbf{DGLC}(C, L) \cong \mathbf{DGC}(C, U^c L).$$

In particular, there is a coalgebra isomorphism, $U^c(\mathbb{L}^c(V)) \cong T^c(V)$. If $(T^c(V), d)$ is a differential graded coalgebra such that, for every n , the part $d_n: \otimes^n V \rightarrow V$ of the differential vanishes on the (p, q) -shuffles with $p + q = n$, then d makes of $(\mathbb{L}^c(V), d)$ a differential graded Lie coalgebra.

Clearly, the dual of a differential Lie coalgebra is a differential Lie algebra. Moreover, the following holds.

LEMMA 6.1: *For any finite type vector space, V , we have $(\mathbb{L}^c(V))^{\sharp} = \widehat{\mathbb{L}}(V^{\sharp})$.*

Proof. From $\mathbb{L}^c(V) = \varinjlim_n (\mathbb{L}^c)^{\leq n}(V)$, we deduce

$$\begin{aligned} \operatorname{Hom}(\varinjlim_n (\mathbb{L}^c)^{\leq n}(V), \mathbb{Q}) &= \varprojlim_n \operatorname{Hom}((\mathbb{L}^c)^{\leq n}(V), \mathbb{Q}) \\ &= \varprojlim_n \mathbb{L}(V^{\sharp}) / \mathbb{L}^{>n}(V^{\sharp}) = \widehat{\mathbb{L}}(V^{\sharp}). \quad \blacksquare \end{aligned}$$

Let $f: (T^c(V), d) \rightarrow (T^c(W), d)$ be a morphism of differential graded coalgebras. We denote by d_1 the linear part of the differentials, $d_1(V) \subset V$, $d_1(W) \subset W$, and we write $f = f_0 + f_1 + \cdots$ with $f_i(T^c)^r(V) \subset (T^c)^{r-i}(W)$.

PROPOSITION 6.2: *With the above notations, suppose $f_0: (V, d_1) \xrightarrow{\sim} (W, d_1)$ is a quasi-isomorphism. Then:*

- (i) *f is a quasi-isomorphism.*
- (ii) *Moreover, if f and the differentials d vanish on the decomposables for the shuffle product, then the induced map $f: (\mathbb{L}^c(V), d) \xrightarrow{\sim} (\mathbb{L}^c(W), d)$ is also a quasi-isomorphism.*

Proof. (i) First of all, by the Künneth Theorem,

$$f_0^{\otimes n}: (T^n(V), d_1) \rightarrow (T^n(W), d_1)$$

is a quasi-isomorphism for each n . Therefore, by the five lemma,

$$f_{\leq q} = f_0 + \cdots + f_q: (T^{\leq q}(V), d) \rightarrow (T^{\leq q}(W), d)$$

is a quasi-isomorphism for all $q \geq 1$. This implies the surjectivity of $H(f)$. To prove the injectivity suppose $H(f)(\alpha) = 0$. Write $\alpha = [a]$, $a \in T^{\leq q}(V)$ and $f(a) = db$, $b \in T^{\leq r}(W)$, for some $r \geq q$. Then since $f_{\leq r}$ is a quasi-isomorphism, we have $\alpha = 0$.

(ii) Recall that $\mathbb{L}^c(V)$ is a retract of $T^c(V)$, obtained by identifying $\mathbb{L}^c(sV)$ with the image of the first Eulerian idempotent $e^{(1)}$, i.e., see [14, Theorem 4.2.3],

$$\mathbb{L}^c(sV) = e^{(1)}T^c(sV).$$

Since this retraction is natural with respect to linear maps, it commutes with the differentials d_1 and with f_0 . Therefore,

$$\mathbb{L}^c(f_0): (\mathbb{L}^c(V), d_1) \rightarrow (\mathbb{L}^c(W), d_1)$$

is a retract of $T^c(f_0)$ and is thus a quasi-isomorphism. We deduce in the same way that $f: (\mathbb{L}^c(V), d) \xrightarrow{\sim} (\mathbb{L}^c(W), d)$ is a quasi-isomorphism. $\blacksquare$

The Quillen correspondence $\mathcal{L}$ and $\mathcal{C}$ (generalized by Neisendorfer in [24]) between **DGL** and **CDGC** dualizes in a pair of adjoint functors between **DGLC** and the category **CDGA** of augmented commutative differential graded algebras, that we can summarize in the following commutative diagram [28, Theorem 4.17]:

$$\begin{array}{ccc} \text{CDGC} & \begin{array}{c} \xrightarrow{\mathcal{L}} \\ \xleftarrow{\mathcal{C}} \end{array} & \text{DGL} \\ \downarrow (-)^\# & \begin{array}{c} \mathcal{A} \\ \mathcal{E} \end{array} & \uparrow (-)^\# \\ \text{CDGA} & \begin{array}{c} \xleftarrow{\mathcal{A}} \\ \xrightarrow{\mathcal{E}} \end{array} & \text{DGLC} \end{array}$$

Moreover, the adjunction maps, $\mathcal{L}\mathcal{C}(L) \xrightarrow{\sim} L$ and $C \xrightarrow{\sim} \mathcal{C}\mathcal{L}(C)$, are quasi-isomorphisms; see [24, Proposition 4.1].

We recall the definition of the functors $\mathcal{E}$ and $\mathcal{A}$.

On the one hand, let (A, d) be an augmented cdga. The differential on the bar construction $BA = T^c(s\overline{A})$ preserves the ideal generated by the shuffle products and induces a differential d on the indecomposables $B^+A/B^+A \cdot B^+A$, which is the free Lie coalgebra $\mathbb{L}^c(s\overline{A})$. By definition $\mathcal{E}(A) = (\mathbb{L}^c(s\overline{A}), d)$.

On the other hand, let (E, d) be a differential graded Lie coalgebra. Then, $\mathcal{A}(E) = (\wedge sE, D)$ with

$$D(sx) = \frac{1}{2} \sum_i (-1)^{|x_i|} s x_i \wedge s x'_i - s dx,$$

with $\Delta x = \sum_i x_i \otimes x'_i$. In particular, if $(\wedge Z, d)$ is the minimal model of $\mathcal{A}(E)$, then $Z \cong sH(E, d)$.

In fact, there are quasi-isomorphisms, $\mathcal{A}\mathcal{E}(A) \xrightarrow{\sim} A$ and $E \xrightarrow{\sim} \mathcal{E}\mathcal{A}(E)$.

Recall now that an A_∞ -**algebra structure** on a graded vector space A consists in a sequence of linear maps $m_n: A^{\otimes n} \rightarrow A$, $n \geq 1$, of degree $2 - n$ such that for all $r, s, t \geq 1$

$$\sum_{r+s+t=n} (-1)^{r+st} m_{r+t+1}(id^{\otimes r} \otimes m_s \otimes id^{\otimes t}) = 0.$$

Now, each map m_n defines a canonical map of degree 1,

$$(12) \quad d_n: (sA)^{\otimes n} \rightarrow sA, \quad d_n = sm_n(s^{-1})^{\otimes n},$$

that extends into a coderivation on the coalgebra $T^c(sA)$. The degree 1 map $d = \sum_{n \geq 1} d_n$ is a coderivation and the identity

$$d_{r+t+1}(id^{\otimes r} \otimes d_k \otimes id^{\otimes t}) = sm_{r+t+1}(id^{\otimes r} \otimes m_k \otimes id^{\otimes t})(s^{-1})^{\otimes r+t+k}$$

implies that

$$\sum_{r+k+t=n} d_{r+t+1}(id^{\otimes r} \otimes d_k \otimes id^{\otimes t}) = 0$$

and therefore $(T^c(sA), d)$ is a differential graded coalgebra.

In fact, if we define $m_n = (-1)^{\frac{n(n-1)}{2}} s^{-1} d_n s^{\otimes n}$, we have that $(A, \{m_n\})$ is an A_∞ -algebra if and only if $(T^c(sA), d)$ is a differential graded coalgebra.

It is well known that A is an A_∞ -algebra if and only if $(T^c(sA), d)$ is a differential graded coalgebra. Moreover, in the case A is a differential graded algebra, we recover the bar construction on A , $(T^c(sA), d)$.

A **morphism** of A_∞ -algebras $f: A \rightarrow B$ consists of a sequence of morphisms $f_n: A^{\otimes n} \rightarrow B$ such that the induced morphism

$$F: (T^c(sA), d) \rightarrow (T^c(sB), d)$$

is a morphism of differential graded coalgebras.

An A_∞ -algebra structure on A is **commutative** (or a C_∞ -algebra) if the differential d vanishes on the shuffle products in $T^c(sA)$. In this case, d induces a differential on the quotient $\mathbb{L}^c(sA)$, and we have a differential graded Lie coalgebra, denoted $(\mathbb{L}^c(sA), d)$. When f is a morphism between commutative A_∞ -algebras, that preserves the shuffle products, then f restricts to a morphism of Lie coalgebras, $(\mathbb{L}^c(sA), d) \rightarrow (\mathbb{L}^c(sB), d)$.

A **unital** A_∞ -algebra is an A_∞ -algebra endowed with an element 1_A of degree 0 such that $m_1(1_A) = 0$, $m_2(1_A, a) = a = m_2(a, 1_A)$ for all $a \in A$ and such that for all $i > 2$ and all $a_1, \dots, a_i \in A$, the product $m_i(a_1, \dots, a_i)$ vanishes if one of the a_i equals 1_A . A morphism of A_∞ -algebras, $f: A \rightarrow B$ is unital if $f_1(1_A) = 1_B$ and for $n > 1$ $f_n(a_1, \dots, a_n) = 0$ if one of the a_i equals 1_A . A unital A_∞ -algebra is **augmented** if it is endowed with a unital morphism $\varepsilon: A \rightarrow \mathbb{Q}$ such that $\varepsilon(1_A) = 1$. The augmentation ideal $\overline{A} = \ker \varepsilon$ inherits then an A_∞ -algebra structure. A morphism of augmented A_∞ -algebras is a unital morphism $f: A \rightarrow B$ such that $\varepsilon_B \circ f = \varepsilon_A$. It induces a morphism between the augmentation ideals.

Note that the functor $\mathcal{E}$ can be extended to any augmented commutative A_∞ -algebra A as the graded Lie coalgebra $\mathcal{E}(A) = (\mathbb{L}^c(s\overline{A}), d)$.

In our setting, a **transfer CDGA diagram** is a diagram of the form

$$\mathfrak{D}: \quad \phi \circlearrowleft (A, d) \xrightleftharpoons[i]{p} (V, d)$$

where (A, d) is an augmented CDGA, (V, d) is an augmented (cochain) differential graded vector space, p and i are quasi-isomorphisms preserving the augmentations, $pi = \text{id}_V$, and ϕ is a chain homotopy between id_A and ip , i.e., $\phi d + d\phi = \text{id}_A - ip$, which satisfies $\phi i = p\phi = \phi^2 = 0$.

The Homotopy Transfer Theorem (see [20, §9.4] for example) states that there is an A_∞ -algebra structure $\{m_k\}_{k \geq 1}$ in (V, d) such that $m_1 = d$.

The formula of Kontsevich and Soibelman for the higher operations m_k is defined in terms of trees. A **planar tree** T is a directed, simply connected graph which can be embedded in the plane. The valence of a vertex is the number of edges having this vertex as the source. A **leaf** is a vertex of valence 0. A tree is **rooted** if there is a unique vertex (the root) with valence 1 which is the target of no edge. A rooted tree is binary if every vertex except the root and the leaves has valence 2. For any $k \geq 2$ denote by $\mathcal{T}_k$ the set of isomorphism classes of planar rooted binary trees with k leaves.

We associate to each tree $T \in \mathcal{T}_k$ a linear map $m_T: V^{\otimes k} \rightarrow V$ as follows: label the root by p , each internal edge by ϕ , each internal vertex by m , and each leaf by i . Then, m_T is defined as the composition of the different labels moving down from the leaves to the root.

The transferred C_∞ -algebra structure in V is given by $\{m_k\}_{k \geq 1}$, where $m_1 = d$ and, for $k \geq 2$,

$$m_k = \sum_{T \in \mathcal{T}_k} m_T.$$

A consequence of the Homotopy Transfer Theorem is the following:

THEOREM 6.3: *The diagram $\mathfrak{D}$ induces quasi-isomorphisms of differential graded Lie coalgebras,*

$$(\mathbb{L}^c(sA), d) \xrightleftharpoons[I]{P} (\mathbb{L}^c(sV), d), \quad (\mathbb{L}^c(s\overline{A}), d) \xrightleftharpoons[I]{P} (\mathbb{L}^c(s\overline{V}), d),$$

where $PI = \text{id}$, $I_1 = si$, $P_1 = sp$ and $d_1 = sd$.

In particular, $\mathcal{E}(A, d) = (\mathbb{L}^c(s\overline{A}), d)$.

Proof. Use the Homotopy Transfer Theorem together with [9, Theorem 12] to obtain quasi-isomorphisms of C_∞ -algebras

$$(T^c(sA), d) \xrightleftharpoons[I]{P} (T^c(sV), d), \quad (T^c(s\bar{A}), d) \xrightleftharpoons[I]{P} (T^c(s\bar{V}), d),$$

where the differentials d in the tensor coalgebras of the right sides are defined as in equation (12). Finally, quotient in both sides by decomposables and apply Proposition 6.2. ■

Remark 6.4: Alternatively, Theorem 6.3 can be proved using Theorem 1.3 of [2] specialized to the cooperad of Lie coalgebras.

7. The cosimplicial structure via a transfer

Let K be a finite simplicial complex. We denote by $C^*(K)$ the dual of the chain complex of K , with rational coefficients, and by $A_{PL}(K)$ the algebra of rational PL-forms on K , [12, 29].

THEOREM 7.1 ([10, 11, 15]): *There is a CDGA transfer diagram of the form*

$$\mathfrak{D}: \quad \phi \circlearrowleft A_{PL}(K) \xrightleftharpoons[i]{p} C^*(K).$$

In particular, there is a simplicial CDGA transfer diagram

$$\mathfrak{D}_\bullet: \quad \phi_\bullet \circlearrowleft A_{PL}(\Delta^\bullet) \xrightleftharpoons[i_\bullet]{p_\bullet} C^*(\Delta^\bullet),$$

where $p_\bullet$, $i_\bullet$ and $\phi_\bullet$ preserve the simplicial structure.

The transfer diagram $\mathfrak{D}_\bullet$ is explicitly described, for instance, in [15, §3]. As usual,

$$A_{PL}(\Delta^n) = \Lambda(t_0, \dots, t_n, dt_0, \dots, dt_n) / \left(\sum t_i - 1, \sum dt_i \right)$$

and the maps $p_\bullet$ and $i_\bullet$ are defined as follows:

Let $\alpha_{i_0 \dots i_k}$ be the basis for $C^*(\Delta^n)$ defined by

$$\langle \alpha_{i_0 \dots i_k}, a_{j_0 \dots j_k} \rangle = \begin{cases} (-1)^{\frac{k(k-1)}{2}} & \text{if } (j_0, \dots, j_k) = (i_0, \dots, i_k), \\ 0 & \text{otherwise.} \end{cases}$$

Then, $i_n(\alpha_{i_0\dots i_k})$ is the Whitney elementary form $\omega_{i_0\dots i_k}$ defined by

$$\omega_{i_0\dots i_k} = k! \sum_{j=0}^k (-1)^j t_{i_j} dt_{i_0} \cdots \widehat{dt_{i_j}} \cdots dt_{i_k}.$$

Since

$$d(\omega_{i_0\dots i_k}) = \sum_q \omega_{qi_0\dots i_k},$$

the map $i_n: C^*(\Delta^n) \rightarrow A_{PL}(\Delta^n)$ is a morphism of cochain complexes. The map $p_n: A_{PL}(\Delta^n) \rightarrow C^*(\Delta^n)$ is defined by

$$p_n(\omega) = \sum_{k=0}^n \sum_{i_0 < \dots < i_k} \alpha_{i_0\dots i_k} I_{i_0\dots i_k}(\omega),$$

with

$$I_{i_0\dots i_k}(t_{i_1}^{a_1} \dots t_{i_k}^{a_k} dt_{i_1} \dots dt_{i_k}) = \frac{a_1! \dots a_k!}{(a_1 + \dots + a_k + k)!},$$

and 0 otherwise. In particular, $I_{i_0\dots i_k}(\omega_{i_0\dots i_k}) = 1$.

Since the morphisms $p_\bullet$ and $i_\bullet$ preserve the simplicial structures, they preserve sub-simplicial complexes. So, if K is a finite simplicial complex, then $K \subset \Delta^n$ and the diagram $\mathfrak{D}$ is obtained by restriction.

Theorem 6.3 implies the following:

COROLLARY 7.2: *For a finite simplicial complex K , there is a quasi-isomorphism of differential graded Lie coalgebras,*

$$I: (\mathbb{L}^c(sC^*(K)), d) \xrightarrow{\simeq} (\mathbb{L}^c(sA_{PL}(K)), d),$$

such that $I_1 = si$ and d_1 in both sides is the suspension of the differential on cochains and forms on K respectively.

Let a_0 be a base point. The associated augmentation of $A_{PL}(K)$ is defined by

$$\varepsilon(t_0) = 1, \quad \varepsilon(t_i) = 0 \quad \text{for } i \neq 0.$$

An augmentation in the cochain complex $C^*(K)$ is also defined by $\varepsilon(a_0) = 1$ and $\varepsilon(a_i) = 0$ for $i > 0$. In the transfer diagram the morphisms p_n and i_n preserve the augmentations because $p_n(t_0) = a_0$ and $i_n(a_0) = t_0$. Therefore, the morphism I induces a quasi-isomorphism

$$(\mathbb{L}^c(s\overline{C}^*(K)), d) \xrightarrow{\simeq} \mathcal{E}(A_{PL}(K)).$$

THEOREM 7.3: *With the above notations:*

- (i) *The dual of the simplicial differential graded Lie coalgebra $(\mathbb{L}^c(sC^*(\Delta^\bullet)), D)^\sharp$ is a sequence of compatible models of Δ .*
- (ii) *The differential graded Lie algebras $(\widehat{\mathbb{L}}^c(sC^*(K)), d)^\sharp$ and $\mathfrak{L}(K)$ are isomorphic.*

Proof. (i) Note first that, by construction, the linear part of the differential is the good one. Thus by Theorem 2.8, we need to check both that, for all $n \geq 0$, each vertex in $(\mathbb{L}^c(sC^*(\Delta^\bullet)), D)^\sharp$ is a Maurer–Cartan element, and that the cofaces are as in (7).

On the one hand, by naturality it is enough to check that the only vertex a_0 in $(\mathbb{L}^c(sC^*(\Delta^0)), D)^\sharp$ is a Maurer–Cartan element. Note that

$$A_{PL}(\Delta^0) = \mathbb{Q} \cdot e = C^*(\Delta^0),$$

with $e^2 = e$ the unit of the algebra. Now in the bar construction [12, Page 269],

$$d([se|se]) = -[se].$$

The anti-symmetrisation of the diagonal gives

$$\Delta_L([se|se]) = 2[se, se].$$

In the dual Lie algebra this gives $d(a_0) = -\frac{1}{2}[a_0, a_0]$.

On the other hand, a simple inspection shows that the faces on $C^*(\Delta^\bullet)$ become the usual cofaces when taking the dual of the free Lie coalgebra.

(ii) This is a direct consequence of the fact that $K \subset \Delta^n$ for some n and $\mathfrak{L}(K)$ is isomorphic to a sub-DGL $(\widehat{\mathbb{L}}(V), \partial) \subset \mathfrak{L}_n$, where (V, ∂_1) is the suspension of the chain complex of K . ■

Next, we connect the functor $\mathfrak{L}$ with the classical Quillen functor

$$\lambda: \mathbf{SimpSet}_1 \rightarrow \mathbf{DGL}$$

from the category of 2-reduced simplicial sets [26] and with Sullivan minimal models [29].

THEOREM 7.4: *Let K be a connected finite simplicial complex with Sullivan minimal model $(\wedge V, d)$. Then:*

- (i) *For any vertex a , $(\mathfrak{L}(K), \partial_a)$ is quasi-isomorphic to $(\mathcal{E}(\wedge V, d))^\sharp$.*
- (ii) *In particular, if K is 1-connected, then $(\mathfrak{L}(K), \partial_a)$ is quasi-isomorphic to $\lambda(K)$.*

Proof. (i) We have a sequence of quasi-isomorphisms

$$\mathcal{E}(\wedge V, d) \xrightarrow{\cong} \mathcal{E}(A_{PL}(K)) \xleftarrow{\cong} (\mathbb{L}^c(s\overline{C}^*(K)), d).$$

By taking duals we get a series of DGL quasi-isomorphisms

$$\mathcal{E}(\wedge V, d)^\# \xleftarrow{\cong} \mathcal{E}(A_{PL}(K))^\# \xrightarrow{\cong} (\mathbb{L}^c(s\overline{C}^*(K)), d)^\#.$$

Now we remark that $(\mathbb{L}^c(s\overline{C}^*(K)), d)^\# = (\mathfrak{L}(K)/(a_0), \overline{\partial})$ and recall from Proposition 5.3 that $(\mathfrak{L}(K)/(a_0), \overline{\partial}) \cong (\mathfrak{L}(K), \partial_{a_0})$.

(ii) When K is 1-connected, the minimal model $(\wedge V, d)$ of K is of finite type. Thus, it is the dual of a differential graded coalgebra $(\wedge V, d)^\#$, and

$$\mathcal{L}((\wedge V, d)^\#) = (\mathcal{E}(\wedge V, d))^\#.$$

The result is then a direct consequence of a theorem of Majewski [21] who connects $\lambda(K)$ with $\mathcal{L}((\wedge V, d)^\#)$ by a sequence of quasi-isomorphisms. ■

8. Representability of the Quillen realization functor

We have proved that $\mathfrak{L}(K)$ is homotopy equivalent to the Quillen construction $\lambda(K)$, when K is 2-reduced of finite type. We study now the connection between the realization functor $\langle - \rangle$ introduced in Section 4 with the composite of the functors,

$$\mathcal{C}^* = (-)^\# \circ \mathcal{C}: \mathbf{DGL}_f \rightarrow \mathbf{CDGA} \quad \text{and} \quad \langle - \rangle_S: \mathbf{CDGA} \rightarrow \mathbf{SimpSet},$$

where $\mathbf{DGL}_f$ is the full subcategory of $\mathbf{DGL}$ of finite type DGL's (L, ∂) with $L = L_{\geq 0}$. The first functor is the classical cochain functor defined on the category of Lie algebras of finite type and the second one is the Sullivan's realization functor defined by

$$\langle A \rangle_S = \mathbf{CDGA}(A, A_{PL}(\Delta^\bullet)).$$

The conditions L of finite type and $L = L_{\geq 0}$ are necessary for the chain coalgebra $\mathcal{C}(L)$ to be of finite type.

THEOREM 8.1: *Let (L, d) be a finite type DGL with $L = L_{\geq 0}$ for $q < 0$. Then, there is a homotopy equivalence of simplicial sets,*

$$\langle L \rangle \simeq \langle \mathcal{C}^*(L) \rangle_S.$$

Proof. Given the simplicial differential graded Lie coalgebra

$$\mathfrak{L}_\bullet^c = (\mathbb{L}^c(sC^*(\Delta^\bullet)), D)$$

of Theorem 7.3 and $L \in \mathbf{DGL}_f$, we define the simplicial set

$$\langle L \rangle'_\bullet := \mathbf{DGLC}(L^\sharp, \mathfrak{L}_\bullet^c).$$

A coalgebra morphism $L^\sharp \rightarrow \mathfrak{L}_n^c$ is completely determined by its projection on the indecomposables $L^\sharp \rightarrow sC^*(\Delta^n)$ and an algebra morphism $\mathfrak{L}_n \rightarrow L$ by its restriction $s^{-1}C_*(\Delta^n) \rightarrow L$. Since L is finite type, $\langle L \rangle'_n = \langle L \rangle_n$. Also, as taking dual is compatible with the simplicial structure, the dual process induces a bijection of simplicial sets

$$(-)^\sharp: \langle L \rangle'_\bullet \rightarrow \langle L \rangle_\bullet.$$

The quasi-isomorphism $I: \mathfrak{L}_\bullet^c \rightarrow \mathcal{E}(A_{PL}(\Delta^\bullet))$ of Theorem 7.3 induces a morphism of simplicial sets

$$\Psi: \mathbf{DGLC}(L^\sharp, \mathfrak{L}_\bullet^c) \rightarrow \mathbf{DGLC}(L^\sharp, \mathcal{E}A_{PL}(\Delta^\bullet)).$$

On the other hand, the functor $\mathcal{A}$ is equipped with a natural evaluation morphism,

$$\varepsilon_A: \mathcal{A}\mathcal{E}(A, d) = (\wedge s\mathbb{L}^c(s\overline{A}), \partial) \rightarrow (A, d),$$

which vanishes on $(\mathbb{L}^c)^{\geq 2}(\overline{A})$. We then obtain a sequence of simplicial maps,

$$\begin{array}{ccc} \mathbf{DGLC}(L^\sharp, \mathfrak{L}_\bullet^c) & \xrightarrow{\Psi} & \mathbf{DGLC}(L^\sharp, \mathcal{E}A_{PL}(\Delta^\bullet)) \\ & & \downarrow \mathcal{A} \\ & & \mathbf{CDGA}(\mathcal{A}(L^\sharp), \mathcal{A}\mathcal{E}A_{PL}(\Delta^\bullet)) \\ & & \downarrow \varepsilon_* \\ & & \mathbf{CDGA}(\mathcal{C}^*(L), A_{PL}(\Delta^\bullet)) = \langle \mathcal{C}^*(L) \rangle_S. \end{array}$$

Recall now that in the transfer,

$$i_n(\alpha_{0\dots n}) = \omega_{0\dots n} = n! dt_1 \cdots dt_n.$$

Therefore, the linear part of $\mathfrak{L}_n^c \rightarrow \mathcal{E}A_{PL}(\Delta^n)$ is i_n . As $\pi_n \langle \mathcal{C}^*(L) \rangle$ can be identified with the linear maps $\mathrm{Hom}(H_{n-1}(L), \mathbb{Q}) \rightarrow \mathbb{Q} \alpha_{0\dots n}$ (see [13, §1.7]), this composition induces an isomorphism on the homotopy groups. ■

Recall now that a group G is **$\mathbb{Q}$ -complete** if each G^n/G^{n+1} is a $\mathbb{Q}$ -vector space and $G = \varprojlim_n G/G^n$. Here $G = G^1 \supset G^2 \supset \cdots$ is the lower central series of G . That is, G^2 is the subgroup generated by the commutators $[a, b]$ and G^n is the subgroup generated by the iterated commutators $[g_1, [g_2, [\cdots [g_{n-1}, g_n] \cdots]]$. A nilpotent space X is called a rational nilpotent space if its fundamental group is $\mathbb{Q}$ -complete, if each $\pi_n(X)$ is a finite dimensional $\mathbb{Q}$ -vector space for $n \geq 2$, and if $\pi_1(X)$ acts nilpotently on $\pi_n(X)$ for $n \geq 2$.

PROPOSITION 8.2: *Let K be a finite connected simplicial complex. Then, the simplicial set $\langle \mathfrak{L}(K), \partial_a \rangle$ is the projective limit of a tower of rational nilpotent spaces.*

Proof. Write $(\mathfrak{L}(K), \partial_a) = (\widehat{\mathbb{L}}(V), \partial)$. Then

$$(\mathfrak{L}(K), \partial_a) = \varprojlim_n (\widehat{\mathbb{L}}(V)/\mathbb{L}^{>n}, \overline{\partial}).$$

Since the realization functor is a right adjoint, it commutes with limits and

$$\langle \mathfrak{L}(K), \partial_a \rangle = \varprojlim_n \langle \widehat{\mathbb{L}}(V)/\mathbb{L}^{>n}, \overline{\partial} \rangle.$$

Now, since $(\widehat{\mathbb{L}}(V)/\mathbb{L}^{>n}, \overline{\partial})$ is a finite-dimensional nilpotent Lie algebra, its geometric realization is a rational nilpotent space in view, for instance, of Theorem 8.1. ■

9. The Malcev completion of the fundamental group

This section is entirely devoted to the proof of the following result.

THEOREM 9.1: *Let K be a connected, finite simplicial complex. Then, $H_0(\mathfrak{L}(K), \partial_a)$ is the Malcev Lie completion of the fundamental group $\pi_1(K)$.*

Proof. Let $(\wedge V, d)$ be the minimal Sullivan model of K . Since K is connected,

$$V = V^{>0}.$$

The graded vector space

$$L_k(\wedge V, d) = (V^{k+1})^\sharp$$

equipped with the quadratic part of the differential d inherits the structure of a graded Lie algebra [13, §2], whose component $L_0(\wedge V, d)$ is the Lie algebra associated to the Malcev completion of $\pi_1(K)$ [13, Theorem 7.5].

Suppose first that V is a finite type vector space. In that case $(\wedge V, d)$ is the dual of a commutative differential graded coalgebra (C, d) . Thus, by [24],

$$H_0(\mathcal{L}(C, d)) \cong L_0(\wedge V, d).$$

On the other hand, the natural map $\mathcal{L}(C, d) \rightarrow (\mathcal{E}(\wedge V, d))^\#$ is the inclusion

$$\varphi: (\mathbb{L}(s^{-1}\overline{C}), d) \hookrightarrow (\widehat{\mathbb{L}}(s^{-1}\overline{C}), d).$$

Finally, since $(s^{-1}V)^\#$ injects in the homology of the two DGL's, and is in an isomorphism in both cases, the injection φ is a quasi-isomorphism.

In the general case, $(\wedge V, d)$ is the increasing union of finite type CDGA's of the form $(\wedge V(n), d)$, i.e.,

$$(\wedge V, d) = \varinjlim_n (\wedge V(n), d),$$

where $L_0(\wedge V(n), d) = L_0(\wedge V, d)/L_0^n(\wedge V, d)$,

$$\begin{aligned} L_0^2(\wedge V, d) &= [L_0(\wedge V, d), L_0(\wedge V, d)], \\ L_0^n(\wedge V, d) &= [L_0(\wedge V, d), L_0^{n-1}(\wedge V, d)], \quad n > 2. \end{aligned}$$

Moreover,

$$L_0(\wedge V, d) = \varprojlim_n L_0(\wedge V, d)/L_0^n(\wedge V, d).$$

We write

$$(\widehat{\mathbb{L}}(W(n)), \partial) = \mathcal{E}(\wedge V(n), d)^\#.$$

It follows that $\mathcal{E}(\wedge V, d) = \varinjlim_n \mathcal{E}(\wedge V(n), d)$ and

$$(\mathfrak{L}_K, \partial_a) \cong \varprojlim_n (\mathcal{E}(\wedge V(n), d))^\# = \varprojlim_n (\widehat{\mathbb{L}}(W(n)), \partial).$$

By construction $W(n) = W(n)_{\geq 0}$ is a finite type graded vector space. We write

$$Y_n = \widehat{\mathbb{L}}(W(n))_0, \quad X_n = \widehat{\mathbb{L}}(W(n))_1 \quad \text{and} \quad X_n^s = \mathbb{L}^s(W(n))_1.$$

We denote by $\rho_n: X_n \rightarrow X_{n-1}$ and $\sigma_n: Y_n \rightarrow Y_{n-1}$ the surjective morphisms induced by the inclusions $\wedge V(n-1) \hookrightarrow \wedge V(n)$. For $r < q$ we also denote by $\rho_{qr}: X_q \rightarrow X_r$ the composition $\rho_{r+1} \circ \cdots \circ \rho_q$.

We consider then the short exact sequences of towers of vector spaces

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & \partial(X_n) & \longrightarrow & \partial(X_{n-1}) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & Y_n & \longrightarrow & Y_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 \cdots & \longrightarrow & H_0(\widehat{\mathbb{L}}(W(n)), \partial) & \longrightarrow & H_0(\widehat{\mathbb{L}}(W(n-1)), \partial) & \longrightarrow & \cdots \\
 & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & &
 \end{array}$$

This induces an exact sequence

$$0 \rightarrow \varprojlim_n \partial(X_n) \rightarrow \varprojlim_n Y_n \rightarrow \varprojlim_n H_0(\widehat{\mathbb{L}}(W(n)), \partial) \rightarrow \varprojlim_n^1 \partial(X_n).$$

Since every $\rho_n: X_n \rightarrow X_{n-1}$ is surjective, the induced map $\partial(X_n) \rightarrow \partial(X_{n-1})$ is also surjective and so $\varprojlim_n^1 \partial(X_n) = 0$.

Now, by the next Lemma, $\varprojlim_n \partial(X_n) = \partial(\varprojlim_n X_n)$. Therefore,

$$H_0(\mathfrak{L}(K), \partial_a) \cong \varprojlim_n H_0(\widehat{\mathbb{L}}(W(n)), \partial) = \varprojlim_n L_0(\wedge V, d) / L_0^n(\wedge V, d)$$

is the Malcev Lie completion of $\pi_1(K)$. $\blacksquare$

LEMMA 9.2: *With the notation in the previous proof, let $a = \varprojlim (a_n) \in Y$. If for each n , $a_n = \partial c_n$ for some c_n , then there is an element $b = \varprojlim (b_n) \in X$ such that $\partial b_n = a_n$ for all n .*

Proof. The element a_1 can be written as $a_1 = \sum_{q=p}^{\infty} a_1^q$ with $a_1^q \in \mathbb{L}^q(W(1))_0$ and $a_1^p \neq 0$. For fixed n we denote by $f(n)$ the maximum integer such that there is an element $e \in \widehat{\mathbb{L}}^{\geq f(n)}(W(n))_1$ with $\partial e = a_n$. Since $d \circ \rho_{n1}(e) = a_1$, we have $f(n) \leq p$. As we can replace $e_1, \dots, e_{n-1}$ by the images of e_n , the function $f(n)$ is decreasing and we denote by r its limit.

We construct a sequence $\beta^r = (\beta_p^r)$ with the following properties:

- (i) $\beta_p^r \in X_p^r$ and $\rho_p(\beta_p^r) = \beta_{p-1}^r$.
- (ii) There exist elements $e_p \in X_p$ with $\partial e_p = a_p$ and $e_p - \beta_p^r \in X_p^{>r}$.

Suppose we have constructed $\beta_1^r, \dots, \beta_{n-1}^r$ with the two preceding properties for $p < n$ and

- (iii) for all q there exists an element $e_q \in X_q^{\geq r}$ with $\partial e_q = a_q$ and $\rho_{q(n-1)}(e_q) - \beta_{n-1}^r \in X_{n-1}^{>r}$.

Then, we denote by $q_{n-1}^r: X_{n-1} \rightarrow X_{n-1}^r$ the projection on the component of length degree r and we write

$$W_q = q_n^r \rho_{qn} ([\partial^{-1}(a_q) \cap X_q^{\geq r}] \cap (q_{n-1}^r \circ \rho_{q(n-1)})^{-1}(\beta_{n-1}^r)).$$

The W_q form a decreasing sequence of affine subspaces of X_n^r . Since X_n^r is finite dimensional and the W_q are non empty, their intersection is not empty and we choose an element β_n^r in this intersection. This ends the construction of $\beta^r = (\beta_n^r)$.

Now consider $\alpha' = \alpha - \partial\beta^r$. We proceed in the same way for α' and note that by construction there is an element $e'_n = e_n - \beta_n^r \in \mathbb{L}^{\geq r+1}$ with $\partial e'_n = \alpha'_n$. This gives an element β^{r+1} , and by induction elements β^q for $q \geq r$. Now

$$\sum_q \beta^q = \left(\sum_q \beta_n^q \right)$$

is a well defined element in $\varprojlim_n \widehat{\mathbb{L}}(W(n))$ and by construction

$$\partial \left(\sum_q \beta^q \right) = a. \quad \blacksquare$$

References

- [1] R. Bandiera, *Descent of Deligne–Getzler ∞ -groupoids*, <https://arxiv.org/abs/1705.02880>.
- [2] A. Berglund, *Homological perturbation theory for algebras over operads*, *Algebraic and Geometric Topology* **14** (2014), 2511–2548.
- [3] A. Berglund, *Rational homotopy theory of mapping spaces via Lie theory for L_∞ -algebras*, *Homology, Homotopy and Applications* **17** (2015), 343–369.
- [4] U. Buijs, Y. Félix, A. Murillo and D. Tanré, *Maurer–Cartan elements in the Lie models of finite simplicial complexes*, *Canadian Mathematical Bulletin* **60** (2017), 470–477.
- [5] U. Buijs, Y. Félix, A. Murillo and D. Tanré, *Homotopy theory of complete Lie algebras and Lie models of simplicial sets*, *Journal of Topology* **11** (2018), 799–825.
- [6] U. Buijs, Y. Félix, A. Murillo and D. Tanré, *The infinity Quillen functor, Maurer–Cartan elements and DGL realizations*, <https://arxiv.org/abs/1702.04397>.
- [7] U. Buijs and A. Murillo, *The Lawrence–Sullivan construction is the right model for I^+* , *Algebraic and Geometric Topology* **13** (2013), 577–588.

- [8] U. Buijs and A. Murillo, *Algebraic models of non-connected spaces and homotopy theory of L_∞ algebras*, *Advances in Mathematics* **236** (2013), 60–91.
- [9] X. Z. Cheng and E. Getzler, *Transferring homotopy commutative algebraic structures*, *Journal of Pure and Applied Algebra* **212** (2008), 2535–2542.
- [10] J. L. Dupont, *Simplicial de Rham cohomology and characteristic classes*, *Topology* **15** (1976), 233–245.
- [11] J. L. Dupont, *Curvature and Characteristic Classes*, *Lecture Notes in Mathematics*, Vol. 640, Springer, Berlin–New York, 1978.
- [12] Y. Félix, S. Halperin and J.-C. Thomas, *Rational Homotopy Theory*, *Graduate Texts in Mathematics*, Vol. 205, Springer, New York, 2001.
- [13] Y. Félix, S. Halperin and J.-C. Thomas, *Rational Homotopy Theory. II*, World Scientific, Hackensack, NJ, 2015.
- [14] B. Fresse, *Théorie des opérades de Koszul et homologie des algèbres de Poisson*, *Annales Mathématiques Blaise Pascal* **13** (2006), 237–312.
- [15] E. Getzler, *Lie theory for nilpotent L_∞ algebras*, *Annals of Mathematics* **170** (2009), 271–301.
- [16] V. Hinich, *Descent of Deligne groupoids*, *International Mathematics Research Notices* **5** (1997), 223–239.
- [17] M. Kontsevich, *Deformation quantization of Poisson manifolds*, *Letters in Mathematical Physics* **66** (2003), 157–216.
- [18] R. Lawrence and D. Sullivan, *A formula for topology/deformations and its significance*, *Fundamenta Mathematicae* **225** (2014), 229–242.
- [19] A. Lazarev and M. Markl, *Disconnected rational homotopy theory*, *Advances in Mathematics* **283** (2015), 303–361.
- [20] J. L. Loday and B. Valette *Algebraic Operads*, *Grundlehren der mathematischen Wissenschaften*, Vol. 346, Springer, Heidelberg, 2012.
- [21] M. Majewski, *Rational homotopical models and uniqueness*, *Memoirs of the American Mathematical Society* **682** (2000).
- [22] M. Manetti, *Lectures on deformations of complex manifolds*, *Rendiconti di Matematica e delle sue Applicazioni. Serie VII* **24** (2004), 1–183.
- [23] W. Michaelis, *Lie coalgebras*, *Advances in Mathematics* **38** (1980), 1–54.
- [24] J. Neisendorfer, *Lie algebras, coalgebras, and rational homotopy theory for nilpotent spaces*, *Pacific Journal of Mathematics* **74** (1978), 429–460.
- [25] P.-E. Parent and D. Tanré, *Lawrence–Sullivan models for the interval*, *Topology and its Applications* **159** (2012), 371–378.
- [26] D. Quillen, *Rational homotopy theory*, *Annals of Mathematics* **90** (1969), 205–295.
- [27] D. Robert-Nicoud, *Representing the deformation ∞ -groupoid*, *Algebraic and Geometric Topology* **19** (2019), 1453–1476.
- [28] D. Sinha and B. Walter, *Lie coalgebras and rational homotopy theory, I: Graph coalgebras*, *Homology, Homotopy and Applications* **13** (2011), 263–292.
- [29] D. Sullivan, *Infinitesimal computations in topology*, *Institut des Hautes Études Scientifiques. Publications Mathématiques* **47** (1977), 269–331.
- [30] D. Tanré, *Homotopie rationnelle: modèles de Chen, Quillen, Sullivan*, *Lecture Notes in Mathematics*, Vol. 1025, Springer, Berlin, 1983.