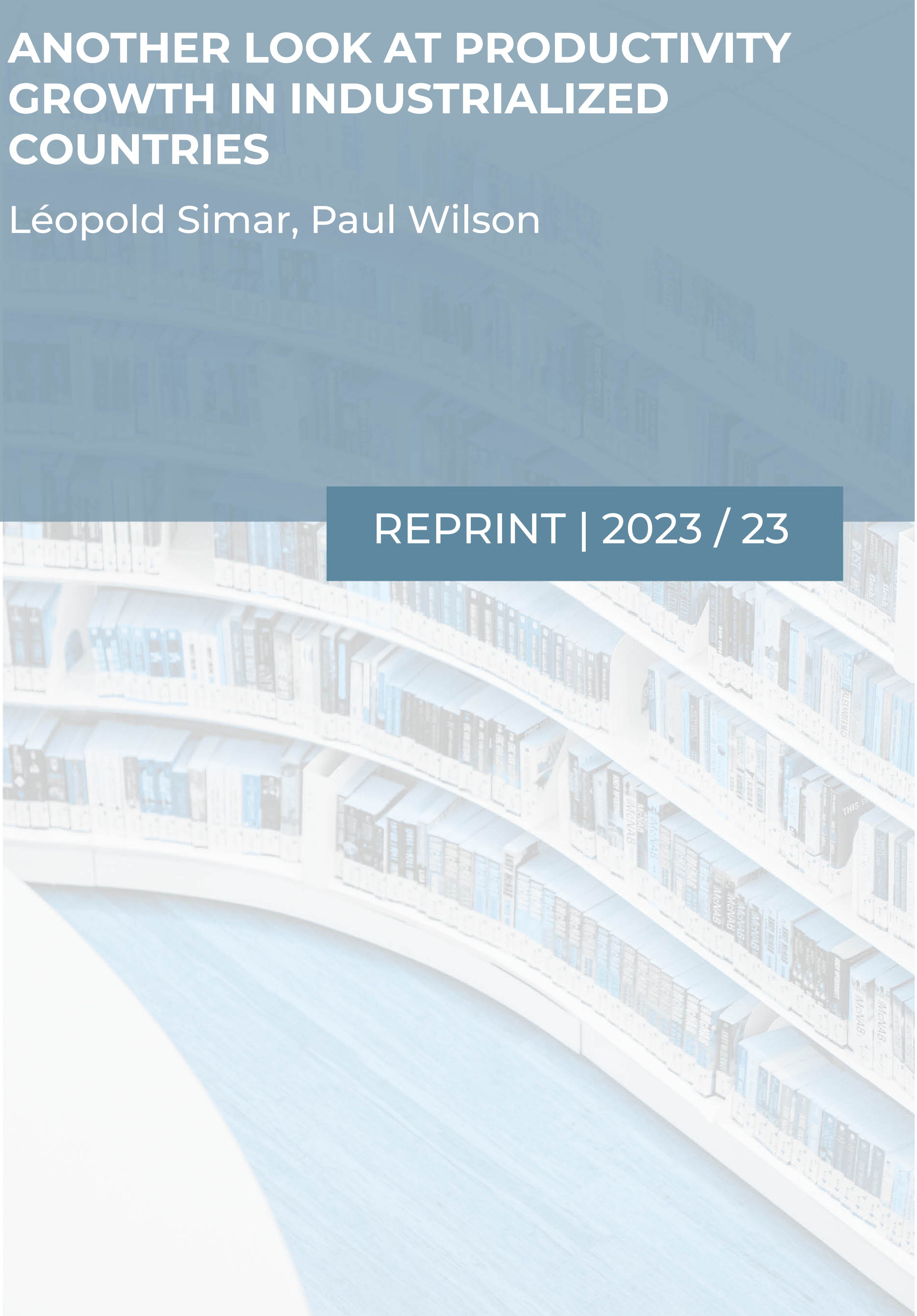
ANOTHER LOOK AT PRODUCTIVITY GROWTH IN INDUSTRIALIZED COUNTRIES

Léopold Simar, Paul Wilson

REPRINT | 2023 / 23



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Journal of Productivity Analysis
https://doi.org/10.1007/s11123-023-00689-w

Another look at productivity growth in industrialized countries

Léopold Simar[1] · Paul W. Wilson[2]

Accepted: 10 July 2023

Abstract
This paper discusses early work on productivity change by Färe and Grosskopf and their co-authors. We illustrate the use of recent statistical results by revisiting and updating Färe et al. (1994, *AER*). We analyze 17 OECD countries, estimating productivity change and its sources as measured by Malmquist indices and various decompositions. We describe and use recent theoretical results to make inferences, thereby quantifying what is learned from the data as opposed to merely providing point estimates.

Keywords DEA · Efficiency · Productivity · Malmquist index · Nonparametric inference

1 Introduction

This paper is based on a presentation we made at a plenary session honoring Rolf Färe and Shawna Grosskopf ("F&G" hereafter) at the 17th European Workshop on Efficiency and Productivity Analysis (EWEPA XVII) held in Porto, Portugal 27–29 June 2022. As noted in our presentation, their work has motivated a good deal of our own work, and without their pioneering work on the measurement of efficiency and productivity, we might have worked on other topics. As such, some of our own success is due to their success, and we are grateful for their contributions to the literature on performance-benchmarking.

Among F&G's work with various co-authors, two papers on measuring productivity change using Malmquist indices —Färe et al. (1992) and Färe et al. (1994)—are foundational. Malmquist indices were introduced by Caves et al. (1982) to measure changes in productivity over time, and are based on the ideas of Malmquist (1953). However, Malmquist indices received little attention until the work by F&G as seen in Fig. 1. The figure gives the number of "hits," presumably papers, using the keywords "Productivity" and "Malmquist Index" for each year 1980–2021. The number of papers with these keywords ranged from 1 to 9 each year from 1980 through 1991, but increases exponentially afterward.

Färe et al. (1992) examined productivity changes among 42 Swedish pharmacies, and showed how productivity change measured by Malmquist indices can be decomposed into measures of changes in technical efficiency and changes in technology. Färe et al. (1994) examined productivity changes among 17 members of the Organization for Economic Co-operation and Development (OECD) during 1979–1988, and decomposed Malmquist indices into measures of changes in technical efficiency, scale efficiency and technology. A search on Google Scholar indicates 1,747 citations of Färe et al. (1992) and 6,939 citations of Färe et al. (1994) up to 19 April 2023. Figure 2 shows the growth of citations of Färe et al. (1994) for each year 1994–2022. Although the citations peaked at 501 in 2015, the paper has received more than 300 citations in each subsequent year. Together, the two papers received 7,891 citations through the end of 2021, amounting to 50.3 percent of the works represented in Fig. 1. Apparently, about half of all papers examining productivity change using Malmquist indices during 1980–2021 cite Färe et al. (1992) or Färe et al. (1994). Clearly, F&G and their co-authors have been influential.

Both Färe et al. (1992) and Färe et al. (1994) rely on earlier work that developed methods for estimating

✉ Léopold Simar
leopold.simar@uclouvain.be

✉ Paul W. Wilson
pww@clemson.edu

[1] Institut de Statistique, Biostatistique, et Sciences Actuarielles (ISBA), LIDAM, Université Catholique de Louvain-la-Neuve, Voie du Roman Pays 20, B 1348 Louvain-la-Neuve, Belgium

[2] Department of Economics and School of Computing, Clemson University, Clemson, SC, USA

Published online: 18 August 2023

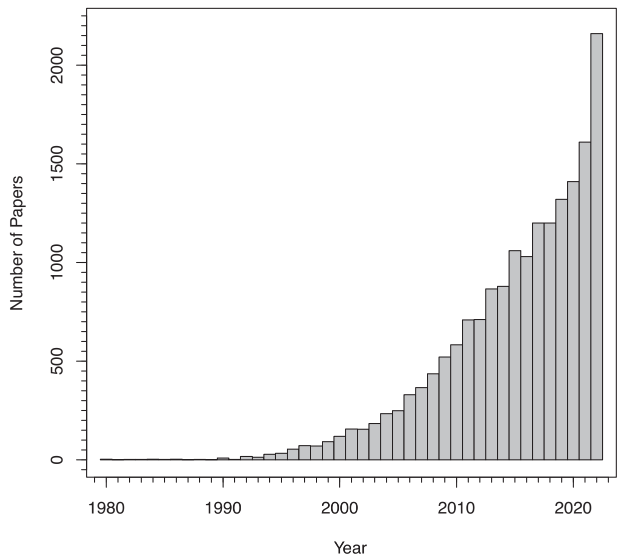

Fig. 1 Google Scholar Hits using Keywords "Productivity" and "Malmquist Index", 1980–2022, 19 April 2023

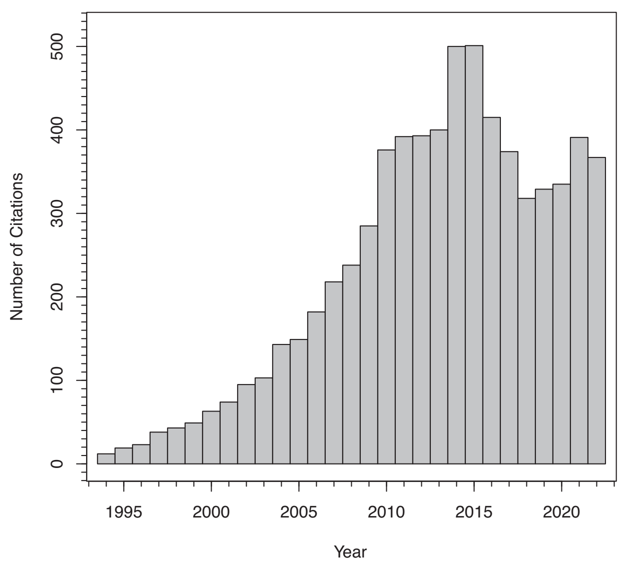

Fig. 2 Citations for FGNZ, 1994–2022, Google Scholar Search, 19 April 2023

efficiency, including the work by Farrell (1957), popularized later by Charnes et al. (1978) as "data envelopment analysis" (DEA). F&G, together with C. A. Knox Lovell, extended these ideas in Färe et al. (1985), showing that efficiency can be estimated in input, output or hyperbolic directions under either weak or strong disposability assumptions.

Most works in the efficiency and productivity literature from the mid-1990s and before, as well as too many later works, are a-statistical, make no attempt at or mention of *inference*, and confound unobservables and observables. Truth cannot be learned from data in any interesting setting, and while estimates by themselves may convey descriptive information about a sample, they provide no meaningful information about the population from which the sample is drawn. Learning can happen only when valid inference is made, and inference requires a coherent, well-defined statistical model. This is not a criticism of the work of F&G and others up to the mid-1990s, but quite the contrary. The nonparametric estimators that are used in studies of efficiency and productivity involve some complication, and not all problems can be solved at once. F&G and others were clever to give us what is familiar today, and as noted earlier, without their work, much of our own would not be possible.

But, more is needed. Fortunately, by now there are numerous statistical results that can be used to make inferences when examining efficiency and productivity. In this paper, we revisit Färe et al. (1994) and show how recent results can be used to analyze changes in productivity among OECD countries and to provide some quantification about what can be learned from the data.

We proceed as follows. In the next section, we present a dynamic, nonparametric statistical model. In Section 3 we discuss the relevant estimators and their properties. Section 4 addresses some misconceptions related to our analysis that appear in the literature. Section 5 discusses the data and the results of our analysis, and compares our results to those obtained by Färe et al. (1994). Summary and conclusions are provided in Section 6.

2 A dynamic, nonparametric statistical model of production

Let $x \in \mathbb{R}^p_+$ and $y \in \mathbb{R}^q_+$ denote vectors of *fixed* input and output quantities, and let $X \in \mathbb{R}^p_+$ and $Y \in \mathbb{R}^q_+$ denote *random* vectors of input and output quantities. Throughout, vectors are assumed to be column-vectors, as opposed to row-vectors. The set

$$\Psi^t := \{(x, y) | x \text{ can produce} y \text{ at time } t\} \tag{2.1}$$

gives the set of feasible combinations of inputs and outputs at time t. The *efficient frontier* of Ψ^t, i.e., the *technology*, is given by

$$\Psi^{t\partial} := \left\{(x, y) | (x, y) \in \Psi^t,\ (\gamma x, \gamma^{-1} y) \notin \Psi^t\ \forall\ \gamma \in (0, 1)\right\}. \tag{2.2}$$

Note that in any realistic setting, neither Ψ nor $\Psi^{t\partial}$ are observed.

Various economic assumptions regarding Ψ^t can be made; the assumptions of Shephard (1970) and Färe et al. (1988) are often used and are used here.

Assumption 2.1 Ψ^t is closed and strictly convex.

Assumption 2.2 $(x, y) \notin \Psi^t$ if $x = 0$, $y \geq 0, y \neq 0$; i.e., all production requires use of some inputs.

Assumption 2.3 For $\widetilde{x} \geq x$, $\widetilde{y} \leq y$, if $(x, y) \in \Psi^t$ then $(\widetilde{x}, y) \in \Psi^t$ and $(x, \widetilde{y}) \in \Psi^t$; i.e., both inputs and outputs are strongly disposable.

Here and throughout, inequalities involving vectors are defined on an element-by-element basis, as is standard. Assumption 2.2 rules out free lunches, while Assumption 2.3 imposes weak monotonicity on the frontier. Assuming that Ψ^t is closed ensures that $\Psi^{t\partial} \subset \Psi^t$, i.e., the technology is part of the production set Ψ^t.

The Farrell (1957) input and output measures of efficiency at time t are defined by

$$\theta(x, y|\Psi^t) := \inf\{\theta|(\theta x, y) \in \Psi^t\} \tag{2.3}$$

and

$$\lambda(x, y|\Psi^t) := \sup\{\lambda|(x, \lambda y) \in \Psi^t\} \tag{2.4}$$

which measure efficiency (respectively) in terms of the amount by which input levels can be scaled downward by the same factor without reducing output levels, or the amount by which output levels can be increased by the same factor without increasing input levels. Clearly, for all $(x, y) \in \Psi^t$, $\theta(x, y|\Psi^t) \leq 1$ and $\lambda(x, y|\Psi^t) \geq 1$. The hyperbolic graph measure of efficiency at time t introduced by Färe et al. (1985) is given by

$$\gamma(x, y|\Psi^t) := \inf\{\gamma > 0|(\gamma x, \gamma^{-1} y) \in \Psi^t\}, \tag{2.5}$$

providing a measure of the amount by which input levels can be feasibly, proportionately scaled downward while simultaneously scaling output levels upward by the same proportion. By construction, $\gamma(x, y|\Psi^t) \leq 1$ for $(x, y) \in \Psi^t$. As discussed by Simar and Wilson (2019), Kneip et al. (2021) and others, measuring efficiency along a hyperbolic path avoids some difficulties in dynamic settings, as will be seen below.

Next, define the operator $\mathcal{C}(\cdot)$ so that

$$\mathcal{C}(\Psi^t) := \{(x, y)|x = a\widetilde{x},\ y = a\widetilde{y} \text{ for some } (\widetilde{x}, \widetilde{y}) \in \Psi^t \text{ and any } a \in \mathbb{R}^1_+\} \tag{2.6}$$

is the conical hull of the set Ψ^t.[1] The frontier of $\mathcal{C}(\Psi^t)$ is the set

$$\mathcal{C}^\partial(\Psi^t) := \{(x, y)|(x, y) \in \mathcal{C}(\Psi^t),\ (\gamma x, \gamma^{-1} y) \notin \mathcal{C}(\Psi^t)\ \forall\ \gamma \in (0, 1)\}, \tag{2.7}$$

and $\mathcal{C}^\partial(\Psi^t) \subset \mathcal{C}(\Psi^t)$. If $\mathcal{C}(\Psi^t) = \Psi^t$, then the technology $\Psi^{t\partial}$ at time t exhibits globally constant returns to scale (CRS). Otherwise, $\Psi^t \subset \mathcal{C}(\Psi^t)$ and $\Psi^{t\partial}$ is said to exhibit variable returns to scale (VRS), with returns to scale either increasing, constant, or decreasing depending on the particular region of the frontier.

Now consider a sample $\mathcal{X}_n = \{(X_i^1, Y_i^1),\ (X_i^2, Y_i^2)\}_{i=1}^n$ of input-output combinations for n countries observed in periods $t = 1$ and 2. To simplify notation, define $Z_i^t := (X_i^t, Y_i^t)$ for $t \in \{1, 2\}$ so that $\mathcal{X}_n$ is represented by $\mathcal{X}_n = \{Z_i^1,\ Z_i^2\}_{i=1}^n$. The hyperbolic Malmquist index

$$\mathcal{M}_i := \left(\frac{\gamma(Z_i^2|\mathcal{C}(\Psi^1))}{\gamma(Z_i^1|\mathcal{C}(\Psi^1))} \times \frac{\gamma(Z_i^2|\mathcal{C}(\Psi^2))}{\gamma(Z_i^1|\mathcal{C}(\Psi^2))}\right)^{1/2} \tag{2.8}$$

measures country i's change in productivity between periods 1 and 2, and is the geometric mean of two ratios, each providing a measure of productivity change, in the first case using the boundary of $\mathcal{C}(\Psi^1)$ as a benchmark, and in the second case using the boundary of $\mathcal{C}(\Psi^2)$ as a benchmark. Clearly, $\mathcal{M}_i > (=$ or $<)$ 1 if productivity increases (remains unchanged or decreases) between periods 1 and 2. As in Simar and Wilson (2019) and Kneip et al. (2021), the Malmquist index here is defined in terms of hyperbolic measures as opposed to input- or output-oriented measures to avoid numerical difficulties, while Färe et al. (1992) defined their Malmquist index in terms of input distance functions and Färe et al. (1994) defined theirs in terms of output distance functions. See Simar and Wilson (2019), Kneip et al. (2021), as well as Zofío and Lovell (2001), Johnson and McGinnis (2009) and Russell (2018) for additional discussion of the advantages of working in the hyperbolic direction when using Malmquist indices. In particular, use of hyperbolic measures helps ensure that all of the components of productivity change obtained via various decompositions of Malmquist indices are well-defined.

A number of different decompositions of Malmquist indices have been proposed in attempts to identify the sources of any changes in productivity. Färe et al. (1992, equation 10) propose the input-oriented analog of

$$\mathcal{M}_i = \underbrace{\frac{\gamma(Z_i^2|\mathcal{C}(\Psi^2))}{\gamma(Z_i^1|\mathcal{C}(\Psi^1))}}_{:=\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)} \times \underbrace{\left[\frac{\gamma(Z_i^2|\mathcal{C}(\Psi^1))}{\gamma(Z_i^2|\mathcal{C}(\Psi^2))} \times \frac{\gamma(Z_i^1|\mathcal{C}(\Psi^1))}{\gamma(Z_i^1|\mathcal{C}(\Psi^2))}\right]^{1/2}}_{:=\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)} \tag{2.9}$$

and remark (p. 90) that "the quotient outside the bracket measures the change in technical inefficiency and the ratios inside the bracket measure the shift in the frontier between periods" 1 and 2. But as noted by Simar and Wilson (2019), this is true if and only if (iff) the technology is one of

[1] Note that this is a pointed cone (i.e., $\mathcal{C}(\Psi^t)$ includes $\{(0, 0)\}$). Moreover, conical hulls are necessarily convex; e.g., see Hiriart-Urruty and Lemaréchal (1993, pp. 101–102).

globally constant returns to scale. Färe et al. (1994) correct this by decomposing the output-oriented analog of $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ to obtain the output-oriented analog of

$$\mathcal{M}_i = \underbrace{\frac{\gamma(Z_i^2|\Psi^2)}{\gamma(Z_i^1|\Psi^1)}}_{:=\mathcal{E}_2(Z_i^1,Z_i^2|\Psi^1,\Psi^2)} \times \underbrace{\left[\frac{\gamma(Z_i^2|\mathcal{C}(\Psi^2))/\gamma(Z_i^2|\Psi^2)}{\gamma(Z_i^1|\mathcal{C}(\Psi^1))/\gamma(Z_i^1|\Psi^1)}\right]}_{:=\mathcal{S}_1(Z_i^1,Z_i^2|\Psi^1,\Psi^2)} \times \mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2). \tag{2.10}$$

The term $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ in (2.10) gives a measure of change in technical efficiency under *either* variable or constant returns to scale since efficiency is measured in terms of Ψ^1 and Ψ^2 as opposed to the conical hulls of Ψ^1 and Ψ^2 as in $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ appearing in (2.9). Färe et al. (1994, footnote 16) refer to $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ as a measure of "pure efficiency change." The ratio in the denominator of $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$' measures the distance between the projection of (Z_i^1) onto $\Psi^{1\partial}$ and the projection of (Z_i^1) onto $\mathcal{C}^\partial(\Psi^1)$, providing a measure of the scale efficiency of country i in period 1. The ratio in the numerator of $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ provides a similar measure of scale efficiency in period 2, and hence the ratio of ratios that defines $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ provides a measure of any change in the scale efficiency of country i from period 1 to period 2.[2] It is easy to see that both the numerator and the denominator of $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ must be less than 1, and that $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2) > (=, <)$ 1 iff scale efficiency for country i increases (remains unchanged, decreases) from period 1 to period 2.

The term $\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ that appears in both (2.9) and (2.10) measures change in technology, but this term is based on the conical hulls of Ψ^1 and Ψ^2. As such, it measures change in technology iff the technology is globally CRS in both periods 1 and 2. Otherwise, under variable returns to scale, it is possible for the conical hulls to remain unchanged while the technology shifts upward or downward in regions where the technology $\Psi^{t\partial}$ is not coincident with $\mathcal{C}^\partial(\Psi^t)$. This problem is addressed by Ray and Desli (1997), who propose the output-oriented analog of

$$\mathcal{M}_i = \mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2) \times \underbrace{\left[\frac{\gamma(Z_i^2|\Psi^1)}{\gamma(Z_i^2|\Psi^2)} \times \frac{\gamma(Z_i^1|\Psi^1)}{\gamma(Z_i^1|\Psi^2)}\right]^{1/2}}_{:=\mathcal{T}_2(Z_i^1,Z_i^2|\Psi^1,\Psi^2)} \times \underbrace{\left[\frac{\gamma(Z_i^2|\mathcal{C}(\Psi^1))/\gamma(Z_i^2|\Psi^1)}{\gamma(Z_i^1|\mathcal{C}(\Psi^2))/\gamma(Z_i^1|\Psi^2)} \times \frac{\gamma(Z_i^2|\mathcal{C}(\Psi^2))/\gamma(Z_i^2|\Psi^2)}{\gamma(Z_i^1|\mathcal{C}(\Psi^1))/\gamma(Z_i^1|\Psi^1)}\right]^{1/2}}_{:=\mathcal{S}_2(Z_i^1,Z_i^2|\Psi^1,\Psi^2)}. \tag{2.11}$$

The term $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ in (2.11) measures any change in technology between periods 1 and 2 regardless of whether returns to scale are constant or variable. This term consists of a geometric mean of two ratios. The first ratio gives a measure of any shift in the technology Ψ^∂ relative to country i's position in period 2 while the second ratio gives a measure of any shift in the technology relative to country i's position in period 1. Either of these ratios is greater than (equal to, less than) 1 iff the technology shifts outward (remains unchanged, shifts inward).

However, the term $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ appearing in (2.11) is problematic. Ray and Desli write (p. 1036) that $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ "is a geometric mean of the ratios of scale efficiencies of the two bundles using in turn the VRS technologies from the two periods as the benchmark. In that sense, it is more in the spirit of a Fisher index." Färe et al. (1997, p. 1042) criticize the measure, and in particular note that the term "may incorrectly identify the scale properties of the underlying technology." while providing an illustrative example in their footnote 7.

Indeed, the term $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ defined by (2.11) can be written as

$$\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2) = \left[\frac{\gamma(Z_i^2|\mathcal{C}(\Psi^1))/\gamma(Z_i^2|\Psi^1)}{\gamma(Z_i^1|\mathcal{C}(\Psi^2))/\gamma(Z_i^1|\Psi^2)} \times \mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)\right]^{1/2}. \tag{2.12}$$

The meaning of $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ is clear and intuitive as discussed above, but the first ratio inside the parentheses in (2.12) is less so. The numerator of this ratio measures scale efficiency in period 1, but from the viewpoint of the firm's location in period 2. Similarly, the denominator measures scale efficiency in period 2, but relative to the firm's location in period 1. This is curious since the term $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ included inside the brackets in (2.12) provides a measure of scale efficiency. Lovell (2003, p. 442) suggests that "the qualifier 'change' [when describing $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$] refers to the quantity vectors but not to the technologies."

Gilbert and Wilson (1998), Simar and Wilson (1998a) and Wheelock and Wilson (1999) use the output-oriented analog of

$$\mathcal{M}_i = \mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2) \times \mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2) \times \mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2) \times \underbrace{\left[\frac{\gamma(Z_i^1|\mathcal{C}(\Psi^1))/\gamma(Z_i^1|\Psi^1)}{\gamma(Z_i^1|\mathcal{C}(\Psi^2))/\gamma(Z_i^1|\Psi^2)} \times \frac{\gamma(Z_i^2|\mathcal{C}(\Psi^1))/\gamma(Z_i^2|\Psi^1)}{\gamma(Z_i^2|\mathcal{C}(\Psi^2))/\gamma(Z_i^2|\Psi^2)}\right]^{1/2}}_{:=\mathcal{S}_3(Z_i^1,Z_i^2|\Psi^1,\Psi^2)}. \tag{2.13}$$

The first three terms on the right-hand side of (2.13) appear in the decompositions discussed above and give measures of changes in technical efficiency, technology and scale efficiency. The fourth term, $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, consists of a geometric mean of two ratios, each resembling the ratio that defines $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ in (2.10), but with some important differences.

[2] Country i is scale-efficient in period 1 if $\gamma(Z_i^1|\mathcal{C}(\Psi^1)) = \gamma(Z_i^1|\Psi^1)$. Otherwise, the country is scale-inefficient. See Wheelock and Wilson (1999) for discussion.

Recall that $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ measures the change in scale efficiency of *the producer*. This could improve if the producer moves closer to the most efficient scale size in period 2, or it could improve if the firm does not move between periods 1 and 2, but the technology changes so that $\Psi^{2\partial}$ is closer to $\mathcal{C}^{\partial}(\Psi^2)$ than $\Psi^{1\partial}$ is to $\mathcal{C}^{\partial}(\Psi^1)$.

Now consider the first ratio of ratios in the definition of $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ in (2.13). Here, the firm's position is fixed at its location in period 1; the ratio can differ from 1 iff the distance between the projection of (Z_i^1) onto $\Psi^{1\partial}$ and $\mathcal{C}^{\partial}(\Psi^1)$ is different from the projection of (Z_i^1) onto $\Psi^{2\partial}$ and $\mathcal{C}^{\partial}(\Psi^2)$ along the hyperbolic path through (Z_i^1). The second ratio in $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ provides a similar measure relative to the firm's position in period 2, and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ is the geometric mean of these two measures. As such, $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ provides a measure of change in returns to scale *of the technology* at points where producer i is projected onto the frontiers from its position in period 1 and in period 2. Other decompositions are possible, and we do not attempt to give an exhaustive treatment here. See Simar and Wilson (2019) for additional discussion and references.

As discussed in Kneip et al. (2021), we define the Malmquist index in (2.8) in terms of hyperbolic distances to avoid difficulties arising from issues of existence and identification when decomposing productivity change into its component sources. These difficulties may arise when input or output-oriented distance measures are used and the technology shifts between periods one and two causing the cross-period measures used to define $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ to be undefined. This may happen, for example, if the point Z_i^1 cannot be projected onto the frontier of Ψ^2 in the input or output direction, or if the point Z_i^2 cannot be projected onto the frontier of Ψ^1. Under the assumptions of the model introduced below, this is avoided when hyperbolic measures are used.

It is important to remember that all of the quantities and model features defined so far are unobservable, and hence must be estimated. In addition, inference is needed in order to know what might be learned from any estimates computed from data. And as noted earlier, inference is possible only within the context of a well-defined, coherent statistical model, and some additional assumptions are needed to complete the statistical model. The following assumptions are analogous to Assumptions 3.1–3.4 of Kneip et al. (2015). In order to draw upon previous results, the assumptions below are stated in terms of the input-oriented measure of efficiency without loss of generality.[3]

[3] The assumptions can also be stated in terms of the output, hyperbolic and directional measures of efficiency, and the results of Kneip et al. (2015) extend to those measures after trivial (but tedious) changes in notation in Kneip et al. (2015).

Assumption 2.4 (i) The random variables (X, Y) possess a joint density f^t with support $\mathcal{D}^t \subset \Psi^t$; and (ii) f^t is continuously differentiable on $\mathcal{D}^t$.

Assumption 2.5 (i) $\mathcal{D}^{t*} := \{\theta(x, y|\Psi^t)x,\ y)|(x, y) \in \mathcal{D}^t\} \subset \mathcal{D}^t$; (ii) $\mathcal{D}^{t*}$ is compact; and (iii) $f^t(\theta(x, y)x, y) > 0$ for all $(x, y) \in \mathcal{D}^t$.

The next two assumptions are needed when DEA estimators are used. Assumption 2.6 imposes some smoothness on the frontier. Kneip et al. (2008) require only two-times differentiability to establish the existence of a limiting distribution for VRS-DEA estimators, but the stronger assumption that follows is needed to establish results on moments of the DEA estimators.

Assumption 2.6 $\theta(x, y|\Psi^t)$ is three times continuously differentiable on $\mathcal{D}^t$.

Recalling that the strong (i.e., free) disposability assumed in Assumption 2.3 implies that the frontier is weakly monotone, the next assumption strengthens this by requiring the frontier to be strictly monotone with no constant segments. This is also needed to establish properties of moments of the DEA estimators.

Assumption 2.7 $\mathcal{D}^t$ is *almost strictly convex*; i.e., for any $(x, y), (\widetilde{x}, \widetilde{y}) \in \mathcal{D}^t$ with $(\frac{x}{\|x\|}, y) \neq (\frac{\widetilde{x}}{\|\widetilde{x}\|}, \widetilde{y})$, the set $\{(x^*, y^*)| (x^*, y^*) = (x, y) + \alpha((\widetilde{x}, \widetilde{y}) - (x, y))$ for some $0<\alpha<1\}$ is a subset of the *interior* of $\mathcal{D}^t$.

Assumptions 2.1–2.7 comprise a statistical model similar to the one defined in Kneip et al. (2015) and where DEA estimators have desirable properties. However, two additional, important assumptions are needed to obtain asymptotic properties of DEA estimators derived by Kneip et al. (2021) of the Malmquist index defined in (2.8) as well as of DEA estimators of the various components of Malmquist indices presented above. These assumptions appear as Assumptions 3.1 and 3.2 in Kneip et al. (2021). Since these assumptions involve considerable technical detail and require additional notation, we do not state them formally here but refer the reader to Kneip et al. (2021) and the additional discussion given in Simar and Wilson (2019, Appendix A, Section A.1). Assumption 3.1 of Kneip et al. (2021) involves the support $\mathcal{D}^t \subset \Psi^t$ of the density f^t and ensures that distances $\theta(x, y|\mathcal{C}(\Psi^t))$ and $\gamma(x, y|\mathcal{C}(\Psi^t))$ to the conical hull of Ψ^t are identified by data and can be estimated consistently. The assumption also ensures that these efficiency scores as well as $\theta(x, y|\Psi^t)$ and $\gamma(x, y|\Psi^t)$ are bounded away from zero so that their logarithms are finite. In addition, the assumption is necessary to guarantee existence of moments of the various distance measures. Assumption 3.2 of Kneip et al. (2021) Involves the supports $\mathcal{D}^1$ and $\mathcal{D}^2$ of the densities f^1 and f^2 for periods one and two, and ensures that the cross-period efficiencies needed to define Malmquist indices and some of its components in the various decompositions discussed above are identified and can be estimated consistently from data.

The model described here is fully nonparametric. No functional-form assumptions are made. Indeed, the underlying functional forms may be infinitely parameterized, i.e., nonparametric. If the underlying functional forms involve a finite number of parameters, the specific form remains unknown, and avoiding functional-form assumptions has the advantage of eliminating the risk of mis-specification in this regard.

Nonetheless, the model described here does not allow for random noise, which some might view as a restrictive assumption. However, in the context of Färe et al. (1994), highly aggregated data are used, which might reasonably be expected to reduce any noise present in the data. Alternatively, one can allow for noise and remain almost fully nonparametric using the ideas of Simar and Wilson (2023b). But, this is not the focus of the present paper.

3 Hyperbolic DEA estimators and their properties

Farrell (1957) proposed estimating Ψ^t by the convex hull of the free-disposal hull of observed input-output pairs in period t given by

$$\widehat{\Psi}_n^t := \{(x,y) \in \mathbb{R}^{p+q} | y \leq \boldsymbol{Y}^t\boldsymbol{\omega},\ x \geq \boldsymbol{X}^t\boldsymbol{\omega},\ \boldsymbol{i}_n'\boldsymbol{\omega} = 1,\ \boldsymbol{\omega} \in \mathbb{R}_+^n\}, \tag{3.1}$$

where $\boldsymbol{X}^t = \left(X_1^t, \ \ldots\ , X_n^t\right)$ and $\boldsymbol{Y}^t = \left(Y_1^t, \ \ldots\ , Y_n^t\right)$ are $(p \times n)$ and $(q \times n)$ matrices of input and output vectors in period t, respectively; $\boldsymbol{i}_n$ is an $(n \times 1)$ vector of ones, and $\boldsymbol{\omega}$ is a $(n \times 1)$ vector of weights. Substituting $\widehat{\Psi}_n^t$ for Ψ^t in (2.3) and (2.5) leads to the VRS-DEA estimators of $\theta(x,y)|\Psi^t)$, $\lambda(x,y)|\Psi^t)$ and $\gamma(x,y)|\Psi^t)$, i.e.,

$$\theta(x,y|\widehat{\Psi}_n^t) = \min_{\theta,\boldsymbol{\omega}}\{\theta | y \leq \boldsymbol{Y}^t\boldsymbol{\omega},\ \theta x \geq \boldsymbol{X}^t\boldsymbol{\omega},\ \boldsymbol{i}_n'\boldsymbol{\omega} = 1,\ \boldsymbol{\omega} \in \mathbb{R}_+^n\}, \tag{3.2}$$

$$\lambda(x,y|\widehat{\Psi}_n^t) = \min_{\lambda,\boldsymbol{\omega}}\{\theta | \lambda y \leq \boldsymbol{Y}^t\boldsymbol{\omega},\ x \geq \boldsymbol{X}^t\boldsymbol{\omega},\ \boldsymbol{i}_n'\boldsymbol{\omega} = 1,\ \boldsymbol{\omega} \in \mathbb{R}_+^n\} \tag{3.3}$$

and

$$\gamma(x,y|\widehat{\Psi}_n^t) = \min_{\gamma,\boldsymbol{\omega}}\{\gamma | \gamma^{-1} y \leq \boldsymbol{Y}^t\boldsymbol{\omega},\ \gamma x \geq \boldsymbol{X}^t\boldsymbol{\omega},\ \boldsymbol{i}_n'\boldsymbol{\omega} = 1,\ \boldsymbol{\omega} \in \mathbb{R}_+^n\}. \tag{3.4}$$

The first two estimators can be computed using standard linear programming algorithms (e.g., the simplex method), but (3.4) is a *nonlinear* program. Consequently, until recently, few studies have used the hyperbolic estimator. A solution is provided by Wilson (2011), who gives a numerical method based on bisection that can be used to compute estimates $\gamma(x,y|\widehat{\Psi}_n^t)$ to arbitrary degrees of accuracy. The *FEAR* package for *R* described by Wilson (2008) uses this method.[4]

The conical DEA (CDEA) estimator $\mathcal{C}(\widehat{\Psi}^t)$ of $\mathcal{C}(\Psi^t)$ discussed by Kneip et al. (2021) is obtained by dropping the constraint $\boldsymbol{i}_n'\boldsymbol{\omega} = 1$ in (3.1). This leads to the CDEA estimators $\theta(x,y|\mathcal{C}(\widehat{\Psi}^t))$, $\lambda(x,y|\mathcal{C}(\widehat{\Psi}^t))$ and $\gamma(x,y|\mathcal{C}(\widehat{\Psi}^t))$ of $\theta(x,y|\mathcal{C}(\Psi^t))$, $\lambda(x,y|\mathcal{C}(\Psi^t))$ and $\gamma(x,y|\mathcal{C}(\Psi^t))$ obtained by dropping the constraint $\boldsymbol{i}_n'\boldsymbol{\omega} = 1$ in (3.2)–(3.4). Both $\theta(x,y|\mathcal{C}(\widehat{\Psi}^t))$ and $\lambda(x,y|\mathcal{C}(\widehat{\Psi}^t))$ can be computed using standard linear programming methods, and $\gamma(x,y|\mathcal{C}(\widehat{\Psi}^t))$ can be computed as $\left[\theta(x,y|\mathcal{C}(\widehat{\Psi}^t))\right]^{1/2}$.

Asymptotic properties of the CDEA estimator $\gamma(x,y|\mathcal{C}(\widehat{\Psi}^t))$ of $\gamma(x,y|\mathcal{C}(\Psi^t))$, under appropriate assumptions, are established by Kneip et al. (2021). Specifically, Kneip et al. establish consistency and existence of a non-degenerate limiting distribution with rate of convergence n^κ under Assumptions 2.1–2.7 where $\kappa = 2/(p+q+1)$. Consequently, under VRS, the CDEA estimator has the same convergence rate as the VRS-DEA estimator. In addition, Kneip et al. (2021) establish properties of the first two moments of $\gamma(x,y|\mathcal{C}(\widehat{\Psi}^t))$ as well as of $\log\gamma(x,y|\mathcal{C}(\widehat{\Psi}^t))$.

Now consider a firm operating at observed, fixed points $(x^1,\ y^1)$ and $(x^2,\ y^2)$ in periods 1 and 2. From (2.8), the Malmquist index for this firm is

$$\mathcal{M} = \left[\frac{\gamma\left(x^2,y^2|\mathcal{C}(\Psi^1)\right)}{\gamma\left(x^1,y^1|\mathcal{C}(\Psi^1)\right)} \times \frac{\gamma\left(x^2,y^2|\mathcal{C}(\Psi^2)\right)}{\gamma\left(x^1,y^1|\mathcal{C}(\Psi^2)\right)}\right]^{1/2}, \tag{3.5}$$

and this is estimated by

$$\widehat{\mathcal{M}} = \left[\frac{\gamma\left(x^2,y^2|\mathcal{C}(\widehat{\Psi}_{n_1}^1)\right)}{\gamma\left(x^1,y^1|\mathcal{C}(\widehat{\Psi}_{n_1}^1)\right)} \times \frac{\gamma\left(x^2,y^2|\mathcal{C}(\widehat{\Psi}_{n_2}^2)\right)}{\gamma\left(x^1,y^1|\mathcal{C}(\widehat{\Psi}_{n_2}^2)\right)}\right]^{1/2} \tag{3.6}$$

using the data $\mathcal{X}_{n_1}^1 := \{\left(X_i^1, Y_i^1\right\}_{i=1,\ldots,n_1}$ and $\mathcal{X}_{n_2}^2 := \{\left(X_i^2, Y_i^2\right\}_{i=1,\ldots,n_2}$. Under Assumptions 2.1–2.7 and Assumptions 3.1–3.2 of Kneip et al. (2021), Theorem 3.3 of Kneip et al. (2021) establishes the existence of a non-

[4] Färe et al. (1985) make no mention of how (3.4) might be computed. Ray (2001) proposes what amounts to a delta-method approximation that can be computed using linear programming methods, but this can introduce substantial errors. Wilson (2011) provides a simple algorithm for computing (3.4) that does not rely on non-linear programming, and this method is implemented in the *FEAR* package for *R* described by Wilson (2008). See Wilson (2011) for additional discussion.

degenerate limiting distribution as well as the convergence rate for the estimator in (3.6) of the Malmquist index for a given firm observed in periods 1 and 2. Using these results, inference about the unobserved, true Malmquist index $\mathcal{M}$ can be made using the subsampling methods described by Simar and Wilson (2011).

Researchers often also report geometric means of estimates of Malmquist indices. Consider the ith observation (Z_1^1, Z_2^2) and the estimator

$$\widehat{\mathcal{M}}_i := \left[\frac{\gamma\left(Z_i^2|\mathcal{C}(\widehat{\Psi}_{n_1}^1)\right)}{\gamma\left(Z_i^1|\mathcal{C}(\widehat{\Psi}_{n_1}^1)\right)} \times \frac{\gamma\left(Z_i^2|\mathcal{C}(\widehat{\Psi}_{n_2}^2)\right)}{\gamma\left(Z_i^1|\mathcal{C}(\widehat{\Psi}_{n_2}^2)\right)}\right]^{1/2} \tag{3.7}$$

of the "true," but unobserved Malmquist index $\mathcal{M}_i$ in appearing in (2.8). The geometric mean of such estimates over the n observations for producers observed in both periods is

$$\widehat{\mathcal{M}}_n := \left(\prod_{i=1}^{n} \widehat{\mathcal{M}}_i\right)^{1/n}, \tag{3.8}$$

which can be viewed as an estimator of

$$\mathcal{M}_n := \left(\prod_{i=1}^{n} \mathcal{M}_i\right)^{1/n}. \tag{3.9}$$

Now define

$$\mu_{\mathcal{M}} := E(\log \mathcal{M}_i) \tag{3.10}$$

where the expectation is over (X, Y) in both periods. Kneip et al. (2021, Remark 3.2) use Jensen's inequality to demonstrate that $E(\mathcal{M}_n) > \exp(\mu_{\mathcal{M}})$, but Theorem 3.7 of Kneip et al. (2021) shows that the additional bias, $(E(\mathcal{M}_n) - \exp(\mu_{\mathcal{M}}))$, is asymptotically negligible. Consequently, inference for $\widehat{\mathcal{M}}_n$ follows from inference for $\log \widehat{\mathcal{M}}_n$, and hence $\exp(\mu_{\mathcal{M}})$ is the quantity of interest when considering geometric means of Malmquist index estimates.

Theorem 3.7 of Kneip et al. (2021) establishes that under Assumptions 2.1–2.7 and Assumptions 3.1–3.2 of Kneip et al. (2021), an asymptotic $(1-\alpha)\times 100$-percent confidence interval for $\exp(\mu_{\mathcal{M}})$ is given by

$$\left[\widehat{\mathcal{M}}_n \pm z_{1-\frac{\alpha}{2}} \frac{\exp\left(\widehat{\mu}_{\mathcal{M},n}\right)\widehat{\sigma}_{\mathcal{M},n}}{\sqrt{n}}\right] \tag{3.11}$$

when $\kappa > 1/2$, where $\widehat{\sigma}_{\mathcal{M},n}$ is the sample variance of the estimates $\log\left(\widehat{\mathcal{M}}_i\right)$, $i = 1, \ldots, n$.

But, as explained by Kneip et al. (2021), the bias in $\widehat{\mathcal{M}}_n$ must be dealt with if $\kappa \leq 1/2$. Theorems 4.2 and 4.3 of Kneip et al. (2021) provide CLTs for such cases. Under Assumptions 2.1–2.7 and Assumptions 3.1–3.2 of Kneip et al. (2021), Theorem 4.2 of Kneip et al. (2021) ensures that

$$\left[\widehat{\mathcal{M}}_n - \widehat{B}_{n,\kappa}^{*} \pm z_{1-\frac{\alpha}{2}} \frac{\exp\left(\widehat{\mu}_{\mathcal{M},n}\right)\widehat{\sigma}_{\mathcal{M},n}}{\sqrt{n}}\right]. \tag{3.12}$$

is an asymptotically correct $(1-\alpha)$ confidence interval for $\exp(\mu_{\mathcal{M}})$ when $\kappa \geq 2/5$, where $\widehat{B}_{n,\kappa}^{*}$ is the generalized jackknife estimator of bias described by Kneip et al. (2021). Alternatively, Theorem 4.3 of Kneip et al. (2021) establishes that under the same assumptions, an asymptotically correct $(1-\alpha)$ confidence interval for $\exp(\mu_{\mathcal{M}})$ is given by

$$\left[\widehat{\mathcal{M}}_{n_\kappa} - \widehat{B}_{n,\kappa}^{*} \pm z_{1-\frac{\alpha}{2}} \frac{\exp\left(\widehat{\mu}_{\mathcal{M},n}\right)\widehat{\sigma}_{\mathcal{M},n}}{n^{\kappa}}\right] \tag{3.13}$$

when $\kappa < 1/2$, where $\widehat{\mathcal{M}}_{n_\kappa}$ is a sample mean over a random subset of the n estimates $\widehat{\mathcal{M}}_i$, i.e.,

$$\widehat{\mathcal{M}}_{n_\kappa} := \left(\prod_{i=1}^{n_\kappa} \widehat{\mathcal{M}}_i\right)^{1/2} \tag{3.14}$$

where $n_\kappa = \min(\lfloor n^{2\kappa} \rfloor,\ n)$ and the $\widehat{\mathcal{M}}_i$ are computed using all of the n observations from both periods.

Kneip et al. (2021) note that when $(p+q) = 4$, either of the intervals in (3.12) or (3.13) can be used to make inference about $\exp(\mu_{\mathcal{M}})$. However, for theoretical reasons, one should expect the interval in (3.13) to provide better approximations in finite samples than the interval in (3.12) when $(p+q) = 4$, and this is confirmed by Monte Carlo experiments. See Kneip et al. (2021) for details and further discussion.

Just as the Malmquist index in (2.8) can be estimated by replacing $\mathcal{C}(\Psi^1)$ and $\mathcal{C}(\Psi^2)$ with the corresponding estimators $\mathcal{C}(\widehat{\Psi}_{n_1}^1)$ and $\mathcal{C}(\widehat{\Psi}_{n_2}^2)$, so too can the various components appearing in the decompositions in (2.9)–(2.11) and (2.13) be estimated. Simar and Wilson (2019) derive results for estimators of the components analogous to the results of Kneip et al. (2021) for Malmquist indices, thereby enabling inference (e.g., estimation of confidence intervals) for each of the components of the various decompositions. See Simar and Wilson (2019) for details.

4 Misconceptions in the Literature

As noted earlier the production set at time t, i.e., Ψ^t, appearing in (2.1) is unobserved, and consequently must be estimated from data. The VRS-DEA estimator of Ψ^t is given by $\widehat{\Psi}_n^t$ in (3.1). These are not the same. The estimator $\widehat{\Psi}_n^t$ is a random set, since it is a set defined by the random sample

$\mathcal{X}_n^t = \{(X_i^t, Y_i^t)\}_{i=1}^n$ which contains random variables (X_i^t, Y_i^t). When observed data are available, these can be used in conjunction with (3.1) to compute an *estimate* $\widehat{\Psi}_n^t$, which is a realization of the estimator in (3.1).[5]

Färe et al. (1994) use notation different from ours. Nonetheless, Färe et al. (1994, equation 1, page 68) give an expression for the production set at time t by writing what appears here in (2.1) using their notation. But then, in their equation (14) on page 73, Färe et al. (1994) state that "The reference (or frontier) technology in period t is constructed from the data as" followed by an expression equating $\widehat{\Psi}_n^t$ with the right-hand side of (3.1) after dropping the constraint $\boldsymbol{i}_n'\boldsymbol{\omega} = 1$. Färe et al. (1994) continue, writing "which exhibits constant returns to scale and strong disposability of inputs and outputs (see Färe et al.,1985 for details). Following Sidney Afriat (1972), the assumption of constant returns to scale may be relaxed to ... allow for variable returns to scale" by adding the constraint $\boldsymbol{i}_n'\boldsymbol{\omega} = 1$ to the right-hand side of their equation (14), which amounts to equating the right-hand side of our (3.1) with Ψ^t defined in (2.1).

We do not mean this as a criticism of Färe et al. (1994); indeed, many papers from the early years of efficiency and productivity analysis make similar statements. Nonetheless, such statements are now known to be incorrect, and have unfortunately resulted in confusion that persists even now. To be clear, Ψ_n^t is unknown and must therefore be estimated from data. Anything computed from data is an estimate. Estimates provide little information about what is estimated until statistical inference is made.

Additional problems arise in the discussion of distance functions by Färe et al. (1994), who define output distance functions for periods t and $t+1$ in their equations (2) and (3) using the Shephard (1970) metric. These are reciprocals of the output-oriented Farrell (1957) in (2.4). But Färe et al. (1994, equation 16, page 75) equates the reciprocal of the distance function defined in their equation (2) with the right-hand side of our (3.3) without the constraint $\boldsymbol{i}_n'\boldsymbol{\omega} = 1$. But this is not correct; the authors' distance function defined in their equation (2) is unobserved, while the expression in their equation (16) is an *estimator* (or an *estimate* when used to compute a value from observed data).

Färe et al. (1994) go on to write later on the same page (i.e., page 75), "In order to calculate changes in scale efficiency, we also calculate distance functions under variable returns to scale by adding" the restriction $\boldsymbol{i}_n'\boldsymbol{\omega} = 1$. Apart from the fact that distance functions must be *estimated* and cannot be *calculated* (only their estimates can be calculated), the statement is at odds with the characterization of the authors' equation (14), where the "reference technology in period t" is constructed to exhibit CRS. Either $\Psi^{t\partial}$ exhibits CRS or it does not. From the discussion above in Sections 2 and 3, it is clear that the Malmquist index in (2.8) is defined in terms of the conical hull of the production set, and is estimated using CDEA estimators regardless of whether $\Psi^{t\partial}$ exhibits globally CRS. As seen from the discussion in Section 3 and in Kneip et al. (2021), this has important implications for how inferences are made.[6]

An additional source of confusion in the literature (but not in Färe et al., 1994) arises from the characterization of DEA estimators as "models." For examples, see Charnes et al. (1978, 1981) and Banker et al. (1984), as well as numerous recent papers in this journal (e.g., Chiu et al., 2022; Mendelová, 2022; Naderi, 2022; Sarac et al., 2022; and Walheer, 2022), *European Journal of Operational Research* (e.g., Gerami et al., 2022; Kalinichenko et al., 2022; Radovanović et al., 2022; Taleba et al., 2022; and Yu et al., 2022), *Annals of Operations Research* (e.g., Bou-Hamad et al., 2022; Henriques et al., 2022; Hu and Kim, 2022; Pachar et al., 2022; and Zhang and Liao, 2022), *Operations Research* (e.g., Adelman, 2022; Chen and Zhu, 2011; Chen and Delmas, 2012; Cook and Zhu, 2011; and Podinovski and Bouzdine-Chameeva, 2013) as well as others. A *model* is a set of assumptions, while DEA is a class of nonparametric envelopment estimators. The distinction is more than a semantic one. Estimates are computed from data, but reveal nothing about quantities or features that are estimated. Only statistical inference is informative in any interesting problem, and inference is impossible without a well-defined, coherent statistical model describing the underlying data-generating process. In addition, inference requires knowledge of the statistical properties of the estimators being used.

Some readers will object. One might claim, "I have observations on all the firms in the industry that I am examining, and therefore I do not need inference." But, what if an entrepreneur starts a new firm in the same industry? A new producer might out-perform all of the existing producers, and a second new producer might out-perform the first one. All that one can hope for is valid inference, and this requires a well-defined model such as the one described above in Section 3.

Again, none of what is written here should be construed as criticism of the work by F&G and their co-authors. Much was unknown or little-understood in the 1990s when Färe

[5] See, for example, Spanos (2019) for detailed discussion of the difference between *estimators* and *estimates*. Korostelev et al. (1995a, b) are the first papers to analyze the differences and relations between Ψ^t and estimators of Ψ^t.

[6] In their Table 3, Färe et al. (1994) report technical efficiency estimates for each of the 17 countries examined in 1979, 1983 and 1988. The technical efficiency estimates are computed while imposing CRS. As such, the estimates are valid iff globally CRS holds in each year, while the VRS-DEA estimators are consistent regardless of whether CRS holds.

Table 1 Geometric Mean Productivity Change and Its Components for OECD Countries, 1979–1988, Hyperbolic Direction

Period	$\widehat{\mathcal{M}}_n$	$\widehat{\mathcal{E}}_{1,n}$	$\widehat{\mathcal{E}}_{2,n}$	$\widehat{\mathcal{T}}_{1,n}$	$\widehat{\mathcal{T}}_{2,n}$	$\widehat{\mathcal{S}}_{1,n}$	$\widehat{\mathcal{S}}_{2,n}$	$\widehat{\mathcal{S}}_{3,n}$
1979–1980	1.00197*	1.00003*	1.00069*	1.00194*	0.99976*	0.99933*	1.00151*	1.00218*
1980–1981	0.99938	0.99900	0.99869	1.00038	0.99997	1.00031	1.00073	1.00041
1981–1982	1.00068	1.00887	1.00786	0.99188	0.99191	1.00100	1.00096	0.99996
1982–1983	1.00683***	0.99107***	0.99895***	1.01589***	1.00868***	0.99211***	0.99921***	1.00715***
1983–1984	1.01111***	0.98957***	0.99335***	1.02177***	1.01800***	0.99620***	0.99988***	1.00370***
1984–1985	1.00741***	0.99693***	0.99795***	1.01051***	1.00958***	0.99898***	0.99990***	1.00092***
1985–1986	1.00338**	1.00003**	0.99904**	1.00335**	1.00470**	1.00099**	0.99964**	0.99865**
1986–1987	1.00515***	1.00512***	0.99906***	1.00002***	1.00546***	1.00607***	1.00062***	0.99459***
1987–1988	1.00883***	1.00616***	1.00030***	1.00265***	1.00773***	1.00585***	1.00079***	0.99497***
1979–1988	1.04719***	0.99662***	0.99585***	1.05074***	1.04521***	1.00077***	1.00607***	1.00530***

NOTE: The column labeled $\widehat{\mathcal{M}}_n$ gives geometric means for estimates of the Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_{1,n}$, $\widehat{\mathcal{E}}_{2,n}$, $\widehat{\mathcal{T}}_{1,n}$, $\widehat{\mathcal{T}}_{2,n}$, $\widehat{\mathcal{S}}_{1,n}$, $\widehat{\mathcal{S}}_{2,n}$ and $\widehat{\mathcal{S}}_{3,n}$ give geometric means for estimates of the components $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

et al. (1994) appeared in print. The same is true today, although arguably more is known now than was known 25 or 30 years ago. Part of the learning that has occurred is due to the early work of F&G and its stimulus for those who followed. Such is the nature of science.

5 Data and results of estimation and inference

Färe et al. (1994) estimate productivity growth and its components using a sample of 17 OECD countries over the period 1979–1988. Their data are taken from the Penn World Tables (PWT) (Mark 5) described by Summers and Heston (1991). The PWT project and data have evolved over the intervening years. The PWT data have been revised, updated and expanded to cover more years and more countries. We use the current version of the PWT data (version 10.0) described by Feenstra et al. (2015), but we use the same variables (over the same years) used by Färe et al. (1994). Specifically, the data we use consist of observations on $p = 2$ inputs (labor and capital stock) and $q = 1$ output (real gross domestic product). In terms of the identifiers in the PWT data, our input variables correspond to *emp* (number of persons engaged, in millions) and *rnna* (capital stock at constant 2017 national prices, in millions of 2017 US dollars). Our output variable corresponds to *rgdpna* (real gross domestic product at constant 2017 national prices, in millions of 2017 US dollars). Because PWT version 10.0 data differ from the data available to Färe et al. (1994), and because we work in the hyperbolic direction whereas Färe et al. (1994) work in the output direction, our quantitative results differ from theirs. We make inferences within the context of the statistical model defined above in Section 3, and thus are able to quantify what is learned from the data.

Ordinarily, one should test the convexity of Ψ^t using the test of Kneip et al. (2016), but in the application here, convexity seems likely to hold, and in any case there are too few observations to make the test. Consequently, we begin by estimating Malmquist indices and components of the various decompositions discussed in Section 2 for each country in the 10 pairs of years 1979–1980, 1980–1981, …, 1987–1988 and 1979–1988. Geometric means of these estimates are reported in Table 1. Recall that values equal to (greater than, less than) 1 indicate no change (increases, decreases). Significant differences from 1 at levels .1, .05 and .01 are indicated by one, two or three asterisks, respectively. Significance is determined by using the central limit thoerem results of Kneip et al. (2021), i.e., by looking at whether confidence intervals estimated using (3.12) cover 1 (lack of significance), or whether lower bounds are greater than 1 or upper bounds are less than 1.

The results in Table 1 indicate that on average, there were no significant changes in mean productivity measured by $\widehat{\mathcal{M}}_n$ from 1979 to 1982, but mean productivity significantly increased in the following years. Mean productivity also significantly increased over the entire sample period from 1979 to 1988. Färe et al. (1994) report a value of 1.0070 for mean productivity change from 1979 to 1988 in their Table 4, whereas our estimate, 1.04719, is larger due to the data revisions discussed above. Moreover, our estimate is statistically significant at .01. Results for the components of productivity change are mixed. In terms of the decomposition in (2.10) used by Färe et al. (1994), for 1979 to 1988, there is no evidence of change in mean (pure) efficiency $\widehat{\mathcal{E}}_{2,n}$, and the estimate $\widehat{\mathcal{S}}_{1,n}$ for mean change in scale efficiency, though statistically significant at .01, is small in magnitude and not economically significant. The Färe et al. (1994) measure of mean technical change, $\widehat{\mathcal{T}}_{1,n}$ is significantly greater than 1 for 1979 to 1988, but as discussed in Section 2, this measure assumes CRS. The measure $\widehat{\mathcal{T}}_{2,n}$ allows for either VRS or CRS, but is insignificant for 1979–1988. From 1979 to 1988, the data provide little evidence on the sources of productivity improvement. This stands in contrast to the discussion of Färe et al. (1994, page 78) where mean productivity growth over 1979–1988 is attributed to "innovation" rather than "improvements in efficiency." Without inference, it is impossible to know what is learned from the data.

Tables 2, 3, 4, 5 show individual estimates of Malmquist indices and components in the decompositions in (2.9)–(2.11) and (2.13) for each of the 17 countries examined by Färe et al. (1994) for 1979–1980, 1983–1984 and 1987–1988. Significant differences from 1 are again indicated by asterisks, with significance determined by estimating confidence intervals for the individual quantities using the sub-sampling methods of Simar and Wilson (2011). [7] Estimates of mean productivity change are significantly different from 1 in all but one case (Germany, 1979–1980), although a number of estimates are numerically close to 1 and perhaps not economically significant. Most of the estimates of mean changes in the various components of productivity change are also significantly different from 1, even though some indicate small changes. Table 5 presents similar estimates for 1979–1988, and all of the estimates for mean productivity change are significant. Three countries (Canada, Greece and Japan) exhibit declines in productivity, but the others exhibit improvements.

Kumar and Russell (2002) and others in the literature on economic development focus on technological "catch-up," (i.e., movements toward or away from the frontier). The estimates in Table 5 provide some insight on this catching-up among the 17 OECD countries examined. The estimates of technical change in Table suggest that the frontier shifted upward between 1979 and 1988. At the same time, some countries—Australia, Canada, Denmark, Greece, Japan and Sweden—became more inefficient. Nonetheless, Australia, Denmark and Sweden increased their productivity, while productivity decreased in Canada, Greece and Japan. As the frontier shifted upward, the evidence suggests that Australia, Denmark and Sweden followed, but did not keep up with the advancing frontier, while Canada, Greece and Japan moved downward, away from the advancing frontier. The case of Greece is dramatic—from 1979 to 1988, its productivity declined by about 4.5 percent, and its efficiency declined by about 11 percent.

Recall the discussion of the advantages of working in the hyperbolic direction discussed earlier in Section 2. To illustrate, Table 6 presents estimates for 1979–1988 obtained using the output orientation, as opposed to the hyperbolic direction used to obtain the estimates in Table 1. Table 6 reveals that geometric means $\widehat{\mathcal{T}}_{2,n}$, $\widehat{\mathcal{S}}_{2,n}$ and $\widehat{\mathcal{S}}_{3,n}$ are unavailable because one or more of the individual estimates in the geometric means are undefined in the output direction. This is due to shifts in the estimated frontiers, resulting in one or more of the cross-period estimates not being defined. Färe et al. (1994) use the output orientation, but do not encounter this problem since their decomposition in (2.10) does not involve distances from observations in one period to the frontier of the production set (as opposed to its conical hull) in the other period. But the decompositions in (2.11) and (2.13) are affected when the output orientation is used. Table 7 presents individual estimates for 1979–1988 in the output direction, revealing that the problem is caused by Ireland, where the estimates $\widehat{\mathcal{T}}_2$, $\widehat{\mathcal{S}}_2$ and $\widehat{\mathcal{S}}_3$ are undefined. Comparison of the results in Table 6 with those in Table 1 reveals that the output-oriented estimates in Table 6 are in some cases numerically larger than the corresponding hyperbolic estimates in Table 1. This is not unusual due to reasons related to the shape of the frontiers in each period; see Wilson (2011) for discussion.

Returning to the hyperbolic direction and exploiting the updates in the PWT data since the Mark 5 version used by Färe et al. (1994), Tables 8–9 present results for 1951–2019 analogous to those for 1979–1988 in Tables 1 and Table 5. Comparison across the two sets of results indicate that the estimates for 1951–2019 are in many cases farther from 1 and more often are significantly different from 1 than are the estimates for 1979–1988. This is to be expected—one has a better chance of observing changes when looking over longer periods than over shorter periods.

6 Summary and conclusions

We are grateful to F&G for their pioneering work in the analysis of efficiency and productivity, which has motivated a good deal of our own work. We have in this paper revisited one of their influential papers and have used updated data and recently developed statistical tools to re-examine productivity changes among the 17 OECD countries examined by Färe et al. (1994). Today, much more is possible than when the authors of Färe et al. (1994) wrote their paper in the early 1990s. By now, we know the statistical properties of DEA estimators, including rates of convergence, limiting distributions, properties of their

[7] The theory developed by Kneip et al. (2021) explicitly allows for dependence of a given country's input-output quantities across time, but require independence of the country's input-output quantities from those of other countries in other periods. Cross-sectional, spatial dependence and how to deal with it in the present context remain unresolved issues left for future research. In Table 2, dependence across time is preserved in the bootstrap world by drawing pairs of input-output observations for specific countries.

Table 2 Productivity Change and Its Components for OECD Countries, 1979–1980, Hyperbolic Direction

Country	$\widehat{\mathcal{M}}$	$\widehat{\mathcal{E}}_1$	$\widehat{\mathcal{E}}_2$	$\widehat{\mathcal{T}}_1$	$\widehat{\mathcal{T}}_2$	$\widehat{\mathcal{S}}_1$	$\widehat{\mathcal{S}}_2$	$\widehat{\mathcal{S}}_3$
Australia	1.00146***	0.99852***	1.00015	1.00294***	1.00292***	0.99837***	0.99839***	1.00002
Austria	1.00114***	0.99221***	0.99632***	1.00900***	1.00751***	0.99587***	0.99734***	1.00147***
Beglium	1.02257***	1.01204***	1.01262***	1.01041***	1.01012***	0.99943**	0.99971***	1.00028
Canada	0.99125***	1.00000	1.00000	0.99125***	0.99154***	1.00000	0.99971***	0.99971**
Denmark	0.99588***	0.99239***	0.98908***	1.00351***	1.00132***	1.00335***	1.00554***	1.00219**
Finland	1.00827***	1.01817***	1.01876***	0.99028***	0.98572***	0.99942***	1.00405***	1.00463***
France	1.00584***	0.99548***	0.99955***	1.01041***	1.00533***	0.99593***	1.00096***	1.00506***
Germany	0.99969	1.00295***	1.00429***	0.99675***	0.99812***	0.99867***	0.99730***	0.99862*
Greece	0.99553***	0.98528***	0.98542***	1.01041***	1.00963***	0.99986	1.00063**	1.00077**
Ireland	1.00944***	1.01427***	1.00000	0.99523***	0.98111***	1.01427***	1.02888***	1.01440***
Italy	1.00894***	0.99855***	1.00000	1.01041***	1.00846***	0.99855**	1.00048***	1.00194***
Japan	0.98646***	0.99614***	1.00078	0.99028***	0.98585***	0.99537***	0.99984	1.00449***
Norway	1.00944***	1.00000	1.00000	1.00944***	1.00789***	1.00000	1.00154	1.00154***
Spain	1.01379***	1.00983***	1.01889***	1.00392***	1.00171***	0.99111***	0.99330***	1.00221***
Sweden	1.00289***	0.99256***	0.99303***	1.01041***	1.00995***	0.99952***	0.99998	1.00046**
United Kingdom	0.98931***	0.99272***	0.99357***	0.99657***	0.99840***	0.99914**	0.99731***	0.99817***
United States	0.99228***	1.00000	1.00000	0.99228***	0.99105***	1.00000	1.00124***	1.00124***

NOTE: The column labeled $\widehat{\mathcal{M}}$ gives estimates of the Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_1$, $\widehat{\mathcal{E}}_2$, $\widehat{\mathcal{T}}_1$, $\widehat{\mathcal{T}}_2$, $\widehat{\mathcal{S}}_1$, $\widehat{\mathcal{S}}_2$ and $\widehat{\mathcal{S}}_3$ give estimates of the components $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

Table 3 Productivity Change and Its Components for OECD Countries, 1983–1984, Hyperbolic Direction

Country	$\widehat{\mathcal{M}}$	$\widehat{\mathcal{E}}_1$	$\widehat{\mathcal{E}}_2$	$\widehat{\mathcal{T}}_1$	$\widehat{\mathcal{T}}_2$	$\widehat{\mathcal{S}}_1$	$\widehat{\mathcal{S}}_2$	$\widehat{\mathcal{S}}_3$
Australia	1.00933***	0.99015***	0.99072***	1.01936***	1.01885***	0.99943**	0.99994	1.00051*
Austria	1.00051***	0.97522***	0.97798***	1.02593***	1.02255***	0.99719***	1.00048**	1.00330***
Beglium	1.01195***	0.98638***	0.99039***	1.02593***	1.02213***	0.99595***	0.99965***	1.00371***
Canada	1.01500***	0.99540***	0.99809***	1.01969***	1.01627***	0.99730***	1.00066***	1.00337***
Denmark	1.01240***	0.99346***	0.99584***	1.01907***	1.01376***	0.99762***	1.00283***	1.00523***
Finland	0.99924***	0.97994***	0.98877***	1.01969***	1.00733***	0.99107***	1.00323***	1.01227***
France	1.01016***	0.98464***	0.99437***	1.02593***	1.01535***	0.99021***	1.00052**	1.01041***
Germany	1.00886***	0.99126***	0.99591***	1.01775***	1.01539***	0.99533***	0.99765***	1.00233***
Greece	1.01071***	0.98517***	0.98823***	1.02593***	1.02261***	0.99690***	1.00014	1.00324***
Ireland	1.02243***	1.00336***	1.00000	1.01900***	1.00019***	1.00336***	1.02223***	1.01881***
Italy	1.01575***	0.99008***	1.00000	1.02593***	1.01560***	0.99008***	1.00015***	1.01017***
Japan	1.00183***	0.98248***	0.98295***	1.01969***	1.01903***	0.99952	1.00017***	1.00065***
Norway	1.02335***	1.00000	1.00000	1.02335***	1.02032***	1.00000	1.00297**	1.00297***
Spain	1.01223***	0.99130***	1.00284***	1.02111***	1.01667***	0.98850***	0.99282***	1.00437***
Sweden	1.01671***	0.99101***	0.99593***	1.02593***	1.02136***	0.99506***	0.99951***	1.00447***
United Kingdom	1.00071***	0.98324***	0.98526***	1.01777***	1.01590***	0.99795***	0.99978*	1.00184***
United States	1.01810***	1.00000	1.00000	1.01810***	1.04327***	1.00000	0.97587***	0.97587***

NOTE: The column labeled $\widehat{\mathcal{M}}$ gives estimates of the Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_1$, $\widehat{\mathcal{E}}_2$, $\widehat{\mathcal{T}}_1$, $\widehat{\mathcal{T}}_2$, $\widehat{\mathcal{S}}_1$, $\widehat{\mathcal{S}}_2$ and $\widehat{\mathcal{S}}_3$ give estimates of the components $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

moments, and we have available now CLT results for making inference about mean efficiency, mean changes in productivity, etc. Using these new tools, it is now possible to quantify what is learned from data when analyzing efficiency and productivity, rather than merely reporting point estimates as was done exclusively until recently.

Table 4 Productivity Change and Its Components for OECD Countries, 1987–1988, Hyperbolic Direction

Country	$\widehat{\mathcal{M}}$	$\widehat{\mathcal{E}}_1$	$\widehat{\mathcal{E}}_2$	$\widehat{\mathcal{T}}_1$	$\widehat{\mathcal{T}}_2$	$\widehat{\mathcal{S}}_1$	$\widehat{\mathcal{S}}_2$	$\widehat{\mathcal{S}}_3$
Australia	1.00115***	0.99903**	0.99401***	1.00212***	1.00708***	1.00504***	1.00010*	0.99508***
Austria	1.01207***	1.01010***	1.00647***	1.00195***	1.00541***	1.00360***	1.00015	0.99656***
Beglium	1.01346***	1.01148***	1.00631***	1.00195***	1.00764***	1.00514***	0.99946***	0.99435***
Canada	1.00178***	0.99557***	0.99453***	1.00625***	1.00664***	1.00105	1.00066***	0.99961
Denmark	0.99677***	0.99675***	0.99164***	1.00002	0.99990	1.00515***	1.00527***	1.00012
Finland	1.00649***	1.00024	0.99326***	1.00625***	1.00910***	1.00703***	1.00418***	0.99718***
France	1.01797***	1.01599***	1.00471***	1.00195***	1.01262***	1.01122***	1.00057***	0.98947***
Germany	1.01299***	1.00865***	1.00499***	1.00431***	1.00975***	1.00363***	0.99822*	0.99461***
Greece	1.01101***	1.00904***	1.00718***	1.00195***	1.00475***	1.00185***	0.99906***	0.99722***
Ireland	1.02135***	1.02258***	1.00000	0.99880***	1.00028***	1.02258***	1.02107***	0.99852***
Italy	1.01527***	1.01329***	1.00000	1.00195***	1.01516***	1.01329***	1.00011***	0.98699***
Japan	1.00813***	1.00187***	1.00163	1.00625***	1.00626***	1.00024	1.00023***	0.99998
Norway	0.99932***	1.00000	1.00000	0.99932***	0.99732***	1.00000	1.00200***	1.00200**
Spain	1.00849***	1.00997***	0.99885***	0.99853***	1.01013***	1.01113***	0.99953***	0.98852***
Sweden	1.00550***	1.00354***	0.99910***	1.00195***	1.00647***	1.00444***	0.99993*	0.99551***
United Kingdom	1.01140***	1.00706***	1.00267***	1.00431***	1.00932***	1.00438***	0.99940*	0.99504***
United States	1.00733***	1.00000	1.00000	1.00733***	1.02382***	1.00000	0.98389***	0.98389***

NOTE: The column labeled $\widehat{\mathcal{M}}$ gives estimates of the Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_1$, $\widehat{\mathcal{E}}_2$, $\widehat{\mathcal{T}}_1$, $\widehat{\mathcal{T}}_2$, $\widehat{\mathcal{S}}_1$, $\widehat{\mathcal{S}}_2$ and $\widehat{\mathcal{S}}_3$ give estimates of the components $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

Table 5 Productivity Change and Its Components for OECD Countries, 1979–1988, Hyperbolic Direction

Country	$\widehat{\mathcal{M}}$	$\widehat{\mathcal{E}}_1$	$\widehat{\mathcal{E}}_2$	$\widehat{\mathcal{T}}_1$	$\widehat{\mathcal{T}}_2$	$\widehat{\mathcal{S}}_1$	$\widehat{\mathcal{S}}_2$	$\widehat{\mathcal{S}}_3$
Australia	1.04103***	0.99015***	0.98764***	1.05139***	1.05837***	1.00253***	0.99592***	0.99340***
Austria	1.06861***	1.00247***	0.99964	1.06597***	1.06663***	1.00283***	1.00221*	0.99938
Beglium	1.09469***	1.01391***	1.01066***	1.07967***	1.08431***	1.00322***	0.99893***	0.99572***
Canada	0.99061***	0.97504***	0.98852***	1.01597***	1.00081	0.98637***	1.00130**	1.01514***
Denmark	1.04967***	0.99392***	0.99355***	1.05609***	1.03101***	1.00037	1.02470***	1.02432***
Finland	1.01106***	0.99661***	1.00978***	1.01450***	0.96612***	0.98696***	1.03638***	1.05007***
France	1.08024***	1.00490***	1.00695***	1.07498***	1.07191***	0.99796***	1.00081***	1.00286***
Germany	1.03273***	0.99242***	0.99839	1.04063***	1.04613***	0.99402***	0.98879***	0.99474
Greece	0.95575***	0.89095***	0.89024***	1.07274***	1.07843***	1.00079	0.99550***	0.99472***
Ireland	1.09412***	1.05162***	1.00000	1.04042***	0.91740***	1.05162***	1.19264***	1.13410***
Italy	1.08414***	1.00414***	1.00000	1.07967***	1.07827***	1.00414***	1.00544***	1.00129***
Japan	0.98673***	0.98553***	0.98005***	1.00121***	1.00368	1.00559***	1.00312***	0.99754
Norway	1.06616***	1.00000	1.00000	1.06616***	1.05100***	1.00000	1.01443**	1.01443**
Spain	1.08751***	1.02285***	1.03911***	1.06321***	1.06634***	0.98435***	0.98146***	0.99706**
Sweden	1.06860***	0.99248***	0.99293***	1.07669***	1.07831***	0.99955	0.99805***	0.99850**
United Kingdom	1.07589***	1.03465***	1.04038***	1.03985***	1.04301***	0.99449***	0.99149***	0.99698*
United States	1.02824***	1.00000	1.00000	1.02824***	1.14828***	1.00000	0.89546***	0.89546***

NOTE: The column labeled $\widehat{\mathcal{M}}$ gives estimates of the Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_1$, $\widehat{\mathcal{E}}_2$, $\widehat{\mathcal{T}}_1$, $\widehat{\mathcal{T}}_2$, $\widehat{\mathcal{S}}_1$, $\widehat{\mathcal{S}}_2$ and $\widehat{\mathcal{S}}_3$ give estimates of the components $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1, \Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

The statistical results that have been developed in recent years involve some complication. This is because the underlying problem is complicated; boundaries are often problematic in econometrics and mathematical statistics. Nonetheless, this paper as well as Simar and Wilson (2023a) provide templates for how empirical researchers

Table 6 Mean Productivity Change and Its Components for OECD Countries, 1979–1988, Output Direction

Period	$\widehat{\mathcal{M}}_n$	$\widehat{\mathcal{E}}_{1,n}$	$\widehat{\mathcal{E}}_{2,n}$	$\widehat{\mathcal{T}}_{1,n}$	$\widehat{\mathcal{T}}_{2,n}$	$\widehat{\mathcal{S}}_{1,n}$	$\widehat{\mathcal{S}}_{2,n}$	$\widehat{\mathcal{S}}_{3,n}$
1979–1980	1.00394	1.00006	1.00186	1.00388***	—	0.99820***	—	—
1980–1981	0.99877	0.99800	0.99728	1.00077	—	1.00072	—	—
1981–1982	1.00135	1.01782***	1.01627***	0.98382***	—	1.00153***	—	—
1982–1983	1.01370***	0.98223***	0.99829	1.03204***	—	0.98390***	—	—
1983–1984	1.02234***	0.97924***	0.98713***	1.04401***	—	0.99201***	—	—
1984–1985	1.01487***	0.99388***	0.99615***	1.02112***	—	0.99772***	—	—
1985–1986	1.00676**	1.00005	0.99793**	1.00671***	—	1.00212***	—	—
1986–1987	1.01032***	1.01027***	0.99761	1.00005***	—	1.01269***	—	—
1987–1988	1.01774***	1.01236***	1.00009	1.00531	—	1.01226***	—	—
1979–1988	1.09661***	0.99325	0.99241	1.10406***	—	1.00084**	—	—

NOTE: The column labeled $\widehat{\mathcal{M}}_n$ gives geometric means for estimates of a Malmquist index defined in terms of output distance functions, analogous to the hyperbolic Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_{1,n}$, $\widehat{\mathcal{E}}_{2,n}$, $\widehat{\mathcal{T}}_{1,n}$, $\widehat{\mathcal{T}}_{2,n}$, $\widehat{\mathcal{S}}_{1,n}$, $\widehat{\mathcal{S}}_{2,n}$ and $\widehat{\mathcal{S}}_{3,n}$ give geometric means for estimates of output-oriented analogs of the components $\mathcal{E}_1(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

Table 7 Productivity Change and Its Components for OECD Countries, 1979–1988, Output Direction

Country	$\widehat{\mathcal{M}}$	$\widehat{\mathcal{E}}_1$	$\widehat{\mathcal{E}}_2$	$\widehat{\mathcal{T}}_1$	$\widehat{\mathcal{T}}_2$	$\widehat{\mathcal{S}}_1$	$\widehat{\mathcal{S}}_2$	$\widehat{\mathcal{S}}_3$
Australia	1.08373***	0.98039***	0.97592***	1.10541***	1.12207***	1.00458***	0.98966***	0.98515***
Austria	1.14192***	1.00494***	1.00050	1.13630***	1.14230***	1.00445**	0.99917	0.99475***
Beglium	1.19834***	1.02801***	1.02104***	1.16569***	1.17321***	1.00682***	1.00036	0.99358***
Canada	0.98131***	0.95070***	0.97682***	1.03219***	1.00133	0.97326***	1.00325**	1.03081***
Denmark	1.10180***	0.98787***	0.98873***	1.11533***	1.07195***	0.99913	1.03956***	1.04047***
Finland	1.02223***	0.99323***	1.02154***	1.02920***	0.92989***	0.97229***	1.07613***	1.10680***
France	1.16691***	1.00982***	1.01402***	1.15557***	1.14804***	0.99586***	1.00239**	1.00656***
Germany	1.06654***	0.98489***	0.99693	1.08290***	1.09521***	0.98793***	0.97683***	0.98876*
Greece	0.91346***	0.79378***	0.79774***	1.15077***	1.15597***	0.99504***	0.99057***	0.99550***
Ireland	1.19711***	1.10591***	1.00000	1.08247***	—	1.10591***	—	—
Italy	1.17537***	1.00831***	1.00000	1.16569***	1.16100***	1.00831***	1.01238***	1.00404***
Japan	0.97363***	0.97127***	0.95920***	1.00243***	1.00879	1.01258***	1.00620***	0.99370
Norway	1.13670***	1.00000	1.00000	1.13670***	1.10404***	1.00000	1.02959***	1.02959***
Spain	1.18267***	1.04622***	1.07960***	1.13042***	1.13587***	0.96908***	0.96443***	0.99520**
Sweden	1.14191***	0.98503***	0.98813***	1.15927***	1.15679***	0.99686***	0.99899*	1.00214
United Kingdom	1.15754***	1.07051***	1.08244***	1.08129***	1.08975***	0.98898***	0.98130***	0.99224**
United States	1.05728***	1.00000	1.00000	1.05728***	1.15275***	1.00000	0.91718***	0.91718***

NOTE: The column labeled $\widehat{\mathcal{M}}$ gives estimates of a Malmquist index defined in terms of output distance functions, analogous to the hyperbolic Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_1$, $\widehat{\mathcal{E}}_2$, $\widehat{\mathcal{T}}_1$, $\widehat{\mathcal{T}}_2$, $\widehat{\mathcal{S}}_1$, $\widehat{\mathcal{S}}_2$ and $\widehat{\mathcal{S}}_3$ give estimates of output-oriented analogs of the components $\mathcal{E}_1(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2 | \Psi^1, \Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

might proceed. Kneip et al. (2021, Section 5) provide an algorithmic description about how one can make inference about changes in productivity using Malmquist indices, and all of the computation in this paper is done using code that is made available in the *FEAR* software package introduced by Wilson (2008) for the *R* programming language.[8]

While a number of papers (e.g., Jeon and Sickles, 2006 and Badunenko et al., 2014) have relied on methods proposed by Simar and Wilson (1998b, 1999) to make inference about productivity change measured by Malmquist indices, these methods are now outdated. The methods proposed by Simar and Wilson (1998b, 1999) were preliminary, first attempts in a nascent literature on statistical methods for use with nonparametric efficiency estimators. Today, much more is known, and researchers

[8] The *FEAR* package is available at https://pww.people.clemson.edu/Software/FEAR/fear.html for free use for academic research.

Table 8 Mean Productivity Change and Its Components for OECD Countries, 1951–2019, Hyperbolic Direction

Period	$\widehat{\mathcal{M}}_n$	$\widehat{\mathcal{E}}_{1,n}$	$\widehat{\mathcal{E}}_{2,n}$	$\widehat{\mathcal{T}}_{1,n}$	$\widehat{\mathcal{T}}_{2,n}$	$\widehat{\mathcal{S}}_{1,n}$	$\widehat{\mathcal{S}}_{2,n}$	$\widehat{\mathcal{S}}_{3,n}$
1951–1959	1.04992**	1.01267	1.01318	1.03678***	1.03722	0.99950	0.99908***	0.99958***
1959–1969	1.08889***	1.01371	1.00529	1.07416***	1.07184	1.00838***	1.01056***	1.00216***
1969–1979	1.05717***	1.01808**	1.01436**	1.03839**	1.02512	1.00366	1.01666***	1.01295***
1979–1989	1.05445***	0.99221***	0.99189*	1.06273***	1.05638	1.00033**	1.00634***	1.00601***
1989–1999	1.06944***	0.96783***	0.97805***	1.10499***	1.08840*	0.98955*	1.00462**	1.01524*
1999–2009	1.01673***	0.99253***	0.99412*	1.02439*	1.01073	0.99840	1.01189***	1.01351***
2009–2019	1.02816**	0.93866***	0.96019***	1.09535***	1.05705	0.97757***	1.01300***	1.03623***
1951–2019	1.34567***	0.93501***	0.95673***	1.43921***	0.92528***	0.97729*	1.52011***	1.55543***

NOTE: The column labeled $\widehat{\mathcal{M}}_n$ gives geometric means for estimates of the Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_{1,n}$, $\widehat{\mathcal{E}}_{2,n}$, $\widehat{\mathcal{T}}_{1,n}$, $\widehat{\mathcal{T}}_{2,n}$, $\widehat{\mathcal{S}}_{1,n}$, $\widehat{\mathcal{S}}_{2,n}$ and $\widehat{\mathcal{S}}_{3,n}$ give geometric means for estimates of the components $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

Table 9 Productivity Change and Its Components for OECD Countries, 1951–2019, Hyperbolic Direction

Country	$\widehat{\mathcal{M}}$	$\widehat{\mathcal{E}}_1$	$\widehat{\mathcal{E}}_2$	$\widehat{\mathcal{T}}_1$	$\widehat{\mathcal{T}}_2$	$\widehat{\mathcal{S}}_1$	$\widehat{\mathcal{S}}_2$	$\widehat{\mathcal{S}}_3$
Australia	1.34693***	0.91550***	0.88653***	1.47124***	0.98159**	1.03268***	1.54782***	1.49884***
Austria	1.43521***	0.97900***	0.97723***	1.46600***	0.67633***	1.00181	2.17148***	2.16756***
Beglium	1.57273***	0.96698***	0.93230***	1.62643***	1.00663	1.03720***	1.67582***	1.61572***
Canada	1.17888***	0.85290***	0.86209***	1.38219***	1.04148	0.98935***	1.31300***	1.32714***
Denmark	1.32803***	0.93005***	0.98573***	1.42791***	0.57659***	0.94352***	2.33660***	2.47647***
Finland	1.11063***	0.81705***	1.00000	1.35931***	0.36595***	0.81705***	3.03490***	3.71445***
France	1.52920***	1.03488***	1.15083***	1.47766***	1.35220***	0.89925***	0.98268	1.09278***
Germany	1.19594***	0.87938***	0.86873***	1.35997***	1.35731***	1.01226	1.01425***	1.00196*
Greece	1.65737***	1.12846***	1.11407***	1.46870***	0.65713***	1.01292**	2.26389***	2.23502***
Ireland	1.51677***	1.00000	1.00000	1.51677***	0.41153***	1.00000	3.68568***	3.68568***
Italy	1.29610***	0.86298***	1.02038***	1.50188***	1.31641***	0.84575***	0.96491***	1.14090***
Japan	1.05149***	0.81781***	0.81313***	1.28574***	1.37036***	1.00577***	0.94366***	0.93825***
Norway	1.52105***	1.02726***	0.96088***	1.48068***	0.59891***	1.06909***	2.64309***	2.47229***
Spain	1.27354***	0.89702***	0.94754***	1.41976***	1.00090	0.94668***	1.34284***	1.41848***
Sweden	1.53164***	1.03713***	0.99348**	1.47680***	0.87511***	1.04394***	1.76171***	1.68756***
United Kingdom	1.12899***	0.81999***	0.82172***	1.37684***	1.33477***	0.99789	1.02935***	1.03152***
United States	1.40236***	1.00000	1.00000	1.40236***	2.90103***	1.00000	0.48340***	0.48340***

NOTE: The column labeled $\widehat{\mathcal{M}}$ gives estimates of the Malmquist index defined in (2.8). Columns labeled $\widehat{\mathcal{E}}_1$, $\widehat{\mathcal{E}}_2$, $\widehat{\mathcal{T}}_1$, $\widehat{\mathcal{T}}_2$, $\widehat{\mathcal{S}}_1$, $\widehat{\mathcal{S}}_2$ and $\widehat{\mathcal{S}}_3$ give estimates of the components $\mathcal{E}_1(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{E}_2(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{T}_1(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{T}_2(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{S}_1(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$, $\mathcal{S}_2(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$ and $\mathcal{S}_3(Z_i^1, Z_i^2|\Psi^1,\Psi^2)$ (respectively) defined in (2.9)–(2.13). Significant differences from 1 at levels .1, .05 or .01 are indicated by one, two or three asterisks, respectively.

should use the methods developed by Kneip et al. (2008, 2011, 2015, 2016, 2021, 2022), Simar and Wilson (2011), Daraio et al. (2018) and others. These more recent methods are accompanied by rigorous theoretical justifications.

Following Färe et al. (1994), we have assumed here that production sets are strictly convex, implying VRS. As noted above, this should be tested when sufficient data are available, and Kneip et al. (2016) provide a test of CRS versus VRS under appropriate assumptions, including convexity of the production set. Kneip et al. (2016) provide a test of convexity of the production set, although here in each year there are too few observations to obtain meaningful results from the test. Several recent papers (e.g., Apon et al., 2015; Daraio et al., 2021; O'Loughlin and Wilson, 2021; Wilson and Zhao, 2022; Simar and Wilson, 2023a) have tested and rejected convexity of production sets in various applications, and the assumption should be tested when sample sizes of 80–100 or more are available. Both of these tests are implemented in *FEAR*. In cases where convexity is rejected, recent results provided by Kneip et al. (2022) enable testing of constant versus non-constant returns to scale as well as estimation and inference about productivity changes using Malmquist indices.

The early work by F&G and their contemporaries has spawned a large and diverse literature. The work by Färe et al. (1994) that has been revisited here in particular inspired (either in whole or in part) many subsequent papers by others as indicated by the citations counts in Fig. 2. Among these are papers in the economic development literature by Kumar and Russell (2002) and others; the banking literature by Wheelock and Wilson (1999) and others; growth and grown accounting (e.g., Jeon and Sickles, 2006); local public finance and regional science (e.g., Domazlicky and Weber, 1997 and Boissoa et al., 2000); agricultural economics (e.g., Le et al., 2019); heathcare (e.g., Şahin and İlgün, 2019) and many others. We should all be grateful to F&G for giving us much to consider.

Acknowledgements A preliminary version of the work in this paper was presented at the 17th European Workshop on Efficiency and Productivity (EWEPA XVII) held in Porto, Portugal 27–29 June, 2022.

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