

GEOMETRICAL CHARACTERISATION FOR STRUCTURAL ROBUSTNESS

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“Les structures sont fainéantes, mais bienveillantes”

[Jean-François Rondeaux]

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ABSTRACT

Assessing a structure's intrinsic level of robustness is a widespread problem in the field of structural engineering and has taken on even greater importance since the attacks of September 11, 2001. Although numerous robustness approaches can be found in the literature, there is still today no consensus on which threshold makes a given structure robust or not, nor on the way to calculate this threshold. Moreover, all the existing methods are based on a numerical assessment of the structure's aptitude for robustness once its design has been fixed.

This thesis is about the development of a new type of approach to structural robustness, working on geometrical considerations only. It allows the integration of some constitutive dimensions of structural robustness – such as the aptitude for force redistributions – at the earliest steps of the design process. Based on graphic statics and strut-and-tie modelling, the method develops a visual robustness indicator showing structural aptitude for force redistributions. When implemented into parametrical software such as Rhino and its graphic algorithm editor Grasshopper, the approach also turns into an interactive tool. This interactivity may help project authors to understand the impact of design decisions on the robustness performance sought for (in terms of aptitude for force redistributions). Combined with Maxwell's theorem of *load paths*, the method also allows the comparison of the efficiency of various design options in terms of the volume of material required for their implementation.

The principles of the proposed geometrical method are developed in this thesis for two-dimensional structures. These principles are then illustrated by the analysis of four case studies. Two concern arched structures, the first in pure compression, the second in compression and bending. Two sets of walls are then studied, the first made of an ideal material with identical compressive and tensile strengths, the second intended to prepare the analysis of reinforced concrete structures.

Keywords: structural robustness, structural design, graphic statics, strut-and-tie modelling, force redistributions, Lower Bound Theorem of Plastic Theory, structural optimisation, static indeterminacy.

RÉSUMÉ

Déduire le niveau de robustesse d'une structure donnée est un problème bien connu dans le champ de l'ingénierie structurale, qui a pris une importance d'autant plus grande depuis les attaques terroristes du 11 septembre 2001. Bien qu'il existe dans la littérature de nombreuses approches visant à caractériser la robustesse structurale, il n'y a à ce jour pas de consensus sur une valeur limite au-delà de laquelle une structure donnée peut être jugée comme robuste, ni sur la manière de calculer cette limite. De plus, toutes les méthodes existantes visant à qualifier la robustesse d'une structure sont basées sur une analyse numérique de cette structure une fois son design fixé.

Cette recherche développe une nouvelle approche de robustesse structurale, uniquement basée sur des considérations géométriques. Cette approche permet d'intégrer certaines dimensions constitutives de la robustesse – comme la capacité à redistribuer les efforts – dès les premières esquisses du projet d'architecture. Basée sur la statique graphique et la modélisation bielles-et-tirants, la méthode développe un indicateur visuel de robustesse montrant les capacités structurales à redistribuer les efforts. Implémentée dans un logiciel de conception paramétrique tel que Rhino et son éditeur graphique d'algorithmes Grasshopper, l'approche peut également servir comme outil interactif de conception. Cette interactivité permet ainsi aux auteurs de projet de comprendre l'impact de leurs choix de design sur la performance de robustesse recherchée (en termes de capacité à redistribuer les efforts). Combinée avec le théorème des *load paths* de Maxwell, la méthode permet aussi de comparer l'efficacité de diverses options de design en termes de volume de matière nécessaire à leur implémentation.

Le fonctionnement de la méthode géométrique proposée est développé dans cette thèse pour des structures en deux dimensions, à travers l'analyse de quatre études de cas. Les deux premières concernent l'étude de structures en arc, l'une en compression pure, l'autre en flexion composée. Deux séries de murs sont ensuite analysés, les premiers étant faits d'un matériau idéalisé ayant la même résistance en traction et en compression, les seconds préparant l'analyse des structures en béton armé.

Mots-clés : robustesse structurale, conception structurale, statique graphique, modélisation bielles-et-tirants, redistribution des efforts, Théorème Statique de la Plasticité, optimisation structurale, hyperstaticité.

TABLE OF CONTENTS

Introduction.....	21
General background	21
Objectives of the research	22
Organisation of the content	23
Part 1 – The issue of structural robustness.....	25
1.1. Definitions of structural robustness	27
1.2. Parameters of structural robustness.....	29
1.2.1. Parameters linked with design considerations.....	30
1.2.2. Parameters linked with design for damage	31
1.2.3. Parameters linked with material properties	33
1.2.4. Parameters linked with secondary structural elements....	33
1.2.5. Parameters linked with human intervention.....	34
1.3. Approaches of structural robustness	35
1.3.1. Probabilistic approaches.....	36
1.3.2. Risk-based approaches	37
1.3.3. Deterministic approaches	37
1.3.4. Energetic approaches.....	38
1.4. Limitations of existing approaches.....	39
1.5. Outcome	40
Part 2 – Exploring geometry and forces with graphic statics..	41
2.1. Modelling structural behaviour	43
2.1.1. Continuous stress fields.....	43
2.1.2. Strut-and-tie networks	43
2.1.3. Discontinuous stress fields.....	44
2.1.4. Conditions of application	45
2.2. Graphic statics and reciprocal diagrams	46

2.3.	Constraint-based graphic statics	49
2.4.	Contemporary structural design tools.....	52
2.4.1.	Geogebra	52
2.4.2.	StructureFit.....	53
2.4.3.	Combinatorial Equilibrium Modelling.....	54
2.5.	Outcome	55
Part 3 – Optimisation approaches for structural design.....		57
3.1.	Theory and concepts	59
3.1.1.	The optimisation problem	59
3.1.2.	The structural model.....	60
3.1.3.	The solving method.....	61
3.2.	Brief review of optimisation methods.....	63
3.2.1.	Graphical optimisation method based on a bar model.....	63
3.2.2.	Numerical optimisation methods based on a bar model...	66
3.2.3.	Numerical optimisation methods based on a cell model...	72
3.3.	Outcome	75
Part 4 – A geometrical approach for robust and efficient structural design		77
4.1.	Preliminary hypotheses	79
4.2.	Building admissible geometrical domains.....	81
4.3.	Admissible geometrical domains and sizing variation	83
4.4.	Admissible geometrical domains and shape variation	84
4.5.	Admissible geometrical domains and topology variation	87
4.5.1.	The issue of sub-domains interdependency	89
4.5.2.	The issue of unbounded number of solutions	91
4.5.3.	The issue of static indeterminacy	92
4.6.	Developed computer model	94
4.7.	Structural applications	96
4.8.	Summary and conclusion.....	98

Part 5 – Shape variation for fixed topologies: the case of arches in compression	101
5.1. Preliminary hypotheses	103
5.1.1. Material behaviour	103
5.1.2. Structural response.....	103
5.1.3. The thrust line theory	103
5.2. Robustness and efficiency assessment	104
5.2.1. Considered variation problem	104
5.2.2. Robustness level analysis	106
5.2.3. Efficiency analysis	108
5.2.4. Measuring structural safety.....	109
5.3. Influence of design parameters on stability and robustness	112
5.3.1. Increase of applied external load.....	113
5.3.2. Decrease of masonry compressive strength	114
5.3.3. Modification of voussoir stereotomy.....	116
5.3.4. Modification of arch geometry	118
5.3.5. Displacement of supports	119
5.4. Analysis of real masonry arches.....	120
5.4.1. Experimental testing	120
5.4.2. Geometrical assessment.....	122
5.4.3. Comparison of results.....	122
5.5. Reinterpretation of ancient safety assessment methods.....	123
5.5.1. Analysis of main historical contributions	124
5.5.2. Comparison of results.....	130
5.6. Summary and conclusion	131
Part 6 – Shape variation for fixed topologies: the case of arches in bending.....	135
6.1. Preliminary hypotheses	137
6.1.1. Description of the structure	137
6.1.2. Description of the load cases.....	139

6.2.	Structural analyses	140
6.2.1.	Geometrical approach.....	140
6.2.2.	Deterministic approach	143
6.2.3.	Energetic approach.....	144
6.3.	Comparison of results.....	145
6.4.	Summary and conclusion.....	147
Part 7 – Shape variation of predefined topologies: the case of ideal walls		149
7.1.	Wall with one opening	151
7.1.1.	Preliminary hypotheses	151
7.1.2.	Robustness level analysis.....	151
7.1.3.	Efficiency analysis	154
7.1.4.	Stiffness analysis.....	154
7.1.5.	Comparison of results.....	155
7.2.	Wall with two openings.....	156
7.2.1.	Preliminary hypotheses	156
7.2.2.	Robustness level analysis.....	156
7.2.3.	Efficiency analysis	159
7.2.4.	Stiffness analysis.....	159
7.2.5.	Comparison of results.....	160
7.2.6.	Robustness level analysis for design alternatives	161
7.3.	Summary and conclusion.....	162
Part 8 – Topology variation of predefined shapes: the case of reinforced concrete structures		165
8.1.	Preliminary hypotheses	167
8.1.1.	Material behaviour	167
8.1.2.	Structural modelling.....	168
8.2.	Pure geometrical approach.....	170
8.2.1.	Description of the structure	170
8.2.2.	Construction of the parametrical force diagram.....	171

8.2.3.	Generation of the admissible geometrical domain.....	172
8.2.4.	Efficiency analysis	174
8.2.5.	Drawbacks.....	177
8.3.	Combined analytical/geometrical approach.....	177
8.3.1.	Description of the structure	177
8.3.2.	Gauss-Jordan elimination.....	178
8.3.3.	Building the convex hull of admissible scalings	180
8.3.4.	Efficiency analysis	183
8.3.5.	Generation of the admissible geometrical domain.....	183
8.4.	Application to an idealised shear wall	185
8.5.	Summary and conclusion	189
General conclusion.....		193
Results		193
Remaining issues and limitations.....		195
Perspectives.....		196
Publications		199
References.....		201

INTRODUCTION

General background

Structural robustness is defined in the Eurocodes as a structure's aptitude to survive unforeseen or unexpected events, without being damaged to an extent that is disproportionate to the original cause [EN1991]. This property is fundamental in ensuring human safety. In Europe, robustness became a compulsory legal requirement after the terrorist attacks of September 11, 2001 [GULVANESESIAN2006]. Many types of robustness assessment approaches are found nowadays in the literature [SORENSEN2012, CAVACO2013]. These approaches are all based on a numerical analysis of the structure's aptitude for strength and force redistributions; these aptitudes are calculated for given failure scenarios and for a design that has been determined beforehand. This leads to two main drawbacks. Firstly, predefining potential failure scenarios does not allow to cover the entire range of events that could happen during the structure's lifetime. Secondly, applying numerical methods to previously designed structures makes these methods difficult to appropriate by project authors: the methods do not allow the interaction with the design while seeking for optimal robustness performance.

Graphic statics is a powerful tool for structural design and analysis [ZALEWSKI1975, 1998]. Indeed, the reciprocity between graphic statics' form and force diagrams allows for a constant and simultaneous control of both the geometry and the load-bearing behaviour of the structure. In the context of Plastic Theory, for a material with sufficient ductile behaviour, graphic statics can be employed for defining strut-and-tie networks that model the structure's internal behaviour [MUTTONI1997]. Such approaches based on the Lower Bound Theorem of Plastic Theory take advantage of the fact that any statically admissible force distribution (i.e. respecting equilibrium, boundary and yield conditions) leads to a lower bound estimation of the structure's load bearing capacity [HEYMAN2008]. Going back to graphic statics, this means that any strut-and-tie network respecting equilibrium, boundary and yield conditions will represent a valid and safe structural model for internal forces distribution. The 21st century reinterpretation of graphic statics within interactive parametrical software has allowed its extension. In particular, graphic statics has been coupled to a set of geometrical constraints, which allow the conditional manipulation

of the nodes in both reciprocal diagrams [FIVET2013]. These constraints are used to define positional and proximity relationships between the diagrams' nodes.

Structural optimisation is a widespread concern in the field of structural engineering. Considering that structural behaviour can be modelled by a network of nodes and bars, optimising this network consists in finding the particular configuration of its nodes and bars that: (a) minimises or maximises a specific performance (usually minimises structural volume or weight); (b) respects specific constraints on structural behaviour (usually on stresses, displacements, buckling loads or member sizes) [ROZVANY2014]. In practice, the process of optimising structures can result in finding the best cross-sectional dimensions (sizing optimisation) or/and the best nodal positions (shape optimisation) or/and the best nodal connections (topology optimisation) in order to reach the structural objective.

Objectives of the research

The aim of this thesis is to explore theoretical bases for a complementary approach to structural robustness, allowing to overcome two main drawbacks of the current robustness approaches: damage dependency and non-interaction with structural design. Based on a geometrical rather than a numerical assessment of structural robustness, the approach combines graphic statics and strut-and-tie modelling to notions of structural optimisation. Given that material behaviour may be considered as perfectly ductile, the proposed geometrical method helps assessing structural robustness in terms of the following parameters: strength, multiple load paths and redundancy. In particular, the method helps finding a set of statically admissible force distributions acting within given structures. In the context of the Lower Bound Theorem of Plastic Theory, such force distributions are intuitively supposed to represent structural alternatives to redistribute loads in case of unexpected damage. This set of potential force redistributions can be depicted graphically within graphic statics' force diagram, by drawing the set of positions/connections its nodes can take in order to respect envelope, equilibrium, boundary and yield conditions. The resulting shape is called *admissible geometrical domain*, and is used by the present approach as an early indicator of robust design: the larger the domain, the more potential force distributions, the more robust the structure.

Taking back notions of structural optimisation methods, points included within the admissible geometrical domain can further be analysed in terms of efficiency, using Maxwell’s load path formula [1890]. The method consists in assessing the volume of material required for the implementation of specific strut-and-tie networks modelling force distributions. Supposing that distributions associated to smaller volumes of material will occur first (as they require a minimum amount of strain energy to be activated), Maxwell’s formula allows the ranking of admissible force distributions in terms of possibility of occurrence.

Organisation of the content

The present work is subdivided into eight main parts.

The three first parts respectively review the state of the art on the questions of structural robustness, graphic statics and structural modelling, and structural optimisation:

- *Part 1* presents the issue of structural robustness. In particular, it details the list of robustness-dependent parameters established by Knoll & Vogel [2009]. It then reviews the main approaches that are frequently met in the literature for structural robustness assessment;
- *Part 2* reviews the main existing approaches for modelling structural behaviour: continuous stress fields, strut-and-tie networks, and discontinuous stress fields. It then briefly explains how graphic statics’ reciprocal diagrams can be built to obtain internal forces within strut-and-tie networks. Is then developed the approach of constraint-based graphic statics [FIVET2013a], one of graphic statics’ recent developments. A few I.T. tools implementing structural design through graphic statics are finally evoked;
- *Part 3* reviews the main concepts concerning structural optimisation. It then briefly details the major types of methods that can be found nowadays for structural optimisation.

Part 4 then details how the concepts developed in the three foregoing parts can be combined, in order to obtain a geometrical approach for robust and efficient structural design. This fourth part first details how *admissible geometrical domains* can be built to indicate graphically the level of robustness of given structures in terms of possible force redistributions. It then develops how sizing, shape and topology optimisation problems can

be taken into account through the method, in order to model various material behaviours.

The last four parts of this thesis develop case study examples. These illustrate the functioning of the proposed geometrical approach of robustness for several structural types and materials:

- *Part 5* concerns the study of masonry arches in pure compression, whose behaviour can be modelled by a strut-and-tie network with fixed funicular topology and varying size and shape. In this part of the thesis, the impact of several design parameters on the robustness level of masonry arches in terms of potential force redistributions are highlighted (such as applied external load, material compressive strength, or arch stereotomy and geometry). Results are then compared to those that have been obtained experimentally through arch failure testing. The proposed geometrical approach for robustness assessment is finally used to reinterpret ancient theories about arch safety measurement;
- *Part 6* concerns the study of a steel arch in compression and bending, whose behaviour can be modelled by a strut-and-tie network with fixed funicular topology and varying shape. In this sixth part of the thesis, results given by the geometrical method for robustness assessment are compared to those obtained with classical numerical approaches in a perspective of validation;
- *Part 7* concerns the study of continuous walls made of a material with idealised properties, whose possible behaviours can be modelled by presupposing strut-and-tie networks with fixed topology and varying shape. The analyses comprise robustness (in terms of aptitude for force redistributions) and efficiency assessment through admissible geometrical domains, as well as stiffness assessment;
- *Part 8* finally concerns the study of statically indeterminate reinforced concrete walls, whose behaviour can be modelled by a strut-and-tie network with fixed shape and varying topology. An alternative methodology for drawing admissible geometrical domains is also presented, combining geometrical and analytical developments. This case study further helps assessing the level of structural robustness (in terms of aptitude for force redistributions) associated to several rebar layouts.

PART 1 – THE ISSUE OF STRUCTURAL ROBUSTNESS

Robustness qualifies the property of a structure to survive unforeseen or unusual circumstances. This aptitude is fundamental in ensuring the security of people in case of accident or terrorist attack. After some theoretical developments of the notion (Section 1.1), this first part of the thesis details the list of parameters established by Knoll & Vogel [2009] that constitute the performance of robustness (Section 1.2). Approaches to structural robustness currently found in the literature are then reviewed and commented (Section 1.3). Section 1.4 finally highlights the limitations that have been raised about the existing approaches, and that will be used to set the bases of a new robustness approach.

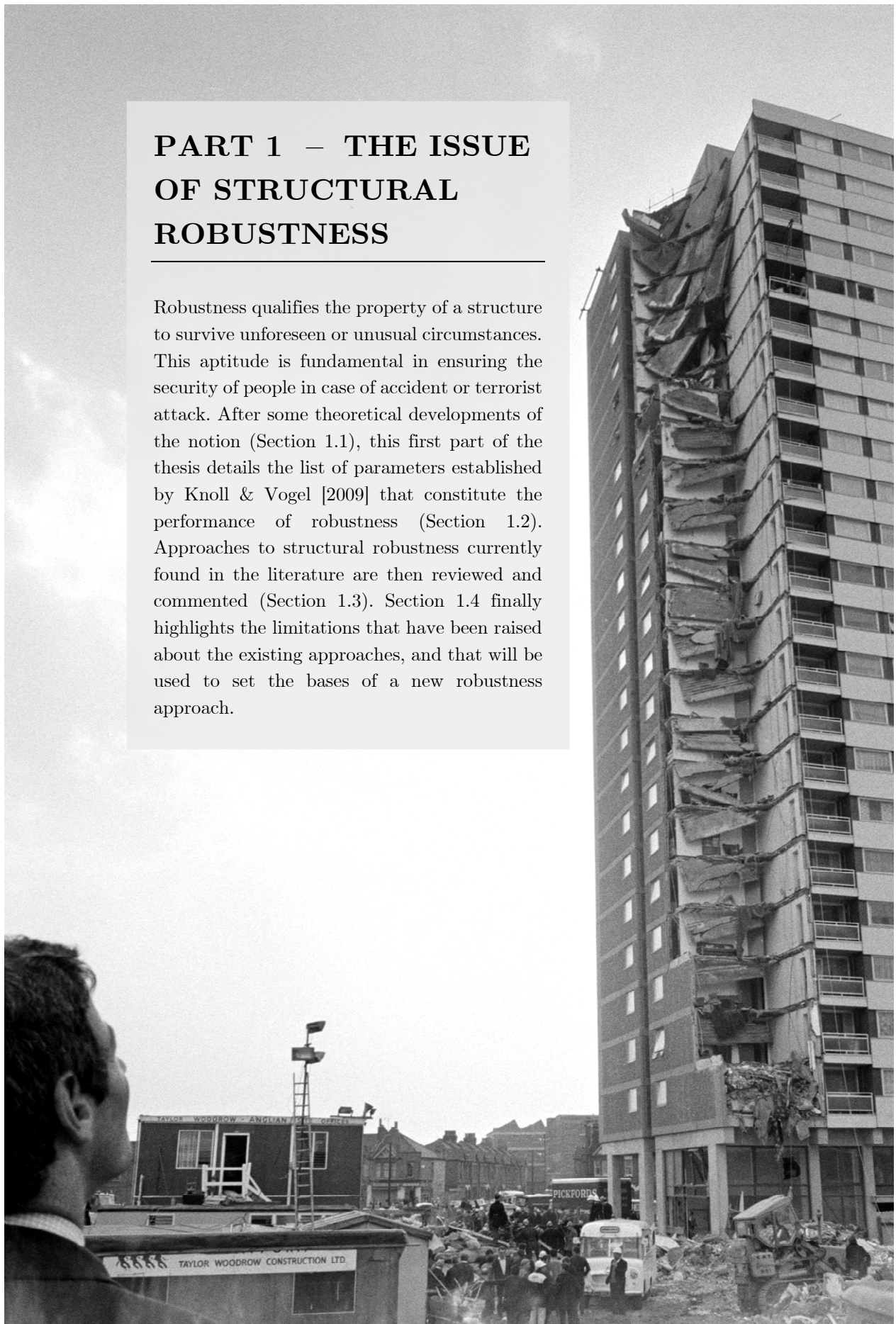


Figure 1.1 – Ronan Point collapse, London, UK, 1968 [GUARDIAN2018]

1.1. Definitions of structural robustness

“Robustness is the property of systems that enables them to survive unforeseen or unusual circumstances.” [KNOLL2009].

“Robustness is the ability of a structure to withstand events like fire, explosions, impact or the consequences of human error, without being damaged to an extent disproportionate to the original cause.” [EN1991]

In 1968, a gas explosion occurred in the recently inaugurated Ronan Point tower block in Newham, near London [GUARDIAN2018]. The strike of a match on the 18th floor provoked a blast, which destroyed the façade walls of an apartment, and created a chain reaction leading to the collapse of a whole corner of the tower, all over its 22 storeys (Figure 1.1). The extent of the incident was judged disproportional to the original cause, as four people found death and seventeen were injured in the event.

After Ronan Point collapse, structural engineers of that time started to study new measures and techniques to prevent the same type of incident to happen again. English building regulations were completely revised, to include concerns about *disproportionate damage to accidental loading* [GRIFFITHS1968]. Many papers were published in the 1970’s about *progressive collapse*, its causes, and techniques to avoid it [SOMES1972, BREEN1975, ELLINGWOOD1978].

After a small decrease of interest in the 1980’s, the concerns of progressive collapse and structural robustness were put back on stage in the 1990’s, after the bombings of the A.P. Murrah building in Oklahoma City in 1995 and of the US embassies in Nairobi and Dar-Es-Salaam in 1998 [VAL2006]. The focus was then turned on the development of explicit techniques for robust structural design of new structures, and on the analysis of existing ones.

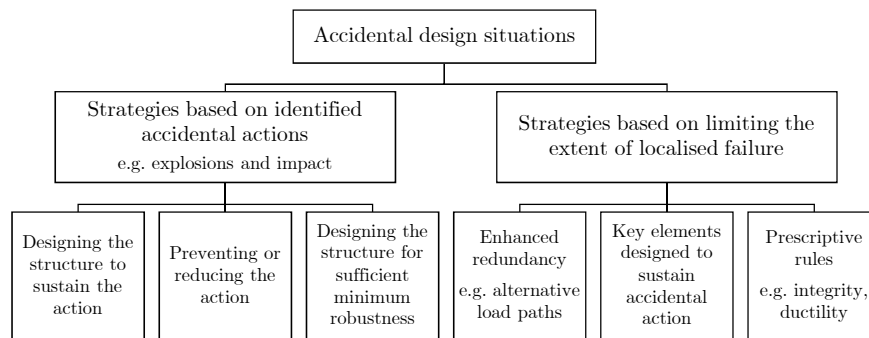
Structural robustness became a compulsory legal requirement in many European countries after the terrorist attacks of September 11, 2001 [GULVANESESSIAN2006]. Robustness was integrated to the Eurocodes as one of the *strategies for safeguarding buildings and other civil engineering works against identifiable and unidentifiable accidental actions* [EN1990, EN1991]. As detailed in the Eurocodes, these safeguard strategies can be of two types: based on identified accidental actions, and based on limiting the extension of localised failure (Figure 1.2).

The strategies based on identified accidental actions consist in:

- preventing the accidental action from occurring; reducing its probability of occurrence, or limiting its magnitude through an appropriate design;
- protecting the structure against accidental actions to reduce their effect;
- ensuring that the structure possesses sufficient robustness, through ductility and redundancy. Ductility can be brought by the use of building materials that can absorb strain energy without rupture, and redundancy by enabling the transfer of actions to alternative load paths.

The strategies to limit the extent of localised failure are:

- designing the structure so that failure due to an accidental action does not spread to a significant part of the overall structure;
- designing key elements, which are elements specifically conceived to withstand particular damage scenarios;
- applying prescriptive design/detailing rules that provide acceptable robustness to the structure.



*Figure 1.2 - Strategies to be considered for accidental design situations
(adapted from [EN1991])*

Eurocodes also develop *safety categories* (Figure 1.3), in which structures are classified according to their type, size, occupancy, exposure and other criteria [VAL2006]. This set of parameters gives the potential consequences of hazard on the considered structures [EN1991, GULVANESESSIAN2006]. Safety categories influence the method used for structural robustness analysis, as well as the safety measures that should be taken in case of damage. *Class 1* concerns structures for which accidental actions must not

be considered explicitly in the analysis. *Class 2* and *class 3* relate to structures for which only a simple robustness analysis should be performed, for identified accidental loads. *Class 4* applies to structures for which an extensive study of potential accidents should be performed, using dynamic and non-linear analyses.

Class	Consequences	Examples
1	Limited	Low rise buildings
2	Medium (lower group)	Buildings up to four storeys
3	Medium (upper group)	Buildings up to fifteen storeys
4	Large	High rise buildings

Figure 1.3 – Safety categories (adapted from [GULVANESSION2006])

The main vision proposed by the Eurocodes is then to accept the occurrence of local damage due to accidental actions, as long as they allow the preservation of the global load-bearing capacity of the structure for such an amount of time that emergency measures can be taken. Eurocodes also provide general guidelines to design structures for robustness [GULVANESSION2006].

1.2. Parameters of structural robustness

In their book *Design for robustness*, Knoll and Vogel detail a list of sixteen parameters that can be implemented to increase the robustness performance of a given structure [KNOLL2009]:

- four that concern pure design considerations: strength, multiple load paths and redundancy, structural integrity and solidarization, stiffness;
- four that are related to damage-oriented design thinking: post-buckling resistance, second line of defence, the zipper stopper, structural fuses;
- two that are then linked with material properties: ductility, strain hardening;
- three that relate to the use of secondary structural elements: sacrificial and protective devices, the knock-out scenario, mechanical devices;

- three that depend on human intervention: warning, active intervention and rescue; sample testing; monitoring, quality control, correction and prevention.

These parameters can (or must, in some cases) be combined in order to reach a sufficient robustness performance. Their range of action is said to depend on the type of event as well as on the type of structural response, as highlighted by the authors in the following questions:

- is the control put on the event load or on deformation;
- is the event repetitive;
- what are the physical limits of the event in terms of forces, impact, energy, deformation, movement, duration, etc;
- what will be the conditions following the event, in terms of structural function and further events?

1.2.1. Parameters linked with design considerations

Strength

The first strategy for robust design is strength, which is reached when applying safety factors on material behaviour to oversize the structure. This strategy is for example employed for structural cases for which the use of ductile materials is not possible, or for structural elements which possess a brittle behaviour (as slender compressed members). The strength parameter is implicitly taken into account in the Eurocodes calculations. Strength is said to be the most economic and easiest parameter to implement to enhance robustness. It is also the oldest, as it was already used in the ancient times through the use of corner stones, buttresses or other massive additional elements.

Multiple load paths and redundancy

The second robustness parameter takes into account the fact that multiple load paths for the internal forces can be mobilised initially in a given structure, with forces being carried in each of them. This means that, if one of these paths becomes impracticable due to any reason, the others paths can continue carrying the internal forces. The load bearing capacity of the structure is thus preserved. Another aspect associated to multiple load paths is redundancy, through which static indeterminacy is obtained by placing overabundant load bearing elements in the structural configuration. Examples of this parameter are statically indeterminate truss structures, for which the overabundance of diagonal bars provides as much alternative load paths as there are supernumerary bars. For multiple load paths and

redundancy to be preliminarily activated, some conditions however have to be met:

- after damage, the remaining paths need to possess sufficient strength (see previous parameter) – or to be able to deform without failure – in order to withstand an increased load;
- the structure needs to possess sufficient ductile properties (see Section 1.2.3) in order to enable force redistributions after damage. Otherwise, damage of any part of the structure would lead to a global premature zipper style failure (see Section 1.2.2);
- after damage, the remaining structure must conserve its overall stability.

Structural integrity and solidarization

Classically, structures are built by placing elements on top of each other, in order to transfer the applied loads by contact and gravity. However, this work scheme may become invalid when horizontal loads appear in the structure, as seismic shaking or movements. To overcome this problem, structural integrity can also be brought to the structure by tying structural elements together, as this may prevent horizontal displacements and disintegration. This makes the structure more rigid, which then has to possess sufficient strength (see previous parameter) to resist the applied loadings, especially at the connections. Structural integrity is also associated to *solidarization* and coherence, meaning that no individual member in a series will collapse on its own, but it will work as a whole with the other elements to increase structural resistance.

Stiffness

A last robustness parameter linked with design considerations is those of stiffness, which is used to avoid large structural deformations. The classical example of this parameter is the mixed shear wall/frame used in seismic design: the stiff brittle wall will sustain shear forces, while the ductile frame will resist moments. Thus, the total structure combines these two behaviours.

1.2.2. Parameters linked with design for damage

Post-buckling resistance

A first parameter linked with design for damage concerns post-buckling resistance. This allows local or partial buckling of thin-walled elements, under smaller loads than the limit one. A necessary condition is that the

structure still possesses sufficient residual strength (see Section 1.2.1) after buckling.

Second line of defence

A second parameter is the implementation of structural shapes that possess a second line of defence, i.e. a configuration which conserves resistance after deformation. The approach is then similar to the one of multiple load paths and redundancy (see Section 1.2.1), except that the associated load path(s) will only be activated once the structure is damaged. The classical illustration of this property are guardrails along roadways: if guardrails are subject to excessive loading along the direction perpendicular to their axis, a kinematic mechanism involving plastic hinges is formed. The structure then bends to reach a new geometry, which conserves resistance thanks to its new behaviour in tension. In the same way, frame structures can be designed in such a way that the loss of a column turns the floors to work in a hammock-like way, thus avoiding progressive collapse. However, this strategy is said to be rather expensive, comparing for example to providing excess strength by oversizing the strategic parts of the structure (see Section 1.2.1). A latent condition for the implementation of a second line of defence is the ductile behaviour of the constitutive material (see Section 1.2.3), which is necessary for the structure to find a second load path. Also, for allowing the structural elements to turn into an alternative behaviour, connections and assemblies have to be carefully (over)sized in order to ensure continuity and strength after damage.

The zipper stopper

A third parameter is the zipper stopper, which is a necessary property to avoid specific types of progressive failure (i.e. zipper effect or domino effect). The aim is to limit failure propagation, by placing “strong points” – more resistant elements – between fragile ones. A classical structural example in which a zipper stopper approach can be implemented is the case of power lines, for which every n^{th} element is replaced with a more ductile and resistant one. Another example is the fabric roofs, which are usually made of brittle material. The structure can then be optimised by adding a mesh of stronger fabric as well as cables attached to the initial one. This will limit potential damage to a predefined area.

Structural fuses

The last parameter in this category is the placement of structural fuses, which are portions of the structure designed to carry a predefined amount of load. When reaching this threshold, fuses will deform and thus limit the

forces in the load path of which they are part. The structure then needs to possess a ductile behaviour (see Section 1.2.3), enabling the redistribution of potential load increments from the fuse to other load paths. Another condition for structural fuses to be activated is of course that fuse deformation remains in an acceptable range, as well as the deformation of the global structure after the load increment.

1.2.3. Parameters linked with material properties

Ductility

The first material property enhancing robustness is ductility, which is said of materials that can absorb strain energy to postpone the triggering of failure. As previously mentioned, ductility is a necessary parameter for other robustness properties to be implemented, such as multiple load paths and redundancy, second line of defence or structural fuses. Ductility is defined by the following equality; the largest the ratio, the more ductile the material:

$$\text{Ductility} = \frac{\text{maximum deformation}}{\text{maximum elastic deformation}} \quad \text{Eq 1.1}$$

It should be noted that material ductility could be negatively affected by weak connections, by local weakening due to connection holes, by corrosion, by fatigue cycles, or by instabilities in slender compressed members.

Strain hardening

The second material property increasing robustness performance is strain hardening, which is found in materials for which stiffness can increase after deformation (typically steel). Strain hardening provides reserve strength for the structure before reaching large deformations. Another advantage of this property is that the structure will deform significantly before breaking, providing a visual warning of failure's imminent occurrence.

1.2.4. Parameters linked with secondary structural elements

Sacrificial and protective devices

When a structural element cannot be provided with sufficient robustness due to financial or practical reasons, another strategy can be the addition of sacrificial and protective devices. These sub-structures or secondary elements are designed to protect the main structure in case of impact, and will serve as buffers by absorbing excess loads. The classical example of

sacrificial and protective devices are the columns in car parks, for which it is more efficient to build a secondary protective structure surrounding them than to oversize their intrinsic dimensions. Another example of sacrificial device are offshore platforms, for which damage loads are potentially large and hard to predict (collision of a ship, of an iceberg, etc). A solution can be the addition of artificial islands, which will absorb or deflect potential damage loads. The design of the structure may thus only be conceived for resisting wind, waves and seism [DAMGAARD1993].

The knock-out scenario

Another strategy that can be implemented when sufficient strength or ductility cannot be provided is the knock-out scenario. This robustness parameter is typically used for structures risking damage through impact or collision. The approach consists in designing particular structural elements that will be knocked out in case of incident, thus attenuating the effect of the blast and preserving the remaining structure. To implement the knock-out scenario, some precautions however have to be taken: the blown elements must not hang or fly away on the remaining structure.

Mechanical devices

A last type of secondary elements that can be added to protect the main structure are mechanical devices. These can be used to control the movement of forces, or to absorb and dissipate excess energy in the structure. Mechanical devices can be either active – if they require (electrical) power to work – or passive. The latter is said to be privileged, as these devices do not necessitate human intervention nor any source of power, which may be cut in case of extreme event. However, for both active and passive types, mechanical devices need to be looked after and maintained, in order to avoid corrosion or other types of alteration.

1.2.5. Parameters linked with human intervention

Warning, active intervention and rescue

The first set of parameters that depend on human intervention concerns warning, active intervention and rescue. Warning is about the physical signs of danger that can be observed on the structure and which mean that something is going wrong, such as cracks, deformations, discoloration or leakage. Sometimes these signs are not directly visible for the eye and can only be detected using instrumentation, such as corrosion or loss of tension. Some signs also depend on other senses, such as abnormal sounds or smells. Active intervention concerns two types of measures. The first are the

emergency measures that can be taken in case of expected damage, such as the placement of auxiliary supports in order to gain time for people evacuation. The second type of intervention measures aims at reducing the applied loads, such as taking off the snow on a roof, or limiting the size/weight of vehicles crossing a bridge. Active intervention thus depends on the awareness of an imminent failure. Finally, rescue measures concern the structure's evacuation, in which the time between the manifestation of danger and the complete evacuation (or the implementation of the auxiliary system) is a crucial parameter to ensure people's life. These three robustness parameters are linked to the efficiency of human intervention, which, as we know, can undergo shortcomings: inattention, lack of communication, absence, incompetence, inefficiency or inadequacy of the procedure in case of imminent failure, etc. In addition, the actions to implement in case of imminent damage depend on the situation. Thus the answer to the question "what to do?" is not always straightforward.

Sample testing

A second parameter associated to human intervention is the physical testing of structural samples. Testing enables to compare the real material behaviour to the one expected when modelling the structure using mathematical or physical models. Indeed, the more complex and innovative the structure, the greater the gap between the expected and the real structural performances.

Monitoring, quality control, correction and prevention

A last parameter in this category concerns monitoring, quality control, correction and prevention. This concerns the discovering of anomalies through the regular inspection of the structure, its verification to check its good state and ageing, or the realisation of non-destructive testing. The eventual defects have to be interpreted then eliminated, which may reveal expensive.

1.3. Approaches of structural robustness

Four major types of robustness assessment approaches can be found in the literature: probabilistic, risk-based, deterministic and energetic [CAVACO2013, SORENSEN2012]. These approaches aim at characterising robustness by a numerical analysis of the capacities of resistance (or the possibility of force redistribution) of a structure once its design has been fixed.

1.3.1. Probabilistic approaches

Progressive collapse characterises a situation in which a local failure in a building triggers a chain reaction creating damage disproportionate to the original cause. This kind of failure is typical of structures that do not possess sufficient robustness properties. The probability of progressive collapse $P(f)$ is detailed by Val and Val [2006] as:

$$P(f) = P(f/DA_i)P(D/A_i)P(A_i) \quad \text{Eq 1.2}$$

where $P(A_i)$ is the probability of the occurrence of an accidental event A_i , $P(D/A_i)$ is the probability of local damage D given the occurrence of A_i , and $P(f/DA_i)$ is the probability of progressive collapse given the occurrence of D due to A_i . The probability of progressive collapse $P(f)$ can then be minimised by decreasing the value of any of the right members of the equation, depending on the method.

As term $P(A_i)$ in Eq 1.2 does not rely on any structural property, another option to assess robustness is to employ a formula based on a structural approach only. The probability of progressive collapse $P(F)$ can then be calculated independently of the accidental action A_i , as in the following simplified equation [GSA2003, DOD2005]:

$$P(F) = P(F/D)P(D) \quad \text{Eq 1.3}$$

Another probabilistic approach is the one of Frangopol and Curley [1987], which evaluate a reliability-based index of robustness β_r as:

$$\beta_r = \frac{\beta_{\text{intact}}}{\beta_{\text{intact}} - \beta_{\text{damaged}}} \quad \text{Eq 1.4}$$

where β_{intact} is the reliability index of the intact system, and β_{damaged} the one of the same system when damaged. Individual reliability indices can be calculated based on:

$$\beta_i = \frac{M_i}{\sigma(M_i)} \quad \text{Eq 1.5}$$

$$M_i = S_i - Q_i$$

in which M_i is the mean value of the performance function of a structural element i . This function is defined as the difference between the random strength of a member S_i and the random load effect in the same member

Q_i . $\sigma(M_i)$ is the standard deviation of the performance function M_i . Eq 1.5 takes values between 0 (totally damaged structure) and ∞ (intact structure).

1.3.2. Risk-based approaches

A robustness index I_{rob} is proposed by Baker [2008] to measure the fraction of total system risk resulting from direct consequences:

$$I_{rob} = \frac{R_{dir}}{R_{dir} - R_{ind}} \quad Eq\ 1.6$$

where R_{dir} is the risk of failure caused by local damage, and R_{ind} is the risk of indirect consequences due to this local damage. A robust structure is then the one for which indirect risks do not contribute significantly to the total system risk, i.e. those for which the result of Eq 1.6 is close to 1. On the opposite, a non-robust structure has a value for I_{rob} close to 0.

1.3.3. Deterministic approaches

Based on the *Residual Influence Factor* used in the offshore industry, the strength redundant factor R showing the reserve resistance of a given structure is defined by Frangopol and Curley [1987] as:

$$R = \frac{L_{intact}}{L_{intact} - L_{damaged}} \quad Eq\ 1.7$$

where L_{intact} is the load bearing capacity of the structure while intact, and $L_{damaged}$ the same capacity after structural damage under determined failure scenarios. The strength redundant factor assesses that robust structures possess properties of redundancy, meaning that failure of a critical component will not lead to the collapse of the whole structure. Robust structures lead to a result for Eq 1.7 close to ∞ , as $L_{damaged}$ will be close to L_{intact} , making the denominator of the fraction tend to 0. On the opposite, if $L_{damaged}$ is much smaller than L_{intact} , the denominator of the fraction will be close to L_{intact} . This means that index R will have a value close to 1, indicating non-redundant structural properties.

1.3.4. Energetic approaches

Energetic approaches classically consist in calculating the deformation energy (work of failure) of a structure led to collapse. This is done by integrating the space below the curve that characterises the structure's force-displacement relationship up to the collapse point (Figure 1.4).

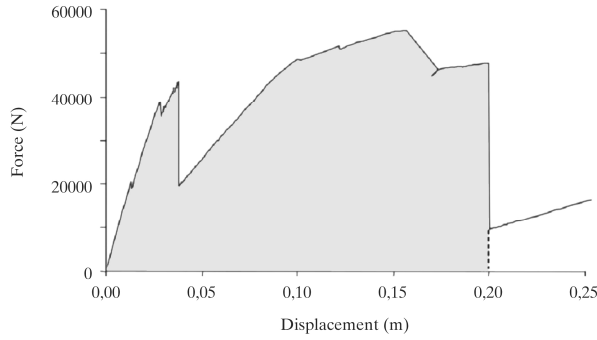


Figure 1.4 – Force-displacement graph of a given structure and associated deformation energy in grey [SMITH2006]

The approach developed by Smith [2006] proposes to use as a robustness indicator the structure's total work of damage, which is the work necessary to destroy a sufficient number of members such as to lead the structure to collapse. Another energetic method is proposed by Starossek and Haberland [2011]. In this approach, the energy-based measure of robustness R_e is defined as:

$$R_e = 1 - \max(E_{r,j}/E_{f,k}) \quad Eq\ 1.8$$

where $E_{r,j}$ is the energy released by the initial failure of an element j , and which is available for the next structural element k to fail according to a scenario of progressive collapse. $E_{f,k}$ is the energy required for this next element k to fail, i.e. the energy required for failure to progress. R_e then shows whether the energy brought by the accidental event - and which triggered failure - is absorbed or not by the failure. A R_e measure close to 1 is associated to a structure with perfect robustness properties; values between 0 and 1 indicate relative robustness; values beyond 0 correspond to a progression of failure and complete collapse.

1.4. Limitations of existing approaches

In practice, it can be noted that numerous structures nowadays are still prone to collapse through a deficit of robustness, even if they have been extensively verified by structural engineers [CAVACO2013]. Indeed, these verifications can only be employed to assess robustness for chosen failure scenarios (see Section 1.3). This, of course, does not allow coverage of the entire range of unexpected damage that may occur during a structure’s lifetime. Probabilistic and risk-based approaches have also been judged to be of limited practical interest because they require dedicated methods, and can since then only be adopted by specialists [SORENSEN2012].

A second drawback of the robustness approaches currently found in the literature is that they only apply to structures whose design has been fixed beforehand. The appropriation of those approaches by project authors then becomes difficult, insofar as the related methods usually totally ignore design concerns. This makes these approaches inconvenient when it comes to interacting with the design and the related robustness performance.

Furthermore, some of the dimensions that were said to be key parameters of structural robustness (as the implementation of *multiple load paths and redundancy*, see Section 1.2) are difficult to assess with classical numerical tools, since analysis software is usually not designed for this purpose.

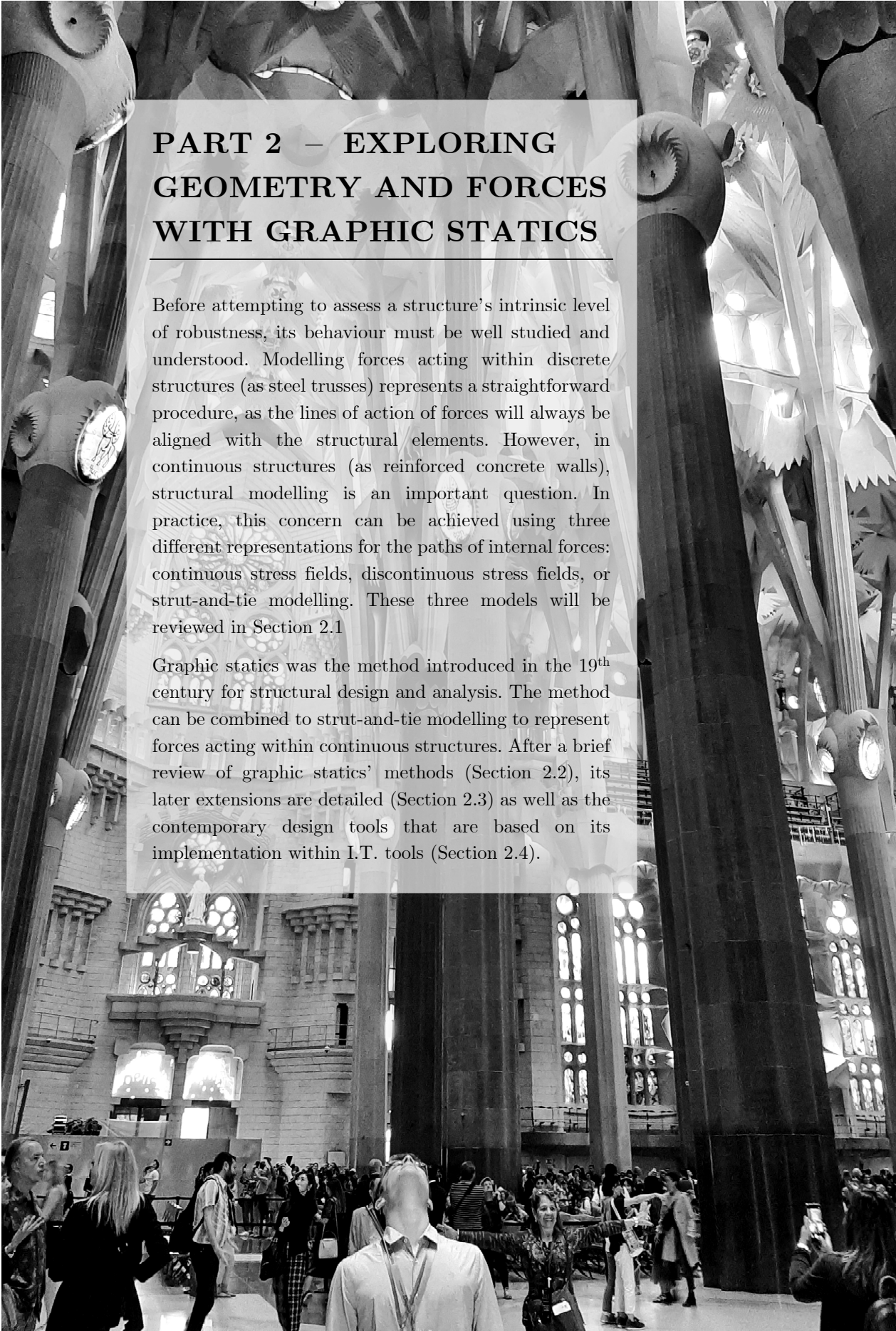
Finally, there is still today no agreement on the definition of structural robustness, on the methods to implement for its characterisation, nor on the evaluation of what presents sufficient robustness properties and what does not [DEBIAGI2016]. The definitions of structural robustness that can be found in the literature propose a good conceptual approach to the required performance, as long as this performance remains qualitative. However, when it comes to characterise a structure in particular, the lack of quantitative approaches can unease the distinction between what is robust and what is not [STAROSSEK2011]. Indeed, all the existing definitions of structural robustness bring forward the idea that “damage should not be disproportionate to the original cause”, without defining explicitly what “disproportionate” means [VAL2006]. This can be problematic as robustness is a property required by European design laws [EN1990, 1991], which should then be absolute: objective, universal, and based on independent indices. Hence, the question of the quantitative assessment of robustness is a topical concern of many conferences and workshops dealing with structures (*IABSE*, for example).

1.5. Outcome

Following the limitations outlined in Section 1.4, the aim of the present thesis is to develop a complementary damage-independent and design-friendly approach to structural robustness. For this purpose, the proposed approach is based on:

- a geometrical definition of structural robustness, based on three of the sixteen parameters developed by Knoll & Vogel [2009] (see Section 1.2). In particular, robustness is seen as the aptitude of a given structure to redistribute internal forces, considering that its constitutive material presents sufficient properties of *ductility* and *strength*. This definition then favours design options that allow the structure to find multiple *alternative load paths* in case of damage;
- a geometrical methodology to assess such properties of robustness, based on graphic statics and strut-and-tie modelling (see Part 2). Both these methods have been selected for their facility of use in the context of early stage structural design;
- a geometrical indicator of structural robustness according to the aforementioned parameters, called the *admissible geometrical domain*. Based on graphical constraints, this indicator depicts the structure's aptitude for strength and redundancy (considering sufficient material ductility).

Before developing the proposed geometrical approach for robustness assessment (see Part 4), Part 2 will first review the main hypotheses and functioning of graphic statics and strut-and-tie modelling.

A black and white photograph of the interior of the Sagrada Família in Barcelona. The image shows the massive, tree-like columns of Antoni Gaudí's design, which branch out at the top like trees. The floor is a light-colored, polished surface. In the foreground, a crowd of people is visible, some looking up at the columns. The background shows more of the church's interior, including arched windows and balconies. The overall atmosphere is one of awe and grandeur.

PART 2 – EXPLORING GEOMETRY AND FORCES WITH GRAPHIC STATICS

Before attempting to assess a structure's intrinsic level of robustness, its behaviour must be well studied and understood. Modelling forces acting within discrete structures (as steel trusses) represents a straightforward procedure, as the lines of action of forces will always be aligned with the structural elements. However, in continuous structures (as reinforced concrete walls), structural modelling is an important question. In practice, this concern can be achieved using three different representations for the paths of internal forces: continuous stress fields, discontinuous stress fields, or strut-and-tie modelling. These three models will be reviewed in Section 2.1

Graphic statics was the method introduced in the 19th century for structural design and analysis. The method can be combined to strut-and-tie modelling to represent forces acting within continuous structures. After a brief review of graphic statics' methods (Section 2.2), its later extensions are detailed (Section 2.3) as well as the contemporary design tools that are based on its implementation within I.T. tools (Section 2.4).

*Figure 2.1 – Jonas Feron at the Sagrada Familia, A. Gaudi, Barcelona, Spain,
under construction [picture by the thesis author]*

2.1. Modelling structural behaviour

Modelling forces acting within continuous structures typically consists in employing one of the three following representations: continuous stress fields (Figure 2.2a), strut-and-tie networks (Figure 2.2b), or discontinuous stress fields (Figure 2.2c).

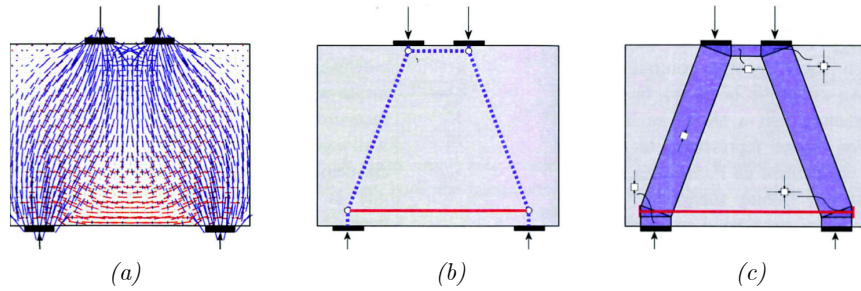


Figure 2.2 – Models representing stresses acting within continuous structures: (a) continuous stress fields, (b) strut-and-tie networks, (c) discontinuous stress fields [MUTTONI2011]

2.1.1. Continuous stress fields

The *flow of forces* is a principle based on the analogy between the path of forces acting within continuous structures and patterns of flows in liquids [ZALEWSKI1975]. The flow of forces is used to derive *continuous stress fields* within continuous structures (Figure 2.2a), which give the direction of principal internal stresses: compressive in blue, and tensile in red. This internal stress organisation will always be such that it minimises the energy stored by the structure. In practice, continuous stress fields can easily be obtained by a finite element analysis of the structure, using commercial software or other tools such as those developed by EPFL’s IBETON lab [IMESH, ICONCRETE].

2.1.2. Strut-and-tie networks

Another approach for modelling forces acting within continuous structures is to use the truss analogy [MUTTONI1997, 2007], which was early introduced by Ritter [1899] then Mörsch [1912]. The truss analogy is a structural abstraction that consists in replacing the principal stresses of a continuous stress field by their resultant normal force [LIANG2000]. This set of forces gives what is called a *strut-and-tie network*, with struts giving

the structure's resultant compressive forces (in blue in Figure 2.2b) and ties the tensile ones (in red in Figure 2.2b). Hence, using strut-and-tie modelling enables to analyse continuous structures in the same way than discrete ones, i.e. as a composition of bar elements linked with nodes representing hinges [SCHLAICH1987]. To be applicable, the model however needs to satisfy the conditions further detailed in Section 2.1.4.

2.1.3. Discontinuous stress fields

Originally, *discontinuous stress fields* were introduced for the design of reinforced concrete structures [DRUCKER1961]. As strut-and-tie networks, discontinuous stress fields are structural abstractions. They consist in replacing strut-and-tie networks' resultant forces by fields, whose thickness is proportional to material strength σ [MARTI1985]. In practice, discontinuous stress fields can present three different shapes: fan (Figure 2.3a), bottle (Figure 2.3b) or prism (Figure 2.3c) [SCHLAICH1987]. However, for the purpose of the present work, only prism-type fields will be considered, as they are easier to implement. Indeed, material strength σ is in that case kept constant throughout the field. To be applicable, discontinuous stress fields however need to satisfy the conditions further detailed in Section 2.1.4. Stresses acting within fields' nodes should also be verified for nodes on which both compressive and tensile forces are applied (see [MUTTONI2007] for further information).

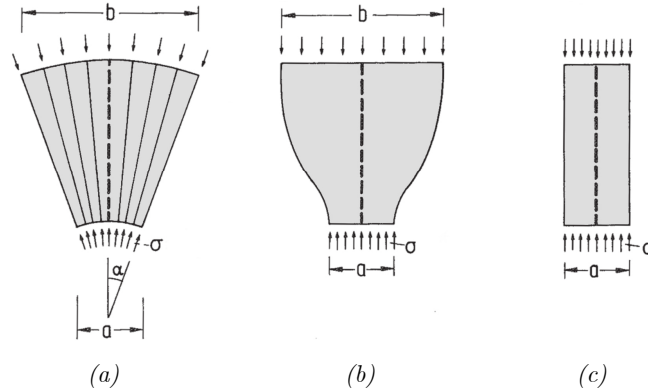


Figure 2.3 – Types of discontinuous stress fields: (a) fan, (b) bottle, (c) prism [SCHLAICH1987]

In practice, strut-and-tie networks and discontinuous stress fields may be used for both the design of new structures as well as the analysis of existing ones [MUTTONI2015].

2.1.4. Conditions of application

For the previous models to be employed, structural materials are supposed to present sufficient *ductile* properties. This ductility may either result in an elasto-plastic or rigid-plastic material behaviour (Figure 2.4), without having any influence on the results. This means that the structural response may be characterised by material ultimate limit strength σ_e only.

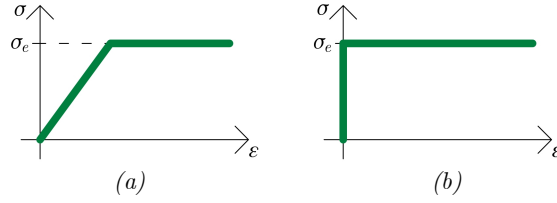


Figure 2.4 – Stress-strain diagrams of idealised material behaviours:
(a) elasto-plastic, (b) rigid-plastic

For steel structures, assessing such a behaviour may not be problematic (as long as the implemented structural model is relatively close to the elastic solution), because steel usually presents good properties of ductility in both compression and tension. The assumption may also be valid for masonry structures under compressive loads, as masonry is usually considered to have infinite compressive strength; hence, it has a perfect rigid behaviour (see Section 5.1.1). However, for materials like reinforced concrete, which ductility in compression is relatively low, such an approximation may not be straightforward. This also applies to timber structures, for which strength decreases as the angle between internal forces and fibre direction increases, and which have a fragile failure mode in tension. Hence, in order to be considered as ductile, real materials may see their ultimate limit strength σ lowered to an effective value σ_e using appropriate reduction factors depending on the material (see for example Eq 8.1 in Part 8 for reinforced concrete [KOSTIC2009]).

Another hypothesis that may be taken on material behaviour is the consideration of a *fully-stressed design*. This means that each section of material will either be fully solicited at its effective strength σ_e (if comprised within a field), either be free of any constraint (if not). When modelling structural behaviour with prismatic discontinuous stress fields (Figure 2.3c), this hypothesis serves for deriving field area A_i as:

$$A_i = \frac{F_i}{\sigma_e} \quad \text{Eq 2.1}$$

with F_i the magnitude of the axial force in bar i of the strut-and-tie network. All the structural parts not included in a stress field are then unloaded. Such a condition ensures that the chosen structural working is not too distant from the elastic solution, i.e. it is not associated to a too large plastic deformation.

In adequate conditions of ductility, the *Lower Bound Theorem of Plastic Theory* can be employed for early structural design [HEYMAN1995]. This theorem states that a given structure will stand as long that there exists a statically compatible distribution of its internal forces, i.e. a strut-and-tie model or a discontinuous stress field for which:

- each node is in equilibrium (*equilibrium condition*);
- constraints and displacements are respected at nodes where external or support loads are applied (*boundary condition*);
- material strength is not exceeded in any bar or field (*yield condition*).

In other words, any strut-and-tie network or discontinuous stress field respecting these three conditions will represent a valid and safe model for characterising the path of internal forces within the considered structure.

Moreover, for the sake of the present work, an additional condition has been added on the structural model: each node and bar of the network or field must not get out of the border of the structure (*envelope condition*) [MUTTONI2007]. This condition is necessary for global structural equilibrium: it ensures that there are sufficient amounts of structural material such as to undertake the internal forces.

2.2. Graphic statics and reciprocal diagrams

Graphic statics was the method introduced in the 19th century for the design and analysis of any type of structure composed of any kind of material [MAXWELL1864, CULMANN1866]. It is a simple and intuitive tool that relies on geometrical manipulations only. Hence, it does not require the use of any calculation. It is also supposed that visual results are more easily appropriated than mathematical quantities [MACKINLAY1986]. This makes those results easy to apprehend and understand.

The principle of graphic statics lies in the representation of two reciprocal diagrams:

- a *form diagram* (Figure 2.5a, always represented on the left in the present dissertation), which depicts structural geometry as a network of bars L_i . Note that these bars can either explicitly represent bar positions in discrete structures (e.g. trusses), either model the path of internal forces acting within continuous structures as a strut-and-tie network (e.g. walls, arches or plates, see Section 2.1.2);
- a *force diagram* (Figure 2.5b, always represented on the right in the present dissertation), in which the length of each bar F_i gives the magnitude of the axial force in the corresponding bar L_i of the form diagram.

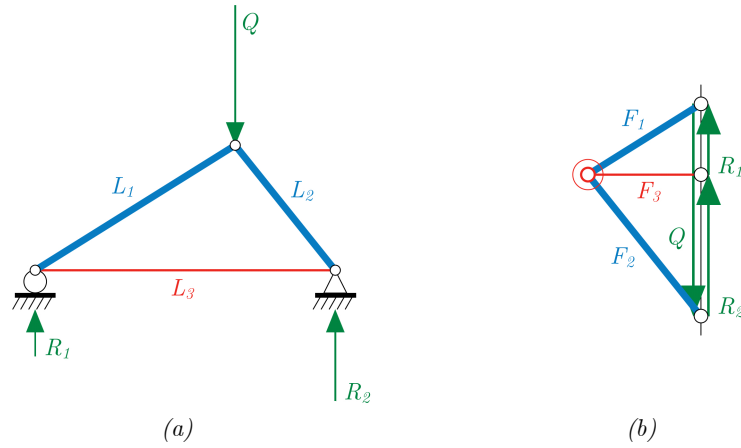


Figure 2.5 – Graphic statics' reciprocal diagrams: (a) form diagram, (b) force diagram

To be reciprocal, form and force diagrams have to satisfy the following conditions [ZALEWSKI1998, ALLEN2009]:

- *parallelism*: each bar L_i of the form diagram is related to a unique reciprocal bar F_i in the force diagram, such that the difference in angle between them is always the same. A common practice is to choose an angle between L_i and F_i equal to zero, so that both bars of a reciprocal pair are parallel;
- *translational equilibrium*: all the bars that connect in a node in the form diagram have reciprocal bars that constitute a closed polygon in the force diagram. Indeed, every closed polygon corresponds to a vector sum at the reciprocal node equal to zero. This rule also stands for the external forces;

- *rotational equilibrium*: all the bars belonging to a same polygon in the form diagram must cross in a unique node in the force diagram. This rule is implicitly verified for internal forces if the previous rules are respected. For external forces however, rotational equilibrium is to be carefully verified. If the structure has one external load and two supports, the condition turns that the three corresponding loads must have lines of action crossing in a same point in the form diagram (or at infinity if these loads are parallel, cfr. Figure 2.5). If there are more than three external loads/reaction forces, Culmann lines have to be drawn [CULMANN1866]. These ensure that each pair of external forces have lines of action crossing in a same point with the resultant of the remaining external loads.

Assessing that the three aforementioned conditions are satisfied, an additional rule can help to determine force direction, i.e. whether a reciprocal pair of bars is in compression (blue) or in tension (red) [CREMONA1867]. To determine force direction within the bars that connect at node n_i in the form diagram, one must first take the index of these bars in a clockwise direction (or counter-clockwise, eventually), cfr. Figure 2.6. Force diagram's reciprocal face t_i is then read through a path that follows the same order of bar indices. Taking back this path to form diagram's node n_i , force direction associated to each bar can be determined: compressive if the force direction in the force diagram goes towards node n_i in the form diagram, and tension if it pulls away from node n_i .

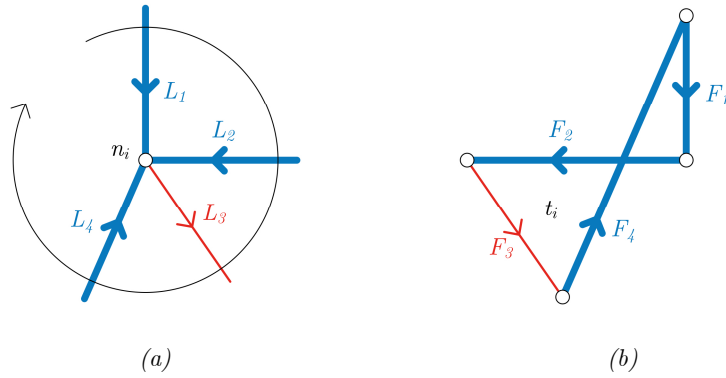


Figure 2.6 – Determining force direction using graphic statics:
(a) form diagram's node i , (b) partial reciprocal force diagram

Moreover, the conditions on the strut-and-tie network mentioned in Section 2.1.4 also have to be verified. As *equilibrium* has already been considered explicitly and *boundary* implicitly in the foregoing conditions on graphic statics' reciprocal diagrams, the two following remain:

- *envelope*: each bar L_i of the form diagram has to remain entirely within the defined structural border;
- *yield*: material effective strength σ_e must not be exceeded in any part of the structure. This consists in verifying that the magnitude F_i of each axial force respects:

$$F_i \leq F_{max} = \sigma_e A_{max} \quad Eq\ 2.2$$

with A_{max} the maximum section that can be given to the network's bars (i.e. to the associated discontinuous stress field, see Parts 5 & 7) while respecting the conditions of Section 2.1.4.

2.3. Constraint-based graphic statics

Graphic statics can be coupled to a set of geometrical constraints and operations, which allow the conditional manipulation of the nodes in both reciprocal diagrams [FIVET2013a, 2013b, 2014]. Since the only parameters implemented within graphic statics are nodal positions in both form and force diagrams, structural constraints can be defined as equalities or inequalities involving nodal coordinates only.

Two types of primitive constraints on nodal coordinates have been defined by Fivet. The first is named *Laterality*, and governs affine relationships. The *Laterality*[$n_A n_B n_C n_D$] constraint is satisfied if node n_A is on the left or in front of node n_B according to the direction from nodes n_C to n_D :

$$\begin{aligned} Laterality[n_A n_B n_C n_D] &\leftrightarrow Laterality[n_B n_A n_D n_C] \\ &\leftrightarrow Laterality[n_C n_D n_B n_A] \leftrightarrow Laterality[n_D n_C n_A n_B] \\ &\Leftrightarrow (x_B - x_A)(y_D - y_C) - (y_B - y_A)(x_D - x_C) \geq 0 \end{aligned} \quad Eq\ 2.3$$

Within graphic statics' reciprocal diagrams, *Laterality* constraints result in lines relative to which nodes of the form and/or force diagrams must be located (dashed segments in Figure 2.7 and Figure 2.8).

The second primitive constraint is called *Proximity*, and governs metric relationships. *Proximity*[$n_A n_B n_C n_D$] is satisfied if the distance from nodes

n_A to n_B is smaller or equal to the distance from nodes n_C to n_D :

$$\begin{aligned}
& Proximity[n_A n_B n_C n_D] \leftrightarrow Proximity[n_B n_A n_C n_D] \\
& \leftrightarrow Proximity[n_A n_B n_D n_C] \leftrightarrow Proximity[n_B n_A n_D n_C] \\
& \Leftrightarrow \left[(x_B - x_A)^2 + (y_B - y_A)^2 \right] - \left[(x_D - x_C)^2 + (y_D - y_C)^2 \right] \leq 0 \quad Eq\ 2.4
\end{aligned}$$

Proximity can for example be used to constrain bar length in the force diagram to remain smaller than a certain value. This is done by compelling one extreme node of the considered bar to remain within a circle of a given radius and centred on the other node (dashed curves in Figure 2.7 and Figure 2.8).

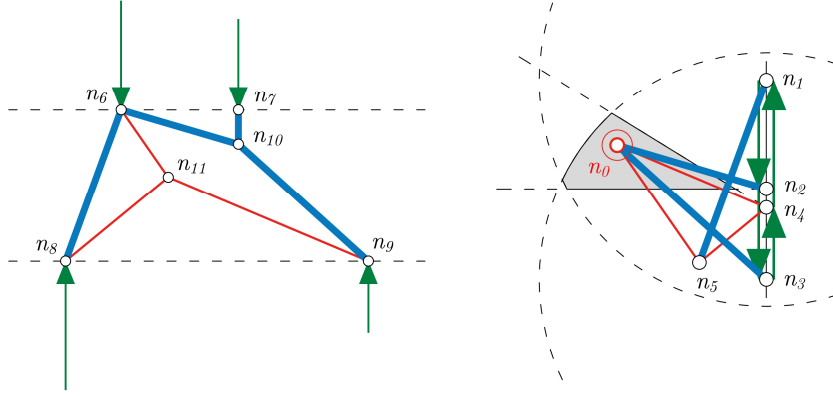


Figure 2.7 – Possible position for node n_0 respecting the constraints of Eq 2.7

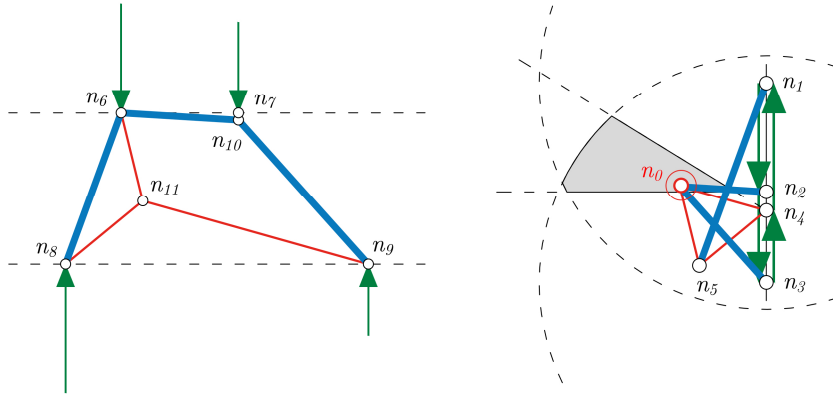


Figure 2.8 – Another admissible solution for node n_0 respecting the constraints of Eq 2.7

Laterality and *Proximity* constraints can be combined with unions, intersections and negations to create more complex statements. For instance, graphic statics' rule of parallelism between reciprocal bars (see previous section) can be translated in a *Parallelism* constraint, defined as the intersection of two *Laterality* constraints with opposite directions:

$$\begin{aligned}
& \text{Parallelism}[n_A n_B n_C n_D] \\
& \Leftrightarrow \text{Laterality}[n_A n_B n_C n_D] \wedge \text{Laterality}[n_A n_B n_D n_C] \\
& \Leftrightarrow (x_B - x_A)(y_D - y_C) - (y_B - y_A)(x_D - x_C) = 0 \quad \text{Eq 2.5}
\end{aligned}$$

An Equidistance constraint can also be defined, as the intersection of two opposite *Proximity* constraints:

$$\begin{aligned}
& \text{Equidistance}[n_A n_B n_C n_D] \\
& \Leftrightarrow \text{Proximity}[n_A n_B n_C n_D] \wedge \text{Proximity}[n_C n_D n_A n_B] \\
& \Leftrightarrow [(x_B - x_A)^2 + (y_B - y_A)^2] - [(x_D - x_C)^2 + (y_D - y_C)^2] = 0 \quad \text{Eq 2.6}
\end{aligned}$$

A combination of primitive constraints can also be implemented to represent conditions on graphic statics' reciprocal diagrams. The example of Figure 2.7 illustrates such diagrams obtained when combining the constraints given in Eq 2.7. The first four lines of Eq 2.7 guarantee equilibrium through *Parallelism* of reciprocal pairs of segments. The next two lines constrain force magnitudes in the bars of the force diagram below 10kN, using *Proximity* constraints. The last three lines ensure that the strut-and-tie network describing the form diagram is not higher than 1m, through a set of *Laterality* constraints:

$$\begin{aligned}
& \text{Parallelism}[n_6 n_8 n_1 n_5] \wedge \text{Parallelism}[n_6 n_{10} n_0 n_2] \\
& \wedge \text{Parallelism}[n_7 n_{10} n_2 n_3] \wedge \text{Parallelism}[n_9 n_{11} n_3 n_4] \\
& \wedge \text{Parallelism}[n_{10} n_{11} n_0 n_3] \wedge \text{Parallelism}[n_6 n_{12} n_0 n_5] \\
& \wedge \text{Parallelism}[n_{11} n_{12} n_0 n_4] \wedge \text{Parallelism}[n_8 n_{12} n_4 n_5] \\
& \wedge \text{Proximity}[n_5 n_1 n_1 n_3] \wedge \text{Proximity}[n_5 n_3 n_1 n_3] \\
& \wedge \text{Proximity}[n_0 n_1 n_1 n_3] \wedge \text{Proximity}[n_0 n_3 n_1 n_3] \\
& \wedge \text{Laterality}[n_{10} n_7 n_7 n_6] \wedge \text{Laterality}[n_{11} n_7 n_7 n_6] \\
& \wedge \text{Laterality}[n_{12} n_7 n_7 n_6] \wedge \text{Laterality}[n_{10} n_8 n_8 n_9] \\
& \wedge \text{Laterality}[n_{11} n_8 n_8 n_9] \wedge \text{Laterality}[n_{12} n_8 n_8 n_9] \quad \text{Eq 2.7}
\end{aligned}$$

The shaded area in the force diagram around node n_0 (highlighted in red in Figure 2.7) is called n_0 's solution space. This space defines the set of admissible positions node n_0 can take such that all the constraints of Eq 2.7 are satisfied, given that some other nodal positions remain fixed. In this example, the set of positions $\{n_1\ n_2\ n_3\ n_4\ n_5\ n_6\ n_7\ n_8\ n_9\}$ are considered as constant, and $\{n_0\ n_{10}\ n_{11}\}$ are variables. Figure 2.8 then shows another position of node n_0 within its solution space, thus also verifying Eq 2.7.

2.4. Contemporary structural design tools

Graphic statics was progressively abandoned in the 1970's due to the emergence of computers and fast numerical calculations. However, the method has made a come-back nowadays in the context of design-oriented I.T. tools, associating the power of geometry with the speed of computer calculations. This combination allows the integration of design and analysis of structures in a unique process [MUELLER2013a], as graphic statics supplies real-time and transparent information about the geometry being drawn and the associated forces [ZALEWSKI1998]. This section reviews a few of the new structural design tools that have emerged in the last ten years.

2.4.1. Geogebra

Geogebra [GEOGEBRA] is an I.T. tool for user-friendly mathematics. It works on with algebraic functions as well as with geometrical objects, such as points, lines, vectors or surfaces. Geogebra allows the dynamic manipulation of these geometrical objects, showing directly the impact of changes on the shape being drawn. In the context of graphic statics, Geogebra can be used to construct reciprocal diagrams in an interactive way. This procedure has for example been implemented by ETHZ's Block Research Group [BLOCK] within an online teaching platform named eEQUILIBRIUM [EQUILIBRIUM]. This platform enables graphic statics' reciprocal diagrams to be manipulated dynamically, allowing the interactive exploration of the relationship between form and forces with real-time visual feedback. Through the proposed applets, eEQUILIBRIUM develops examples that teach structural design and analysis through interactive drawings (Figure 2.9). These drawings help better understanding structural behaviour, as they allow the exploration of structural design as well as the forces associated to it.

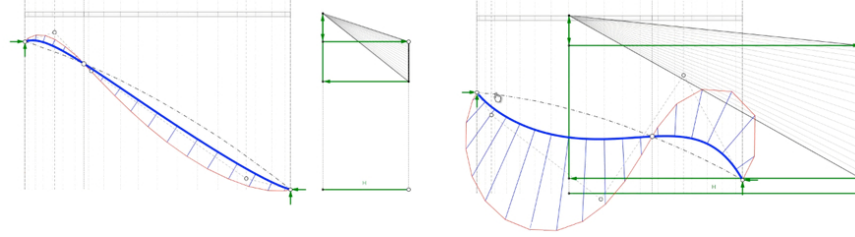


Figure 2.9 – Interactive structural design using Geogebra [EQUILIBRIUM]

2.4.2. StructureFit

StructureFit [STRUCTUREFIT] is a structural design tool that allows the quick and interactive exploration of structural design alternatives [MUELLER2013a, 2013b, LEE2015]. After having drawn an initial structure, the tool allows to derive the design by automatically generating alternative shape scenarios. This process is evolutionary: the design(s) preferentially chosen in step i are used as seeds for generating the alternative designs of step $i+1$ (highlighted by grey frames in Figure 2.10). The tool also gives the relationship between a chosen shape and its structural performance in terms of required quantity of material (as further detailed in Part 3). Considering that an initial design has a performance of 1 in terms of quantity of material used, each alternative design is then assessed with a number (given under each shape) which represents its relative performance compared to the initial shape. Hence, by combining these questions of shape and associated performance, StructureFit helps integrating architects and engineers' work into a common tool.



Figure 2.10 – StructureFit's interface [STRUCTUREFIT]

2.4.3. Combinatorial Equilibrium Modelling

Graphic statics can also be extended for early design exploration of structures in the third dimension [DACUNTO2016, JASIENSKI2014, 2016]. 3D graphic statics has been implemented in a framework called *Combinatorial Equilibrium Modelling* (CEM), based on Rhino and its algorithm editor Grasshopper [RHINO, GRASSHOPPER]. The tool is based on three reciprocal diagrams [OHLBROCK2015, 2016]:

- a topological diagram based on Graph Theory [ESCHRIG2011], which gives the qualitative behaviour of the structure by describing the connectivity between structural elements (Figure 2.11a);
- graphic statics' form and force diagrams, giving the quantitative behaviour of the structure (Figure 2.11b and c respectively).

Alternative 3D structural equilibriums can then be explored by manipulating CEM's topological diagram. These manipulations account for the addition or removal of bars (Figure 2.12), the inversion of structural behaviour from compression to tension and vice-versa (Figure 2.13), and the modification of force distribution within bar elements (Figure 2.14).

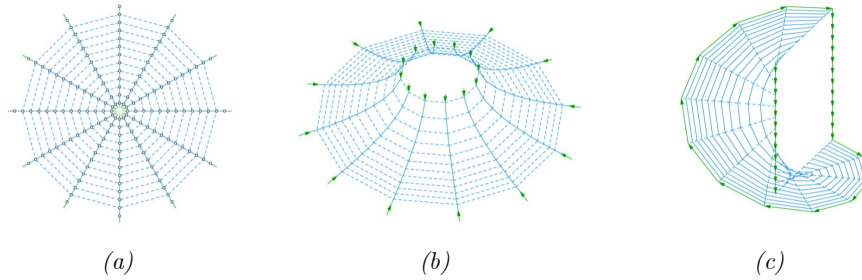


Figure 2.11 – *Combinatorial Equilibrium Modelling: (a) topological diagram, (b) form diagram, (c) force diagram [OHLBROCK2016]*

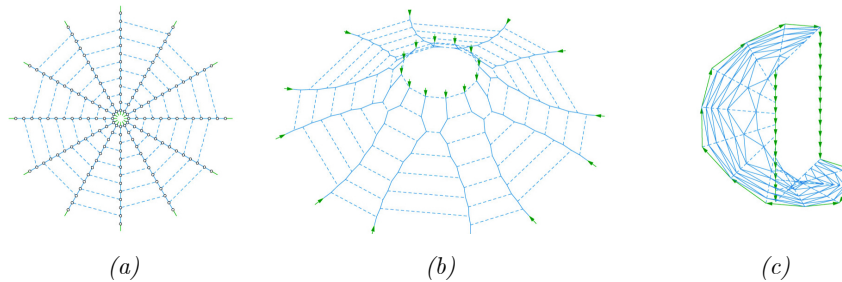


Figure 2.12 - *Alternative equilibrium state after bar removal: (a) topological diagram, (b) form diagram, (c) force diagram [OHLBROCK2016]*

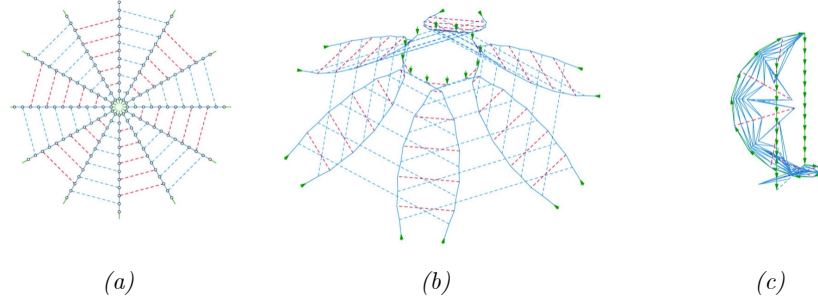


Figure 2.13 – Alternative equilibrium state after behaviour inversion: (a) topological diagram, (b) form diagram, (c) force diagram [OHLBROCK2016]

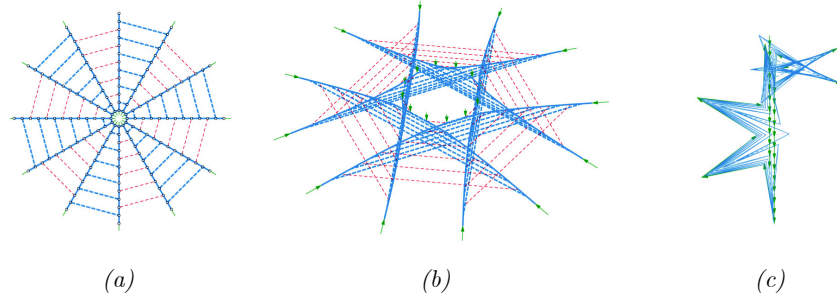


Figure 2.14 – Alternative equilibrium state after force redistribution: (a) topological diagram, (b) form diagram, (c) force diagram [OHLBROCK2016]

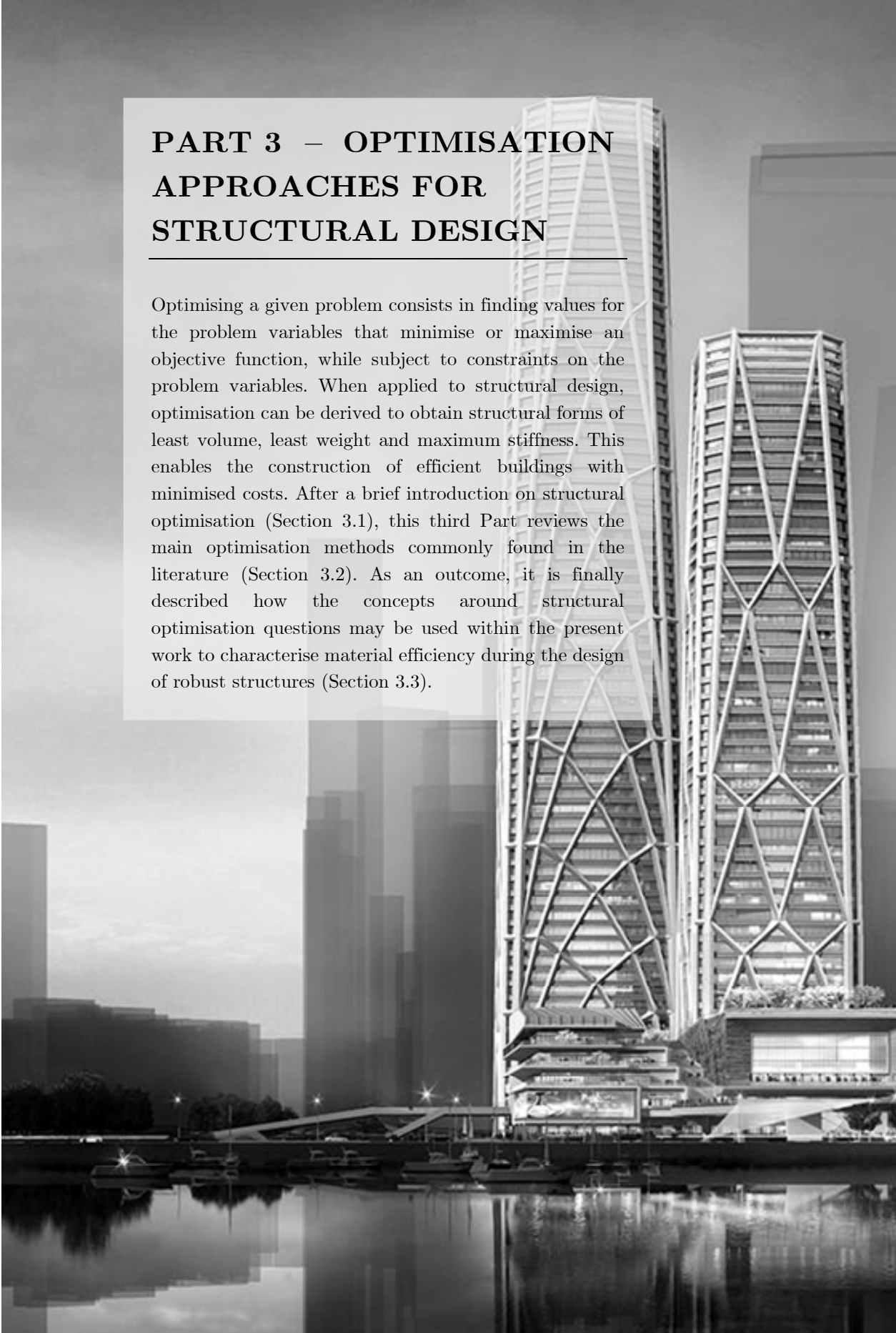
2.5. Outcome

Based on graphic statics and strut-and-tie modelling, constrained solution spaces defined by Fivet [2013a] present the advantages of:

- making visible the structural problem as well as its degrees of freedom;
- allowing the manipulation of the structural parameters (i.e. nodes of graphic statics' reciprocal diagrams) within their range of admissible values.

For these reasons, constraint-based graphic statics represents a powerful development of graphic statics' initial method in the context of structural design. Hence, Fivet's method has been chosen as a source of inspiration to build the bases of the proposed geometrical approach for robustness assessment (see Part 4). This complementary approach to structural robustness is also inspired from the way the design tools developed in Section 2.4 are made interactive, in order to develop a user-friendly tool.

Part 2 has also shown that, within continuous structures, there are infinite ways of building and manipulating strut-and-tie networks modelling structural behaviour. Hence, Part 3 will now detail how some procedures taking inspiration from structural optimisation methods can be employed to assess the pertinence and efficiency of such strut-and-tie networks.



PART 3 – OPTIMISATION APPROACHES FOR STRUCTURAL DESIGN

Optimising a given problem consists in finding values for the problem variables that minimise or maximise an objective function, while subject to constraints on the problem variables. When applied to structural design, optimisation can be derived to obtain structural forms of least volume, least weight and maximum stiffness. This enables the construction of efficient buildings with minimised costs. After a brief introduction on structural optimisation (Section 3.1), this third Part reviews the main optimisation methods commonly found in the literature (Section 3.2). As an outcome, it is finally described how the concepts around structural optimisation questions may be used within the present work to characterise material efficiency during the design of robust structures (Section 3.3).

Figure 3.1 – CITIC Financial Center, Skidmore Owings & Merrill, Shenzhen, China, under construction [SOM]

3.1. Theory and concepts

Consider the truss structure depicted in Figure 3.2, which can represent a discrete structure in itself, or be used to model force trajectories acting within continuous structures (cfr. Section 2.1.2).

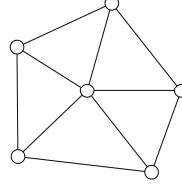


Figure 3.2 – Considered truss structure

Optimising this structure consists in the mathematical problem of minimising or maximising an objective function subject to constraints on the structural response (see Section 3.1.1) [ROZVANY2014]. A structural model also has to be chosen (see Section 3.1.2), before the optimisation problem to be treated with a specific solving method (see Section 3.1.3).

Optimisation is a powerful tool for efficient structural design process [ROTHWELL2017]. It allows to take structural parameters altogether as design constraints (such as stress limitations, stiffness or buckling), while seeking for the best structural layout. Thus the critical design parameters must not be known in advance.

3.1.1. The optimisation problem

For truss structures, the objective function is commonly taken as the structural volume V (Eq 3.1), or the associated weight W (Eq 3.2). These can be minimised using:

$$\min V = \sum_{i=1}^n L_i A_i \quad \text{Eq 3.1}$$

$$\min W = \rho V = \sum_{i=1}^n \rho_i L_i A_i \quad \text{Eq 3.2}$$

with ρ material density, L the length and A the cross section of each individual bar i of the truss. Another optimisation criterion, also referred as the *minimum compliance problem*, consists in minimising deflection δ

(or maximising stiffness $k = P/\delta$, or minimising strain energy $E = P.\delta$) for a given volume of material V [BAKER1992]:

$$\min \delta = \sum_{i=1}^n \frac{p_i P_i L_i}{E_i A_i} \quad \text{Eq 3.3}$$

with L , E and A respectively the length, Young modulus and cross-sectional area of each bar i of the truss, P the internal force in each member i due to the applied external loads, and p the same force due to an external virtual unit load.

Concerning optimisation constraints, these can be as varied as stresses, displacements, buckling loads, member sizes, or cross-section dimensions.

3.1.2. The structural model

Two types of models are commonly employed for the solving of classical optimisation problems: the *bar model* (Figure 3.3a), and the *cell model* (Figure 3.3b).

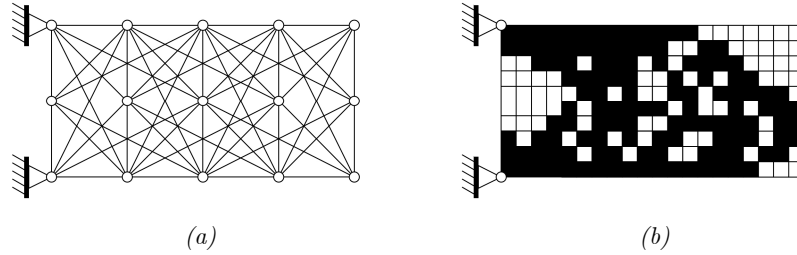


Figure 3.3 – Common models for structural optimisation: (a) bar model, (b) cell model

The bar model (Figure 3.3a) consists in assimilating structural behaviour to a grid of nodes linked by bars. These nodes are said to be highly connected: at first, each node of the grid is joined to the others with a *distance of connection* of d cells (Figure 3.4). This highly connected structure forms the *initial truss*, representing “the discrete version of structural universe” [HAJELA1995]. The initial truss may be considered as a truss structure in itself, or be seen as a strut-and-tie network modelling the path of internal forces within continuous structures (see Section 2.1.2).

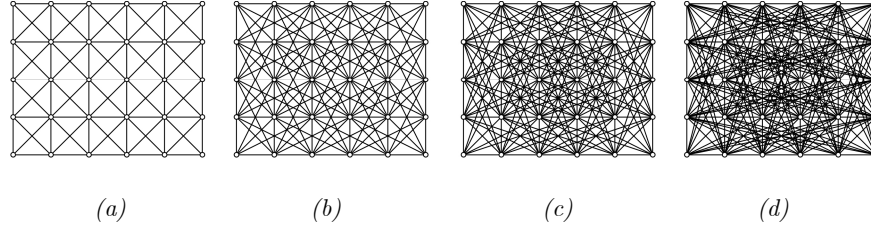


Figure 3.4 – Initial trusses with various distances of connection d : (a) $d=1 \times 1$, (b) $d = 2 \times 2$, (c) $d = 3 \times 3$, (d) $d = 5 \times 4$ (adapted from [ROZVANY2014])

The cell model (Figure 3.3b) consists in transforming the optimisation problem into a material distribution problem within continuous structures [LIANG2001]. Continuous design domains are thus converted into sets of (square) cells, which are all attributed a binary value: plain (black cells in Figure 3.3b) or void (white cells in Figure 3.3b).

3.1.3. The solving method

Consider again the structure of Figure 3.2, which is based a bar model. Optimisation of this structure – using either Eq 3.1, Eq 3.2, or Eq 3.3 as they provide equivalent results [BAKER2012] – may be performed using three different methods [ROTHWELL2017]:

- *sizing optimisation* (Figure 3.5a), which is the easiest optimisation method to deal with. It consists in dimensioning the structure's bars, i.e. giving them the smallest cross section possible $A_{min,i}$ (i.e. the optimisation variables), in order to withstand the applied loads (i.e. the optimisation constraints). The structure is considered as fully stressed, meaning that its constitutive bars are constrained homogeneously throughout all their cross-section with a stress corresponding to the resisting one (see Section 2.1.4). In practice, sizing optimisation may result in choosing an appropriate cross-section from a given book or stock;
- *shape optimisation* (Figure 3.5b), which consists in finding the optimal position of the truss' nodes in space. This type of optimisation method is achieved by changing nodes' x- and y- – and z- for 3D structures – coordinates in space (i.e. the optimisation variables), in order to withstand the applied loads (i.e. the optimisation constraints). Shape optimisation methods are also sometimes called geometry optimisation methods [ROZVANY2014]. However, in the present work, the latter terminology will never be used, as the word “geometry” is already

referring to Euclidian geometry in the context of graphic statics (see Part 2);

- *topology optimisation* (Figure 3.5c), which is achieved by changing bar connections between structural nodes (i.e. the optimisation variables), in order to withstand the applied loads (i.e. the optimisation constraints). This type of optimisation method is said to represent the most complex and challenging one [HAJELA1995]. Indeed, as trusses are discrete structures, the number of possible topological changes for a given scenario is large.

Layout optimisation (Figure 3.5d) is the terminology used to represent the simultaneous optimisation of structural size (dimension of cross sections), shape (nodal positions) and topology (nodal connections).

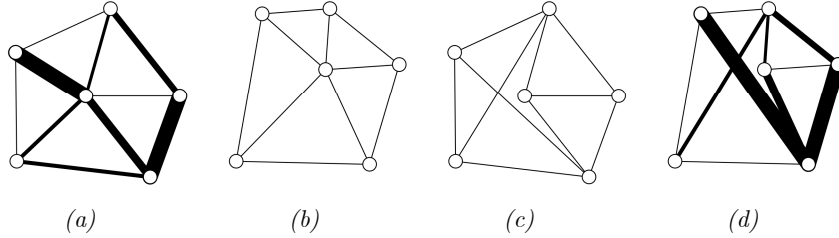


Figure 3.5 – Schematic representation of (a) sizing, (b) shape, (c) topology and (d) layout optimisation problems applied to the structure of Figure 3.2

If representing structural behaviour with a cell model, the optimisation method simply consists in changing the binary value of several cells from plain to void (and reciprocally).

In practice, such optimisation methods are processed through the use of an algorithm, which is a research and comparison procedure of admissible solutions until an optimal one is found [KOSTIC2009]. Most of these algorithms are based on finite element analyses, coupled with mathematical programming techniques [LIANG2001]. The resulting optimal design will then be free of any material part inefficient to carry the applied loads [LIANG2000].

Next section will now review the major optimisation solving methods commonly found in the literature:

- a graphical optimisation method, based on a bar model (Section 3.2.1);
- numerical optimisation methods, based on a bar model (Section 3.2.2) or on a cell model (Section 3.2.3).

3.2. Brief review of optimisation methods

3.2.1. Graphical optimisation method based on a bar model

The premises of structural optimisation were developed in the late 19th and early 20th centuries by J.C. Maxwell [1890] and A.G. Michell [1904]. These first works were of course purely geometrical, as computers had not been invented yet. Structural optimisation was therefore implemented through graphic statics' reciprocal diagrams (see Section 2.2).

Consider the structure depicted in Figure 3.6. Figure 3.6a gives a set of external loads P_i in equilibrium, as well as their relative position d_i in both x- and y- directions from the point of application of load P_2 . Figure 3.6b gives a possible strut-and-tie network supporting the external loads P_i . Figure 3.6c is the reciprocal force diagram associated to the network. L represents bar lengths in the form diagram. F then gives these lengths in the force diagram, i.e. the magnitude of the associate axial force. c indices correspond to struts, and t 's to ties.

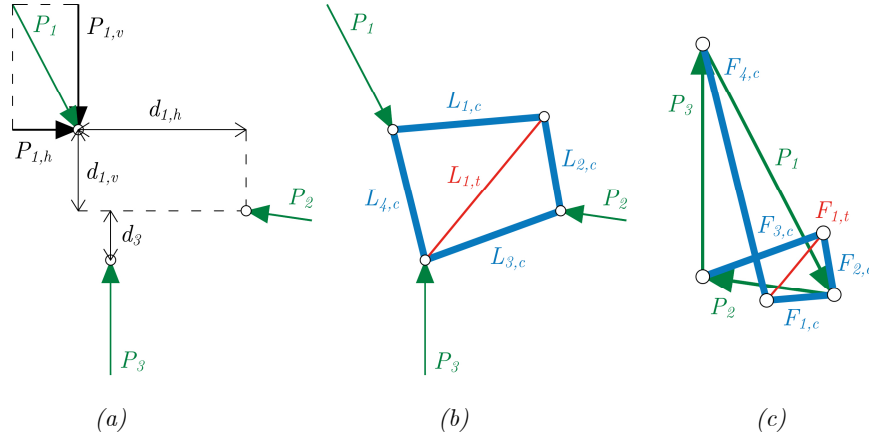


Figure 3.6 – (a) System of loads in equilibrium, (b) form diagram depicting one possible strut-and-tie network of internal forces, (c) corresponding force diagram

Maxwell's theorem then states that:

“In any system of points in equilibrium in a plane under the action of repulsions and attractions, the sum of the products of each attraction multiplied by the distance of the points between which it acts, is equal to the sum of the products of the repulsions multiplied each by the distance of the points between which it acts.” [MAXWELL1890]

When translated in terms of graphic statics' reciprocal diagrams, this theorem can be turned into the following formulas [MICHELL1904]:

$$\sum_i L_{t,i} F_{t,i} - \sum_j L_{c,j} F_{c,j} = \sum_k \vec{P}_k \cdot \vec{d}_k = \text{constant} \quad \text{Eq 3.4}$$

$$\begin{aligned} \sum_i L_{t,i} F_{t,i} + \sum_j L_{c,j} F_{c,j} &= \sum_i L_{t,i} A_i \frac{F_{t,i}}{A_i} + \sum_j L_{c,j} A_j \frac{F_{c,j}}{A_j} \\ &= \sum_i V_{t,i} \sigma_{t,i,e} + \sum_j V_{c,j} \sigma_{c,j,e} = V_t \sigma_{t,e} + V_c \sigma_{c,e} \\ &= V \sigma_e \quad (\text{if } \sigma_{t,e} = \sigma_{c,e}) \end{aligned} \quad \text{Eq 3.5}$$

with A_i the section that has to be given to each individual bar i in order to be fully stressed by a constraint σ_e , which represents material's ultimate limit strength supposing an idealised ductile behaviour (see Section 2.1.4). $L_{t,i} F_{t,i}$ is called the *load path* in tension, and $L_{c,i} F_{c,i}$ the *load path* in compression.

Eq 3.4 states that the difference between the load path in tension and the load path in compression is equal to a value which depends on the intensity and position of external loads only. Hence, this value will remain constant as long as the external loads are remained unchanged. This means that the result of Eq 3.4 is independent of the strut-and-tie network drawn to support the external loads [MICHELL1904], as well as of the chosen structural material(s).

Eq 3.5 then states that the sum of the load paths in tension and in compression – the total load path – is directly proportional to the quantity of material required to build the corresponding strut-and-tie network, hence to the weight of this network. In terms of graphic statics, assuming a fully stressed design, this means that any structure can be optimised by finding nodal positions and bar connections that minimise the product of the length of reciprocal pairs of bars. As these lengths are the only parameters required for optimisation, the result of applying Eq 3.5 to a given strut-and-tie network is also independent of the material chosen for its implementation.

Note that Eq 3.5 does not take buckling into account. Indeed, bar elements are employed in the present work to represent resultant trajectories of internal forces acting within continuous structures (see Section 2.1). Hence, buckling will not represent a preponderant failure mode, and can be neglected here. However, Eq 3.5 could have been weighted if buckling of

struts were to be considered, thus favouring short compressive members [NIELSEN2017, TRUSSPATH].

Theoretical applications of the geometrical optimisation process have been developed by Michell [1904], for the design of minimum weight trusses subject to a single point load. The *Michell truss* (Figure 3.7) represents the least volume design of a structure subject to a vertical point load at one end and clamped at the other.

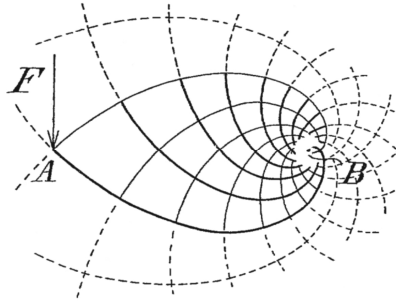


Figure 3.7 – Optimal truss layout under single point load [MICHELL1904]

The Michell truss is composed of two series of curves that cross each other at right angles. Even if this optimal truss design corresponds in practice to an infeasible result, the solution developed by Michell was a keystone in the field of optimal structural layout development [LIANG2001]. Moreover, the shape of the Michell truss corresponds to the flow of forces that would occur within continuous structures if subjected to the same loading scheme [ZALEWSKI1975, 1996].

Practical applications of the geometrical structural optimisation method have recently been extensively studied by the members of *Skidmore, Owings & Merrill* [SOM], a renowned architecture and engineering office based in Chicago, USA. Members of SOM are using Maxwell's equations to develop optimal geometries for the design of real buildings, such as least weight cable structures, least weight bracing systems for high-rise buildings (Figure 3.1), or for the spanning system of the *Exchange House* in London, UK (Figure 3.8). SOM's work has led to many publications, such as [BAKER1992, BEGHINI2011, BAKER2012, MAZUREK2012, STROMBERG2012, BAKER2013, BEGHINI2013a and 2013b, MAZUREK2013, BEGHINI2014a, 2014b and 2015].



Figure 3.8 – SOM’s Exchange House in London [SOM]

3.2.2. Numerical optimisation methods based on a bar model

The bases for numerical optimisation of bar models have been developed by Dorn & al. [1964]. To overcome the impracticalities of Michell trusses, the authors proposed a *Ground Structure Method* (GSM), which was originally developed for the topology optimisation of steel structures. The method is based on a bar model with a $H \times L$ grid of nodes of fixed position (Figure 3.9), which is then optimised using the minimal weight criterion as detailed in Eq 3.2.

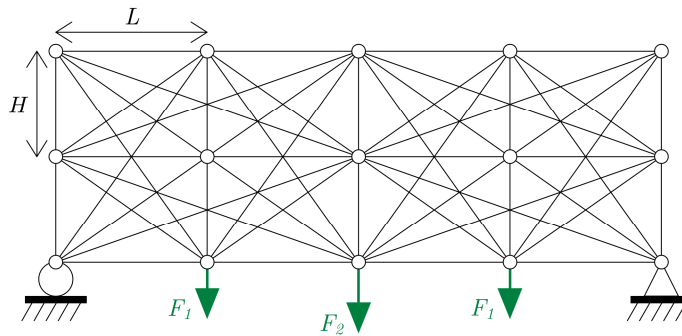


Figure 3.9 – Initial truss with two simple supports and three point loads applied at the bottom chord (adapted from [DORN1964])

As the truss’ nodes are meant to remain of fixed position by hypothesis, optimisation is only performed on the size and topology of the truss. In practice, this consists in iteratively removing the non-optimal or less

constrained bar elements of the initial structure (while guaranteeing its equilibrium), and sizing the remaining bars. Results of the optimisation process give truss topologies with minimal weight W for different ratios between the applied loads (F_1/F_2 in Figure 3.10) and the structural dimensions (H/L in Figure 3.10).

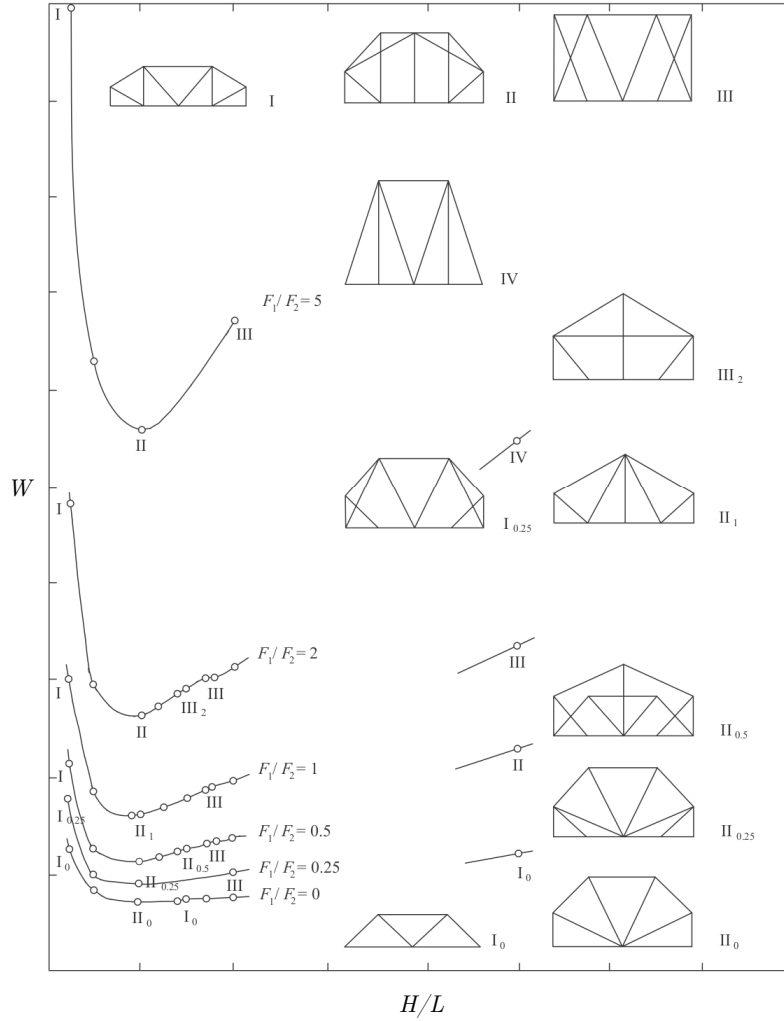


Figure 3.10 – Optimal truss topologies in terms of weight W for various force and dimension ratios [DORN1964]

A main drawback of the Ground Structure Method lies in the fact that the result of the optimisation procedure highly depends on the dimensions H and L of the grid of nodes, as well as on the distance of connection d that has been considered for the bars (see Section 3.1.2) [ROZVANY2014].

Indeed highly refined initial trusses, with small values for H and L and large values for d , will give better results in term of resultant optimisation. However, initial problems with those characteristics will necessitate larger calculation time and power for the optimal solution to be reached. Another limitation of the method lies in the discreteness of the initial truss in itself: as the optimal layout cannot be found by a continuous treatment of the design variables, the resultant solution may not correspond to the global minimum [HAJELA1995]. Finally, the method only accounts for bar removal, as the reintroduction of a deleted element in the structure is not made possible.

Weight optimisation of a structure modelled by bar elements has been further studied by Kirsch [1990], using linear programming solving methods. Similarly to Dorn & al's work, Kirsch defines a *fixed geometry problem* (FGP), which consists in optimising bar connectivity (topology) and cross-section (sizing) for fixed nodal locations (shape). He then details a *fixed topology problem* (FTP), allowing the optimisation of nodal positions (shape) and bar cross-sections (sizing). Applying successively FGP and FTP to the initial truss then allows to find *multiple optimal topologies*, which represent the set of local solutions to the general optimisation problem. These topologies can either correspond to statically determinate or indeterminate structures.

Hajela & Lee [1995] have proposed to combine a *Genetic Algorithm* with the Ground Structure Method in a new search procedure. Assuming a fully connected initial truss, Eq 3.1 can be rewritten as:

$$\min V = \sum_{i=1}^n t_i L_i A_i \quad \text{Eq 3.6}$$

with variable t_i accounting for the presence ($t_i = 1$) or absence ($t_i = 0$) of each bar element. This new problem formulation allows to take stresses, displacements and buckling constraints into account. The associated method performs successive topology and sizing optimisations, while accounting for removal but also addition of bars. Hence, the procedure allows to overcome the dimensional problems of Ground Structure Method's initial truss, which were due to the intrinsic discontinuous nature of the considered design space.

Another development of the Ground Structure Method has been introduced by Xu & al. [2003]. Topology and sizing optimisation of structural weight for truss structures with distance of connection $d=1$ have been coupled to

partial shape optimisation, allowing nodes on a same column or row to move accordingly in the x- or y- direction. Depending on the studied structural problem, this can result in the suppression of some columns or rows of cells from the initial truss, in order to reach an alternative optimal truss topology (Figure 3.11).

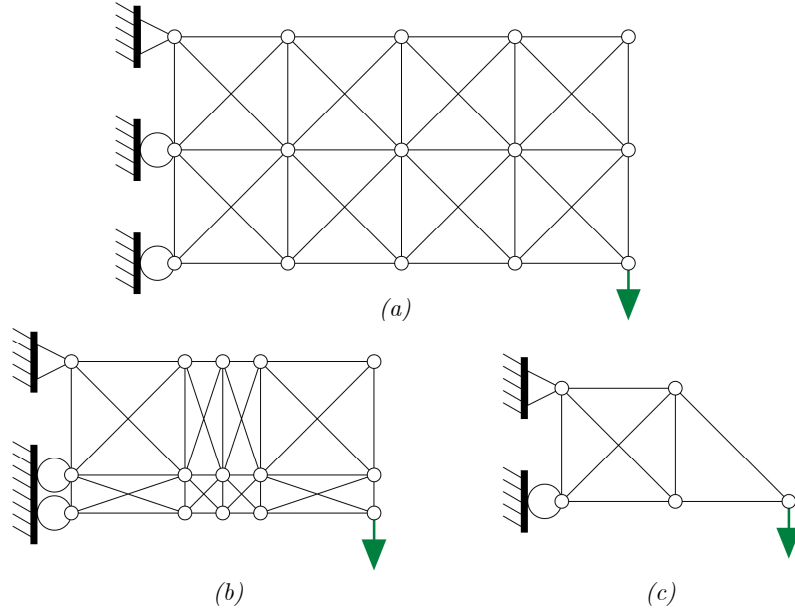


Figure 3.11 – Alternative Ground Structure Method: (a) initial truss, (b) optimisation process, (c) result of optimisation (adapted from [XU2003])

Another approach to structural optimisation is to consider structural nodes as vertices and bars as edges, in the sense of Graph Theory [WEST2001]. Topological graphs can then be combined to the Ground Structure Method to reduce the complexity of the optimisation problem [KAWAMOTO2004].

An extension of the concepts introduced within the Ground Structure Method has been proposed by Latteur [2000, 2016]. As there often exist numerous parameters that have to be taken into account when dealing with optimisation problems, Latteur proposed to combine these parameters into dimensionless *morphologic indicators* in order to ease the optimisation process. These indicators depend on:

- structural shape, obtained from the structure's height and length;
- cross-sectional shape, obtained from section area, inertia and height;

- a combination of several parameters, that gives indicators such as the buckling indicator, the self-weight indicator, the volume indicator or the displacement indicator.

Achtziger [2007] has proposed another extension of the Ground Structure Method, allowing simultaneous topology and shape optimisation of 2D structures subjected to a single load case. As considering nodal positions as design variables consequently turns the optimisation process into a highly nonlinear problem (and so computationally expensive), optimisation is here based on a sparse initial truss. Deformation energy $f^T u$ is then minimised according to a fixed volume constraint:

$$\begin{aligned} & \min (f^T u) \\ \text{supposing} \quad & K(a)u = f \\ & \sum_{i=1}^n L_i A_i \leq V \end{aligned} \tag{Eq 3.7}$$

with f the vector of external loads, a the vector of cross-sectional areas, u the vector of bar elongations, K the stiffness matrix of the structure in global reduced coordinates, L the length and A the cross-sectional area of bar i , and V the total structural volume.

The *Growing Ground Structure Method* (GGSM) has been developed by Hagishita & al. [2009]. A set of rules, called *growth strategies*, have been added to the original Ground Structure Method. Using linear programming, these rules allow the adding of new bars through the optimisation process, making the final result less dependent on structural parameters H , L and d . For this purpose, five growth strategies have been developed by the authors:

- *GSTRG1*, for the addition of a single bar. The strategy first finds *candidate bars*, i.e. bars that do not already exist between a pair of nodes. The potential strain is calculated for each candidate bar; the one having the largest potential strain is chosen as those to be added. This strategy is said to require very small computational cost, as nodal displacements for the new truss design do not have to be calculated. Hence, no additional structural analysis is needed for a bar to be added;
- *GSTRG2*, which is also used for the addition of a single bar. In this strategy, the displacement of the truss after the addition of a single bar is calculated for every candidate bar. The one leading to

the minimal displacement is chosen as those to be added;

- *GSTRG3*, which works in the same way that *GSTRG2*, but for which the objective function is minimising the weight of the truss instead of its displacement;
- *GSTRG4*, which is used for the removal of a single bar. The existing bar that decreases the less the effectiveness of the truss while deleted is chosen as the one to be removed from the truss;
- *GSTRG5*, which is used for the addition of a new node as well as of j new bars that connect this new node to the existing ones. The number and location of these j new bars is chosen as to maximise the efficiency of the new truss design.

These five growth strategies can be combined to generate three different optimisation algorithms, which all account for volume minimisation but which are used alternatively, depending on the chosen design constraints. These constraints are chosen as the cross-sectional areas under a single applied load in *Algorithm 1*, the cross-sectional areas and the nodal coordinates under a single applied load in *Algorithm 2*, and the stress constraints considering multiple load cases in *Algorithm 3* (Figure 3.12).

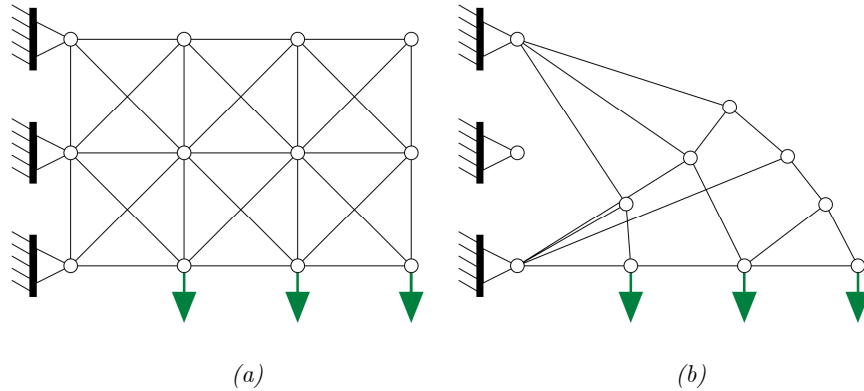


Figure 3.12 – Growing Ground Structure Method: (a) initial truss, (b) result of optimisation using Algorithm 3 considering fixed left and bottom nodes (adapted from [HAGISHITA2009])

Extensions of the Ground Structure Method have been implemented for the design of continuous reinforced concrete structures. The method of Biondini & al. [1996] accounts for a fully connected initial truss subject to a minimum volume objective. Optimisation is then performed using a corrective coefficient favouring bars of shortest length. In the same vein, Ali & White [1997, 2001] have proposed an adapted Ground Structure

Method for reinforced concrete design, in which the objective function is the minimisation of rebar quantity. For this purpose, an additional coefficient has been added to the original method to promote horizontal and vertical rebars while minimising their length.

The approach developed by Kostic [2009] for optimal concrete design is based on the combination of an initial truss with non-linear finite element analyses. The structure is modelled with a distributed orthogonal rebar layout that undertakes tensile forces. This rebar layout defines a grid of square elements cut into four triangular finite elements, in which compressive fields may take place (Figure 3.13a). An objective function derived from Eq 3.3 then allows the minimisation of the initial stress field in terms of deformation energy, while respecting compatibility of displacements (Figure 3.13b). The resultant optimal continuous stress field is then transformed into a strut-and-tie network (Figure 3.13c). Kostic's method also accounts for reaching near-optimal topologies, used to better describe the real behaviour of concrete structures within which internal forces tend to occupy the whole continuous structural geometry.

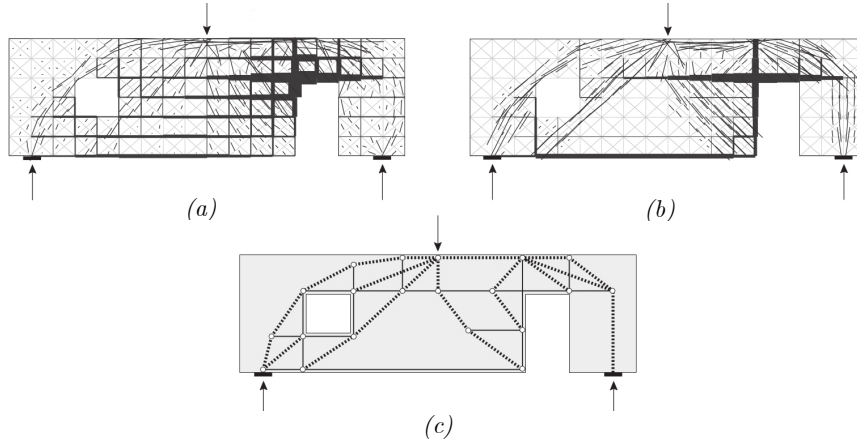


Figure 3.13 – Optimisation of reinforced concrete structures: (a) initial stress field, (b) optimal stress field, (c) corresponding strut-and-tie network
[KOSTIC2009]

3.2.3. Numerical optimisation methods based on a cell model

When implemented within a structural model made of cells, the *Genetic Algorithm* method (GA) introduced by Holland [1975] consists in converting the design domain composed of plains and voids (Figure 3.14a) into a binary domain (Figure 3.14b). This is simply done by giving a value

of 1 to cells corresponding to material elements, and 0 for the ones corresponding to voids. This binary domain is then finally turned into a linear binary string, as seen on Figure 3.14c.

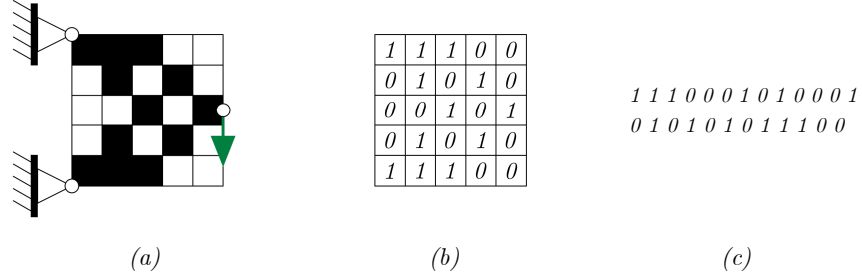


Figure 3.14 – Structural optimisation using cell elements: (a) cell design domain, (b) binary design domain, (c) associated binary string (adapted from [JAKIELA1999])

The binary string resulting from this procedure can be assimilated to a chromosome, which gives the structure’s genetic information [JAKIELA1999]. Optimisation is then performed through an evolutionary computational method, in which the chromosomes representing the design domain are associated to organisms involved in a process of natural selection (principle of *survival for the fittest*). The idea is to breed two *parent designs* with favourable characteristics, by manipulating their associated chromosomes. This generates two *offspring designs* that have fitter characteristics than their parents [HAJELA1995]. The best offspring designs are then iteratively selected to become the next generation of parents, thus improving the resultant structural design while guaranteeing diversity. For the evolution process to occur, i.e. for the structural optimisation process to be performed, two kinds of genetic manipulations on parent designs are proposed by Jakiela & al. [1999]:

- crossover, in which offspring is obtained from a combination of two parent genes. In practice, this consists in swapping parent chromosomes at one or more locations, to obtain two new design possibilities that have conserved traits of their parents:

Parent chromosomes	1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
Crossover operation	1 1 1 1 1 1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
Resulting offspring chromosomes	1 1 1 1 1 1 1 1 1 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1 1 1

- mutation, in which offspring is obtained by changing the value of one or more gene of a parent design:

<i>Parent chromosome</i>	1 1 0 0 1 0 1 1 1 0 0 1 1 1 0
<i>Resulting offspring chromosome</i>	1 1 0 1 1 0 1 1 1 0 0 1 1 0 0

The optimal “structural chromosome” is obtained by iteratively applying crossover and mutation operations to the initial design domain, selecting at each step i the most performant offspring designs that will become the parent designs at step $i+1$. However, results of genetic-based optimisation process also depend on the size of the cells into which the design domain is discretised (Figure 3.15). This discretisation will also influence computation time.

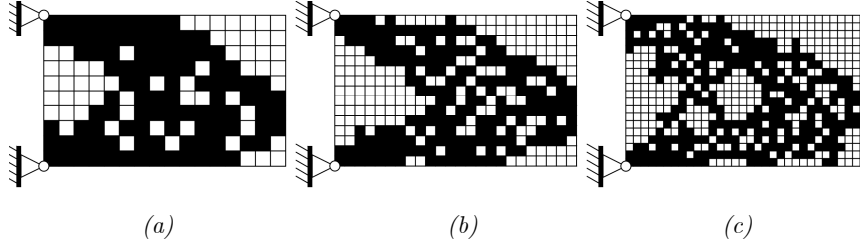


Figure 3.15 – Results of the genetic optimisation process for various design domain discretisation: (a) 16x10 cells, (b) 24x15 cells, (c) 32x20 cells (adapted from [JAKIELA1999])

Bendsøe & al. [1988, 1995] have developed the *Homogenization-Based Optimisation* method (HBO), based on the optimisation of structural models made of cells. This method turns the layout optimisation problem of a continuous structure into a sizing problem only. It is based on a relaxation process, considering optimisation as the cutting of small holes within a structure initially continuous. Compliance is taken as the objective function according to Eq 3.3, while a constraint is fixed on material volume. The method is said to propose a very good solution to the optimisation problem, but its very complex mathematical formulation may be difficult to be apprehended by practitioners [LIANG2001]. Furthermore, as only material volume has been considered as a problem constraint, the final result of the optimisation process directly depends on the particular value chosen for this volume, and is therefore not optimal for other values.

An *Evolutionary Structural Optimisation* method (ESO) was also proposed by Xie & al. [XIE2005, HUANG2010], for the optimisation of 2D and 3D structures subjected to multiple loadings. The structure to be analysed is

first meshed using square (in 2D) or cubic (in 3D) elements. A finite element analysis is then performed to obtain the stress associated to the von Mises criterion acting within each discrete element. All the elements within which the acting stresses are smaller than a given threshold are then removed from the initial domain. Alternatively, new cell elements can be added near elements of high sensitivity. The method accounts for the design of pure tensile or compressive shapes, and allows structural optimisation under multiple constraints (typically minimising compliance under volume but also displacement constraints).

Talischi & al. [2012a, 2012b] have proposed to discretise the initial design domain into polygonal cell elements instead of square ones. This allows to overcome numerical instabilities as well as meshing complex structures. Using *PolyMesher*, a mesh generator written in Matlab, the design domain is first discretised into an *unstructured mesh* composed of Voronoi tessellations. Another Matlab code called *PolyTop* then performs optimisation by simply solving a distributed sizing optimisation problem.

Liang [2000, 2001] has applied cell discretisation to the optimal topology design of reinforced concrete structures. Liang believes that this technique better models the continuous nature of concrete members than ground structures. The objective function is taken as minimisation of displacements (Eq 3.3), as concrete structures only allow limited plastic deformation. Liang developed a *Performance-Based Optimisation* method (PBO), in which the structure's performance index is calculated at each iteration by scaling the thickness of each cell element in order to obtain the efficiency of the associated material layout. Elements with the lowest virtual strain energy are then removed from the design. Once the optimal topology is found, it is turned into a strut-and-tie network that can be sized using classical Eurocode methods.

3.3. Outcome

Part 3 has briefly detailed some concepts of structural optimisation (the optimisation problem, the structural model and the solving method), which algorithms will further be used within the present work.

Concerning structural models, it has been shown that these may classically result in discretising the structure into cells, as commonly implemented within finite element models. But this may also mean assimilating

structural behaviour to a mesh of bar elements, i.e. a strut-and-tie network that may be drawn within graphic statics' interface (cfr. Part 2).

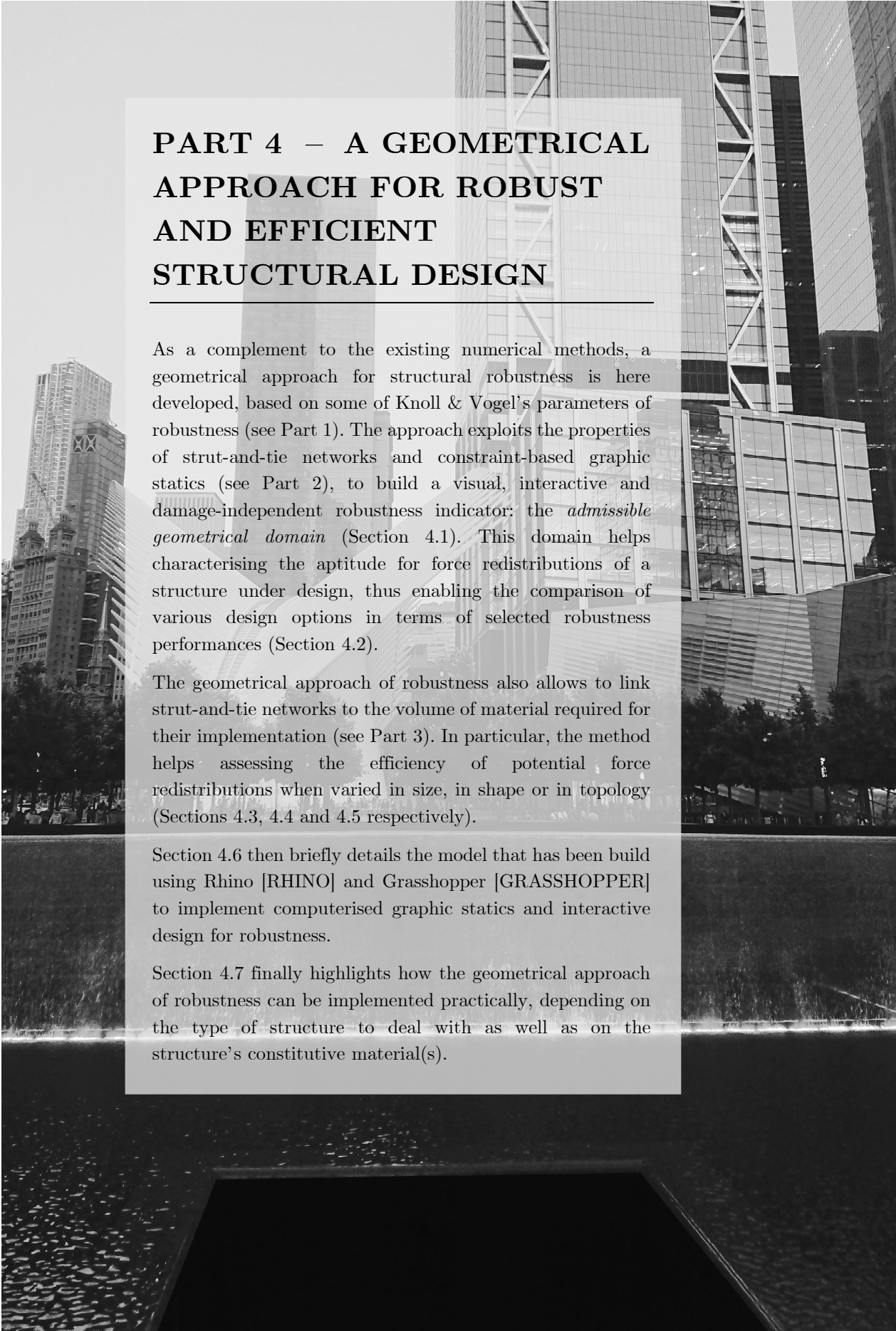
The volume of such a network (i.e. the optimisation problem) may then be easily calculated from graphic statics' reciprocal diagrams, using Maxwell [1890]'s theorem of load paths. Load path values give an image of the efficiency associated to given strut-and-tie networks. Hence, these values may help characterising the volume of material that is required within the strut-and-tie network's bars in order to carry the applied loads.

The total load path value may finally be varied graphically by applying the following solving methods: modifying the strut-and-tie network's size and/or shape and/or topology. These procedures may help finding the truss layout that minimises the quantity of material required for structural implementation (or, when using strut-and-tie networks to model structural behaviour within continuous elements, find the most efficient structural behaviour, i.e. the elastic solution). Load path assessment may also allow ranking various truss solutions regarding efficiency in terms of the necessary volume of material (or, within continuous elements, rank admissible plastic force distributions according to their distance to the elastic solution).

Now that:

- the concept of structural robustness has been introduced (Part 1);
- the models that will be used throughout the present work to represent stress states acting within continuous structures have been presented, as well as a way to manipulate these stress states graphically (Part 2);
- the methodology that will be employed here to assess the efficiency of such stress states in terms of the volume of material required for their implementation has been highlighted (Part 3);

Part 4 will now develop how particular parameters of structural robustness may be assessed through a geometrical approach based on graphic statics. This approach implements sets of strut-and-tie networks representing possible force distributions acting within given structures. Moreover, Part 4 also details how the efficiency of such force distributions can be calculated from the associated strut-and-tie network using Maxwell [1890]'s theorem of load paths.



PART 4 – A GEOMETRICAL APPROACH FOR ROBUST AND EFFICIENT STRUCTURAL DESIGN

As a complement to the existing numerical methods, a geometrical approach for structural robustness is here developed, based on some of Knoll & Vogel's parameters of robustness (see Part 1). The approach exploits the properties of strut-and-tie networks and constraint-based graphic statics (see Part 2), to build a visual, interactive and damage-independent robustness indicator: the *admissible geometrical domain* (Section 4.1). This domain helps characterising the aptitude for force redistributions of a structure under design, thus enabling the comparison of various design options in terms of selected robustness performances (Section 4.2).

The geometrical approach of robustness also allows to link strut-and-tie networks to the volume of material required for their implementation (see Part 3). In particular, the method helps assessing the efficiency of potential force redistributions when varied in size, in shape or in topology (Sections 4.3, 4.4 and 4.5 respectively).

Section 4.6 then briefly details the model that has been build using Rhino [RHINO] and Grasshopper [GRASSHOPPER] to implement computerised graphic statics and interactive design for robustness.

Section 4.7 finally highlights how the geometrical approach of robustness can be implemented practically, depending on the type of structure to deal with as well as on the structure's constitutive material(s).

*Figure 4.1 – Reflecting Absence (9/11 Memorial), Michael Arad, New York,
USA, 2011 [picture by the thesis author]*

4.1. Preliminary hypotheses

A geometrical approach for partial robustness assessment is here developed as a complement to those of Section 1.3. Based on graphic statics instead than on numerical analyses, this approach presents the advantage to be damage independent, as well as to provide a direct and visual insight on several robustness properties at the earliest steps of the design process. Such robustness properties have then been selected among those of Section 1.2, for their link with questions of structural design as well as for their capacity to be easily represented by geometrical design constraints within graphic statics' interface (see Section 2.3):

- *multiple load paths and redundancy*, emphasizing aptitude for force redistributions. This parameter assesses that several load paths can be activated in a structure: if one is disabled due to some damage, the structure has the possibility of finding other ones to keep on withstanding the applied loads. The extent to which alternative load paths may be mobilised within the structure's geometry can be represented within graphic statics' force diagram by *Laterality* constraints, related to envelope conditions represented by dashed lines E_i in Figure 4.2, right. Such lines give the limits within which force diagram's mobile nodes (highlighted in red by convention) have to stay for form diagram's associated strut-and-tie network (Figure 4.2, left) to remain within the given structural limits;
- *ductility*, by supposing an idealised material behaviour (see Section 2.1.4). Ductility is a necessary parameter to allow force redistributions (i.e. nodal displacements within graphic statics' reciprocal diagrams), and so to implement multiple load paths and redundancy. In such conditions, the Lower Bound Theorem of Plastic Theory may be applied. This places the proposed geometrical approach of robustness on the safe side;
- *strength*, which defines the value of material effective strength σ_e through the use of reduction factors (see Section 2.1.4). In the context of constraint-based graphic statics (see Section 2.3), this strength parameter may be turned into a *Proximity* constraint. Such constraints are depicted by the dashed circular boundaries C_{ni} in Figure 4.2's force diagram, whose radius equals material admissible force F_{max} (see Eq 2.2 in Section 2.2). Constraining magnitudes of axial forces F_i in the force diagram below their maximum value F_{max} thus ensures that yield is not reached in any part of the structure.

Following these, and in the light of the conditions given in Section 2.1.4, the following hypothesis will be taken into account in the development of the present work for assessing structural robustness geometrically:

“Given that its constitutive materials present sufficient properties of ductility and strength, a structure’s level of robustness may be assessed during its design from the point of view of force redistributions. The amplitude of such force redistributions may be described within graphic statics’ force diagram using admissible geometrical domains.”

Next Section will now detail how admissible geometrical domains can be obtained from Fivet [2013]’s solution spaces.

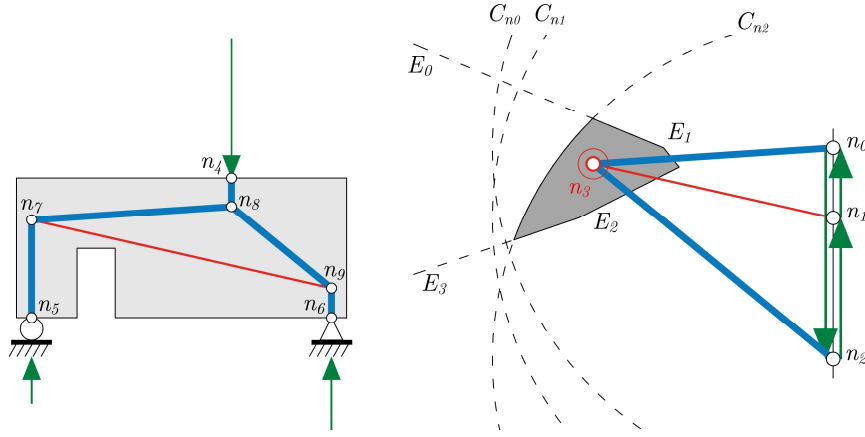


Figure 4.2 – Admissible geometrical domain representing the set of plastic solutions to the structural problem

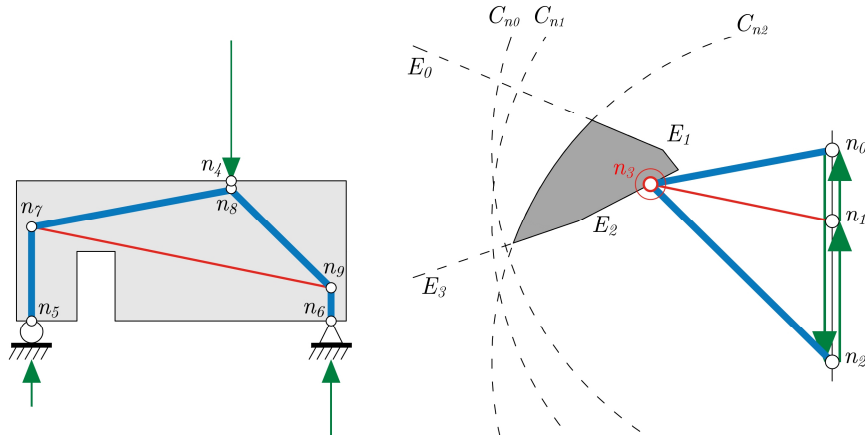


Figure 4.3 – Another possible force distribution within the domain

4.2. Building admissible geometrical domains

When turned into graphical constraints, the set of robustness properties selected in Section 4.1 gives geometrical representations of solution spaces linked with admissible force distributions within the structure. For example, Figure 4.2 and Figure 4.3's reciprocal diagrams depict two different strut-and-tie networks respecting the same set of structural constraints E_i and C_{ni} . Hence, these networks represent two admissible force redistributions within the depicted structure.

The set of admissible force redistributions associated to the pictured strut-and-tie network is represented by the dark grey area in both Figure 4.2 and Figure 4.3's force diagrams. In practice, these solution spaces represent the set of positions that can be taken by force diagram's mobile node n_3 (highlighted in red to indicate mobility), with the other nodes of the force diagram remaining fixed. These allowable positions are such that form diagram's associated strut-and-tie network is valid, i.e. it respects the conditions of Section 2.1.4. These conditions lead that nodes n_7 , n_8 and n_9 may be moved along a vertical direction only, while the other nodes of the form diagram remain fixed (unless modifying external load configuration).

In the context of robustness assessment, solution spaces associated to force diagram's mobile nodes are called *admissible geometrical domains*. Such domains are intuitively supposed to give the set of statically admissible plastic solutions to the structural problem. Going back to Figure 4.2 and Figure 4.3, the pictured strut-and-tie networks represent two among the set of statically admissible force distributions within the depicted structure. Hence, both the associated stress states correspond to alternative load paths that can be implemented within the structure in case of unexpected damage. In contrast to the existing robustness approaches developed in Section 1.3, potential damage must not be presupposed through the proposed geometrical approach of robustness. Indeed, it is here considered that the largest the size of the admissible geometrical domain, the more admissible force redistributions, the more robust the structure. Hence, the approach helps comparing some dimensions of robustness for various structural scenarios, by the size of their relative admissible geometrical domain.

In this perspective, the proposed geometrical approach of robustness can further be used by practitioners to gain insight on the redistribution properties of the design they are currently working on. In particular, the approach helps the comparison of design alternatives, as well as the

orientation of design choices during early conception. For example, Figure 4.4 shows the impact of Figure 4.2's door displacement on the size of the admissible geometrical domain associated to force diagram's mobile node n_3 . As it possesses a larger admissible geometrical domain, the design of Figure 4.2 should then be favoured in order to maximise the possibility of force redistributions within the structure, hence to enhance its level of robustness. Note that the relative shape and position of the domain in space may also be used as indicators of structural efficiency (see Section 4.4).

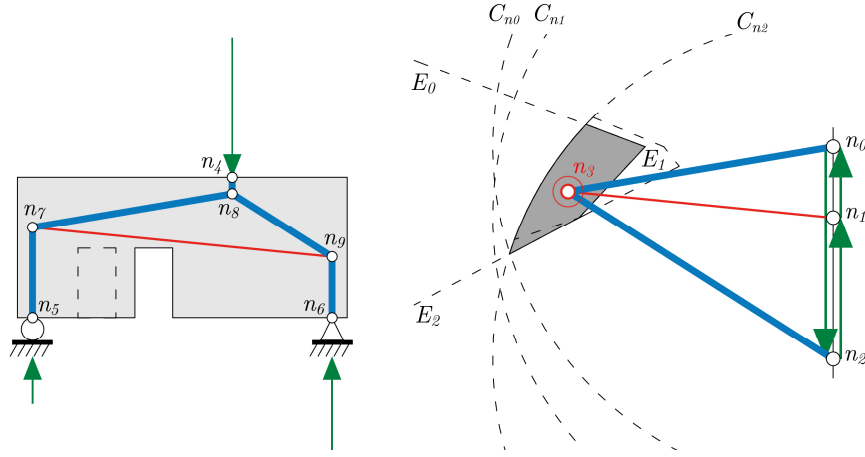


Figure 4.4 – Admissible geometrical domain corresponding to a design alternative

The proposed geometrical approach for robustness assessment partially answers the two main limitations of the existing robustness approaches (see Sections 1.4 and 1.5). Indeed, although it only involves some of the dimensions of structural robustness assessment (see Section 4.1), the proposed approach is damage-independent, and it may already be employed for early structural design. Admissible geometrical domains, which are the indicators of robustness associated to the approach, help to assess and compare the robustness properties of various design options in terms of possibilities of finding alternative load paths.

It has been shown how the issue of structural robustness (see Part 1) can be associated to graphic statics and geometrical thinking (see Part 2), to develop a complementary approach to structural robustness based on a graphic indicator: the admissible geometrical domain. It has been developed how this approach can be used to gain insight on the structure's performance in terms of potential force redistributions while seeking for its design. The following sections will now detail how the proposed geometrical

approach of robustness can be crossed with notions of structural optimisation (see Part 3), in order to gain insight on the efficiency of each potential force redistribution in terms of the volume of material required for its implementation. In particular, next sections will detail how the three types of optimisation methods can be related to the geometrical assessment of structural robustness: Section 4.3 deals with sizing variation, Section 4.4 with shape variation, and Section 4.5 with topology variation.

4.3. Admissible geometrical domains and sizing variation

Implementing sizing optimisation in a given structure is an easy process. In a geometrical thinking, this simply consists in turning the strut-and-tie network describing the structure's internal stress state into a discontinuous stress field (see Section 2.1.3). This is done by giving physical dimensions to the strut-and-tie network, i.e. giving the network's bars i an area A_i which is proportional to the axial force F_i they withstand (see Eq 2.1 in Section 2.1.4). For example, Figure 4.5a gives a possible strut-and-tie network within the previously considered structure. In Figure 4.5b, this network has been turned further into a discontinuous stress field.

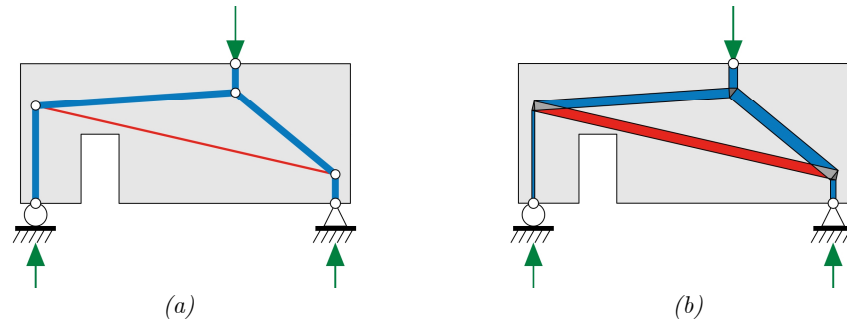


Figure 4.5 – Sizing optimisation: (a) original strut-and-tie network, (b) associated discontinuous stress field

When considering a hypothesis of fully-stressed design (see Section 2.1.4), the size of the discontinuous stress fields is also inversely proportional to material effective strength σ_e . Hence, for a given strut-and-tie network, the size of the corresponding discontinuous stress field may be varied by changing the value of σ_e , with this size decreasing with the increase of σ_e . Increasing σ_e will also affect the size of the admissible geometrical domain

in the force diagram. As σ_e also serves to derive the radius F_{max} of the circular boundaries of the domain (see Section 4.1), increasing the structure's level of robustness by oversizing (i.e. by providing extra strength $\Delta\sigma_e$) then enlarges the radius of the domain's circular boundaries.

4.4. Admissible geometrical domains and shape variation

To combine admissible geometrical domains with graphic shape variation, each possible force distribution may be associated to its load path value. These values, obtained from Eq 3.5 of Section 3.2.1, give the efficiency of the corresponding strut-and-tie networks in terms of required quantity of material. As each possible strut-and-tie network is associated to a particular position of node n_3 within its admissible geometrical domain, each point of the domain may then be correlated to its load path value. As depicted on Figure 4.6, these values can further be converted into a colour scheme, which respects the following convention: the smaller the load path value, the lighter the associated colour (yellow). Reciprocally, points associated to the largest load path values are depicted with darkest colours (blue). The load path assessment of each point of the domain then gives a visual result of the efficiency of the corresponding force distributions in terms of the volume of material required for their implementation. Hence, the method helps ranking solutions to the structural problem in terms of performance.

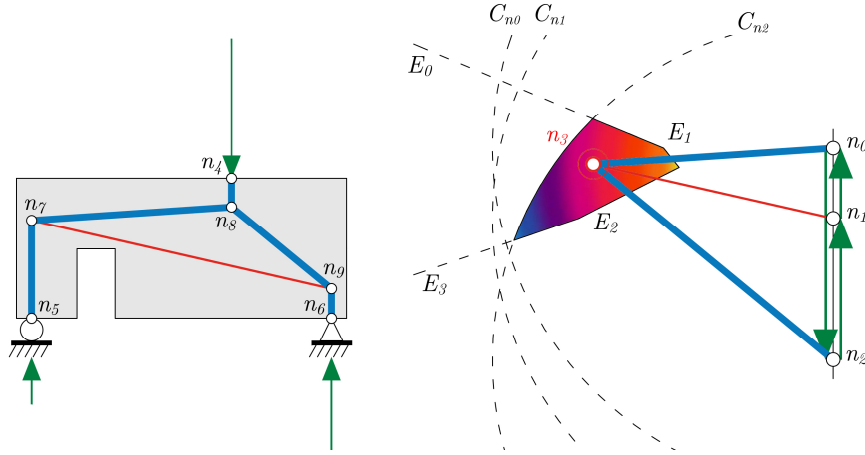


Figure 4.6 – Shape optimisation: colour scheme associated to the load path value of each point of the admissible geometrical domain

These considerations enable the proposed approach for structural robustness to account for graphic shape optimisation of solicited structural volume. Indeed, the location of node n_3 giving the strut-and-tie network of minimum load path value can be directly found by looking at its domain's colour scheme. For the previous structural example, this particular location is the rightmost point of the domain, associated to a yellow value (Figure 4.7). Amongst the set of admissible plastic solutions, this location of node n_3 gives the strut-and-tie network that minimises internal forces, i.e. the elastic solution to the structural problem (for which a minimum of strain energy is required for implementation). Implementing alternative strut-and-tie network distributions close to the elastic solution then ensures that material ductility is not overpassed, i.e. that too large plastic deformations are not occurring.

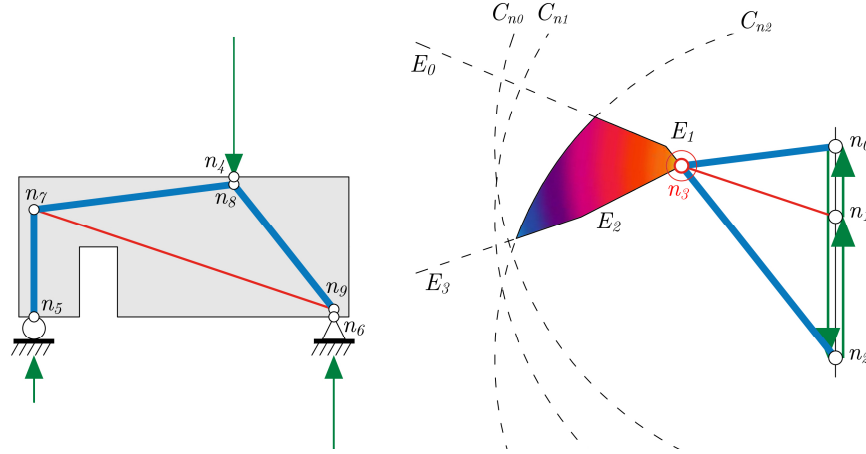


Figure 4.7 – Shape optimisation: strut-and-tie network associated to the minimum load path value

Locations of node n_3 associated to larger load path values will then correspond to other admissible force redistributions that may be reached if the minimal one cannot be activated, for example due to some structural damage. These alternative load path configurations can be ranked according to their associated efficiency value in terms of the volume of material required for their implementation. As the structure may first choose to redistribute loads according to paths whose related volume is minimal (and so for which efficiency is maximal), paths of increasing volume may then be successively chosen if the minimal one is not implementable, and so on until the strut-and-tie network associated to node n_3 's “worst” location in terms of associated volume is reached. This location,

depicted in Figure 4.8, is related to the strut-and-tie network with the maximal volume of material required for its implementation. Hence, this network represents the least efficient solution to redistribute loads within the structure. Moreover, no scaling of the external loads is possible for this location of node n_3 : a small increase of the external loads would indeed lead the structure to collapse through violation of the yield condition.

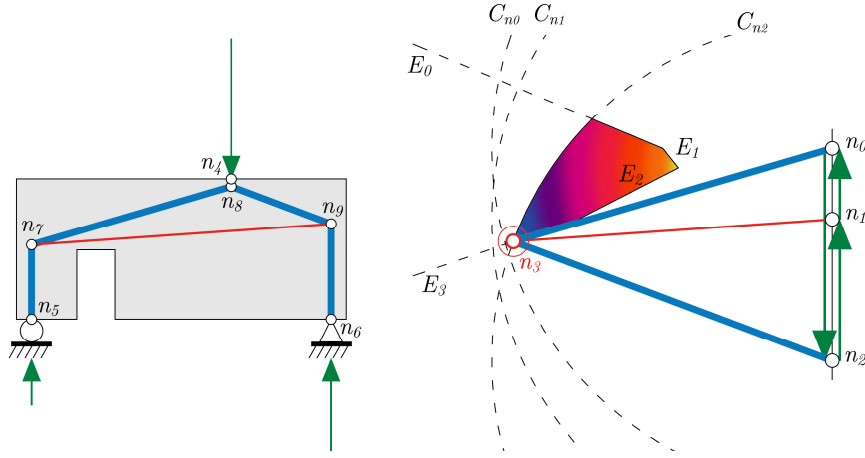


Figure 4.8 – Shape optimisation: strut-and-tie network associated to the maximum load path value

In this context, it is proposed that the theoretical “best design option” is the one that:

- maximises the size of the admissible geometrical domain, thus providing the best robustness performance in terms of aptitude for force redistributions;
- minimises the load path value of the domain’s internal points, thus providing the highest structural efficiency in terms of the volume of material required for the implementation of the associated strut-and-tie networks.

The second feature is linked to the relative shape of the admissible geometrical domain, as well as to its position from the polygon representing the external load equilibrium in graphic statics’ force diagram. An admissible geometrical domain placed relatively close to the equilibrium polygon of the external loads is indeed related to strut-and-tie networks having shorter bars, and so leading to smaller load path values. This means that the associated structure will be more efficient and so cheaper to build, hence representing a more economically suitable option.

In the present work, the term *variation* will then be used to define a combination of:

- a geometrical assessment of structural robustness, by drawing the structure's admissible geometrical domain depicting its set of possible force redistributions;
- a shape optimisation, by finding the most efficient point of the admissible geometrical domain. This point is associated to the strut-and-tie network with minimal load path, i.e. the network that requires the smallest volume of material to be implemented;
- a load path assessment of the strut-and-tie networks associated to all the other points of the admissible geometrical domain. These networks correspond to the set of plastic solutions for internal forces, and are alternative load paths associated to potential force redistributions in case of damage.

In this context, the purpose of the design process may not be about optimising the structure for a required performance (e.g. minimal volume). It is first about maximising the set of plastic solutions to the structural problem, and then about ranking these solutions by efficiency. In other words, this means that the optimal structural behaviour associated to the most efficient way to distribute internal forces will not be considered as pertinent within this thesis if it is the only one that can be implemented. Indeed, this would reduce the admissible geometrical domain to a unique point, meaning that the associated structure does not have any aptitudes for force redistributions nor strength. Hence, this structure would not be considered as robust according to the chosen definition (see Section 4.1).

4.5. Admissible geometrical domains and topology variation

It has been shown in Section 4.3 that performing sizing variation on a given structure is a straightforward process. It has also been detailed in Section 4.4 how shape variation can be performed graphically, by assessing the set of admissible positions for force diagram's mobile nodes, and by further calculating the load path value associated to each of these positions to indicate efficiency in terms of the activated volume of material. The remaining problem then concerns the obtaining, by geometrical means, of the set of strut-and-tie network topologies that may be implemented within a given structure. This means obtaining all the possible manners in which

the strut-and-tie network's nodes may be linked together to model structural behaviour (see Section 3.1.3).

Figure 4.9 depicts an alternative strut-and-tie network topology that may be drawn to model the path of internal forces within the continuous structure considered previously. For this alternative topology, the admissible geometrical domain may also be obtained by performing a shape variation, i.e. by assessing the set of admissible positions for force diagram's mobile nodes. As Figure 4.9's force diagram presents two mobile nodes n_3 and n_4 , as highlighted in red in the figure, the resulting domain is composed of two interdependent parts (i.e. two sub-domains), one per mobile node. Moving these nodes in the force diagram corresponds in practice to moving nodes n_8 , n_9 and n_{10} of the form diagram along a vertical direction and node n_{11} in both horizontal and vertical directions, such as to respect the conditions of Section 2.1.4. Concerning the other nodes of the form diagram, these must remain fixed unless modifying external load configuration.

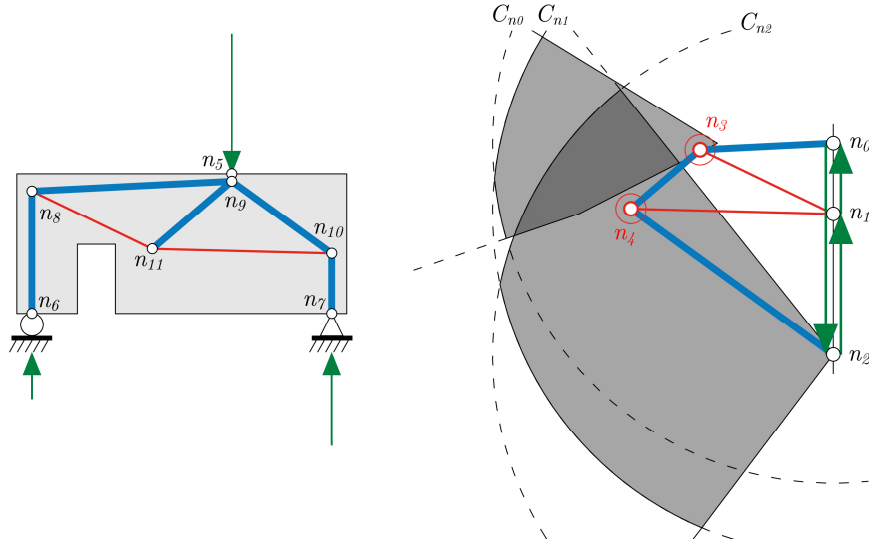


Figure 4.9 – Alternative strut-and-tie network topology, with admissible geometrical domain composed of two sub-domains

Finding alternative/more complex strut-and-tie network topologies to describe the structural problem brings on three collateral issues. First, the resultant sub-domains make unclear the way their relative size should be used to characterise structural robustness in terms of aptitude for force redistributions. Secondly, in the context of Plastic Theory (see Section 2.1.4), there exists in practice an infinite number of strut-and-tie network

topologies that may be drawn within a given structural geometry to characterise its internal behaviour. Finally, the set of resulting topologies may turn to be statically indeterminate, hence difficult to deal with using geometrical means. These three issues will be discussed in the next three subsections.

4.5.1. The issue of sub-domains interdependency

When modelling the path of internal forces within continuous structures, the topology given to the strut-and-tie network may often result in a force diagram characterised by more than one mobile node, as for example depicted in Figure 4.9. The degree of freedom of each mobile node will then be associated to its own relative admissible geometrical domain, i.e. its sub-domain. The total admissible geometrical domain may then be build as a combination of a sub-domains, with a the number of mobile nodes within the structure's force diagram.

A particular node's sub-domain is obtained in the present work by considering that all the other mobile nodes of the force diagram are being moved in any of their admissible range of positions. In other words, this means that a node's sub-domain characterises all the admissible positions of this node, considering that all the other mobile nodes of the force diagram may take any position within their relative sub-domain. Hence, sub-domains are interdependent: if any of the mobile nodes were to be fixed within its relative sub-domain, the other mobile nodes' sub-domains would be restricted to a specific geometry, which depends on the location where the fixed node was placed. This fact is illustrated by Figure 4.10, for which the location of Figure 4.9's node n_3 has been fixed within its own sub-domain. Hence, there remains only one mobile node within the structure's force diagram: n_4 . This impacts form diagram's strut-and-tie network, meaning that nodes n_8 and n_9 are kept fixed, while nodes n_{10} and n_{11} are still able to move within structural geometry. The sub-domain associated to these structural constraints, i.e. node n_4 's sub-domain, is therefore restricted in size accordingly to the location of node n_3 . Indeed, it is limited to Figure 4.10's plain dark grey zone, while nodal locations within the hatched grey part of node n_4 's sub-domain may not be reached by the node anymore. Reciprocally, note that node n_4 could have been fixed within its own sub-domain, leading to a sub-domain for n_3 that would have then been restricted in size.

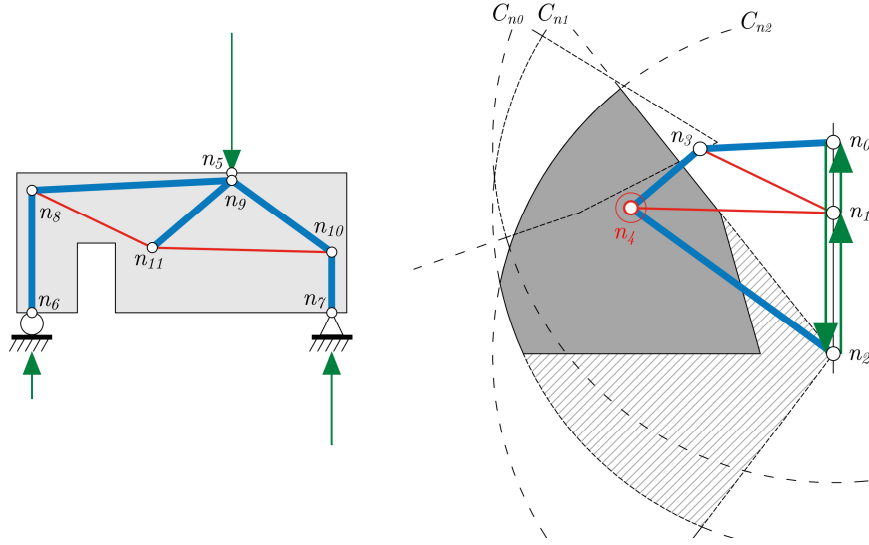


Figure 4.10 – Node n_4 's restricted sub-domain, obtained when fixing position of node n_3 at the depicted location

Having multiple sub-domains to depict the level of robustness (in terms of aptitude for force redistributions) associated to complex strut-and-tie network topologies leads to additional questions concerning the methods developed in Sections 4.2 and 4.4.

First, considering that the admissible geometrical domain is composed of multiple sub-parts, what is the resultant domain size that should be used to characterise structural robustness? Is it the total “relative” area, obtained by summing each sub-domain area? Is it the total “absolute” area, in which sub-domain intersections are counted only once? Is it one of both these areas, divided by the total number of sub-domains? Is it the perimeter associated to one of these areas? Or one of these areas divided by its perimeter? This question remains open at the actual state of the research, especially since there is for now no other geometrical method for robustness assessment to which these statements could be compared. Hence, domain size cannot be used yet to describe structural robustness quantitatively. Nevertheless, admissible geometrical domains may still be useful for qualitative comparison of structural designs in terms of aptitude for force redistribution, through relative size, shape and position of the domain within graphic static's force diagram.

Another question that arises when the admissible geometrical domain is composed of multiple independent parts is those of representing load path

values within the domain's surface, as done in Section 4.4. Going back to Figure 4.10, sub-domain interdependency means that load path values can be depicted within node n_i 's sub-domain when node n_j is fixed, giving a relative image of structural efficiency. However, this efficiency cannot be depicted globally, as each nodal position is in practice associated to a range of load path values, which are reached when the other nodes of the strut-and-tie network are moving within their own sub-domain.

A last collateral question associated to the representation of structural behaviour using complex strut-and-tie network topologies is the one of characterising the dimension of the admissible geometrical domain. For simple topologies, e.g. the one of Figure 4.8, the domain has been obtained when moving force diagram's highlighted red node in both $-x$ and $-y$ directions. This has led to two-dimensional admissible geometrical domains, which area might then be expressed in square units of force within graphic statics' force diagram (e.g. N^2). For the strut-and-tie network topology of Figure 4.10, the (sub-)domain(s) is (are) also depicted by two-dimensional surface(s), if seen in a physical way. However, in a conceptual sense, these domains are not two- but four-dimensional. Indeed, these domains are obtained from the displacement of two interdependent mobiles nodes in two directions. More generally, taking a as the number of mobile nodes within a structure's associated force diagram and ν as the number of directions along which these nodes may be moved (i.e. 2 in the present case), it comes that the dimension of the structural problem may be obtained as $\nu.a = 2.2 = 4$. Hence, the unit expressing the size of the admissible geometrical domain may be set equal to forces at the power of $\nu.a$, i.e. $N^{\nu.a}$.

4.5.2. The issue of unbounded number of solutions

Finding the strut-and-tie network shape associated to a minimal volume of material to support the applied load is a well-known and widely studied problem, as detailed in Part 3. However, when it comes to assessing the whole range of possible strut-and-tie network topologies that may depict the behaviour of continuous structures, the problem turns to have an infinite number of solutions. Indeed, in the context of Plastic Theory, there are infinite ways of putting nodes as well as bars linking those nodes to model structural behaviour. Indeed, any strut-and-tie network could be drawn to model structural behaviour, as long as it respects equilibrium, boundary, yield and envelope conditions. For example, for the structure considered in the previous figures, the strut-and-tie network topology

depicted in Figure 4.2 is a valid solution to redistribute forces, as well as the one depicted in Figure 4.9, but also many more...

Moreover, the total level of robustness of a given structure (in terms of aptitude for force redistributions) should logically be a function of the whole set of possible network topologies that may be drawn within its geometry. Hence, a structure's associated level of robustness should be given by the cumulative sum of the domain sizes of its associated possible topologies. This particular issue will be further discussed in Part 8 of the present thesis.

4.5.3. The issue of static indeterminacy

In practice, some strut-and-tie network topologies modelling structural behaviour may result to be statically indeterminate. Their degree of static indeterminacy can be obtained using Maxwell's rigidity number N , which is calculated for two-dimensional structures from the following equation as adapted by Calladine [1978]:

$$N = 2n - b - r = m - s \quad \text{Eq 4.1}$$

with n the number of nodes in the network, b its number of bars, and r the number of reaction forces (distinguished in their $-x$ and $-y$ components). m is the number of independent kinematic mechanisms, and s the number of independent self-stress states (i.e. the number of stress distributions in equilibrium with themselves). n is multiplied by 2 in Eq 4.1 to account for the fact that each node has two degrees of freedom in two dimensions. Supposing that no mechanisms can occur ($m = 0$), Eq 4.1 can be rewritten as:

$$s = b + r - 2n = h_e + h_i \quad \text{Eq 4.2}$$

with h_e the degree of external static indeterminacy, i.e. the number of independent external self-stress states, which is given by the number of excess reaction forces for equilibrium to be uniquely determined. h_i is the degree of internal static indeterminacy, i.e. the number of independent internal self-stress states. In mathematical terms, this gives:

$$h_e = r - 3 \quad \text{Eq 4.3}$$

$$h_i = b + 3 - 2n \quad \text{Eq 4.4}$$

If a given strut-and-tie network topology presents static indeterminacy (whether external and/or internal), this means that the associated

structure's stress state cannot be determined by means of equilibrium considerations only. Several force distributions are then possible within the same topology, as long as these distributions respect the conditions of Section 2.1.4. In the context of graphic statics, this means that the force diagram associated to such a strut-and-tie network (Figure 4.11a) cannot be drawn unequivocally. If $h_e > 0$, there will be an infinite number of manners to draw the polygon associated to the external loads (Figure 4.11b). Furthermore, if $h_i > 0$, this means that h_i parts of the force diagram can be scaled independently, as long as force intensities are kept below their limit value F_{max} in all bars (Figure 4.11c).

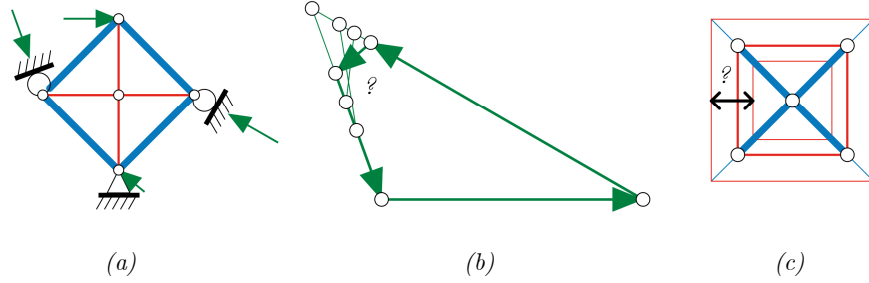


Figure 4.11 – Structure with $h_e = h_i = 1$: (a) form diagram, (b) indeterminate external force diagram, (c) indeterminate internal force diagram

However, a general methodology for drawing the set of possible force diagrams in case of static indeterminacy has been obtained by Rondeaux [2019]. The methodology combines graphic statics with Plastic Theory, and results in the following steps:

- the degree of static indeterminacy s of the structure is obtained by adding its degrees of external and internal indeterminacies, i.e. h_e and h_i . Hence, for the structure of Figure 4.11, it comes $s = 1 + 1 = 2$;
- the h_e partial external force diagrams (Figure 4.12a) as well as the h_i partial internal force diagrams (Figure 4.12b) are drawn at an arbitrary scale. Both these sets of partial force diagrams are independent and statically determinate;
- by performing a vector addition, the h_e and h_i statically determinate partial force diagrams are assembled with the partial force diagram of the applied loads (Figure 4.12c), to form the global force diagram. As each of these partial force diagrams can be drawn at an independent scale, they correspond to degrees of freedom of the global force diagram;

- using Rhino [RHINO] and its parametric algorithm editor Grasshopper [GRASSHOPPER] (further developed in next Section), the global force diagram is consequently constructed as a parametrical force diagram, in which each degree of freedom corresponds to an offset (Figure 4.12d). The parametrical force diagram then depends on x_s parameters, characterising the position of the s mobile nodes in the global force diagram (highlighted in red). For the example of Figure 4.12, position of x_1 gives the value of the external load at support R , and position of x_2 gives the internal force within bar b . Corresponding nodes are highlighted in red in Figure 4.12d, and are constrained along the line of action (dashed) of the force of which they are endpoints.

The issue of static indeterminacy will be further discussed in Part 8 of the present thesis.

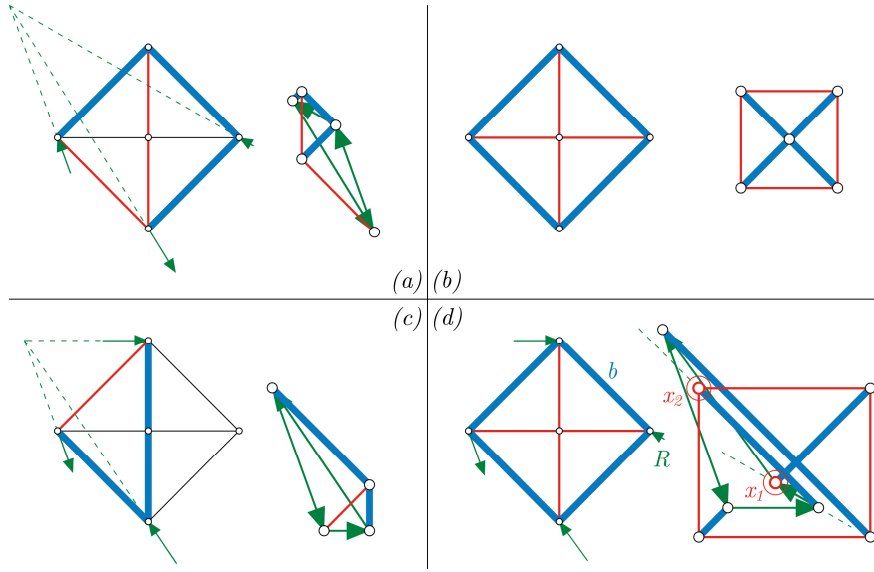


Figure 4.12 – Construction of the parametrical force diagram: (a) partial reciprocal diagrams associated to h_e , (b) partial reciprocal diagrams associated to h_i , (c) partial reciprocal diagrams associated to Q , (d) global parametrical reciprocal diagrams

4.6. Developed computer model

In practice, the proposed geometrical approach for robust structural design has been computerised using Rhino [RHINO] and Grasshopper

[GRASSHOPPER]. Rhino is a CAD software, which presents the advantage of being associated to the graphic and parametric algorithm editor Grasshopper. Grasshopper is based on a visual implementation of IronPython. Code lines are included within boxes named *components*, which may then simply be linked together to perform standard numerical operations. Data processed through the component boxes may take the form of strings, numbers, Booleans, geometrical objects, etc; these are included within lists or data trees and may be updated or varied using slider components.

A given Rhino drawing can then be easily manipulated and modified using Grasshopper's components and sliders. In a perspective of structural design, this means that:

- design constraints can be associated to Grasshopper's sliders, hence the design can be easily updated;
- in the context of graphic statics, a structure's associated force diagram can easily be drawn, as well as linked with *Laterality* and *Proximity* constraints (see Section 2.3);
- admissible geometrical domains depicting structural robustness in terms of force redistributions can automatically be obtained and updated with the design;
- load path values corresponding to admissible strut-and-tie network solutions can be automatically computed. The best network in terms of the minimal volume of material required for its implementation can also be found using the Goat component for optimisation [GOAT].

Hence, through their visual and interactive aspects, Rhino and Grasshopper are pertinent and efficient tools for structural design, as well as for structural robustness assessment in terms of aptitude for force redistributions.

However, Grasshopper presents the main inconvenient of being time-consuming, as it is based on an interpreted computer language, which calculation speed becomes even worse when implemented within Grasshopper's component boxes. This makes the tool unpractical when designing large/complex structures, for which the number of operations to be solved is high. This issue is further illustrated by Part 8's case studies.

4.7. Structural applications

In two dimensions, a way to classify structures is to separate (Figure 4.13):

- those within which the strut-and-tie network modelling the path of internal forces is topologically defined. This is for example the case of truss structures (usually in steel or timber), for which internal forces can only be transferred through the truss' bars. This is also the case of arched structures (usually in masonry, in concrete or in steel), within which forces will flow following a funicular path;
- those within which the strut-and-tie network modelling the path of internal forces is not topologically defined. In the context of Plastic Theory, any strut-and-tie network respecting equilibrium, boundary and yield conditions is a valid solution to the structural problem (cfr. Section 2.1.4). In other words, this means that there potentially exists an infinite number of strut-and-tie network topologies that could be drawn to model the behaviour of such structures. This is typically the case of wall structures (in reinforced concrete, in timber or in masonry), or other types of structures such as reinforced concrete beams and frames.

Considering that the conditions of Section 2.1.4 are respected, the next four parts of the present work will now concern the application of the proposed geometrical approach for robustness assessment to four case study structures composed of various materials. These four case studies are depicted in the bottom cells of Figure 4.13.

Part 5 concerns the geometrical robustness assessment of simple masonry arches. By hypothesis, masonry is a material that cannot undertake tensile forces; hence, masonry arches have a behaviour in pure compression. As a result, the strut-and-tie network describing such a behaviour is topologically defined: it can only consist in a funicular curve, i.e. a thrust line. Designing masonry arches for robustness then only consists in a combined shape/sizing variation of a funicular topology.

Part 6 then aims to validate the proposed approach for topologically defined strut-and-tie networks, through the analysis of a real arched structure: Rome's *Ponte della Musica*. For this purpose, the structure's level of robustness (in terms of aptitude for force redistributions) is first assessed using the proposed geometrical approach, considering three different damage scenarios. These same scenarios are then evaluated using

deterministic and energetic approaches adapted from the literature (see Sections 1.3.3 and 1.3.4 respectively), in order to observe if robustness performances assessed geometrically have the same evolution than those evaluated numerically.

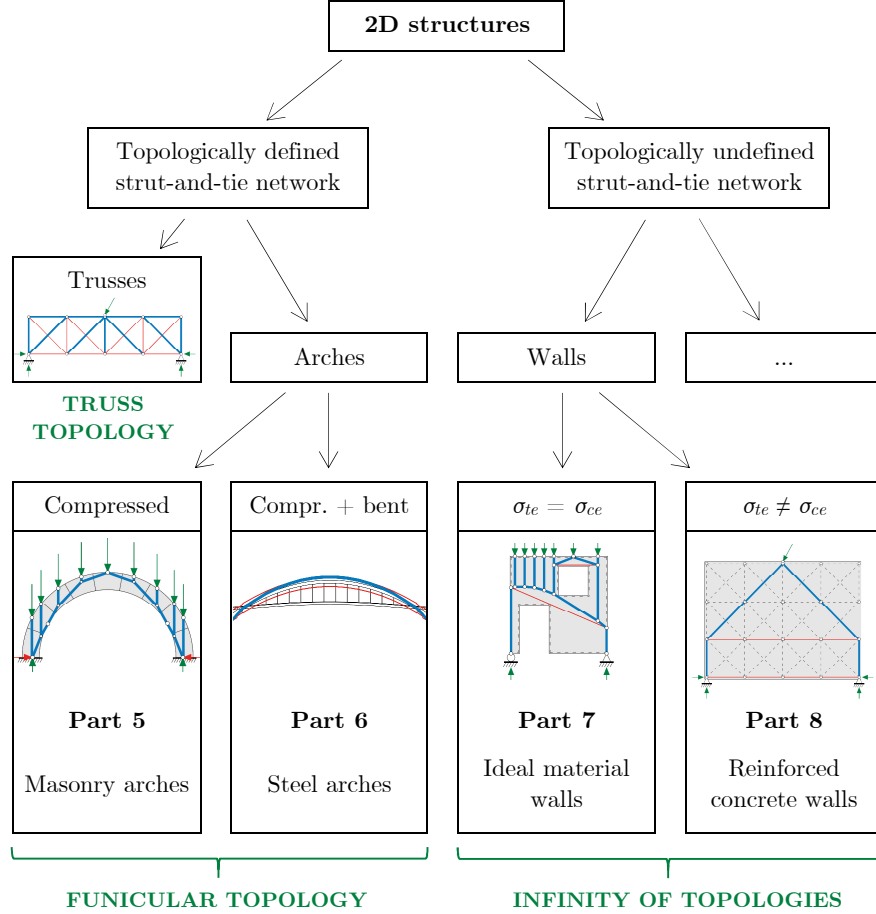


Figure 4.13 – Considered structural types in two dimensions

Part 7 concerns the geometrical robustness assessment of walls composed of an ideal material, having identical compressive and tensile strengths. As mentioned previously, the number of topologies that may be taken by the strut-and-tie network to model the path of internal forces within such structures is infinite. Hence, to deal with this issue, it has been chosen in Part 7 to predefine some strut-and-tie network topologies within the considered walls, which are then varied in shape to assess the structures'

level of robustness in terms of possible force redistributions. The analyses also comprise efficiency and stiffness assessment.

Part 8 finally explores another way to deal with the issue of the infinite number of strut-and-tie network topologies that may exist within a given structure to model its internal behaviour. It will be chosen in this last Part to assess the aptitude for force redistributions by changing the topology of a strut-and-tie network whose shape is given by a fixed and regular mesh of nodes. The method is illustrated through the geometrical robustness assessment of reinforced concrete walls. As concrete cannot undertake tensile forces, rebars may be placed within specific parts of the structure in order to activate pertinent structural behaviours. This means that candidate locations may be fixed beforehand for the (struts and) ties composing the structural model. Hence, this fixes the strut-and-tie network's shape, reducing the structural problem to a topology variation only. Part 8 also illustrates how static indeterminacy may practically be taken into account within graphic statics' reciprocal diagrams, hence for the drawing of admissible geometrical domains.

4.8. Summary and conclusion

Part 4 of this thesis has set the bases of a geometrical approach for robustness assessment. In particular, it has shown how (constraint-based) graphic statics and strut-and-tie networks can be combined to represent graphically the set of potential force distributions within two-dimensional structures composed of an ideal ductile and uniformly resistant material. This set of possible force distributions is proposed to represent alternative paths that can be mobilised within the structure in the context of the Lower Bound Theorem of Plastic Theory.

Alternative force distributions are related to strut-and-tie networks of various topologies and shapes. Nodes of such networks are associated to a solution space depicting their degree of freedom. The set of these spaces is called *admissible geometrical domain*, and is used as a graphical robustness indicator within the proposed method: the larger the domain, the more potential force distributions, the more robust the structure. Hence, robustness is here associated to structural capacity to find alternative load paths in case of unexpected damage, without assessing what this damage could be (increase of applied load, loss of structural strength, support settlement, etc).

Moreover, force redistributions can be associated to their load path value, which gives the efficiency of the corresponding strut-and-tie network in terms of the volume of material required for its implementation. Load path assessment then helps ranking potential force distributions in terms of possibility of occurrence. Indeed, from the analyses led by Baker & al., distributions with the lowest load path values will first be implemented within the structure, as these correspond to a minimum of strain energy needed. Such a level of efficiency may be depicted within the admissible geometrical domain by a colour scheme.

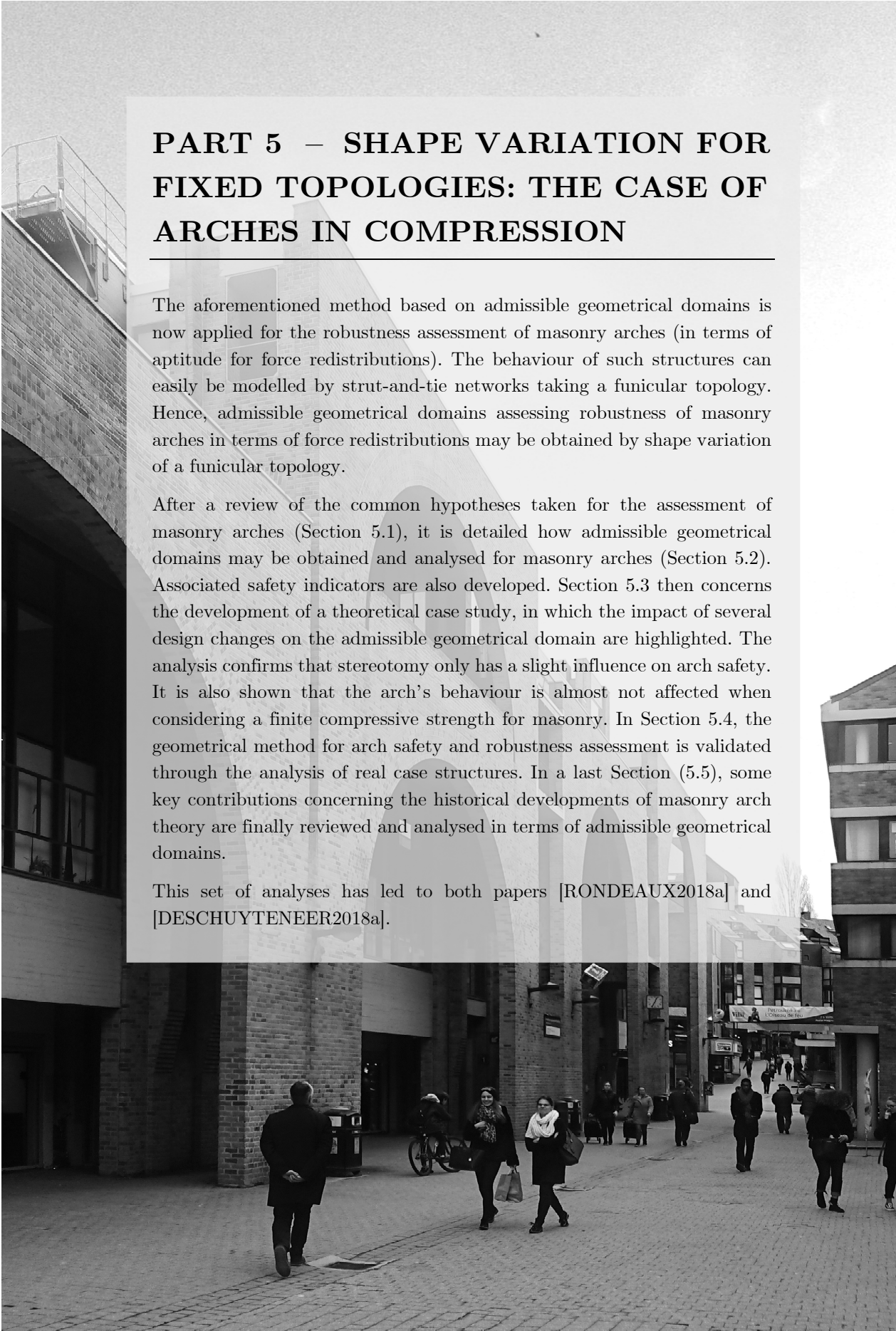
As it is visual, the proposed geometrical approach suits well for early design exploration. Indeed, the impact of design manipulations can directly be seen on the size of the admissible geometrical domain, hence on the redistribution properties of the structure under design. Moreover, when implemented by means of computers and dynamic geometry, the proposed approach can be parameterised. In particular, within the present thesis, the CAD software Rhino [RHINO] and its associated graphic algorithm editor Grasshopper [GRASSHOPPER] have been employed to turn the developed methods into a parametrical tool for efficient design for robustness. Using Grasshopper, structural form and behaviour can be managed with sliders, with which one can play to interact with the design and the possibilities for force redistributions associated to robustness performances.

However, Part 4 has also shown that assessing potential force redistributions associated to robustness remains a complex question, and this is no exception for the developed geometrical approach. In particular, Part 4 has raised the following questions, which can not yet be answered unequivocally:

- how can the infinity of topologies that may be taken by the strut-and-tie network be dealt with?
- are results pertinent if only a few topologies amongst the possible ones are considered to build a structure's admissible geometrical domain?
- when the admissible geometrical domain is composed of multiple sub-domains, what is the characteristic size that has to be taken into account to depict the level of robustness in terms of potential force redistributions?
- what is the physical meaning of representing domain sizes with units in N^x when $x \geq 2$?
- as detailed in Section 4.7, the next four Parts of this thesis concern structural applications of the proposed geometrical method, for

which the constitutive material has been idealised. What happens then if the method were to be used for robustness assessment of real structures composed of real materials? This last question will be later discussed in the general conclusion.

Furthermore, there still does not exist a way of proving that the size of the admissible geometrical domain is actually proportional to the robustness performance of a given structure in terms of aptitude for force redistributions.

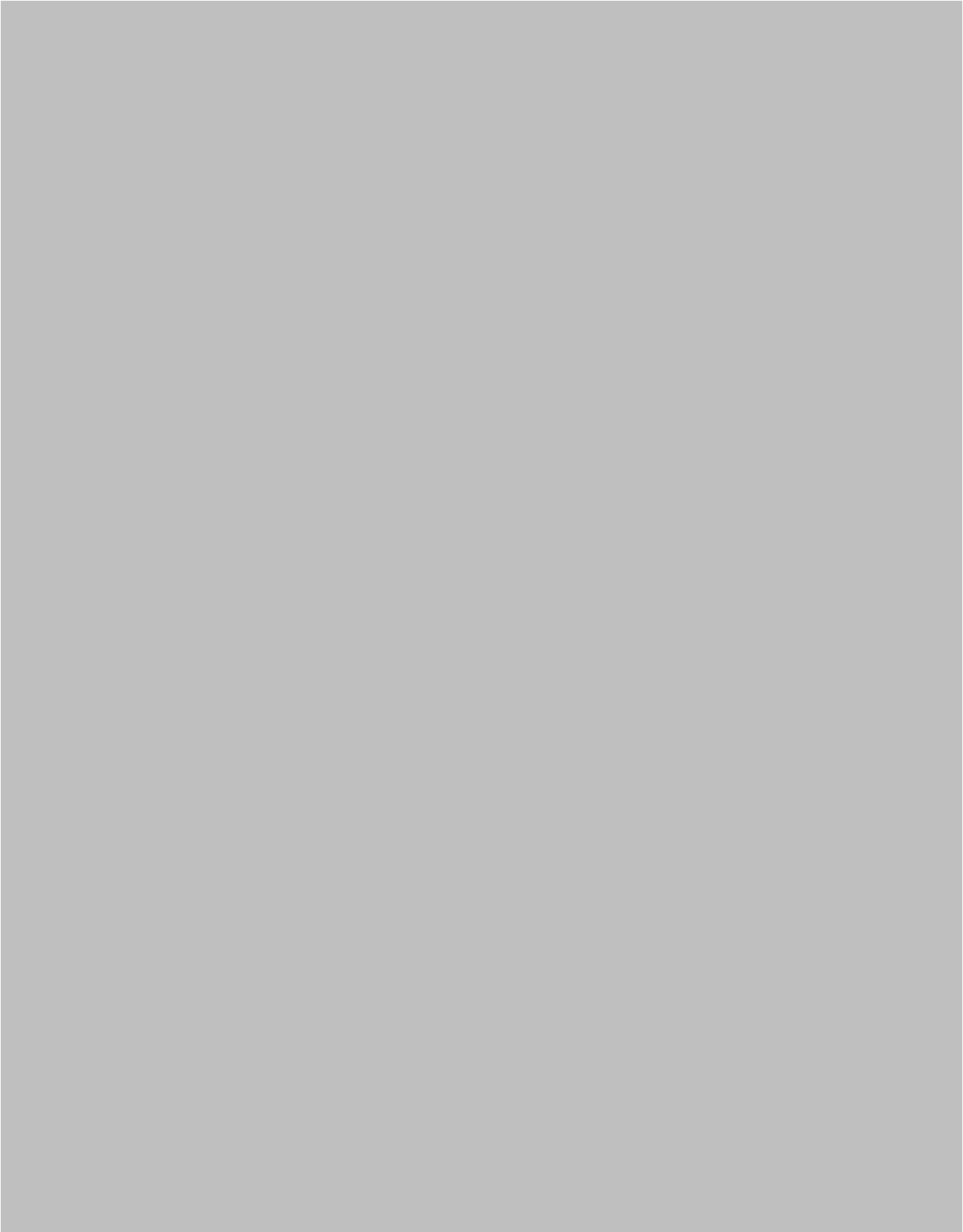


PART 5 – SHAPE VARIATION FOR FIXED TOPOLOGIES: THE CASE OF ARCHES IN COMPRESSION

The aforementioned method based on admissible geometrical domains is now applied for the robustness assessment of masonry arches (in terms of aptitude for force redistributions). The behaviour of such structures can easily be modelled by strut-and-tie networks taking a funicular topology. Hence, admissible geometrical domains assessing robustness of masonry arches in terms of force redistributions may be obtained by shape variation of a funicular topology.

After a review of the common hypotheses taken for the assessment of masonry arches (Section 5.1), it is detailed how admissible geometrical domains may be obtained and analysed for masonry arches (Section 5.2). Associated safety indicators are also developed. Section 5.3 then concerns the development of a theoretical case study, in which the impact of several design changes on the admissible geometrical domain are highlighted. The analysis confirms that stereotomy only has a slight influence on arch safety. It is also shown that the arch's behaviour is almost not affected when considering a finite compressive strength for masonry. In Section 5.4, the geometrical method for arch safety and robustness assessment is validated through the analysis of real case structures. In a last Section (5.5), some key contributions concerning the historical developments of masonry arch theory are finally reviewed and analysed in terms of admissible geometrical domains.

This set of analyses has led to both papers [RONDEAUX2018a] and [DESCHUYTENEER2018a].



*Figure 5.1 – Halles universitaires of the Université catholique de Louvain, Yves
Lepère, Louvain-la-Neuve, Belgium, 1976 [picture by the thesis author]*

5.1. Preliminary hypotheses

5.1.1. Material behaviour

The present Part of the thesis takes into account the three fundamental hypotheses on the behaviour of masonry structures, as given by Méry [1840]. First, masonry is supposed to have a null tensile strength, meaning that forces acting within masonry structures will be considered as of compression only. An infinite compressive strength is also taken into account, meaning that masonry can be considered as perfectly ductile in compression. This makes the Lower Bound Theorem of Plastic Theory (see Section 2.1.4) of application in the context of the present investigations. Hence, strut-and-tie networks representing the internal behaviour of masonry arches will in practice result in funicular shapes only. Finally, the last hypothesis taken on the behaviour of masonry arches is that no sliding can occur between arches' *voussoirs*.

5.1.2. Structural response

When assessing the structural response of an existing building, an engineer may focus on three main requirements: stiffness, which relates to the serviceability of the building; strength and stability, which are linked to structural safety at ultimate limit state [HEYMAN2008]. For the case of masonry structures however, the stiffness requirement can be neglected. As the sections are generally large, and as a rigid material behaviour can be considered, working deflections are indeed usually not occurring [HEYMAN1995]. In the same vein, the strength requirement can also be considered as negligible, as the mean acting stresses are typically relatively low in comparison to the (infinite) compressive strength of masonry. Local crushing of the material is then hardly affecting global structural integrity. For masonry structures, the focus can then be exclusively put on ultimate limit states related to stability requirements. This means avoiding a loss of static equilibrium of the global structure, as well as of any of its *voussoirs* (considered as rigid bodies).

5.1.3. The thrust line theory

The classical approach for evaluating the safety of masonry arches is based on the concept of thrust line, defined as “the *locus* of points where the resultant force passes through the joints between the *voussoirs*”

[OCHSENDORF2002]. As it only deals with equilibrium considerations, the stability of masonry arches is then ensured by the existence of at least one thrust line lying entirely within the masonry envelope [HUERTA2001].

Moseley [1833] and Méry [1840] were the first to develop a graphical method for constructing the thrust line. They correlated the geometry of the thrust line with limit states occurring when the thrust line reaches the masonry envelope in such an amount of points that the creation of a collapse mechanism is allowed. However, it was Milanković [1907] that was the first to determine the correct analytical description of the thrust line for a semi-circular arch made of radial *voussoirs*. Following these considerations, Heyman [1995] integrated the thrust line theory within the framework of Plastic Theory, considering that masonries made of rigid blocks connected by soft joints behave as a whole like a perfectly plastic material. Supposing such a behaviour, the Lower Bound Theorem of Plastic Theory may then be applied. It results that a given arch will stand as long as there exists a statically compatible distribution of internal forces (i.e. a thrust line) lying entirely within the masonry envelope and respecting yield conditions.

5.2. Robustness and efficiency assessment

5.2.1. Considered variation problem

From the hypotheses of Section 5.1.1, it comes that the external loads Q applied to masonry structures can only transfer to their supports through a pure compressive pattern of internal forces. Going back to notions of structural optimisation (see Part 3), this means that the topology of internal forces within masonry arches may only be represented by a resultant funicular curve. Note that, in practice, the different bars constituting the funicular curve could have been subdivided into multiple compression patterns, as done by Akbarzadeh & al. [2014]. So could have been the vertical struts, linking the external forces to the funicular curve within the arch's form diagram. However, these alternative funicular topologies will not be considered in the present work, as they are only subdivisions of the resultant funicular curve.

Following this hypothesis, the only way to redistribute forces within masonry structures is then through shape variation, represented in the force diagram by the displacement of funicular curve's pole. This displacement is illustrated by Figure 5.2, where the pole (highlighted in red) may move

for the funicular curve to transfer from the dark blue configuration to the light blue one, while the other nodes of the force diagram remain fixed. In the form diagram, pole displacement results in moving the end nodes of each segment of the funicular curve along a vertical direction, in order to respect the conditions of Section 2.1.4 on strut-and-tie networks.

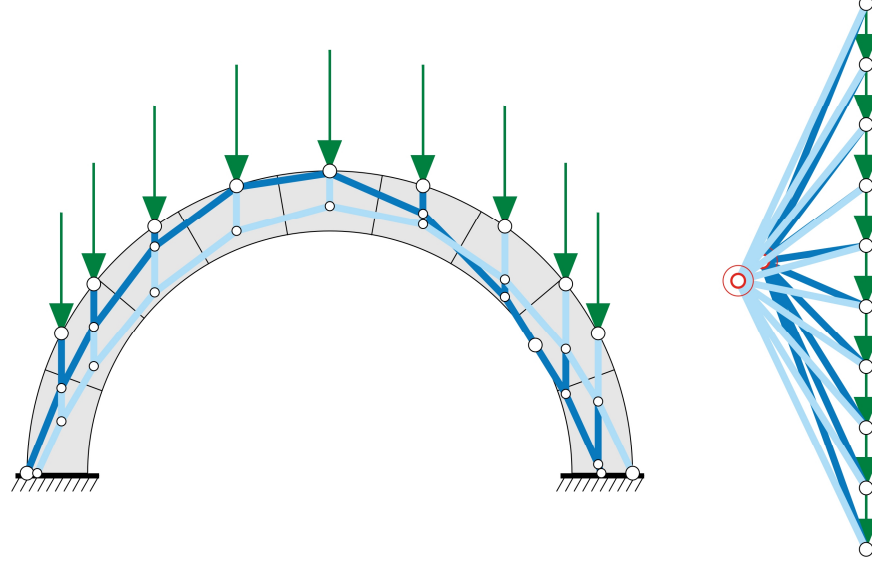


Figure 5.2 – Shape variation problem through pole displacement

Moreover, if chosen to consider a finite compressive strength for masonry, the thrust line then has to be given a certain thickness, i.e. its sizing has to be performed. Considering a fully-stressed design (see Section 2.1.4), the thickness t_i of each section i of the thrust line may be obtained from:

$$t_i = \frac{F_i}{\sigma_e d} \quad \text{Eq 5.1}$$

with F_i the magnitude of the axial force (i.e. bar length) in the force diagram which is associated to the section L_i of the thrust line in the form diagram. σ_e is masonry's effective ductile strength, and d is the arch's depth in the third dimension. This “thick funicular” then constitutes a rectangular discontinuous stress field (see Section 2.1.3), the whole surface of which has to remain within the masonry envelope to guarantee equilibrium (Figure 5.3). Values for F_i (thus those for t_i) depend on the position of the funicular curve's pole in the force diagram, hence on the shape of the thrust line. Therefore, robustness assessment of masonry structures in terms of force redistributions may be performed through sizing and shape variations.

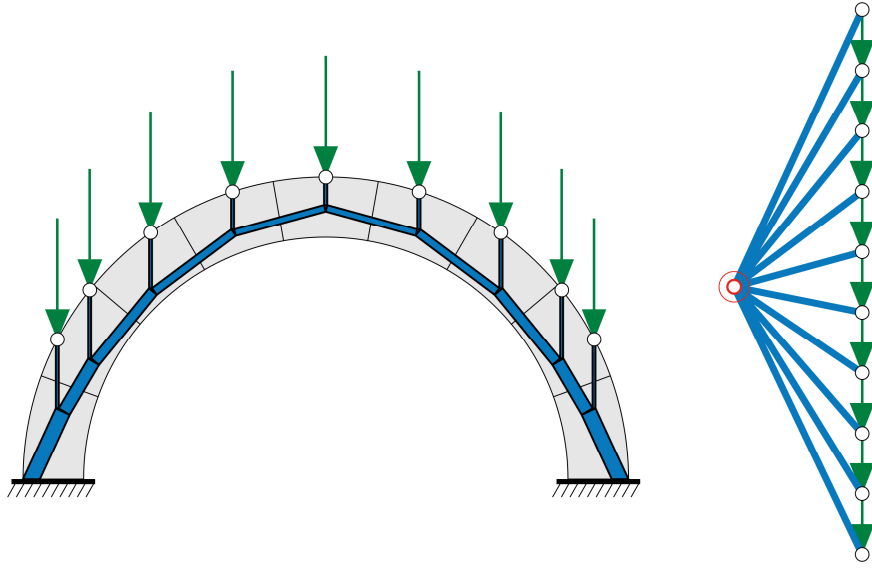


Figure 5.3 – Sizing variation problem through thrust line thickness assessment

5.2.2. Robustness level analysis

Following the approach developed in Part 4, the admissible geometrical domain can be drawn for the considered arch, here through a discrete method. First, as a funicular shape can be defined and parameterised by three points only, three particular radial segments of the arch have been chosen for the purpose of the analysis: two at the abutments, and one at the arch crown (in red in Figure 5.4's form diagram). These particular segments have then been discretised into $w = 3$ sub-segments. $w^3 = 27$ different funicular shapes may then be obtained from the combination of three out of the total set of sub-segment endpoints. The obtained funicular shapes have then been sorted between those respecting the equilibrium and envelope conditions of Section 2.1.4 (the valid ones, in blue) and those that do not (the invalid ones, in grey). Valid funicular shapes' reciprocals have finally been drawn into a common force diagram, from which the admissible geometrical domain is obtained as the surface containing the set of valid poles (in red).

The admissible geometrical domain depicts the set of statically admissible positions for the pole of the thrust line in the force diagram. Hence, it represents the level of robustness of the considered structure in terms of aptitude for force redistributions. Points of the domain will correspond in the form diagram to thrust lines lying entirely within the masonry envelope,

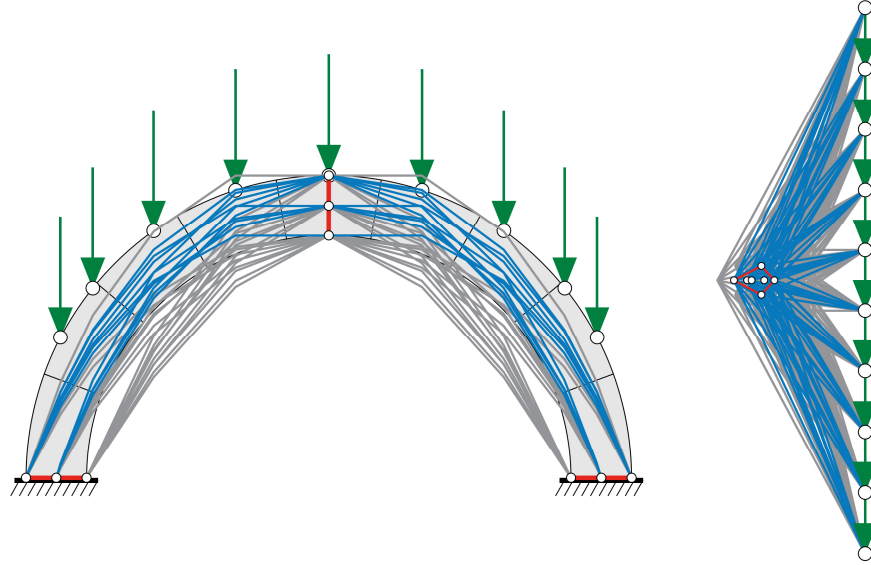


Figure 5.4 – Admissible geometrical domain for discretisation $w = 3$

depicting all the possible equilibrium configurations of the considered structure. For semi-circular arches with both clamped abutments and a high degree of discretisation (typically $w = 20$), the admissible geometrical domain takes the shape of an eye with four “corners” (Figure 5.5). Rondeaux [2019] has shown that corners of a given structure’s domain correspond to particular limit states of the structural problem in the sense of Plastic Theory, while other points on the edge of the domain are associated to isostatic states. In the case of arched structures, specific positions of the pole within its admissible geometrical domain will correspond, in the form diagram, to thrust lines reaching the masonry envelope in a defined amount of points. In particular, corners points of the domain can be associated to limit state equilibriums, i.e. to thrust lines reaching the masonry envelope in four points in the form diagram (Figure 5.5a and b) – or three due to the symmetry of the problem (Figure 5.5c). This will cause a mechanism to form, as four hinges means the release of four degrees of freedom to the original embedded problem with six degrees of freedom. The other points on the border of the domain can then be associated to limit state thrust lines touching the masonry envelope in two points in the form diagram (Figure 5.5d). Inner points of the domain finally correspond to safe shape variations of the thrust line in the form diagram,

touching the masonry envelope in 0 or 1 points depending on vertical translation (Figure 5.5e).

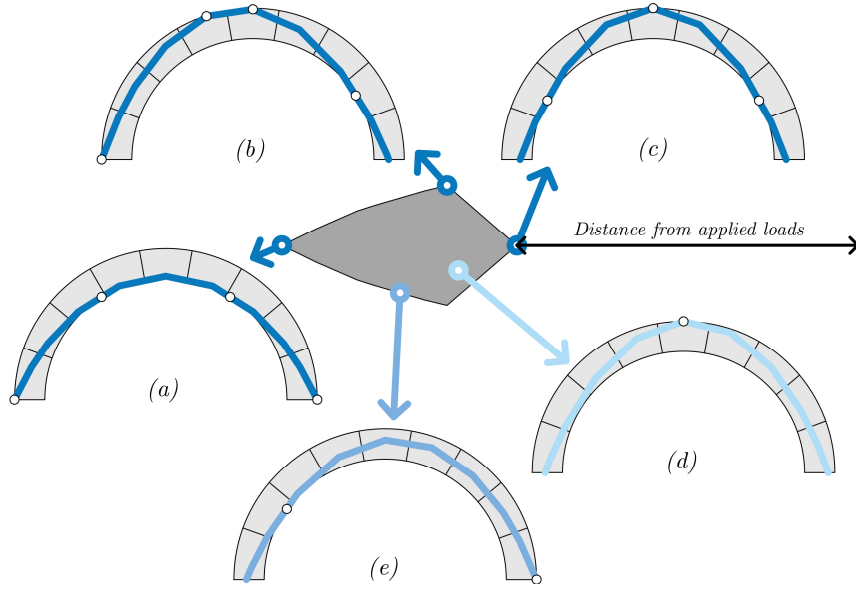


Figure 5.5 – Admissible geometrical domain for a semi-circular arch, and associated equilibrium configurations

5.2.3. Efficiency analysis

Using Eq 3.5 (see Section 3.2.1), the load path values can be obtained for each point of the admissible geometrical domain. These values give the efficiency of each possible thrust line distribution in terms of the volume of material required for their implementation. As detailed in Section 4.4, this set of values can further be turned into a colour scheme, characterising the inner parts of the domain (Figure 5.6). Load path values may also be expressed as a graph with numerical values. For semi-circular masonry arches, curves of same load path values will take the form of straight lines, their associated numerical value being reported on the top axis of Figure 5.6. The bottom horizontal axis of Figure 5.6 gives values for the horizontal thrust at the abutments, associated with each position of the pole within the admissible geometrical domain. This value is simply obtained from the horizontal distance between each point of the domain and the line of external load equilibrium in the force diagram. It can be concluded that values for the horizontal thrust increase with the load path values.

Concerning the numerical values given on the left and right vertical axes of Figure 5.6, these will be further explained in Section 5.3.2.

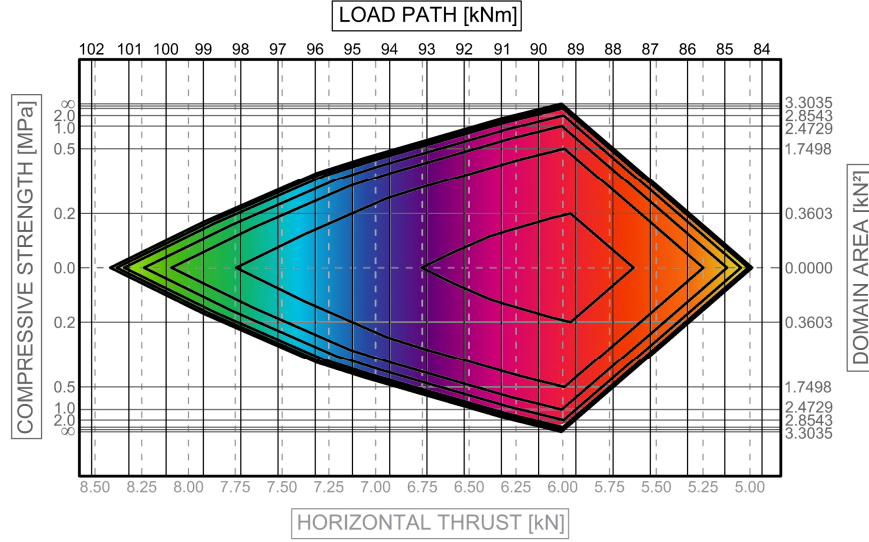


Figure 5.6 – Graph linking load path values, horizontal thrusts, domain sizes and masonry compressive strength for semi-circular arches

5.2.4. Measuring structural safety

As developed in the Eurocodes [EN1990], the most common approach for evaluating the structural response is to consider structures in terms of limit states, intended as states beyond which they no longer fulfil the relevant design criteria. This can be formulated as:

$$E_{d,dst} = \gamma_{dst} \cdot E_{dst} < E_{d,std} = \frac{E_{stb}}{\gamma_{stb}} \quad Eq 5.2$$

with E_{dst} the characteristic value related to the effects of destabilising actions, and E_{stb} the same value for stabilising actions. Design values for these actions, with d indices, are obtained by applying the corresponding safety factors γ .

However, for masonry arches, the question of measuring the structural safety as in Eq 5.2 is not straightforward. The general expression of the safety factor as a ratio between stabilising and destabilising actions induces a difficulty for the choice of an appropriate characteristic value, especially for arches under self-weight only. Heyman [1995] proposed instead a geometrical safety factor, defined as the ratio between the actual thickness

of the *voussoirs* and the minimum one, i.e. the one for which only one thrust line can be drawn and the admissible geometrical domain is reduced to a sole point. This way of defining structural safety is interesting, because it only deals with the geometrical characteristics of the arch rather than with material strength. Assuming for instance a geometrical safety factor of three means that it would be possible to find a thrust line contained in the middle third of the arch. This corresponds to the classical design criterion for masonry structures, enabling to avoid tensile forces when considering linearly distributed stresses over the whole cross section [EN1996, HUREZ2009]. However, Heyman's approach might be uncomfortable for practical use in case of irregular geometries. Indeed, the method would first require the identification of the critical section to which apply the safety factor, which is not always easy to determine. Another limitation of Heyman's method is that it cannot easily be connected to the magnitudes of the external loads acting on the structure. This is not an issue for existing structures under self-weight only, but it becomes for instance a clear drawback when evaluating the response of an existing structure under a modified external loading. Finally, the approach cannot easily be linked with the safety factor formulations commonly used by structural engineers, as those of Eq 5.2.

To overcome these limitations, indicators based on values obtained from arches' admissible geometrical domains have been developed. These indicators give information about structural safety, through stability and equilibrium, as well as on robustness, through the possibility of finding alternative load paths in the structure.

Stability assessment implies to compare the acting or destabilising value of a chosen parameter with the stabilising one. For masonry arches, the stabilising thrust line is defined as the one corresponding to the minimum horizontal thrust H_{min} that has to be mobilised at the arch's supports in order for the thrust line to lie entirely within the masonry envelope. This value H_{min} can be measured in the force diagram by the distance between the rightmost point of the domain and the line of external load equilibrium (Figure 5.7). In the same way, H_{max} is the distance between the leftmost point of the domain and the line of external load equilibrium, characterising the magnitude of the maximum horizontal thrust the arch can stand without being destabilised. The other points of the domain then correspond to alternative statically admissible configurations for the thrust line, showing the reserve structural safety.

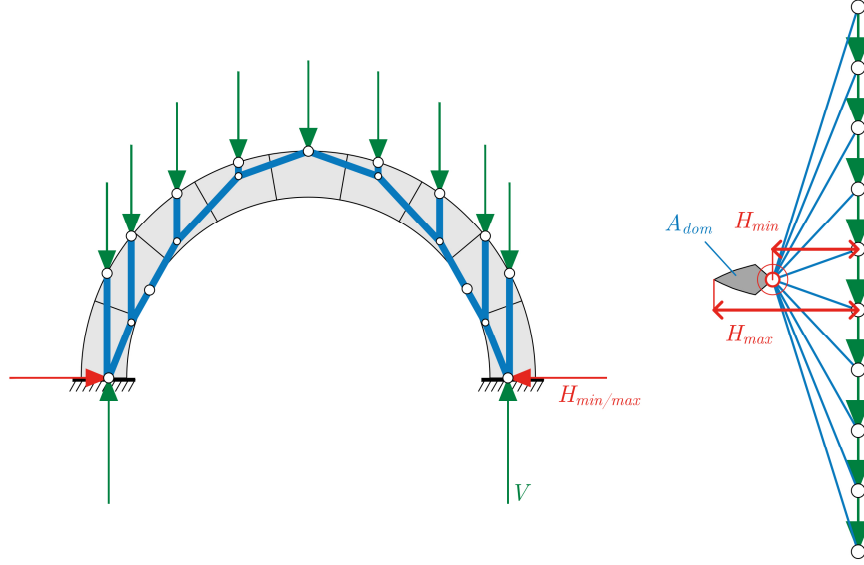


Figure 5.7 – Assessing H_{min} , H_{max} and A_{dom} graphically

The overall aptitude of a given arch to undertake horizontal thrusts is then represented in the force diagram by the horizontal range of the admissible geometrical domain. In terms of safety indicator, taking inspiration from Eq 5.2, a stability index λ can be defined as:

$$\lambda = \frac{H_{max}}{H_{min}} \quad Eq\ 5.3$$

Note that the previous equation may only be employed for the stability assessment of two-dimensional structures that may not buckle in a direction perpendicular to their plane. Indeed, if out of plane buckling was to be considered, increasing the extrados' height would lead in practice to increasing the arch's slenderness, hence decreasing its stability. This result would then be contradictory with Eq 5.3, where increasing slenderness leads to a smaller value for H_{min} , hence to an increased arch stability. The stability index proposed in Eq 5.3 may also present greater pertinence when applied to symmetrical structural cases, for which the resultant admissible geometrical domain is also symmetrical. Indeed, this index is a measure of the domain's horizontal length; hence, it does not enable to take eventual domain asymmetry in the vertical direction into account.

As mentioned previously, the area A_{dom} of the admissible geometrical domain may be taken as a geometrical criterion to give a measure of the arch's aptitude for plastic redistributions. These possible distributions

correspond to the set of thrust lines lying entirely within the masonry envelope. This aptitude for force redistributions then contributes to the performance of robustness of the considered structure, as defined in Section 4.1. Consider $A_{dom,i}$ as the domain area of an intact structure. In case of local failure, this domain area may reduce to a smaller value $A_{dom,f}$, as some thrust line configurations may not be possible anymore. Taking inspiration from [FRANGOPOL1987], a robustness index ρ can then be given by:

$$\rho = \frac{A_{dom,i}}{A_{dom,i} - A_{dom,f}} \quad Eq\ 5.4$$

The ratio between domain areas before and after the occurrence of a given damage then represents the arch's aptitude to survive damage. A value for ρ close to infinity indicates a larger structural aptitude for re-equilibrating the applied forces, by finding an alternative path for the thrust line after damage. On the contrary, a value for ρ close to 1 indicates a poor capacity for force redistributions under the considered damage. Going back to Plastic Theory [HEYMAN1995], this means that the arch may only collapse if its admissible geometrical domain is an empty set of points.

5.3. Influence of design parameters on stability and robustness

Consider the semi-circular masonry arch composed of nine radial *voussoirs* depicted in the previous Figures of this fifth Part. This arch has an internal radius of 2m, and is 0.5m thick and 0.5m deep. It is first supposed that the *voussoirs* have an infinite compressive strength, and that the structure is loaded by its self-weight only (18kN/m³). The arch has then been analysed geometrically using a Rhino [RHINO] and Grasshopper [GRASSHOPPER] model (see Section 4.6). This allows the parameterisation of the structure, hence the design to be manipulated. The impact of design changes on the admissible geometrical domain (hence on the safety and robustness levels in terms of aptitude for force redistributions) can then be seen directly. In particular, the model gives domain changes and associated safety factors for an increasing external load at the arch crown (Section 5.3.1), a decreasing material strength (Section 5.3.2), and alternative *voussoir* stereotomies (Section 5.3.3). Other arch shapes have also been analysed (Section 5.3.4). A last series of analyses finally gives the impact of support displacement on the admissible geometrical domain (Section 5.3.5).

5.3.1. Increase of applied external load

The structure is now considered to be loaded by its self-weight, as well as by a vertical point load P applied at the arch crown. This load P first has a magnitude of 0kN, which is progressively increased. The impact of this load increment has an influence of the admissible geometrical domain, as depicted in Figure 5.8. Indeed, larger values for P lead to domains translating to the left, increasing the values obtained for H_{min} and H_{max} . These domains also have an area A_{dom} that reduces homothetically with the increase of P . This means that a load increment:

- decreases the structure's level of robustness in terms of aptitude for force redistributions as domain area decreases;
- decreases structural efficiency in terms of the associated volume of material, as increasing values for the horizontal thrust are associated to increasing load path values (see Section 5.2.3).

The maximum point load that can be applied at the arch crown is found when the corresponding admissible geometrical domain is reduced to a sole point, which means $A_{dom} = 0$ and $H_{min} = H_{max}$. This case is reached for $P = 16.16\text{kN}$.

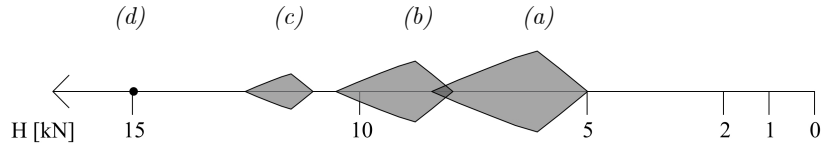


Figure 5.8 – Admissible geometrical domains for increasing values of load P [kN] at the arch crown, namely (a) 0, (b) 5, (c) 10 and (d) 16.16kN

Numerical values for H_{min} , H_{max} and A_{dom} for different values of P can be found in Figure 5.9.

P	0	5	10	16.16
H_{min}	4.988	7.950	11.027	14.973
H_{max}	8.418	10.525	12.519	14.973
A_{dom}	3.300	1.772	0.593	0

Figure 5.9 – Numerical values of H_{min} [kN], H_{max} [kN] and A_{dom} [kN²] for increasing values of P [kN] at the arch crown

5.3.2. Decrease of masonry compressive strength

Another series of analyses have been carried out considering a finite compressive strength for masonry. Thrust forces have thus been converted into discontinuous stress fields using Eq 5.1, to be drawn with a thickness proportional to the axial load transferred. Assuming no shear failure, the arch's equilibrium condition becomes that the whole surface of the discontinuous stress field must remain within the masonry envelope. Admissible geometrical domains have been drawn for material compressive strengths σ_e decreasing from infinity to 10 and 1MPa [TRAUTZ1998], considering $P = 0\text{kN}$ (Figure 5.10), $P = 5\text{kN}$ (Figure 5.11), then $P = 10\text{kN}$ (Figure 5.12). Corresponding numerical values for H_{min} , H_{max} , λ and A_{dom} are further given in Figure 5.13.

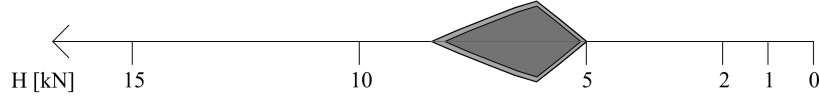


Figure 5.10 – Admissible geometrical domains for $P = 0\text{kN}$ and decreasing values for σ_e , namely ∞ , 10 and 1MPa

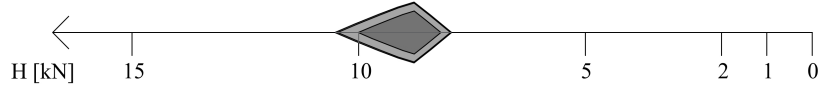


Figure 5.11 – Admissible geometrical domains for $P = 5\text{kN}$ and decreasing values for σ_e , namely ∞ , 10 and 1MPa



Figure 5.12 – Admissible geometrical domains for $P = 10\text{kN}$ and decreasing values for σ_e , namely ∞ , 10 and 1MPa

For each of these three load cases, it can be observed that domain areas A_{dom} fall with the decrease of material strength σ_e , as corresponding discontinuous stress fields tend to grow larger; so does the structure's level of robustness in terms of aptitude for force redistributions. But this diminution is only marginal when material strength decreases from infinity to 10 MPa, leading to domain boundaries that are nearly superimposed for these two cases. In opposition, the reduction in domain area for $\sigma_e = 1\text{MPa}$ is clearly observable, cfr. domains depicted in dark grey in the three

previous figures. Such results validate the common use of an infinite compressive strength for masonry, since stability index λ is almost not affected when reducing σ_e from infinity to 10MPa (less than 2% for each load case). On the contrary, a compressive strength of 1MPa reduces λ to a larger extent (around 10% for each load case), emphasizing the important role played by the joints on the overall stability.

Value	$P = 0$			$P = 5$			$P = 10$		
	$\sigma_e = \infty$	$\sigma_e = 10$	$\sigma_e = 1$	$\sigma_e = \infty$	$\sigma_e = 10$	$\sigma_e = 1$	$\sigma_e = \infty$	$\sigma_e = 10$	$\sigma_e = 1$
H_{min}	4.988	4.998	5.122	7.950	7.976	8.187	11.027	11.070	11.462
H_{max}	8.418	8.407	8.118	10.525	10.465	10.010	12.519	12.430	11.779
λ	1.688	1.682	1.585	1.324	1.312	1.223	1.135	1.123	1.028
A_{dom}	3.300	3.182	2.446	1.772	1.670	0.848	0.593	0.502	0.027

Figure 5.13 – Numerical values for H_{min} [kN], H_{max} [kN], λ [/] and A_{dom} [kN²] for various values of P [kN] and σ_e [MPa]

Figure 5.14 gives the graph depicting the complete curves for H_{min} , H_{max} and A_{dom} as a function of the point load P applied at the arch crown. These curves are represented for the three considered material strengths: $\sigma_e = \infty$ (dark grey), $\sigma_e = 10$ MPa (medium grey) and $\sigma_e = 1$ MPa (light grey).

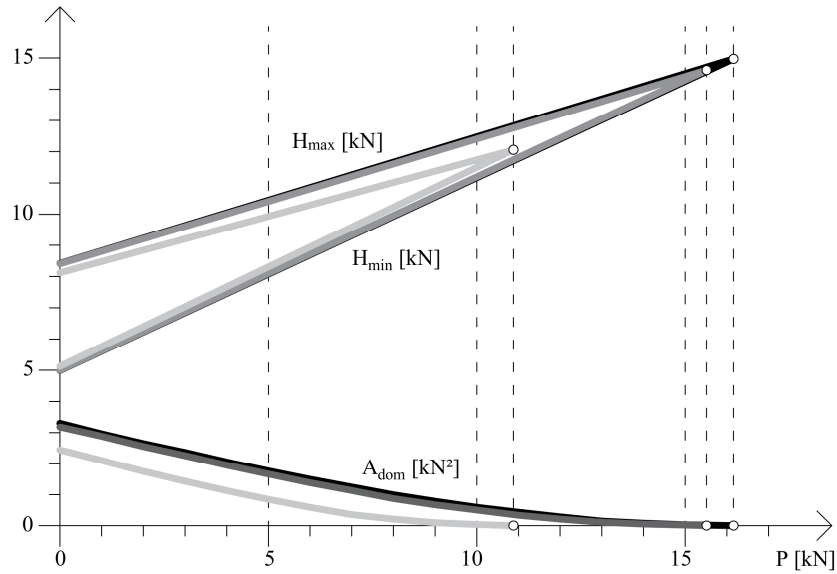


Figure 5.14 – Graph for H_{min} [kN], H_{max} [kN] and A_{dom} [kN²] for various values of P [kN] and σ_e [MPa]

As illustrated in the previous Figures, results for $\sigma_e = \infty$ and $\sigma_e = 10\text{MPa}$ give very close curves. Moreover, a larger applied load and/or a smaller material strength give smaller admissible geometrical domains, hence lower levels of robustness in terms of aptitude for force redistributions. This logically means that the structure loses its aptitude to redistribute loads when the external load increases, or when the strength of its constitutive material decreases. A point load increment also leads to a linear increase of H_{min} and H_{max} , even if at different rates. The gap between curves for H_{min} and H_{max} is also decreasing with the decrease of material strength.

The previous results for H_{min} and H_{max} can be translated in terms of stability indices, giving curves for λ as a function of the applied load P (Figure 5.15). These curves are logically decreasing with the point load increment and/or the diminution of material compressive strength.

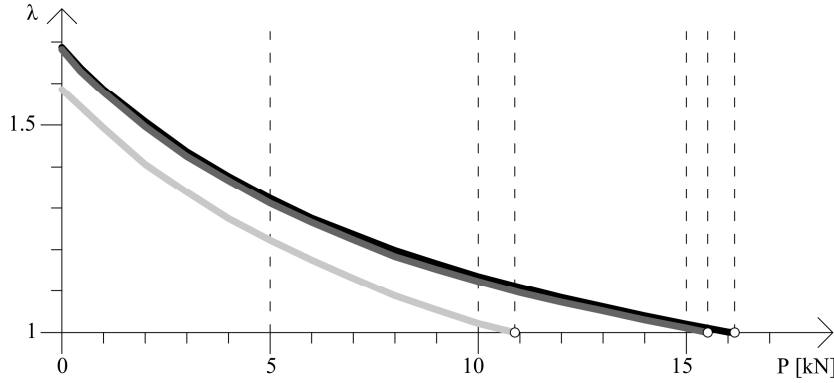


Figure 5.15 – Graph for λ [//] for various values of P [kN] and σ_e [MPa]

5.3.3. Modification of voussoir stereotomy

Considering that no sliding may occur between the arch's *voussoirs*, two other *voussoir* shapes have been taken into account for the purpose of the analysis: *voussoirs* with vertical joints and constant width L (Figure 5.16a), and *voussoirs* with vertical joints and constant weight, i.e. constant area (Figure 5.16b). Corresponding numerical values for H_{min} , H_{max} , λ and A_{dom} are further given in Figure 5.17.

Figure 5.18 then gives results for H_{min} , H_{max} and A_{dom} as a continuous graph for $\sigma_e = \infty$. Dark grey curves correspond to radial *voussoirs*, medium grey ones to vertical *voussoirs* with same width, and light grey ones to vertical *voussoirs* with same weight. Results show that the shape of the *voussoirs*

has nothing but a very small influence on the arch's properties of stability and of robustness in terms of aptitude for force redistributions (less than 5%). However, results also confirm Makris' [2013] conclusion that the radial pattern for joints is not conservative nor safe, as it leads to a slightly larger ultimate value for the external load P (i.e. the one for which $H_{max} = H_{min}$ and for which A_{dom} is reduced to one sole point). This shows that geometrical hypotheses have a slight influence on the construction of the thrust line, hence on the geometrical assessment of structural safety and of robustness of masonry arches in terms of aptitude for force redistributions.

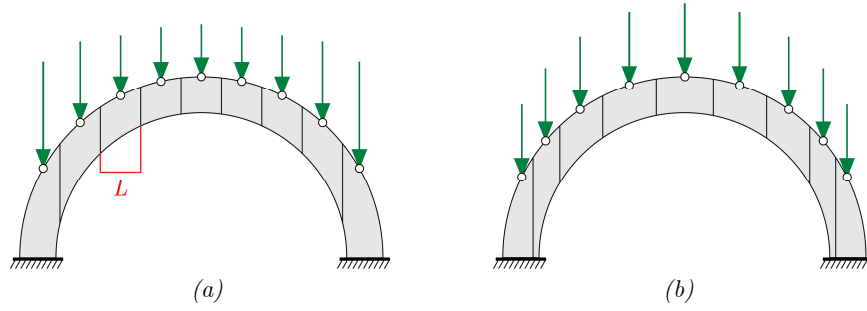


Figure 5.16 – Alternative voussoir shapes: (a) vertical with same width, (b) vertical with same weight

Stereo- tomy	Value	$P = 0$			$P = 5$			$P = 10$		
		$\sigma_c = \infty$	$\sigma_c = 10$	$\sigma_c = 1$	$\sigma_c = \infty$	$\sigma_c = 10$	$\sigma_c = 1$	$\sigma_c = \infty$	$\sigma_c = 10$	$\sigma_c = 1$
Radial	H_{min}	4.988	4.998	5.122	7.950	7.976	8.187	11.027	11.070	11.462
	H_{max}	8.418	8.407	8.118	10.525	10.465	10.010	12.519	12.430	11.779
	λ	1.688	1.682	1.585	1.324	1.312	1.223	1.135	1.123	1.028
	A_{dom}	3.300	3.182	2.446	1.772	1.670	0.848	0.593	0.502	0.027
Vert. same width	H_{min}	4.839	4.854	4.976	7.795	7.819	8.022	10.988	11.034	11.425
	H_{max}	8.331	8.331	8.124	10.506	10.443	9.951	12.446	12.380	11.667
	λ	1.722	1.717	1.633	1.348	1.336	1.241	1.133	1.122	1.021
	A_{dom}	3.544	3.441	2.677	1.852	1.754	0.940	0.533	0.454	0.016
Vert. same weight	H_{min}	5.016	5.033	5.142	7.989	8.018	8.249	11.080	11.122	11.531
	H_{max}	8.416	8.404	8.115	10.492	10.435	9.991	12.471	12.422	11.708
	λ	1.678	1.670	1.578	1.313	1.306	1.211	1.126	1.117	1.015
	A_{dom}	3.213	3.107	2.348	1.704	1.601	0.794	0.550	0.453	0.009

Figure 5.17 – Numerical values for H_{min} [kN], H_{max} [kN], λ [/] and A_{dom} [kN²] for various values of P [kN] and σ_c [MPa] and different voussoir shapes

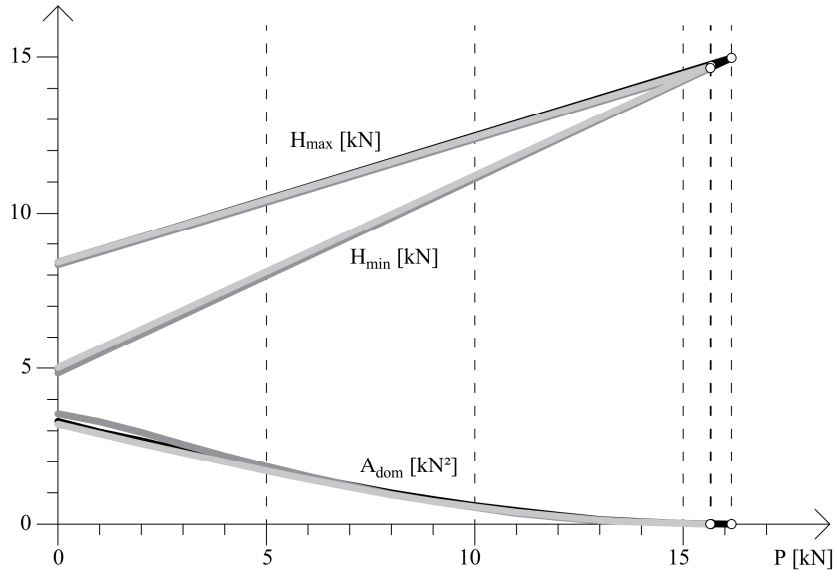


Figure 5.18 – Graph for H_{min} [kN], H_{max} [kN] and A_{dom} [kN²] for various values of P [kN] and different voussoir shapes, considering $\sigma_e = \infty$

5.3.4. Modification of arch geometry

The model has been extended to obtain the admissible geometrical domain associated to any arch geometry, as for example the one depicted in Figure 5.19.

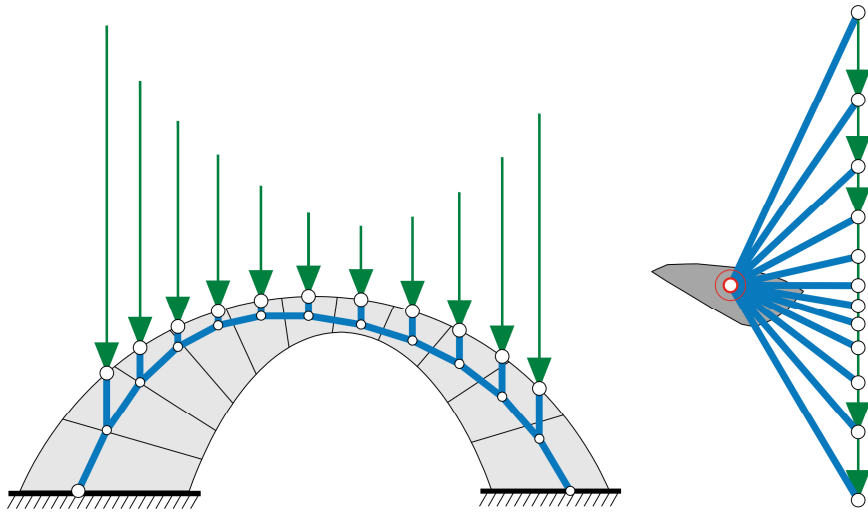


Figure 5.19 – Admissible geometrical domain for a complex arch geometry

This extension enables the analysis of a wider range of structural geometries in pure compression, regarding their stability and level of robustness in terms of aptitude for force redistributions. But, as this is not the purpose of the present work, this aspect has not been further developed here.

5.3.5. Displacement of supports

The proposed method may also be used to analyse residual stability and aptitude for force redistributions of arched structures after geometrical displacements. For example, Figure 5.20 represents a semi-circular arch whose right supports has slid. A crack has thus formed between the central *voussoir* and the one to its right, and a plastic hinge has appeared in the cross-section of the crack. As a consequence, corresponding thrust lines must pass through the hinge in order to reach an equilibrium configuration.

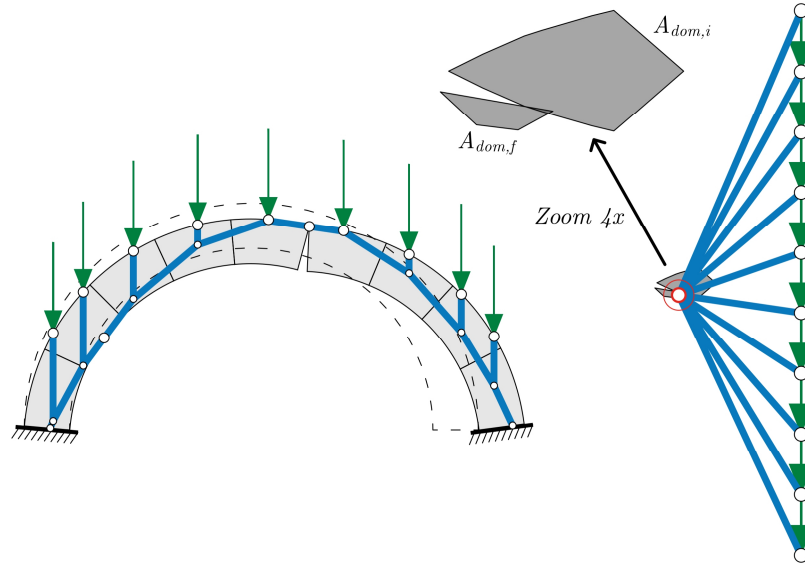


Figure 5.20 – Admissible geometrical domain after support displacement

Taking back Eq 5.4, the geometrical robustness index ρ can be calculated by comparing domain areas before ($A_{dom,i}$) and after ($A_{dom,f}$) the crack occurred. This defines the arch's aptitude to survive support sliding by finding alternative force distributions:

$$\rho = \frac{A_{dom,i}}{A_{dom,i} - A_{dom,f}} = \frac{3.300}{3.300 - 0.427} = 1.149 \quad Eq\ 5.5$$

with $A_{dom,i}$ the domain area as found in Figure 5.17 for $P = 0\text{kN}$, $\sigma_e = \infty$ and radial *voussoirs*. The same kind of analysis could also be performed to evaluate structural performance against other types of geometrical modifications, such as *voussoir* sliding or support settlement.

5.4. Analysis of real masonry arches

5.4.1. Experimental testing

Results from Sections 5.3.1 and 5.3.2 have been validated by experimental analyses led on real arches [SCHLIT2018], using the equipment of UCLouvain's *Laboratoire Essais Mécaniques, Structures et génie Civil* [LEMSC]. Two series of semi-circular arches with different material strengths and dimensions have been considered, as detailed in Figure 5.21. In particular, arches made of *Ytong* blocks of a low compressive strength (3MPa) have been studied comparatively to arches made of very resistant cobblestones (98MPa). This difference in material strength helps verifying, through the experimental tests, the common hypothesis of supposing an infinite compressive strength for masonry.

	<i>Ytong</i> blocks	<i>Blanc de Bierges</i> cobblestones
Material strength [MPa]	3	98
Number of <i>voussoirs</i>	11	9
Extrados diameter [m]	2.291	1.016
Extrados height [m]	0.905	0.349
Thickness [m]	0.219	0.080

Figure 5.21 – Arches' characteristics

These two types of arches have each been implemented with three types of *voussoir* interfaces: clean interfaces, interfaces made of 2mm *Teflon* sheets, and classic lime mortar joints (3 parts of sand for 1 part of binder made of 2/3 lime and 1/3 cement). The six material/joints combinations have then been pushed to failure, by applying an increasing load P at the arch crown using a hydraulic jack. Figure 5.22 gives the mean values for the maximum applicable load P_u recorded for each scenario (considering two drying times for the lime mortar for the tests led on the *Ytong* block arches).

	<i>Ytong</i> blocks	<i>Blanc de Bierges</i> cobblestones
Clean	3.78	0.10
<i>Teflon</i> sheets	2.52	0.07
Lime mortar (1 day drying)	2.22	0.33
Lime mortar (4 days drying)	4.93	(not recorded)

Figure 5.22 – Mean values for the maximum applicable load P_u [kN] recorded for each material/joints combination

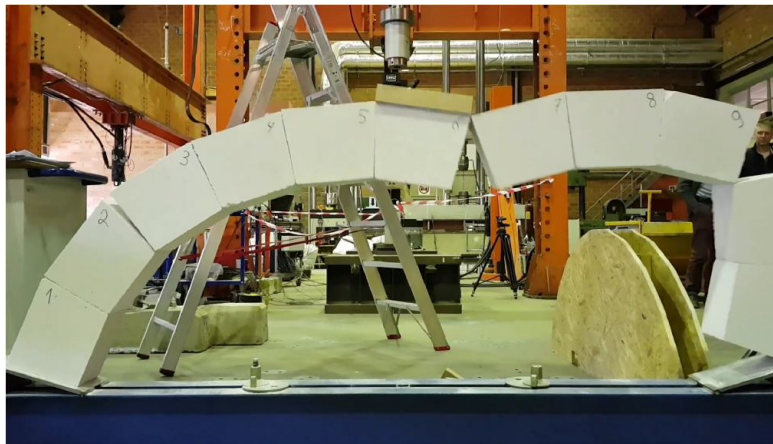


Figure 5.23 – Failure of an arch made of *Ytong* blocks with clean joints [SCHLIT2018]

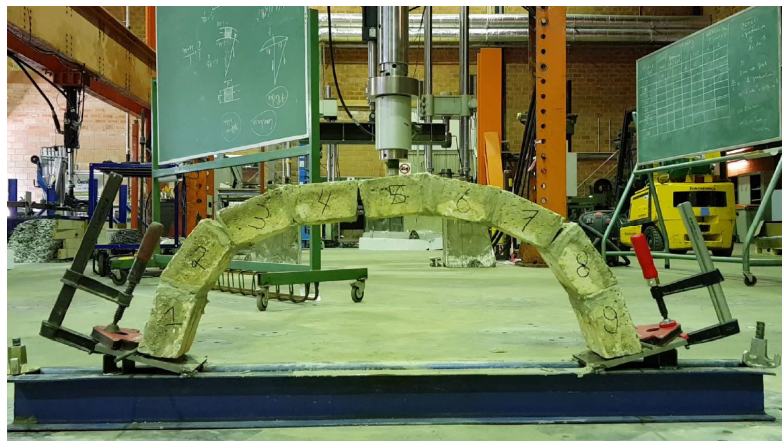


Figure 5.24 – Failure of an arch made of *Blanc de Bierges* cobblestones with one day old lime mortar joints [SCHLIT2018]

Figure 5.23 depicts the failure of an *Ytong* block arch with clean joints. It can clearly be seen that failure occurs when four hinges have formed between the *voussoirs*, here between *voussoirs* 2-3, 6-7, 9-10 and at the right support. The same type of result is then given in Figure 5.24 for the *Blanc de Bierges* cobblestone arch, where hinges can be seen at the left support and between blocks 2-3, 4-5 and 7-8.

5.4.2. Geometrical assessment

Both *Ytong* block and cobblestone arches have then been modelled and analysed using the geometrical method detailed in Section 5.3. For this purpose, arches' dimensions have been assimilated as purely semi-circular. Moreover, both arches have been supposed to be made of a continuous material all over their surface, their strength being described by a unique value (namely 3 or 98MPa depending on the case, cfr. Figure 5.21). The maximum applicable load P_u has then been found for each case as the one reducing the admissible geometrical domain to a sole point. Such values for P_u are given in the following table:

	<i>Ytong</i> blocks	<i>Blanc de Bierges</i> cobblestones
Geometrical approach	4.95	0.87

Figure 5.25 – Values for the maximum applicable load P_u [kN] obtained geometrically

5.4.3. Comparison of results

Analysing and comparing the previous experimental and geometrical results, it can be deduced that:

- the nature of its joints has a great influence on an arch's ultimate strength P_u , cfr. Figure 5.22. The common hypothesis of neglecting sliding between arch *voussoirs* is then only valid for structures whose joints are in a good state of conservation. Otherwise, the structure's ultimate load could be overestimated. This highlights the importance of joint refurbishment of real masonry structures;
- considering joints in a good state of conservation, here modelled by four days old lime mortar, experimental results for the *Ytong* block arches give a very close value to the one obtained geometrically regarding the maximum applicable load P_u at the arch crown

(4.93kN vs. 4.95kN, cfr. Figure 5.22 and Figure 5.25 respectively). This validates the common hypothesis of taking masonry compressive strength as equal to infinity, even if this value is much lower in reality. The example also illustrates the fact that the safety assessment of masonry arches is more a question of stability than of strength (cfr. Section 5.1.2);

- no pertinent results could be extracted from the tests led on the *Blanc de Bierges* cobblestone arches, and this for two reasons. First, the experimental results obtained for clean and *Teflon* sheet joints are not representative. As cobblestones are intrinsically of irregular geometries, this lead to imperfect blocky interfaces. This causes hinges to form in some parts of the interface under self-weight only, whereas blocks should have been perfectly in contact to match the numerical model. Secondly, for the sake of the geometrical analysis, arch geometries have been assimilated to perfect semi-circular ones. However, this approximation may not be considered as pertinent due to the relatively small size of the *Blanc de Bierges* cobblestones.

5.5. Reinterpretation of ancient safety assessment methods

Some key historical theories concerning the assessment of the structural behaviour of masonry arches have been reviewed, in order of appearance. Because of the low stresses usually soliciting masonry structures, authors have always considered strength requirement as a secondary issue compared to overall stability. Therefore they have linked arches' safety to their geometry, proposing different ways to define minimum values for the arch thickness (expressed here as a ratio between *voussoir* thickness t_i and extrados radius R_e). Taking inspiration from the PhD dissertation of Ochsendorf [2002], Section 5.5.1 first details the values for minimal arch radius obtained by five major ancient authors. Supposing 21 *voussoirs*, these arches of supposed minimal radius are also analysed in terms of admissible geometrical domains. Note that the corresponding force diagrams have all been drawn at a normalised scale for the vector of external loads, for easier domain comparison. Section 5.5.2 then reviews the values obtained for H_{min} , H_{max} , λ and A_{dom} from the abovementioned five theories.

5.5.1. Analysis of main historical contributions

Leonardo da Vinci (1452-1519)

Da Vinci's theory for the assessment of the minimal arch radius was to consider each single *vousoir* of the arch as a wedge [BENVENUTO1991]. He then developed a simple measuring arrangement made of ropes and pulleys that characterises the horizontal thrust effect at the springings of the arch (Figure 5.26). Da Vinci's proposal that "*the arch will not break if the outer arc chord does not touch the inner arc*" can actually be considered as a first attempt to define a thrust line lying entirely within the masonry envelope. According to Benvenuto, this assumption probably comes from assimilating the arch to a system of two rigid bars.



Figure 5.26 – da Vinci's intimations for an experimental estimate of the thrust on the abutment piers [BENVENUTO1991]

Assuming a symmetric arch, this means that self-weight is transmitted from the crown down to the supports along two straight lines intersecting the extrados at the crown and abutments. Considering that both lines have to remain entirely within the masonry envelope to ensure equilibrium, a minimal arch thickness $t_{davinci}$ can be deduced from the arch's external radius R_e :

$$t_{davinci} = 0.293R_e \quad \text{Eq 5.6}$$

An arch whose radius equals $t_{davinci}$ is depicted in Figure 5.27's form diagram. Following the abovementioned rule, the arch's self-weight has been applied at the crown, and the funicular curve depicting the path of forces acting within the arch is composed of two segments only. The admissible geometrical domain given in Figure 5.27's force diagram then corresponds to the one that would have been obtained if the arch were to be discretised into 21 *voussoirs*, i.e. for which self-weight could have been divided in 21 vectors, turning the thrust line into a funicular topology composed of 22 segments. As this domain is composed of multiple points,

this means that arch thickness $t_{davinci}$ is in fact larger than the real minimum arch thickness, which is supposed to lead to an admissible geometrical domain reduced to one sole point. Moreover, it can be noticed that the pole (i.e. force diagram's highlighted red node) of the “minimal” funicular curve as predicted by da Vinci does not belong to the admissible geometrical domain corresponding to the 21 *voussoirs* arch. This fact also confirms that the author’s prediction for minimal arch thickness is quite far from reality.

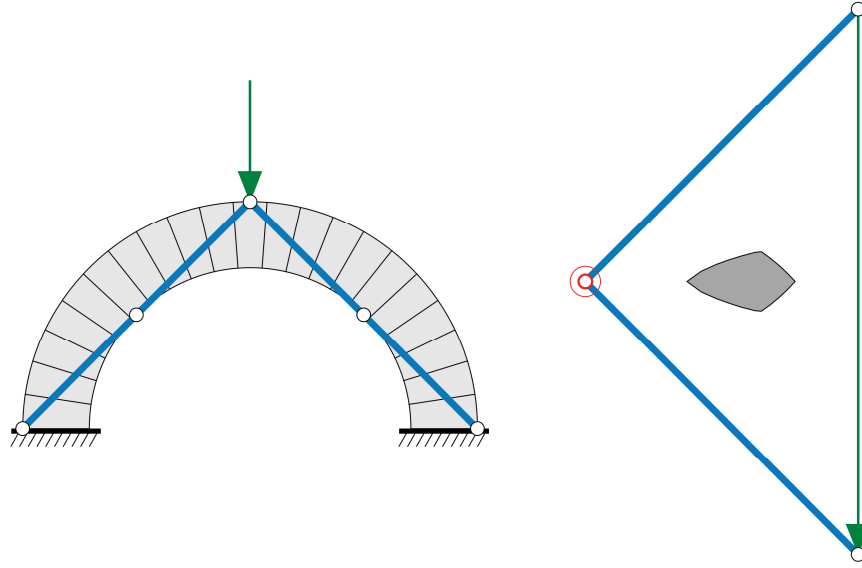


Figure 5.27 – Admissible geometrical domain for a 21-vousoir arch with a thickness $t = t_{davinci}$

Honoré Fabri (1607-1668)

Fabri’s mechanical model for arch safety assessment follows a similar rule to those of da Vinci, reducing the thrust line to two straight lines intersecting the extrados at the crown and at the abutments [FABRI1669, KURRER2008]. To determine the thickness of the abutments, and so of the arch in itself, Fabri considered that each thrust line segment must be tangent to a circle whose diameter is equal to the distance between the two left (or right) ends of both abutments (Figure 5.29). Fabri’s minimal arch thickness t_{fabri} is then given by:

$$t_{fabri} = 0.343R_e \quad Eq\ 5.7$$

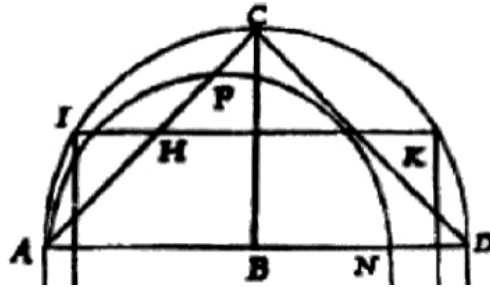


Figure 5.28 – Fabri’s model for arch stability assessment [BENVENUTO1991]

An arch whose thickness equals t_{fabri} is depicted in Figure 5.29's form diagram. Following the abovementioned rule, the arch's self-weight is applied at the crown, and the funicular curve depicting the path of forces within the arch is composed of two segments that are tangent to the circle passing by the endpoints at the same side of the abutments. As done in Figure 5.27, Figure 5.29's admissible geometrical domain corresponds to the one that would have been obtained if the arch were to be discretised into 21 *voussoirs*. Observing this domain, the same conclusions than for da Vinci's hypothesis can be made: minimum arch thickness has been overestimated by Fabri.

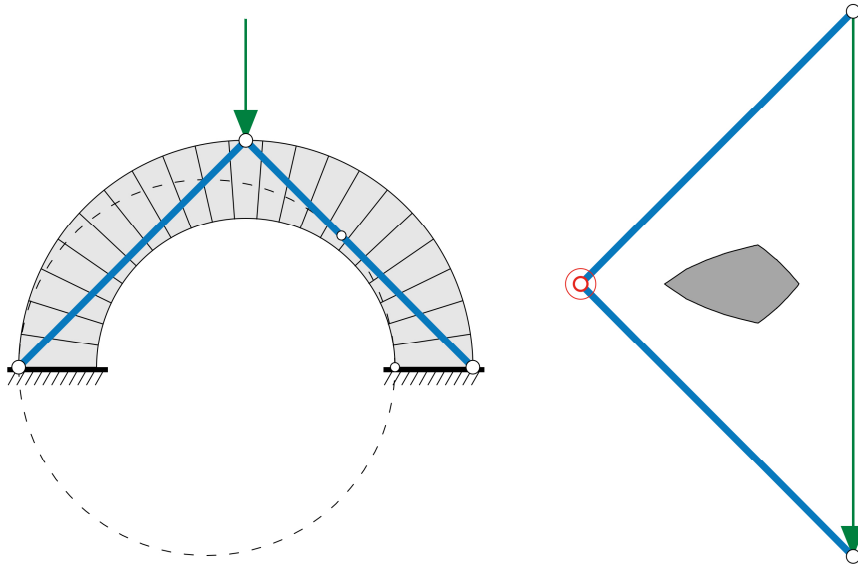


Figure 5.29 – Admissible geometrical domain for a 21-voussoir arch with a thickness $t = t_{abri}$

Philippe de la Hire (1640-1718) & Bernard Forest de Bélidor (1698-1761)

Philippe de la Hire proposed what is now referred as a kinematic ultimate load theory. His first reasoning was based on the wedge and cranked lever mechanical principles, inherited from the five simple machines tradition [KURRER2008]. In this reasoning, he considered the arch and its abutments as a four-hinge mechanism in equilibrium. He then corrected his theory in 1712, introducing the idea that friction in the joints is such that sliding cannot occur [HEYMAN1998]. The goal was to understand how to design the abutments in order to ensure the stable behaviour of the arch.

Bélidor [1729] reused de la Hire's theory, to which he added three simplifying assumptions for practical purposes. First, he considered series of adjacent blocks as a sole rigid stone. He then assumed that arch failure would occur at a 45° angle, dividing the arch in four equal *voussoirs* (Figure 5.30).

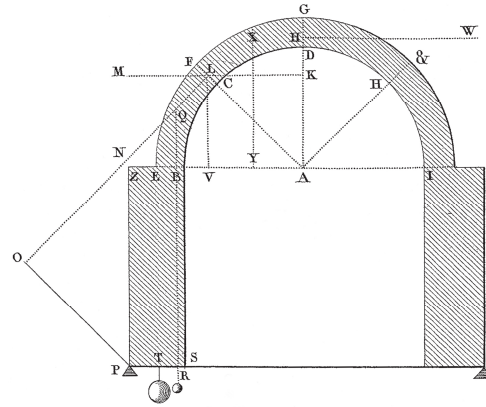


Figure 5.30 – Bélidor's model for arch stability assessment [BÉLIDOR1729]

Bélidor finally supposed that the two crown *voussoirs* were acting as wedges on the two others, considered as monolithically bound with the piers. He recognised that this last hypothesis puts his theory “on the safe side”, since he considered that no friction may occur between the *voussoirs*. Following these hypotheses, Bélidor's minimum arch thickness is given by:

$$t_{\text{bélidor}} = 0.261R_e \quad \text{Eq 5.8}$$

An arch whose radius equals $t_{\text{bélidor}}$ is depicted in Figure 5.31's form diagram. Following the abovementioned rule, the arch's self-weight can be

divided into four vectors. Forces acting within the *voussoirs* may be determined in terms of direction (perpendicular to the arch's section) and position (middle point of the 45° cross sections), turning the associated thrust line into a 5-segments funicular. As done for the previous authors, Figure 5.29's admissible geometrical domain corresponds to the one that would have been obtained if the arch were to be discretised into 21 *voussoirs*. Note this time that the pole associated to Bélidor's funicular curve lies within the admissible geometrical domain, meaning that the solution for “minimal” arch thickness developed by the author is getting closer to the real minimum one.

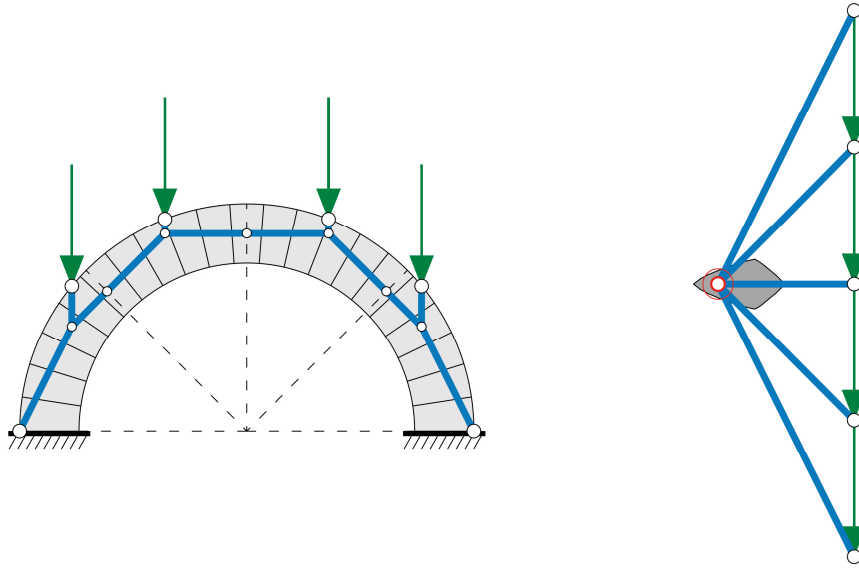


Figure 5.31 – Admissible geometrical domain for a 21-voussoir arch with a thickness $t = t_{\text{bélidor}}$

Pierre Couplet (1670-1743)

Couplet [1731] further developed de la Hire and Bélidor's kinematic methods. He thought of the same collapse mechanism, in which the arch has been divided into four equal rigid bodies. But Couplet considered instead that the failure mechanism would occur with a rotation of *voussoirs* around specific points of the intrados and extrados (Figure 5.32).

Couplet was the first to articulate explicitly the three basic assumption for masonry structures: infinite material compressive strength, null material tensile strength, and infinite friction between *voussoirs*. Considering these, Couplet's minimum arch thickness is given by:

$$t_{\text{couplet}} = 0.096R_e \quad \text{Eq 5.9}$$

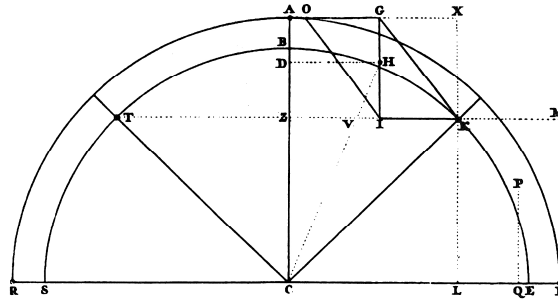


Figure 5.32 – Couplet's model for arch stability assessment [COUPLET1731]

An arch whose radius equals $t_{couplet}$ is depicted in Figure 5.33's form diagram. Following the abovementioned rule, the arch's self-weight can also be divided into four vectors. The funicular curve depicting the path of internal forces is thus composed of five segments. Forces acting within the *voussoirs* can be determined in terms of direction (perpendicular to the arch's section) and position (touching the extrados at the crown and the intrados for the intermediate segments). Since thickness value $t_{couplet}$ is not conservative, no thrust line lying entirely within the arch's geometry can be found. As a consequence, the admissible geometrical domain corresponding to the associated 21-*voussoirs* arch is an empty set of points.

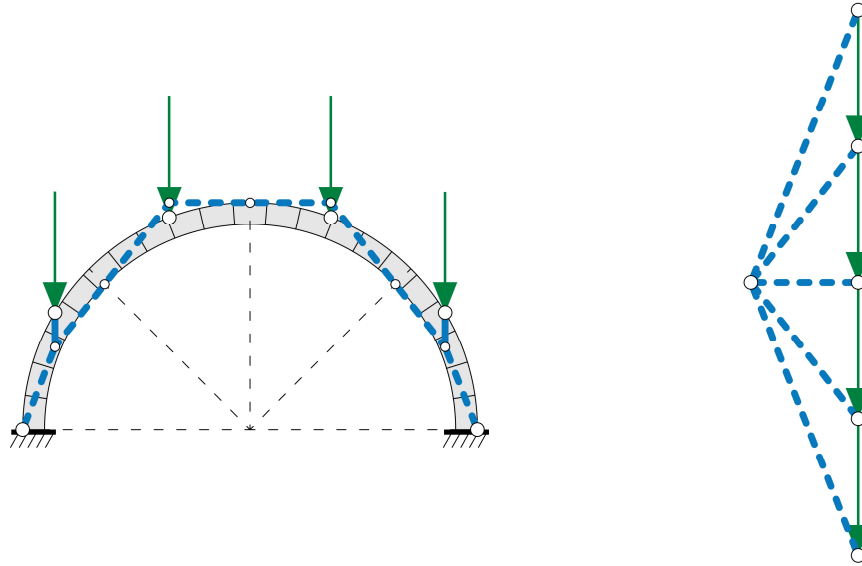


Figure 5.33 – Null admissible geometrical domain for a 21-*voussoir* arch with a thickness $t = t_{couplet}$

Milutin Milanković (1879-1958)

The correct analytical description of the thrust line for a semi-circular arch made of radial *voussoirs* has been given by Milanković [1907]. In terms of ratio between the arch's minimal thickness and its external radius, this gives:

$$t_{\text{milankovic}} = 0.1075R_e \quad \text{Eq 5.10}$$

As this formulation is the correct one, the admissible geometrical domain corresponding to the equivalent 21-*voussoirs* arch is logically reduced to a sole point (highlighted in red in Figure 5.34's force diagram).

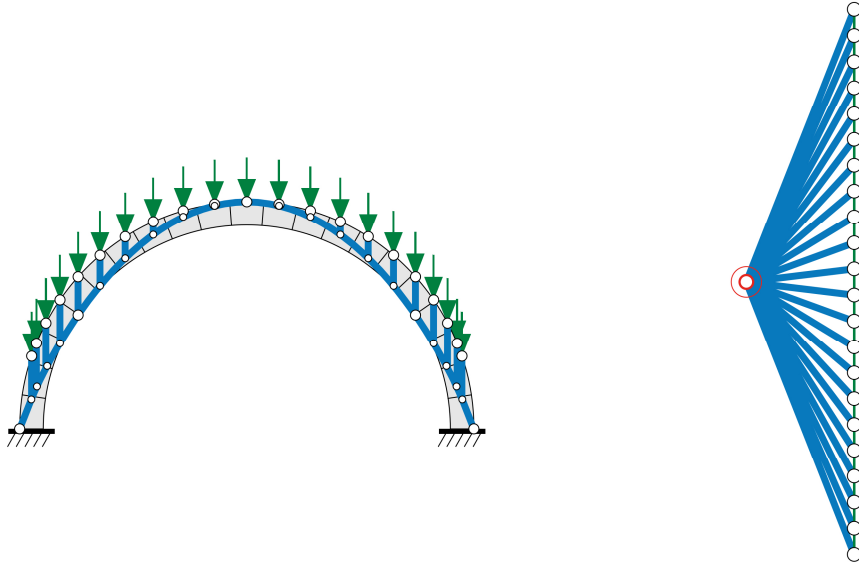


Figure 5.34 – Admissible geometrical domain for a 21-voussoir arch with a thickness $t = t_{\text{milankovic}}$

5.5.2. Comparison of results

Values obtained in the previous Section for minimum arch thickness ratios t_i/R_e according to the five considered authors are summarised in Figure 5.35. Are also given values for $H_{\min}$, $H_{\max}$, λ and A_{dom} (see Section 5.2.4) for these five cases, obtained for a corresponding 21-*voussoirs* arch.

	t_i/R_e [/]	H_{min} [kN]	H_{max} [kN]	λ [/]	A_{dom} [kN ²]
da Vinci	0.293	10.82	29.39	2.72	113.5
Fabri	0.343	16.04	53.81	3.36	498.2
Bélidor	0.261	8.42	19.48	2.31	38.99
Couplet	0.096	/	/	/	/
Milanković	0.1075	1.26	1.26	1	0

Figure 5.35 – Values for t_i/R_e , H_{min} , H_{max} , λ and A_{dom} according to the considered key historical contributions

The comparison of these results brings forward some important conclusions about the respective reliability of classical masonry arch theories. In particular, the analyses have confirmed that Milanković’s method leads to the correct analytical description of the thrust line, and that Couplet’s approach is unsafe. The method of admissible geometrical domains also provides a significant contribution to construction history. It completes the theories with a more actual formulation of structural safety, which nowadays hardly includes the issue of masonry arches.

5.6. Summary and conclusion

Part 5 of this thesis has explored geometrically the aptitude for force redistributions and safety assessment of masonry arches. As this type of structure may only withstand compressive forces, internal force distributions may be modelled by funicular topologies (i.e. thrust lines). These funicular topologies may then be varied in size and shape, in order to assess the range of potential force redistributions within the arch.

As it is based on the Lower Bound Theorem of Plastic Theory, the proposed methodology based on admissible geometrical domains is safe. Therefore, it is suited for practical applications. The visual and interactive environment in which the method is developed also gives an immediate and intuitive insight on structural safety when dealing with historical structures. In particular, Part 5 has shown that:

- an increase of applied load at the arch crown reduces the area of the admissible geometrical domain; hence, this also reduces the structure’s level of robustness in terms of global strength and aptitude for force redistributions. Furthermore, this load increase

also leads to a domain translation, making the associated load path values increase and thus structural efficiency decrease (i.e. more structural material must logically be activated for the arch to continue withstanding this increasing load);

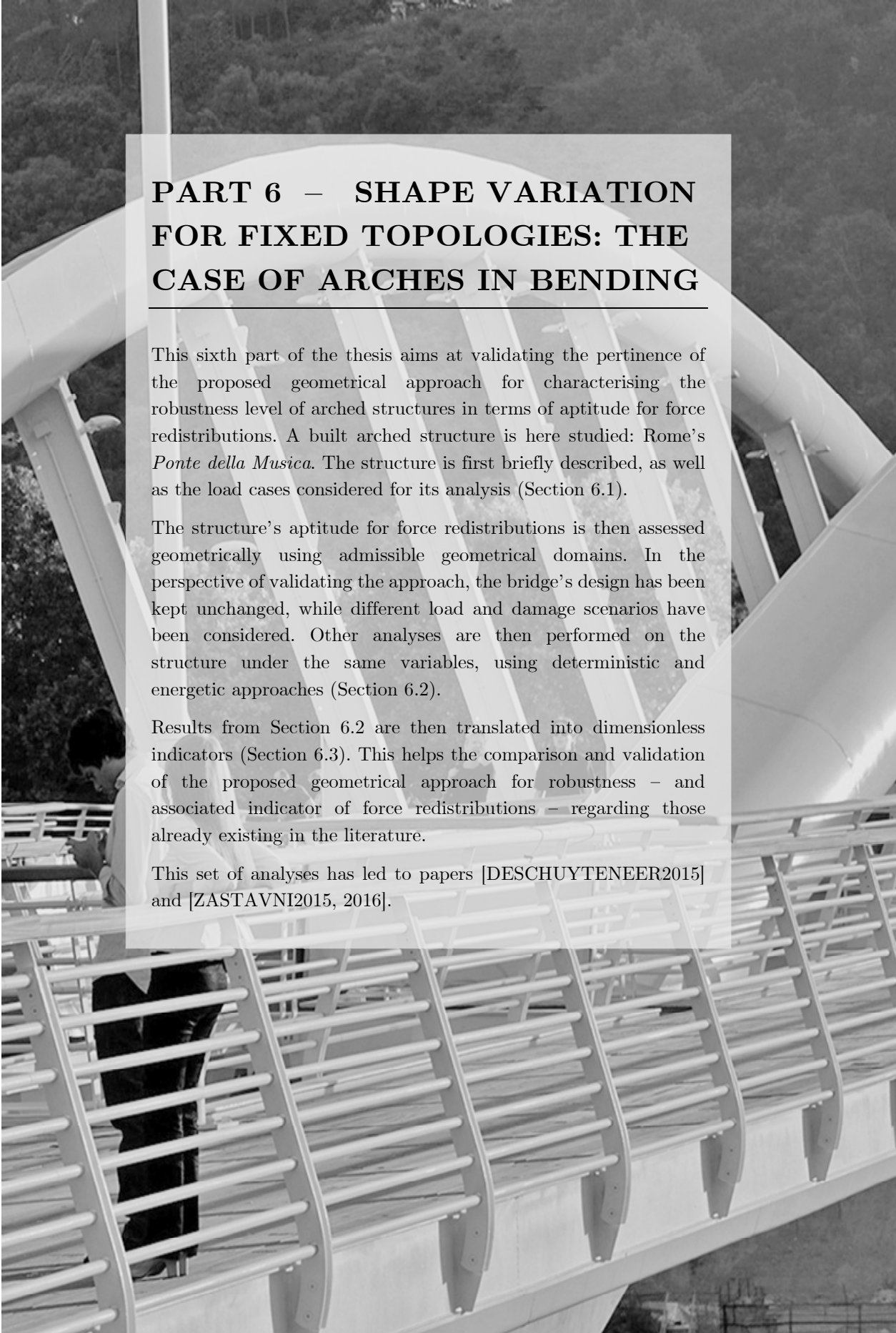
- a decrease of masonry effective compressive strength σ_e from ∞ to 1MPa does not lead to a significant reduction of the arch's level of robustness in terms of global strength and aptitude for force redistributions. This phenomenon has been further proved by experimental testing on real masonry arches. Hence, this confirms that considering an infinite compressive strength for masonry is a pertinent hypothesis;
- *voussoir* shape seems to have a low influence on the arch's level of robustness in terms of aptitude for force redistributions. It has also been shown that the radial pattern for joints is less safe than the vertical one (even if at a relatively low extent), as demonstrated theoretically by Makris [2013];
- considering undamaged masonry arches, the nature of the *voussoir* joints has a great influence on stability and therefore on force redistributions associated to robustness, as highlighted by experimental testing on real masonry arches. Hence, for real structures, the common hypothesis of considering that no sliding may occur between the arch's *voussoirs* is in practice only valid if joints are in a good state of conservation.

Part 5 has also outlined how the geometrical approach for robustness assessment could be further employed for dealing with more complex arch geometries (Section 5.3.4), as well as with unknown material behaviour laws (Section 5.3.2). Interesting perspectives of development of the proposed method may then be expected, in order to account for the geometrical robustness analysis of existing medieval masonry vaults, buttressed arches or unreinforced concrete shells. Finally, for the sake of the exercise, ancient masonry arch theories have been reinterpreted and compared using admissible geometrical domains.

The present Part of the thesis has only dealt with relatively simple arched structures, whose force diagram may be described with a unique pole, associated to a unique domain. However, in practice, a pure compressive behaviour may also be reached by subdivision of the funicular curve or subdivision of the struts linking the applied loads with the funicular curve, as developed by Block Research Group [AKBARZADEH2014]. This would lead to a force diagram characterised by multiple nodes, which would

themselves be characterised by their own sub-domain. Hence, as mentioned in Section 4.5.1, what information should then be used to characterise structural aptitude for force redistributions? The same question also applies to complex arch geometries, e.g. geometries composed of multiple arches or buttresses.

The developments of Part 5 may also be refined regarding the hypotheses taken on material behaviour (see Section 5.1.1), i.e. by considering a real behaviour for masonry. Section 5.3.2 has already shown how a finite material compressive strength may be taken into account within the model. This model could also be improved to account for the fact that, in practice, masonry blocks may undertake small tensile forces (while the joints can not). This means adapting the equilibrium conditions of the arches, by allowing the thrust line to overcome the masonry envelope at any location except at the joints. Finally, *voussoir* sliding may be accounted for within the structural model. This may be done by taking inspiration from the works of Foce and Aita [2003].



PART 6 – SHAPE VARIATION FOR FIXED TOPOLOGIES: THE CASE OF ARCHES IN BENDING

This sixth part of the thesis aims at validating the pertinence of the proposed geometrical approach for characterising the robustness level of arched structures in terms of aptitude for force redistributions. A built arched structure is here studied: Rome's *Ponte della Musica*. The structure is first briefly described, as well as the load cases considered for its analysis (Section 6.1).

The structure's aptitude for force redistributions is then assessed geometrically using admissible geometrical domains. In the perspective of validating the approach, the bridge's design has been kept unchanged, while different load and damage scenarios have been considered. Other analyses are then performed on the structure under the same variables, using deterministic and energetic approaches (Section 6.2).

Results from Section 6.2 are then translated into dimensionless indicators (Section 6.3). This helps the comparison and validation of the proposed geometrical approach for robustness – and associated indicator of force redistributions – regarding those already existing in the literature.

This set of analyses has led to papers [DESCHUYTENEER2015] and [ZASTAVNI2015, 2016].

Figure 6.1 – Ponte della Musica, Powell-Williams, Rome, Italy, 2011
[POWELLWILLIAMS]

6.1. Preliminary hypotheses

6.1.1. Description of the structure

The *Ponte della Musica* (Figure 6.1) is located in Rome, Italy. It was built in 2011 by the architects Powell-Williams [POWELLWILLIAMS], and the engineers C. Lotti & Associati and Buro Happold [BUROHAPPOLD]. The bridge is used as a footbridge, but has also been designed to carry busses and trams. It is a hybrid typology between an arch bridge and a bow-string bridge, with a clear span of 130m and a total length of approximately 145m. The bridge is composed of (Figure 6.2, Figure 6.3):

- a 20m wide steel deck;
- two series of twelve UB610x305x179 S355 steel hangers;
- two 1.4m diameter and 55mm thick tubular steel arches;
- C35/45 concrete foundation blocks whose section varies from $1.77 \times 1.95\text{m}^2$ to $1.77 \times 2.20\text{m}^2$, into which the tubular steel arches are embedded.

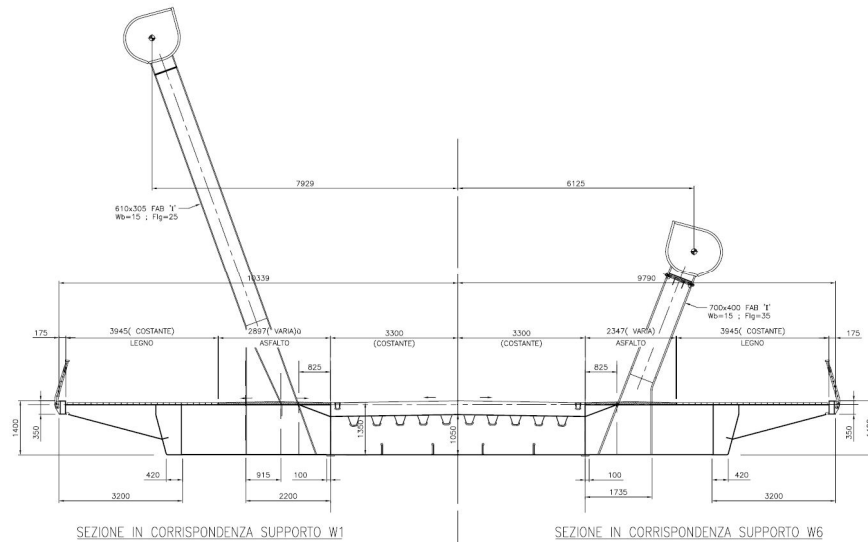


Figure 6.2 – Cross-section of the Ponte della Musica [BUROHAPPOLD]

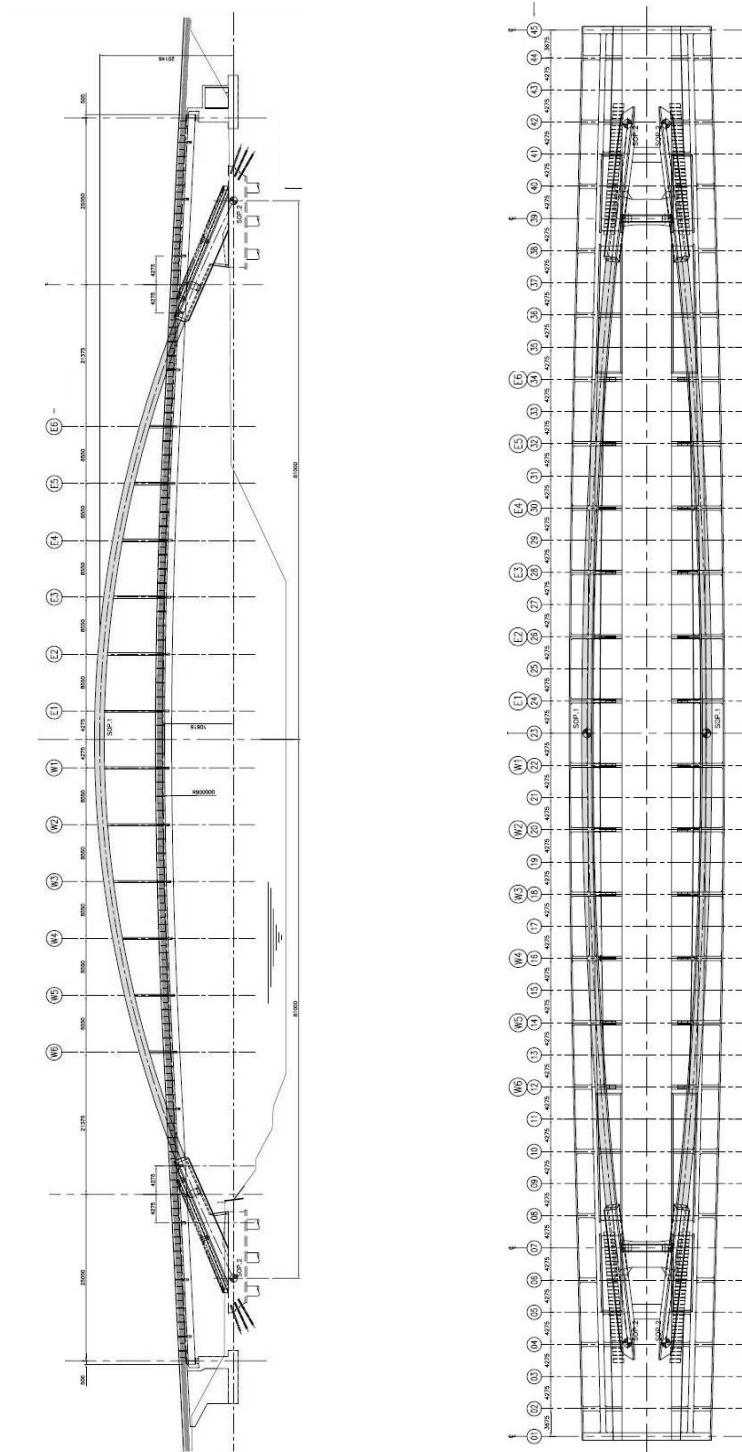


Figure 6.3 – Elevation and plan of the Ponte della Musica [BUROHAPPOLD]

6.1.2. Description of the load cases

The aim of the analyses further described in Section 6.2 is to study the possible redistributions of bending moments along the structure. For this purpose, only the bridge's arches have been considered for the analysis, i.e. the steel tubes and their concrete abutments, neglecting the role of the deck and hangers. Moreover, as the structure presents a longitudinal symmetry, only one of the two arches supporting half of the deck has been taken into account.

For the purpose of the analyses, it has been considered in this sixth Part of the thesis that the bridge's design remains unchanged. However, two configurations of live load and three damage scenarios for the steel hangers have been applied to the structure. This makes six load cases (Figure 6.4):

- *LC1*: symmetrical live load (over the whole deck), intact structure;
- *LC2*: symmetrical live load (over the whole deck), damage of the four central hangers;
- *LC3*: symmetrical live load (over the whole deck), damage of the three left hangers;
- *LC4*: asymmetrical live load (over the left part of the deck only), intact structure;
- *LC5*: asymmetrical live load (over the left part of the deck only), damage of the four central hangers;
- *LC6*: asymmetrical live load (over the left part of the deck only), damage of the three left hangers.

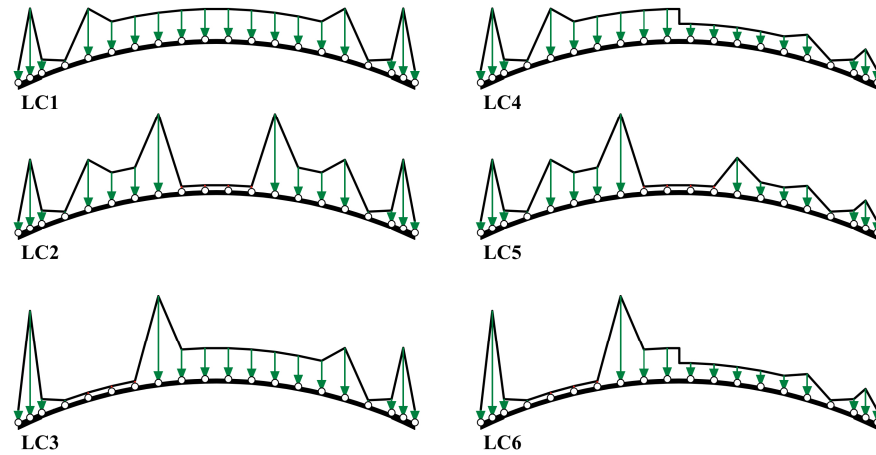


Figure 6.4 – Six load cases considered for the bridge's geometrical robustness assessment

The assumption has been made that the weight of the damaged hangers is neglected. Arch sections for which hangers are damaged are then only loaded by the arch's self-weight in itself, as the self-weight of the deck and the live load will be undertaken by the nearer undamaged hanger. This explains the peak loads that can be seen on Figure 6.4 next to the damaged areas.

6.2. Structural analyses

The bridge's level of robustness in terms of aptitude for force redistributions has first been analysed using admissible geometrical domains (Section 6.2.1). Taking back the methods of Sections 1.3.3 and 1.3.4, deterministic and energetic analyses have also been performed on the arch (Sections 6.2.2 and 6.2.3 respectively). The whole set of results has then been turned into dimensionless indices for better comparison (Section 6.3).

6.2.1. Geometrical approach

To perform the proposed geometrical analysis, the bridge's form and force diagrams have first been drawn (Figure 6.5). As for the case study detailed in Part 5, forces acting within the arch may be modelled by a funicular strut-and-tie network topology in equilibrium, i.e. a thrust line. This results in a force diagram within which all the segments are crossing in one sole point, named pole of the funicular curve. As both steel and concrete sections of the arch are embedded, shape variations of the thrust line are allowed, enabling force redistributions within the structure.

However, for the present structure, tensile forces may also be allowed within the arch, as steel and reinforced concrete can resist both compression and tension. In particular, tensile forces may happen within the arch when the resultant funicular curve modelling the path of internal forces presents a large eccentricity from the arch's neutral axis. This induces a global behaviour in compression and bending. Going back to the structure's reciprocal diagrams (Figure 6.5), this means that the condition of non-failure of the arch has to be adapted. Here, this condition turns that the funicular curve needs to remain within the envelope of resisting bending moments (in red in the Figure) to ensure safety; hence, it is not restricted to remain within the arch's geometry (as this was the case for structures in pure compression like those of Part 5).

To obtain the envelope of resisting bending moments, moments associated to a randomly chosen thrust line have first been calculated. The moment M_{ij} acting in section i of the arbitrary thrust line j is given by:

$$M_{ij} = F_{ij}d_{ij} \quad \text{Eq 6.1}$$

with F_{ij} the length of force diagram's bar corresponding to segment i of thrust line j , and d_{ij} the vertical eccentricity in the form diagram between segment i of thrust line j and the neutral axis of the arch (Figure 6.5).

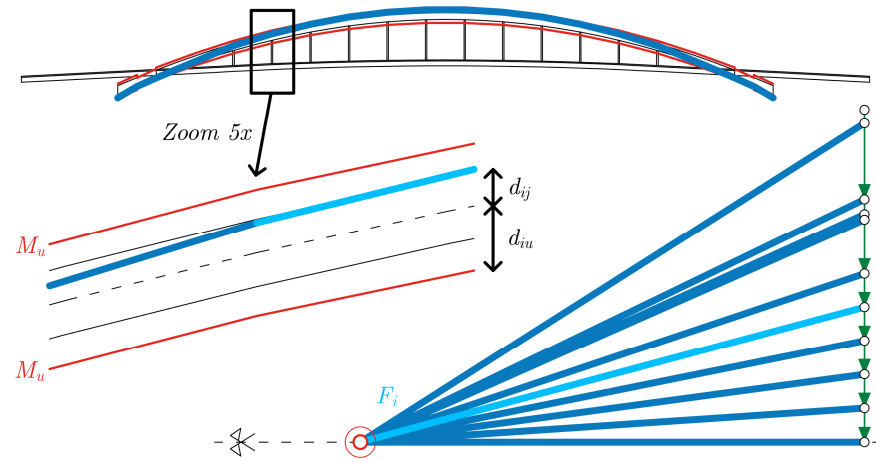


Figure 6.5 – Form and force diagrams associated to the studied structure

The geometry of the thrust line has then been iterated until reaching its limit position u . This particular position is obtained when the value of the acting bending moment M_{ij} reaches those of the resisting moment M_u in one of the section i of thrust line u , with $M_{u,s} = 32.2\text{MNm}$ in the steel part of the arch and $M_{u,c} = 24.9$ to 40.6MNm in the concrete foundation blocks (depending on the section). The thickness of the envelope of bending moments at each section i of the arch has then been extracted from:

$$2d_{iu} = \frac{M_u}{F_{iu}} \quad \text{Eq 6.2}$$

Corresponding admissible geometrical domains have finally been drawn for the six load cases of Section 6.1.2, by varying the position of the thrust line within the envelope of bending moments (Figure 6.6). Hence, domains depict the possibility of redistributing bending moments within the arch. Numerical values for domain areas are given in Figure 6.7.

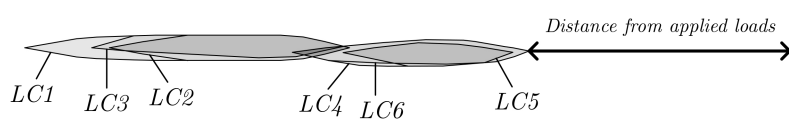


Figure 6.6 – Admissible geometrical domains obtained for the six considered load cases

A_{dom} [MN ²]	Symmetrical live load	Asymmetrical live load
Intact structure	LC1: 4.50	LC4: 2.78
Failure of 4 central hangers	LC2: 3.15	LC5: 1.73
Failure of 3 left hangers	LC3: 2.77	LC6: 2.76

Figure 6.7 – Corresponding domain areas for the six considered load cases

Comparing this set of values, it can be seen that:

- for a same level of damage, cases with a symmetrical live load present larger domains than those with an asymmetrical one ($LC1 > LC4$; $LC2 > LC5$; $LC3 > LC6$). Hence, the structure may present better robustness properties in terms of aptitude for force redistributions when loaded symmetrically. This might be due to the transversal symmetry of the bridge;
- for a same type of loading, the domain area obtained for the intact configuration is always the greatest ($LC1 > LC2$ and $LC3$; $LC4 > LC5$ and $LC6$). The structure's level of robustness in terms of aptitude for force redistributions then logically seems to decrease with damage;
- for cases with a symmetrical loading, the domain shows less reduction compared to the intact configuration when the four central hangers are damaged ($LC2 > LC3$). This might also be due to the transversal symmetry of the bridge;
- on the opposite, cases for which the load is asymmetrical present smaller domain reduction compared to the intact configuration when the left three hangers are damaged ($LC5 < LC6$).

Also, the difference of position between the domains obtained from symmetrical and asymmetrical live loads demonstrates the bridge's aptitude to redistribute bending moments on both ends of its arch, even when subjected to an asymmetrical damage. This faculty enables the domain reduction to be less significant than when no force redistribution is allowed.

6.2.2. Deterministic approach

The bridge has then been modelled using Robot Structural Analysis [ROBOT], a finite element program, in order to perform a deterministic analysis of the bridge (cfr. Section 1.3.3). Values for the arch's ultimate limit load L have been obtained, with this ultimate load expressed as the number of times (λ) the live load of the bridge may be multiplied in order to reach the bending strength of the arch. Such values for λ have been found for the six considered load cases, as detailed in Figure 6.8.

In a real scenario, redistributions between bending moments at the supports and in the central part of the arch are allowed. To model these redistributions, progressive changes of support conditions have been introduced in the finite element model, in order to equilibrate bending moments between the arch and the supports. These changes in support conditions are represented by a parameter R_y , which models the rotational stiffness of the supports. Values for R_y are given in brackets in Figure 6.8, and concern cases with an asymmetrical damage and/or loading only (LC1 to LC6).

To be consistent with the analysis and with real occurrences in the structural behaviour, a slight asymmetry on the applied loads has been introduced for the cases with symmetrical damage and loading (LC1 and LC2). This avoids overvaluing the structural behaviour of the arch under such load cases. Compressive forces are indeed dominant for these cases, and the strength capacity is maximized far beyond asymmetrical scenarios (where bending forces are governing). Hence an ideal symmetrical scenario is likely to be contested by geometrical defects, asymmetrical settings or temperature loadings.

λ [/] (R_y [MNm/°])	Symmetrical live load	Asymmetrical live load
Intact structure	LC1: 6.15	LC4: 1.79 (15)
Failure of 4 central hangers	LC2: 2.93	LC5: 1.23 (29)
Failure of 3 left hangers	LC3: 1.71 (8)	LC6: 1.40 (49)

Figure 6.8 – Ultimate limit loads for the six considered load cases

Comparing the set of results given in Figure 6.8, the same observations can be made than for Section 6.2.1.

6.2.3. Energetic approach

The energetic approach used for the present analysis has been adapted from the one described in Section 1.3.4. While these of Section 1.3.4 consisted in obtaining the structure's deformation energy until failure, the alternative approach considered here aims to calculate such deformation energy after the failure of the weakest structural element only.

First, the structure has been discretised into a mesh of bar elements using GMSH [GMSH]. Using a Matlab code [MATLAB] developed by Jean-François Remacle (UCL-IMMC-MEMA), local stiffness matrices k_j have been obtained for each of the j bars of the system. The global stiffness matrix K has then been assembled. Displacements $\mathbf{u}$ of the structure's nodes in the $-x$ and $-y$ directions have been obtained by solving:

$$[\mathbf{K}] (\mathbf{u}) = (\mathbf{q}) \quad \text{Eq 6.3}$$

where $\mathbf{q}$ corresponds to the vector of external loads. The total deformation energy U of the system has then been computed as:

$$U = \sum_{j=1}^b U_j = \sum_{j=1}^b (\mathbf{u}_j^T [\mathbf{K}_j] (\mathbf{u}_j)) \quad \text{Eq 6.4}$$

The weakest bar i is defined as the one that reduces stiffness the most. The two previous equations have then been solved again with element i removed from the structure, thus obtaining an alternative value U_i from Eq 6.4. The difference of deformation energies ΔU in each structural element is finally obtained from:

$$\Delta U = U - U_i \quad \text{Eq 6.5}$$

Considering the six damage scenarios mentioned in 6.1.2, differences of deformation energies ΔU have been computed as:

ΔU [MJ]	Symmetrical live load	Asymmetrical live load
Intact structure	LC1: 0.540	LC4: 0.437
Failure of 4 central hangers	LC2: 0.663	LC5: 0.543
Failure of 3 left hangers	LC3: 1.827	LC6: 1.936

Figure 6.9 – Difference of deformation energies for the six considered load cases

It can also be considered that the loads applied on the structure are not the ones in service, but those that push the structure to yield. To obtain these ultimate loads, service live loads have simply been multiplied by the ponderation coefficients λ found via the deterministic approach, as given in Figure 6.8. Associated deformation energies then become:

ΔU_λ [MJ]	Symmetrical live load	Asymmetrical live load
Intact structure	LC1: 20.42	LC4: 1.40
Failure of 4 central hangers	LC2: 5.70	LC5: 0.82
Failure of 3 left hangers	LC3: 5.34	LC6: 3.79

Figure 6.10 – Difference of deformation energies for the six load cases pushed to ultimate limit state

For the set of values for ΔU of Figure 6.10, the same observations than for Sections 6.2.1 and 6.2.2 can be made, except that the value found for *LC6* is greater than these of *LC4*. For the values for ΔU_λ of Figure 6.9 however, the previous observations cannot be reached and have to be modified:

- for a symmetrical damage, the structure will still present larger values for ΔU_λ for symmetrical loadings than for asymmetrical ones (*LC1* > *LC4*; *LC2* > *LC5*). However, in case of an asymmetrical damage, the opposite relation is observed (*LC3* < *LC6*);
- for a same type of loading, the value for ΔU_λ for an intact configuration is this time the smallest, and the one for an asymmetrical damage turns to be the greatest (*LC1* < *LC2* < *LC3*; *LC4* < *LC5* < *LC6*).

As deformations have been considered without any possibility of force redistributions within the structure, a greater value for $\Delta U_{(\lambda)}$ does not mean that the structure is behaving in a more robust way. It just means that it deforms more. This leads to question the relevance of the energetic indicator for robustness assessment.

6.3. Comparison of results

It can be seen that the geometrical, deterministic and energetic approaches involve highly different units:

- the geometrical approach compares forces, redistributing load paths while loadings are maintained;

- the deterministic approach compares forces, with load paths unchanged and loadings multiplied to reach the yielding of the arch;
- the energetic approach considers forces and displacements, with load paths unchanged and loadings multiplied (or not) to reach the yielding of the arch.

In order to properly compare the 3 (+1) sets of results obtained previously, corresponding values have been turned into dimensionless indices that describe residual robustness levels of the structure after the considered damage scenarios. In particular, two types of indices have here been proposed:

$$R_1 = \frac{\nu_{\text{intact}}}{|\nu_{\text{intact}} - \nu_{\text{damaged}}|} \quad [1 \dots \infty] \quad \text{Eq 6.6}$$

$$R_2 = 1 - \frac{1}{R_1} \quad [0 \dots 1] \quad \text{Eq 6.7}$$

with v the values to be compared, i.e. the ones found in Figure 6.7 to Figure 6.10. Note that Eq 6.6. takes the same form as Eq 1.7 given in the deterministic approach of Section 1.3.3.

Values for ratios R_1 and R_2 can be found in Figure 6.11 and Figure 6.12 respectively. Both these ratios show how each scope of assessment affects the associated measure of robustness when the structure is damaged. For R_1 indices, values for v_{damaged} moving towards 0 lead to an index close to 1, showing a non-robust behaviour of the structure. On the opposite, a structure behaving in a robust way according to the measured performance is characterised by a value v_{damaged} close to v_{intact} , leading to an index close to infinity. The same reasoning may be applied to R_2 indices.

R_1 []		Geometrical (service)	Energetic (service)	Deterministic (yield)	Energetic (yield)
Sym load	4 central	3.35	4.37	1.91	1.39
	3 left	2.61	0.42	1.39	1.35
Asym load	4 central	2.64	4.10	3.20	2.42
	3 left	148.81	0.29	4.59	-0.58

Figure 6.11 – Values for robustness index R_1

R_2 [/]		Geometrical (service)	Energetic (service)	Deterministic (yield)	Energetic (yield)
Sym load	4 central	0.70	0.77	0.48	0.28
	3 left	0.62	-1.38	0.28	0.26
Asym load	4 central	0.62	0.76	0.69	0.59
	3 left	0.99	-2.43	0.78	2.71

Figure 6.12 – Values for robustness index R_2

Looking at the values presented in Figure 6.11 and Figure 6.12, it can be observed that some indices obtained with the energetic method under asymmetrical structural damage belong to a forbidden range of values (in bold). Indeed, the energetic method measures deformations, which seem to become larger under asymmetrical damage (as LC3 $\gg$ LC1 and LC6 $\gg$ LC4 in Figure 6.9, as well as LC6 $\gg$ LC4 in Figure 6.10). Hence, the global deformation energy may increase greatly for LC6 (and LC3 in some cases), whereas robustness in itself does not. The reason for this may be that structural behaviour moves from a stiff compression behaviour towards a bending behaviour under asymmetrical damage, which may imply wide deformations but not necessarily high redistributions of forces through a change of geometry.

Nevertheless, separating the other results of Figure 6.11 and Figure 6.12 between service and yield states, similar evolutions of indices may globally be observed between the considered approaches. This fact highlights the relevance of such indices, although they compare different dimensions of structural robustness.

6.4. Summary and conclusion

Part 6 of this thesis has explored the geometrical robustness assessment of a steel arch with reinforced concrete abutments. For arched structures, internal forces may be modelled by compressed funicular topologies, which can then be varied in shape in order to assess the range of potential force redistributions within the structure. While Part 5 of the present work had considered masonry structures behaving in pure compression, Part 6 has detailed a case study example built in steel and reinforced concrete, which is then also allowed to withstand tensile forces. For the latter, this means that the funicular topology modelling internal stress behaviour may be

allowed to trespass the structural envelope, inducing a global structural behaviour in compression and bending. In particular, Part 6 has shown that:

- intact structures are logically more robust than damaged ones in terms of aptitude for force redistributions, for a same loading scheme;
- symmetrical arches with a symmetrical loading seem to present better robustness properties in terms of aptitude for force redistributions than the ones with an asymmetrical loading. Such symmetrical arches are also less affected by a symmetrical damage in terms of force redistributions associated to robustness. Similarly, symmetrical arches with an asymmetrical loading seem to conserve better redistribution properties under asymmetrical damage than under symmetrical one.

Results from the geometrical, deterministic and energetic robustness indices proposed in Part 6 do not explicitly converge. Indeed, it has been shown that these three methods are based on different approaches to the question of structural robustness: they implement different robustness parameters, and are based on different indicators. However, when translated into dimensionless ratios, those three series of robustness indicators show similar progressions among equal scenarios. This confirms the intuition that the proposed geometrical approach is relevant to assess some constitutive dimensions of structural robustness.

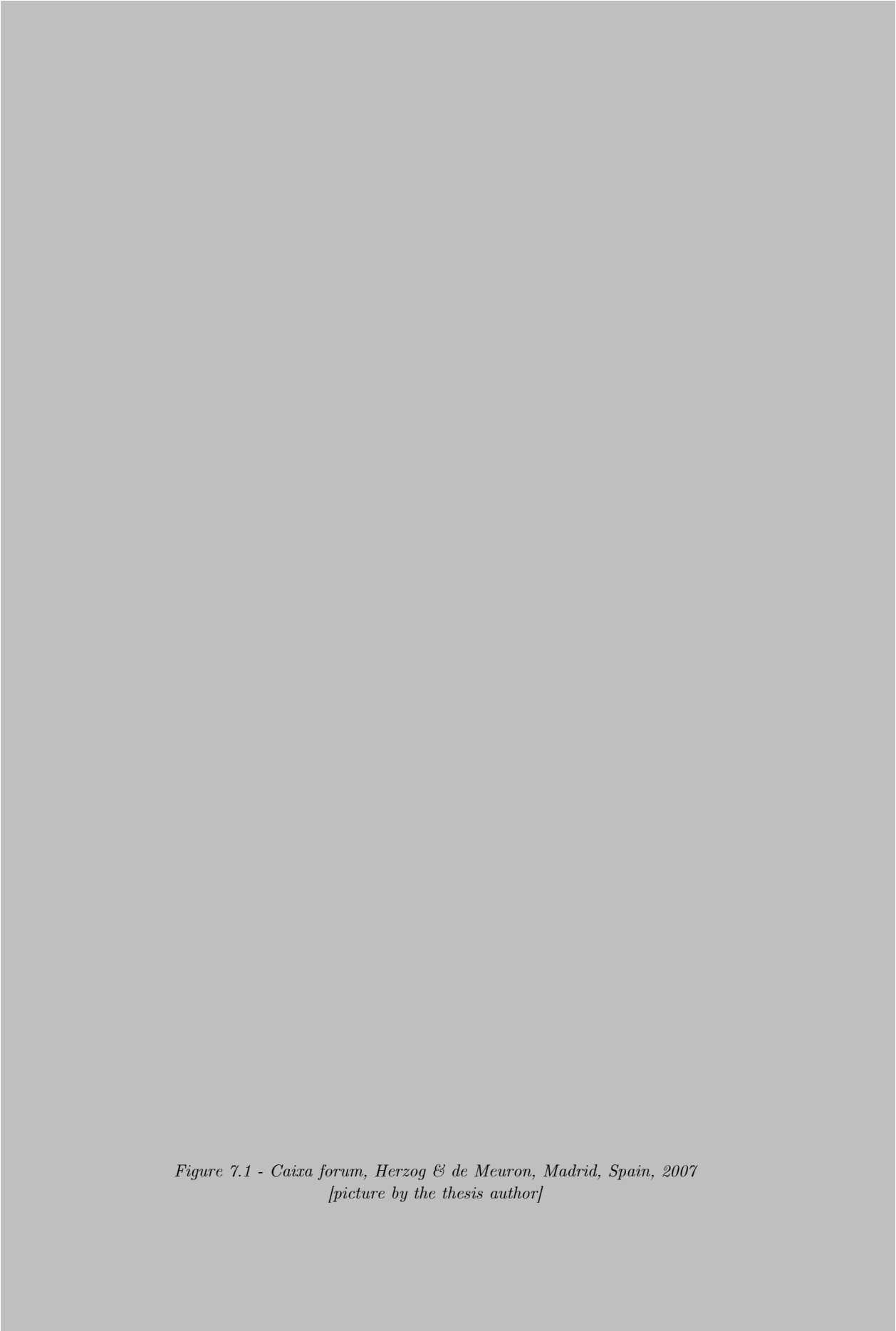


PART 7 – SHAPE VARIATION OF PREDEFINED TOPOLOGIES: THE CASE OF IDEAL WALLS

Part 7 of the thesis now concerns the robustness assessment (in terms of aptitude for force redistributions) of structural cases for which the strut-and-tie network describing structural behaviour may take an infinite number of different topologies in the context of Plastic Theory. To overcome this issue, it has been chosen in the present Part to assess structural robustness levels associated to strut-and-tie network topologies that have been selected by the designer beforehand. The aim of the analyses is then to perform shape variation on such predefined topologies, in order to compare their relative level of robustness (in terms of aptitude for force redistributions), efficiency and stiffness.

The aforementioned analyses are led on two geometries of walls: the first is rectangular with one opening, the second square with two openings (Sections 7.1 and 7.2 respectively). For the second case, several design alternatives have also been considered. These highlight the impact of support displacement on the associated level of robustness in terms of aptitude for force redistributions.

This set of analyses has led to papers [DESCHUYTENEER2015, 2017] and [ZASTAVNI2015, 2016].



*Figure 7.1 - Caixa forum, Herzog & de Meuron, Madrid, Spain, 2007
[picture by the thesis author]*

7.1. Wall with one opening

7.1.1. Preliminary hypotheses

The first structural example is adapted from a case study structure developed by Schlaich & al [1987]. It is composed of a simple rectangular wall of length $L = 7\text{m}$, height $h = 4.7\text{m}$ and depth $d = 0.4\text{m}$ (Figure 7.2). This wall has a 1.5m square opening at the bottom left corner. The structure is considered to be simply supported at both extremities; hence it is externally statically determinate. The wall is composed of an ideal uniform material, with a perfect ductile and isotropic behaviour. Both tensile and compressive material strengths σ_{te} and σ_{ce} are equal to 17MPa .

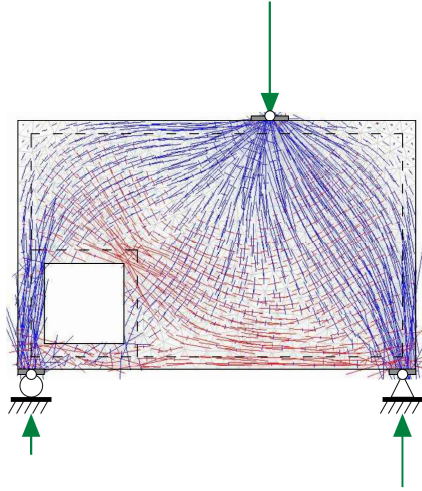


Figure 7.2 – Wall with one opening, and associated continuous stress field

The aim of the analysis is to define strut-and-tie network topologies of internal forces that can undertake a 3MN vertical point load applied on the top of the wall, at $3/5$ of its length. For this purpose, it has been considered that the bars composing the networks are constrained to remain within a certain distance from the border of the wall and its opening, i.e. within a cover $e = 25\text{cm}$ (depicted by the dashed lines in Figure 7.2 and following).

7.1.2. Robustness level analysis

Taking inspiration from the continuous stress fields acting within the wall (Figure 7.2), obtained using EPFL's iMesh and iConcrete applets [IMESH, ICONCRETE], four strut-and-tie network topologies have been established

to describe the structure's internal behaviour. These topologies take the form of a funicular network (Figure 7.3), a catenary network (Figure 7.4), a two-ties network (Figure 7.5), and a truss network (Figure 7.6).

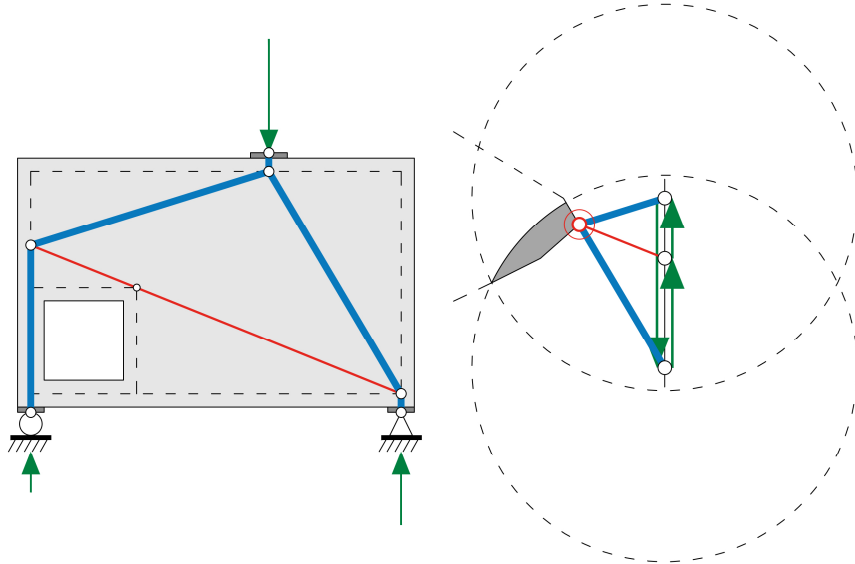


Figure 7.3 – Wall with one opening: funicular strut-and-tie network topology

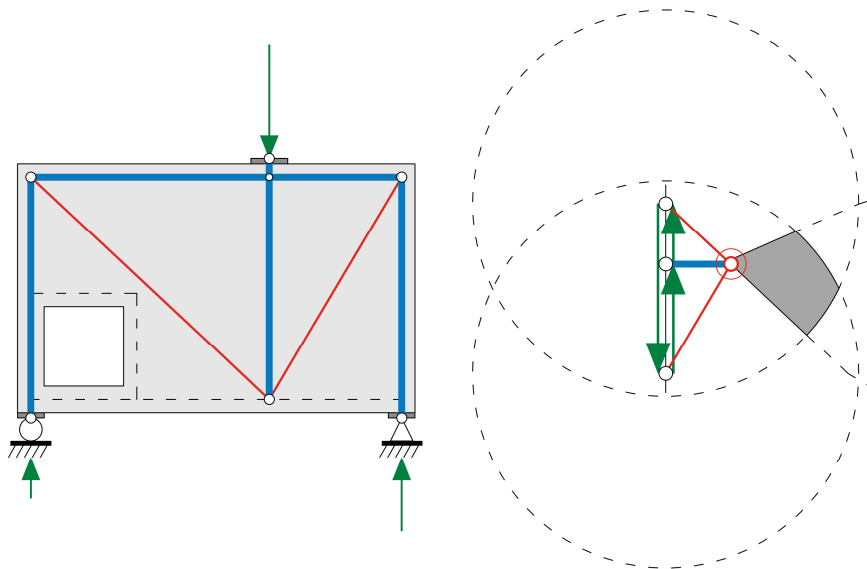


Figure 7.4 – Wall with one opening: catenary strut-and-tie network topology, considering a continuous upper strut and a middle strut of fixed vertical orientation

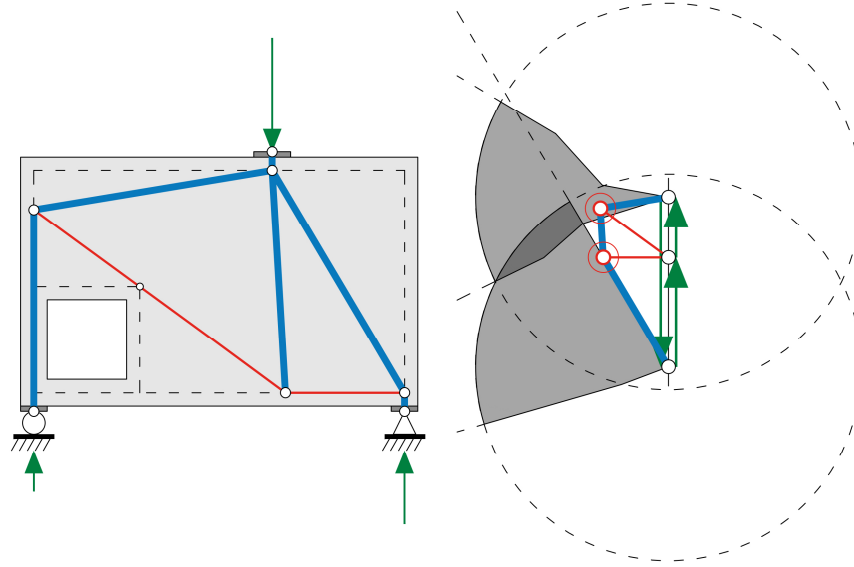


Figure 7.5– Wall with one opening: two-ties strut-and-tie network topology

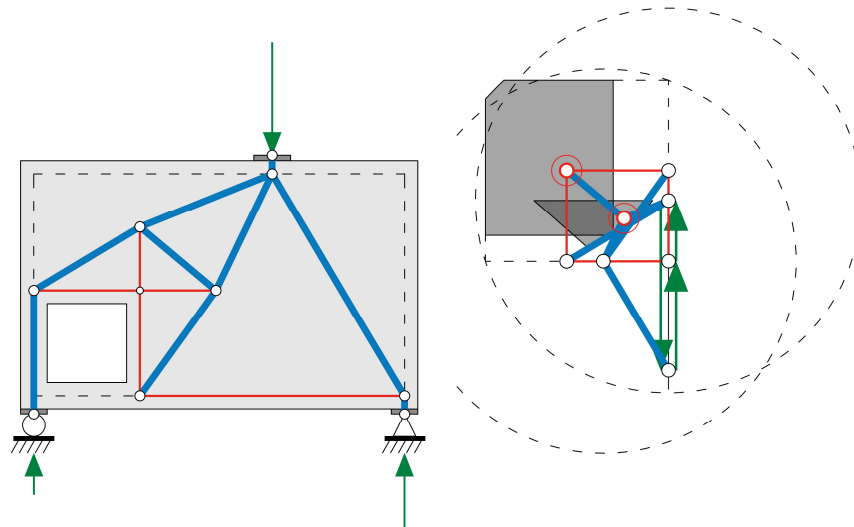


Figure 7.6– Wall with one opening: truss strut-and-tie network topology, considering that ties keep an orthogonal orientation

Admissible geometrical domains corresponding to force diagram's highlighted red nodes have been obtained for the four considered topologies through shape variation. Corresponding domain sizes are given in the second column of Figure 7.7, in terms of total measured area. Linear borders of the domains ensure that internal forces remain within structural

geometry and its cover $e = 25\text{cm}$. Considering that strut-and-tie networks are associated to prismatic discontinuous stress fields (see Section 2.1.3) that take rectangular sections, it comes that the largest dimensions of the field's sections is a length of two times the cover e and a depth equal to the wall's depth d . Hence, circular domain borders limiting internal forces to their maximum value F_{max} have a radius r which is given by:

$$r = \sigma_e 2 e d = 17\text{MPa} \times 2 \times 0.25\text{m} \times 0.4\text{m} = 3.4\text{MN} \quad \text{Eq 7.1}$$

7.1.3. Efficiency analysis

Taking back Maxwell's theorem of load paths as detailed in Section 3.2.1, values for the load path have been calculated for each point of the four sets of admissible geometrical domains depicted in the previous subsection. Considering a fully-stressed design (cfr. Section 2.1.4), optimising the shape of the strut-and-tie network topologies means finding the particular point position within their relative domain that leads to a minimal active quantity of material. These specific point positions are the ones already depicted in the previous Figures. The corresponding minimal load path values are given in the third column of Figure 7.7.

7.1.4. Stiffness analysis

The optimal shape (in terms of the volume of material required for its implementation) of each of the four strut-and-tie network topologies have been further analysed in terms of stiffness. For this purpose, those optimal shape networks have been modelled in Robot Structural Analysis [ROBOT], a finite element software. From the hypotheses taken in Section 7.1.2, each bar composing these strut-and-tie networks has been modelled with a rectangular section of $2e \times d$ (i.e. $50\text{cm} \times 40\text{cm}$), and is composed of a material whose strength equals 17MPa in both compression and tension. A Young modulus also had to be chosen for the networks' material; it was given a value of 35000MPa (i.e. Young's modulus for fibre-reinforced concrete). From these analyses, stiffness values k have been obtained as:

$$k = \frac{F}{f} \quad \text{Eq 7.2}$$

with F the value of the applied external load (3MN), and f the maximum displacement of the network's nodes found via the finite element analysis. Resulting values for k are given in the last column of Figure 7.7.

7.1.5. Comparison of results

Comparing the values obtained from robustness (in terms of aptitude for force redistributions), efficiency and stiffness analyses, it can be seen that the two-ties strut-and-tie network topology presents the best results whatever the analysis (Figure 7.7). This topology indeed leads to the largest admissible geometrical domain, hence to the best robustness properties in terms of aptitude for force redistributions. The two-ties strut-and-tie network topology also leads to the shape of minimal load path, which is also the stiffest. This meets the observation of Baker & al. [2012], which states that the optimal network is always the lightest and the stiffest at the same time. The optimal two-ties strut-and-tie network shape thus represents the structural behaviour that would be primarily activated in the structure, as it minimises deformation energy. These results are validated by the fact that the optimal two-ties strut-and-tie network shape in terms of material volume is the one which is the closest to the structure's continuous stress field model (see Figure 7.2).

STN topology	A_{dom} [MN ²]	LP_{min} [MNm]	k [MN/m]
Funicular (Figure 7.3)	0.712	38.57	697.67
Catenary (Figure 7.4)	3.261	55.37	576.92
Two-ties (Figure 7.5)	12.656	36.69	1 000.00
Truss (Figure 7.6)	7.084	42.66	882.35

Figure 7.7 – Domain sizes, minimal load paths and associated stiffness values for the four considered strut-and-tie network topologies

A return has been made on the original analyses developed by Schlaich & al [1987]. The authors considered a two-ties as well a truss strut-and-tie network topology (even if more complex) to model the path of internal forces within the wall. They then assumed that each strut-and-tie network topology would carry half the applied load when implemented simultaneously. However, as revealed by the present analyses, these two topologies of strut-and-tie networks are not equivalent, as the two-ties one is both more stiff and efficient in terms of active material volume. Hence, the two strut-and-tie network topologies from the original example may not support the same proportion of the external load when implemented simultaneously. Indeed, in practice, the stiffer of these two considered paths (i.e. the two-ties strut-and-tie network topology) may carry a greater part of the external load, to a proportion relative to its stiffness.

7.2. Wall with two openings

7.2.1. Preliminary hypotheses

The same type of analyses are now performed for a more complex case study. The window of Figure 7.2 has been turned into a door, and a new window has been added on the top right part of the structure (Figure 7.8).

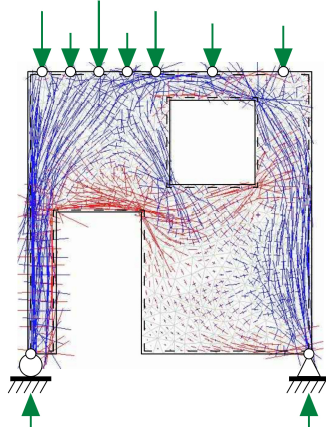


Figure 7.8 – Wall with two openings, and associated continuous stress field

This second wall is 10m long, 10m high and 50cm thick. The door measures 3x5m and is located at 1m from the left end of the wall. The window is 3x3m wide, and is located at 1m from the top of the wall and 2m from its right border. Seven external loads whose magnitude oscillates between 125 and 237.5kN are applied on top of the wall. A cover $e = 10\text{cm}$ has been considered (dashed lines), within which internal forces must remain. The wall is composed of the same material than these of Section 7.1.

7.2.2. Robustness level analysis

Five strut-and-tie network topologies have been analysed in terms of admissible geometrical domains: two funicular networks (Figure 7.9 and Figure 7.10), a catenary network (Figure 7.11), a fan-like network (Figure 7.12), and a truss network (Figure 7.13). The depicted domains, whose relative sizes are given in the second column of Figure 7.14 in terms of total measured area, show the allowable shape variations of the five considered networks' lowest tie, represented in the force diagram by the associated highlighted red node.

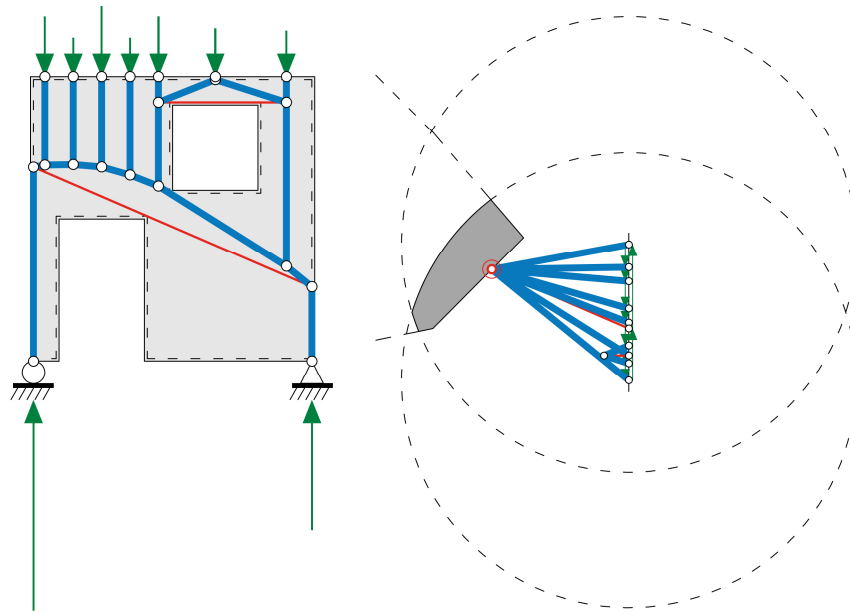


Figure 7.9 – Wall with two openings: first funicular strut-and-tie network topology

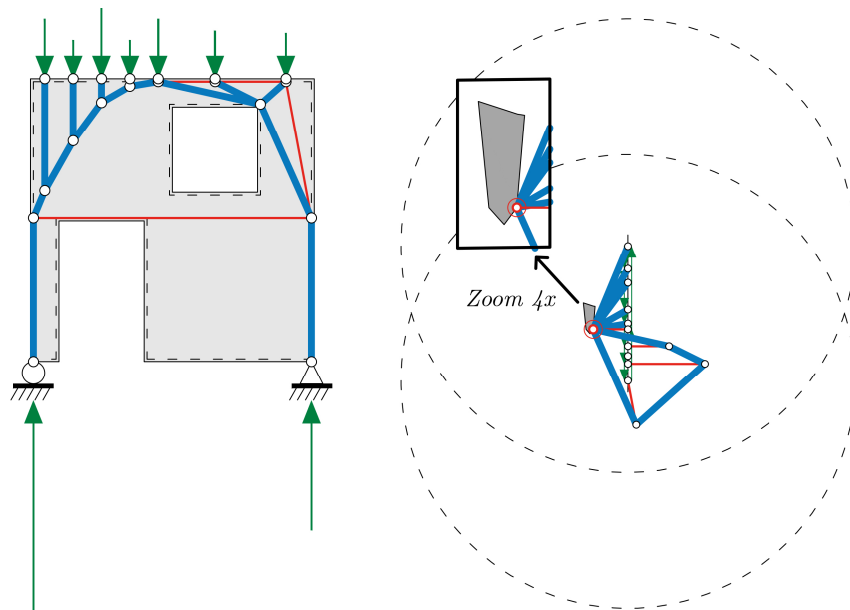


Figure 7.10 – Wall with two openings: second funicular strut-and-tie network topology

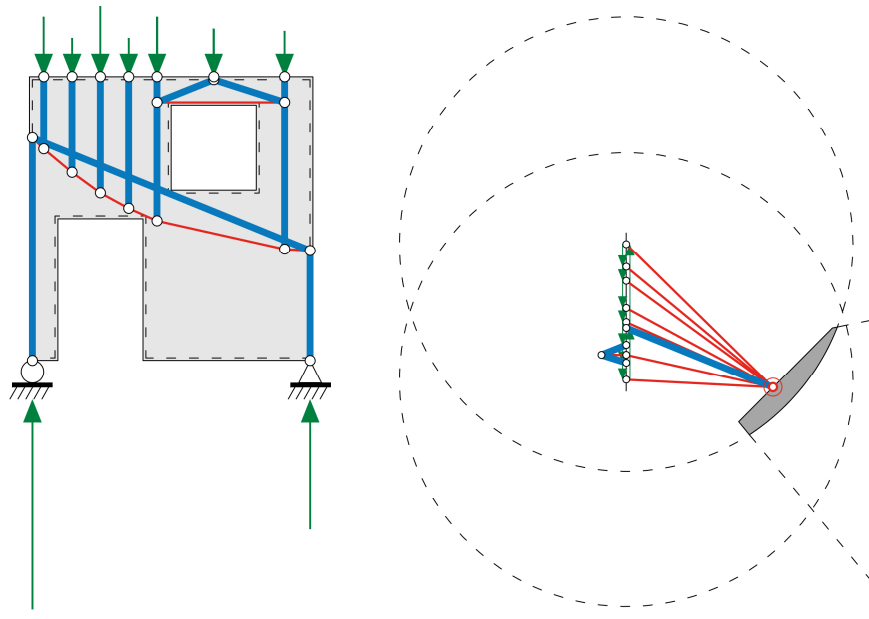


Figure 7.11 – Wall with two openings: catenary strut-and-tie network topology

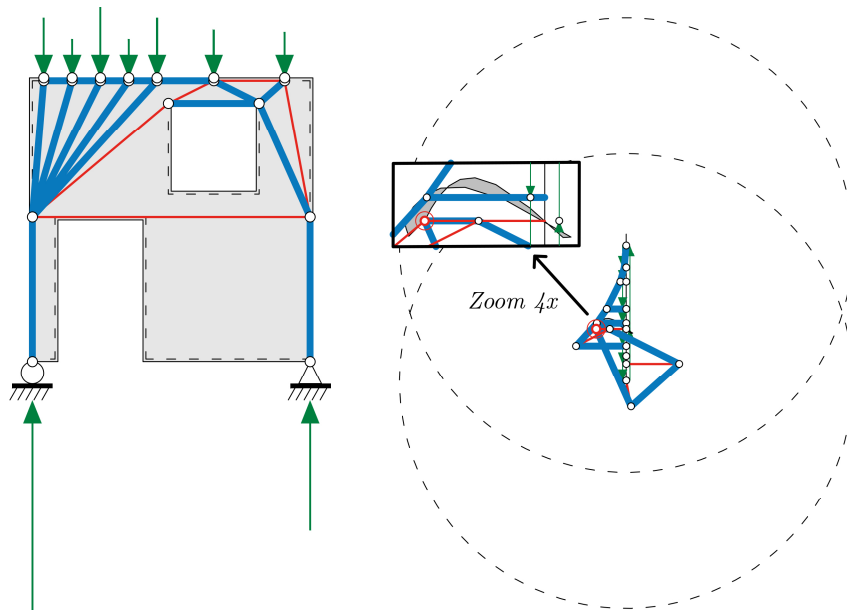


Figure 7.12 – Wall with two openings: fan-like strut-and-tie network topology

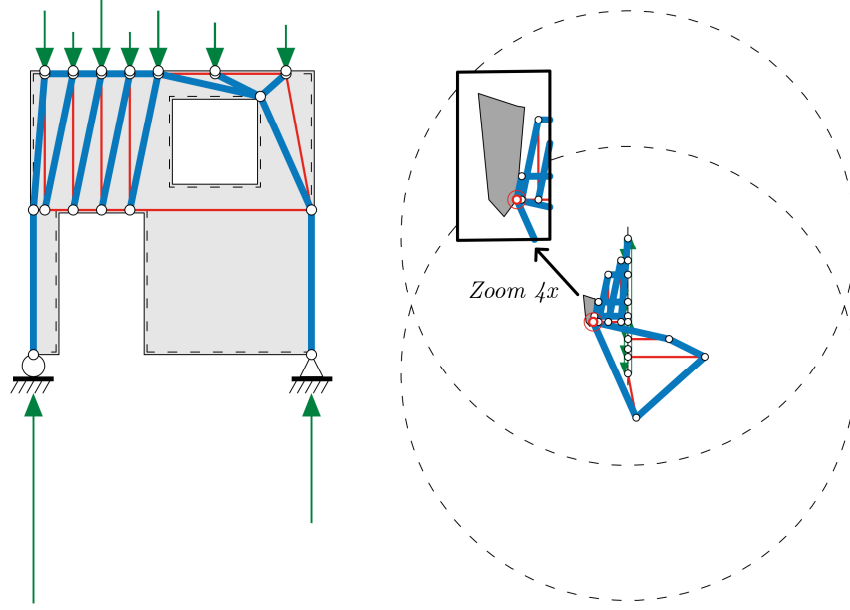


Figure 7.13 – Wall with two openings: truss strut-and-tie network topology, considering that vertical ties keep the same orientation

Linear borders of the domains ensure that internal forces remain within the chosen geometry and its cover $e = 10\text{cm}$. Following the same reasoning than in Section 7.1.2, the radius r of the circular borders of the domain limiting internal forces to their maximum value F_{max} is given by:

$$r = \sigma_e 2 e d = 17\text{MPa} \times 2 \times 0.1\text{m} \times 0.5\text{m} = 2\text{ MN} \quad \text{Eq 7.3}$$

7.2.3. Efficiency analysis

Following the same principle that in Section 7.1.3, shape optimisation has been performed on the five considered strut-and-tie network topologies, in order to obtain their minimal load path configuration. These particular strut-and-tie network shapes are the ones already depicted in the previous Figures. The numerical values associated to the volume of material required for their implementation are given in the third column of Figure 7.14.

7.2.4. Stiffness analysis

Following the same methodology than in Section 7.1.4, stiffness values have been calculated with Robot Structural Analysis for the optimal shapes (in terms of the volume of material required for their implementation)

associated to each of the five considered strut-and-tie network topologies. From the hypotheses taken in Section 7.2.2, each bar composing the strut-and-tie networks has been modelled with a rectangular section of $2e \times d$ (i.e. 20cm x 50cm), and is composed of a material whose strength equals 17MPa in both compression and tension. A Young modulus had to be chosen for the networks' material; it was also given a value of 35000MPa. Force F considered for the application of Eq 7.2 to this second structural case is the total external load (1195MN). The obtained stiffness values are given in the last column of Figure 7.14.

7.2.5. Comparison of results

Comparing the values obtained for robustness (in terms of aptitude for force redistributions), efficiency and stiffness analyses, it can be seen that the first funicular strut-and-tie network topology presents the best results according to the geometrical analysis (Figure 7.14). However, it is the fan-like strut-and-tie network topology that presents the best performances in terms of both stiffness and associated active volume of material. As for the case study example of Section 7.1, this means that the lightest strut-and-tie network is also the stiffest, thus the most efficient in terms of the volume of material required for its implementation. But this network is not necessarily the one presenting the best robustness properties in terms of aptitude for force redistributions. Hence, the most efficient strut-and-tie network topology in terms of the volume of material required for its implementation is not necessarily the safest. Therefore, this means that robustness properties may have to be designed specifically for a given structure, by implementing alternative structural mechanisms characterised by large admissible geometrical domains.

STN topology	A_{dom} [MN ²]	LP_{min} [MNm]	k [GN/m]
Funicular (Figure 7.9)	0.475	419.36	50.42
Funicular (Figure 7.10)	0.019	266.69	77.60
Catenary (Figure 7.11)	0.183	452.18	42.23
Fan-like (Figure 7.12)	0.008	258.36	134.27
Truss (Figure 7.13)	0.019	374.10	76.11

Figure 7.14 – Domain sizes, minimal load paths and associated stiffness values for the five considered strut-and-tie network topologies

7.2.6. Robustness level analysis for design alternatives

In this subsection, the first funicular strut-and-tie network topology as depicted in Figure 7.9 has been further analysed. The aim of the analysis is to view the impact of design manipulations on the structure's admissible geometrical domain, hence on its performance of robustness in terms of aptitude for force redistributions. In particular, the following design alternatives have been considered:

- original design (Figure 7.15a);
- displacement of the left support (Figure 7.15b);
- displacement of the right support (Figure 7.15c);
- cantilever structure (Figure 7.15d).

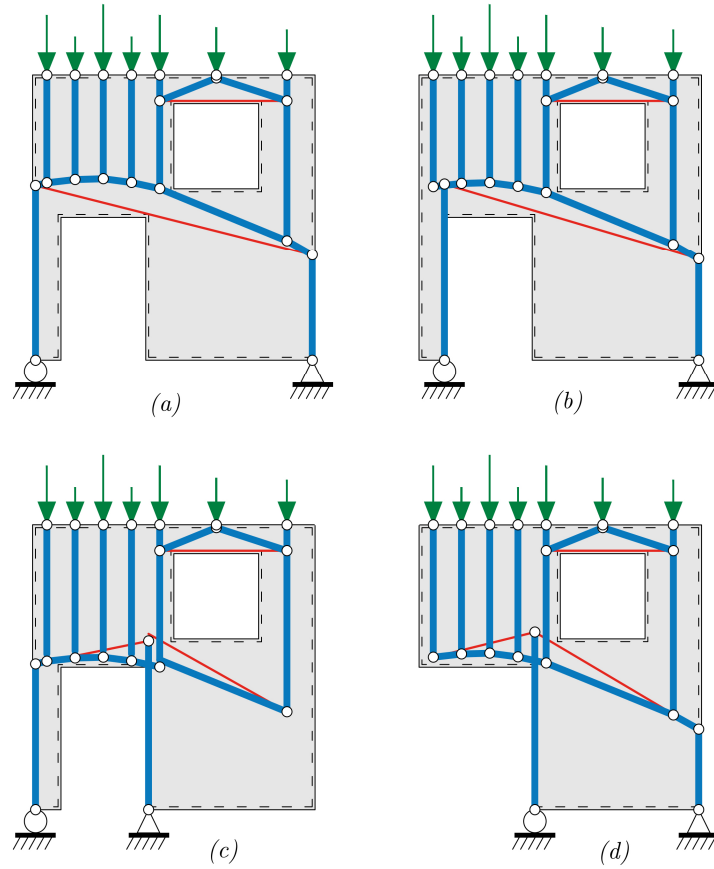


Figure 7.15 – Design alternatives for the first funicular topology

Using the geometrical method for robustness assessment, admissible geometrical domains have been drawn for these four design alternatives (Figure 7.16). Linear borders of these domains ensure that internal forces are kept within the cover $e = 10\text{cm}$, while circular borders guarantee that internal force magnitudes are kept below 2MN (see Eq 7.3). The analyses show that design alternative (b) results in the largest admissible geometrical domain. Hence, among the four considered design alternative, is it the one that should be favoured in order to enhance robustness properties in terms of aptitude for force redistributions. On the contrary, design alternative (d), with the cantilever supports, is the one that offers the less possibilities for force redistributions. Hence, its associated level of robustness is the lowest.

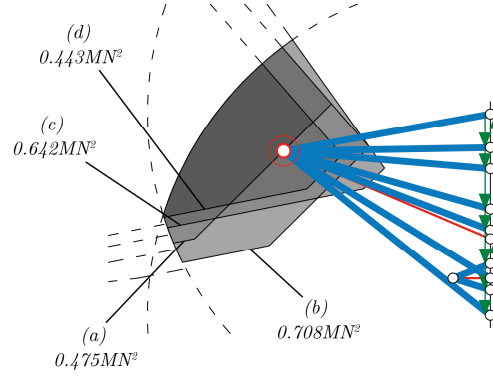


Figure 7.16 – Admissible geometrical domains for the four considered design alternatives

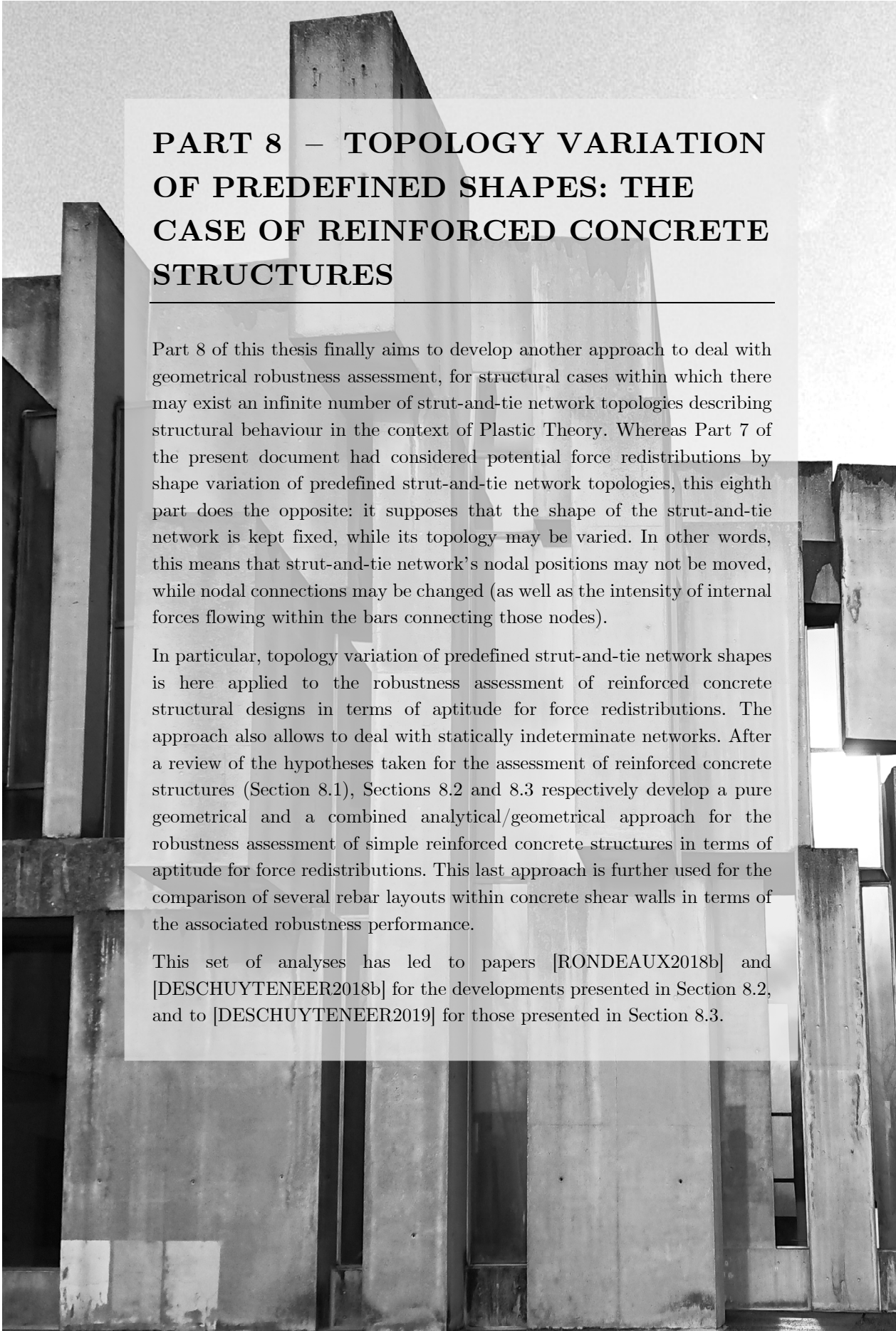
7.3. Summary and conclusion

Part 7 of this thesis has explored the geometrical robustness assessment of two walls, composed of an ideal material with similar properties in tension and compression. In the context of Plastic Theory, as there exists an infinity of strut-and-tie network topologies that may be implemented within such structures to model internal behaviour, the amount of these topologies has here been reduced to a few predefined ones for the purpose of the analyses. These strut-and-tie network topologies have then been varied in shape to obtain their corresponding admissible geometrical domain. These considered strut-and-tie network topologies have also been analysed regarding stiffness and efficiency in terms of required volume of material. In particular, Part 7 has shown that:

- the lightest strut-and-tie network confirms to be the stiffest. However, it is not necessarily the one presenting the largest admissible geometrical domain, hence the best robustness properties in terms of aptitude for force redistributions. This highlights the fact that the focus must be put on efficiency as well as on robustness performance while seeking for a design;
- the analyses reveal to be pertinent for comparing predefined strut-and-tie network topologies, both in terms of force redistributions associated to structural robustness (by the size of the admissible geometrical domain) and efficiency (by minimising load paths and maximising stiffness).

However, as the considered strut-and-tie network topologies are only preselected cases amongst the set of valid ones, resultant admissible geometrical domains are not characteristic of the global structural aptitude to redistribute internal forces. Indeed, in practice, there would exist numerous other topologies for force redistribution within given structures. In a same way, the minimum load path values found via shape optimisation of a particular strut-and-tie network topology may be useful for taking particular design decisions (e.g. the placement of rebars in concrete structures, see Part 8). But these values will not be representative of the absolute easiest way to distribute internal forces within the structure. In the context of geometrical robustness assessment, global structural performance in terms of aptitude for force redistributions may then not be characterised pertinently when studying admissible geometrical domains associated to particular strut-and-tie network topologies. Indeed, such a strategy only gives relative performances associated to robustness, but which may not describe well the overall structural performance in terms of aptitude for force redistributions.

Furthermore, the ideal material behaviour that has been considered within this seventh Part of the thesis may in practice only be suited for the design of steel structures, as steel has a ductile behaviour and similar compressive and tensile strengths. Hence, how may the proposed methodology be adapted to characterise real structures in terms of aptitude for force redistributions associated to robustness ?

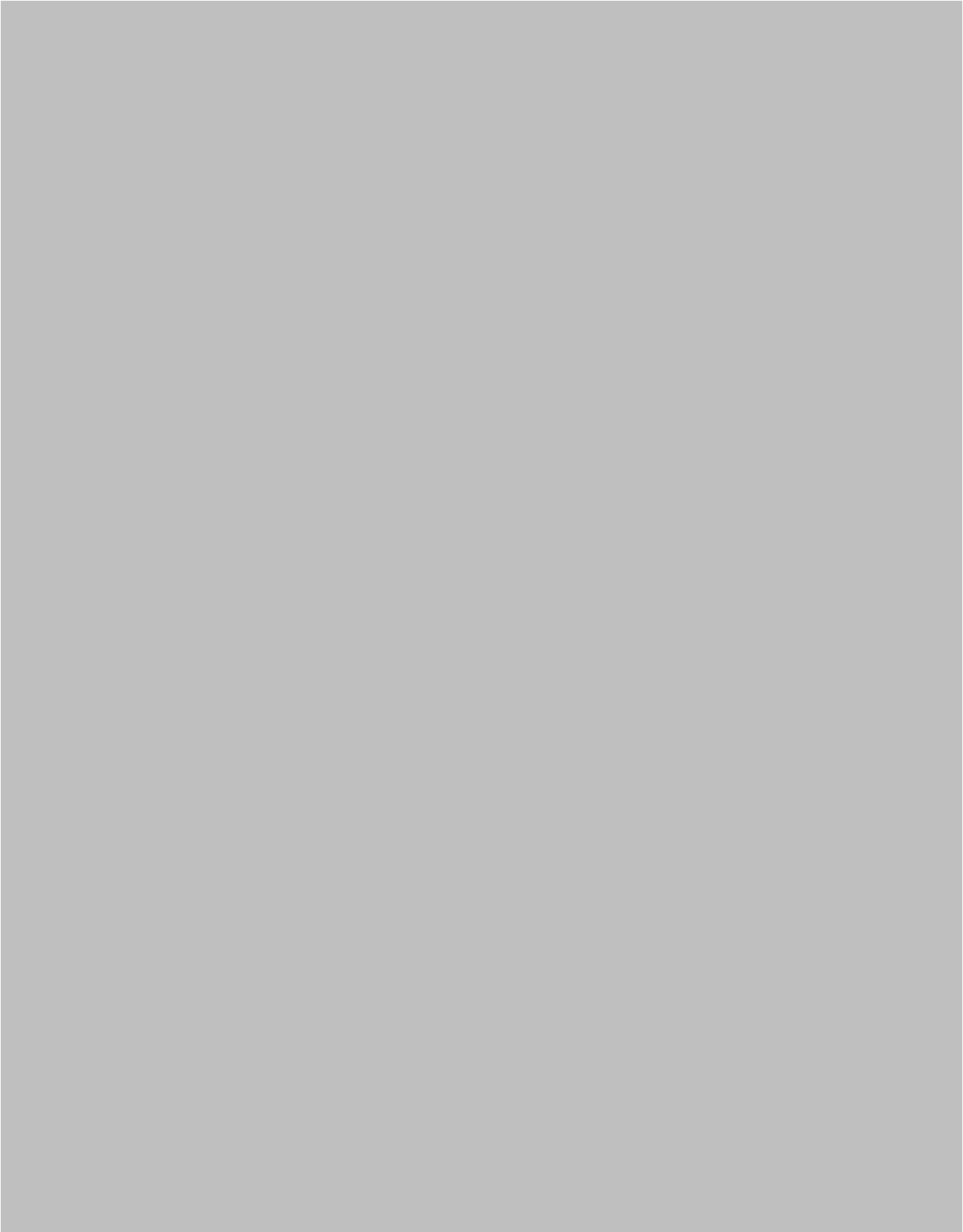


PART 8 – TOPOLOGY VARIATION OF PREDEFINED SHAPES: THE CASE OF REINFORCED CONCRETE STRUCTURES

Part 8 of this thesis finally aims to develop another approach to deal with geometrical robustness assessment, for structural cases within which there may exist an infinite number of strut-and-tie network topologies describing structural behaviour in the context of Plastic Theory. Whereas Part 7 of the present document had considered potential force redistributions by shape variation of predefined strut-and-tie network topologies, this eighth part does the opposite: it supposes that the shape of the strut-and-tie network is kept fixed, while its topology may be varied. In other words, this means that strut-and-tie network's nodal positions may not be moved, while nodal connections may be changed (as well as the intensity of internal forces flowing within the bars connecting those nodes).

In particular, topology variation of predefined strut-and-tie network shapes is here applied to the robustness assessment of reinforced concrete structural designs in terms of aptitude for force redistributions. The approach also allows to deal with statically indeterminate networks. After a review of the hypotheses taken for the assessment of reinforced concrete structures (Section 8.1), Sections 8.2 and 8.3 respectively develop a pure geometrical and a combined analytical/geometrical approach for the robustness assessment of simple reinforced concrete structures in terms of aptitude for force redistributions. This last approach is further used for the comparison of several rebar layouts within concrete shear walls in terms of the associated robustness performance.

This set of analyses has led to papers [RONDEAUX2018b] and [DESCHUYTENEER2018b] for the developments presented in Section 8.2, and to [DESCHUYTENEER2019] for those presented in Section 8.3.



*Figure 8.1 – Wotruba Kirche, Fritz Wotruba, Vienna, Austria, 1976
[picture by the thesis author]*

8.1. Preliminary hypotheses

8.1.1. Material behaviour

As detailed in Section 2.1.4, the method developed in the present work for geometrical robustness assessment takes into account strong hypotheses on material behaviour. Indeed, this behaviour has been considered as ideal, i.e. perfectly plastic. Concerning reinforced concrete structures, it may also be supposed an isotropic and uniformly strong material behaviour. This means that questions of concrete cracking and minimal/maximal quantities of steel reinforcement may not be considered within the analyses.

However, in reality, such an approximation of material behaviour may not be straightforward for reinforced concrete elements, as concrete usually presents a brittle behaviour in compression and has negligible strength in tension. In order to consider concrete as a ductile material, an effective compressive strength σ_{ce} may be obtained from the following formula [KOSTIC2009]:

$$\sigma_{ce} = f_{ck} \eta_{fc} \eta_{\epsilon} \eta_D \eta_{LCF} \eta_{LTL} \eta_t \quad \text{Eq 8.1}$$

with f_{ck} concrete's characteristic compressive strength on a cylindrical sample (fractile 5%). A series of strength reduction factors then decrease the initial value of f_{ck} in order to approximate a ductile behaviour, according to several parameters:

- $\eta_{fc} = (\frac{30}{f_{ck}})^{\frac{1}{3}} \leq 1$ for the augmentation of brittleness with strength;
- η_{ϵ} for the presence of lateral deformations, which is equal to 1 in case of axial loading;
- η_D for the influence of transverse cracking due to pretensioning (not applicable here);
- η_{LCF} for fatigue due to cyclic loading (not applicable at ultimate limit state);
- η_{LTL} for the time of application of external loads (not applicable at ultimate limit state);
- η_t for the increase of strength with time (majoring coefficient, thus neglected here).

Eq 8.1 thus simplifies in:

$$\sigma_{ce} = f_{ck} \min[1, (\frac{30}{f_{ck}})^{\frac{1}{3}}] \quad \text{Eq 8.2}$$

The design value for concrete compressive strength under axial loads σ_{ced}^* is then given by:

$$\sigma_{ced}^* = \alpha_{cc} \frac{\sigma_{ce}}{\gamma_c} = 0.85 \frac{\sigma_{ce}}{1.5} \quad Eq\ 8.3$$

with α_{cc} a coefficient taking into account the long term effects on strength, and γ_c the partial safety factor on material behaviour.

Concerning rebars, it can be considered that steel already possesses sufficient ductile properties. Hence, steel tensile strength σ_{ted} may be obtained from the following classical formulation:

$$\sigma_{ted} = f_{yd} = \frac{f_{yk}}{\gamma_s} = \frac{f_{yk}}{1.15} \quad Eq\ 8.4$$

with f_{yd} steel's design strength, f_{yk} its characteristic strength, and γ_s the partial safety factor on material behaviour.

8.1.2. Structural modelling

Part 8 considers that the behaviour of continuous reinforced concrete elements may be depicted by trusses that represent an equivalent meshing of the structure. Hence, structural behaviour may be modelled by ground structures, composed of a regular mesh of bars (see Figure 8.2 and Section 3.2.2). In particular, taking inspiration from the work of Kostic [2009] (see Figure 3.13 in Section 3.2.2), such ground structures may take the form of a regular and orthogonal mesh, whose geometry is then close to the way in which rebars are usually placed within reinforced concrete structures. As it is based on a layout of nodes and bars, a ground structure can easily be turned further into a (statically indeterminate) strut-and-tie model. This network can then be analysed using graphic statics, i.e. a reciprocal force diagram and its associated admissible geometrical domain can be built.

As developed in Part 4 and 7, alternative force distributions may be reached in structures made of an ideal material by changing nodal locations within the strut-and-tie network describing the structure's internal state of stress. This changes the strut-and-tie network's shape. In the present Part, another manner to redistribute forces within given (reinforced concrete) structures is explored. It has here been chosen to change nodal connections in the strut-and-tie network, while its shape remains fixed. In other words, this means fixing the network's nodal locations while changing nodal

connections (in the classical terms of bar presence/absence, but also in terms of the force intensity carried within the implemented bars).

However, in order to model reinforced concrete structures using strut-and-tie networks, two more conditions than those detailed in Section 2.1.4 have to be taken into account. First, *compatibility of deformations* has to be ensured in every part of the structure [SCHLAICH1987]. This means that the angle α between any pair of compressed and tensed bars of the network has to verify the following inequality, as required by the Eurocode related to concrete design [EN1992]:

$$\tan(\alpha) \geq 0,5 \Leftrightarrow \alpha \geq 26.5^\circ \quad \text{Eq 8.5}$$

Considering square-based cells, this means that the ground structures giving valid strut-and-tie networks to model the behaviour of reinforced concrete elements are those for which the distance of connection is equal to 1 or 2 (Figure 8.2).

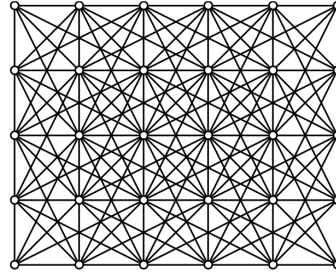


Figure 8.2 – Ground structure with distance of connection $d=2$, giving potential strut-and-tie network bar locations to model reinforced concrete structures

The second additional condition concerns the *serviceability* of reinforced concrete structures, by limiting deformations and crack opening [KOSTIC2009]. However, as the focus of the present work is to characterise structural behaviour at ultimate limit state, this last consideration has not been taken into account, and cracking of concrete has been neglected.

As ground structures give strut-and-tie networks that present internal static indeterminacy (eventually added of some external static indeterminacy, depending on the number and type of supports), topology variation results in a scaling of independent fractions of the associated force diagram (cfr. Section 4.5.3). To do so, two types of approaches can be followed: either by discretisation of scaling values (Section 8.2), either by analytical pre-obtaining of limit scaling values (Section 8.3).

8.2. Pure geometrical approach

8.2.1. Description of the structure

The approach presented here follows the preliminary work of Rondeaux [2017], which accounted for dealing with internal static indeterminacy in the context of graphic statics' reciprocal diagrams. This work has here been extended to external static indeterminacy. Thus the structure considered for the sake of the present study takes inspiration from the one detailed in the abovementioned paper. It consists in a reinforced concrete square shear wall whose diagonal measures 2.1m, as depicted in Figure 8.3. The wall is loaded at the top by a horizontal point load $Q = 0.23\text{MN}$, and is supported by two rollers and a pin support. Hence, the structure's degree of external static indeterminacy is given by $h_e = r - \mathcal{P} = 4 - 1 = 1$ (cfr. Eq 4.3). The strut-and-tie network that has been chosen to model the resultant path of internal forces takes the form of a square set at an angle of 45° , with two diagonals that measure $2 \times 1\text{m}$ each (dashed lines in Figure 8.3). This strut-and-tie network is placed within the structure with a cover $e = 5\text{cm}$. Dashed lines represent candidate positions for concrete struts or steel ties, i.e. each line can either be turned into a strut or a tie. Hence, the degree of internal static indeterminacy of the network is given by $h_i = b + \mathcal{P} - 2n = 8 + 3 - 2 \cdot 5 = 1$ (cfr. Eq 4.4). Note that the distance of connection d of the strut-and-tie network, kept simple for the sake of the explanation, is here only equal to 1; but it could take a value of 2 for more complex networks.

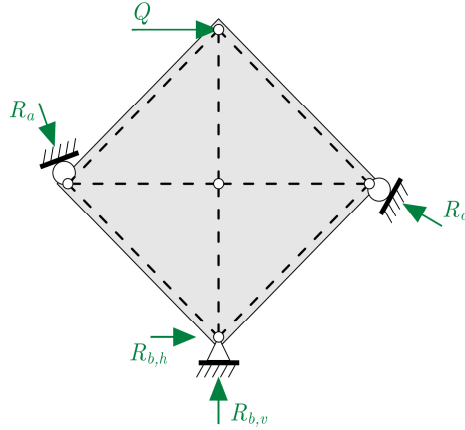


Figure 8.3 – Structure under consideration, and associated strut-and-tie network

Before assessing the structure's aptitude for force redistributions using admissible geometrical domains, material type and effective strength need to be set. Supposing a C25/30 class for concrete and BE500s steel, it comes, from Eq 8.2 to Eq 8.4:

$$\sigma_{ced}^* = \frac{0.85 \times 25\text{MPa}}{1.5} \min[1, (\frac{30\text{MPa}}{25\text{MPa}})^{\frac{1}{3}}] \approx 14\text{MPa} \quad \text{Eq 8.6}$$

$$\sigma_{ted} = \frac{500\text{MPa}}{1.15} = 435\text{MPa} \quad \text{Eq 8.7}$$

Next subsection will now detail how the structure's reciprocal diagrams can be constructed while accounting for scaling of the force diagram.

8.2.2. Construction of the parametrical force diagram

For statically indeterminate structures like multi-supported reinforced concrete elements, constructing graphic statics' reciprocal diagrams requires to manipulate force diagrams that cannot be unequivocally defined for a given external loading. However, as detailed in Section 4.5.3, a structure's parametrical force diagram can easily be obtained using Rhino [RHINO] and Grasshopper [GRASSHOPPER]. Going back to the structure depicted in Figure 8.3, the associated parametrical force diagram can be drawn by setting as much mobile nodes in the force diagram that there are degrees of static indeterminacy within the associated strut-and-tie network in the form diagram. For the present case, this means that the structure's force diagram can be parameterised by two mobile nodes, as $s = h_i + h_e = 1 + 1 = 2$ (cfr. Eq 4.2). These nodes are named x_1 and x_2 , and are highlighted in red in Figure 8.4.

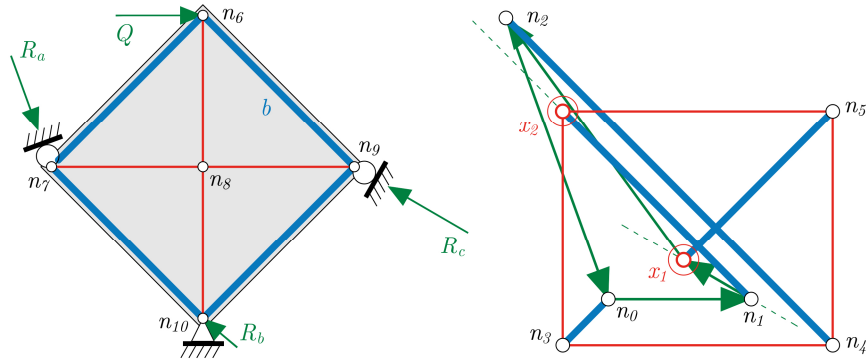


Figure 8.4 – Graphic statics' parameterised reciprocal diagrams

In particular, node x_1 is associated to the scaling of the partial external force diagram. This parameter has here been chosen as the position of reaction force R_c 's end node along the force's line of action. Moving node x_1 along this line will then lead to a force redistribution between the structure's three supports. This means that reaction forces R_a and R_c may vary in intensity only (as these are associated to roller supports), whereas R_b may vary in intensity and orientation (because of the pin support it is related to). Hence, location of node n_2 in the force diagram may vary with the displacement of the parameterised node x_1 , while nodes n_0 and n_1 remain fixed.

In the same way, node x_2 is responsible for the scaling of the partial internal force diagram. This parameter has here been chosen as the position of one of bar b 's end node in the force diagram, along a line following the bar's direction. Moving node x_2 along this line then leads to force redistributions between the strut-and-tie network's eight bars in the form diagram (i.e. Figure 8.3's eight dashed lines). Hence, position of nodes n_3 , n_4 and n_5 may vary with the displacement of the parameterised node x_2 .

Note that the manipulation of force diagram's parametrical nodes x_1 and x_2 does not affect the position of form diagram's nodes (n_6 to n_{10}). Indeed, these nodes always remain of fixed position by hypothesis: only the topology of the considered strut-and-tie network may vary within this eight part of the thesis, while the network's shape remains unchanged.

8.2.3. Generation of the admissible geometrical domain

In a statically indeterminate strut-and-tie network for which the topology is fixed, redistributing internal forces means changing the fraction of the applied load carried by each bar L_i of the strut-and-tie network. In other words, this means changing the position of parametrical force diagram's mobile nodes (highlighted in red) along their line of action, in order to modify the magnitudes of the associated axial forces F_i . Following this approach, admissible geometrical domains depicting the range of possible force redistributions within the structure have been obtained using Grasshopper [GRASSHOPPER], by sweeping the whole range of admissible positions for mobile nodes x_1 and x_2 in the parametrical force diagram (cfr. Section 4.5.3). Valid combinations of locations for nodes x_1 and x_2 are defined as those for which there is no yielded bar in the strut-and-tie network. For compressed paths, this means that each axial force magnitude $F_{c,i}$ in the force diagram has to verify the following equation:

$$\begin{aligned}
F_{c,i} &\leq F_{c,max} = \sigma_{ced}^* A_{c,max} \\
&= 14\text{MPa} \times \pi \times (5\text{cm} \times 10 \frac{\text{mm}}{\text{cm}})^2 = 110.0\text{kN} \quad \text{Eq 8.8}
\end{aligned}$$

with material effective strength σ_{ced}^* as given by Eq 8.6. $A_{c,max}$ is the maximum area that can be allocated for the concrete struts, which is only limited for those placed near the border of the wall. Supposing that struts are associated to prismatic discontinuous stress fields (see Section 2.1.3) that take circular sections, it comes that their radius must be smaller or equal to the 5cm cover.

In the same way, each bar length $F_{t,i}$ in the force diagram associated to a tensile path has to verify the following equation:

$$\begin{aligned}
F_{t,i} &\leq F_{t,max} = \sigma_{ted} A_{t,max} \\
&= 435\text{MPa} \times 4 \times \pi \times (\frac{12\text{mm}}{2})^2 = 196.8\text{kN} \quad \text{Eq 8.9}
\end{aligned}$$

with material effective strength σ_{ted} as given by Eq 8.7. $A_{t,max}$ is the maximum area that can be allocated for the steel ties, which is fixed by the number and diameter chosen for rebars. The above value has been obtained for 4Ø12 steel rebars.

For the present case study, force diagram's parameters are x_1 governing reaction force in support R_c , and x_2 governing force intensity inside bar b , as detailed in Figure 8.4. As done in Part 5 for masonry arches, the admissible geometrical domain associated to the structure's aptitude for force redistributions has here been obtained by discretisation. The level of discretisation w of the solution has been chosen as the number of different values that may be taken for each parameter x_s within its range of admissible ones. Combining these values together, this means that w^s scalings of the force diagram are candidate solutions to the structural problem. These solutions are then sorted between those in which all bars verify Eq 8.8 or Eq 8.9 (the valid ones) and those trespassing material strength (the invalid ones). Strut-and-tie networks associated to valid combinations of internal forces have finally been drawn into a common force diagram, from which the admissible geometrical domain is obtained as the set of sub-domains containing each bar's valid end nodes (Figure 8.5). Note that these sub-domains have only been depicted for nodes that are not linked to an external load or a support reaction, hence for nodes x_2 , n_3 , n_4 and n_5 only (highlighted in red in Figure 8.5). Indeed, domains are used within the present work to depict redistributions of internal forces (see Section 4.1), without considering redistributions of support reactions.

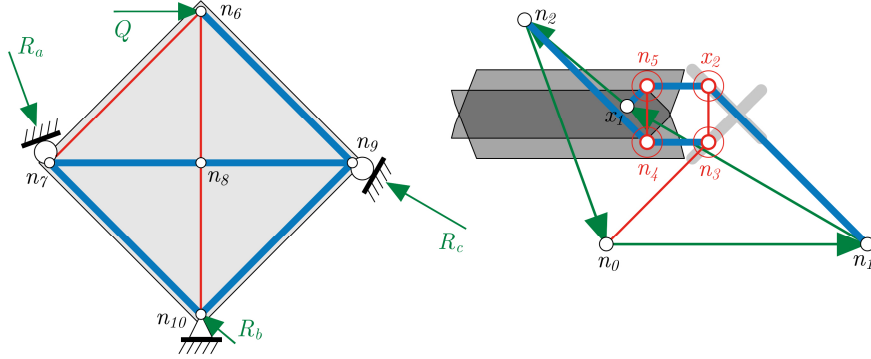


Figure 8.5 – Admissible geometrical domain for discretisation $w = 100$

Note that sub-domains associated to force diagram's nodes x_2 and n_3 take the shape of a line instead of a surface. This is because the shape of the strut-and-tie network describing structural behaviour may not be varied according to the approach followed in the present Part (topology variation for a fixed shape). Hence, some nodal positions may only be moved along particular directions unless violation of the conditions of Section 2.1.4 on the strut-and-tie network.

Also, as mentioned in Section 4.5.1, the structure's associated sub-domains are interdependent. Hence, the dimension of the structural problem may be set equal to $\nu \cdot a = 2 \cdot 2 = 4$, with a the number of mobile nodes within the structure's force diagram, i.e. the number of parameters that may be set to draw the structure's parametrical force diagram (cfr. Figure 8.4). In the present case, a is also equal to the degree of static indeterminacy s of the considered structure. Hence, the size of the admissible geometrical domain depicting the structure's level of robustness in terms of aptitude for force redistributions may be expressed in N^4 .

8.2.4. Efficiency analysis

Once parameterised, the structure's force diagram can further be employed for the efficiency analysis of the considered strut-and-tie network topologies in terms of the volume of material required for their implementation. In particular, optimisation and variation have here been performed according to four concerns that may be relevant for the design of reinforced concrete structures: minimisation of the total volume of material (load path), minimisation of the volume of ties, minimisation of the volume of struts, and homogenisation of the compressive forces.

Minimisation of the total volume of material

Using Eq 3.5 from Section 3.2.1, load path values have been obtained for each point of the admissible geometrical domain. These values give the efficiency of each possible force distribution within the strut-and-tie network shape in terms of the volume of material required for their implementation. This set of values has further been turned into a colour scheme, as depicted in Figure 8.6. The force distribution associated to the minimum load path value is the one that has been depicted in Figure 8.6. It gives $LP_{min} = 0.819MNm$, for $F_{Rc} = 0.299MN$ and $F_b = 0.212MN$.

Minimisation of the volume of ties

The minimal volume of ties, corresponding to the minimal load path in tension $LP_{t,min}$, has been obtained by applying Eq 3.5 to bars in tension only (Figure 8.7). This gives $LP_{t,min} = 0.23MNm$, for $F_{Rc} = 0.23MN$ and $F_b = 0.163MN$.

Minimisation of the volume of struts

The minimal volume of struts, corresponding to the minimal load path in compression $LP_{c,min}$, has been obtained by applying Eq 3.5 to bars in compression only. This gives $LP_{c,min} = 0.519MNm$, for $F_{Rc} = 0.299MN$ and $F_b = 0.212MN$. Hence, the associated strut-and-tie network corresponds to these already depicted in Figure 8.6.

Homogenisation of the compressive forces

For each valid force distribution j , the mean value of the compressive forces $F_{c,mean,j}$ has been obtained from:

$$F_{c,mean,j} = \frac{\sum_i F_{ci,j}}{i} \quad Eq\ 8.10$$

with i the number of compressed bars in force distribution j . A tolerance value t_j is then obtained by penalising each bar accordingly to the difference between the magnitude of its corresponding compression force $F_{ci,j}$ and the mean compressive force $F_{c,mean,j}$:

$$t_j = \sum_i [F_{ci,j} (F_{ci,j} - F_{c,mean,j})] \quad Eq\ 8.11$$

Force distribution j associated to the minimum value for t_j is then selected, as it is the one leading to compressed elements having the most similar internal force (Figure 8.8). This leads to $F_{c,mean,j} = 0.175MN$, for $F_{Rc} = 0.310MN$ and $F_b = 0.175MN$.

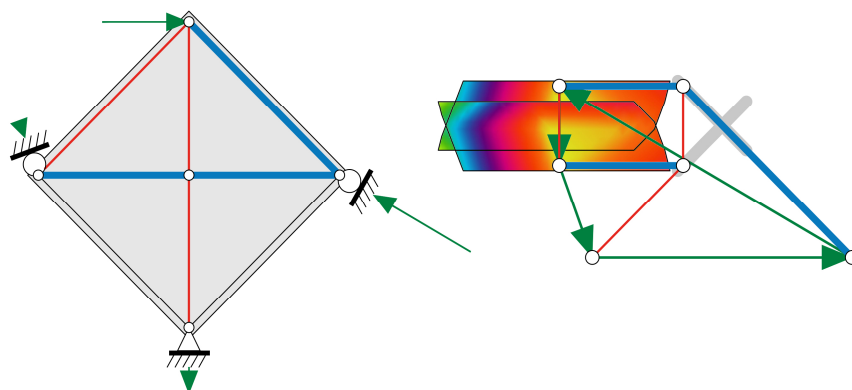


Figure 8.6 – Load path values and strut-and-tie network scaling associated to the minimum load path value, as well as to the minimal one for compressed bars only

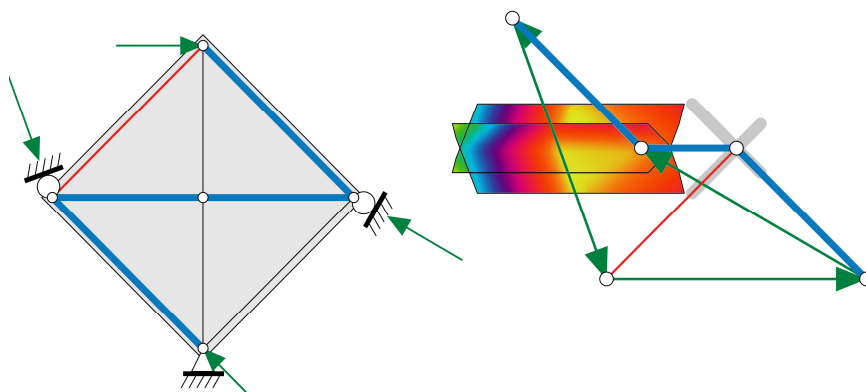


Figure 8.7 – Strut-and-tie network scaling associated to the minimum load path value for tensed bars only

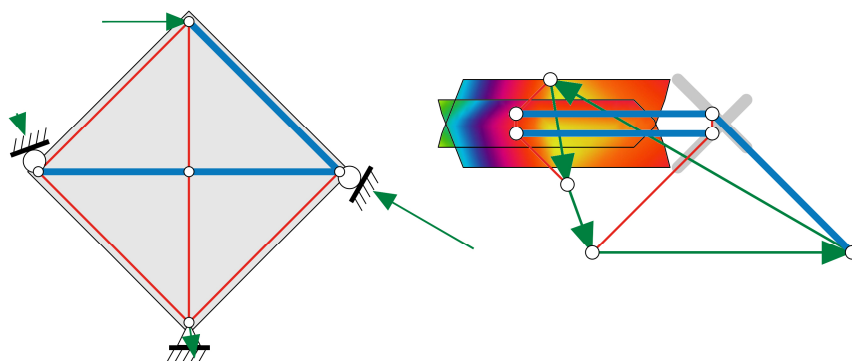


Figure 8.8 – Strut-and-tie network scaling associated to the mean value for compressive forces

8.2.5. Drawbacks

The methodology for structural analysis presented in this Section has given a conceptual approach for dealing with static indeterminacy while assessing the level of robustness of concrete structures regarding aptitude for force redistributions. However, the considered structural problem is for now formulated in such a way that it takes a time proportional to w^s to be solved, with w the level of discretisation and s the degree of static indeterminacy (see Section 8.2.3). Hence, the proposed process quickly becomes time-consuming when seeking for a refined and more accurate solution for the admissible geometrical domain (i.e. when increasing the level of discretisation w) and/or when increasing the complexity of the structure under consideration (i.e. when increasing its degree of static indeterminacy s , typically when $s > 2$).

Next Section will now detail an alternative methodology for the assessment of potential force redistributions within given structures that reduces calculation times. It is based on the observation that admissible geometrical domains associated to topology variation of orthogonal strut-and-tie networks are always polygonal and convex. This means that all the points comprised within the domain's boundaries will always correspond to valid solutions, and reciprocally that no valid point may be found outside the domain's polygonal boundary. Hence, to characterise a given structural problem in terms of aptitude for force redistributions, it may be sufficient to study its limit force distributions, which are associated to the corner points of the admissible geometrical domain. To do so, an analytical pre-study may first be performed on the structure under consideration.

8.3. Combined analytical/geometrical approach

8.3.1. Description of the structure

The structure considered within the proposed combined methodology consists in a reinforced concrete deep beam, which is 2.1m long, 1.1m high and 20cm thick (Figure 8.9). The wall is loaded on the top by a vertical point load $Q = 10kN$. It is simply supported; hence, its degree of external static indeterminacy is given by $h_e = r - 3 = 3 - 3 = 0$ (cfr. Eq 4.3). The strut-and-tie network modelling the resultant path of internal forces acting within the structure is composed of 11 candidate bars L_i (dashed lines in Figure 8.9), which are placed with a cover $e = 5cm$ from the edge of the

wall, and which will later represent potential locations for struts or ties. These 11 bars take the form of two squares of 1m wide, with two diagonals that do not intersect into a node. Hence, the degree of internal static indeterminacy is given by $h_i = b + 3 - 2n = 11 + 3 - 2 \cdot 6 = 2$ (cfr. Eq 4.4). It then comes that the total degree of static indeterminacy of the structure is given by $s = h_i + h_e = 2$. Note that the distance of connection d of the strut-and-tie network has been kept to 1 for the sake of the analysis, but it could have taken a value of 2.

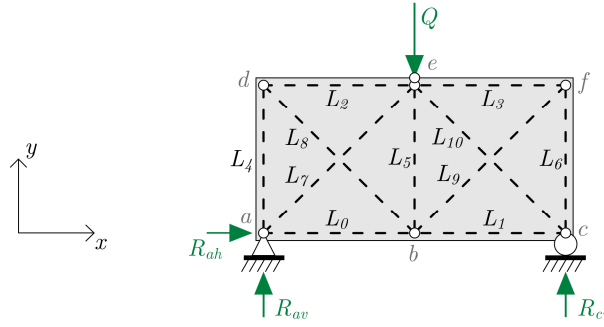


Figure 8.9 – Structure under consideration, and associated strut-and-tie network

Supposing that the structure is made of C25/30 concrete and BE500s steel, it comes that material strengths are defined by $\sigma_{ced}^* = 14MPa$ for concrete (cfr. Eq 8.6) and $\sigma_{ted} = 435MPa$ for steel (cfr. Eq 8.7). To speed up the robustness analysis of the structural problem in terms of aptitude for force redistributions, an analytical pre-processing of structural equilibrium has first been performed. This allows the obtention of the limit internal force combinations corresponding to the corner points of the structure's admissible geometrical domain, which are the only points that are required to draw this domain (as it is, as mentioned in the previous subsection, a polygonal and convex set of solutions).

8.3.2. Gauss-Jordan elimination

The structure's equilibrium equations are first written as an augmented matrix, under the form $(A / b) x = 0$. The system possesses 12 lines, one for each of the 6 nodes' $-x$ and $-y$ directions of the strut-and-tie network depicted in Figure 8.9. Matrix A has 14 columns, one for each of the 11 bars L_i of the strut-and-tie network, and one for each of the three support reactions R_i . b is a column vector that contains the external load Q . x is the vector containing the values for internal forces F_i and for support reactions R_i :

$$\left[\begin{array}{cccccccccccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0.71 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
-1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & -0.71 & 0.71 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0.71 & 0.71 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0.71 & 0 & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & -0.71 & 0 & 0 & 0.71 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 0 & -0.71 & 0 & 0 & -0.71 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & -0.71 & 0 & 0 & 0 & 0 & 0
\end{array} \right] \begin{array}{c} F_0 \\ F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ R_{ah} \\ R_{av} \\ R_{cv} \\ 10 \end{array} = 0$$

L_i
 R_i
 Q

To obtain such equilibrium equations under a simplified form, Gauss-Jordan elimination has been performed on matrix $A|b$ to transform the above system into a reduced row echelon form [STATLECT]. For an initial matrix A of size $M \times N$ ($N \geq M$), this means performing numerical operations on the rows of $A|b$ such as to transform the M first columns of matrix A into an identity matrix. In the present case, columns corresponding to the indeterminate unknowns F_0 and F_1 have first been placed as the last two columns of matrix A (and the vector of unknowns x has been adapted in consequence). Hence, the system becomes:

$$\left[\begin{array}{cccccccccccccccc|cc}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0.5 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0.5 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0.5 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & -1 & 1 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0.5 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1.41 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1.41 & 0 & -0.71 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1.41 & -0.71 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1.41 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & -0.5 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & -0.5
\end{array} \right] \begin{array}{c} F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \\ F_7 \\ F_8 \\ F_9 \\ F_{10} \\ R_{ah} \\ R_{av} \\ R_{cv} \\ F_0 \\ F_1 \\ 10 \end{array} = 0$$

a
 b
 c

8.3.3. Building the convex hull of admissible scalings

To guarantee admissible sets of internal force distributions within the chosen strut-and-tie network shape, material strength must not be exceeded in any part of the structure. This means that the following inequality has to be verified for each bar F_i of the network:

$$-F_{c,max} \leq F_i \leq F_{t,max} \quad \text{Eq 8.12}$$

where :

$$\begin{cases} F_{c,max} = \sigma_{ced}^* A_{c,max} = 14\text{MPa} \times \pi \times (5\text{cm} \times 10\frac{\text{mm}}{\text{cm}})^2 = 110.0\text{kN} \\ F_{t,max} = \sigma_{ted} A_{t,max} = 435\text{MPa} \times 2 \times \pi \times (\frac{12\text{mm}}{2})^2 = 98.4\text{kN} \end{cases}$$

with σ_{ced}^* material effective strength in compression and σ_{ted} those in tension, as given by Eq 8.6 and Eq 8.7 respectively. $A_{c,max}$ is the maximum area that can be allocated for the concrete struts. Supposing that struts are associated to prismatic discontinuous stress fields (see Section 2.1.3) that take circular sections, it comes that their radius must be smaller or equal to the $e = 5\text{cm}$ cover. In a same way, $A_{t,max}$ is the maximum area that can be allocated for the steel ties, which is given by the number and diameter of rebars. The above value has been obtained for 2Ø12 steel rebars.

Based on the last columns of the reduced row echelon matrix, the first line of Eq 8.12 can be rewritten for bars $F_{i=2...10}$, as:

$$-F_{c,max} \leq F_i = -(aF_0 + bF_1 + cQ) \leq F_{t,max} \quad \text{Eq 8.13}$$

with coefficients a , b , and c as respectively given by the three last columns of the reduced row echelon matrix. Note that the same reasoning could have been applied for constraining support reactions R_i between specific values. However, for the sake of the present example, forces at the supports have been kept unbounded.

Eq 8.13 gives 11 pairs of inequalities that must be verified by any force distribution to be admissible regarding yield. From a geometrical point of view, these inequalities may be drawn as halfspaces (the dashed lines in Figure 8.10) within a graph whose dimension equals the structure's degree of static indeterminacy s , i.e. 2. The $-x$ axis of Figure 8.10 stands for the force within bar F_0 ; $-y$ for those in F_1 . The light grey surface comprised within the dashed lines gives the structure's convex hull of admissible scalings [QHULL, BARBER1996]. It depicts the set of valid combinations (F_0, F_1) that are solution to the structural problem, considering that 2Ø12 steel rebars have been placed at each location L_i of Figure 8.9.

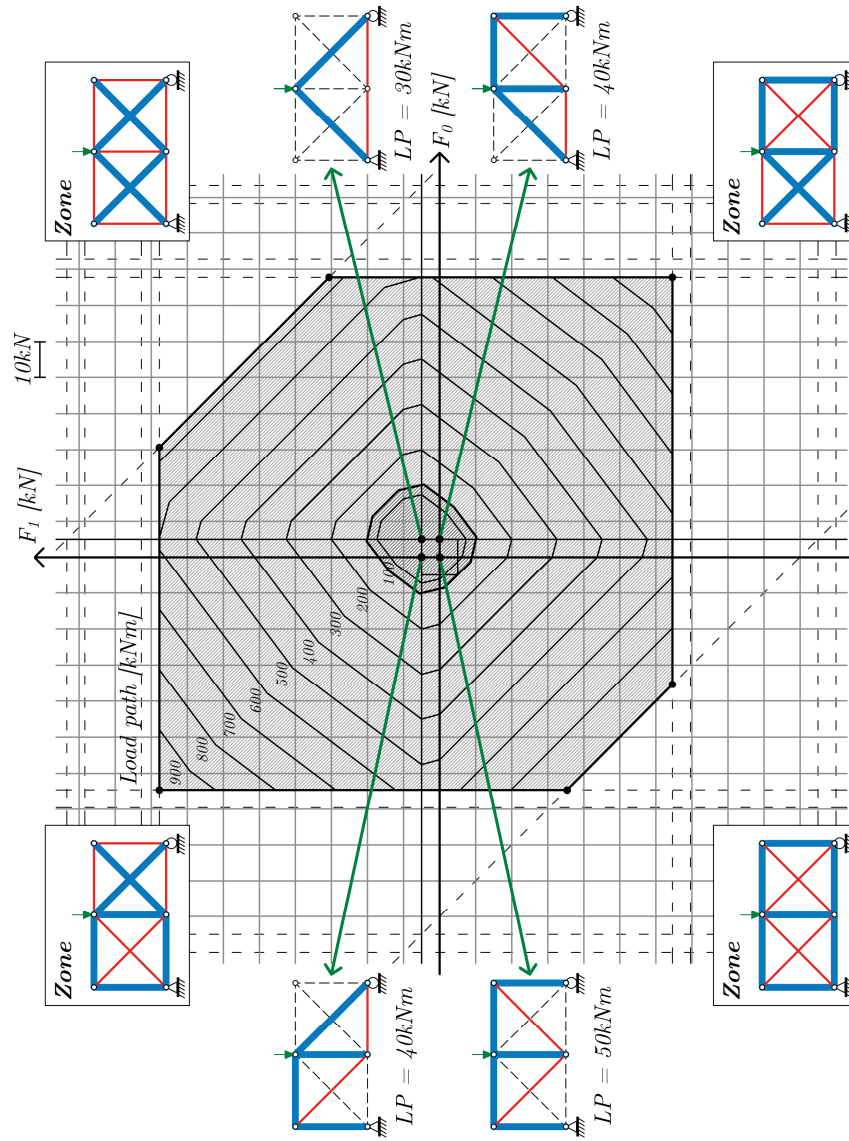


Figure 8.10 – Convex hull of admissible scalings associated to the structural problem, and corresponding load path values

It has been shown in Section 8.3.2 that the structure under consideration is $s = 2$ times internally statically indeterminate; hence, two indeterminate unknowns F_0 and F_1 have been chosen. Knowing that those internal forces can be of two types (either tension or compression), this means that there are $2^s = 2^2 = 4$ main patterns for force distributions within the structure: both F_0 and F_1 in tension, both F_0 and F_1 in compression, F_0 in tension

and F_I in compression, F_θ in compression and F_I in tension. These four possible force combinations have been defined in Figure 8.10's convex hull by four corner zones, which help relating rebar locations to levels of structural robustness in terms of aptitude for force redistributions.

In particular, these four zones may further help project authors in identifying valid “families” of structural workings. For example, Figure 8.10's convex hull has been obtained for a structure within which tensile forces may be activated in any bar L_i of the initial strut-and-tie network shape (see Figure 8.9). It may be considered instead that the rebar layout is limited to bars L_θ , L_I , L_8 , and L_9 only, which corresponds to a classical reinforcement scheme within concrete beams. This reduced rebar layout would then allow force distributions according to “family 1”, occurring when bars L_θ and L_I are tensed with a force $F_\theta = F_I = 5\text{kN}$ (diagram associated to the load path value of 30 in Figure 8.10). This rebar layout would also allow force distributions according to an independent structural behaviour, i.e. “family 2”, occurring when bars L_θ and L_I are not loaded, meaning that tensile forces are mobilised within bars L_8 and L_9 only (diagram associated to the load path value of 50 in Figure 8.10). Going back to Figure 8.10's convex hull of admissible scalings, this means that combining independent families of structural workings associated to their own rebar layout, which may not be robust in themselves unless oversizing of the rebars, helps enhancing the structure's overall performance of robustness by allowing internal forces to redistribute from one “family” to the other. In other words, this means that the structure's convex hull of admissible scalings may turn from a unique point to an area, depicted here by the square formed between the four central nodes of Figure 8.10's hull. Combining rebar layouts associated to “family 1” and “family 2” then allows hybrid structural workings, e.g. those depicted by the two diagrams associated to a load path value of 40 in Figure 8.10.

Hence, when designing structures for robustness, convex hulls of admissible scalings may help project authors in selecting pertinent independent rebar layout “families” to be combined together in order to enhance structural aptitude for force redistributions. The convex hull of admissible scalings may also help project authors in ranking possible “families” of structural workings according to their efficiency in terms of the volume of material required for their implementation, as well as selecting those presenting the best properties of efficiency.

8.3.4. Efficiency analysis

Using Eq 3.5 from Section 3.2.1, the structure's convex hull of admissible scalings can further be analysed: the efficiency in terms of the volume of material required for the implementation of force distributions associated to each point coordinate (F_0, F_1) can be assessed. Hence, corresponding load path values have been depicted directly within the hull. These values are given by the polygonal curves within Figure 8.10's grey surface, which link point coordinates associated to a same performance in terms of the volume of material required for their implementation.

The point associated to the minimum load path value (30kNm) is obtained for $F_0 = F_1 = 5kN$, leading to a statically determinate force distribution. Three other statically determinate force distribution can also be reached: for $F_0 = 5kN$ and $F_1 = 0kN$; for $F_0 = 0kN$ and $F_1 = 5kN$; for $F_0 = 0kN$ and $F_1 = 0kN$. These four particular determinate force combinations correspond to the four black points depicted at the centre of the convex hull of admissible scalings.

Figure 8.10's convex hull of admissible scalings also helps knowing whether a given force distribution is relevant or penalised by the adjunction of an additional self-stress state. Relevant combinations have been defined here as the ones at least as efficient as the one for which forces F_0 and F_1 are opposite to those in the most efficient strut-and-tie network. For the present structure, the combination of internal forces leading to maximal efficiency (load path = 30kNm) is the one for which $F_0 = F_1 = 5kN$. Thus pertinent combinations of internal forces are the ones at least as efficient as the solution for which $F_0 = F_1 = -5kN$. The associated solution sub-surface is given by the dark grey area in Figure 8.10.

8.3.5. Generation of the admissible geometrical domain

The six corner points of Figure 8.10's hull give the six limit combinations of internal forces (F_0, F_1) . These correspond to the six maximal scalings of the structure's force diagram, for which the corresponding strut-and-tie network topology has reached yielding in two bars (as $s = 2$). Hence, these six limit force distributions are the ones associated to the corner points of the admissible geometrical domain, which may also be drawn using Rhino [RHINO] and Grasshopper [GRASSHOPPER]. This domain is composed of 8 linear sub-domains (some are partly superimposed), as depicted in grey in Figure 8.11's parametrical force diagram. These eight sub-domains are associated to force diagram's eight mobile nodes, i.e. to every node of the

force diagram except the three central ones. Indeed, the latter are linked to support reactions, which are statically determinate. Hence, these nodes must remain of fixed position unless violating the conditions of 2.1.4. In particular, Figure 8.11's reciprocal diagrams give force distribution associated to $F_0 = 7.775\text{kN}$ and $F_1 = -6.458\text{kN}$. This solution yields bar F_7 in compression (in light blue) and bar F_9 in tension (in orange).

As for the structural example of Section 8.2, the structure's sub-domains are lines instead of surfaces, because the shape of form diagram's strut-and-tie network describing structural behaviour may not be varied according to the followed approach. Hence, nodal positions may only be displaced in the force diagram along particular linear directions, unless violation of the conditions of Section 2.1.4 on the strut-and-tie network.

Looking at the structure's force diagram, it may seem that the depicted sub-domains are in one dimension. However, as mentioned in Section 4.5.1, the structure's associated sub-domains are interdependent, which adds extra dimensions to the structural problem. For the present case, it then comes that the problem's dimension is equal to $\nu \cdot a = 2 \cdot 1 = 2$, with a the number of parameters that need to be set to draw the structure's parametrical force diagram, i.e. the structure's degree of static indeterminacy s . Hence, the size of the admissible geometrical domain depicting the structure's level of robustness in terms of aptitude for force redistributions may be expressed in square units of force (N^2).

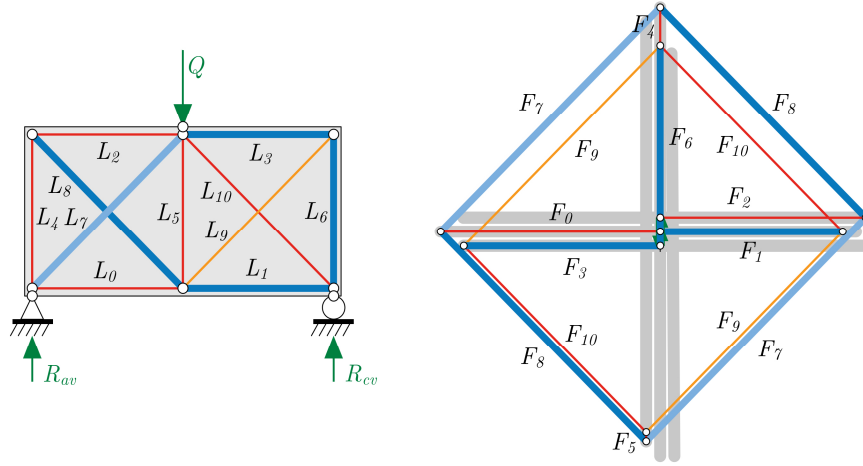


Figure 8.11 – Admissible geometrical domain and possible limit combination of internal forces

8.4. Application to an idealised shear wall

While the pure geometrical method developed in Section 8.2 allowed to deal with structures with one or two degrees of static indeterminacy, the combined analytical/geometrical approach presented in the previous Section allows to analyse structures presenting a degree of static indeterminacy comprised between two and approximately twelve.

Consider the C25/30 reinforced concrete shear wall of Figure 8.12, which is 4.1m long, 1.1m high and 20cm thick. It is loaded on the top by a point load $Q = 10kN$, applied at a 30° angle from vertical. The structure is pinned at both ends. Hence it is one time externally statically indeterminate: $h_e = r - 3 = 4 - 3 = 1$ (cfr. Eq 4.3). A 2Ø12 BE500s rebar layout has been placed within the wall according to the layout described in Section 8.1.2, with a cover $e = 5cm$. This layout takes the form of four squares of 1m wide crossed by two diagonals (dashed lines in Figure 8.12). The resultant strut-and-tie network modelling structural behaviour is composed of 21bars. Hence, the degree of internal static indeterminacy of the wall is given by $h_i = b + 3 - 2n = 21 + 3 - 2 \cdot 10 = 4$ (cfr. Eq 4.4). It then comes $s = h_e + h_i = 5$.

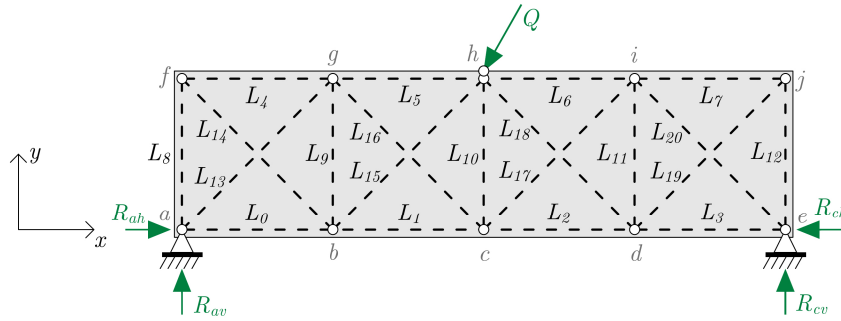


Figure 8.12 – Structure under consideration and associated strut-and-tie network

Given concrete and steel strengths as given by Eq 8.12, the structure's convex hull of admissible scalings may be determined. As shown in Section 8.3.3, the dimension of this hull is equal to the structure's degree of static indeterminacy s . As $s = 5$ in the present case, the associated convex hull may not be represented physically. Nevertheless, it may still be obtained automatically using the Qhalf function from Qhull algorithm [QHULL]. Feeding Qhalf with five-dimensional halfspace equations, this gives the 176 nodal coordinates (x, y, z, a, β) that correspond to the corners of the hull. These coordinates give the 176 limit combinations of internal forces within the five bars of the network $(F_0, F_1, F_2, F_3, F_4)$ chosen as indeterminate

unknowns. In particular, reciprocal diagrams associated to Figure 8.13 give the internal force distribution for which $F_0 = -5.14\text{kN}$, $F_1 = 4.74\text{kN}$, $F_2 = 9.84\text{kN}$, $F_3 = 4.13\text{kN}$ and $F_4 = -6.458\text{kN}$. This solution yields bar F_{18} in compression (in light blue) and bars F_2 , F_{10} , F_{11} and F_{14} in tension (in orange). The 176 limit scalings of the force diagram can further be used to obtain the endpoints of the admissible geometrical domain, as depicted in grey in Figure 8.13's force diagram. This domain is composed of 16 sub-domain segments (some are partly superimposed), associated to the 16 force diagram's nodes that are not linked to an external load or a support reaction. Indeed, by hypothesis, domains are used to represent redistributions of internal forces (see Section 4.1), without considering redistributions of support reactions.

In the present case, the structure's sub-domains are also lines instead of surfaces, because the shape of form diagram's strut-and-tie network may not be varied according to the followed approach. However, it comes that the problem's dimension is equal to $\nu.a = 1.5 = 5$ (see Section 4.5.1), where a is also equal to the structure's degree of static indeterminacy s . Hence, the size of the admissible geometrical domain depicting the structure's robustness in terms of aptitude for force redistributions may be expressed in N^5 . Following these, the domain's cumulative length may be measured as equal to 2287.81kN^5 . According to the proposed approach, this value indicates the structure's level of robustness in terms of aptitude for force redistributions, considered that tensile forces may be implemented in any candidate bar L_i of form diagram's strut-and-tie network (dashed lines in Figure 8.12). In other words, the domain's size indicates structural performance for force redistribution, when typical rebar layout in concrete (bottom ties to undertake bending moments and stirrups to undertake shear forces) has been improved with additional rebars.

As the five indeterminate unknowns (F_0 , F_1 , F_2 , F_3 , F_4) may each be of two types (either tension or compression), this means there are $2^s = 2^5 = 32$ main patterns for force distributions, each defined by a particular zone of the 5-dimensional hull. As mentioned in Section 8.3.3, these zones may help project authors in identifying families of structural workings, depending on the location chosen for rebars. Indeed, bar locations L_i where no rebars have been placed may only be associated to compressive paths, thus limiting the size of the convex hull. Hence, this reduces the admissible geometrical domain's cumulative length, which affects the structure's level of robustness (in terms of aptitude for force redistributions).

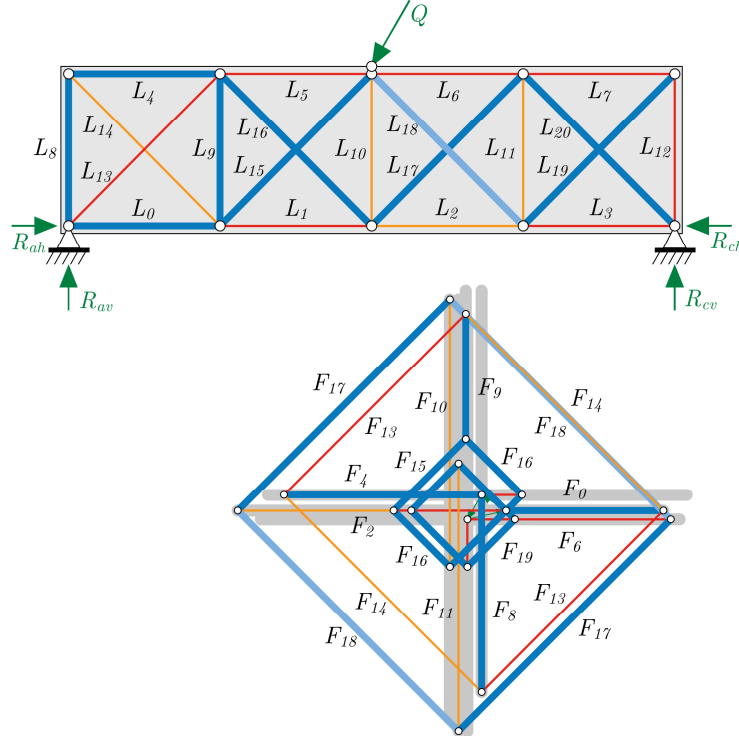


Figure 8.13 – Admissible geometrical domain and possible limit combination of internal forces for complete rebar layout

For example, Figure 8.14's force diagram gives the admissible geometrical domain that may be obtained when the rebar layout is limited to the orthogonal – horizontal and vertical – bars of the strut-and-tie network (in red in the figure). This means that the network's diagonal bars may only be compressed. The resulting domain has a cumulative length of 1174.66 kN^5 , meaning that the associated rebar layout reduces the structure's aptitude for force redistributions to approximately 51% of its initial one.

Considering now that the rebar layout is limited to the bottom chord of the strut-and-tie network as well as to some vertical and diagonal bars (in red in Figure 8.15's form diagram), the domain's cumulative length reduces further to 69.27 kN^5 , thus approximately 3% of the initial one. Comparing domain sizes for several rebar configurations may then help highlighting the loss or gain in the aptitude to redistribute forces. Hence, the method may help designing pertinent rebar layouts such as to reach the associated robustness performance.

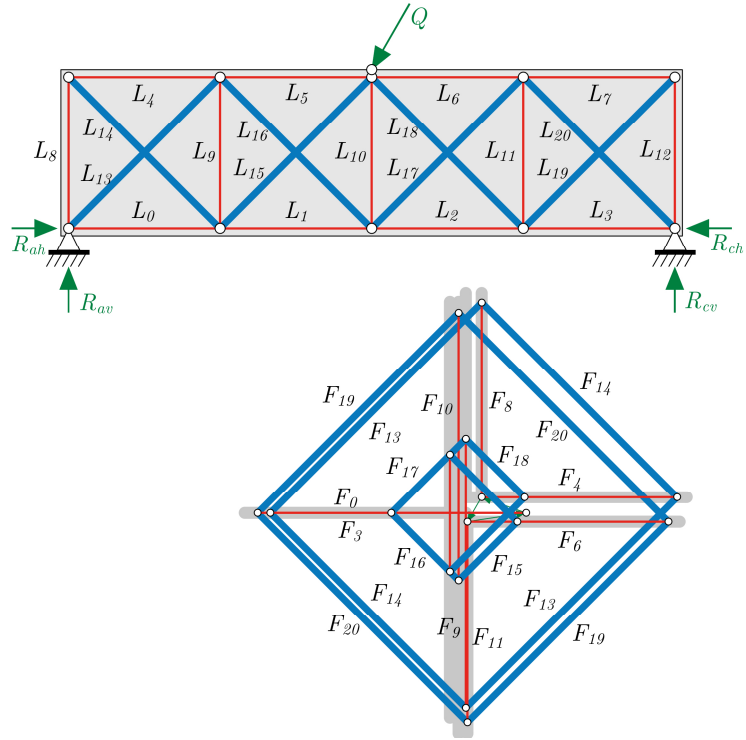


Figure 8.14 – Admissible geometrical domain for an orthogonal rebar layout

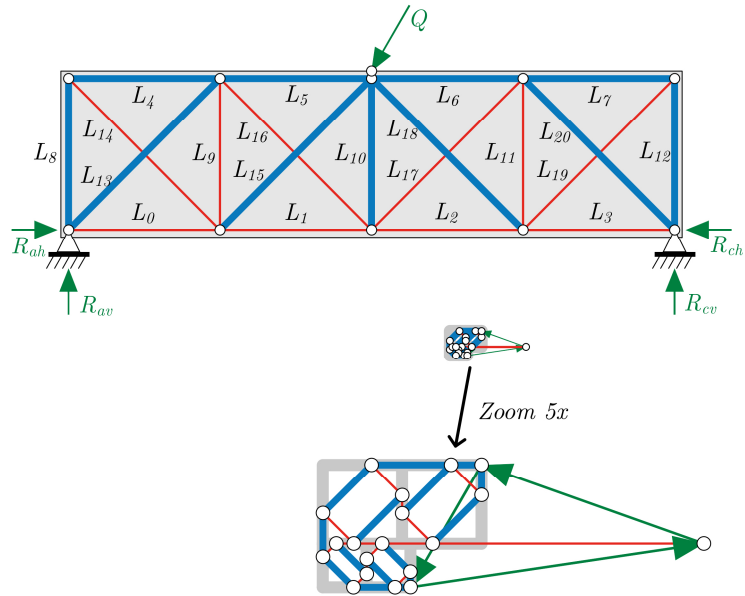


Figure 8.15 – Admissible geometrical domain for a bottom + stirrups rebar layout

As a further analysis, ranges of load path values associated to particular rebar layouts may also be assessed. This would bring a complementary information to project authors on the efficiency and associated cost of the reinforcement scheme under design. For the structural cases depicted in Figure 8.14 and Figure 8.15, force distributions associated to the minimal load path value both turn to be the Howe truss. This means that the two proposed rebar layouts are as efficient as the other at service state.

8.5. Summary and conclusion

Part 8 of this thesis has explored another manner to reduce the complexity of topology and shape variation problems within continuous structures, this time by fixing shape and making topology vary. In particular, the focus has here been put on the geometrical robustness assessment of reinforced concrete shear walls. This type of structure may be modelled by an orthogonal mesh of fixed nodes, within which bars may be placed (or not) to form various strut-and-tie network topologies. As the resulting topology may present static indeterminacy, associated graphic statics' force diagram has been parameterised under Rhino and Grasshopper in order to account for the scaling of its individual parts. In particular, Part 8 has shown that:

- through the proposed approach, the pertinence of several rebar layouts within concrete elements may be compared regarding their robustness performance in terms of potential force redistributions;
- the approach allows to overcome three out of the four limitations highlighted by Kostic [2009] while using an initial grid composed of square cells with two diagonals. The two first limitations respectively concerned external and internal static indeterminacy, which have been integrated in the present approach. The third limitation was about obtaining the most efficient topology for the path of internal forces (one of the aims of Kostic's research), but also some alternative ones having a relatively high level of efficiency in terms of the volume of material they activate. According to Kostic, this ensures that the structure will present a good behaviour towards cracking at serviceability. This third point may find a solution through geometrical efficiency assessment using load paths: curves of same load path values within the hull (Figure 8.10) can easily give which distribution of internal forces is the most efficient regarding active material volume, and which ones present similar performances. However, the last limitation

mentioned by Kostic has not been overcome through the proposed geometrical approach of robustness: the results proposed here are also highly dependent on the size of the initial grid;

- the use of convex hulls to depict limit solutions for the scaling of the individual parts of the force diagram works well for strut-and-tie networks with two or three degrees of static indeterminacy. Indeed, the convex hull of admissible scalings will in that case correspond to a two or three-dimensional graph respectively, which is then easy to represent and understand. Indeed, the convex hull enables to see directly the set of valid combinations of internal forces that are solution of the structural problem. These topologies are also automatically sorted by main patterns (families) of force distributions, given by the convex hull's zones. In the context of reinforced concrete design, the convex hull of admissible scalings also helps to link a structure's aptitude for force redistributions to the location of its rebar layout. This gives a useful tool for practitioners when designing reinforced concrete structures for robustness;
- through the convex hull of admissible scalings, each possible strut-and-tie network topology can also be associated visually to its corresponding load path value, using curves of same load path values.

Depicting aptitude for force redistributions through convex hulls of admissible scalings then allows to obtain an "absolute" index for geometrical robustness assessment. Indeed, the convex hull helps solving the limitations due to the superimposition of admissible geometrical sub-domains for multiple dimension structural problems (see Section 4.5.1). However, when the structure presents a degree of static indeterminacy equal or larger than four, the shape of the convex hull may still be calculated analytically, but not be drawn geometrically/physically. Hence, the approach loses the visual and interactive aspect that was to be favoured for design purposes.

Also, even if partially simplified through the analytical pre-treatment of limit solutions, the proposed geometrical approach for robustness assessment still requires large calculation times when implemented within Rhino and Grasshopper, even to deal with simplistic structures. Indeed, the largest structures that can be analysed with the proposed tool using the combined analytical/geometrical method are composed of 3x3 or 4x2 cells. If the approach was totally made analytical through more efficient coding and was run on powerful computers, calculation times may be improved. But the significance of this improvement may still be limited, as the total

number of operations to be run to solve the structural problem would remain high in any case.

Another issue arising from this problem of calculation time is that, in practice, the size of the strut-and-tie network mesh should be reduced to better represent the behaviour of real reinforced concrete structures (e.g. one square of the mesh should be 10-15cm large, as this distance usually represents the spacing of a rebar layout). Hence, meshes modelling real reinforced concrete structures should be composed of many cells to represent structural behaviour pertinently. However, as mentioned before, this increases the mesh's static indeterminacy, and further increases calculation times.

Finally, in practice, the approximation of material behaviour to those of an ideal one (ductile, uniform and isotropic) may not be accurate for reinforced concrete structures. Specific requirements concerning reinforced concrete structures have also been neglected within the model, such as cracking of concrete or control of the minimal/maximal quantity of steel rebars. At the actual state of the research, this limits the area of application of the proposed method. Furthermore, the consideration of such questions would make the developed model even more complex, which may increase calculation times even more.

In conclusion, all these limitations still make the proposed approach currently impractical and inefficient, furthermore when it comes to the robustness assessment of real (reinforced concrete) structures...

GENERAL CONCLUSION

Results

“(...) Redundancy has to be associated to ductility and over-strength to turn into robustness (...) In the future, the possibility to implement the approach in the preliminary design stages has to be considered (...) A design procedure able both to tackle random damages and to optimise the structure is a sustainable strategy.” [DEBIAGI2016]

This thesis has explored some theoretical methods for a complementary geometrical approach of structural robustness. This approach aims to evaluate constitutive elements of the performance of robustness, such as strength, ductility, and the aptitude for force redistributions through multiple load paths and redundancy. Based on graphic statics’ reciprocal diagrams and strut-and-tie modelling, the approach gives:

- internal forces associated to designed structural shapes;
- the set of permissible positions for the nodes of the force diagram (as developed by Fivet [2013a]), which are here associated to potential plastic force redistributions within the considered structure. This set of permissible positions is called *admissible geometrical domain*, which size is used as a geometrical robustness indicator. Indeed, force distributions are supposed to represent alternative load paths that can be activated in the structure in case of an unexpected event;
- the efficiency associated to each potential force redistribution, in terms of required volume of material [BAKER2012]. Hence, the method gives which force distributions will preferentially be activated within the structure, i.e. those close to the elastic solution which is associated to the minimal volume configuration.

The proposed geometrical approach of structural robustness presents the following advantages:

- the approach can already be employed at the design stage, when models are still abstract and when material behaviour laws may be idealised. Hence, the method could be used by practitioners to gain insight into the redistribution properties of the design they are currently working on;

- as it is based on graphic statics, the approach is visual and intuitive. Hence, results can easily be appropriated and interpreted;
- when implemented into parametrical software, the approach turns out to be interactive. In particular, through coding in Rhino and Grasshopper, each design parameter can be associated to a slider, with which one can play to interact with the design: number, position, intensity and direction of applied loads; number, type and position of supports; structural dimensions; number, size and position of eventual openings; material type and strength; location of rebar layout within reinforced concrete elements;... Hence, the method helps assessing and comparing the influence of each design parameter on the level of robustness associated to possible force redistributions, as well as on structural efficiency in terms of the required volume of material;
- the approach relies on a damage-independent robustness indicator, thus damage that could potentially happen during the structure's lifetime does not have to be predicted (loss of a structural member, support displacement, increase of applied loads, etc);
- the approach enables dealing with both externally and internally statically indeterminate strut-and-tie networks.

The proposed geometrical approach for robustness assessment works well for the design of structures that may be modelled with a topologically defined strut-and-tie network, e.g. Part 5's arches. Indeed, for such structures, the path that is taken by the applied loads throughout the structure is explicitly given by a unique funicular topology. The associated level of robustness in terms of aptitude for force redistributions is then obtained through shape variation. In particular, it has been shown that the proposed approach can provide pertinent information about the parameters that influence both robustness (in terms of aptitude for force redistributions) and material efficiency of arched structures in compression.

Even if there does not yet exist another approach to which the developed one could be explicitly compared to, Part 6 has shown that the proposed geometrical approach for robustness assessment may be compared with the existing deterministic and energetic ones if associated results are turned into dimensionless indices. Indeed, the whole set of results shows similar progressions among equal scenarios of loading and damage. This confirms the intuition that force redistributions associated to the performance of robustness may be proportional to the size of the admissible geometrical domain.

Remaining issues and limitations

However, when dealing with structures for which the strut-and-tie network modelling internal behaviour is not topologically defined (i.e. it may not be described by a unique topology), the geometrical approach for structural robustness proposed in this thesis still presents issues and limitations.

The first issue is about the general fact that there does not exist an other way than discretisation to deal with the infinity of strut-and-tie networks that may be drawn within a continuous structure to model its internal behaviour, and the present approach makes no exception to this fact. It has been tried to model force redistributions by changing the shape of a fixed topology of bar elements (Part 7), as well as by changing the bar topology of a network of fixed nodes (Part 8). Hence, the admissible geometrical domains developed in the present work as robustness indicators may only give relative structural performances for force redistributions associated to predefined strut-and-tie networks. Also, the variation of both topology and shape at the same time should be considered to model internal forces more exhaustively.

The second issue is related to the multiple sub-domains that may be obtained while aiming to characterise force redistributions within structural cases modelled using complex strut-and-tie networks, i.e. networks having more than one mobile node within graphic statics' force diagram. This leads to questions which have not found an answer yet. First, which information about the size of the admissible geometrical (sub-)domain(s) should be taken into account as a robustness parameter? Also, what is the pertinence of having domain units in N^x when $x \geq 2$? Finally, the work on the meaning, the use and the management of convex hulls to depict the set of admissible scalings within statically indeterminate structures should be further pursued, especially to make their use practical for design purposes.

Concerning the limitations of the proposed approach, it has been shown that implementing a structural design tool within Rhino and Grasshopper's interfaces presents some advantages of simplicity, visualisation and interactivity (see Results section above). However, from a practical point of view, such an implementation of the method requires enormous calculation times regarding the number of solutions to be dealt with. This makes the approach impractical when it comes to designing complex structural cases for robustness.

A second limitation comes from the fact that the supposition of an ideal material strength (ductile, isotropic and uniformly resistant) may not be valid for real materials, except for steel. Furthermore, the developed methods have considered simplifying assumptions on structural behaviour, e.g. neglecting tensile strength and sliding for masonry arches, or neglecting cracking and minimal/maximal quantity of steel rebars within reinforced concrete structures. The proposed approach should then be improved by using more realistic behaviour laws in order to be applicable for robust structural design of real structures.

A last limitation is about the set of robustness parameters that have been taken into account within the proposed approach. Indeed, strength, ductility and multiple load paths and redundancy only represent three out of the set of sixteen parameters detailed by Knoll and Vogel [2009] as constitutive of the performance of robustness. May some other parameters be taken into account within the proposed method, even if these are farther from design issues? How could this be done graphically? In that case, what should become the definition of structural robustness presented in Part 4?

Perspectives

The present thesis has shown that the topical question of structural robustness assessment still remains complex, even through geometrical means. Indeed, lots of questions on that subject still need to be answered, and various parameters that were for now neglected should be introduced within the developed methods to improve them. In particular, the work proposed here could be followed according to two main directions.

A first perspective would be to make the approach applicable to a whole set of real materials. As mentioned before, the proposed methods are for now simplified regarding material behaviour, which may then in practice only be pertinent for steel structures. Regarding masonry structures for example, the model should be improved in order to account for *voussoir* sliding, as well as to allow low tensile forces to appear within the masonry blocks. The option of adding reinforcement at specific places of the structure could also be studied, in order to allow larger tensile forces to be activated. The geometrical approach of robustness proposed in this work could also be extended in order to be applicable to a broader range of structural materials. For example, it could account for the robust design of timber elements. For such structures, both tensile and compressive

strengths have to be calculated continuously, as they depend on the angle between bars of the strut-and-tie network and wood fibre direction. Hence, the model could account for parallel sizing, shape and topology variation processes. Inspiring work concerning the assessment of discontinuous stress field within wooden assemblies already exists, to which one could refer [FIVET2012, MISSON2017, DHEUR2019].

The second direction of improvement of the present work may be the coding of the developed Grasshopper tool into another computer language, such as C++. Even if this would probably cause the loss of the visual and interactive facets of the initial procedure, this could decrease calculation times in order to enable robust design of more complex structural cases. Indeed, once improved in terms of calculation speed and solving algorithm, the geometrical method for robustness assessment could be extended in order to deal with a broader range of structural types in two dimensions (as well as in three dimensions, eventually). Concerning arched structures, the proposed geometrical approach could be employed to study the level of robustness of more complex masonry arch designs in terms of aptitude for force redistributions. The approach could also be further used for the analysis of ancient complex geometries in terms of force redistributions associated to robustness (e.g. the cross-section of a church, a bridge, etc). Concerning reinforced concrete structures, improvement of calculation times and solving algorithm could enable using a more pertinent size for the nodal grid at the basis of the strut-and-tie network, i.e. putting those nodes every 10...15cm as to correspond to the usual way of placing steel reinforcement in concrete. Improved calculation times could also serve to refine distances of connection between the network's nodes, i.e. considering grids for which $d > 1$. Finally, to be coherent with real design scenarios, the approach should be extended in order to consider structural cases with a more complex design: several external loads, multiple supports, complex geometries, holes modelling openings (doors or windows), etc.

PUBLICATIONS

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