

Part III

Institutional Asset Liability Management in complete markets.

Chapter 8

Dynamic assets allocation under VaR constraint with stochastic interest rates.

In Anglo-Saxon countries, the benefits of a defined contribution plan are not necessary guaranteed and may only depend on the performance of investments. In this situation, the pension managers adapt their assets allocation in order to keep their exposure to market risk to a level in agreement with affiliates' risk aversion. The topic of this chapter is precisely to propose a dynamic investment strategy maximizing the expected utility of wealth, under a bounded shortfall risk, which is measured by the so called Value-at-Risk (VaR) criterion. The contribution of our work with respect to the existing literature is to study this issue in a market allowing for stochastic term structures of interest rates. And in particular when interest rates are ruled by a Hull & White model.

8.1 Introduction.

The issue of optimal assets allocation for an investor with given utility function and a fixed initial endowment is one of the classical problems in financial mathematics. Since Merton's pioneering work (1969, 1971), many attempts have been made to solve the assets allocation problem in a framework that allows more realistic market models. In particular, stochastic term structures for the interest rates were introduced. General non explicit results were first inferred by Karatzas et al. (1987), Karatzas (1989). Afterwards, many authors have obtained closed form solutions for some term structure models. E.g. Korn and Kraft (2001) have investigated the case where interest rates follow a Vasicek and Ho & Lee model, using stochastic control techniques. Sørensen (1999) studied a similar problem by the martingale method.

On another side, the value at risk based risk management emerges over the last decades as the industry standard and remains today a key tool for assets allocation. The interested reader can refer to Artzner et al. (1999) for details about pros and cons of this risk measure. Basak and Shapiro (2001) and Gabih et al. (2005) have embedded this risk management criterion into an utility maximization problem and provided a comprehensive analysis of VaR investment strategies. Static version of this issue and other risk measures have been considered by Emmer et al. (2000, 2001). Gundel and Weber (2005) analyzes optimal portfolio choice

when asset prices are semi martingales.

However, at our knowledge, papers treating the issue of utility maximization under a VaR constraint only consider deterministic interest rates. The contribution of our research is precisely to extend this problem to a market allowing stochastic term structures of interest rates. We have decided to work with the Hull and White model, due to its ability to match perfectly the observed yield curve and to its analytical tractability.

The outline of this chapter is as follows. Sections 2 and 3 respectively present the financial market and asset allocation issues. We first establish the terminal wealth, maximizing the investor's utility without risk management constraint. Based upon those results, the VaR optimal terminal wealth is inferred. In section 4, we build investment strategies hedging the VaR and non VaR optimal terminal wealth. Section 5 illustrates numerically our results and the last section concludes.

8.2 The financial market.

This section develops the market structure of our model and the dynamics of interest rates and asset values. The financial market is defined on a complete probability space $(\Omega, \mathcal{F}, P)$ where $\mathcal{F}$ is the filtration generated by a 2-dimensional standard Brownian motion $W_t^P = (W_t^{r,P}, W_t^{S,P})'$:

$$\mathcal{F} = (\mathcal{F}_t)_t = \sigma \left\{ (W_u^{r,P}, W_u^{S,P})' : u \leq t \right\}.$$

The two Wiener processes $W_t^{r,P}$ and $W_t^{S,P}$ are independent. A direct consequence of the completeness of the financial market is the existence of an unique equivalent measure under which the discounted prices of assets are martingales. This risk neutral measure is denoted by Q . An investor can purchase three assets: cash, zero coupon bonds, and stocks. The return of cash is the risk free rate r_t and is driven by a Hull & White model:

$$dr_t = a(b^P(t) - r_t)dt + \sigma_r dW_t^{r,P}. \quad (8.2.1)$$

The speed of mean reversion a and the volatility σ_r of r_t are constant. The level of mean reversion $b^P(t)$ is a function of time, chosen such that modeled bond prices match real bond prices. The market price of risk associated to r_t is constant and denoted λ_r . The dynamics of r_t may then be rewritten

$$\begin{aligned} dr_t &= a(b^P(t) - \sigma_r \frac{\lambda_r}{a} - r_t)dt + \sigma_r (dW_t^{r,P} + \lambda_r dt) \\ dr_t &= a(b(t) - r_t)dt + \sigma_r dW_t^{r,Q}, \end{aligned} \quad (8.2.2)$$

where $W_t^{r,Q}$ is a Wiener process under Q . If $f(0, t)^1$ is the instantaneous forward rate at time 0 for a maturity t , $b(t)$ is defined as follows:

$$b(t) = \frac{1}{a} \frac{\partial}{\partial t} f(0, t) + f(0, t) + \frac{\sigma_r^2}{2a^2} (1 - e^{-2at}).$$

¹If $R(t, T, T + \Delta)$ is the forward rate as seen at time t for the period between time T and time $T + \Delta$, the instantaneous forward rate $f(t, T)$ is equal to the following limit $f(t, T) = \lim_{\Delta \rightarrow 0} R(t, T, T + \Delta)$. If $P(0, t)$ is the price of a zero coupon bond maturing at time t , one has that $f(0, t) = -\frac{\partial}{\partial t} \log P(0, t)$ (see Hull (1997), for further details).

In appendix A, more details on the calibration of $f(0, t)$ are given. The second available asset on the market is a zero coupon bond of maturity T_P whose price at time t is denoted $P(t, T_P)$. The price $P(t, T_P)$ (see appendix B, for details) obeys to the dynamics,

$$\begin{aligned} \frac{dP(t, T_P)}{P(t, T_P)} &= r_t dt - \sigma_r B(t, T_P) \left(dW_t^{r, P} + \lambda_r dt \right) \\ &= r_t dt - \sigma_r B(t, T_P) dW_t^{r, Q}, \end{aligned} \quad (8.2.3)$$

where $B(t, T_P)$ is a function of the maturity T_P :

$$B(t, T_P) = \frac{1}{a} \left(1 - e^{-a(T_P - t)} \right).$$

In the sequel of this work, the risk premium of the bond is denoted by $\nu_P(t, T_P) = -\sigma_r B(t, T_P) \lambda_r$. The last kind of assets considered is a stock. Its price process S_t is modeled by a geometric Brownian motion and is correlated with the interest rates fluctuations

$$\begin{aligned} \frac{dS_t}{S_t} &= r_t dt + \sigma_{Sr} \left(dW_t^{r, P} + \lambda_r dt \right) + \sigma_S \left(dW_t^{S, P} + \lambda_S dt \right) \\ &= r_t dt + \sigma_{Sr} dW_t^{r, Q} + \sigma_S dW_t^{S, Q}, \end{aligned}$$

where constants σ_{Sr} , σ_S and λ_S are respectively the correlation between stocks and interest rates, the intrinsic volatility of stocks and the market price of risk. For convenience, the stocks risk premium is noted $\nu_S = \sigma_{Sr} \lambda_r + \sigma_S \lambda_S$.

The market completeness implies the existence of a unique state price process $\left(\frac{dQ}{dP} \right)_t$ given by

$$\left(\frac{dQ}{dP} \right)_t = \exp \left(-\frac{1}{2} \int_0^t \|\Lambda\|^2 du - \int_0^t \Lambda' dW_u^P \right),$$

where $\Lambda = (\lambda_r, \lambda_S)'$. The unique deflator $H(t, s)$ at time t , for a cash flow paid at time $s \geq t$ is defined by

$$\begin{aligned} H(t, s) &= \frac{\exp \left(-\int_0^s r_u du \right) \left(\frac{dQ}{dP} \right)_s}{\exp \left(-\int_0^t r_u du \right) \left(\frac{dQ}{dP} \right)_t} \\ &= \exp \left(-\int_t^s r_u du - \frac{1}{2} \int_t^s \|\Lambda\|^2 du - \int_t^s \Lambda' dW_u^P \right). \end{aligned}$$

8.3 Optimization problems.

We consider an investor endowed with an initial wealth x at time t . The portfolio process, $\pi_t = (\pi_t^S, \pi_t^P)'$ is the vector of amounts of money invested in stocks and bonds and is assumed to be self financing. The investor's wealth process, denoted X_t , is then ruled by the following dynamics;

$$\begin{aligned} dX_t &= (X_t - \pi_t^P - \pi_t^S) r_t dt + \frac{\pi_t^P}{P(t, T_P)} dP(t, T_P) + \frac{\pi_t^S}{S_t} dS_t \\ &= (r_t X_t + \pi_t^S \nu_S + \pi_t^P \nu_P(t, T_P)) dt + \pi_t^S \sigma_S dW_t^{S, P} \\ &\quad + (\pi_t^S \sigma_{Sr} - \pi_t^P \sigma_r B(t, T_P)) dW_t^{r, P}. \end{aligned}$$

Let T be the investment time horizon. The investor aims to determine the investment strategy maximizing the expected utility arising from terminal wealth $U(X_T)$. We have decided to limit the scope of our research to power utility functions (CRRA) with a risk aversion parameter $\gamma \in (-\infty, 1) \setminus \{0\}$. The case $\gamma = 0$ corresponds to the logarithmic utility and is not developed in this chapter. The value function at time t is therefore defined as follows:

$$V(t, x) = \max_{X_T \in \mathcal{A}_t(x)} \mathbb{E} \left(\frac{X_T^\gamma}{\gamma} \mid \mathcal{F}_t \right), \quad (8.3.1)$$

where $\mathcal{A}_t(x)$ is the set of replicable wealth processes with or without bounded shortfall risks. The next two subsections respectively present optimal terminal wealth without and with VaR constraint.

8.3.1 Optimization without bound on shortfall risks.

In the setting of complete markets, it can be proved that a random process X_T is replicable by an adapted, self financed, investment strategy if and only if the expectation of the deflated terminal wealth is at most equal to the current wealth x (for details see Dana & Jeanblanc 2004, chapter 4). Without VaR limit, the set of admissible controls of (8.3.1) $\mathcal{A}_t(x)$ is given by

$$\mathcal{A}_t(x) = \{X_T \mid \mathbb{E}(H(t, T)X_T \mid \mathcal{F}_t) \leq x\}. \quad (8.3.2)$$

This constraint is known in the literature as the budget constraint. If $y_t \in \mathbb{R}_+$ is the Lagrange multiplier coupled to this, the Lagrangian of the optimization program is defined by

$$\mathcal{L}(t, x, X_T, y_t) = \mathbb{E} \left(\frac{X_T^\gamma}{\gamma} + y_t (x - H(t, T)X_T) \mid \mathcal{F}_t \right),$$

and the value function may be reformulated as follows;

$$V(t, x) = \inf_{y_t} \left(\sup_{X_T} \mathcal{L}(t, x, X_T, y_t) \right).$$

The optimal terminal wealth X_T^* and Lagrange multiplier y_t^* are such that

$$V(t, x) = \mathcal{L}(t, x, X_T^*, y_t^*).$$

Deriving the Lagrangian with respect to X_T leads to the optimal wealth for a given y_t^* ,

$$X_T^* = (y_t^* H(t, T))^{\frac{1}{\gamma-1}}.$$

Whereas y_t^* is such that the budget constraint is binding:

$$\begin{aligned} x &= \mathbb{E}(H(t, T)X_T^* \mid \mathcal{F}_t) \\ &= \mathbb{E} \left(y_t^{*\frac{1}{\gamma-1}} H(t, T)^{\frac{\gamma}{\gamma-1}} \mid \mathcal{F}_t \right). \end{aligned}$$

We now state a proposition that allows us to calculate the expectation of X_T^* in function of y_t^* .

Proposition 8.3.1. $\frac{X_T^*}{y_t^{*1/(\gamma-1)}}$ is a lognormal random variable under P :

$$\ln \left(\frac{X_T^*}{y_t^{*1/(\gamma-1)}} \right) \sim N \left(\mu^{X_T^*}(t, T), \sigma^{X_T^*}(t, T) \right)$$

where

$$\begin{aligned} \mu^{X_T^*}(t, T) = & \frac{1}{1-\gamma} \left[\ln \left(\frac{P(0, t)}{P(0, T)} \right) + \left(\frac{\sigma_r^2}{2a^2} + \frac{\lambda_r \sigma_r}{a} + \frac{1}{2} (\lambda_r^2 + \lambda_S^2) \right) (T - t) \right] + \\ & \frac{1}{1-\gamma} \left[\left(r_t - f(0, t) - \frac{\sigma_r^2}{2a^2} - \frac{\lambda_r \sigma_r}{a} \right) B(t, T) - \frac{\sigma_r^2}{4a} e^{-2at} B(t, T)^2 \right] \end{aligned} \quad (8.3.3)$$

$$\left(\sigma^{X_T^*}(t, T) \right)^2 = \frac{1}{(1-\gamma)^2} \int_t^T (\sigma_r B(u, T) + \lambda_r)^2 + \lambda_S^2 du. \quad (8.3.4)$$

Proof. When the dynamics of interest rates is driven by a Hull and White model, one can show that the integral $\int_t^T r_s ds$ is a Gaussian variable under P (see appendix C for a proof):

$$\begin{aligned} \int_t^T r_u du = & \ln \left(\frac{P(0, t)}{P(0, T)} \right) + (r_t - f(0, t)) B(t, T) + \\ & \left(\frac{\sigma_r^2}{2a^2} + \frac{\lambda_r \sigma_r}{a} \right) ((T - t) - B(t, T)) - \\ & \frac{\sigma_r^2}{4a^3} (e^{-at} - e^{-aT})^2 + \sigma_r \int_t^T B(u, T) dW_u^{r, P}. \end{aligned} \quad (8.3.5)$$

By definition of X_T^* , $\ln \left(\frac{X_T^*}{y_t^{*1/(\gamma-1)}} \right)$ is then equal to:

$$\frac{1}{\gamma-1} \ln (H(t, T)) = \frac{1}{1-\gamma} \left(\int_t^T r_u du + \frac{1}{2} \int_t^T \|\Lambda\|^2 du + \int_t^T \Lambda' dW_u^P \right). \quad (8.3.6)$$

Since $W_t^{r, P}$ and $W_t^{S, P}$ are independent, the sum of stochastic integrals present in (8.3.6) is a normal random variable whose variance is given by (8.3.14) whereas the expectation of (8.3.6) is well (8.3.13). □

The lognormal property of $\left(\frac{X_T^*}{y_t^{*1/(\gamma-1)}} \right)$ entails that the expected terminal wealth is

$$\mathbb{E} (X_T^* | \mathcal{F}_t) = y_t^{* \frac{1}{\gamma-1}} \exp \left(\mu^{X_T^*}(t, T) + \frac{(\sigma^{X_T^*}(t, T))^2}{2} \right).$$

In order to compute the Lagrange multiplier such that the budget constraint is binding, one needs to value the expectation of deflated optimal wealth. This point requires a forward

change of measure. The expectation of a discounted payoff under Q is equal to the price of a zero coupon bond times the expected payoff under the forward measure, F_T :

$$\begin{aligned}\mathbb{E}(H(t, T)X_T^*|\mathcal{F}_t) &= \mathbb{E}^Q\left(e^{-\int_t^T r_u du} X_T^*|\mathcal{F}_t\right) \\ &= P(t, T)\mathbb{E}^{F_T}(X_T^*|\mathcal{F}_t) .\end{aligned}$$

F_T is the measure obtained by choosing a zero coupon bond $P(t, T)$ as numeraire (see Shreve 2004, chapter 9 for further details) and its Radon-Nykodym derivative is defined by:

$$\frac{\left(\frac{dF_T}{dQ}\right)_T}{\left(\frac{dF_T}{dQ}\right)_t} = \exp\left(-\frac{1}{2}\int_t^T (\sigma_r B(s, T))^2 ds - \int_t^T \sigma_r B(s, T) dW_s^{r, Q}\right). \quad (8.3.7)$$

And under F_T , the following random process

$$\begin{aligned}dW_s^{r, F_T} &= dW_s^{r, P} + \lambda_r ds + \sigma_r B(s, T) ds \\ dW_s^{r, F_T} &= dW_s^{r, Q} + \sigma_r B(s, T) ds\end{aligned} \quad (8.3.8)$$

is a Brownian motion. The expectation of X_T^* under F_T may be inferred from the following proposition:

Proposition 8.3.2. $\frac{X_T^*}{y_t^{*1/(\gamma-1)}}$ is a lognormal random variable under the forward measure F_T :

$$\ln\left(\frac{X_T^*}{y_t^{*1/(\gamma-1)}}\right) \sim N\left(\mu^{X_T^*, F_T}(t, T), \sigma^{X_T^*, F_T}(t, T)\right)$$

where

$$\begin{aligned}\mu^{X_T^*, F_T}(t, T) &= \frac{1}{1-\gamma} \left[\ln\left(\frac{P(0, t)}{P(0, T)}\right) - \left(\frac{\sigma_r^2}{2a^2} + \frac{\lambda_r \sigma_r}{a} + \frac{1}{2}(\lambda_r^2 + \lambda_S^2)\right)(T-t) \right] \\ &\quad + \frac{1}{1-\gamma} \left[\left(r_t - f(0, t) + \frac{\sigma_r^2}{2a^2} + \frac{\lambda_r \sigma_r}{a}\right) B(t, T) \right] \\ &\quad + \frac{1}{1-\gamma} \left[\frac{\sigma_r^2}{2a} \left(1 - \frac{1}{2}e^{-2at}\right) B(t, T)^2 \right]\end{aligned} \quad (8.3.9)$$

$$\left(\sigma^{X_T^*, F_T}(t, T)\right)^2 = \frac{1}{(1-\gamma)^2} \int_t^T (\sigma_r B(u, T) + \lambda_r)^2 + \lambda_S^2 du. \quad (8.3.10)$$

Proof. Combining eq. (8.3.5), eq. (8.3.6) and eq. (8.3.8) leads after calculations to the desired result. □

The lognormal property of $\left(\frac{X_T^*}{y_t^{*1/(\gamma-1)}}\right)$ under F_T implies that the expected, deflated terminal wealth is worth:

$$\mathbb{E}(H(t, T)X_T^*|\mathcal{F}_t) = y_t^{*\frac{1}{\gamma-1}} P(t, T) \exp\left(\mu^{X_T^*, F_T}(t, T) + \frac{(\sigma^{X_T^*, F_T}(t, T))^2}{2}\right). \quad (8.3.11)$$

The Lagrange multiplier y_t^* binding the budget constraint is therefore given by

$$y_t^* = x^{\gamma-1} \left(P(t, T) \exp \left(\mu^{X_T^*, F_T}(t, T) + \frac{(\sigma^{X_T^*, F_T}(t, T))^2}{2} \right) \right)^{1-\gamma}.$$

The investment strategy replicating the process X_T^* will be established in section 8.4.

8.3.2 Optimization with bounded shortfall risks.

The Value at Risk is originally defined by the ϵ quantile of the risk variable (here the terminal wealth). Bounding this risk measure is then equivalent to bounding the shortfall probability. If $q > 0$ is some shortfall level and $\epsilon \in [0, 1]$ is the maximum shortfall probability accepted by the investor, then the VaR constraint is defined as follows

$$P(X_T \leq q) \leq \epsilon.$$

The set of admissible controls for (8.3.1) $\mathcal{A}_t(x)$ is now delimited by the budget and VaR constraints:

$$\mathcal{A}_t(x) = \{X_T \mid \mathbb{E}(H(t, T)X_T \mid \mathcal{F}_t) \leq x, \quad P(X_T \leq q)\}. \quad (8.3.12)$$

Basak and Shapiro (2001) characterize the optimal terminal wealth of this problem with deterministic interest rates. Adding stochastic interest rates modifies the structure of the deflator. But the proposition of Basak and Shapiro remains valid given that the optimal terminal wealth is built pointwise and that $\forall \omega \in \Omega$, $X_T^\epsilon(\omega)$ depends only on the realization $H(t, T)(\omega)$ of the deflator and not on its structure. We reproduce here without proof their main result:

Proposition 8.3.3. *At time t , the VaR optimal terminal wealth, noted X_T^ϵ is:*

$$X_T^\epsilon = \begin{cases} \max \left(q, (y_t^\epsilon H(t, T))^{\frac{1}{\gamma-1}} \right) & H(t, T) < h \\ (y_t^\epsilon H(t, T))^{\frac{1}{\gamma-1}} & h \leq H(t, T) \end{cases},$$

where h is the ϵ percentile of the deflator, $P(H(t, T) > h) = \epsilon$, and y_t^ϵ is the Lagrange multiplier such that the budget constraint is binding $\mathbb{E}(H(t, T)X_T^\epsilon) = x$.

We refer the interest reader to the PhD thesis of Gandy (2005), for a complete discussion over the existence of the solution. It can be proved in a similar way to proposition 8.3.1 that the distribution of the deflator is lognormal :

Proposition 8.3.4. *$H(t, T)$ is a lognormal random variable under P :*

$$\ln(H(t, T)) \sim N(\mu^H(t, T), \sigma^H(t, T))$$

where

$$\begin{aligned} \mu^H(t, T) = & - \left[\ln \left(\frac{P(0, t)}{P(0, T)} \right) + \left(\frac{\sigma_r^2}{2a^2} + \frac{\lambda_r \sigma_r}{a} + \frac{1}{2} (\lambda_r^2 + \lambda_S^2) \right) (T - t) \right] \\ & - \left[\left(r_t - f(0, t) - \frac{\sigma_r^2}{2a^2} - \frac{\lambda_r \sigma_r}{a} \right) B(t, T) - \frac{\sigma_r^2}{4a} e^{-2at} B(t, T)^2 \right] \end{aligned} \quad (8.3.13)$$

$$(\sigma^H(t, T))^2 = \int_t^T (\sigma_r B(u, T) + \lambda_r)^2 + \lambda_S^2 du. \quad (8.3.14)$$

The percentile h is then easily computable. We rewrite the VaR optimal wealth X_T^ϵ as a sum of options payoffs depending on $X_T^{*,\epsilon} = (y_t^\epsilon H(t, T))^{\frac{1}{\gamma-1}}$:

$$\begin{aligned} X_T^\epsilon &= q + (X_T^{*,\epsilon} - q)_+ - (K - X_T^{*,\epsilon})_+ - 1_{X_T^{*,\epsilon} < K} (q - K) \quad \text{if } K < q \\ &= X_T^{*,\epsilon} = X_T^* \quad \text{if } K \geq q \end{aligned}$$

where K is a constant that is worth $(y_t^\epsilon h)^{\frac{1}{\gamma-1}}$. Note that if $K \geq q$ the VaR constraint is not binding. In the sequel of the chapter, one will then assume that $K < q$. The following proposition will help us to calculate the expectation of X_T^ϵ :

Proposition 8.3.5. *Let define the following variables:*

$$\begin{aligned} d_2^{X_T^{*,\epsilon}, q} &= \frac{\ln\left(\frac{q}{y_t^{\frac{1}{\gamma-1}}}\right) - \mu^{X_T^*}(t, T)}{\sigma^{X_T^*}(t, T)} \quad , \quad d_1^{X_T^{*,\epsilon}, q} = d_2^{X_T^{*,\epsilon}, q} - \sigma^{X_T^*}(t, T) \quad , \\ d_2^{X_T^{*,\epsilon}, K} &= \frac{\ln\left(\frac{K}{y_t^{\frac{1}{\gamma-1}}}\right) - \mu^{X_T^*}(t, T)}{\sigma^{X_T^*}(t, T)} \quad , \quad d_1^{X_T^{*,\epsilon}, K} = d_2^{X_T^{*,\epsilon}, K} - \sigma^{X_T^*}(t, T) \quad . \end{aligned}$$

The expected VaR terminal wealth is given by:

$$\begin{aligned} \mathbb{E}(X_T^\epsilon | \mathcal{F}_t) &= q + \mathbb{E}\left((X_T^{*,\epsilon} - q)_+ | \mathcal{F}_t\right) - \mathbb{E}\left((K - X_T^{*,\epsilon})_+ | \mathcal{F}_t\right) \\ &\quad - \mathbb{E}\left(1_{X_T^{*,\epsilon} \leq K} | \mathcal{F}_t\right) (q - K). \end{aligned} \quad (8.3.15)$$

If $\Phi(\cdot)$ is the cumulative distribution of a standard normal random variable, conditional expectations present in eq. (8.3.15) are:

$$\mathbb{E}\left((X_T^{*,\epsilon} - q)_+ | \mathcal{F}_t\right) = y_t^{\frac{1}{\gamma-1}} e^{\mu^{X_T^*}(t, T)} + \frac{(\sigma^{X_T^*}(t, T))^2}{2} \Phi(-d_1^{X_T^{*,\epsilon}, q}) - q \Phi(-d_2^{X_T^{*,\epsilon}, q})$$

$$\mathbb{E}\left((K - X_T^{*,\epsilon})_+ | \mathcal{F}_t\right) = K \Phi(d_2^{X_T^{*,\epsilon}, K}) - y_t^{\frac{1}{\gamma-1}} e^{\mu^{X_T^*}(t, T)} + \frac{(\sigma^{X_T^*}(t, T))^2}{2} \Phi(d_1^{X_T^{*,\epsilon}, K})$$

$$\mathbb{E}\left(1_{X_T^{*,\epsilon} \leq K} | \mathcal{F}_t\right) = \Phi(d_2^{X_T^{*,\epsilon}, K})$$

Proof. The distribution of $X_T^{*,\epsilon}$ being lognormal by proposition 8.3.1, one can apply proposition 8.6.1 (appendix D) to value the expectations involved in equation (8.3.15). $\square$

As in the previous subsection, the calculation of the expected deflated terminal wealth is done by a change of measure from Q to F_T , the T -forward measure as defined by eq. (8.3.7). We can state the following proposition:

Proposition 8.3.6. *Let define the following variables:*

$$\begin{aligned} d_2^{X_T^{*,\epsilon},q,F_T} &= \frac{\ln\left(\frac{q}{y_t^{\frac{1}{\epsilon-1}}}\right) - \mu^{X_T^{*,F_T}}(t,T)}{\sigma^{X_T^{*,F_T}}(t,T)}, & d_1^{X_T^{*,\epsilon},q,F_T} &= d_2^{X_T^{*,\epsilon},q,F_T} - \sigma^{X_T^{*,F_T}}(t,T), \\ d_2^{X_T^{*,\epsilon},K,F_T} &= \frac{\ln\left(\frac{K}{y_t^{\frac{1}{\epsilon-1}}}\right) - \mu^{X_T^{*,F_T}}(t,T)}{\sigma^{X_T^{*,F_T}}(t,T)}, & d_1^{X_T^{*,\epsilon},K,F_T} &= d_2^{X_T^{*,\epsilon},K,F_T} - \sigma^{X_T^{*,F_T}}(t,T). \end{aligned}$$

The expected VaR terminal wealth under F_T is given by:

$$\begin{aligned} \mathbb{E}^{F_T}(X_T^\epsilon|\mathcal{F}_t) &= q + \mathbb{E}^{F_T}\left((X_T^{*,\epsilon} - q)_+|\mathcal{F}_t\right) - \mathbb{E}^{F_T}\left((K - X_T^{*,\epsilon})_+|\mathcal{F}_t\right) \\ &\quad - \mathbb{E}^{F_T}\left(1_{X_T^{*,\epsilon} \leq K}|\mathcal{F}_t\right)(q - K). \end{aligned} \quad (8.3.16)$$

If $\Phi(\cdot)$ is the cumulative distribution of a standard normal random variable, conditional expectations present in eq. (8.3.16) are:

$$\begin{aligned} \mathbb{E}^{F_T}\left((X_T^{*,\epsilon} - q)_+|\mathcal{F}_t\right) &= y_t^{\frac{1}{\epsilon-1}} e^{\mu^{X_T^{*,F_T}}(t,T) + \frac{(\sigma^{X_T^{*,F_T}}(t,T))^2}{2}} \Phi(-d_1^{X_T^{*,\epsilon},q,F_T}) - q\Phi(-d_2^{X_T^{*,\epsilon},q,F_T}) \\ \mathbb{E}^{F_T}\left((K - X_T^{*,\epsilon})_+|\mathcal{F}_t\right) &= K\Phi(d_2^{X_T^{*,\epsilon},K,F_T}) - y_t^{\frac{1}{\epsilon-1}} e^{\mu^{X_T^{*,F_T}}(t,T) + \frac{(\sigma^{X_T^{*,F_T}}(t,T))^2}{2}} \Phi(d_1^{X_T^{*,\epsilon},K,F_T}) \\ \mathbb{E}^{F_T}\left(1_{X_T^{*,\epsilon} \leq K}|\mathcal{F}_t\right) &= \Phi(d_2^{X_T^{*,\epsilon},K,F_T}). \end{aligned}$$

Proof. The distribution of $X_T^{*,\epsilon}$ being lognormal under F_T by proposition 8.3.2, one can apply proposition 8.6.1 (appendix D) to value the expectations involved in equation (8.3.16). $\square$

According to previous results, the expected, deflated VaR terminal wealth is worth:

$$\begin{aligned} \mathbb{E}(H(t,T)X_T^\epsilon|\mathcal{F}_t) &= P(t,T)q + P(t,T)\mathbb{E}^{F_T}\left((X_T^{*,\epsilon} - q)_+|\mathcal{F}_t\right) - \\ &\quad P(t,T)\mathbb{E}^{F_T}\left((K - X_T^{*,\epsilon})_+|\mathcal{F}_t\right) - \\ &\quad P(t,T)\mathbb{E}\left(1_{X_T^{*,\epsilon} \leq K}|\mathcal{F}_t\right)(q - K). \end{aligned} \quad (8.3.17)$$

Contrary to the optimization without bound on shortfall risks, the Lagrange multiplier y_t^ϵ binding the budget constraint is here not analytically calculable. In the examples developed in section 8.5, we have computed it by the Newton Raphson method. The investment strategy replicating the VaR process X_T^ϵ will be established in the next section.

8.4 Investment strategies.

Firstly, we present the investment policy replicating the optimal terminal wealth of an investor maximizing his utility without risk management constraint.

8.4.1 Optimization without bound on shortfall risk.

So that we can find the asset allocation hedging X_T^* , we need to define a new stochastic process $V_s = \mathbb{E}^{F_T}(X_T^*|\mathcal{F}_s)$ with $s \geq t$. Note that by the Tower property for conditional expectation, V_t is a martingale under F_T . A relatively long calculation, summarized in appendix E, leads to the following dynamics for V_s :

Proposition 8.4.1. *For $s \geq t$, V_s is solution of the S.D.E. :*

$$\begin{aligned} dV_s = & \frac{1}{1-\gamma} V_s (\lambda_r \sigma_r B(s, T) + \sigma_r^2 B(s, T)^2) ds + \\ & \frac{1}{1-\gamma} V_s ((\lambda_r + \sigma_r B(s, T)) dW_s^{r, Q} + \lambda_S dW_s^{S, Q}) . \end{aligned} \quad (8.4.1)$$

Notice that the expectation of deflated terminal wealth is related to the current wealth as follows:

$$X_t^* = \mathbb{E}(H(t, T)X_T^*|\mathcal{F}_t) = P(t, T)\mathbb{E}^{F_T}(X_T^*|\mathcal{F}_t) = P(t, T)V_t .$$

The dynamic of X_t^* is then inferred by the following proposition:

Proposition 8.4.2. *The dynamics of $X_t^* = P(t, T)V_t$ is given by:*

$$\begin{aligned} d(P(t, T)V_t) = & r_t V_t P(t, T)dt + \frac{1}{1-\gamma} \lambda_S V_t P(t, T) dW_t^{S, Q} + \\ & \left(\frac{1}{1-\gamma} (\lambda_r + \sigma_r B(t, T)) - \sigma_r B(t, T) \right) V_t P(t, T) dW_t^{r, Q} . \end{aligned} \quad (8.4.2)$$

Proof. According to eq. (8.2.3) and proposition 8.4.1, one can apply the Itô's lemma to get the result;

$$d(P(t, T)V_t) = P(t, T)dV_t + dP(t, T)V_t + d\langle P(t, T), V_t \rangle ,$$

where $\langle P(t, T), V_t \rangle$ is the quadratic covariation between $P(t, T)$ and V_t . □

Finally, we get the optimal investment strategy:

Proposition 8.4.3. *At time t , the optimal amount of stocks is*

$$\pi_t^S = \frac{1}{1-\gamma} \frac{\lambda_S}{\sigma_S} X_t^* , \quad (8.4.3)$$

and the optimal amount of bonds is

$$\pi_t^P = X_t^* \frac{B(t, T)}{B(t, T_P)} + \frac{1}{1-\gamma} \frac{X_t^*}{B(t, T_P)} \left(\frac{\lambda_S \sigma_{Sr}}{\sigma_S \sigma_r} - \frac{\lambda_r}{\sigma_r} - B(t, T) \right) . \quad (8.4.4)$$

Proof. The asset allocation is obtained by comparison of eq. (8.4.2) and of dynamics of the investor's portfolio under Q :

$$dX_t^* = rX_t^* dt + \pi_t^S \sigma_S dW_t^{S, Q} + (\pi_t^S \sigma_{Sr} - \pi_t^P \sigma_r B(t, T_P)) dW_t^{r, Q} .$$

□

8.4.2 Optimization with bounded shortfall risk.

Calculating the VaR optimal asset allocation strategy requires to differentiate the expectation of deflated terminal wealth, eq (8.3.17). The following proposition will help us to perform this task.

Proposition 8.4.4. *The differential of expectations appearing in equation (8.3.17) are given by:*

$$\begin{aligned}
d\mathbb{E}^{F_T} \left((X_T^{*,\epsilon} - q)_+ | \mathcal{F}_t \right) &= \Phi(-d_1^{X_T^{*,\epsilon}, q, F_T}) \frac{1}{1-\gamma} V_t (\lambda_r \sigma_r B(t, T) + \sigma_r^2 B(t, T)^2) dt + \\
&\quad \Phi(-d_1^{X_T^{*,\epsilon}, q, F_T}) \frac{1}{1-\gamma} V_t (\lambda_r + \sigma_r B(t, T)) dW_t^{r, Q} + \\
&\quad \Phi(-d_1^{X_T^{*,\epsilon}, q, F_T}) \frac{1}{1-\gamma} V_t \lambda_S dW_t^{S, Q}, \\
d\mathbb{E}^{F_T} \left((K - X_T^{*,\epsilon})_+ | \mathcal{F}_t \right) &= -\Phi(d_1^{X_T^{*,\epsilon}, K, F_T}) \frac{1}{1-\gamma} V_t (\lambda_r \sigma_r B(t, T) + \sigma_r^2 B(t, T)^2) dt \\
&\quad -\Phi(d_1^{X_T^{*,\epsilon}, K, F_T}) \frac{1}{1-\gamma} V_t (\lambda_r + \sigma_r B(t, T)) dW_t^{r, Q} \\
&\quad -\Phi(d_1^{X_T^{*,\epsilon}, K, F_T}) \frac{1}{1-\gamma} V_t \lambda_S dW_t^{S, Q}, \\
d\mathbb{E}^{F_T} \left(1_{X_T^{*,\epsilon} < K} (q - K) | \mathcal{F}_t \right) &= -(q - K) \frac{\varphi(d_2^{X_T^{*,\epsilon}, K, F_T})}{\sigma^{X_T^{*,\epsilon}, F_T}(t, T)} \frac{1}{1-\gamma} (\lambda_r \sigma_r B(t, T) + \sigma_r^2 B(t, T)^2) .dt \\
&\quad -(q - K) \frac{\varphi(d_2^{X_T^{*,\epsilon}, K, F_T})}{\sigma^{X_T^{*,\epsilon}, F_T}(t, T)} \frac{1}{1-\gamma} (\lambda_r + \sigma_r B(t, T)) dW_t^{r, Q} \\
&\quad -(q - K) \frac{\varphi(d_2^{X_T^{*,\epsilon}, K, F_T})}{\sigma^{X_T^{*,\epsilon}, F_T}(t, T)} \frac{1}{1-\gamma} \lambda_S dW_t^{S, Q}.
\end{aligned}$$

Where $\Phi(\cdot)$ and $\varphi(\cdot)$ denote respectively the cumulative distribution and the density function of a standard normal random variable.

The proof is detailed in appendix F. Based upon those last results, we are now able to establish the S.D.E. ruling X_t^ϵ :

Proposition 8.4.5. *Let define a random variable C_t as:*

$$C_t = \Phi(-d_1^{X_T^{*,\epsilon}, q, F_T}) V_t + \Phi(d_1^{X_T^{*,\epsilon}, K, F_T}) V_t + \frac{\varphi(d_2^{X_T^{*,\epsilon}, K, F_T})}{\sigma^{X_T^{*,\epsilon}, F_T}(t, T)} (q - K). \quad (8.4.5)$$

The derivative of $X_t^\epsilon = \mathbb{E}(H(t, T) X_T^\epsilon | \mathcal{F}_t)$ is then:

$$\begin{aligned}
d\mathbb{E}(H(t, T) X_T^\epsilon | \mathcal{F}_t) &= \mathbb{E}(H(t, T) X_T^\epsilon | \mathcal{F}_t) \left(r_t dt - \sigma_r B(t, T) dW_t^{r, Q} \right) + \\
&\quad \frac{1}{1-\gamma} P(t, T) (\lambda_r + \sigma_r B(t, T)) C_t dW_t^{r, Q} + \\
&\quad \frac{1}{1-\gamma} P(t, T) \lambda_S C_t dW_t^{S, Q}. \quad (8.4.6)
\end{aligned}$$

Proof. Relation (8.4.6) is a direct consequence of the Itô's lemma,

$$\begin{aligned} d\mathbb{E}(H(t, T)X_T^\epsilon|\mathcal{F}_t) &= P(t, T)d\mathbb{E}^{F^T}(X_T^\epsilon|\mathcal{F}_t) + \mathbb{E}^{F^T}(X_T^\epsilon|\mathcal{F}_t)dP(t, T) \\ &\quad + d\langle P(t, T), \mathbb{E}^{F^T}(X_T^\epsilon|\mathcal{F}_t) \rangle, \end{aligned}$$

of proposition 8.4.4 and of equation (8.2.3). $\square$

Finally, we are able to determine the optimal investment policy hedging the VaR terminal wealth.

Proposition 8.4.6. *Let C_t be defined by eq. (8.4.5). The optimal amount of stocks is*

$$\pi_t^S = \frac{1}{1-\gamma} \frac{\lambda_S}{\sigma_S} P(t, T) C_t, \quad (8.4.7)$$

and the amount of bonds is

$$\pi_t^P = X_t^\epsilon \frac{B(t, T)}{B(t, T_P)} + \frac{1}{1-\gamma} \frac{C_t P(t, T)}{B(t, T_P)} \left(\frac{\lambda_S \sigma_{Sr}}{\sigma_S \sigma_r} - \frac{\lambda_r}{\sigma_r} - B(t, T) \right). \quad (8.4.8)$$

Proof. The asset allocation is obtained by comparison of eq. (8.4.6) and of dynamics of the investor's portfolio under Q :

$$dX_t^\epsilon = rX_t^\epsilon dt + \pi_t^S \sigma_S dW_t^{S, Q} + (\pi_t^S \sigma_{Sr} - \pi_t^P \sigma_r B(t, T_P)) dW_t^{r, Q}.$$

$\square$

The comparison of investment strategies without (eq. (8.4.3) (8.4.4)) and with a bounded value at risk (eq. (8.4.7) (8.4.8)) are quite similar. In the constrained case, X_t^* is in fact replaced by $P(t, T)C_t$. Furthermore, we infer from the definition of C_t that $\lim_{q \rightarrow K} P(t, T)C_t = P(t, T)V_t = X_t^*$.

8.5 Examples.

This section is devoted to numerical examples. We have fitted the restricted exponential model introduced in appendix A to the curve of Belgian zero coupon rates, on 31/12/2006. The parameters are shown in table 8.5.1. Other market parameters take values presented in table 8.5.2.

Table 8.5.1: Expo model; parameters:

n	3	β_0	0.0434
θ_1	0.4443	β_1	0.0118
θ_2	0.6837	β_2	-0.0082
θ_3	0.1544	β_3	-0.0094

Table 8.5.2: Other market parameters:

a	0.15	λ_S	0.15
σ_r	0.015	σ_S	0.20
λ_r	-0.10	σ_{Sr}	-0.01

The investor's time horizon is set to $T = 10$ years. His wealth is $x = 10$ at $t = 0$. The bond maturity is $T_p = 10$. The chosen maximum shortfall probability is worth $\epsilon = 5\%$. Figure 8.5.1 presents the expected terminal wealth at time $t = 0$, for different risk aversion parameters γ and different shortfall levels $q = \{8, 10, 12\}$. Without surprise, the higher is the shortfall trigger q , the lower is the expected terminal wealth. Whereas increasing γ raises the mean terminal wealth.

Figure 8.5.1: Expected terminal wealth.

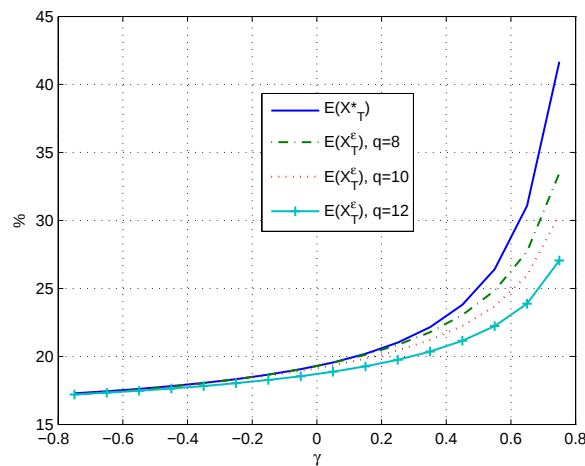


Figure 8.5.2 presents the optimal investment strategies at time $t = 0$ with and without VaR constraint, for different risk aversion parameters γ and a shortfall level q fixed to 10 (the initial wealth). The comparison of the VaR and unconstrained solutions reveals that the higher is γ , the lower is the fraction of wealth invested in risky assets, and in particular in stocks. Note that amounts held in bonds are quite similar in both cases.

Figure 8.5.2: Optimal asset allocation by preference factors.

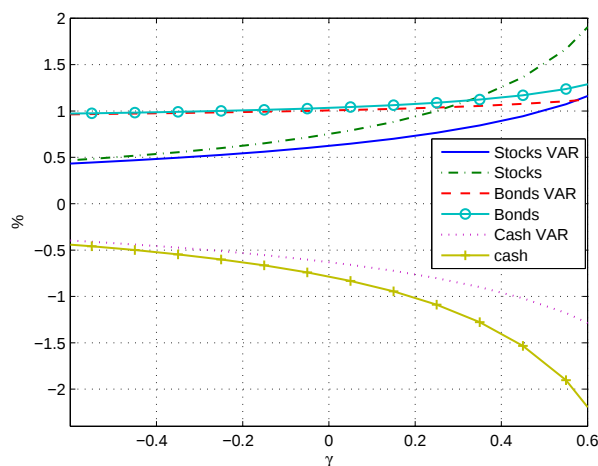
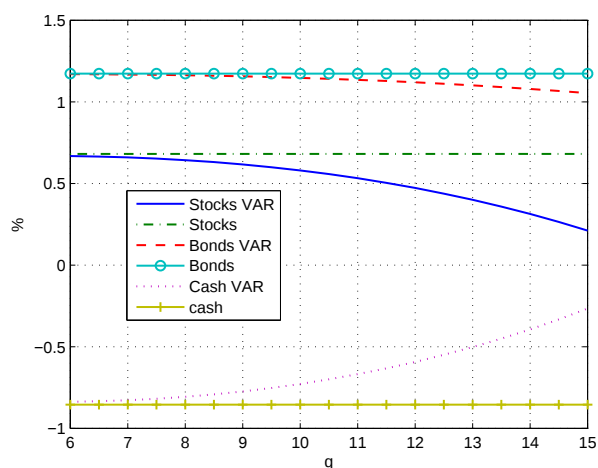


Figure 8.5.3 shows the optimal asset allocation in function of q , for a γ equal to -0.10. It reveals that increasing the trigger q leads to a massive reduction of stocks held in portfolio with respect to the unconstrained investment strategy. E.g. if $q = 15$, it is recommended to invest 21.22% of fund in stocks instead of 68.18%. Increasing q has less impact on the amount of bonds purchased: when $q = 15$, 105.4% of portfolio is invested in bonds instead of 117.3% in the unconstrained case.

Figure 8.5.3: Optimal asset allocation by Shortfalls q .

The graph 8.5.4 illustrates the influence of bond maturities T_P on the optimal asset mix. γ is still equal to -0.10 whereas q is set to 10. The fraction of wealth invested in stocks is independent of T_P (it is a direct consequence of formulas (8.4.3) and (8.4.7)) whereas the evolution of position in bonds in portfolio is similar in both cases: the amount of bonds held

in portfolio decreases when T_p increases.

Figure 8.5.4: Optimal asset allocation by bond maturities T_p .

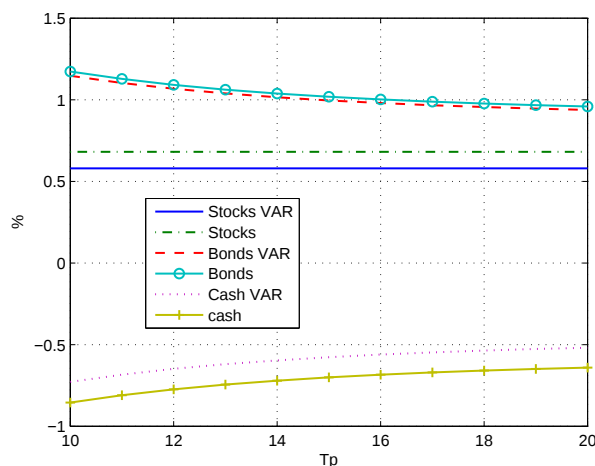
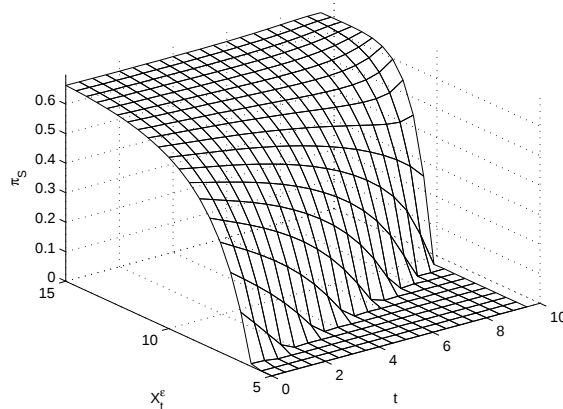


Figure 8.5.5: Stocks percentage, in function of time and wealth.



Finally, figures 8.5.5, 8.5.6, 8.5.7 illustrate the dependence between the VaR investment strategies, the residual time horizon and the available wealth. γ is still worth -0.10 whereas q is set to 10 and $T_p = 15$. When the wealth X_t^e falls below the admissible shortfall level q , the amount of stocks hold in portfolio decreases. But this decrease is sharper for a small residual time horizon than for a big one. If X_t^e is far below q , the fund manager stops to invest in stocks and only keeps a long position in cash and bonds. Remark that in case of long positions in cash, the amount of cash purchased is inversely proportional to the remaining time horizon. In case of short positions in cash, the amount of cash borrowed is inversely proportional to the residual time horizon.

Figure 8.5.6: Bonds percentage, in function of time and wealth.

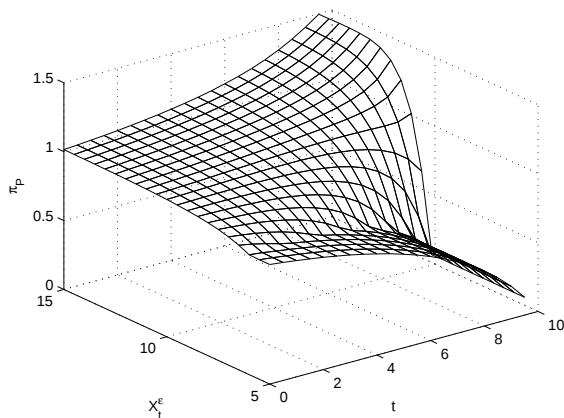
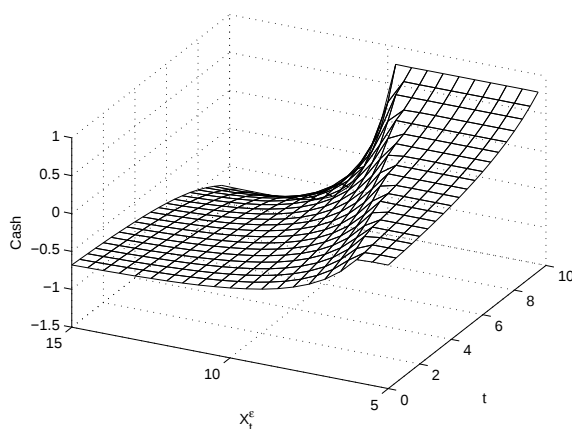


Figure 8.5.7: Cash percentage, in function of time and wealth.



8.6 Conclusions.

The drawbacks of the value at risk are well known: it is not a coherent risk measure and ignores the severity of losses. However, this measure of shortfall risks remains an important management tool for hedge funds due to its easiness of interpretation and of calculation. Moreover, banks monitor their VaR on a daily basis and have to disclose it to regulators. In this context, investors are still interested to determine the dynamic asset allocation which maximizes their utility from wealth at some horizon but with a bounded VaR. The contribution of this chapter is precisely to revisit this issue and to consider a market with stochastic interest rates, ruled by a Hull and White model. This model perfectly matches the observed yield curve and is analytically tractable. The market considered is complete and made up of three assets: cash,

zero-coupon bonds and stocks.

We have first determined the optimal wealth without VaR constraint, and his expectation under real and forward measures. Those results allow us then to reformulate the VaR constrained wealth process as a sum of options payoffs. Next, we have found closed form expressions for investment strategies hedging those wealth processes. Finally, we illustrate our results by a numerical application. For the chosen set of parameters, this example reveals that imposing a VaR constraint reduces rather the amount of stocks than the amount of bonds held in portfolio. One has also observed that the higher is the maturity date of the zero coupon, the lower is the part of wealth devoted to purchase bonds. A future research could be to extend the analysis to a risk management model limiting the expected losses rather the probability of losses. This alternative model remedies indeed the main shortcomings of VaR risk management.

Appendix A.

The instantaneous forward rate, $f(0, t) = -\frac{\partial}{\partial t} \log P(0, t)$ plays a crucial role for defining the mean reversion level $b(t)$ (see eq (8.2.2)). In examples presented in section 8.5, we have fitted a continuous function (a sum of exponentials) on the observed curve of instantaneous forward rates:

$$f(t, t+s) = \beta_0 + \sum_{i=1}^n \beta_i \exp(-\theta_i s)$$

where n is a constant determining the number of parameters $\beta_{0...n}$ and $\theta_{1...n}$. This model, known as the restricted exponential model was developed by Cairns (1998) and further details may be found in Cairns (2004).

Appendix B.

The Hull and White model belongs to the category of no arbitrage model. Furthermore, the price of a zero coupon bond is an affine function of interest rates:

$$P(t, T) = \exp(A(t, T) - B(t, T)r_t), \quad (8.6.1)$$

where

$$B(t, T) = \frac{1}{a}(1 - e^{-a(T-t)}),$$

$$A(t, T) = \log \frac{P(0, T)}{P(0, t)} + B(t, T)f(0, t) - \frac{\sigma_r^2}{4a^3}(1 - e^{-a(T-t)})^2(1 - e^{-2at}).$$

For further details on affine models, we refer to Duffie (2001).

Appendix C.

In this appendix, we review the demonstration of relation (8.3.5). As mentioned in section 8.2, the dynamics of risk free rate under the risk neutral measure is given by:

$$dr_u = a(b(u) - r_u) du + \sigma_r dW_u^{r,Q}. \quad (8.6.2)$$

Consider a process Z_u defined by:

$$Z_u = e^{au} (b(u) - r_u), \quad (8.6.3)$$

taking into account (8.6.2), the differential of Z_u is so that

$$\begin{aligned} dZ_u &= ae^{au} (b(u) - r_u) du + e^{au} b(u) du - e^{au} dr_u \\ &= e^{au} b(u) du - e^{au} \sigma_r dW_u^{r,Q}, \end{aligned}$$

and then the process Z_u may be rewritten as the following sum of integrals:

$$Z_u = Z_t + \int_t^u e^{as} b(s) ds - \int_t^u e^{as} \sigma_r dW_s^{r,Q}. \quad (8.6.4)$$

From relation (8.6.3), we know that

$$r_u = b(u) - e^{-au} Z_u, \quad Z_t = e^{at} (b(t) - r_t). \quad (8.6.5)$$

It suffices therefore to combine (8.6.4) (8.6.5) to get that

$$r_u = e^{-a(u-t)} r_t + \int_t^u ae^{-a(u-s)} b(s) ds + \int_t^u e^{-a(u-s)} \sigma_r dW_s^{r,Q}. \quad (8.6.6)$$

The short term rate r_u is hence Gaussian under Q and

$$\int_t^u ae^{-a(u-s)} b(s) ds = \left[e^{-a(u-s)} f(0, s) \right]_{s=t}^{s=u} + \int_t^u \frac{\sigma_r^2}{2a} e^{-a(u-s)} (1 - e^{-2as}) ds.$$

Integrating expression (8.6.6) and taking into account that $dW_u^{r,Q} = dW_u^{r,P} + \lambda_r du$, lead to the desired result.

Appendix D.

This section presents three propositions that are used to calculate the expected VaR terminal wealth eq. (8.3.15) and the expected VaR terminal wealth under forward measure eq. (8.3.16). Let ZK , ZC be constant parameters and $Z(t, T)$ be a random variable such that $\frac{Z(t, T)}{ZC}$ is lognormal:

$$\ln \left(\frac{Z(t, T)}{ZC} \right) \sim N(\mu^Z(t, T), \sigma^Z(t, T))$$

where $\mu^Z(t, T)$ and $\sigma^Z(t, T)$ are function of times t and T . Furthermore, we define $d_1^{Z, ZK}$ and $d_2^{Z, ZK}$ as follows:

$$\begin{aligned} d_1^{Z, ZK} &= \frac{\ln \left(\frac{ZK}{ZC} \right) - \mu^Z(t, T)}{\sigma^Z(t, T)} - \sigma^Z(t, T), \\ d_2^{Z, ZK} &= \frac{\ln \left(\frac{ZK}{ZC} \right) - \mu^Z(t, T)}{\sigma^Z(t, T)}. \end{aligned}$$

If $\Phi(\cdot)$ denotes the cumulative distribution of a standard normal random variable, we can state the following proposition:

Proposition 8.6.1. *Under the previous assumptions, we have that:*

$$\begin{aligned} a) \quad & \mathbb{E}((Z(t, T) - ZK)_+ | \mathcal{F}_t) = ZC e^{\mu^Z(t, T) + \frac{(\sigma^Z(t, T))^2}{2}} \Phi(-d_1^{Z, ZK}) - ZK \Phi(-d_2^{Z, ZK}) , \\ b) \quad & \mathbb{E}((ZK - Z(t, T))_+ | \mathcal{F}_t) = ZK \Phi(d_2^{Z, ZK}) - ZC e^{\mu^Z(t, T) + \frac{(\sigma^Z(t, T))^2}{2}} \Phi(d_1^{Z, ZK}) , \\ c) \quad & \mathbb{E}(1_{Z(t, T) \leq ZK} | \mathcal{F}_t) = \Phi(d_2^{Z, ZK}) . \end{aligned}$$

The proof directly results from the lognormality of $\frac{Z(t, T)}{ZC}$. E.g. statement a) can be rewritten as:

$$\begin{aligned} \mathbb{E}((Z(t, T) - ZK)_+ | \mathcal{F}_t) &= \int_{ZC e^{\mu^Z + \sigma^Z \cdot u} \geq ZK}^{+\infty} ZC e^{\mu^Z + \sigma^Z u} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du \\ &\quad - ZK \underbrace{\int_{ZC e^{\mu^Z + \sigma^Z \cdot u} \geq ZK}^{+\infty} \frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi}} du}_{=\Phi(-d_2^{Z, ZK})} . \end{aligned}$$

A change of variable, $s = u - \sigma^Z$, in the first integral leads to the desired result. Statements b) and c) are proved in the same way.

Appendix E.

In this section, the dynamics of V_s , for $s \geq t$, is established. By definition, we have that:

$$\begin{aligned} V_s &= \mathbb{E}^{F_T} \left(y_t^{*\frac{1}{\gamma-1}} H(t, T)^{\frac{1}{\gamma-1}} | \mathcal{F}_s \right) \\ &= y_t^{*\frac{1}{\gamma-1}} H(t, s)^{\frac{1}{\gamma-1}} \mathbb{E}^{F_T} \left(H(s, T)^{\frac{1}{\gamma-1}} | \mathcal{F}_s \right) . \end{aligned} \tag{8.6.7}$$

And, in a similar way to proposition 8.3.2, it can be demonstrated that $H(s, T)^{\frac{1}{\gamma-1}}$ is lognormal under F_T :

$$\mathbb{E} \left(H(s, T)^{\frac{1}{\gamma-1}} | \mathcal{F}_s \right) = \exp \left(\mu^{X_T^*, F_T}(s, T) + \frac{(\sigma^{X_T^*, F_T}(s, T))^2}{2} \right) . \tag{8.6.8}$$

Developing all terms of (8.6.7) leads to the following expression for V_s :

$$\begin{aligned}
V_s = & y_t^{*\frac{1}{\gamma-1}} \exp \left[\frac{1}{1-\gamma} \left[\int_t^s r_u du - \frac{1}{2} \int_t^s (\lambda_r^2 + \lambda_S^2) du + \int_t^s \lambda_r dW_u^{r,Q} + \int_t^s \lambda_S dW_u^{S,Q} \right] \right] \\
& \exp \left[\frac{1}{1-\gamma} \ln \left(\frac{P(0,s)}{P(0,T)} \right) + \frac{\gamma}{(1-\gamma)^2} \left(\frac{1}{2} \frac{\sigma_r^2}{a^2} + \frac{\lambda_r \sigma_r}{a} + \frac{1}{2} \lambda_r^2 + \frac{1}{2} \lambda_S^2 \right) (T-s) \right] \\
& \exp \left[\frac{1}{1-\gamma} r_s B(s,T) - \frac{1}{1-\gamma} f(0,s) B(s,T) - \frac{\gamma}{(1-\gamma)^2} \left(\frac{1}{2} \frac{\sigma_r^2}{a^2} + \frac{\lambda_r \sigma_r}{a} \right) B(s,T) \right] \\
& \exp \left[\frac{1}{4} \frac{\sigma_r^2}{a} \left(\frac{1-2\gamma}{(1-\gamma)^2} - \frac{1}{1-\gamma} e^{-2as} \right) B(s,T)^2 \right]. \quad (8.6.9)
\end{aligned}$$

The random variable V_s depends then on time s and on two stochastic processes $Z_{1,s}$, $Z_{2,s}$ defined hereafter:

$$Z_{1,s} = \int_t^s r_u du + \int_t^s \lambda_r dW_u^{r,Q} + \int_t^s \lambda_S dW_u^{S,Q}$$

$$Z_{2,s} = r_s B(s,T),$$

which are solutions of the next S.D.E. :

$$dZ_{1,s} = r_s + \lambda_r dW_s^{r,Q} + \lambda_S dW_s^{S,Q}$$

$$dZ_{2,s} = ab(s)B(s,T)ds - r_s ds + \sigma_r B(s,T) dW_s^{r,Q}.$$

Using the Itô's lemma allows us to differentiate V_s :

$$\begin{aligned}
dV_s = & \frac{\partial V_s}{\partial s} ds + r_s \frac{\partial V_s}{\partial Z_{1,s}} ds + (ab(s)B(s,T) - r) \frac{\partial V_s}{\partial Z_{2,s}} ds + \frac{\partial V_s}{\partial Z_{1,s}} \lambda_r dW_s^{r,Q} \\
& + \frac{\partial V_s}{\partial Z_{1,s}} \lambda_S dW_s^{S,Q} + \frac{\partial V_s}{\partial Z_{2,s}} \sigma_r B(s,T) dW_s^{r,Q} + \frac{1}{2} \frac{\partial^2 V_s}{\partial^2 Z_{1,s}} d\langle Z_{1,s}, Z_{1,s} \rangle \\
& + \frac{1}{2} \frac{\partial^2 V_s}{\partial^2 Z_{2,s}} d\langle Z_{2,s}, Z_{2,s} \rangle + \frac{\partial^2 V_s}{\partial Z_{1,s} \partial Z_{2,s}} d\langle Z_{1,s}, Z_{2,s} \rangle. \quad (8.6.10)
\end{aligned}$$

The partial derivatives involved in the Itô's equation are easily inferred from eq. (8.6.9):

$$\begin{aligned}
\frac{\partial V_s}{\partial s} = & V_s \left[-\frac{1}{1-\gamma} f(0,s) a B(s,T) - \frac{\gamma}{(1-\gamma)^2} \left(\frac{\sigma_r^2}{2a^2} + \frac{\lambda_r \sigma_r}{a} \right) a B(s,T) \right] \\
& \left[-\frac{1}{2} \frac{1}{1-\gamma} (\lambda_r^2 + \lambda_S^2) - \frac{1}{1-\gamma} \frac{\partial f(0,s)}{\partial s} B(s,T) \right] \\
& \left[\frac{\sigma_r^2}{2a} \frac{1}{1-\gamma} B(s,T) \left(e^{-2as} - \frac{1-2\gamma}{(1-\gamma)} e^{-a(T-s)} \right) \right],
\end{aligned}$$

$$\frac{\partial V_s}{\partial Z_{1,s}} = \frac{\partial V_s}{\partial Z_{2,s}} = V_s \frac{1}{1-\gamma},$$

$$\frac{\partial^2 V_s}{\partial Z_1^2} = \frac{\partial^2 V_s}{\partial Z_2^2} = \frac{\partial^2 V_s}{\partial Z_1 \partial Z_2} = V_s \frac{1}{(1-\gamma)^2}.$$

And quadratic variations are given by:

$$\begin{aligned} d\langle Z_1, Z_1 \rangle &= (\lambda_r^2 + \lambda_S^2), \\ d\langle Z_2, Z_2 \rangle &= \sigma_r^2 B(s, T)^2, \\ d\langle Z_1, Z_2 \rangle &= \lambda_r \sigma_r B(s, T). \end{aligned}$$

Developing eq. (8.6.10) leads to the desired result.

Appendix F.

We just prove the first statement of proposition 8.4.4, the two others being demonstrated in a similar way. According to the definition of V_s , one has the following equality:

$$y_t^{\frac{1}{\gamma-1}} = V_t \exp \left(-\mu^{X_T^*, F_T}(s, T) - \frac{(\sigma^{X_T^*, F_T}(s, T))^2}{2} \right).$$

The expectation $\mathbb{E}^{F_T} \left((X_T^{*, \epsilon} - q)_+ | \mathcal{F}_t \right)$ may then be rewritten as follows:

$$\mathbb{E}^{F_T} \left((X_T^{*, \epsilon} - q)_+ | \mathcal{F}_t \right) = V_t \Phi(-d_1^{X_T^{*, \epsilon}, q, F_T}) - q \Phi(-d_2^{X_T^{*, \epsilon}, q, F_T}).$$

Where $d_1^{X_T^{*, \epsilon}, q, F_T}$ and $d_2^{X_T^{*, \epsilon}, q, F_T}$ are also reformulated in terms of V_t :

$$\begin{aligned} d_1^{X_T^{*, \epsilon}, q, F_T} &= \frac{\ln \left(\frac{q}{V_t} \right) - \frac{1}{2} (\sigma^{X_T^*, F_T}(t, T))^2}{\sigma^{X_T^*, F_T}(t, T)}, \\ d_2^{X_T^{*, \epsilon}, q, F_T} &= \frac{\ln \left(\frac{q}{V_t} \right) + \frac{1}{2} (\sigma^{X_T^*, F_T}(t, T))^2}{\sigma^{X_T^*, F_T}(t, T)}. \end{aligned}$$

In order to lighten notations, we denote $\mathbb{E}^{F_T} \left((X_T^{*, \epsilon} - q)_+ | \mathcal{F}_t \right)$ by Z_t . The dynamic of Z_t is obtained by the Itô's lemma with V_t as state variable:

$$\begin{aligned} dZ_t &= \frac{\partial Z_t}{\partial t} + \frac{1}{1-\gamma} (\lambda_r \sigma_r B(t, T) + \sigma_r^2 B(t, T)^2) V_t \frac{\partial Z_t}{\partial V_t} dt \\ &\quad + \frac{1}{2} \frac{\partial^2 Z_t}{\partial V_t^2} \frac{1}{(1-\gamma)^2} V_t^2 \left((\lambda_r + \sigma_r B(t, T))^2 + \lambda_S^2 \right) dt + \\ &\quad \frac{1}{1-\gamma} (\lambda_r + \sigma_r B(t, T)) V_t \frac{\partial Z_t}{\partial V_t} dW_t^{r, Q} + \frac{1}{1-\gamma} \lambda_S V_t \frac{\partial Z_t}{\partial V_t} dW_t^{S, Q}. \end{aligned} \quad (8.6.11)$$

Taking into considerations the following relations:

$$\begin{aligned} \frac{\partial d_1^{X_T^{*, \epsilon}, q, F_T}}{\partial V_t} &= \frac{\partial d_2^{X_T^{*, \epsilon}, q, F_T}}{\partial V_t} = -\frac{1}{\sigma^{X_T^*, F_T}(t, T) V_t}, \\ V_t \varphi(-d_1^{X_T^{*, \epsilon}, q, F_T}) - q \varphi(-d_2^{X_T^{*, \epsilon}, q, F_T}) &= 0, \\ \frac{\partial d_1^{X_T^{*, \epsilon}, q, F_T}}{\partial t} &= \frac{\partial d_2^{X_T^{*, \epsilon}, q, F_T}}{\partial t} - \frac{\partial}{\partial t} \sigma^{X_T^*, F_T}(t, T). \end{aligned}$$

the partial derivatives of Z_t are after calculations:

$$\begin{aligned}\frac{\partial Z_t}{\partial t} &= V_t \varphi(-d_1^{X_T^{*,\epsilon}, q, F_T}) \frac{\partial}{\partial t} \sigma^{X_T^{*, F_T}}(t, T), \\ \frac{\partial Z_t}{\partial V_t} &= \Phi(-d_1^{X_T^{*,\epsilon}, q, F_T}), \\ \frac{\partial^2 Z_t}{\partial V_t^2} &= \varphi(-d_1^{X_T^{*,\epsilon}, q, F_T}) \frac{1}{\sigma^{X_T^{*, F_T}}(t, T) V_t}.\end{aligned}$$

It remains to develop eq. (8.6.11) to infer the dynamic of $\mathbb{E}^{F_T} \left((X_T^{*,\epsilon} - q)_+ | \mathcal{F}_t \right)$.