On some aspects of internal waves

Eric Deleersnijder, October 23, 2013

Let t and $\mathbf{x}=(x,y,z)$ denote the time and the position vector, where x and y are horizontal Cartesian coordinates, while z is the vertical coordinate (increasing upwards). The domain of interest is infinite, i.e. $x,y,z \in \,]-\infty,+\infty[$. The velocity is denoted $\mathbf{v}(t,\mathbf{x})=\mathbf{u}(t,\mathbf{x})+w(t,\mathbf{x})\mathbf{e}_z$, where $\mathbf{u}$ and w are the horizontal velocity vector and the vertical velocity component, respectively; $\mathbf{e}_z$ is a vertical unit vector, pointing upward. Upon denoting q and b the reduced pressure and the buoyancy, respectively, the governing equations of the flow in a non-rotating frame of reference read

$$\nabla_h \bullet \mathbf{u} + \frac{\partial w}{\partial z} = 0 \ , \tag{1}$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\bullet\nabla_h\mathbf{u} + w\frac{\partial \mathbf{u}}{\partial z} = -\nabla_h q \ , \tag{2}$$

$$\alpha\left(\frac{\partial w}{\partial t} + \mathbf{u}\bullet\nabla_h w + w\frac{\partial w}{\partial z}\right) = -\frac{\partial q}{\partial z} + b \ , \tag{3}$$

$$\frac{\partial b}{\partial t} + \mathbf{u}\bullet\nabla_h b + w\frac{\partial b}{\partial z} = 0 \ , \tag{4}$$

where α is a dimensionless flag that is set to zero or unity according to whether the hydrostatic equilibrium is assumed to hold valid or not; ∇_h is the horizontal part of the del operator ∇, i.e.

$$\nabla = \nabla_h + \mathbf{e}_z\frac{\partial}{\partial z} \ . \tag{5}$$

Equation (1) is the continuity equation obtained under the Boussinesq approximation. Relations (2) and (3) are the inviscid horizontal and vertical momentum budget equations. Finally, (4) is a thermodynamic equations ensuing from a series of approximations, the details of which are beyond the scope of the present working note.

Now consider an unperturbed, motionless state (whose variables are identified by the superscript “0”):

$$\mathbf{u}^0 = 0 \ , \quad w^0 = 0 \ , \quad q^0 = \frac{N^2 z^2}{2} \ , \quad b^0 = N^2 z \ , \tag{6}$$

where the constant N denotes the Brunt-Väisälä frequency. In this situation, the hydrostatic equilibrium prevails, since

$$\frac{dq^0}{dz} = b^0 \ . \tag{7}$$

Now add perturbations, identified by superscripts “1”, to the unperturbed state variables considered above so that the flow is now characterised by the following variables:

$$
\begin{cases}
\mathbf{u}(t,\mathbf{x}) = \mathbf{u}^0 + \mathbf{u}^1(t,\mathbf{x}) = \mathbf{u}^1(t,\mathbf{x}) \\
w(t,\mathbf{x}) = w^0 + w^1(t,\mathbf{x}) = w^1(t,\mathbf{x}) \\
q(t,\mathbf{x}) = q^0(z) + q^1(t,\mathbf{x}) = \dfrac{N^2 z^2}{2} + q^1(t,\mathbf{x}) \\
b(t,\mathbf{x}) = b^0(z) + q^1(t,\mathbf{x}) = N^2 z + b^1(t,\mathbf{x})
\end{cases}
\tag{8}
$$

Next, substitute (8) into (1)-(4). Assuming that the amplitude of the perturbations is small, products of perturbations may be neglected. Dropping the superscripts "1" for simplicity, the resulting linearised equations are

$$\nabla_h \bullet \mathbf{u} + \frac{\partial w}{\partial z} = 0 \ , \tag{9}$$

$$\frac{\partial \mathbf{u}}{\partial t} = -\nabla_h q \ , \tag{10}$$

$$\alpha \frac{\partial w}{\partial t} = -\frac{\partial q}{\partial z} + b \ , \tag{11}$$

$$\frac{\partial b}{\partial t} + N^2 w = 0 \ . \tag{12}$$

Equations (9)-(12) constitute a linear, homogeneous system of partial differential equations, which admits a plane wave solution of the form

$$(\mathbf{u}, w, q, b) = \mathrm{Re}\left[(\mathbf{U}, W, Q, B) e^{i(\omega t - \mathbf{k}\bullet\mathbf{x})}\right] \tag{13}$$

where $i = \sqrt{-1}$ and the constants $\mathbf{U}$, W, Q and B are the (complex) amplitudes of the horizontal velocity vector, the vertical velocity component, the reduced pressure and the buoyancy; ω is the angular frequency and $\mathbf{k}$ is the wavenumber vector. It is convenient to write the latter as follows

$$\mathbf{k} = \underbrace{k\mathbf{e}_x + l\mathbf{e}_y}_{=\mathbf{k}_h} + m\mathbf{e}_z = \mathbf{k}_h + m\mathbf{e}_z \ , \tag{14}$$

where $\mathbf{k}_h$ is the horizontal wavenumber vector, while m is the vertical wavenumber.

Substituting (13)-(14) into (9)-(12) yields

$$-i\mathbf{k}_h \bullet \mathbf{U} - imW = 0 \ , \tag{15}$$

$$i\omega \mathbf{U} = i\mathbf{k}_h Q \ , \tag{16}$$

$$\alpha i\omega W = imQ + B \ , \tag{17}$$

$$i\omega B + N^2 W = 0 \ . \tag{18}$$

This is an homogeneous system of linear algebraic equations. The latter admits a non-trivial solution if the determinant of its matrix is zero, yielding the dispersion relation

$$\omega^2 = \frac{|\mathbf{k}_h|^2 N^2}{m^2 + \alpha |\mathbf{k}_h|^2} \ . \tag{19}$$

The horizontal and vertical length scales of the wave under consideration may be taken to be $|\mathbf{k}_h|^{-1}$ and m^{-1}, respectively. Then, the corresponding aspect ratio reads

$$\delta = \frac{m^{-1}}{|\mathbf{k}_h|^{-1}} = \frac{|\mathbf{k}_h|}{m} . \tag{20}$$

Substituting (20) into (12), the dispersion relation may be reformulated as follows (Figure 1)

$$\omega^2 = \frac{\delta^2 N^2}{1+\alpha\delta^2} \tag{21}$$

Clearly, the Brunt-Väisälä frequency is the upper bound of the angular frequency of internal waves; the smaller their aspect ratio, the smaller their angular frequency. For small values of the aspect ratio, the angular frequency admits the following asymptotic expression

$$\omega^2 \sim \delta^2 N^2 , \quad \delta \to 0 . \tag{22}$$

This expression is equivalent to that obtained by setting $\alpha = 0$ in (21), which amounts to assuming that the hydrostatic equilibrium holds valid. This is not surprising: the smaller the aspect ratio, the more valid the hydrostatic assumption.

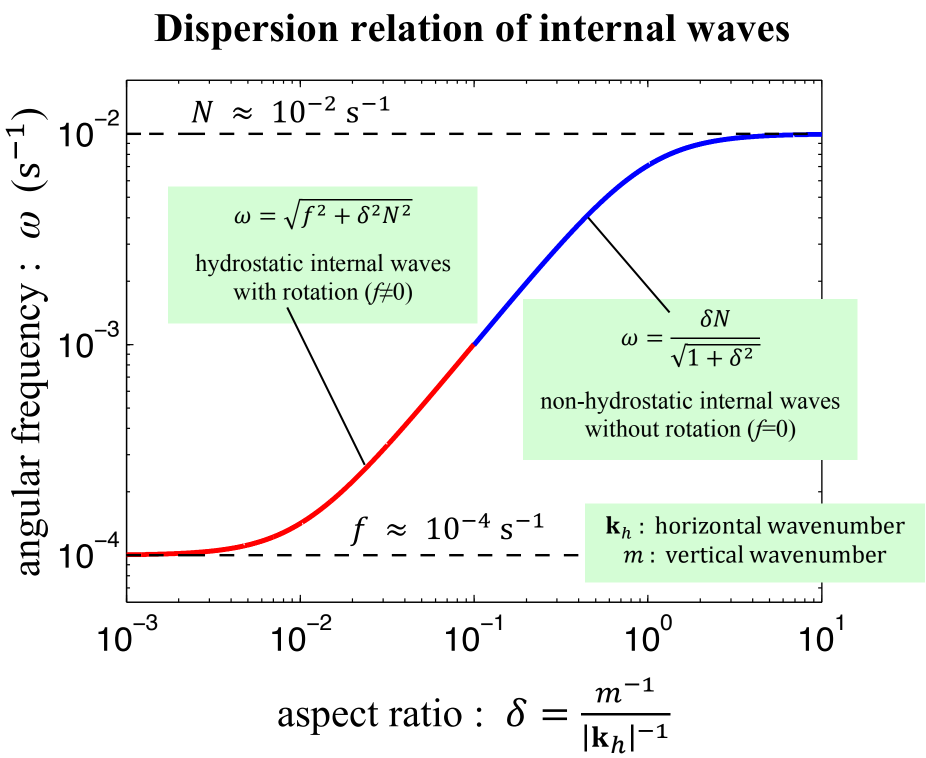

Figure 1. Dispersion relation of non-hydrostatic internal waves in a non-rotating medium (f=0, blue curve) and hydrostatic internal waves in a rotating medium (f≠0, red curve). This graph highlights the crucial importance of the Brunt-Väisälä frequency N and the Coriolis parameter f.

The phase velocity of internal waves is

$$\mathbf{c}_p \equiv \frac{\omega}{|\mathbf{k}|}\frac{\mathbf{k}}{|\mathbf{k}|} = \frac{|\mathbf{k}_h| N}{|\mathbf{k}|^3}\mathbf{k} , \tag{23}$$

while their group velocity is

$$\mathbf{c}_g \equiv \frac{\partial \omega}{\partial k}\mathbf{e}_x + \frac{\partial \omega}{\partial l}\mathbf{e}_l + \frac{\partial \omega}{\partial m}\mathbf{e}_z = \frac{mN}{|\mathbf{k}_h||\mathbf{k}|^3}\left[m\mathbf{k}_h - |\mathbf{k}_h|^2\mathbf{e}_z\right] . \tag{24}$$

Then, it is readily seen than the phase velocity and the group velocity are orthogonal:

$$\mathbf{c}_p \bullet \mathbf{c}_g = 0 . \tag{25}$$

This remarkable property of internal waves in a non-rotating medium is related to their dispersive nature.

Let θ represent the angle of the phase velocity with respect to the horizontal. It is readily understood that this angle is defined by

$$\cos\theta = \frac{|\mathbf{k}_h|}{|\mathbf{k}|} . \tag{26}$$

Next, combining (19) and (26) for non-hydrostatic waves ($\alpha = 1$) yields

$$\omega = \cos\theta\, N , \tag{27}$$

indicating that the frequency depends on the direction of the phase velocity (or the wavenumber vector) but not on the norm of the latter (Figure 2).

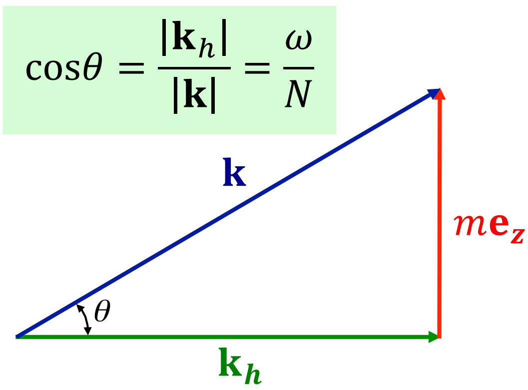

Figure 2. Illustration of a peculiar property of non-hydrostatic ($\alpha = 1$) internal waves in a non-rotating medium: the angular frequency ω depends only on the direction of the phase velocity $\mathbf{c}_p$ or, equivalently, the direction of the wavenumber number vector $\mathbf{k}$, since these vectors satisfy the well-known relation $\mathbf{c}_p = \omega|\mathbf{k}|^{-2}\,\mathbf{k}$.

As long as the angular frequency of the waves is much larger than the Coriolis parameter f, neglecting the Earth's rotation (i.e. setting f=0) is acceptable. However, as the aspect ratio diminishes, so does the angular frequency ω. Therefore, it is also necessary to study hydrostatic ($\alpha = 0$) internal waves in a rotating medium (f≠0). Accordingly, the linearised equations to be dealt with are

$$\nabla_h \bullet \mathbf{u} + \frac{\partial w}{\partial z} = 0 , \tag{28}$$

$$\frac{\partial \mathbf{u}}{\partial t} + f\mathbf{e}_z \times \mathbf{u} = -\nabla_h q \ , \tag{29}$$

$$0 = -\frac{\partial q}{\partial z} + b \ , \tag{30}$$

$$\frac{\partial b}{\partial t} + N^2 w = 0 \ . \tag{31}$$

Introducing into these equations a plane wave solution similar to (13) leads to the system of algebraic equations

$$-i\mathbf{k}_h \bullet \mathbf{U} - imW = 0 \ , \tag{32}$$

$$i\omega\mathbf{U} + f\mathbf{e}_z \times \mathbf{U} = i\mathbf{k}_h Q \ , \tag{33}$$

$$0 = imQ + B \ , \tag{34}$$

$$i\omega B + N^2 W = 0 \ . \tag{35}$$

The associated dispersion relation is (Figure 1)

$$\omega^2 = f^2 + \frac{|\mathbf{k}_h|^2}{m^2} N^2 = f^2 + \delta^2 N^2 \ . \tag{36}$$

The angular frequency decreases as the aspect ratio decreases and, in the limit $\delta \rightarrow 0$, the angular frequency is equal to f.

Since $N^2 >> f^2$, there exist hydrostatic internal waves whose dispersion relation (36) admits the asymptotic expression

$$\omega^2 \sim \delta^2 N^2 \ , \quad \delta^2 N^2 >> f^2 \text{ and } \delta << 1 \ . \tag{37}$$

Such waves are characterised by a small aspect ratio and hardly affected by the Earth's rotation, which is why their dispersion relation is similar to (22). Figure 1 illustrates this: there is a range of values of the aspect ratio δ where the angular frequency of the non-hydrostatic internal waves in a non-rotating medium (blue curve) is similar to the angular frequency of the hydrostatic internal waves on a rotating planet (red curve).