

Distributive politics: Does electoral competition promote inequality ?*

Jean-François Laslier[†]
CNRS and Ecole Polytechnique

Nathalie Picard
THEMA, Université de Cergy-Pontoise

March 2000

Abstract

Within the framework of pure redistribution (dividing one unit of a homogeneous good among identical individuals), the paper analyses the redistributions that arise from Downsian two-party electoral competition. It appears that the strategic behavior of vote-maximizing parties lead them to propose divisions which are not far from the egalitarian one.

*While preparing this article, we benefited from comments of many people: we thank Philippe De Donder, Françoise Forges, Nicolas Gravel, Richard McKelvey, Guillermo Owen, Elena Yanovskaya and specially Michel LeBreton for also having suggested to draw the Lorenz curve. An early version circulated with another title [20]. The current version was completed while Laslier was visiting CORE; he thanks this institution for its hospitality.

[†]Corresponding author: Laboratoire d'Econométrie, 1 rue Descartes, 75005 Paris. e-mail: laslier@poly.polytechnique.fr

1 Introduction

Dividing one unit of a homogeneous good among n identical individuals is probably the simplest distributive question that can be considered. For that problem, any solution from the theories of distributive justice recommends perfect equality, as would do any benevolent social planner. An important question is to what extent that egalitarian ideal point can be approached by actual institutions. Here, we consider the institution of electoral politics. We will use the most classical positive model of electoral politics, that is the downsian model of electoral competition, in which two parties compete for the vote of the electorate in a zero-sum, perfect information game.

It will be seen that perfect egalitarianism is not what comes out of down-sian competition in the redistributive framework, therefore this paper raises the question of the level of inequality generated by down-sian competition for redistribution. The objective of this paper is twofold. Firstly we find which kinds of redistribution schemes will be proposed by down-sians parties in this setting, and secondly we determine to what extent these redistribution schemes are egalitarian or not. If the second part of the objective does not raise theoretical difficulties (there exists standard ways of measuring inequality of a distribution), the first one is uneasy because down-sian games in most setting, including the one we are dealing with, have no pure-strategy equilibria, therefore *the* outcome of electoral competition is not well defined.

Voting theory has found social choice correspondences which place bounds on the possible outcomes of electoral competition. The best known is the Uncovered set, but recent literature has proposed other, more selective, solutions. These solutions (the Minimal Covering set and the Essential set) have been proved to exist in a finite context and to satisfy some nice normative properties, they also have positive political interpretation in the electoral competition game. It is worth trying to see which outcomes these solutions select on specific economic domains, and this is precisely what is done in this paper for the problem of sharing one unit of a homogeneous good. In this classical problem, majority rule is known to behave very badly. First, every alternative is Pareto optimal. Second, the problem is very similar to a spatial voting problem, and thus majority rule is chaotic. Third, chaotic cycles in the space of alternatives can, in this problem, be taken not only as majority cycles but as “almost unanimity” cycles. On the other hand, the mathematical structure of the division problem is quite simple, so that one

can hope to carry the analytical work quite far.

In a two-party electoral competition, each individual votes for the party whose proposal he or she prefers. The objective of a party is to gather as many votes as possible (“plurality game”) or simply to win the election (“tournament game”), in which case the size of the majority does not matter. Here we chose to work with the plurality game, whose payoff function is termed g in the sequel, rather than with the tournament game, whose payoff function $\text{sgn}(g)$ does not take into account the size of the majorities¹.

In the division problem, we first show that the *Uncovered set* gives only trivial indications as to the outcome of the electoral competition (propositions 1 and 3). As a social choice correspondence, this solution is here uninteresting. Since the space of alternatives is a continuum, Dutta’s *Minimal Covering set* is not a priori defined; extending word for word the definition is always possible, but then the existence theorem of Dutta (1988) no longer holds. Nevertheless the ideas which are used to define the Minimal Covering set in the finite context can be used here to exhibit a set which is almost covering. This set, termed *Hex* in the paper, is the set of divisions such that *the share of anyone is never more than twice the average share* (proposition 2).

Note that Banks, Duggan and Le Breton (1998) proved under quite general assumptions that the Uncovered set contains the support of any optimal strategy (if such a strategy exists). Their result does not help in the case considered here because the Uncovered set contains almost all the alternatives. It would be useful to have a similar result for the Minimal Covering set, since one contribution of the present paper is to show the power, in the division problem, of the idea behind the definition of the Minimal Covering set.

A *Bipartisan set* is by definition the support of an optimal strategy in a two-party, zero-sum, electoral competition game (Laffond, Laslier and Le Breton 1993, 1994; Laslier 1999a). This concept was first introduced in the tournament case where it was proved that the optimal strategy is unique. In general, Dutta and Laslier (1999) showed that it is meaningful to extend

¹This is a slight departure from the usual literature since most authors, when they make the distinction, work in the tournament case. In particular, the definition of covering is usually given in the tournament case (Fishburn 1977, Miller 1980, McKelvey 1986, Laslier 1997, Peris and Subiza 1999). For the sake of completeness, some results are also given for the tournament game.

the definition and to consider the largest support, called the *Essential set*. A pure strategy is essential if it is played with some probability in some equilibrium. In economic domains, the set of alternatives is generally infinite and, unless specific conditions are met, finding a Bipartisan set usually requires to compute an optimal (mixed) strategy in a game with discontinuous pay-off. For that class of games, no general existence theorem is available and one has to prove existence directly, by exhibiting an optimal strategy. These technical difficulties make the exercise of computing the Essential set in a given problem a non-trivial one.

To the best of our knowledge, this exercise has not been performed yet. Our setting of pure redistributive politics is already the subject of an early contribution by Shubik (1970), where he makes the point that electoral competition leads to allocations which are unequal compared to the competitive market solution. Myerson (1993) studies various electoral systems, including majority voting, in the framework of the division problem. Like we do in section 3 of this paper when we allow for mixed strategies; he supposes that parties' offers to voters are random variables. In Myerson's model², there is an atomless set of voters and each candidate makes independent offers to every voter, the budget constraint being satisfied on average. We consider a finite number of voters, we allow for any acceptable correlation between offers but for no violation of the budget constraint. As will be seen, the conclusions reached through Myerson's model are not always the same as the ones resulting from a limit operation in our setting. De Donder (2000) reports computer simulations for a taxation problem but contains no analytical statement. The models of Lindbeck and Weibull (1987), Dixit and Londregan (1995) or Coughlin (1992) are further away from ours since they essentially assume away the disequilibrium property of multidimensional pure-strategy electoral competition.

The model of pure redistributive politics is of course a special case of the general spatial model of elections extensively developed by formal political scientists. We can therefore apply here the apparatus which has been developed for this general model: existence of spanning majority cycles, Uncovered set, and more... Unfortunately the general theory of the spatial model is not as developed as the corresponding theory for finite policy spaces. In the division problem, our contribution is threefold:

²As subsequently in the Lizzeri and Persico (1999) and Lizzeri (1999).

First we derive how far one can go by sticking to pure strategies and narrowing the set of possible outcomes on the basis of dominance-like arguments. It turns out that one can go quite far on that basis, and predict that the proposed distributions will be such that no individual receives more than twice the average share.

Second we exhibit optimal strategies. Uniqueness does not hold, but the essential divisions (those which are played with positive probability at equilibrium) all satisfy the “no more than twice the average” property. (proposition 6).

Third we offer a sample of quantitative estimates of how unequal are the distributions proposed at equilibrium. The conclusion which can be drawn from this study is that majority voting in a division problem, although it does not result in the choice of the completely equitable division, generates only a limited level of inequality. This in sharp contrast with the conclusion that would be drawn from considering the top-cycle of the majority relation (that corresponds to studying winning response dynamics in the electoral games) or even uncovered sets (that corresponds to studying dominated strategies).

The paper is organized as follows: Section 2 formalizes the model. Section 3 presents the results obtained in pure undominated strategies. Section 4 is devoted to optimal (mixed) strategies, with an emphasis on the so-called “disk solution”. In section 5, various indices of inequality are computed, and a Lorenz curve is drawn, showing that the strategic behavior of competing vote-maximizers parties lead them to propose divisions which are not too far from the equal one. Section 6 is an appendix which contains some of the proofs.

2 The Model

2.1 The economic framework

The economic framework is simple: one unit of a homogeneous good (for instance one euro) is to be divided among n individuals. Individuals are labelled $i = 1, \dots, n$, and x_i is the fraction of that euro received by individual i . Each individual only cares about the amount that he or she receives. Denote by Δ_n the simplex in $\mathbb{R}^n$, by S^i the apex of Δ_n whose i -th coordinate

is 1 and by Δ_n^b the simplex minus its apexes :

$$\begin{aligned}\Delta_n &= \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^n x_i = 1\} \\ S^i &= (0, \dots, 0, 1, 0, \dots, 0) \\ \Delta_n^b &= \Delta_n \setminus \{S^i : i = 1, \dots, n\}.\end{aligned}$$

2.2 The political framework

The political framework is the pure downsian competition model: two identical parties are perfectly informed of the individual preferences, they compete for the individual votes, and voters vote sincerely on the sole basis of parties' proposals. The political game is therefore based on pairwise comparisons:

Definition 1 Let $g : \Delta_n \times \Delta_n \rightarrow \mathbb{R}$ be defined by:

$$g(x, y) = \sum_{i=1}^n \text{sgn}(x_i - y_i),$$

with $\text{sgn}(u)$ being $+1$ if $u > 0$, 0 if $u = 0$ and -1 if $u < 0$; $g(x, y)$ is called the **plurality** in favor of x against y .

If two divisions x and y are proposed, individual i will vote for x if x_i is larger than y_i and for y if x_i is smaller. If $x_i = y_i$ we can suppose that i casts half of a vote for each, or tosses a coin, or abstains, this is of no real importance, so that $g(x, y)$ is the number of votes for x minus the number of votes for y . The following properties of the plurality are straightforward:

Lemma 1 For any x and y in Δ_n , $g(y, x) = -g(x, y)$, $g(x, y)$ is an integer between $-n + 2$ and $n - 2$. If $g(x, y) = n - 2$ then there exists i such that $x_i < y_i$ and for any $j \neq i$, $x_j > y_j$.

Two-party symmetric electoral competition is studied through two slightly different games, depending on the objectives of the parties.

Definition 2 The **plurality game** is the two-player, symmetric, zero-sum game (Δ_n, g) where Δ_n is the strategy space of both players and the payoff to the pure strategy $x \in \Delta_n$ against another pure strategy $y \in \Delta_n$ is the plurality $g(x, y)$. The **tournament game** is the two-player, symmetric, zero-sum game $(\Delta_n, \text{sgn}(g))$.

2.3 The problem

In the above games, the sets of available strategies are infinite, therefore many questions are here unclear even if they are trivial in the finite case (for instance the existence of undominated strategies). The existence of an equilibrium is not guaranteed by usual theorems³. The results on games of timing do not apply because our problem is essentially multidimensional. Notice moreover that existence theorems based on fixed-points arguments are in general not constructive; and since our goal is to evaluate inequality, we need to know not only that equilibria exist but also how they look like.

3 Results in pure strategies

As usually in Game Theory, when a zero-sum game has no pure-strategy equilibrium, *bounds* can be placed on the possible outcomes of the game by resorting to dominated strategy considerations and by looking for essential strategies. This standard route is followed in this section and the next one.

3.1 Covering in the plurality game

The next definition is borrowed from Dutta and Laslier (1999), it is a variant of the game-theoretical concept of weak dominance and it also slightly differs from the definition used in a similar spatial voting context by McKelvey (1986). A discussion of the concept of covering and its relation with weak dominance is also provided by Duggan and Le Breton (1996), De Donder, Le Breton and Truchon (1997), Peris and Subiza (1999).

Definition 3 *Let x and y in Δ_n , we say that x **covers** y if $g(x, y) > 0$ and for all $z \in \Delta_n$, $g(x, z) \geq g(y, z)$. The **uncovered set**, $UC(\Delta_n)$, is the set of points in Δ_n which are not covered.*

In order to compute the uncovered set, a small preliminary lemma, which describes the covering relation in Δ_n , is useful.

³The payoff function is highly discontinuous. At first glance, existence of an equilibrium does not follow from the existing results on the matter like Dasgupta and Maskin (1986), Mertens (1986) or Reny (1999). Kramer (1978) proves an existence result for the plurality game with an atomless set of voters.

Lemma 2 For any $x, y \in \Delta_n^b$ with $x \neq y$ there exists $z \in \Delta_n^b$ such that $g(x, z) > g(y, z)$ (hence y cannot cover x). Moreover, if $g(x, y) \neq -n + 2$, z can be chosen arbitrarily close to y .

Proof. First suppose $g(x, y) > 0$ and let $z = \epsilon y + (1 - \epsilon)x$, then $g(x, z) = g(x, y) > 0$ and $g(y, z) = g(y, x) < 0$, thus $g(x, z) > g(y, z)$ and z can be close to y .

Second suppose $g(x, y) \leq 0$ and $g(x, y) \neq -n + 2$. Because x and y are two different points in the simplex, there exists i such that $x_i < y_i$. Because y is not an apex, for any j , $y_j < 1$. Let $z_i = y_i - \epsilon$ and for $j \neq i$, $z_j = y_j + \epsilon/(n-1)$, then $\sum_{j=1}^n z_j = 1$ and for ϵ small enough, $z \in \Delta_n^b$. For $\epsilon > 0$ $g(y, z) = n - 2$ and for ϵ small: for any j , $x_j < y_j \Rightarrow x_j < z_j$ and $x_j > y_j \Rightarrow x_j > z_j$. If $x_j = y_j$, $x_j < z_j$ thus :

$$g(x, z) = g(x, y) - \text{Card} \{j : x_j = y_j\} \geq g(x, y).$$

Hence if $g(x, y) \neq -n + 2$, $g(x, z) > g(y, z)$ and z is close to y .

Third suppose $g(x, y) = -n + 2$. One may suppose $x_n > y_n$ and for any $i < n$, $x_i < y_i$. Since x is not an apex, $x_n < 1$ thus one may also suppose $x_{n-1} > 0$. Let $z_{n-1} = x_{n-1} - \epsilon$ and for $i = 1, \dots, (n-2)$, $z_i = y_i + \epsilon/(n-2)$. Let also $z_n = 1 - \sum_{i=1}^{n-1} z_i$. For ϵ small enough, $z \in \Delta_n^b$, moreover:

$$z_n = 1 - (x_{n-1} - \epsilon) - \sum_{i=1}^{n-2} (y_i + \frac{\epsilon}{n-2}) = 1 - x_{n-1} - \sum_{i=1}^{n-2} y_i$$

thus $y_n < z_n < x_n$ and it comes $g(x, z) = -n + 4$, $g(y, z) = -n + 2$ and the result. **QED**

Proposition 1 $UC(\Delta_n) = \Delta_n$.

Proof. First, it is easy to check that a point which is not an apex cannot be covered by an apex. Thus it follows from lemma 1 that $\Delta_n^b \subseteq UC(\Delta_n)$. Second, consider an apex $S_i \in \Delta_n \setminus \Delta_n^b$ and another point $x \in \Delta_n$. To prove that x does not cover S_i it suffices to exhibit a $z \in \Delta_n$ such that $g(x, z) < s(S_i, z)$. Without loss of generality, consider S_1 . Since $x \neq S_1$ there exists $j > 1$ such that $x_j > 0$, write

$$x = (x_1, x_2, \dots, x_p, 0, \dots, 0)$$

with $x_2, \dots, x_p > 0$. For $j \leq p-1$, let $z_j = x_j + \frac{x_p}{p-1}$, and for $j \geq p$, let $z_j = 0$. Then $g(x, z) = 2 - p$ and $g(S_1, z) = 3 - p$. **QED**

This proposition means that, as a solution concept to the social choice problem at hand, the Uncovered Set is practically not selective.

3.2 Minimal covering in the plurality game

A refinement of the Uncovered set is proposed by Laslier and Dutta (1999), following the idea of Dutta (1988).

Definition 4 *Let $x, y \in \Delta_n$ and $X \subseteq \Delta_n$. We say that x **covers** y in X if $g(x, y) > 0$ and for all $z \in X$, $g(x, z) \geq g(y, z)$. A subset X of Δ_n is a **covering set** if for any $y \in \Delta_n \setminus X$ there exists $x \in X$ such that x covers y in X .*

In a finite context, one can prove that there exists a unique smallest (by inclusion) covering set; this set is called the **minimal covering set**. Here the set of alternatives, Δ_n , is infinite, so that the existence proof does not work. It is nevertheless possible to exhibit a non trivial subset of Δ_n which is ‘almost’ covering. This set will play an important role in the next section.

Proposition 2 *Let $Hex = \{x \in \Delta_n : \forall i \in \{1, \dots, n\}, x_i \leq 2/n\}$ and let $Int(Hex)$ be the relative interior of Hex , then for any $y \in \Delta_n \setminus Hex$ there exists $x \in Hex$ such that x covers y in $Int(Hex)$.*

Proof. Let $y \notin Hex$, denote $A = \{i : y_i > 2/n\}$, $B = \{i : y_i = 2/n\}$ and $C = \{i : y_i < 2/n\}$, $a = \text{Card } A$, $b = \text{Card } B$, $c = \text{Card } C$. Then $a + b + c = n$ and $a \geq 1$. Because $\sum_{i=1}^n y_i = 1$, $a + b < c$. Consider $x \in Hex$ such that $x_i = 2/n$ for $i \in A \cup B$ and $x_i > y_i$ for $i \in C$. It is not difficult to check that such an x exists and satisfies $g(x, y) = c - a > 0$. Take $z \in Int(Hex)$, for any $i \in A \cup B$, $z_i < 2/n = x_i \leq y_i$ thus $\text{sgn}(x_i - z_i) = \text{sgn}(y_i - z_i)$. For $i \in C$, $x_i > y_i$. It follows that $g(x, z) \geq g(y, z)$. **QED**

3.3 Sign-covering

For the sake of completeness, we compute in this sub-section the uncovered set for the tournament game, which we call the “sign-uncovered set” and

denote by $SUC(\Delta_n)$. It turns out that this set is not very different from the uncovered set $UC(\Delta_n)$ previously computed. Both contain the whole relative interior of Δ_n . The same point is made in Epstein (1998).

Definition 5 *Let x and y in Δ_n , we say that x **sign-covers** y if $g(x, y) > 0$ and for all $z \in \Delta_n$, $\text{sgn}(g(x, z)) \geq \text{sgn}(g(y, z))$. The **sign-uncovered set** $SUC(\Delta_n)$ is the set of points in Δ_n which are not sign-covered.*

Proposition 3 $SUC(\Delta_n) = \{x \in \Delta_n : \text{Card } \{i : x_i = 0\} < n/2\}$. *This set is a strict subset of Δ_n^b but still larger than the relative interior of Δ_n .*

Proof. Denote $\Delta_n^{bb} = \{x \in \Delta_n : \text{Card } \{i : x_i = 0\} < n/2\}$. We first prove that any point not in Δ_n^{bb} is sign-covered. Let x be such a point. Without loss of generality suppose $x_1 > 0$. Denote $a = \text{Card } \{i : x_i = 0\} \geq n/2 \geq 2$ and define y by: $y_1 = 0$, $y_i = x_i$ if $i > 1$ and $x_i > 0$, and $y_i = x_1/a$ if $x_i = 0$. Then $g(y, x) = a - 1 > 0$. Let z be such that $g(x, z) \geq 0$ and denote $b = \text{Card } \{i : z_i = x_i = 0\}$, then:

$$g(y, z) \geq g(x, z) - 1 + b.$$

If $b = 0$ then $g(z, x) \geq a - (n - a) > 0$, contradicting $g(x, z) \geq 0$, hence $b \geq 1$ and $g(y, z) \geq g(x, z)$. It follows that y sign-covers x . Thus

$$SUC(\Delta_n) \subseteq \Delta_n^{bb}.$$

Conversely, let $x \in \Delta_n^{bb}$ and $y \in \Delta_n$, we will prove that y does not sign-covers x . If $g(x, y) \geq 0$, it is true. Suppose $g(y, x) > 0$. It is sufficient to show that there exists z such that $g(x, z) \geq 0$ and $g(z, y) \geq 0$ with at least one strict inequality. Denote: $A = \{i : x_i > y_i\}$, $a = \text{Card } A$. Let b be the integer part of $(n - 1)/2$ and consider a set B of indices i such that $x_i \leq y_i$ and $\text{Card } B = b$. Since $x \in \Delta_n^{bb}$ we can take B such that $x_i = 0 \Rightarrow i \in B$. Write $C = I \setminus (A \cup B)$, $c = \text{Card } C$, and define z for $0 < \lambda < 1$ and $\epsilon > 0$ by:

- If $i \in A$, $z_i = \lambda x_i + (1 - \lambda)y_i$, thus $y_i < z_i < x_i$.
- If $i \in B$, $z_i = y_i + \epsilon$, thus $x_i \leq y_i < z_i$.
- If $i \in C$, $z_i = x_i - \epsilon$, thus $z_i < x_i \leq y_i$.

Notice that, for $i \in C$, $x_i > 0$, thus if ϵ is small, $z_i \geq 0$ for all i . Also, since $g(y, x) > 0$, y is not an apex and $y_i < 1$ for all i , and $z_i < 1$ for all i if ϵ is small. From the definitions it follows that $g(x, z) = a - b + c$ and $g(z, y) = a + b - c$.

Since $g(y, x) > 0$, $a < n/2$, $c \geq 1$ and thus $g(x, z) \geq 0$. Since $x \neq y$, $a \geq 1$, thus $g(z, y) = 2a + 2b - n \geq 0$. If $g(x, z) = g(z, y) = 0$ then $a = 0$, which is not true. Thus one of the inequalities is strict and we proved that y does not sign-covers x . Thus

$$\Delta_n^{bb} \subseteq SUC(\Delta_n).$$

QED

4 Optimal behavior for parties

4.1 Statement of the problem

From now on, only the plurality game is considered. A (mixed) Nash equilibrium of the game is a pair of minimax probability distributions (p, q) over the strategy set Δ_n . Because the game is zero-sum and symmetric we may consider only symmetric equilibria (p, p) , and such an optimal strategy p is characterized by the fact that p gives non-negative payoff against any pure strategy $x \in \Delta_n$. General theorems that ensure the existence of optimal strategies do not apply directly here because the (pure) strategy space is infinite and the payoff function g is discontinuous. To prove existence, we exhibit one optimal strategy.

It turns out that this problem was among the very first ones considered in Game Theory. The $n = 3$ case was given as the example of a game “in which the psychology of the players plays a fundamental role” in Emile Borel’s course on probability at the university of Paris in 1936-37. Two solutions were given, that we shall call the *hexagonal* and the *disk* solutions. The content of the course was edited by Jean Ville and published in 1938⁴. This problem and similar ones later came to be known as “Colonel Blotto” games. Borel’s two solutions appear in 1950 in an unpublished research memorandum (Gross and Wagner, 1950). This memorandum also mentions the extension to

⁴See Borel and Ville (1938). This is the same fascicule which contains Ville’s now-standard proof of Von Neumann’s minmax theorem.

larger values of n of the disk solution which we use in the present paper. The classical Game Theory textbook of Owen (1982) follows Ville in presenting the disk solution for $n = 3$.

Here, a mixed strategy is a random choice of n numbers x_i , $i = 1 \dots n$. The random variables x_i , $i = 1 \dots n$ cannot be independent since they sum to 1. The following proposition is very useful for finding a solution. It provides a sufficient condition for a mixed strategy to be optimal.

Proposition 4 *Any probability distribution of $x = (x_1, \dots, x_n) \in \Delta_n$ such that each variable x_i ($i = 1, \dots, n$) is uniformly distributed on $[0, 2/n]$ is an optimal strategy.*

Proof. Let p be such a probability distribution. It is enough to prove that the payoff to any pure strategy y against p is negative. Let $y \in \Delta_n$ and let x be randomly chosen according to p . The probability that $x_k < y_k$ is equal to 1 in the case $y_k \geq 2/n$ and to $ny_k/2$ in the case $y_k < 2/n$. The probability that $x_k = y_k$ is 0. It follows that the expected value of the sign of $y_k - x_k$ is $+1$ if $y_k \geq 2/n$ and $ny_k - 1$ if not. Therefore the payoff to y against p is:

$$g(y, p) = \sum_{k=1}^n \min \{1, ny_k - 1\} \leq 0.$$

QED

It will be later shown that the condition that the variable x_i lies, with probability one, in $[0, 2/n]$ is also a necessary condition for equilibrium. It is not known whether it is also necessary that x_i be uniform on $[0, 2/n]$. Following Borel's argument, necessity is proved for strategies which are absolutely continuous with respect to the Lebesgue measure on Δ_n and whose support (in Δ_n) satisfy some additional connectedness property. But, as Gross and Wagner (1950) pointed out, it may well be the case that each marginal is uniform, without the joint distribution being absolutely continuous with respect to the Lebesgue measure on Δ_n or having a connected support.⁵

4.2 The disk solution

In order to present the disk solution for any $n \geq 3$, some geometrical considerations are needed. In the 3-dimensional space consider the sphere of

⁵The conjecture that marginals need to be uniform is still waiting for its complete proof.

radius $1/n$ centered at $O = (0, 0, 0)$. Using spherical coordinates, write points on that sphere as $R = ((1/n) \sin \varphi \cos \theta, (1/n) \sin \varphi \sin \theta, (1/n) \cos \varphi)$. (Picture 1.) On the horizontal plane $\{\varphi = \pi/2\}$ consider the regular n -gon $[P_0, \dots, P_{n-1}]$ defined by the points:

$$P_k = \left(r_n \cos \frac{(2k-1)\pi}{n}, r_n \sin \frac{(2k-1)\pi}{n}, 0 \right) \equiv r_n e^{(2k-1)i\pi/n}$$

for $k = 0, \dots, n-1$, and with

$$r_n = \frac{2}{n \sqrt{1 + \cos \frac{2\pi}{n}}}.$$

[Insert Figure 2 about here]

The largest disk $\mathcal{D}_n$ inside this n -gon is centered at O and has radius

$$\left| \frac{P_k + P_{k+1}}{2} \right| = \frac{r_n}{2} \sqrt{1 + \cos \frac{2\pi}{n}} = \frac{1}{n},$$

therefore $\mathcal{D}_n$ is the projection of the sphere on the horizontal plane.

Lemma 3 *For a point Q inside $[P_0, \dots, P_{n-1}]$ and for $1 \leq k \leq n$ let x_k be the “ k -th height of Q ”, that is the distance from Q to the line (P_{k-1}, P_k) . The vector $x = (x_1, \dots, x_n)$ is in Hex.*

Proof. Let M_k be the projection of Q on the line (P_{k-1}, P_k) , and let P'_k be the middle of the segment $[P_{k-1}, P_k]$. (See Figure 3.) Then $OP'_k = 1/n$ and $x_k = QM_k$. Writing $Q = \rho e^{i\theta}$, $P'_k = r_n e^{2i(k-1)\pi/n}$ one finds:

$$x_k = 1/n - \rho \cos \left(\theta - \frac{2(k-1)\pi}{n} \right).$$

It follows that $0 \leq x_k \leq 2/n$ and $\sum_k x_k = 1$.

QED

[Insert Figure 3 about here]

Lemma 4 *Let R be chosen uniformly on the surface of the sphere centered at 0 with radius $1/n$. Let Q be the projection of R on the horizontal plane and let $x \in \Delta_n$ be defined in the preceding lemma. Each variable x_k is uniformly distributed on $[0, 2/n]$.*

Proof. In fact x_k is the distance from R to a fixed vertical plane tangent to the sphere. Therefore the probability that x_k is less than h is proportional to the surface on the sphere of a cap of height h . The result follows because the surface of a cap is proportional to its height. **QED**

According to the previous lemmas, proposition 4 applies, and we have proved:

Proposition 5 *Let p^* denote the probability distribution over Δ_n induced by the above-mentioned process of choosing a point uniformly on the sphere, projecting it on a plane, and computing its heights with respect to a regular n -gon. Then p^* is an optimal strategy.*

Analytically, p^* is defined as follows: φ and θ are two independent random variables on $[0, \pi]$ and on $[0, 2\pi]$ respectively. The variable φ has density $(1/2) \sin \varphi$ and the variable θ is uniform so that:

$$d(\varphi, \theta) = \frac{1}{4\pi} \sin \varphi \, d\varphi \, d\theta, \quad (1)$$

and for $k = 1, \dots, n$:

$$x_k = (1/n) (1 - \sin \varphi \cos (\theta - 2(k-1)\pi/n)). \quad (2)$$

The support of p^* is the image $\Psi(\mathcal{D}_n)$ of the disk $\mathcal{D}_n$ by the application Ψ which, to a point Q , associates the vector $x = (x_1, \dots, x_n)$ of its heights with respect to the n -gon $[P_0, \dots, P_{n-1}]$. This application is defined, by equation (2) on the whole plane.

$$\Psi : \begin{cases} \mathbb{R}^2 & \longrightarrow \mathbb{R}^n \\ Q & \longmapsto \Psi(Q) = (x_1, \dots, x_n) \end{cases}$$

It is an affine dilatation of scale $\sqrt{n/2}$ (see lemma 5 in the appendix), so $\Psi(\mathcal{D}_n)$ is a disk of radius $(1/n)\sqrt{n/2} = 1/\sqrt{2n}$ centered at the center Ω of Δ_n ,

$$\Omega = \Psi(O) = \left(\frac{1}{n}, \dots, \frac{1}{n}\right).$$

[Insert Figures 4 and 5 about here]

This disk touches the frontiers $\{x_i = 0\}$ and $\{x_i = 2/n\}$ of Hex . Notice that, except for the case $n = 3$, the definition of p^* loses symmetry between the coordinates. For instance the two variables x_k and x_{k+1} cannot take very different values because, for $n > 3$, points in $\mathcal{D}_n$ which are close to $[P_{k-1}, P_k]$ tends to be close to $[P_k, P_{k+1}]$ too. On the other hand, if n is odd then $x_k + x_{k+n/2}$ is equal to $2/n$ with probability one. Of course, relabeling the n individuals provides other solutions, on different disks inside Δ_n whose intersection is the center of Δ_n .

4.3 Characterization of optimal strategies

The existence of the disk solution proves that there exists optimal strategies whose support is included in the set Hex , which means that no individual ever receives more than twice the average share. It will now be proven that such is the case for any optimal strategy. Notice that the existence of the disk solution is used in the proof of the following proposition.

Proposition 6 *The support of any optimal strategy is included in Hex .*

Proof. For a mixed strategy p , denote by F_i^p the following adjusted cumulative density for the variable x_i :

$$F_i^p(a) = p(\{x \in \Delta_n : x_i < a\}) + \frac{1}{2}p(\{x \in \Delta_n : x_i = a\}).$$

Then for all $x \in \Delta_n$, we can write

$$g(x, p) = \sum_{i=1}^n g_i(x_i, p)$$

with

$$g_i(x_i, p) = 2F_i^p(x_i) - 1.$$

Recall that p^* stands for the disk solution, whose margins are uniform. A short computation shows that, for any $x \in \Delta_n$, $g_i(x_i, p^*)$ is equal to $nx_i - 1$

if $0 \leq x_i \leq \frac{2}{n}$ and to 1 if $x_i > \frac{2}{n}$. Let p be another optimal strategy.

$$\begin{aligned}
g(p, p^*) &= \int_{x \in \Delta_n} g(x, p^*) dp(x) \\
&= \int_{x \in \Delta_n} \sum_i g_i(x_i, p^*) dp(x) \\
&= \int_{x \in Hex} \sum_i g_i(x_i, p^*) dp(x) + \int_{x \notin Hex} \sum_i g_i(x_i, p^*) dp(x) \\
&= \int_{x \in Hex} \sum_i (nx_i - 1) dp(x) + \int_{x \notin Hex} \sum_i [g_i(x_i, p^*) - nx_i + 1] dp(x) \\
&\quad + \int_{x \notin Hex} \sum_i (nx_i - 1) dp(x) \\
&= \int_{x \in \Delta_n} \sum_i (nx_i - 1) dp(x) + \int_{x \notin Hex} \sum_i [g_i(x_i, p^*) - nx_i + 1] dp(x)
\end{aligned}$$

The first term is zero. The integrand of the second term, $\sum_i [g_i(x_i, p^*) - (nx_i - 1)]$, is negative everywhere, and strictly negative on the complement of Hex therefore $g(p, p^*) \geq 0$ implies that p gives 0 probability to this set. This proves that $p(Hex) = 1$, but clearly, the support of each variable x_i , considered separately, is included in $[0, 2/n]$ if and only if the support of x is included in Hex . **QED**

To compare with Myerson (1993)'s formulation, notice that in the disk solution, voter's shares are far from being independent: not only their sum is given, but in fact, even for large n , knowing two of them is sufficient to deduce the $n - 2$ remaining ones.

There are as many "disk" solutions as there are ways to order n objects in a circle without taking into account the orientation of the circle, that is $(n - 1)!/2$. These solutions have different supporting disks. Mixing them provides solutions on the union of a finite number of disks, still a two-dimensional manifold. In the $n = 3$ case, Borel noticed another solution whose support is the whole set Hex , in that case an hexagon. When $n = 4$, it is possible to generalize the hexagonal solution and to find a solution whose support is Hex , in that case an icosahedron. This indicates that solutions do not necessarily have two-dimensional support. The hexagonal-icosahedral solution does not generalize to higher dimensions. All the results, positive and negative, concerning these solutions are gathered in Laslier and Picard (1999).

5 Analysis of Inequality

Divisions of the euro are more or less equal, $x = (1/n, \dots, 1/n)$ being of course the most equal one. To quantify how much inequality is involved, we

use in this section standard notions from inequality measurement such as inequality indices and the Lorenz curve. For the “maximal absolute difference” and for the “Gini index”, we perform the computation for the disk solution only. For the variance, the computation is valid for any optimal strategy satisfying the condition of uniform margins. We conclude by computing the Lorenz curve expected at equilibrium.

5.1 Inequality according to the index $r(x)$

For $x \in \Delta_n$, let

$$r(x) = \text{Max}_{1 \leq i \leq n} |x_i - 1/n|.$$

The number $r(x)$ can be interpreted as an inequality index for the distribution x . Notice that, on Δ_n , $r(x)$ goes from 0 to $1 - \frac{1}{n}$. One has:

$$\text{Hex} = \{x \in \Delta_n : r(x) \leq 1/n\}.$$

On Hex , $r(x)$ goes from 0 to $\frac{1}{n}$. The expectation of $r(x)$ according to an equilibrium probability p is an indicator of the inequality one can expect the electoral competition to generate in the division problem. For the probability p^* which defines the disk solution we denote by $\text{ineq}(n)$ this number:

$$\text{ineq}(n) = \int_{\Delta_n} r(x) dp^*(x). \quad (3)$$

This integral is easily computed with the geometrical definition of p^* (proposition 5).

Proposition 7 *For any n , $\text{ineq}(n) = \frac{1}{4} \sin \frac{\pi}{n}$. For n large:*

$$\text{ineq}(n) \simeq \frac{\pi}{4n}.$$

Proof. For $Q = \rho e^{i\theta} \in \mathcal{D}_n$, the smallest value of x_k is obtained for the side $[P_{k-1}, P_k]$ of the n -gon which is the closest to Q , thus for k such that $(2k-3)\pi/n \leq \theta \leq (2k-1)\pi/n$. For instance x_1 is the smallest value when Q is in the sector $-\pi/n \leq \theta \leq \pi/n$.

Suppose first that n is even. Then, for any k , the lines (P_{k-1}, P_k) and $(P_{k-1+n/2}, P_{k+n/2})$ are parallel thus $x_k + x_{k+n/2} = 2/n$. It follows that if $x_k = \min_i \{x_i\}$, then $x_{k+n} = \max_i \{x_i\}$ and thus $r(x) = 1/n - x_k$. Consequently the

integral to be computed is equal to n times the integral of (for instance) $1/n - x_1$ over the sector $-\pi/n \leq \theta \leq \pi/n$. From (2), $1/n - x_1 = (1/n) \sin \varphi \cos \theta$. Since $dp(x) = \sin \varphi d\varphi d\theta / (4\pi)$ we find:

$$\begin{aligned} \text{ineq}(n) &= n \int_{\varphi=0}^{\pi} \int_{\theta=-\pi/n}^{\pi/n} (1/n) \sin \varphi \cos \theta [\sin \varphi d\varphi d\theta / (4\pi)] \\ &= \frac{1}{4\pi} \int_{\varphi=0}^{\pi} \sin^2 \varphi d\varphi \int_{\theta=-\pi/n}^{\pi/n} \cos \theta d\theta \\ &= \frac{1}{4} \sin \pi/n. \end{aligned}$$

Suppose now that n is odd. Then the maximum value of $|x_k - 1/n|$ is attained for a k such that $x_k < 1/n$; in effect if $x_k = \min_i \{x_i\}$ then $\max_i \{x_i\}$ is obtained for $k' = \frac{n+1}{2}$ or $k' = \frac{n+3}{2}$ and in any case $x'_k \leq 2/n - x_k$. Therefore the computation made in the case n even also holds in the case n odd. **QED**

It may be useful to compare this exact result with the one that would come out of Myerson (1993) formulation. In this atomless population model, the average share is one unit per individual. If n units are divided among n individuals, it follows from the above proposition that, in the disk solution, the average absolute inequality tends to $\frac{\pi}{4}$ when n tends to infinity. If one follows Myerson and supposes that the individual shares are independent random variables uniformly distributed on $[0, 2]$ then one can easily compute the expected value of the absolute inequality. Under the independence hypothesis, the state space is the n -cube $[0, 2]^n$ with a uniform probability distribution. For $0 \leq r \leq 2$, the probability that $\max\{x_i - 1 : i = 1, \dots, n\}$ is less than r is exactly r^n , thus the expected value of this maximum is $\int_{r=0}^1 r n r^{n-1} dr = \frac{n}{n+1}$, which tends to 1 when n tends to infinity. This is a case where the infinite-population, independent-drawings model does *not* provides an approximation of the finite population model.

5.2 Inequality according to the Gini index of x

For the disk solution it is possible to compute the expected value of the Gini index of inequality. Let y be an ordered division: $y_1 \leq y_2 \leq \dots \leq y_n$ such that $\sum_i y_i = 1$, the Gini index of y is usually defined as “twice the area between the Lorenz curve and the diagonal”, that is:

$$G(y) = \frac{n-1}{n} - \frac{2}{n} \sum_{i=1}^n \sum_{j=1}^{i-1} y_j.$$

The following equivalent formula is more convenient:

$$G(y) = \frac{n-1}{n} - \frac{2}{n} \sum_{i=1}^{n-1} (n-i)y_i.$$

If $x \in \Delta_n$ the Gini index of x is simply the Gini index of the ordering of x . Denote by $\text{gini}(n)$ the expected value of $G(x)$ according to the probability p^* that defines the disk solution:

$$\text{gini}(n) = \int_{\Delta_n} G(x) dp^*(x).$$

Computation of this number is tedious but possible (see in the appendix) and gives:

Proposition 8 *For any n , $\text{gini}(n) = \frac{1}{2n} \cot \frac{\pi}{2n}$. For n large:*

$$\text{gini}(n) \simeq \frac{1}{\pi}.$$

5.3 Inequality according to the moments of x

Another measure of the inequality in a division x is its variance $(1/n) \sum_i (x_i - 1/n)^2$. More generally, consider the k -th centered moment of x :

$$\mathcal{M}_k(x) = \frac{1}{n} \sum_{i=1}^n (x_i - \frac{1}{n})^k. \quad (4)$$

It turns out that the average of $\mathcal{M}_k(x)$ when x is distributed according to some probability distribution p can be very easily computed if the margins of p are uniform. This case includes the disk solution and all known solutions, and we conjecture that it includes in fact all the optimal strategies. For any distribution p , denote by $\text{var}_k(p, n)$ this average:

$$\text{var}_k(p, n) = \int_{\Delta_n} \mathcal{M}_k(x) dp(x). \quad (5)$$

Proposition 9 *For any strategy p such that each x_i is uniformly distributed on $[0, \frac{2}{n}]$, $\text{var}_k(p, n) = \text{var}_k(n)$ does not depend on p : If k is odd $\text{var}_k(n) = 0$, and if k is even $\text{var}_k(n) = \frac{1}{(k+1)n^k}$.*

Proof. By linearity and symmetry,

$$\int_{x \in \Delta_n} \frac{1}{n} \sum_{i=1}^n (x_i - \frac{1}{n})^k dp(x) = \int_{x \in \Delta_n} (x_1 - \frac{1}{n})^k dp(x).$$

But x_1 is uniformly distributed on the interval $[0, \frac{2}{n}]$, it follows that:

$$\text{var}_k(n) = \int_{a=0}^{2/n} (a - \frac{1}{n})^k (\frac{n}{2}) da.$$

A short computation then shows the result. **QED**

For instance the average variance of x is $\frac{1}{3n^2}$. Denote by $sdev(n)$ the average of the standard deviation. It is not possible to compute exactly this indicator of inequality but from the majoration:

$$sdev(n) = \int \sqrt{\mathcal{M}_2(x)} dp(x) \leq \sqrt{\int \mathcal{M}_2(x) dp(x)}$$

one finds:

$$sdev(n) \leq \frac{1}{n\sqrt{3}}.$$

5.4 The Lorenz curve

Notice that, since we are dividing one unit, *ineq* and *sdev* are percentages of the total sum to be divided. But it might not be very clear whether one can conclude from the previous appraisal through inequality indices that, unambiguously, when n grows larger, the proposed divisions “become more and more egalitarian”. We could also have considered the “relative” standard deviation, *sdev* divided by the mean share $1/n$. To state more clearly the problem, consider situations $y^{(n)} \in \Delta_n$ in which one individual has everything and the $n - 1$ other individuals have nothing. When n is increasing, should one say that the inequality in $y^{(n)}$ is increasing or decreasing? A short computation shows that the standard deviation for $y^{(n)}$ is $\frac{\sqrt{n-1}}{n}$, therefore the standard deviation tends to 0 when n tends to infinity, whereas the “relative” standard deviation, equal to $\sqrt{n-1}$, tends to infinity. The reader will make his or her mind as to the choice of the “good” index according to his or her own intuition of the measurement of inequality in a context where

the size of the population is variable. We think that the situation is best described by the Gini index, according to which the inequality tends neither to 0 or to 1 but is close to .3. To avoid the difficulties attached with the choice of a specific index, we now look at the average Lorenz curve.

For y an ordered division, $y_1 \leq y_2 \leq \dots \leq y_n$ such that $\sum_i y_i = 1$, and for $1 \leq k \leq n$, denote by $c_k(y)$ the k -th partial sum:

$$c_k(y) = \sum_{i=1}^k y_i$$

If x is not ordered, denote by x' the corresponding ordered division, and so denote by $c_k(x')$ the corresponding k -th partial sum. We define the k -th partial sum of p^* , denoted $lor_k(n)$, as the average value of $c_k(x')$:

$$lor_k(n) = \int_{\Delta_n} c_k(x') dp^*(x).$$

Proposition 10 For $0 \leq k \leq n$, $lor_k(n) = \frac{k}{n} - \frac{1}{4} \sin \frac{k\pi}{n}$.

The proof of this proposition is in the appendix. With this result, one can see that the Lorenz curve defined by $lor_k(n)$ is a discrete approximation to the curve:

$$c : \begin{cases} [0, 1] & \longrightarrow [0, 1] \\ t & \longmapsto c(t) = t - \frac{1}{4} \sin \pi t \end{cases}$$

For instance, on average, the 20% poorest individuals receive about 5.3% of the total and the 20% richest ones about 34.7% (notice the symmetry $t - c(t) = (1 - t) - c(1 - t)$). The largest gap $t - c(t)$ is for $t = 1/2$: the poorest half of the population receives on average 25% of the total.

[Insert Figure 6 about here]

6 Appendix

Lemma 5 Ψ is an affine dilatation of scale $\sqrt{n/2}$.

Proof. The fact that Ψ is affine is straightforward in view of the definition $x_k = QM_k$. Moreover for any two points Q_1 and Q_2 ,

$$\| \Psi(Q_1) - \Psi(Q_2) \|^2 = \sum_{k=1}^n \left(\rho_1 \cos(\theta_1 - 2(k-1)\frac{\pi}{n}) - \rho_2 \cos(\theta_2 - 2(k-1)\frac{\pi}{n}) \right)^2$$

and a short computation shows:

$$\begin{aligned} \| \Psi(Q_1) - \Psi(Q_2) \|^2 &= (n/2) [(\rho_1 \cos \theta_1 - \rho_2 \cos \theta_2)^2 + (\rho_1 \sin \theta_1 - \rho_2 \sin \theta_2)^2] \\ &= (n/2) \| Q_1 - Q_2 \|^2. \end{aligned}$$

The lemma follows. **QED**

The next lemma is useful in the analysis of inequality:

Lemma 6 *Suppose (for simplifying notations) that $-\pi/n \leq \theta \leq 0$. Then if n is even:*

$$x_1 \leq x_n \leq x_2 \leq x_{n-1} \leq x_3 \leq \dots \leq x_{(n/2)+1}$$

and if n is odd:

$$x_1 \leq x_n \leq x_2 \leq x_{n-1} \leq x_3 \leq \dots \leq x_{(n+1)/2}.$$

Proof. Take n even. We have to prove that for $k = 1, \dots, n/2$, $x_k \leq x_{n+1-k}$ and that for $k = 2, \dots, n/2$, $x_k \geq x_{n+2-k}$. One has $x_k = 1/n - \rho \cos(\theta - 2(k-1)\pi/n)$ and $x_{n+1-k} = 1/n - \rho \cos(\theta + 2k\pi/n)$. Write $\alpha_k = \frac{2k-1}{n}\pi$ and $\eta = \theta + \pi/n$, then the cosines in x_k and x_{n+1-k} are $\cos(\alpha_{\pm}\eta)$. For $1 \leq k \leq n/2$, the inequality $x_k \leq x_{n+1-k}$ follows easily. The other set of inequalities, as well as the case n odd, can be treated the same way. **QED**

With this lemma one can perform the computation of the Gini index (proposition 8) and of the Lorenz curve (proposition 10):

Proposition 8 For any n , $gini(n) = \frac{1}{2n} \cot \frac{\pi}{2n}$. For n large, $gini(n) \simeq \frac{1}{\pi}$.

Proof. By symmetry, $gini(n)$ is equal to $2n$ times the integral of $G(x)$ on the domain $-\pi/n \leq \theta \leq 0$.

$$gini(n) = 2n \int_{-\pi/n \leq \theta \leq 0} G(x) dp^*(x).$$

On this domain, we know how to order x , by the previous lemma. Therefore in order to compute $gini(n)$ we just need to compute for each k the integral:

$$g_k = \int_{-\pi/n \leq \theta \leq 0} x_k dp^*(x).$$

If we denote by $x_{(1)} = x_1, x_{(2)} = x_n, x_{(3)} = x_2, \dots$ the ordering of x and similarly $g_{(1)} = g_1, g_{(2)} = g_n, g_{(3)} = g_2, \dots$ we have:

$$\begin{aligned} gini(n) &= \frac{n-1}{n} - \frac{2}{n} \sum_{i=1}^n (n-i) \int_{\Delta_n} x_{(i)} dp^* \\ &= \frac{n-1}{n} - 4 \sum_{i=1}^n (n-i) g_{(i)} \end{aligned}$$

Computation of g_k through

$$g_k = \int_{\varphi=0}^{\pi} \int_{\theta=-\pi/n}^0 \frac{1}{n} \left[1 - \sin \varphi \cos \left(\theta - \frac{2(k-1)}{n} \pi \right) \right] \sin \varphi d\varphi d\theta / (4\pi)$$

provides the useful formula:

$$g_k = \frac{1}{2n^2} - \frac{1}{8n} \left(\sin \frac{2k-1}{n} \pi - \sin \frac{2k-2}{n} \pi \right) \quad (6)$$

If n is even then for $k = 1, \dots, n/2$, $x_{(2k-1)} = x_k$ and $x_{(2k)} = x_{n+1-k}$. If n is odd then $x_{(2k-1)} = x_k$ for $k = 1, \dots, (n+1)/2$ and $x_{(2k)} = x_{n+1-k}$ for $k = 1, \dots, (n-1)/2$. For n even:

$$\begin{aligned} \sum_{i=1}^n (n-i) g_{(i)} &= \sum_{k=1}^{n/2} (n - (2k-1)) g_{(2k-1)} + \sum_{k=1}^{n/2} (n-2k) g_{(2k)} \\ &= \sum_{k=1}^{n/2} (n-2k+1) g_k + \sum_{k=1}^{n/2} (n-2k) g_{n+1-k} \\ &= \sum_{k=1}^{n/2} (n-2k+1) \left[\frac{1}{2n^2} - \frac{1}{8n} \left(\sin \frac{2k-1}{n} \pi - \sin \frac{2k-2}{n} \pi \right) \right] \\ &\quad + \sum_{k=1}^{n/2} (n-2k) \left[\frac{1}{2n^2} - \frac{1}{8n} \left(\sin \frac{1-2k}{n} \pi - \sin \frac{-2k}{n} \pi \right) \right] \\ &= \frac{n-1}{4n} - \frac{1}{8n} \sum_{k=1}^{n/2} \left[-(n-2k) \sin \frac{2k-1}{n} \pi + (n-2k) \sin \frac{2k}{n} \pi \right. \\ &\quad \left. + (n-2k+1) \sin \frac{2k-1}{n} \pi - (n-2k+1) \sin \frac{2k-2}{n} \pi \right] \end{aligned}$$

Computing these sums is possible and gives:

$$\sum_{i=1}^n (n-i) g_{(i)} = \frac{n-1}{4n} - \frac{1}{8n} \cot \frac{\pi}{2n}.$$

The result follows easily for n even. Similar computations can be done for n odd. **QED**

Proposition 10 For $0 \leq k \leq n$, $lor_k(n) = \frac{k}{n} - \frac{1}{4} \sin \frac{k\pi}{n}$.

Proof. Like in the computation of the Gini index, the average $lor_k(n) = \int_{\Delta_n} c_k(x') dp^*(x)$ can be computed as $2n$ times the integral on the domain $-\pi/n \leq \theta \leq 0$, thus, with the notation of the previous proof:.

$$lor_k(n) = 2n \sum_{i=1}^k g_{(i)}.$$

The above lemma provides the ordering of x on the considered domain, so that:

$$lor_k(n) = 2n (g_1 + g_n + g_2 + g_{n-1} + g_3 + \dots)$$

(k terms in the sum). We now by equation (6) how to compute g_i so that most terms cancel in this sum; the result follows easily. **QED**

References

- [1] Banks, J., J. Duggan and M. Le Breton (2000) “Bounds for mixed strategy equilibria in infinite games”, *Journal of Economic Theory*, forthcoming.
- [2] Borel, E. and J. Ville (1938) *Application de la théorie des probabilités aux jeux de hasard*, original edition by Gauthier-Villars, Paris, 1938; reprinted at the end of *Théorie mathématique du bridge à la portée de tous*, by E. Borel and A. Chéron, Editions Jacques Gabay, Paris, 1991.
- [3] Coughlin, P. J. (1992) *Probabilistic Voting Theory*, Cambridge: Cambridge University Press.
- [4] Dasgupta, P. and E. Maskin (1986) “The existence of equilibrium in discontinuous economic games, I: theory” *Review of Economic Studies*, 53:1-26.
- [5] De Donder, Ph. (2000) “Majority voting solution concepts and redistributive taxation: Survey and simulations”, *Social Choice and Welfare* forthcoming.
- [6] De Donder, Ph., M. Le Breton and M. Truchon (2000) “Choosing from a weighted tournament”, *Mathematical Social Sciences*, forthcoming.
- [7] Dixit, A. and J. Londregan (1995) “Redistributive politics and economic efficiency” *American Political Science Review* 89:856-866.
- [8] Duggan, J. and M. Le Breton (1996) “Dutta’s minimal covering set and Shapley’s weak saddles” *Journal of Economic Theory* 70:257-265.
- [9] Dutta, B. (1988) “Covering sets and a new Condorcet choice correspondence”, *Journal of Economic Theory* 44:63-80.
- [10] Dutta, B. and J-F. Laslier (1999) “Comparison functions and choice correspondences”, *Social Choice and Welfare* 16:513-532.
- [11] Epstein, D. (1998) “Uncovering some subtleties of the uncovered set: Social choice theory and distributive politics”, *Social Choice and Welfare* 15:81-93.

- [12] Fishburn, P.C. (1977) "Condorcet social choice functions" *SIAM Journal of Applied Mathematics* 33:469-489.
- [13] Gross, O. and R. Wagner (1950) "A continuous Colonel Blotto game" RM-408, Rand Corporation, Santa Monica.
- [14] Kramer, G. (1978) "Existence of an electoral equilibrium" in P.C. Ordeshook (Ed.) *Game Theory and Political Science*, New York University Press, New York, pp. 375-391.
- [15] Laffond, G., J-F. Laslier and M. Le Breton (1993) "The Bipartisan set of a tournament game", *Games and Economic Behavior* 5:182-201.
- [16] Laffond, G., J-F. Laslier and M. Le Breton (1994) "Social choice mediators", *American Economic Review (proc.)* 84:448-453.
- [17] Laslier, J-F (1997) *Tournament Solutions and Majority Voting*, Berlin: Springer-Verlag.
- [18] Laslier, J-F. (1999a) "Aggregation of preferences with a variable set of alternatives", *Social Choice and Welfare* forthcoming.
- [19] Laslier, J-F (1999b) "Interpretation of electoral mixed strategies", *Social Choice and Welfare* forthcoming.
- [20] Laslier, J.-F. and N. Picard (1999) "Dividing one euro, democratically" Working paper of the THEMA No 99-23, Université de Cergy-Pontoise.
- [21] Lindbeck, A; and J. Weibull (1987) "Balanced-budget redistribution as the outcome of political competition", *Public Choice* 52:273-297.
- [22] Lizzeri, A. (1999) "Budget deficit and redistributive politics", *Review of Economic Studies*, 66:909-928.
- [23] Lizzeri, A. and N. Persico (1999) "The provision of public goods under alternative electoral incentives" Working paper in Economic Theory No 99F4, Princeton University.
- [24] McKelvey, R. (1986) "Covering, dominance and institution-free properties of Social Choice", *American Journal of Political Science* 30:283-314.

- [25] Mertens, J.-F. "The minimax theorem for USC-LSC payoff functions" *International Journal of Game Theory* 15:237-250.
- [26] Miller, N. (1980) "A new solution set for tournaments and majority voting: further graph-theoretical approaches to the theory of voting", *American Journal of Political Science* 24:68-96.
- [27] Myerson, R. B. (1993) "Incentives to cultivate favored minorities under alternative electoral systems", *American Political Science Review* 87:856-869.
- [28] Owen, G. (1982) *Game Theory*, second edition, New York: Academic Press.
- [29] Peris, J.E. and B. Subiza (1996) "Condorcet choice correspondences for weak tournaments", *Social Choice and Welfare* 16:217-231.
- [30] Reny, P.J. (1999) "On the existence of pure and mixed strategies Nash equilibria in discontinuous games" *Econometrica* 67:1029-1056.
- [31] Schubik, M. (1970) "Voting, or a price system in a competitive market structure" *American Political Science Review* 64:179-181.

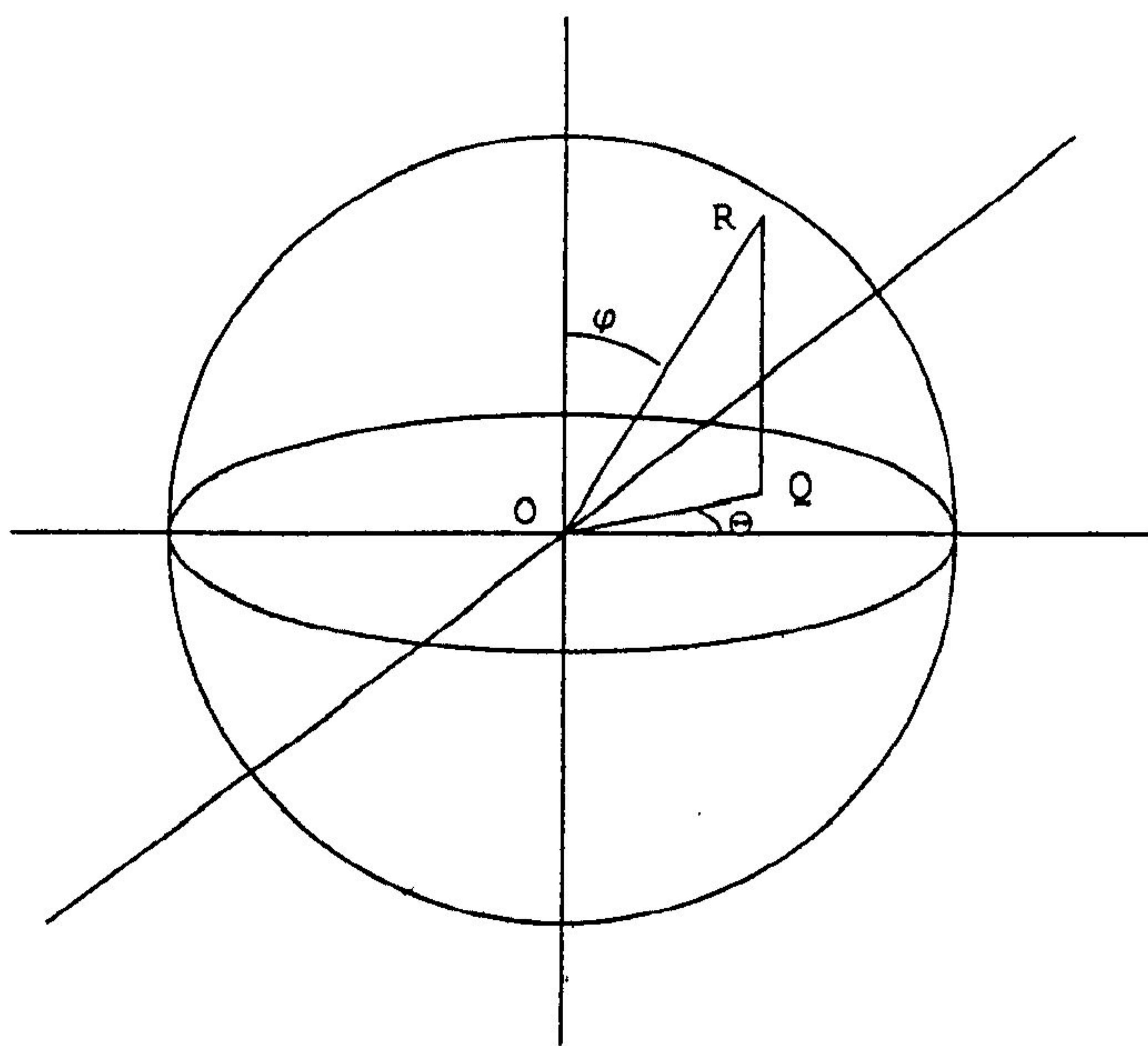


FIGURE 1

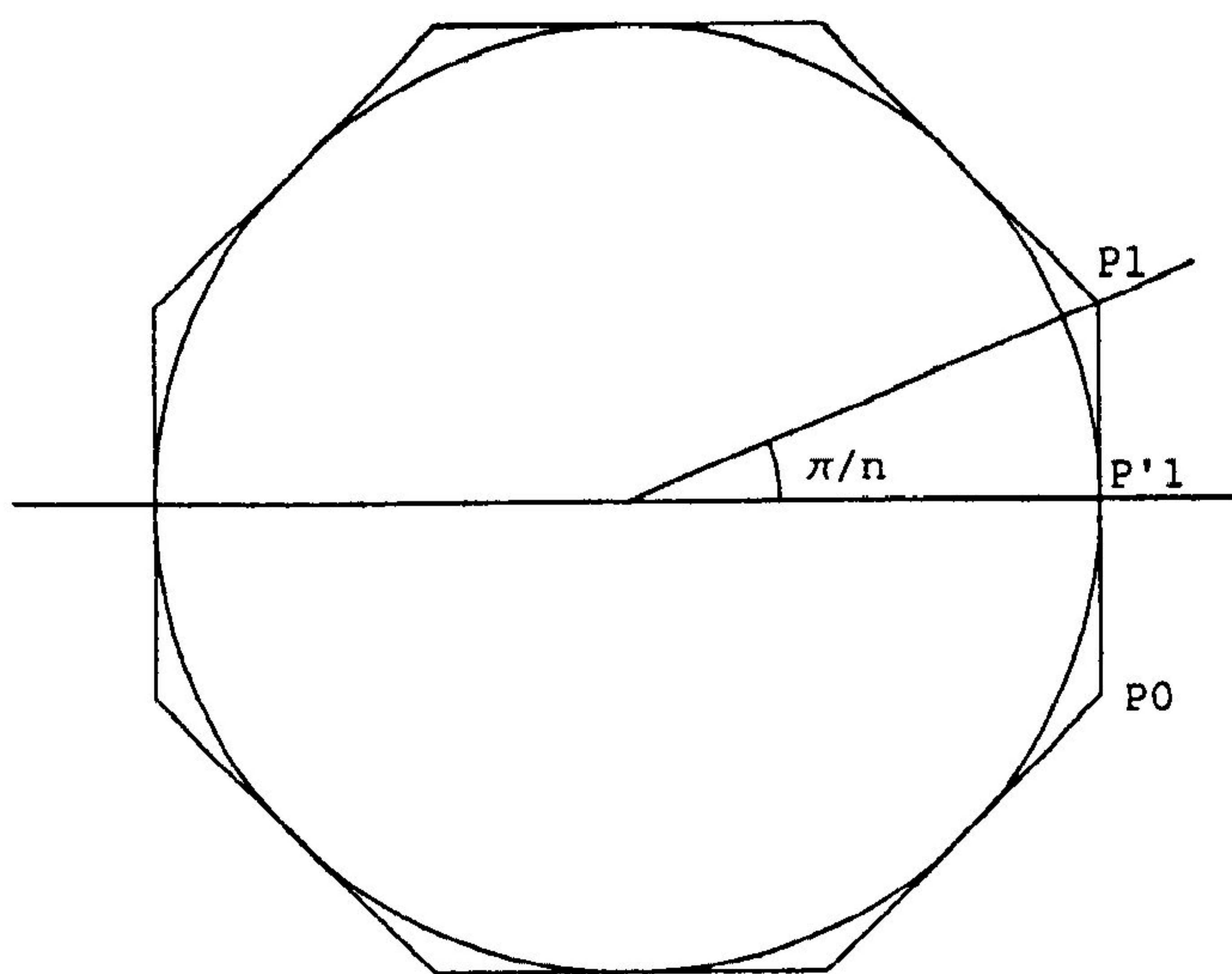


FIGURE 2

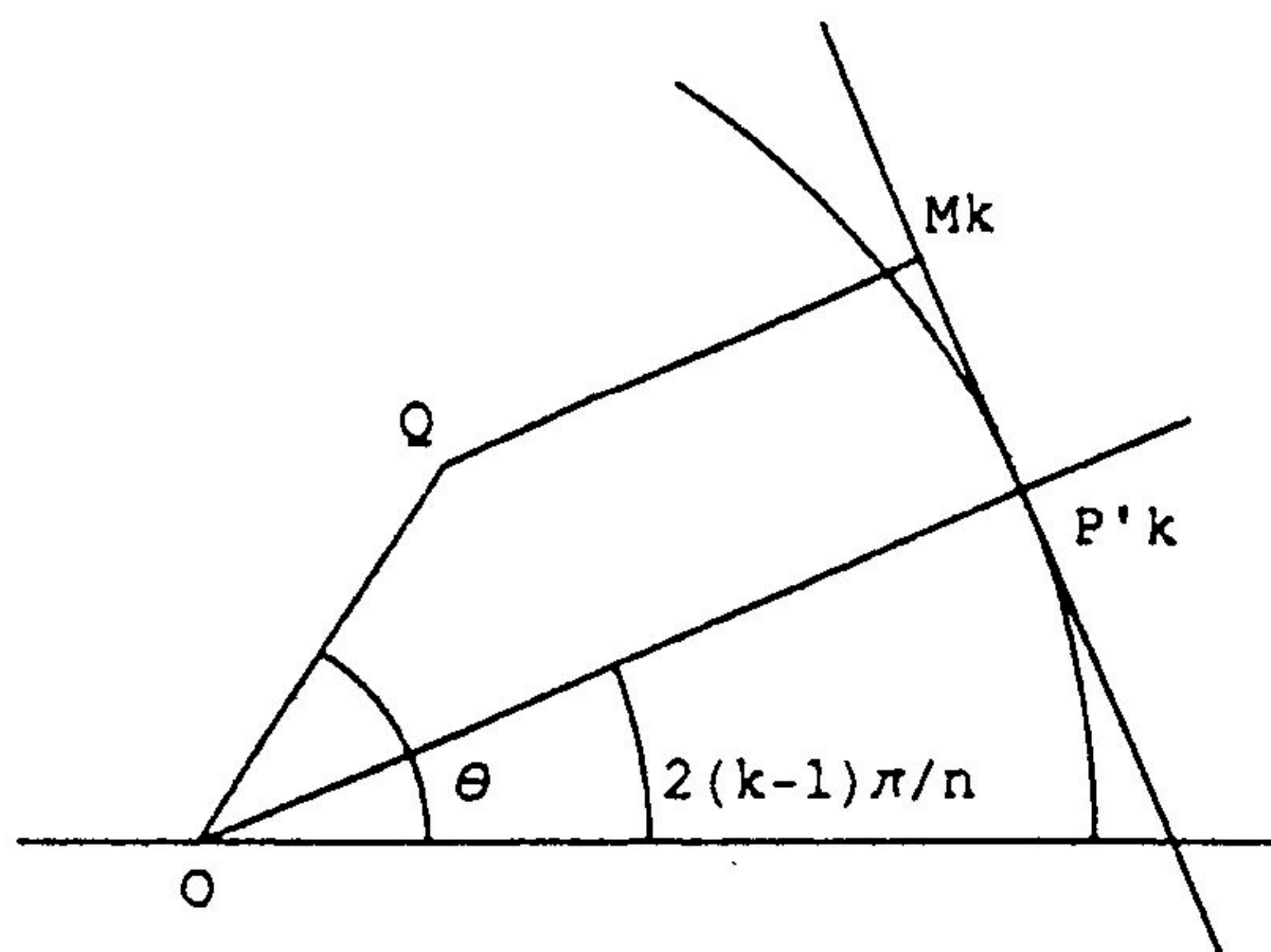


FIGURE 3

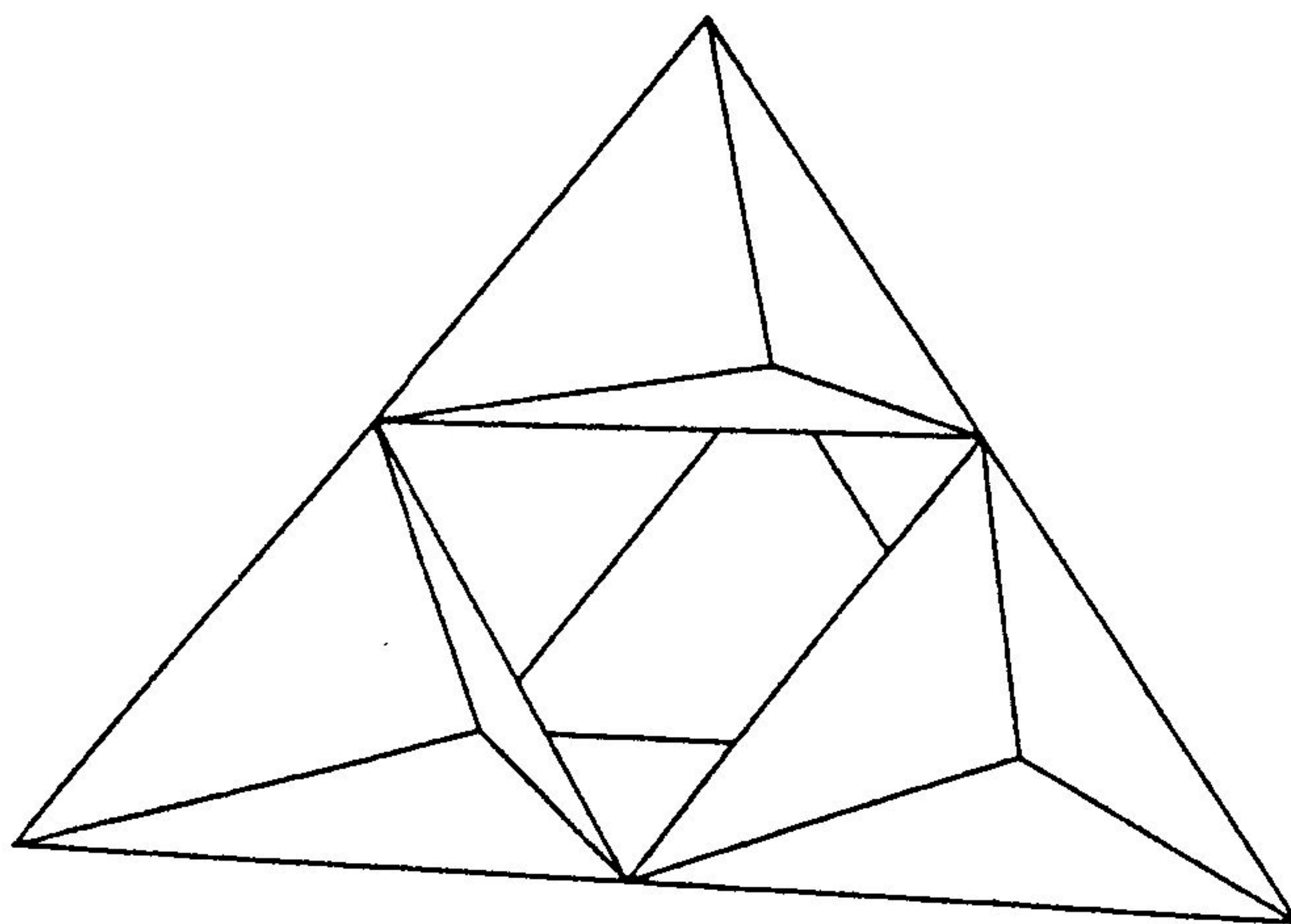


FIGURE 4

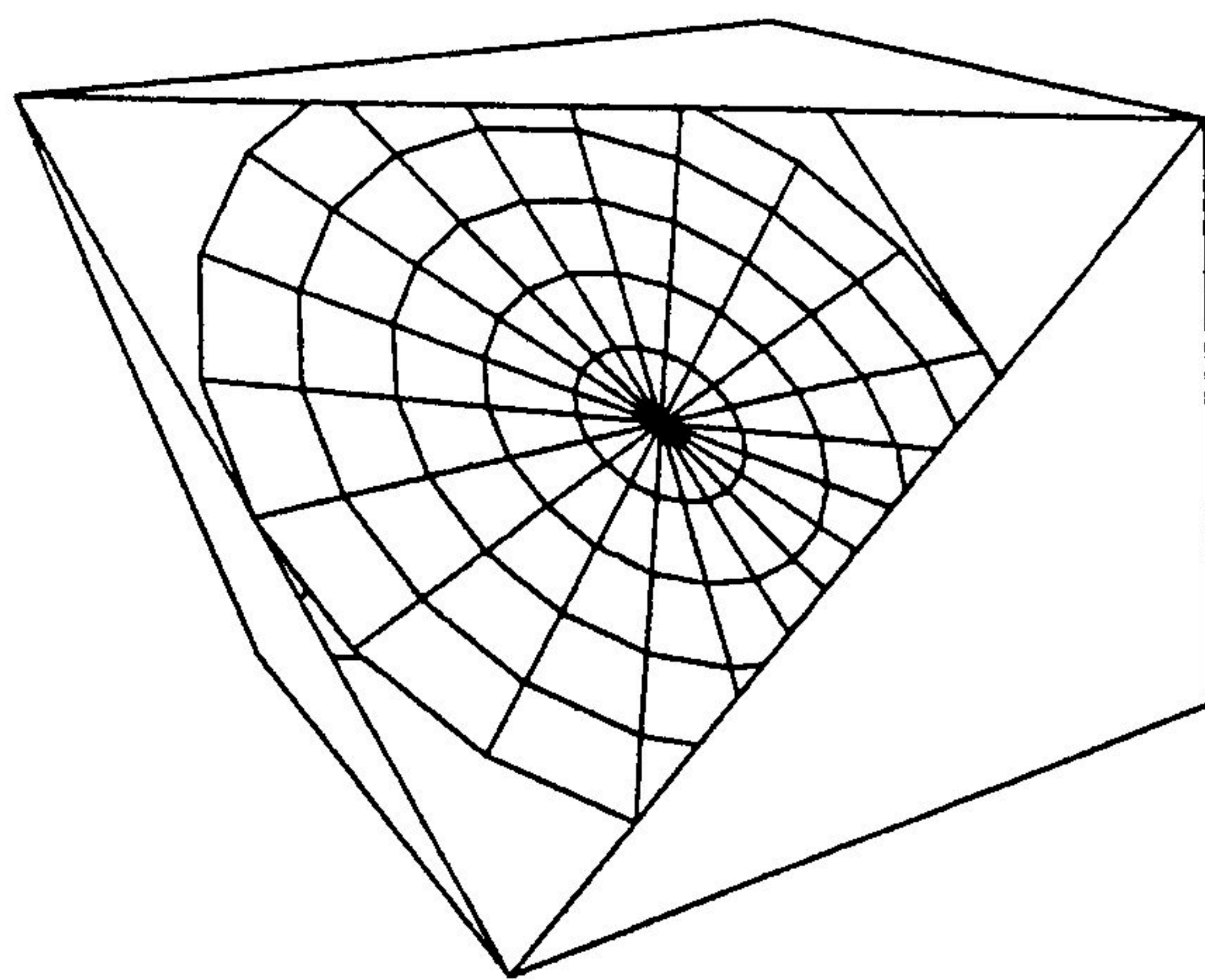


FIGURE 5

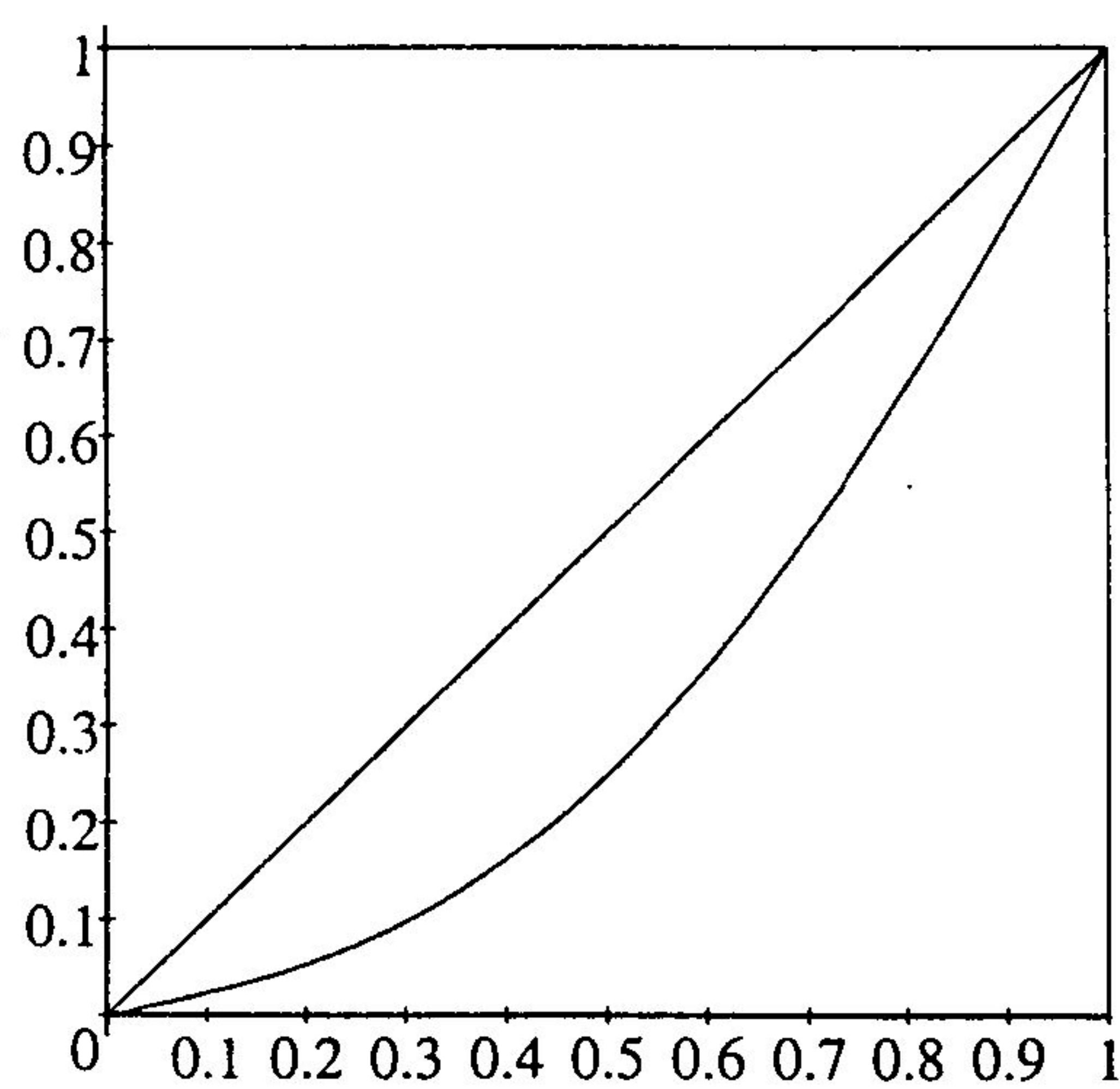


FIGURE 6