

ON 2-VERMA MODULES FOR QUANTUM $\mathfrak{sl}_2$

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ABSTRACT. In this paper we study the superalgebra A_n , introduced by the authors in previous work on categorification of Verma modules for quantum $\mathfrak{sl}_2$. The superalgebra A_n is akin to the nilHecke algebra, and shares similar properties. In particular, we prove a uniqueness result about 2-Verma modules on $\mathbb{k}$ -linear 2-categories.

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1. INTRODUCTION

In previous work [23] the authors gave a categorification of Verma modules for quantum $\mathfrak{sl}_2$ using cohomology rings of infinite-dimensional, complex Grassmannians and their Koszul duals. The categorical $\mathfrak{sl}_2$ -action therein was built from a setup using pullback maps induced by a geometric correspondence with an infinite one-step partial flag variety, as pioneered in the finite case by Chuang–Rouquier [5] and Frenkel–Khovanov–Stroppel [9]. Moreover, in [23], a superalgebra A_n akin to the nilHecke algebra NH_n was introduced, which governs part of the higher structure in this Verma categorification. In fact, the superalgebra A_n is $\mathbb{Z}$ -graded and contains NH_n as a graded subsuperalgebra concentrated in even parity. It also admits a diagrammatic presentation, as NH_n , but contains additionally a set of anticommuting generators, not present in NH_n . The latter allows the introduction of an extra grading, which is a key ingredient in the categorification of Verma modules, and turn A_n into a $\mathbb{Z} \times \mathbb{Z}$ -graded superalgebra.

The superalgebra A_n was obtained as an endomorphism superring of the functors $F^n = F \circ \cdots \circ F$ and $E^n = E \circ \cdots \circ E$ realizing the categorical $\mathfrak{sl}_2$ -action. In contrast, in this paper we describe another way to obtain A_n , namely from an action of the symmetric group S_n on a supercommutative ring R . That is, in the same way as the nilHecke algebra: if R^{S_n} denotes the subsuperring of S_n -invariants, then A_n is the endomorphism superring of R as an R^{S_n} -supermodule, and therefore is isomorphic to an algebra of matrices with coefficients in R^{S_n} .

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(see [18, §3.2]). Additionally, we give an explicit description of R^{S_n} in terms of Schur polynomials in R . These polynomials are indexed by pairs consisting of a partition and a strict partition, and contain commuting and anticommuting variables.

We would also like to mention that the superring R^{S_n} was studied independently by Appel, Egilmez, Hogancamp, and Lauda in [2], where it is given a combinatorial description, different than ours.

As usual, induction and restriction functors give rise to functors $F_n : A_n\text{-smod} \rightarrow A_{n+1}\text{-smod}$ and $E_n : A_{n+1}\text{-smod} \rightarrow A_n\text{-smod}$ on the category of supermodules of $A = \bigoplus_{n \geq 0} A_n$. The latter define a categorical $\mathfrak{sl}_2$ -action and are connected through an exact sequence

$$0 \rightarrow F_{n-1}E_{n-1} \rightarrow E_n F_n \rightarrow q^{-2n} \lambda Q_{n+1} \oplus q^{2n} \lambda^{-1} \Pi Q_{n+1} \rightarrow 0,$$

where Q_n is a functor of tensoring with a polynomial ring in one variable, Π is a parity shift functor, and $q^a \lambda^b$ denotes a shift in the bigrading by (a, b) . The exact sequence above does not split and this is a crucial difference from the finite-dimensional case of [5] and [9]. With the $\mathfrak{sl}_2$ -action above the (suitably defined) Grothendieck group of A is isomorphic to a Verma module for quantum $\mathfrak{sl}_2$.

Moreover, recall that NH_n has certain so-called cyclotomic quotients for all $N \in \mathbb{N}_0$, which are Morita equivalent to cohomology rings of finite-dimensional Grassmannians, and play an important role in the categorification of finite-dimensional representations of $\mathfrak{sl}_2$ (see [5] and [20]). These can be recovered from A_n by using certain differentials d_N on it.

Let us mention that in [2] it was introduced another differential on A_n , whose homology is a quotient of NH_n which is Morita equivalent to the $GL(N)$ -equivariant cohomology of Grassmannians, and also categorify these representations.

Next, assembling the data of the categorification of a Verma module in a 2-category, we give an axiomatic definition of a 2-Verma module. Following the techniques in [28, §5] (see also [5, §5]) we state a uniqueness result for 2-Verma modules whenever they are subcategories of the strict 2-categories of all bigraded, $\mathbb{k}$ -linear, supercategories, with the 1-morphisms being functors and where the 2-morphisms are grading preserving natural transformations.

Remark 1.1. We give a diagrammatic presentation of the superalgebra A_n which is slightly different from the one given in [23]: in this paper “white dots” corresponding to odd elements are placed in the regions of diagrams rather than on the strands. This allows writing elaborate relations involving these generators in a compact form.

Remark 1.2. This paper introduces some of the notions and techniques that are used in a general context, done in the sequel [24], where versions of KLR algebras are constructed which allow categorification of Verma modules for all symmetrizable quantum Kac–Moody algebras.

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2. THE SUPERALGEBRA A_n

2.1. Reminders on super structures. Recall that a *superring* is a $\mathbb{Z}/2\mathbb{Z}$ -graded ring. Let $A = A_0 \oplus A_1$ be a superring. We will call the $\mathbb{Z}/2\mathbb{Z}$ -grading the *parity* and use the notation $p(a)$ to indicate the parity of a homogeneous element $a \in A$. Elements with parity 0 are called *even*, elements of parity 1 are called *odd*. Whenever we refer to an element of A as even or odd, we will always be assuming that it is homogeneous. A *subsuperring* of A is a subring which is itself a superring, that is, the canonical inclusion preserves the parity. The *supercenter* $Z_s(A)$ of A is the set of all elements of A which supercommute with all elements of A , that is $Z_s(A) = \{x \in A \mid xa = (-1)^{p(x)p(a)}ax \text{ for all } a \in A\}$. A left A -*supermodule* M is a $\mathbb{Z}/2\mathbb{Z}$ -graded module over A such that $A_i M_j \subseteq M_{i+j}$ ($i, j \in \mathbb{Z}/2\mathbb{Z}$).

If A has additional gradings, then we say A is a *graded superring* (or multigraded superring). In this context we can speak of (multi)graded supermodules. The notion of (graded, multigraded) *superalgebra* and related structures are defined in the same way.

The (graded) supermodules over a (graded) superring A , together with (degree and) parity preserving morphisms of supermodules form an abelian (graded) supercategory $A\text{-smod}$. In this supercategory, we write $\Pi : A\text{-smod} \rightarrow A\text{-smod}$ for the parity shift functor (i.e. the action of $\mathbb{Z}/2\mathbb{Z}$).

2.2. Supercommutative polynomials, symmetric group action and Demazure operators.

Let $\underline{x}_n = (x_1, \dots, x_n)$ be even variables and $\underline{\omega}_n = (\omega_1, \dots, \omega_n)$ be odd variables, and form the commutative superring $R = \mathbb{Z}[\underline{x}_n] \otimes \bigwedge^\bullet(\underline{\omega}_n)$, where $\bigwedge^\bullet(\underline{\omega}_n)$ is the exterior ring in the variables $\underline{\omega}_n$ and coefficients in $\mathbb{Z}$. Introduce a $\mathbb{Z} \times \mathbb{Z}$ -grading in R , declaring that $\deg(x_i) = (2, 0)$ and $\deg(\omega_i) = (-2i, 2)$. The first grading is referred to as the q -grading and the second as the λ -grading (see [23, §9.1] for details).

Let S_n be the symmetric group on n letters, which we view as being generated as a Coxeter group with generators s_i . These correspond to the simple transpositions $(i \ i+1)$, and we use these two descriptions interchangeably throughout. As explained in [23, §9.3], S_n acts (from the left) on R as follows,

$$(1) \quad \begin{aligned} s_i(x_j) &= x_{s_i(j)}, \\ s_i(\omega_j) &= \omega_j + \delta_{i,j}(x_i - x_{i+1})\omega_{i+1}. \end{aligned}$$

Proposition 2.1. *The assignment in (1) gives R the structure of an S_n -module.*

This action respects the bigrading as well as the parity, as one easily checks.

Using the S_n -action above, we introduce the *Demazure operators* ∂_i on R for all $1 \leq i \leq n-1$ in the usual way, as

$$\partial_i(f) = \frac{f - s_i(f)}{x_i - x_{i+1}}.$$

The operator ∂_i is an even operator, and it is homogeneous of bidegree $\deg(\partial_i) = (-2, 0)$.

From the formula for ∂_i above one sees that $s_i \partial_i(f) = \partial_i(f)$ and $\partial_i(s_i f) = -\partial_i(f)$ for all i , so ∂_i is in fact an operator from R to the subring $R^{s_i} \subset R$ of invariants under the transposition $(i \ i+1)$.

The following is proved in [23, §9.2], and can be viewed as a direct consequence of Proposition 2.1 together with the definition of ∂_i .

Lemma 2.2. *The action of the Demazure operators on R satisfies the Leibniz rule,*

$$\partial_i(fg) = \partial_i(f)g + s_i(f)\partial_i(g),$$

for all $f, g \in R$ and for $1 \leq i \leq n-1$, and the relations

$$\partial_i^2(f) = 0, \quad \partial_i \partial_{i+1} \partial_i(f) = \partial_{i+1} \partial_i \partial_{i+1}(f),$$

$$\partial_i \partial_j(f) = \partial_j \partial_i(f) \quad \text{for } |i - j| > 1,$$

$$x_i \partial_i(f) - \partial_i(x_{i+1}f) = f, \quad \partial_i(x_i f) - x_{i+1} \partial_i(f) = f,$$

and

$$\partial_i(\omega_k f) = \omega_k \partial_i(f) \quad \text{for } k \neq i,$$

$$\partial_i((\omega_i - x_{i+1}\omega_{i+1})f) = (\omega_i - x_{i+1}\omega_{i+1})\partial_i(f),$$

for all $f \in R$ and $1 \leq i \leq n-1$.

For a reduced expression $\vartheta = s_{i_1} \cdots s_{i_r}$ in terms of simple transpositions we put

$$\partial_\vartheta = \partial_{i_1} \cdots \partial_{i_r}.$$

From Lemma 2.2 it follows that this is independent of the choice of the reduced decomposition. As in the polynomial case [22] the action of the Demazure operators on R also satisfies

$$\partial_\vartheta \partial_{\vartheta'} = \begin{cases} \partial_{\vartheta\vartheta'}, & \text{if } \ell(\vartheta\vartheta') = \ell(\vartheta) + \ell(\vartheta'), \\ 0, & \text{else,} \end{cases}$$

where ℓ denotes the length function on S_n .

Definition 2.3. We define A_n to be the bigraded $\mathbb{Z}$ -superalgebra of operators on R generated by the Demazure operators ∂_i for $1 \leq i \leq n-1$, together with multiplication by elements of R . We put $A_0 = \mathbb{Z}$ and define

$$A = \bigoplus_{n \in \mathbb{N}_0} A_n.$$

Recall that the *nilHecke* algebra NH_n is the $\mathbb{Z}$ -algebra generated by $T_1, \dots, T_{n-1}$ and $x_1, \dots, x_n$, with relations

$$(2) \quad T_i^2 = 0, \quad T_i T_j = T_j T_i \text{ if } |i - j| > 1, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1},$$

$$(3) \quad x_i x_j = x_j x_i,$$

$$(4) \quad T_i x_j = x_j T_i \text{ if } j - i \neq 0, 1, \quad T_i x_i - x_{i+1} T_i = 1, \quad T_i x_{i+1} - x_i T_i = -1.$$

This is a graded algebra with $\deg_q(x_i) = 2$ and $\deg_q(T_i) = -2$. We extend this grading to a $\mathbb{Z} \times \mathbb{Z}$ -grading whose first, i.e. q -grading, is the one from before, and whose second, i.e. λ -grading, is trivial.

In [23, §9] we gave a presentation of A_n by generators and relations. In our convention, the notation $A \rtimes B$ for two (graded) R -(super)algebras means we take the free product (i.e. the algebra generated by R -linear combinations of words with characters in A and B together with multiplication given by concatenation and reduction), which we quotient by some specified relations.

Proposition 2.4. *The superalgebra A_n is isomorphic to the bigraded superalgebra*

$$A_n \cong \mathrm{NH}_n \rtimes \bigwedge^\bullet(\underline{\omega}_n)$$

with the relations $x_i \omega_j = \omega_j x_i$ for all i, j , and

$$T_i \omega_j = \omega_j T_i \text{ if } i \neq j, \quad T_i(\omega_i - x_{i+1} \omega_{i+1}) = (\omega_i - x_{i+1} \omega_{i+1}) T_i.$$

For a reduced decomposition $\vartheta = s_{i_1} \cdots s_{i_k} \in S_n$ we put $T_\vartheta = T_{i_1} \cdots T_{i_k}$, which by (2) is a well-defined element of NH_n .

There is a canonical inclusion $\mathrm{NH}_n \hookrightarrow A_n$, given by the inclusion $\mathrm{NH}_n \hookrightarrow \mathrm{NH}_n \rtimes 1 \subset \mathrm{NH}_n \rtimes \bigwedge^\bullet(\underline{\omega}_n)$, and therefore NH_n is a graded subsuperalgebra of A_n concentrated in even parity and with trivial λ -grading.

The following is [23, Proposition 9.1].

Proposition 2.5. *The superalgebra A_n is a free $\mathbb{Z}$ -module with the sets*

$$\begin{aligned} &\{x_1^{k_1} \cdots x_n^{k_n} \omega_1^{\ell_1} \cdots \omega_n^{\ell_n} T_\vartheta : k_i \in \mathbb{N}_0, \ell_i \in \{0, 1\}, \vartheta \in S_n\}, \\ &\{x_1^{k_1} \cdots x_n^{k_n} T_\vartheta \omega_1^{\ell_1} \cdots \omega_n^{\ell_n} : k_i \in \mathbb{N}_0, \ell_i \in \{0, 1\}, \vartheta \in S_n\}, \end{aligned}$$

and

$$\{T_\vartheta x_1^{k_1} \cdots x_n^{k_n} \omega_1^{\ell_1} \cdots \omega_n^{\ell_n} : k_i \in \mathbb{N}_0, \ell_i \in \{0, 1\}, \vartheta \in S_n\},$$

being basis.

Let $\mathrm{NC}_n \subset A_n$ denote the *nilCoxeter algebra*, which is the subalgebra generated by the T_i 's. From Proposition 2.5 it follows at once that we have a decomposition $A_n \cong R \otimes \mathrm{NC}_n$ as a $\mathbb{Z}$ -(super)module.

We introduce the notations $[n] = q^{n-1} + q^{n-3} \cdots + q^{1-n}$ and $[n]! = [n][n-1] \cdots [1]$. Let $\mathrm{grk}_{\mathbb{Z}}(A_n) \in \mathbb{Z}[[q, q^{-1}, \lambda, \lambda^{-1}]]/(\pi^2 - 1)$ denote the graded rank of A_n viewed as a $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ -graded $\mathbb{Z}$ -module, with π being the parity grading (i.e. the $\mathbb{Z}/2\mathbb{Z}$ -degree). Then as a direct consequence of Proposition 2.5 we obtain the following:

Corollary 2.6. *The graded rank of A_n is*

$$\mathrm{grk}_{\mathbb{Z}}(A_n) = q^{-n(n-1)/2} [n]! \prod_{j=1}^n \frac{1 + \pi \lambda^2 q^{-2j}}{1 - q^2},$$

where we interpret the fractions as power sums.

2.2.1. Tight monomials.

Definition 2.7. We say a monomial in A_n is *tight* if it can be written as a word involving only symbols from the alphabet $\{T_1, \dots, T_{n-1}, x_1, \dots, x_n, \omega_1\}$.

Or in other words, a monomial is tight if it can be written without using any ω_i for $i > 1$. Note that for $i > 1$ we have

$$\omega_i = x_{i-1}T_{i-1}\omega_i - T_{i-1}x_i\omega_i,$$

and

$$T_{i-1}\omega_i = -T_{i-1}\omega_{i-1}T_{i-1},$$

and so, ω_i can be written as a combination of tight monomials by recursion.

We now introduce another basis, expressed in terms of tight monomials, which is in some sense more natural from the point of view of categorification, and which will be used in §3. For each $\vartheta \in S_n$ we choose a left-adjusted reduced expression $\vartheta = s_{i_r} \cdots s_{i_1}$, where *left-adjusted* means that $i_r + \cdots + i_1$ is minimal.

Recall that, given a reduced expression, we can obtain all other reduced expressions of the same permutation by applying a sequence of moves $s_i s_{i+1} s_i \leftrightarrow s_{i+1} s_i s_{i+1}$ and $s_i s_j \leftrightarrow s_j s_i$ for $|i - j| > 1$. Moreover, given two reduced expressions $s_{i_r} \cdots s_{i_1}$ and $s_{i'_r} \cdots s_{i'_1}$ such that the second one can be obtained from the first by a single move $s_i s_{i+1} s_i \mapsto s_{i+1} s_i s_{i+1}$ or $s_i s_j \mapsto s_j s_i$ then

$$\min_{t \in \{0, \dots, r\}} s_{i_t} \cdots s_{i_1}(k) \leq \min_{t \in \{0, \dots, r\}} s_{i'_t} \cdots s_{i'_1}(k),$$

for all $k \in \{1, \dots, n\}$. Hence, a reduced expression $s_{i_r} \cdots s_{i_1}$ is left-adjusted if and only if

$$(5) \quad \min_{t \in \{0, \dots, r\}} s_{i_t} \cdots s_{i_1}(k) \leq \min_{t \in \{0, \dots, r\}} s_{i'_t} \cdots s_{i'_1}(k),$$

for all $k \in \{1, \dots, n\}$ and all other reduced expression $s_{i'_r} \cdots s_{i'_1}$ of the same permutation. In this condition, we write $\min_{\vartheta}(k) = \min_{t \in \{0, \dots, r\}} s_{i_t} \cdots s_{i_1}(k)$. Note that a left-adjusted reduced expression always exists and is unique up to distant permutations (i.e. moves $s_i s_j \leftrightarrow s_j s_i$ for $|i - j| > 1$). In particular, we can obtain a left-adjusted reduced expression for any permutation by taking its representative in the coset decomposition

$$(6) \quad S_n = \bigsqcup_{a=1}^n S_{n-1} s_{n-1} \cdots s_a,$$

applied recursively.

Example 2.8. The permutation $(1 \ 3 \ 2 \ 4)$ of $\{1, 2, 3, 4\}$ admits as left-adjusted reduced expression the word $s_1 s_2 s_1 s_3 s_2$ which comes from the summand $S_2 s_3 s_2$ in the first step of the recursive decomposition. Note that $s_1 s_2 s_3 s_1 s_2$ is also left-adjusted while $s_2 s_1 s_2 s_3 s_2$ and $s_2 s_1 s_3 s_2 s_3$ are not.

Suppose $s_{i_r} \cdots s_{i_1}$ is a left-adjusted reduced expression of ϑ . Then we can choose for each $k \in \{1, \dots, n\}$ an index t_k such that

$$s_{i_{t_k}} \cdots s_{i_1}(k) = \min_{\vartheta}(k).$$

Clearly this choice is not necessarily unique and we can have $t_k = t_{k'}$ for $k \neq k'$. However, it defines a partial order on the set $\{1, \dots, n\}$ where we say $k < k'$ if $t_k \leq t_{k'}$. We extend this order arbitrarily and we write $<_t$ for it. There is a bijective map $s : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ which sends $k < k'$ to $s(k) <_t s(k')$, so that $t_{s(k)} \leq t_{s(k')}$. For $k \in \{1, \dots, n+1\}$, we put

$$\vartheta^k = s_{it_{s(k)}} \cdots s_{it_{s(k-1)}},$$

where it is understood that $t_{s(0)} = 0$ and $t_{s(n+1)} = r$. It defines a partition of the reduced expression of ϑ . Moreover, it is constructed such that

$$\vartheta^k \cdot \vartheta^1(s(k)) = \min_{\vartheta}(s(k)).$$

Example 2.9. Consider again $\vartheta = s_1 s_2 s_1 s_3 s_2$ with $i_1 = 2, i_2 = 3, i_3 = 1, i_4 = 2, i_5 = 1$. We can choose $t_1 = 0, t_2 = 0, t_3 = 3$ and $t_4 = 5$ (we could also have chosen $t_1 = 1$ or $t_1 = 2$ and also $t_3 = 4$, changing what follows). Then we can have $s(1) = 1$ (or 2), $s(2) = 2$ (or 1), $s(3) = 3$ and $s(4) = 4$, with $\vartheta^1 = 1, \vartheta^2 = 1, \vartheta^3 = s_1 s_3 s_2$ and $\vartheta^4 = s_1 s_2$.

Now we define the element

$$(7) \quad \theta_a = T_{a-1} \cdots T_1 \omega_1 T_1 \cdots T_{a-1},$$

and we consider the set

$$(8) \quad \left\{ x_1^{k_1} \cdots x_n^{k_n} T_{\vartheta^{n+1}} \theta_{\min_{\vartheta}(s(n))}^{\ell_{\vartheta}(s(n))} T_{\vartheta^n} \cdots \theta_{\min_{\vartheta}(s(2))}^{\ell_{\vartheta}(s(2))} T_{\vartheta^2} \theta_{\min_{\vartheta}(s(1))}^{\ell_{\vartheta}(s(1))} T_{\vartheta^1} : k_i \in \mathbb{N}_0, \ell_i \in \{0, 1\}, \vartheta \in S_n \right\},$$

where it is understood that $\theta_a^0 = 1$ and $\vartheta = \vartheta^{n+1} \cdots \vartheta^1$ is as above. As we will see in Lemma 3.1 ahead, the set in (8) forms a basis of A_n for all involved choices.

Example 2.10. We take $n = 3$. Then (8) can be given by the following elements (we use (6)):

$$\begin{aligned} & p(\underline{x}_3) \omega_1^{\ell_1} \theta_2^{\ell_2} \theta_3^{\ell_3}, \\ & p(\underline{x}_3) \omega_1^{\ell_1} T_1 \omega_1^{\ell_2} \theta_3^{\ell_3}, \\ & p(\underline{x}_3) \omega_1^{\ell_1} \theta_2^{\ell_2} T_2 \theta_2^{\ell_3}, \\ & p(\underline{x}_3) \omega_1^{\ell_1} \theta_2^{\ell_2} T_2 T_1 \omega_1^{\ell_3}, \\ & p(\underline{x}_3) \omega_1^{\ell_1} T_1 \omega_1^{\ell_2} T_2 \theta_2^{\ell_3}, \\ & p(\underline{x}_3) \omega_1^{\ell_1} T_1 \omega_1^{\ell_2} T_2 T_1 \omega_1^{\ell_3} \end{aligned}$$

where $p(\underline{x}_3)$ is a polynomial in the variables x_1, x_2, x_3 , and $(\ell_1, \ell_2, \ell_3) \in \{0, 1\}^3$.

Using this basis, we give another construction of the superalgebra A_n .

Definition 2.11. Let A'_n be the bigraded $\mathbb{Z}$ -superalgebra given by

$$A'_n = \text{NH}_n \rtimes \bigwedge^{\bullet}(\omega_1)$$

with relations

$$(9) \quad \begin{aligned} x_i \omega_1 &= \omega_1 x_i, & T_j \omega_1 &= \omega_1 T_j, \\ T_1 \omega_1 T_1 \omega_1 &= -\omega_1 T_1 \omega_1 T_1, \end{aligned}$$

for $1 \leq i \leq n$ and $2 \leq j \leq n$, and where NH_n is concentrated in parity 0 and λ -degree 0, and ω_1 has parity 1 and degree $(-2, 2)$.

Of course, one can view the monomials in (8) as elements of A'_n .

Proposition 2.12. *The monomials in (8) span A'_n .*

Proof. By the nilHecke relations (4) and because ω_1 commutes with the x_i 's, A'_n is generated by monomials with all x_i 's to the front (this can be proven by induction on the number of T_i 's). Therefore, A'_n is spanned by elements of the form

$$x_1^{k_1} \cdots x_n^{k_n} T_{\vartheta_r} \omega_1 T_{\vartheta_{r-1}} \omega_1 \cdots T_{\vartheta_2} \omega_1 T_{\vartheta_1},$$

where ϑ_i 's are reduced expressions. If $\vartheta_{i+t} \cdots \vartheta_i(1) = 1$, then $\omega_1 T_{\vartheta_{i+t}} \cdots T_{\vartheta_{i+1}} \omega_1 T_{\vartheta_i} \omega_1 = 0$ by (9) and the braid relations (2). Hence, we can assume $r \leq n+1$ and left-adjusted reduced expressions together with the θ_i 's span A'_n . The proof follows by observing that left-adjusted reduced expressions are unique up to permutation of distant crossings, and by observing the θ_i 's anticommutes up to adding elements with fewer number of T_i 's, because of (9) and the braid relations. $\square$

Proposition 2.13. *There is an isomorphism of bigraded superalgebras $A'_n \cong A_n$, sending $x_i \mapsto x_i$, $T_i \mapsto T_i$ and $\omega_1 \mapsto \omega_1$.*

Proof. We prove $T_1 \omega_1 T_1 \omega_1 = -\omega_1 T_1 \omega_1 T_1 \in A_n$. First note that $T_1 \omega_2 \omega_1 = -\omega_1 \omega_2 T_1$ by the last relation of Lemma 2.2. Then we compute

$$\begin{aligned} T_1 \omega_1 T_1 \omega_1 &= T_1 \omega_1 \omega_2 T_1 x_2 - T_1 \omega_1 \omega_2 x_2 T_1, \\ \omega_1 T_1 \omega_1 T_1 &= x_2 T_1 \omega_2 \omega_1 T_1 - T_1 x_2 \omega_2 \omega_1 T_1, \end{aligned}$$

using the last relation of Lemma 2.2 again. By the nilHecke relations (2) we get

$$\begin{aligned} -T_1 \omega_1 \omega_2 T_1 x_2 &= T_1 \omega_1 \omega_2 x_1 T_1 - T_1 \omega_1 \omega_2, \\ -x_2 T_1 \omega_2 \omega_1 T_1 &= T_1 x_1 \omega_2 \omega_1 T_1 - \omega_2 \omega_1 T_1. \end{aligned}$$

Thus, we have a surjective homomorphism $A'_n \twoheadrightarrow A_n$. Since A_n is free and A'_n is generated by the same elements as A_n by Proposition 2.12, we conclude thanks to this surjective morphism that A'_n is also free of the same rank. In particular we have constructed an isomorphism. $\square$

We define recursively $\phi_i \in A'_n$ as follows. We let $\phi_1 = \omega_1$ and

$$\phi_{i+1} = T_i \phi_i T_i x_{i+1} - x_i T_i \phi_i T_i = x_{i+1} T_i \phi_i T_i - T_i \phi_i T_i x_i,$$

for all $1 \leq i \leq n$. A direct consequence of the preceding proposition is the following.

Proposition 2.14. *The isomorphism $A'_n \rightarrow A_n$ in Proposition 2.13 sends ϕ_i to ω_i .*

Note this gives the explicit inverse of the surjection $A'_n \twoheadrightarrow A_n$. As a matter of facts, it is a lengthy, but straightforward computation to check that we have

$$\phi_i \phi_j = -\phi_j \phi_i, \quad \phi_i x_j = x_j \phi_i,$$

and

$$T_j \phi_i = \phi_i T_j + \delta_{ij}(T_i \phi_{i+1} x_{i+1} - x_{i+1} \phi_{i+1} T_i),$$

in A'_n for all $1 \leq i, j \leq n$. This gives an alternative proof of Proposition 2.13, and thus of Proposition 2.12.

2.3. The subsuperring of S_n -invariants and the supercenter of A_n . Let $R^{S_n} \subset R$ be the subsuperring of S_n -invariants. Using $\partial_i^2 = 0$, we can view (R, ∂_i) as being a chain complex with the homological grading being one-half of the q -grading. As explained in [7, §2.1.1] for the case of the so-called odd polynomial rings, the chain complex (R, ∂_i) is contractible for each i and therefore $\ker(\partial_i) = \text{im}(\partial_i)$. Transporting the arguments therein to our case, it is easy to see that

$$R^{S_n} = \bigcap_{i=1}^n \ker(\partial_i) = \bigcap_{i=1}^n \text{im}(\partial_i).$$

This superring was defined the same way in [2, §3.1.1]. It is straightforward to check that the supercenter of A_n coincides with R^{S_n} .

Proposition 2.15. *There is an isomorphism of graded superrings*

$$Z_s(A_n) \cong R^{S_n}.$$

Remark 2.16. The supercenter of A_n is bigger than the center of A_n , for example, ω_n is in $Z_s(A_n)$ but it is not central since it does not commute with for example ω_{n-1} . More generally, the center of A_n is the subsuperring of R^{S_n} consisting of all the elements of even parity.

We are now going to describe R^{S_n} . It is clear that it contains the ring of symmetric polynomials $\mathbb{Z}[\underline{x}_n]^{S_n}$ as a subsuperring concentrated in even parity and λ -degree zero.

Definition 2.17. A *superpartition of type $(n|k)$* is a pair (α, β) , where $\alpha = \{(\alpha_1, \dots, \alpha_n) \in \mathbb{N}_0^n : \alpha_1 \geq \alpha_2 \geq \dots \geq \alpha_n \geq 0\}$ is a partition and $\beta = \{(\beta_1, \dots, \beta_k) \in \mathbb{N}_0^k : 0 \leq \beta_1 < \beta_2 < \dots < \beta_k \leq n, k \leq n\}$ is a (bounded) strict partition¹. We say that a strict partition like β above has k *parts* if $\beta_1 > 0$ and denote by $\mathcal{P}_n$ and respectively by $\mathcal{P}_k^{str}$ the sets of partitions with n parts and the set of strict partitions with k parts.

Remark 2.18. Superpartitions are known in the literature, and can be used for example to study Jack superpolynomials [6, §2.2]. The only difference to our setting is that we restrict the size of the strict partition and require $k \leq n$.

Let $\vartheta_0 \in S_n$ be the longest element.

¹Note that α and β are ordered oppositely.

Definition 2.19. For a superpartition (α, β) we define the *Schur superpolynomials*² on R as

$$\mathcal{S}_{\alpha, \beta}(\underline{x}_n, \underline{\omega}_n) = \partial_{\vartheta_0}(\underline{x}_n^{\delta+\alpha} \omega_\beta).$$

where $\underline{x}_n^{\delta+\alpha} = x_1^{n-1+\alpha_1} x_2^{n-2+\alpha_2} \cdots x_n^{\alpha_n}$ and $\omega_\beta = \omega_{\beta_1} \cdots \omega_{\beta_k}$.

For $\beta = 0$ we recover the usual Schur polynomials on $\mathbb{Z}[\underline{x}_n]$. If we denote by $>$ the lexicographic order on strict partitions, then the polynomials $\mathcal{S}_{\alpha, \beta}(\underline{x}_n, \underline{\omega}_n)$ can be written as

$$\mathcal{S}_{\alpha, \beta}(\underline{x}_n, \underline{\omega}_n) = \mathcal{S}_{\alpha, 0}(\underline{x}_n, \underline{\omega}_n) \omega_\beta + \sum_{\mu > \beta} c_\mu(\underline{x}_n) \omega_\mu,$$

where the coefficients $c_\mu(\underline{x}_n)$ lie in $\mathbb{Z}[\underline{x}_n]$. This follows immediately from the definition of $\mathcal{S}_{\alpha, \beta}(\underline{x}_n, \underline{\omega}_n)$ and the Leibniz rule for the Demazures.

From now on we write $\mathcal{S}_{\alpha, \beta}$ instead of $\mathcal{S}_{\alpha, \beta}(\underline{x}_n, \underline{\omega}_n)$ for the sake of simplicity, if no confusion can arise.

It is striking to look for multiplication rules on Schur polynomials on R that generalize the well-known multiplication rules for ordinary Schur polynomials. For the particular case of α being the zero partition we can give a precise formula (see Proposition 2.22 below).

Let us first look at the case of $\beta = (i)$.

Lemma 2.20. We have $\mathcal{S}_{0, i} = \sum_{\ell \geq i} (-1)^{\ell+i} h_{\ell-i}(\ell, n) \omega_\ell$, where $h_j(\ell, n)$ is the j -th complete homogeneous symmetric polynomial in the variables $(x_\ell, \dots, x_n)$.

Proof. Set $\mathcal{S}_{0, i} = \partial_{\vartheta_0}(\underline{x}_n^\delta \omega_i) = \sum_{\ell \geq i} c_\ell \omega_\ell$, with $c_\ell \in \mathbb{Z}[\underline{x}_n]$. Since $\sum_{\ell \geq i} c_\ell \omega_\ell \in R^{S_n}$, we have for $i \leq j \leq n-1$

$$0 = \partial_j \left(\sum_{\ell \geq i} c_\ell \omega_\ell \right) = -(s_j c_j) \omega_{j+1} + \sum_{\ell \geq i} (\partial_j c_\ell) \omega_\ell.$$

This implies $c_j = \partial_j c_{j+1}$, since $s_j \partial_j f = \partial_j f$ for every $f \in R$. Hence,

$$(10) \quad c_j = \partial_j \partial_{j+1} \cdots \partial_{n-1} c_n.$$

Now, using the following presentation for ∂_{ϑ_0} ,

$$\partial_{\vartheta_0} = \partial_1(\partial_2 \partial_1)(\partial_3 \partial_2 \partial_1) \cdots (\partial_{n-1} \cdots \partial_2 \partial_1),$$

one can see that

$$(11) \quad \mathcal{S}_{0, i} = \sum_{r=i}^{n-1} c_r \omega_r + (-1)^{n-i} \left(\partial_1(\partial_2 \partial_1) \cdots (\partial_{n-2} \cdots \partial_2 \partial_1) s_{n-1} s_{n-2} \cdots s_i \partial_{i-1} \cdots \partial_2 \partial_1(\underline{x}_n^\delta) \right) \omega_n.$$

Using repeatedly

$$\partial_j \partial_{j-1} \cdots \partial_1 (x_1^{m-1} x_2^{m-2} \cdots x_j^{m-j} x_{j+1}^{m-j-1}) = x_1^{m-2} x_2^{m-3} \cdots x_j^{m-j-1} x_{j+1}^{m-j-2}$$

on (11) gives

$$c_n = (-1)^{n-i} x_n^{n-i}.$$

²Similar terminology can be found in the literature, like *supersymmetric polynomials* or *Schur superfunctions* but they are not related to ours. As far as we can tell, our symmetric polynomials on R are new.

Finally, using the formula for c_j in (10) we get

$$c_j = (-1)^{j+i} \hbar_{j-i}(j, n),$$

as claimed. $\square$

For strict partitions β, β' define the product $\beta\beta'$ as the unique strict partition that can be formed from the set $\{\beta\} \cup \{\beta'\}$. Set $\epsilon_{\beta, \beta'}$ as the length of the minimal permutation taking the ordered set $\{\beta\} \cup \{\beta'\}$ to $\beta\beta'$.

Example 2.21. For $\beta = (1, 3, 4)$ and $\beta' = (2, 6)$ we have

$$\{\beta\} \cup \{\beta'\} = \{1, 3, 4, 2, 6\}, \quad \beta\beta' = (1, 2, 3, 4, 6),$$

and $\epsilon_{(1,3,4),(2,6)} = 2$.

Proposition 2.22. The superpolynomials $\mathcal{S}_{0,\beta}$ satisfy the following multiplication rule:

$$\mathcal{S}_{0,\beta}\mathcal{S}_{0,\beta'} = (-1)^{\epsilon_{\beta,\beta'}} \mathcal{S}_{0,\beta\beta'}.$$

Proof. We consider first the case of $\beta = (i), \beta' = (j)$ with $i \neq j$. First note that $\partial_{\vartheta_0}(\underline{x}_n^\delta \hbar_{k-j}(k, n))$ is zero unless $k = j$, and that $\partial_{\vartheta_0}(\omega_i \omega_j) = 0$. Hence,

$$\begin{aligned} \mathcal{S}_{0,i}\mathcal{S}_{0,j} &= \partial_{\vartheta_0}(\underline{x}_n^\delta \omega_i \mathcal{S}_{0,j}) \\ &= \sum_{k \geq j} (-1)^{n-j} \partial_{\vartheta_0}(\underline{x}_n^\delta \hbar_{k-j}(k, n) \omega_i \omega_j) \\ &= \partial_{\vartheta_0}(\underline{x}_n^\delta \omega_i \omega_j) + \sum_{k > j} (-1)^{n-j} \underline{x}_n^\delta \hbar_{k-j}(k, n) \partial_{\vartheta_0}(\omega_i \omega_j) \\ &= \partial_{\vartheta_0}(\underline{x}_n^\delta \omega_i \omega_j) \\ &= (-1)^{\epsilon_{i,j}} \mathcal{S}_{0,ij}. \end{aligned}$$

The same reasoning proves that for general $\beta \in \mathcal{P}_k^{str}$, one has $\mathcal{S}_{0,\beta}\mathcal{S}_{0,\ell} = (-1)^{\epsilon_{\beta,\ell}} \mathcal{S}_{0,\beta\ell}$. The claim now follows by induction on the length of β' , since $\mathcal{S}_{0,\beta}\mathcal{S}_{0,ij} = \mathcal{S}_{0,\beta}\mathcal{S}_{0,i}\mathcal{S}_{0,j}$ for $i < j$. $\square$

Proposition 2.23. The $\mathbb{Z}$ -module $(R^{S_n})_{(\bullet, 2k)} \subset R^{S_n}$ consisting of all elements with λ -degree equal to $2k$ has a $\mathbb{Z}[\underline{x}_n]^{S_n}$ -basis given by

$$\{\mathcal{S}_{0,\nu} : \nu \in \mathcal{P}_k^{str}\}.$$

Proof. Denote by $<$ the lexicographic order on $\mathcal{P}_k^{str}$ and let $z_\nu = b_\nu \omega_\nu + \sum_{\mu > \nu} b_\mu \omega_\mu \in R^{S_n}$, with $b_\rho \in \mathbb{Z}[\underline{x}_n]$, be homogeneous of λ -degree $2k$. It is not hard to see that $b_\nu \in \mathbb{Z}[\underline{x}]^{S_n}$. Moreover, we have

$$(12) \quad z_\nu - b_\nu \mathcal{S}_{0,\nu} = \sum_{\mu > \nu} b'_\mu \omega_\mu \in R^{S_n}$$

for some $b'_\mu \in \mathbb{Z}[\underline{x}_n]$. The claim follows by recursive application this argument to the right-hand-side of (12). $\square$

Corollary 2.24. *There is an isomorphism of bigraded superrings*

$$R^{S_n} \cong \mathbb{Z}[\underline{x}_n]^{S_n} \otimes \bigwedge^\bullet(\mathfrak{S}_{0,1}, \dots, \mathfrak{S}_{0,n}).$$

Proof. This follows directly from Proposition 2.22 and Proposition 2.24. $\square$

Corollary 2.24 together with the well-known fact that elementary symmetric polynomials generate $\mathbb{Z}[\underline{x}_n]^{S_n}$ shows that

$$\mathrm{grk}_{\mathbb{Z}}(R^{S_n}) = \prod_{j=1}^n \frac{1 + \pi \lambda^2 q^{-2j}}{1 - q^{2j}} = q^{-n(n-1)/2} \frac{1}{[n]!} \prod_{j=1}^n \frac{1 + \pi \lambda^2 q^{-2j}}{1 - q^2},$$

with the same notations as in 2.6.

Proposition 2.25. *The Schur superpolynomials form a homogeneous $\mathbb{Z}$ -linear basis of R^{S_n} .*

Proof. We first note that the Schur superpolynomials of R are linearly independent over $\mathbb{Z}$, since

$$(13) \quad \mathfrak{S}_{\alpha,\beta}(\underline{x}_n, \underline{\omega}_n) = \mathfrak{S}_{\alpha}(\underline{x}_n) \omega_{\beta} + \text{higher terms},$$

and the classical fact that the $\mathfrak{S}_{\alpha}(\underline{x}_n)$'s are linearly independent over $\mathbb{Z}$. Using (13) we compute

$$\mathrm{grk}_{\mathbb{Z}}(\mathrm{span}_{\mathbb{Z}}\{\mathfrak{S}_{\alpha,\beta}(\underline{x}_n, \underline{\omega}_n) : \alpha \in \mathcal{P}_n, \beta \in \mathcal{P}_k^{str} = k\}) = \pi^k \lambda^{2k} q^{-n-1} [n-1] \mathrm{grk}_{\mathbb{Z}}(\mathbb{Z}[\underline{x}_n]^{S_n}),$$

which matches the graded rank of $(R^{S_n})_{2k}$ over $\mathbb{Z}$, the latter computed from Proposition 2.24. This shows the Schur superpolynomials span R^{S_n} over $\mathbb{Z}$. $\square$

2.4. Labeled ω_k 's. Define recursively for $a \in \mathbb{N}_0$

$$\omega_k^a = \omega_{k-1}^{a-1} - x_k \omega_k^{a-1},$$

with $\omega_k^0 = \omega_k$ and $\omega_0 = 0$. The degrees of these elements are

$$\deg(\omega_k^a) = (2(a-k), 2).$$

Example 2.26. *We have*

$$\begin{aligned} \omega_4^2 &= \omega_2 - (x_3 + x_4) \omega_3 + x_4^2 \omega_4, \\ \omega_2^3 &= (x_1^2 + x_1 x_2 + x_2^2) \omega_1 - x_2^3 \omega_2, \end{aligned}$$

with degrees $(-4, 2)$ and $(2, 2)$, respectively.

Lemma 2.27. *The following holds in A_n :*

$$\omega_k^a = \sum_{\ell=1}^k (-1)^{a+k+\ell} \hbar_{a+\ell-k}(\ell, k) \omega_{\ell}.$$

Proof. Clearly, the equation holds for $k = 1$, where $\hbar_a(1, 1) = x_1^a$. For $k > 1$ we have that

$$\omega_k^a = (-1)^a x_k^a \omega_k + \sum_{b=0}^{a-1} (-1)^b x_k^b \omega_{k-1}^{a-1-b}.$$

Thus, by induction,

$$\begin{aligned}\omega_k^a &= (-1)^a x_k^a \omega_k + \sum_{b=0}^{a-1} (-1)^{a+k} x_k^b \sum_{\ell=1}^{k-1} (-1)^\ell \hbar_{a-b-k+\ell}(\ell, k-1) \omega_\ell \\ &= (-1)^a x_k^a \omega_k + \sum_{\ell=1}^{k-1} (-1)^{a+k+\ell} \hbar_{a-k+\ell}(\ell, k) \omega_\ell,\end{aligned}$$

which finishes the proof. $\square$

Lemma 2.27 together with Lemma 2.20 implies that $\omega_n^i = \mathcal{S}_{0,n-i} \in R^{S_n}$ for all $0 \leq i < n$. Moreover, the labeled ω_k^a 's interact with the T_i 's in the same way as the unlabeled ω_k 's, which formally translates to the following:

Proposition 2.28. *In A_n we have*

$$\begin{aligned}T_i \omega_k^a &= \omega_k^a T_i, \quad i \neq k, \\ T_i(\omega_i^a - x_{i+1} \omega_{i+1}^a) &= (\omega_i^a - x_{i+1} \omega_{i+1}^a) T_i,\end{aligned}$$

for all $a \in \mathbb{N}_0$ and all $k, i \in \{1, \dots, n\}$.

Proof. The proof follows by induction on a , together with the fact that a symmetric polynomial in x_i, x_{i+1} commutes with T_i , and the relation $T_i(\omega_i - x_{i+1} \omega_{i+1}) = (\omega_i - x_{i+1} \omega_{i+1}) T_i$. $\square$

From the above we can rephrase Proposition 2.24 in terms of the labeled ω_k 's.

Corollary 2.29. *There is an isomorphism of $\mathbb{Z}$ -algebras $R^{S_n} \cong \mathbb{Z}[\underline{x}_n]^{S_n} \otimes \bigwedge^\bullet(\omega_n^0, \omega_n^1, \dots, \omega_n^{n-1})$.*

It can also be useful to invert the formula in Lemma 2.20 and write the ω_k 's as a $\mathbb{Z}[\underline{x}_n]$ -linear combination of the ω_n^a 's (or equivalently, of the $\mathcal{S}_{0,i}$'s).

Proposition 2.30. *We have*

$$\omega_k = \sum_{\ell=0}^{n-k} e_\ell(k+1, n) \omega_n^{n-k-\ell},$$

where $e_\ell(k+1, n)$ is the ℓ -th elementary symmetric polynomial in the variables $(x_{k+1}, \dots, x_n)$.

Proof. One can prove

$$\omega_k^a = \sum_{\ell=0}^{n-k} e_\ell(k+1, n) \omega_n^{n-k-\ell+a}$$

by backwards induction on k , beginning with $k = n$ and descending to zero using the equality

$$\omega_k^a = \omega_{k+1}^{a+1} + x_{k+1} \omega_{k+1}^a.$$

We leave the details to the reader. $\square$

2.5. The R^{S_n} -supermodule structure of R . We introduce some notation and recall some results from the polynomial case. Recall that the Schubert polynomial $\mathfrak{s}_\vartheta$ for $\vartheta \in S_n$ is defined by

$$\mathfrak{s}_\vartheta(\underline{x}_n) = \partial_{\vartheta^{-1}\vartheta_0}(x_1^{n-1}x_2^{n-2}\cdots x_{n-1}),$$

with $\vartheta_0 \in S_n$ denoting again the longest element. It is well-known that both $\{\mathfrak{s}_\vartheta(\underline{x}_n)\}_{\vartheta \in S_n}$ and $\{x_1^{a_1}\cdots x_n^{a_n}\}_{0 \leq a_i \leq n-i}$ give integral basis of $\mathbb{Z}[\underline{x}_n]$ as a left (and as a right) $\mathbb{Z}[\underline{x}_n]^{S_n}$ -module. Let

$$U_n = \text{span}_{\mathbb{Z}}\{\mathfrak{s}_\vartheta(\underline{x}_n) : \vartheta \in S_n\} = \text{span}_{\mathbb{Z}}\{x_1^{a_1}\cdots x_n^{a_n} : 0 \leq a_i \leq n-i\}.$$

Proposition 2.31. *R is a free left and right R^{S_n} -supermodule of graded rank $q^{\frac{n(n-1)}{2}}[n]!$ with homogeneous basis given by the Schubert polynomials $\{\mathfrak{s}_\vartheta(\underline{x}_n)\}_{\vartheta \in S_n}$. The set $\{x_1^{a_1}\cdots x_n^{a_n}\}_{0 \leq a_i \leq n-i}$ is also a basis.*

Proof. We show that the multiplication

$$R^{S_n} \otimes U_n \rightarrow R,$$

is an isomorphism of $\mathbb{Z}$ -modules. In order to do so we need to show that any $f \in R$ can be expressed in the form $f = \sum f_i u_i$, with $f_i \in R^{S_n}$ and $u_i \in U_n$. Having in mind that the result holds in the polynomial case, we only need to prove the case when f has non-zero λ -degree. It is enough to show that any $p(\underline{x}_n)\omega_\mu = p(\underline{x}_n)\omega_{\mu_1}\omega_{\mu_2}\cdots\omega_{\mu_k}$ can be written as a $\mathbb{Z}[\underline{x}_n]$ -linear combination of the $\mathcal{S}_{0,\beta}$'s, since then we can apply the result of the polynomial case and write every such coefficient as a $\mathbb{Z}[\underline{x}]^{S_n}$ -linear combination of Schubert polynomials. To prove this claim we note that, since $\mathcal{S}_{0,\mu} = \omega_\mu + \sum_{\nu > \mu} c_\nu \omega_\nu$ with $c_\mu \in \mathbb{Z}[\underline{x}]$ (cf. (13)), we have

$$p(\underline{x}_n)\omega_\mu - p(\underline{x}_n)\mathcal{S}_{0,\mu} = \sum_{\nu > \mu} c'_\nu \omega_\nu,$$

with $c'_\mu \in \mathbb{Z}[\underline{x}]$. The claim follows by recursive application of this procedure to the $c_\nu \omega_\nu$'s. This shows the multiplication map above is surjective. Since both sides are free $\mathbb{Z}$ -modules, equality between their graded ranks proves it is injective. $\square$

Let $\text{Mat}_{q^{n(n-1)/2}[n]!}(R^{S_n})$ be the algebra of matrices of size $q^{n(n-1)/2}[n]!$ with coefficients in R^{S_n} . Note that we have $\text{grk}_{\mathbb{Z}}(\text{Mat}_{q^{n(n-1)/2}[n]!}(R^{S_n})) = ([n]!)^2 \text{grk}_{\mathbb{Z}}(R^{S_n}) = \text{grk}_{\mathbb{Z}}(A_n)$.

Corollary 2.32. *The action of A_n on R induces an isomorphism*

$$\psi: A_n \xrightarrow{\cong} \text{End}_{R^{S_n}}(R) \cong \text{Mat}_{q^{n(n-1)/2}[n]!}(R^{S_n})$$

of bigraded superalgebras.

Proof. Since A_n acts on R by linearly independent operators and R is free over R^{S_n} (see Proposition 2.31), the map ψ is injective. The surjectivity follows directly from the decomposition $A_n \cong R \otimes \text{NC}_n$ and the definition of the Schubert polynomial (recalling that they are defined by the action of the Demazure operators). $\square$

2.5.1. Idempotents in A_n . Every homogeneous idempotent in A_n is the image of an idempotent in NH_n under the inclusion map introduced right after Proposition 2.4, which is immediate by degree reasons. Denote by x the image of $x \in \text{NH}_n$ in A_n . Let $\underline{x}_n^\delta = x_1^{n-1} x_2^{n-1} \cdots x_{n-1}$ and form the idempotent $e_n = T_{\vartheta_0} \underline{x}_n^\delta$ in the nilHecke algebra, where $\vartheta_0 \in S_n$ is the longest element. This element is an idempotent because

$$T_{\vartheta_0} \underline{x}_n^\delta T_{\vartheta_0} \underline{x}_n^\delta = T_{\vartheta_0} (\underline{x}_n^\delta) T_{\vartheta_0} \underline{x}_n^\delta + \vartheta_0(\underline{x}_n^\delta) T_{\vartheta_0}^2 \underline{x}_n^\delta = T_{\vartheta_0} \underline{x}_n^\delta,$$

since $T_{\vartheta_0}(\underline{x}_n^\delta) = 1$ and $T_{\vartheta_0}^2 = 0$.

Then $A_n e_n$ is a left A_n -supermodule isomorphic to R , which is, up to isomorphism and grading shifts, the unique left, bigraded projective indecomposable supermodule (it is not hard to check that the map sending $1 \in R$ to $e_n \in A_n e_n$ is an isomorphism of left A_n -supermodules). We denote by $P_{(n)}$ the left supermodule $A_n e_n$ with the q -degree shifted down by $n(n-1)/2$. As a consequence of Corollary 2.32 we have that as a supermodule over itself, A_n decomposes into $[n]!$ copies of R , giving a direct sum decomposition of bigraded, left, A_n -supermodules.

Proposition 2.33. *As bigraded left-supermodule over itself, A_n decomposes as*

$$A_n \cong \bigoplus_{[n]!} P_{(n)}.$$

Here, for a Laurent polynomial $f = \sum f_i q^i \in \mathbb{N}[q^{\pm 1}]$ and a bigraded supermodule M , $\bigoplus_f M$ denotes the direct sum $\bigoplus_i q^i M^{\oplus f_i}$, and where q^r is a shift up by r units in the q -degree.

There is an involutive anti-automorphism τ of A_n fixing the generators of A_n , which is homogeneous of degree 0. We write u^τ instead of $\tau(u)$ for $u \in A_n$.

Letting ${}_{(n)}P$ be the right, bigraded projective indecomposable supermodule ${}_{(n)}P = e_n^\tau A_n$ with the q -degree shifted down by $n(n-1)/2n$ we get a similar decomposition of bigraded right A_n -supermodules

$$A_n \cong \bigoplus_{[n]!} {}_{(n)}P.$$

The following corollary to Proposition 2.31 is immediate.

Corollary 2.34. *We have $A_n e_n A_n = A_n$.*

2.6. Affine cellular structure. Affine cellular algebras have been introduced in [17] as a generalization of cellular algebras to include some natural infinite-dimensional algebras, as for example algebras over polynomial rings. In particular, the nilHecke algebra is affine cellular, but not cellular [16, §4]. Graded affine cellularity is a slight extension of affine cellularity to the graded world and was introduced in [16]. Our aim is to (mildly) generalize these notions further. We extend this notion to the case of a superalgebra and introduce the notion of bigraded affine cellular superalgebra. We prove below that the case of A_n is bigraded affine cellular by checking that the arguments given in [16, §4] for the case of the nilHecke algebra extend directly to our case.

(Affine) cellular algebras come equipped with a so-called cellular structure. This additional datum has important consequences for the representation theory of such algebras, e.g. a theory of standard modules. We hope that bigraded cellularity can be useful in the study of infinite-dimensional superalgebras. As in the case of NH_n whose graded affine cellularity is established

without too much effort, and is then used to prove graded affine cellularity of KLR algebras of finite types, we hope that graded affine cellularity of A_n hints at a similar feature of the extended KLR algebras of [24]. Our notion of cellularity works in the finite-dimensional case as well, similar as affine cellularity generalizes cellularity.

We give below the main definitions involved, adapted from [16, §4] to the bigraded case. We say that an *affine superalgebra* is a quotient of a supercommutative algebra (i.e. a polynomial algebra tensored with an exterior algebra).

Definition 2.35. Let C be a graded, unital, $\mathbb{k}$ -superalgebra equipped with a $\mathbb{k}$ -anti-involution τ , where $\mathbb{k}$ is a noetherian domain. A two-sided ideal J in C is called an *affine cell ideal* if the following conditions are satisfied:

- (1) $\tau(J) = J$,
- (2) there exists an affine $\mathbb{k}$ -superalgebra B with a parity preserving $\mathbb{k}$ -involution σ and a free $\mathbb{k}$ -supermodule V of finite rank such that $\Delta := V \otimes_{\mathbb{k}} B$ has a (C, B) -superbimodule structure, with the right B -supermodule structure induced by the regular right B -supermodule structure on B ,
- (3) let $\Delta' := B \otimes_{\mathbb{k}} V$ be the (B, C) -superbimodule with left B -supermodule structure induced by the regular left B -supermodule structure on B and right C -supermodule structure defined by

$$(b \otimes v)c = (s)(\tau(c)(v \otimes b)),$$

where $s : V \otimes_{\mathbb{k}} B \rightarrow B \otimes_{\mathbb{k}} V$, $v \otimes b \rightarrow b \otimes v$; then there is a (C, C) -superbimodule isomorphism $\alpha : J \rightarrow \Delta \otimes_B \Delta'$, such that the following diagram commutes:

$$\begin{array}{ccc} J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta' \\ \downarrow \tau & & \downarrow v \otimes b \otimes b' \otimes w \mapsto w \otimes \sigma(b') \otimes \sigma(b) \otimes v \\ J & \xrightarrow{\alpha} & \Delta \otimes_B \Delta'. \end{array}$$

Definition 2.36. The superalgebra C is called *bigraded affine cellular* if there is a $\mathbb{k}$ -supermodule decomposition

$$C = J'_1 \oplus J'_2 \oplus \cdots \oplus J'_n$$

with $\tau(J'_\ell) = J'_\ell$ for $1 \leq \ell \leq n$, such that, setting $J_m := \bigoplus_{\ell=1}^m J'_\ell$, we obtain an ideal filtration

$$0 = J_0 \subset J_1 \subset J_2 \subset \cdots \subset J_n = C$$

so that each J_m/J_{m-1} is an affine cell ideal of C/J_{m-1} .

The following establishes that the superalgebra A_n is bigraded affine cellular. Let $\{z_\gamma\}_{\gamma \in \Gamma}$ be any $\mathbb{Z}$ -basis of R^{S_n} . Recall $\underline{x}_n^\delta$ and τ from §2.5.1.

Proposition 2.37. *The superalgebra A_n has an affine cellular basis given by one cell J generated by the set*

$$\{T_u z_\gamma \underline{x}_n^\delta e_n T_v^\tau \mid u, v \in S_n, \gamma \in \Gamma\}.$$

Proof. This is the same proof as [16, Theorem 4.8]. Let $J = \text{span}_{\mathbb{Z}}\{T_u z_{\gamma} x_n^{\delta} e_n T_v^{\tau} \mid u, v \in S_n, \gamma \in \Gamma\}$. By comparing (bi)graded dimensions one checks that $J \cong A_n$. The results in §2.3 and §2.5 (in particular, Corollaries 2.32 and 2.34) above imply that J is in fact, a bigraded affine cell ideal. $\square$

2.7. Diagrammatic presentation of the superalgebra A_n . Next, we aim to give the superalgebra A_n a diagrammatic presentation very much in the spirit of [15].

Remark 2.38. A graphical calculus for the superalgebra A_n was already introduced by the authors in [23, §9.1]. However, the one we give here is slightly different: white dots from [23, §9.1] are now placed in the regions of the diagrams rather than on the strands, and can be labeled by non-negative integers. In this presentation we call them *floating dots*.

We use the usual diagrams for the generators of NH_n :

$$\begin{array}{c} \begin{array}{c} | \\ 1 \end{array} \quad \begin{array}{c} | \\ 2 \end{array} \quad \cdots \quad \begin{array}{c} | \\ n \end{array} = 1 \in A_n, \quad \begin{array}{c} | \\ 1 \end{array} \quad \cdots \quad \begin{array}{c} | \\ i-1 \end{array} \quad \begin{array}{c} | \\ \bullet \\ i \end{array} \quad \begin{array}{c} | \\ i+1 \end{array} \quad \cdots \quad \begin{array}{c} | \\ n \end{array} = x_i \in A_n, \\ \\ \begin{array}{c} | \\ 1 \end{array} \quad \cdots \quad \begin{array}{c} | \\ i-1 \end{array} \quad \begin{array}{c} \diagup \\ \diagdown \end{array} \quad \begin{array}{c} | \\ i+1 \end{array} \quad \cdots \quad \begin{array}{c} | \\ n \end{array} = T_i \in A_n. \end{array}$$

and we picture the ω_i 's as *floating dots*:

$$\begin{array}{c} | \\ 1 \end{array} \quad \cdots \quad \begin{array}{c} | \\ \circ^a \\ i \end{array} \quad \begin{array}{c} | \\ i+1 \end{array} \quad \cdots \quad \begin{array}{c} | \\ n \end{array} = \omega_i^a \in A_n.$$

Here $a \in \mathbb{N}_0$ is a non-negative integer and we write $\omega_i^0 = \omega_i$ as a floating dot without label, by convention.

Multiplication is given by stacking diagrams on top of each other where, in our conventions, ab means stacking the diagram for a atop the one for b (thus, we read diagrams from bottom to top).

Relations in A_n acquire a graphical interpretation, as shown in (14) to (19) below, together with the usual height move for distant crossings and dots. Relations (15)-(19) have to be understood as being local. By this we mean that they are embedded in some bigger diagrams where everything matches except a small disk which is represented in the pictures below. Relation (14) means floating dots anticommute, and in particular a diagram containing two floating dots with the same label in the same region is zero.

$$(14) \quad \begin{array}{c} \circ^b \\ \circ^a \end{array} \quad \cdots = - \quad \begin{array}{c} \circ^a \\ \circ^b \end{array} \quad \cdots$$

$$(15) \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = 0 \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagup \\ \diagdown \diagdown \end{array}$$

$$(16) \quad \begin{array}{c} \bullet \\ \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagup \diagdown \\ \diagdown \bullet \end{array} + \begin{array}{c} | \\ | \end{array}$$

$$(17) \quad \begin{array}{c} \diagup \diagdown \\ \bullet \diagup \end{array} = \begin{array}{c} \bullet \diagup \diagdown \\ \diagdown \end{array} + \begin{array}{c} | \\ | \end{array}$$

$$(18) \quad \begin{array}{c} \diagup \diagdown \\ \circ^a \end{array} - \begin{array}{c} \diagup \diagdown \\ \diagdown \bullet \end{array} \circ^a = \begin{array}{c} \circ^a \\ \diagup \diagdown \\ \diagdown \end{array} - \begin{array}{c} \diagup \diagdown \\ \bullet \diagup \end{array} \circ^a$$

$$(19) \quad \begin{array}{c} | \\ \circ^a \end{array} = \begin{array}{c} \circ^{a-1} \\ | \end{array} - \begin{array}{c} | \\ \bullet \end{array} \circ^{a-1}$$

By Propositions 2.4 and 2.28, relations (14) to (19) form a complete set of relations for A_n . As explained in §2.2 (and §2.4 for the labeled floating dots), the superalgebra A_n is bigraded with degrees given by

$$\deg(x_i) = (2, 0), \quad \deg(T_i) = (-2, 0), \quad \deg(\omega_i^a) = (2(a - i), 2).$$

The symmetry τ from §2.5.1 consists in reflecting a diagram around the horizontal axis. Note however that a reflection around a vertical axis is not homogeneous with respect to the q -grading: the q -degree of a floating dot depends on the number of strands at its left, and the reflection can change this number.

Remark 2.39. In fact, one can give a topological definition for those diagrams, where they are actually taken up to isotopy which does not create critical points and preserves the relative height of floating dots. This is possible because of the relations defining A_n , which allow the permutations of distant crossings, dots and floating dots together (except the commutation between floating dots).

Remark 2.40. In the definition of the ω_i^a 's from §2.4, we put $\omega_0^a = 0$. This translates into the diagrammatic framework to kill all diagrams with floating dots in the leftmost region:

$$\begin{array}{c} \circ^a \\ | \end{array} \begin{array}{c} | \\ 1 \end{array} \begin{array}{c} | \\ 2 \end{array} \cdots \begin{array}{c} | \\ n \end{array} = 0.$$

The equations presented in the beginning of §2.2.1 can be generalized for labeled ω_i 's and turned into diagrams to give the following consequence, which will be used in the sequel:

Lemma 2.41. *In A_n we have the local relation*

$$(20) \quad \left| \begin{array}{c} | \\ | \end{array} \right| \left| \begin{array}{c} \circ^a \end{array} \right| = \begin{array}{c} \text{diagram 1} \end{array} - \begin{array}{c} \text{diagram 2} \end{array}$$

Proof. First we compute

$$\begin{array}{c} \text{diagram 1} \end{array} \stackrel{(18)}{=} \begin{array}{c} \text{diagram 2} \end{array} + \begin{array}{c} \text{diagram 3} \end{array} - \begin{array}{c} \text{diagram 4} \end{array} \\ \stackrel{(16),(15)}{=} - \begin{array}{c} \text{diagram 5} \end{array}$$

Then we observe

$$\left| \begin{array}{c} | \\ | \end{array} \right| \left| \begin{array}{c} \circ^a \end{array} \right| \stackrel{(17)}{=} \begin{array}{c} \text{diagram 1} \end{array} - \begin{array}{c} \text{diagram 2} \end{array}$$

and we conclude the proof by combining these two equalities. $\square$

In view of the discussion in §2.2.1, we say a floating dot is *tight* if it is unlabeled and placed directly to the right of the leftmost strand. The element θ_a defined in (7) translates in the diagrammatic framework as a tightened floating dot, i.e.

$$\theta_a = \begin{array}{c} \text{diagram 1} \end{array} \left| \begin{array}{c} | \\ | \end{array} \right| \dots \left| \begin{array}{c} | \\ | \end{array} \right|$$

1 a-1 a a+1 n

With this in mind, the basis (8) has a nice diagrammatic description. Recall it is given by

$$\left\{ x_1^{k_1} \cdots x_n^{k_n} T_{\vartheta^{n+1}} \theta_{\min_{\vartheta}(s(n))}^{\ell_{\vartheta}(s(n))} T_{\vartheta^n} \cdots \theta_{\min_{\vartheta}(s(2))}^{\ell_{\vartheta}(s(2))} T_{\vartheta^2} \theta_{\min_{\vartheta}(s(1))}^{\ell_{\vartheta}(s(1))} T_{\vartheta^1} : k_i \in \mathbb{N}_0, \ell_i \in \{0, 1\}, \vartheta \in S_n \right\},$$

where $\vartheta = \vartheta^n \vartheta^{n-1} \cdots \vartheta^0$ admits a left-adjusted, reduced expression and is partitioned such that each $i \in \{1, \dots, n\}$ attains its minimal position $\min_{\vartheta}(i)$ in $\vartheta^{s^{-1}(i)} \cdots \vartheta^1(i)$.

To explain the diagrammatic presentation of the basis from (8), we first consider elements of S_n using the usual string diagrams. In such a diagram, we can index the strands at the bottom or at the top, counting from the left. We will refer to those indexing by saying a strand is the i -th strand at the bottom (resp. at the top) if it starts (resp. ends) at the i -th position, counting from the left. For example, in the following string diagram, the first strand at the bottom is the third at the top and the first at the top is the second at the bottom:



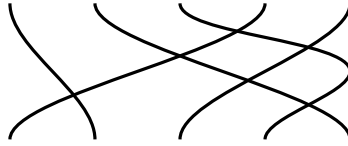
Next, choose a left-adjusted, reduced expression ϑ for each element of S_n . This can be thought as choosing any reduced expression and then pulling all strands as much as possible to the left

using Reidemeister 3 type moves only (the one on the right in (15)). Note that the bijection $s : \{1, \dots, n\} \rightarrow \{1, \dots, n\}$ defined in §2.2.1 tells us in which order (read from bottom to top) each strand attains its leftmost position in the diagram, i.e. the $s(i)$ -th strand at the bottom (or $\vartheta(s(i))$ -th strand at the top) is the i -th strand (from bottom to top) to attain its leftmost position.

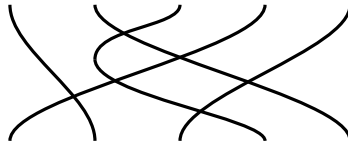
Remark 2.42. In particular, if we consider the left-adjusted reduced expressions constructed through the coset decomposition (6) of S_n , then $s(i)$ is given by looking at the position at the bottom of the i -th strand counting from the right at the top (i.e. $(n - i + 1)$ -th strand counting from the left at the top). Then we have $\vartheta(s(i)) = n - i + 1$.

Clearly, we can view such a string diagram as an element of A_n , using the above diagrammatic presentation. We can now put k_i dots at the top of the string diagram on the i -th strand at the top, which still is an element of A_n . Last, we put tightened floating dots for $\ell_i = 1$ into the string diagram by adding a $\circ$ symbol to the right of the i -th strand at the top, after pulling it to the far left, where it attained already its leftmost position. We do this in order, i.e., we insert these tightened floating dots from bottom to top starting with the smallest $i \in \{1, \dots, n\}$ with $\ell_{\vartheta(s(i))} = 1$. In the case of Remark 2.42, we would do this from ℓ_n to ℓ_1 . By construction, we end with a decorated string diagram describing an element of A_n . The reader should convince himself/herself that the resulting diagram is actually a basis element from the basis from (8).

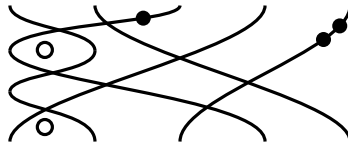
Example 2.43. Consider the following string diagram obtained by taking a reduced expression of the permutation $(1\ 4\ 3\ 5\ 2) \in S_5$:



It is not left-adjusted as we can pull to the left the fourth strand at the bottom by applying two Reidemeister 3 moves. Doing so we get the following left-adjusted reduced presentation



Now, if we take for example $k_1 = k_2 = k_3 = 0$, $k_3 = 1$, $k_5 = 2$, $\ell_3 = \ell_4 = 1$ and $\ell_1 = \ell_2 = \ell_5 = 0$, then we get



where we had to pull the third strand (at the top) to the left in order to put a tight floating dot, the fourth one being already to the left.

2.8. The superalgebra A_n and infinite Grassmannian varieties. In this section, we describe the relation between A_n and the geometry of the Grassmannian varieties used in [23]. For this section, we consider A_n as a superalgebra over $\mathbb{Q}$ rather than $\mathbb{Z}$, and we work with cohomology with rational coefficients.

Recall that the cohomology ring of the Grassmannian $G_{n;N}$ of n -planes in $\mathbb{C}^N$ is generated by the Chern classes $X_{1,n}, \dots, X_{n,n}$ and $Y_{1,n}, \dots, Y_{N-n,n}$, modded out by the Whitney sum formula. More explicitly, we have

$$H(G_{n;N}) \cong \mathbb{Q}[X_{1,n}, \dots, X_{n,n}, Y_{1,n}, \dots, Y_{N-n,n}] / I_{n;N},$$

with $I_{n;N}$ being the ideal generated by all homogeneous terms in the equation

$$(1 + tX_{1,n} + t^2X_{2,n} + \dots + t^nX_{n,n})(1 + tY_{1,n} + \dots + t^{N-n}Y_{N-n,n}) = 1.$$

The same applies for the two-step flag variety

$$G_{n,n+1;N} = \{V_n \subset V_{n+1} \subset \mathbb{C}^N \mid \dim(V_n) = n, \dim(V_{n+1}) = n+1\},$$

with the Chern classes $X_{1,n}, \dots, X_{n,n}, \xi_{n+1}$ and $Y_{1,n+1}, \dots, Y_{N-n-1,n+1}$, resulting in

$$H(G_{n,n+1;N}) \cong \mathbb{Q}[X_{1,n}, \dots, X_{n,n}, \xi_{n+1}, Y_{1,n+1}, \dots, Y_{N-n-1,n+1}] / I_{n,n+1;N},$$

where $I_{n,n+1;N}$ is given by

$$(1 + tX_{1,n} + t^2X_{2,n} + \dots + t^nX_{n,n})(1 + t\xi_{n+1})(1 + tY_{1,n+1} + \dots + t^{N-n-1}Y_{N-n-1,n+1}) = 1.$$

The forgetful maps $G_{n,n+1;N} \rightarrow G_{n;N}$ and $G_{n,n+1;N} \rightarrow G_{n+1;N}$ induce an $(H(G_{n;N}), H(G_{n+1;N}))$ -bimodule structure on $H(G_{n,n+1;N})$ explicitly given by

$$X_{i,n+1} \mapsto X_{i,n} + \xi_{n+1}X_{i-1,n}, \quad Y_{i,n} \mapsto Y_{i,n+1} + \xi_{n+1}X_{i-1,n+1}.$$

A nice exposition on the cohomology rings of the finite Grassmannians and iterated flag varieties for the use of categorification is given in [18, §6]. It is explained there how these rings can be used to construct a categorification of the finite-dimensional representations of quantum $\mathfrak{sl}_2$, using the aforementioned bimodule structure.

The cohomology ring of the infinite Grassmannian $G_{n;\infty}$ of n -planes in $\mathbb{C}^\infty$ is basically given by the limit

$$\lim_{N \rightarrow \infty} H(G_{n;N}).$$

Explicitly, we have

$$H(G_{n;\infty}) \cong \mathbb{Q}[X_{1,n}, \dots, X_{n,n}, Y_{1,n}, \dots] / I_{n;\infty},$$

with infinitely many $Y_{i,n}$, and where $I_{n;\infty}$ is defined by the homogeneous terms in

$$(1 + tX_{1,n} + t^2X_{2,n} + \dots + t^nX_{n,n})(1 + tY_{1,n} + \dots) = 1.$$

Therefore, $H(G_{n;\infty}) \cong \mathbb{Q}[X_{1,n}, \dots, X_{n,n}]$. The same applies for the two-step flag variety

$$G_{n,n+1;\infty} = \{V_n \subset V_{n+1} \subset \mathbb{C}^\infty \mid \dim(V_n) = n, \dim(V_{n+1}) = n+1\},$$

resulting in

$$H(G_{n,n+1;\infty}) \cong \mathbb{Q}[X_{1,n}, \dots, X_{n,n}, \xi_{n+1}, Y_{1,n+1}, \dots] / I_{n,n+1;\infty},$$

where $I_{n,n+1;\infty}$ is given by

$$(1 + tX_{1;n} + t^2X_{2;n} + \cdots + t^nX_{n;n})(1 + t\xi_{n+1})(1 + tY_{1,n+1} + \cdots) = 1.$$

Hence, $H(G_{n,n+1;\infty}) \cong \mathbb{Q}[X_{1,n}, \dots, X_{n,n}, \xi_{n+1}]$.

Again, there are forgetful maps $G_{n,n+1;\infty} \rightarrow G_{n;\infty}$ and $G_{n,n+1;\infty} \rightarrow G_{n+1;\infty}$, which induce an $(H(G_{n;\infty}), H(G_{n+1;\infty}))$ -bimodule structure on $H(G_{n,n+1;\infty})$. It is explicitly described by

$$(21) \quad X_{i,n+1} \mapsto X_{i,n} + \xi_{n+1}X_{i-1,n}, \quad Y_{i,n} \mapsto Y_{i,n+1} + \xi_{n+1}X_{i-1,n+1}.$$

All of this is explained in details in [23, §3].

Then following [23, §3], we define

$$\begin{aligned} \Omega_n &= H(G_{n;\infty}) \otimes \bigwedge^\bullet(s_{1,n}, \dots, s_{n,n}), \\ \Omega_{n,n+1} &= H(G_{n,n+1;\infty}) \otimes \bigwedge^\bullet(s_{1,n+1}, \dots, s_{n+1,n+1}), \end{aligned}$$

where we think of $\bigwedge^\bullet(s_{1,n}, \dots, s_{n,n})$ as the Koszul dual of $\mathbb{Q}[X_{1,n}, \dots, X_{n,n}] \cong H(G_{n;\infty})$. By this we mean we can view $\mathbb{Q}[X_{1,n}, \dots, X_{n,n}]$ as the symmetric algebra of the free vector space generated by $\{X_{1,n}, \dots, X_{n,n}\}$, and then $\bigwedge^\bullet(s_{1,n}, \dots, s_{n,n})$ is the quadratic dual (hence, the Koszul dual) of this symmetric algebra (see [23, §3] for details).

We give $\Omega_{n,n+1}$ an (Ω_n, Ω_{n+1}) -bimodule structure by extending the above actions (21) via

$$s_{i,n} \mapsto s_{i,n+1} + \xi_{n+1}s_{i+1,n+1}.$$

Recall from Corollary 2.29 that $R^{S_n} \cong \mathbb{Q}[\underline{x}_n]^{S_n} \otimes \bigwedge^\bullet(\omega_n^0, \omega_n^1, \dots, \omega_n^{n-1})$. Also recall that $\mathbb{Q}[\underline{x}_n]^{S_n} \cong \mathbb{Q}[e_1(\underline{x}_n), \dots, e_n(\underline{x}_n)]$, so that $R^{S_n} \cong \Omega_n$. Hence Ω_n is Morita-equivalent to A_n by Corollary 2.32.

Proposition 2.44. *There is an isomorphism*

$$R^{S_n} \cong \Omega_n,$$

given by $e_i(\underline{x}_n) \mapsto X_{i,n}$, $\hbar_i(\underline{x}_n) \mapsto (-1)^i Y_{i,n}$ and $\omega_n^{n-i} \mapsto s_{i,n}$.

Proof. The only thing which requires a proof is $\hbar_i(\underline{x}_n) \mapsto (-1)^i Y_{i,n}$, which can be checked by verifying the collections of $e_i(\underline{x}_n)$'s and $(-1)^i \hbar_i(\underline{x}_n)$'s respect an equation of the same kind as the one that defines $I_{n,n+1;\infty}$. $\square$

Remark 2.45. It is also worthwhile to note that ω_k^a corresponds exactly to $s_{k-a,k}$ from [23, §7.2], even when $a \geq k$. This is related to the formulas used in the reference for the categorification of the shifted Verma modules $M(\lambda q^m)$ for $m \geq 0$.

It turns out that $H(G_{n,n+1;\infty})$ can also be described in terms of symmetric polynomials by introducing some extra structure on the nilHecke algebra. First, let us consider the finite case of $H(G_{n;N})$ and $H(G_{n,n+1;N})$, that can be obtained from $H(G_{n;\infty})$ and $H(G_{n,n+1;\infty})$ by modding out $Y_{>N-n,n} = \{Y_{i,n} | i > N - n\}$. Following a classical approach due to Borel [4] (see also [11]), let $C_N = \mathbb{Q}[z_1, \dots, z_N]/(e_i(\underline{z}_N))$ be the coinvariant algebra, which comes with an action of the symmetric group S_N by permutation of variables. Write C_N^n and $C_N^{n,n+1}$ for the rings of invariants

for the actions on C_N of $S_n \times S_{N-n} \subset S_N$ and of $S_n \times S_1 \times S_{N-n-1} \subset S_N$, respectively. Clearly we have

$$C_N^m \cong \mathbb{Q}[e_1(z_1, \dots, z_n), \dots, e_n(z_1, \dots, z_n), \\ e_1(z_{n+1}, \dots, z_N), \dots, e_{N-n}(z_{n+1}, \dots, z_N)] / (e_i(\underline{z}_N)),$$

and

$$C_N^{m,n+1} \cong \mathbb{Q}[e_1(z_1, \dots, z_n), \dots, e_n(z_1, \dots, z_n), \\ z_{n+1}, e_1(z_{n+2}, \dots, z_N), \dots, e_{N-n-1}(z_{n+2}, \dots, z_N)] / (e_i(\underline{z}_N)).$$

There is an isomorphism $H(G_{n;N}) \cong C_N^n$ given by

$$X_{i,n} \mapsto e_i(z_1, \dots, z_n), \quad Y_{i,n} \mapsto e_i(z_{n+1}, \dots, z_N),$$

and an isomorphism $H(G_{n,n+1;N}) \cong C_N^{m,n+1}$ given by

$$X_{i,n} \mapsto e_i(z_1, \dots, z_n), \quad \xi_{n+1} \mapsto z_{n+1}, \quad Y_{i,n+1} \mapsto e_i(z_{n+2}, \dots, z_N).$$

Indeed, a computation shows that killing $e_i(\underline{z}_N)$ comes down to killing the same elements as the homogeneous terms of degree i in the equations that define $I_{n;N}$ and $I_{n,n+1;N}$, through the bijection between symmetric polynomials and Chern classes explained above. Moreover, we recover the $(H(G_{n;N}), H(G_{+1;N}))$ -bimodule structure of $H(G_{n,n+1;N})$ in $C_N^{m,n+1}$ by letting C_N^n and C_N^{m+1} acting by multiplication. This can be checked by verifying the symmetric polynomials respect the following identities

$$e_i(z_1, \dots, z_{n+1}) = e_i(z_1, \dots, z_n) + z_{n+1}e_{i-1}(z_1, \dots, z_n), \\ e_i(z_{n+1}, \dots, z_N) = e_i(z_{n+2}, \dots, z_N) + z_{n+1}e_{i-1}(z_{n+2}, \dots, z_N).$$

We can write³ elements of $C_N^{m,n+1}$ as one strand diagrams on which we can put dots, representing the variable z_n , and “bubbles” on both sides representing the actions of C_N^n and C_N^{m+1} . For example the equation

$$e_i(z_{n+1}, \dots, z_N) = Y_{i,n} = Y_{i,n+1} + \xi_{n+1}Y_{i-1,n+1} = e_i(z_{n+2}, \dots, z_N) + z_{n+1}e_{i-1}(z_{n+2}, \dots, z_N),$$

becomes

$$\begin{array}{c} \textcircled{Y_i} \end{array} \Big| = \Big| \begin{array}{c} \textcircled{Y_i} \end{array} + \begin{array}{c} \bullet \end{array} \begin{array}{c} \textcircled{Y_{i-1}} \end{array}$$

This generalizes for $n_1 \leq \dots \leq n_k \leq N$, with $H(G_{n_1, \dots, n_k; N}) \cong C_N^{n_1, \dots, n_k}$ through the Young subgroup $S_{n_1} \times S_{n_2 - n_1} \times \dots \times S_{N - n_k} \subset S_N$. In addition, we have $C_N^{m,n+1} \otimes_{C_N^{n+1}} C_N^{m+1, n+2} \cong C_N^{m, n+1, n+2}$, yielding diagrams with k strands for $C_N^{m, n+1, \dots, n+k}$.

For $0 \leq i < k$, the Demazure operator ∂_{n+i} acts on $C_N^{m, n+1, \dots, n+k}$ by the divided difference operator

$$\partial_{n+i}(f) = \frac{f - s_{n+i}(f)}{x_{n+i} - x_{n+i+1}}.$$

³This approach is inspired by the lectures given by Webster during the FSMP's Junior Chair [29].

In fact, all bimodule morphisms in $\text{End}_{(C_N^n, C_N^{n+k})\text{-bim}}(C_N^{n,n+1,\dots,n+k})$ are given by multiplications by elements of C_N^n and C_N^{n+k} , multiplications by $z_n, \dots, z_{n+1}$, and the Demazure operators. Therefore we can write $\text{End}_{(C_N^n, C_N^{n+k})\text{-bim}}(C_N^{n,n+1,\dots,n+k})$ as diagrams with bubbles, dots and crossings respecting the nilHecke relations. This construction motivates the next section.

2.8.1. Bubbled nilHecke algebras.

Definition 2.46. The *bubbled nilHecke algebra* is defined as

$$\text{BNH}_n = \text{NH}_n \rtimes \left(\bigotimes_{p=0}^n \mathbb{Z}[X_{1,p}, X_{2,p}, \dots, X_{i,p}, \dots] \otimes \mathbb{Z}[Y_{1,p}, Y_{2,p}, \dots, Y_{i,p}, \dots] \right),$$

with the relations

$$\begin{aligned} X_{i,p} &= X_{i,p-1} + x_p X_{i-1,p-1}, & Y_{i,p-1} &= Y_{i,p} + x_p Y_{i-1,p}, \\ X_{i,0}u &= uX_{i,0}, & Y_{j,n}u &= uY_{j,n}, \end{aligned}$$

for all $i, j, p \geq 1$, $u \in \text{NH}_n$ and where $X_{0,p} = Y_{0,p} = 1$.

The algebra BNH_n admits a diagrammatic description given by all diagrams of NH_n on which we can put labeled bubbles in the regions

$$\left| \begin{array}{c} \cdots \\ 1 \end{array} \right| \left| \begin{array}{c} \textcircled{X_i} \\ p \end{array} \right| \left| \begin{array}{c} \cdots \\ p+1 \end{array} \right| \left| \begin{array}{c} \cdots \\ n \end{array} \right| = X_{i,p} \in \text{BNH}_n, \quad \left| \begin{array}{c} \cdots \\ 1 \end{array} \right| \left| \begin{array}{c} \textcircled{Y_i} \\ p \end{array} \right| \left| \begin{array}{c} \cdots \\ p+1 \end{array} \right| \left| \begin{array}{c} \cdots \\ n \end{array} \right| = Y_{i,p} \in \text{BNH}_n,$$

with the local relations

$$\begin{aligned} (22) \quad & \left| \begin{array}{c} \textcircled{X_i} \\ p \end{array} \right| = \left| \begin{array}{c} \textcircled{X_i} \\ p \end{array} \right| + \left| \begin{array}{c} \textcircled{X_{i-1}} \\ p \end{array} \right| \bullet \\ (23) \quad & \left| \begin{array}{c} \textcircled{Y_i} \\ p \end{array} \right| = \left| \begin{array}{c} \textcircled{Y_i} \\ p \end{array} \right| + \bullet \left| \begin{array}{c} \textcircled{Y_{i-1}} \\ p \end{array} \right| \end{aligned}$$

From this we can deduce the relations

$$\begin{aligned} (24) \quad & \left| \begin{array}{c} \textcircled{Y_i} \\ p \end{array} \right| - \left| \begin{array}{c} \textcircled{Y_{i-1}} \\ p \end{array} \right| \bullet = \left| \begin{array}{c} \textcircled{Y_i} \\ p \end{array} \right| - \left| \begin{array}{c} \textcircled{Y_{i-1}} \\ p \end{array} \right| \bullet \\ (25) \quad & \left| \begin{array}{c} \textcircled{X_{i-1}} \\ p \end{array} \right| \bullet - \left| \begin{array}{c} \textcircled{X_i} \\ p \end{array} \right| = \left| \begin{array}{c} \textcircled{X_{i-1}} \\ p \end{array} \right| \bullet - \left| \begin{array}{c} \textcircled{X_i} \\ p \end{array} \right| \end{aligned}$$

and bubbles float freely in the regions delimited by the strands, commuting with one another (in opposition to the floating dots of A_n).

Definition 2.47. For $M, N \in \mathbb{N}_0 \cup \{\infty\}$, the (M, N) -bubbled nilHecke algebra, (M, N) -BNH $_n$, is defined as the quotient of BNH $_n$ by the ideal generated by $X_{>M,0} = \{X_{i,0} | i > M\}$ and $Y_{n,>N-n} = \{Y_{n,i} | i > N - n\}$.

This means that we kill all bubbles in the leftmost region with X -label greater than M (or none if $M = \infty$) and all bubbles in the rightmost region with Y -label greater than $N - n$.

Let $I_{\infty,\infty}$ denote the ideal in BNH $_n$ obtained by the homogeneous terms in the equation

$$(26) \quad (1 + tX_{1,p} + t^2X_{2,p} + \dots)(1 + tY_{1,p} + t^2Y_{2,p} + \dots) = 1,$$

for all $p \geq 0$.

Proposition 2.48. *There is an isomorphism $(0, \infty)$ -BNH $_n / I_{\infty,\infty} \cong \text{NH}_n$ induced by the inclusion $\text{NH}_n \subset \text{BNH}_n$. Moreover, it sends*

$$X_{i,p} \mapsto e_i(x_1, \dots, x_p), \quad Y_{i,p} \mapsto (-1)^i h_i(x_1, \dots, x_p).$$

Proof. Using (22) and (23) repetitively in BNH $_n$, we can bring all bubbles to the left, up to adding dots. In particular one can show by induction that

$$\begin{aligned} X_{i,p} &= e_i(x_1, \dots, x_p)X_{0,0} + \sum_{r=1}^i e_{i-r}(x_1, \dots, x_p)X_{r,0}, \\ Y_{i,p} &= (-1)^i h_i(x_1, \dots, x_p)Y_{0,0} + \sum_{r=1}^i (-1)^{i-r} h_{i-r}(x_1, \dots, x_p)Y_{r,0}. \end{aligned}$$

By killing $I_{\infty,\infty}$ together with all $X_{i,0}$ for $i > 0$, we kill all $Y_{i,0}$ with $i > 0$. This observation together with the computations above conclude the proof. $\square$

We recall that the cyclotomic nilHecke algebra NH_n^N is the quotient of NH_n by (x_1^N) .

Proposition 2.49. *There is an isomorphism $(0, N)$ -BNH $_n / I_{\infty,\infty} \cong \text{NH}_n^N$ induced by the inclusion $\text{NH}_n \subset \text{BNH}_n$.*

Proof. First we observe that in $(0, \infty)$ -BNH $_n / I_{\infty,\infty}$ we have

$$(-1)^N x_1^N = Y_{N,1} = \sum_{r=0}^{n-1} e_r(x_2, \dots, x_n)Y_{N-r,n},$$

and also we have $Y_{N-n+1+i,n} = h_{N-n+1+i}(x_1, \dots, x_n) \in (0, \infty)$ -BNH $_n / I_{\infty,\infty}$, which is killed in NH_n^N [12, Proposition 2.8]. $\square$

Note that similarly $H(G_{n,n+1;\infty})$ is isomorphic to (n, ∞) -BNH $_1 / I_{\infty,\infty}$ and $H(G_{n,n+1;N}) \cong (n, N - n)$ -BNH $_1 / I_{\infty,\infty}$, if we extend the ground ring to $\mathbb{Q}$.

2.8.2. *Bubbled A_n .* We now define a bubbled version of A_n , in the same spirit as above.

Definition 2.50. The *bubbled superalgebra* BA_n is defined as

$$\text{BA}_n = \text{BNH}_n \rtimes \left(\bigwedge_{p=0}^n \left(\bigwedge^\bullet(\omega_p^0, \omega_p^1, \dots, \omega_p^i, \dots) \wedge \bigwedge^\bullet(\varpi_p^0, \varpi_p^1, \dots, \varpi_p^i, \dots) \right) \right),$$

with the relations

$$\begin{aligned} X_{b,i}\omega_j^a &= \omega_j^a X_{b,i}, & Y_{b,i}\omega_j^a &= \omega_j^a Y_{b,i}, \\ x_i\omega_j^a &= \omega_j^a x_i, & \omega_i^a &= \omega_{i+1}^{a+1} + x_{i+1}\omega_{i+1}^a, \\ T_i\omega_j^a &= \omega_j^a T_i, \text{ if } i \neq j, & T_i(\omega_i^a - x_{i+1}\omega_{i+1}^a) &= (\omega_i^a - x_{i+1}\omega_{i+1}^a)T_i, \end{aligned}$$

and

$$\begin{aligned} X_{b,i}\varpi_j^a &= \varpi_j^a X_{b,i}, & Y_{b,i}\varpi_j^a &= \varpi_j^a Y_{b,i}, \\ x_i\varpi_j^a &= \varpi_j^a x_i, & \varpi_i^a &= \varpi_{i+1}^{a+1} + x_{i+1}\varpi_{i+1}^a, \\ T_i\varpi_j^a &= \varpi_j^a T_i, \text{ if } i \neq j, & T_i(\varpi_i^a - x_i\varpi_{i-1}^a) &= (\varpi_i^a - x_i\varpi_{i-1}^a)T_i, \end{aligned}$$

for all i, j and $a \geq 0$.

The (N, M) -bubbled superalgebra, denoted $(N, M)\text{-BA}_n$, is obtained by killing each $X_{>N,0}$, $Y_{n,>M-n}$, $\omega_0^{\geq N}$ and $\varpi_n^{\geq M-n}$.

Let $J_{\infty,\infty}$ be the two-sided ideal of BA_n generated by $I_{\infty,\infty}$ and, as before, by

$$(1 + t\omega_0^0 + t^2\omega_0^1 + \dots)(1 + t\varpi_0^0 + t^2\varpi_0^1 + \dots) = 1.$$

Hence, ω_0^i and ϖ_0^i are equivalent in the quotient $\text{BA}_n/J_{\infty,\infty}$.

Proposition 2.51. *There is an isomorphism $A_n \cong (0, \infty)\text{-BA}_n/J_{\infty,\infty}$ induced by the inclusion $A_n \subset \text{BA}_n$.*

Proof. The proof follows from the same arguments as in Proposition 2.49 together with the observation that we kill all ω_0^a 's. $\square$

Note also that $\Omega_{n,n+1} \cong (n, \infty)\text{-BA}_1/J_{\infty,\infty}$ if we extend the ground ring to $\mathbb{Q}$.

Floating dots have a nice interpretation in terms of bubbles. Indeed, we can view ω_n^a as corresponding to $X_{n-a,n}$ under Koszul duality, meaning that we can think of ω_n^a as the algebraic dual $X_{n-a,n}^*$, as explained below. Moreover, ω_n^a interacts like $Y_{-n+a,n}$ with the T_i 's (compare (24) with (18)) and and bubble slides ((23) and (19)), where $Y_{-n+a,n}$ possesses a supposed negative index. Of course, the same applies for ϖ_n^a , which corresponds to $Y_{n-a,n}$ and interacts like $X_{-n+a,n}$.

Remark 2.52. In particular, we can identify $s_{i,n} \in \Omega_{n,n+1}$ with ω_n^{n-i} , recovering the fact it behaves like $Y_{-i,n}$, as explained in [23, § 3].

Back to $Y_{i,p}$ in the coinvariant algebra C_N , that is the elementary symmetric polynomial $e_i(z_{p+1}, \dots, z_N) = e_i(z_{p+2}, \dots, z_N) + z_{p+1}e_{i-1}(z_{p+2}, \dots, z_N)$, and acting on it with $s_p \in S_N$,

we get

$$\begin{aligned} s_p(Y_{i,p}) &= e_i(z_{p+2}, \dots, z_N) + z_p e_{i-1}(z_{p+2}, \dots, z_N) \\ &= Y_{i,p} + (z_p - z_{p+1})Y_{i-1,p+1}, \end{aligned}$$

and $s_p(Y_{i,k}) = Y_{i,k}$ for $k \neq p$. This is suggestive in view of our action of the symmetric group on $\omega_j \in R$ from §2.2. In view of the explanation about quadratic duality from [23, § 3.4], we can also give it an interpretation in terms of “duals of $X_{i,p} \in \text{BNH}$ ”.

For this we have to consider the polynomial space $P(\text{BNH}_n)$ of BNH_n , which is given by the subalgebra of BNH_n generated by the x_i 's and all $X_{i,p}$ and $Y_{i,p}$, and extending the scalars to the field of rational fractions $\mathbb{Q}(\underline{x}_n) = \mathbb{Q}(x_1, \dots, x_n)$. Hence, $P(\text{BNH}_n)$ is a $\mathbb{Q}(\underline{x}_n)$ -vector space, admitting as basis $\langle X_{0,p}, X_{1,p}, \dots, Y_{0,p}, Y_{1,p}, \dots \rangle$ for any p .

There is an action of S_n on $P(\text{BNH}_n)$, with s_p exchanging x_p and x_{p+1} and acting invariantly on $X_{i,k}$ and $Y_{i,k}$ whenever $k \neq p$. For $k = p$, we have

$$\begin{aligned} s_p(X_{i,p}) &= X_{i,p} + (x_{p+1} - x_p)X_{i-1,p-1}, \\ s_p(Y_{i,p}) &= Y_{i,p} + (x_p - x_{p+1})Y_{i-1,p+1}. \end{aligned}$$

This action induce an action on the algebraic dual space $P(\text{BNH}_n)^*$ with $s_p f = s_p \circ f \circ s_p$ for any $f \in P(\text{BNH}_n)^* : P(\text{BNH}_n) \rightarrow \mathbb{Q}(\underline{x}_n)$. In particular, we compute

$$\begin{aligned} (s_p X_{i,p}^*)(X_{i+k,p}) &= s_p \circ X_{i,p}^*(s_p(X_{i+k,p})) \\ &= s_p \circ X_{i,p}^*(X_{i+k,p} + (x_{p+1} - x_p)X_{i+k-1,p-1}) \\ &= s_p \circ X_{i,p}^* \left(X_{i+k,p} + (x_{p+1} - x_p) \sum_{\ell=0}^{i-1+k} (-1)^\ell x_p^\ell X_{i+k-1-\ell,p} \right) \\ &= \begin{cases} 0, & \text{if } k < 0, \\ 1, & \text{if } k = 0, \\ (-1)^{k-1} x_{p+1}^{k-1} (x_p - x_{p+1}), & \text{if } k > 0, \end{cases} \end{aligned}$$

for all $k \in \mathbb{Z}$, and where we used the fact $X_{i,p-1} = \sum_{\ell=0}^i (-1)^\ell x_p^\ell X_{i-\ell,p}$, which can be deduced from (22). Then we compute

$$\begin{aligned} (X_{i,p}^* + (x_p - x_{p+1})X_{i+1,p+1}^*)(X_{i+k,p}) &= X_{i,p}^*(X_{i+k,p}) + (x_p - x_{p+1})X_{i+1,p+1}^*(X_{i+k,p}) \\ &= X_{i,p}^*(X_{i+k,p}) + (x_p - x_{p+1})X_{i+1,p+1}^* \left(\sum_{\ell=0}^{i+k} (-1)^\ell x_{p+1}^\ell X_{i+k-\ell,p+1} \right) \\ &= \begin{cases} 0, & \text{if } k < 0, \\ 1, & \text{if } k = 0, \\ (-1)^{k-1} x_{p+1}^{k-1} (x_p - x_{p+1}), & \text{if } k > 0. \end{cases} \end{aligned}$$

Hence, we obtain $s_p(X_{i,p}^*) = X_{i,p}^* + (x_p - x_{p+1})X_{i+1,p+1}^*$ and $s_p(X_{i,k}^*) = X_{i,k}^*$ for $k \neq p$. Then, if we think of ω_p^{p-i} as $X_{i,p}^*$, we recover the action on floating dots,

$$s_p(\omega_p^{p-i}) = \omega_p^{p-i} + (x_p - x_{p+1})\omega_{p+1}^{p-i}, \quad s_p(\omega_k^{k-i}) = \omega_k^{k-i},$$

as expected. The same story applies for ϖ_p^{p-i} and $Y_{i,p}^*$.

2.8.3. Equivariant cohomology. The $GL(N)$ -equivariant cohomologies of the finite Grassmannian varieties and their iterated flags are described by

$$\begin{aligned} H_{GL(N)}^*(G_{n;N}) &\cong \mathbb{Q}[X_{1,n}, \dots, X_{n,n}, Y_{1,N-n}, Y_{N-n,N-n}], \\ H_{GL(N)}^*(G_{n,n+1;N}) &\cong \mathbb{Q}[X_{1,n}, \dots, X_{n,n}, \xi_{n+1}, Y_{1,N-n-1}, Y_{N-n-1,N-n-1}]. \end{aligned}$$

See [10] for details (see also [19, §3] for an exposition in the context of higher representation theory). Therefore, we have

$$H_{GL(N)}^*(G_{n,n+1;N}) \cong (n, N-n)\text{-BNH}_1,$$

and in general equivariant cohomology of iterated flag varieties can be expressed using BNH.

3. CATEGORICAL $\mathfrak{sl}_2$ -ACTION

This section is dedicated to proving that the algebra $A(m)$, which is a generalization of A defined below, categorify the $U_q(\mathfrak{sl}_2)$ Verma module with highest weight λq^m , denoted by $M(\lambda q^m)$ (see [23, §2] for details about those modules and conventions for quantum $\mathfrak{sl}_2$ and Vermas). This will be done using a similar approach as the one used by Kang–Kashiwara in [13] to prove that cyclotomic quotients of KLR algebras categorify the finite-dimensional, irreducible representation of $U_q(\mathfrak{g})$.

Remark 3.1. From now on, we will assume again A_n to be defined over $\mathbb{Z}$ except when specified otherwise (e.g. when we need to compute Grothendieck groups). Also by module we will mean bigraded supermodule, with homomorphisms preserving the degree. In general, we will tend to drop all “super” prefixes, whenever it is clear from the context (e.g. superbimodule, superfunctor, etc.).

For all $m \in \mathbb{Z}$, we define a (super)algebra $A_n(m)$ in the same way we have defined A_n , but with floating dots having minimal label $m+1$ instead of 0. This means a floating dot in $A_n(m)$ can have a negative label whenever $m < -1$. Thus, in $A_n(m)$, the floating dots with minimal label have q -degree given by $\deg_q(\omega_i^{m+1}) = 2 - 2i + 2m$. In this context, we draw a floating dot as unlabeled if it has label $m+1$, and a tight floating dot has also label $m+1$. We also write $\tilde{\omega}_i = \omega_i^{m+1}$. The analogs of the rings R and R^{S_n} from §2.2 and §2.3 are denoted by $R(m)$ and $R^{S_n}(m)$, respectively. We let $A(m) = \bigoplus_{n \in \mathbb{N}} A_n(m)$. Clearly, $A_n(m)$ has similar properties as A_n , and in particular $A_n(-1) = A_n$. There is also an inclusion $A_n(m) \subset A_n(m')$ for $m \geq m'$, sending x_i to x_i , ω_i^a to ω_i^a and T_i to T_i .

3.1. Categorical $\mathfrak{sl}_2$ -commutator. The inclusion of algebras $\iota : A_n(m) \hookrightarrow A_{n+1}(m)$ that adds a vertical strand to the right of a diagram gives rise to an induction functor

$$\text{Ind}_n^{n+1} : A_n(m)\text{-smod} \rightarrow A_{n+1}(m)\text{-smod}.$$

In terms of bimodules, this functor can be viewed as tensoring from the left with the $(A_{n+1}(m), A_n(m))$ -bimodule $A_{n+1}(m)$:

$$\text{Ind}_n^{n+1} = A_{n+1}(m) \otimes_{A_n(m)} (-).$$

Taking its right adjoint gives a restriction functor

$$\text{Res}_n^{n+1} : A_{n+1}(m)\text{-smod} \rightarrow A_n(m)\text{-smod},$$

which consists in seeing an $A_{n+1}(m)$ -module M as an $A_n(m)$ -module ${}_{A_n(m)}M$ with action given by $a \bullet_{{}_{A_n(m)}} (-) = \iota(a) \bullet_{{}_{A_{n+1}(m)}} (-)$. In terms of bimodules, it is given by tensoring with the $(A_n(m), A_{n+1}(m))$ -bimodule $A_n(m)$:

$$\text{Res}_n^{n+1} \cong A_n(m) \otimes_{A_{n+1}(m)} (-),$$

Remark 3.2. The fact that it is an adjunction comes from the usual observation that

$$A_n(m) \otimes_{A_{n+1}(m)} (-) = \text{Hom}_{A_{n+1}(m)}(A_{n+1}(m) {}_{A_n(m)}, -),$$

together with

$$\text{Hom}_{A_{n+1}(m)}(A_{n+1}(m) \otimes_{A_n(m)} M, N) = \text{Hom}_{A_n(m)}(M, \text{Hom}_{A_{n+1}(m)}(A_{n+1}(m), N)),$$

for all $A_n(m)$ -module M and $A_{n+1}(m)$ -module N .

In order to simplify notation we introduce some conventions. We suppose that m is fixed and we write $\otimes_n$ for $\otimes_{A_n(m)}$. We draw $A_n(m)$ as a box

$$A_n(m) = \begin{array}{c} \text{---} \vdots \text{---} \\ | \quad | \quad | \\ \boxed{n} \\ | \quad | \quad | \\ \text{---} \vdots \text{---} \end{array}$$

and we write the tensor product $\otimes_n$ as concatenating boxes on top of each other, reading from right to left.

Example 3.3. We can write

$$A_n(m) \otimes_n A_{n+1}(m) = \begin{array}{c} \text{---} \vdots \text{---} \\ | \quad | \quad | \\ \boxed{n} \\ | \quad | \quad | \\ \text{---} \vdots \text{---} \\ | \quad | \quad | \\ \boxed{n+1} \\ | \quad | \quad | \\ \text{---} \vdots \text{---} \end{array}$$

When $A_{n+1}(m)$ is regarded as an $A_n(m)$ -left module, as an $A_n(m)$ -right module or as an $(A_n(m), A_n(m))$ -bimodule, we draw respectively

$$\begin{array}{c} \text{---} \vdots \text{---} \\ | \quad | \quad | \\ \boxed{n+1} \end{array}, \quad \begin{array}{c} \boxed{n+1} \\ | \quad | \quad | \\ \text{---} \vdots \text{---} \end{array}, \quad \begin{array}{c} \text{---} \vdots \text{---} \\ | \quad | \quad | \\ \boxed{n+1} \\ | \quad | \quad | \\ \text{---} \vdots \text{---} \end{array}$$

Also, for an $A_n(m)$ -module M , writing $M \otimes G$ where $G = \bigoplus_{x \in X} x\mathbb{Z}$ is a free $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -graded abelian group generated by all (homogeneous) x in some countable set X means taking the direct sum

$$M \otimes G \cong \bigoplus_{x \in X} q^{\deg_q(x)} \lambda^{\deg_\lambda(x)} \Pi^{p(x)} M,$$

where we recall Π is the parity shift, q^a is a q -degree shift by a , and λ^b is a λ -degree shift by b . In particular, if we write $M \otimes x$ for $x \in A_{n+1}(m)$, it means we take the tensor product with $\mathbb{Z}x$, that is $q^{\deg_q(x)} \lambda^{\deg_\lambda(x)} \Pi^{p(x)} M$. We also write $M \otimes (x \oplus y) = M \otimes (\mathbb{Z}x \oplus \mathbb{Z}y)$. By abuse of notation, we can assume $\oplus$ is distributive with respect to the composition in $A_{n+1}(m)$.

Example 3.4. We interpret a formula as in Lemma 3.8 below as

$$\begin{aligned} M \otimes \mathbb{Z}[x_{n+1}] T_w(y \oplus z) &= M \otimes (\mathbb{Z}[x_{n+1}] T_w y \oplus \mathbb{Z}[x_{n+1}] T_w z) \\ &= \bigoplus_{i \geq 0} \left(q^{\deg_q(x_{n+1}^i T_w y)} \lambda^{\deg_\lambda(x_{n+1}^i T_w y)} \Pi^{p(y)} M \right. \\ &\quad \left. \oplus q^{\deg_q(x_{n+1}^i T_w z)} \lambda^{\deg_\lambda(x_{n+1}^i T_w z)} \Pi^{p(z)} M \right). \end{aligned}$$

We will now prove that Ind_n^{n+1} and Res_n^{n+1} give rise to a categorical version of the $\mathfrak{sl}_2$ -commutator

$$EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, \quad \frac{1}{q - q^{-1}} = -q(1 + q^2 + \dots),$$

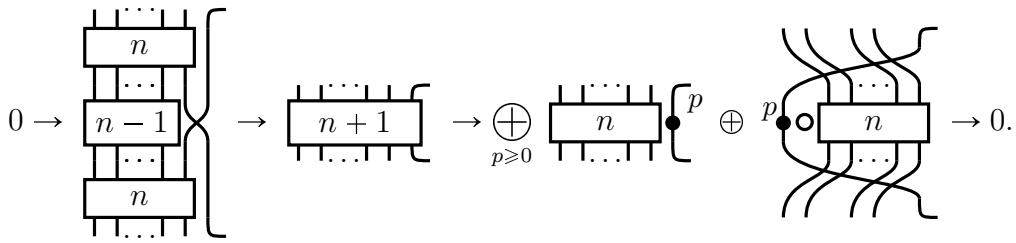
as in [23, §6]. This will be presented in the form of a short exact sequence:

Theorem 3.5. *There is a short exact sequence*

$$\begin{aligned} 0 \rightarrow q^{-2} A_n(m) \otimes_{n-1} A_n(m) \rightarrow A_{n+1}(m) \\ \rightarrow (A_n(m) \otimes \mathbb{Z}[\xi]) \oplus (q^{2m-4n} \lambda^2 \Pi A_n(m) \otimes \mathbb{Z}[\xi]) \rightarrow 0, \end{aligned}$$

of $(A_n(m), A_n(m))$ -bimodules.

Informally, we can interpret the short exact sequence in the theorem above in terms of diagrams as



The rightmost bimodule in the short exact sequence should not be immediately translated from the diagram as depicted, but rather as the quotient of $A_{n+1}(m)$ corresponding to such diagrams, as explained in the proof of Theorem 3.5 below. It is isomorphic to $q^{2m-4n} \lambda^2 \Pi A_n(m) \otimes \mathbb{Z}[\xi]$, since crossings, dots and floatings dots, when added at the top or bottom, can freely slide to the other side. Indeed, in doing so in $A_{n+1}(m)$ can produce remaining terms that are killed when projected onto the quotient (see the proof of Lemma 3.11 below).

The induction and restriction functors still need to be shifted accordingly in order to get an “ $\mathfrak{sl}_2$ -commutator” relation. To this end, we define the functors

$$F_n = \text{Ind}_n^{n+1}, \quad E_n = q^{2n-m} \lambda^{-1} \text{Res}_n^{n+1}, \quad Q_n = - \otimes q \Pi \mathbb{Z}[\xi],$$

where $\deg(\xi) = (2, 0)$ and $p(\xi) = 0$. As a direct consequence of Theorem 3.5 we get the following.

Corollary 3.6. *There is a natural short exact sequence*

$$0 \rightarrow F_{n-1} E_{n-1} \rightarrow E_n F_n \rightarrow q^{m-2n} \lambda Q_{n+1} \oplus q^{2n-m} \lambda^{-1} \Pi Q_{n+1} \rightarrow 0.$$

Remark 3.7. We can speak of a short exact sequence of functors (i.e. natural short exact sequence) since the category of functors between abelian categories is itself abelian (recall we only consider degrees and parity preserving maps, so that our categories of modules are abelian).

The remaining of this subsection is dedicated to the proof of Theorem 3.5.

Lemma 3.8. *As a left $A_n(m)$ -module, $A_{n+1}(m)$ is free with decomposition given by*

$$\bigoplus_{a=1}^{n+1} A_n(m) \otimes \mathbb{Z}[x_{n+1}] T_n \cdots T_a (1 \oplus \theta_a),$$

where $\theta_a = T_{a-1} \cdots T_1 \tilde{\omega}_1 T_1 \cdots T_{a-1}$, and it is understood that $T_n \cdots T_{n+1} = 1$ and $T_1 \cdots T_0 = 1$.

This can be pictured as

$$\boxed{n+1} \cong \bigoplus_{a=1}^{n+1} \bigoplus_{p \geq 0} \boxed{n} \oplus \boxed{n}$$

Proof. Using the third basis from Proposition 2.5 and the right coset decomposition of S_{n+1} , i.e.

$$S_{n+1} = \bigsqcup_{a=1}^{n+1} S_n s_n \cdots s_a,$$

it is not hard to see that $A_{n+1}(m)$ decomposes as left $A_n(m)$ -module as

$$A_{n+1}(m) \cong \bigoplus_{a=1}^{n+1} A_n(m) \otimes \mathbb{Z}[x_{n+1}, \tilde{\omega}_{n+1}] T_n \cdots T_a.$$

Moreover, we have $\deg(\tilde{\omega}_{n+1}) = \deg(T_n \cdots T_1 \tilde{\omega}_1)$ and thus, since $A_{n+1}(m)$ is locally of finite dimension (i.e. finite-dimensional in each bidegree as a $\mathbb{Z}$ -modules), it is enough to show that

$$(27) \quad \bigoplus_{a=1}^{n+1} (A_n(m) \otimes \mathbb{Z}[x_{n+1}] T_n \cdots T_a) \oplus (A_n(m) \otimes \mathbb{Z}[x_{n+1}] T_n \cdots T_1 \tilde{\omega}_1 T_1 \cdots T_{n+1-a})$$

generates $A_{n+1}(m)$. Indeed, it means we can split everything into free $\mathbb{Z}$ -modules of finite rank (one for each bidegree), and then we exhibit a generating family having the same number of

elements as the rank, thus it is a basis. Then the only thing that requires a proof is that (27) generates

$$\bigoplus_{a=1}^{n+1} A_n(m) \otimes \mathbb{Z}[x_{n+1}] \tilde{\omega}_{n+1} T_n \cdots T_a.$$

To this end, we first observe that applying repetitively Lemma 2.41 together with the nilHecke relations (16-17) allows us to write $\tilde{\omega}_{n+1}$ as a linear combination in (27) so that

$$\begin{aligned} \bigoplus_{a=1}^{n+1} A_n(m) \otimes \mathbb{Z}[x_{n+1}] \tilde{\omega}_{n+1} T_n \cdots T_a &\subset \bigoplus_{a=1}^{n+1} \bigoplus_{\ell=1}^{n+1} A_n(m) \otimes \mathbb{Z}[x_{n+1}] T_n \cdots T_\ell T_n \cdots T_a \\ &\quad \oplus A_n(m) \otimes \mathbb{Z}[x_{n+1}] T_n \cdots T_1 \tilde{\omega}_1 T_1 \cdots T_{n+1-\ell} T_n \cdots T_a. \end{aligned}$$

Then, using the braid move (15), we get that $T_n \cdots T_1 \tilde{\omega}_1 T_1 \cdots T_{n+1-\ell} T_n \cdots T_a$ is generated by (27). The same applies for $T_n \cdots T_\ell T_n \cdots T_a$ and this concludes the proof. $\square$

Corollary 3.9. *The family of elements (8) forms a basis for $A_n(m)$.*

Proof. This follows by recursion on n , applying Lemma 3.8 at each step. $\square$

It is not hard to see we can choose any (fixed) position on each strand for placing the dots on the diagrams in the construction of the basis, after inserting the tightened floating dots. In particular, we get the following.

Lemma 3.10. *As a left $A_n(m)$ -module, $A_{n+1}(m)$ is free with decomposition given by*

$$\bigoplus_{a=1}^{n+1} A_n(m) \otimes T_n \cdots T_a (\mathbb{Z}[x_a] \oplus \theta_a^{x_1}),$$

where $\theta_a^{x_1} = T_{a-1} \cdots T_1 \mathbb{Z}[x_1] \tilde{\omega}_1 T_1 \cdots T_{a-1}$.

This can be written as

$$\boxed{n+1} \cong \bigoplus_{a=1}^{n+1} \bigoplus_{p \geq 0} \left(\text{Diagram 1} \right) \oplus \left(\text{Diagram 2} \right)$$

Proof. Again, it is enough to prove that $\bigoplus_{a=1}^{n+1} A_n(m) \otimes T_n \cdots T_a (\mathbb{Z}[x_a] \oplus \theta_a^{x_1})$, yields a generating family, and thus that it generates all $\mathbb{Z}[x_{n+1}] T_n \cdots T_a (1 \oplus \theta_a)$ because of Lemma 3.8. Therefore, we apply the nilHecke relation (17) $n - a + 1$ times on $x_{n+1}^p T_n \cdots T_a$ so that

$$x_{n+1}^p T_n \cdots T_a = x_{n+1}^{p-1} T_n \cdots T_a x_a + R,$$

where $R \in \bigoplus_{b=a+1}^{n+1} A_n(m) \otimes \mathbb{Z}[x_{n+1}] / (x_{n+1})^p T_n \cdots T_b$. Thus, a backward induction on b and an induction on p shows that we can generate all $\mathbb{Z}[x_{n+1}] T_n \cdots T_a 1$. A similar reasoning can be applied for $\mathbb{Z}[x_{n+1}] T_n \cdots T_a \theta_a$, with

$$x_{n+1}^p T_n \cdots T_1 x_1^q \tilde{\omega}_1 T_1 \cdots T_{a-1} = x_{n+1}^{p-1} T_n \cdots T_1 x_1^{q+1} \tilde{\omega}_1 T_1 \cdots T_{a-1} + R$$

where $R \in \bigoplus_{b=1}^{n+1} A_n(m) \otimes \mathbb{Z}[x_{n+1}] T_n \cdots T_b$, and this concludes the proof. $\square$

Lemma 3.11. *As a right $A_n(m)$ -module, $A_{n+1}(m)$ is free with decomposition*

$$\bigoplus_{a=1}^{n+1} (\mathbb{Z}[x_a] \oplus \theta_a^{x_1}) T_a \cdots T_n \otimes A_n(m),$$

and if $y \in \theta_{n+1}^{x_1} \otimes A_n(m)$ then there exists some $w \notin A_n(m) \otimes \theta_{n+1}^{x_1}$ such that $y - w \in A_n(m) \otimes \theta_{n+1}^{x_1}$.

Proof. The decomposition follows from the same arguments as above and thus, we only prove the second claim.

By the Reidemeister 3 move (15), we can freely slide crossings over $T_n \cdots T_1 x_1^p \tilde{\omega}_1 T_1 \cdots T_n$. Because of the nilHecke relations (16-17), we can slide dots over crossings at the cost of adding diagrams with less crossings, such that dots slide over $T_n \cdots T_1 x_1^p \tilde{\omega}_1 T_1 \cdots T_n$ at the cost of adding terms not in $A_n(m) \otimes \theta_{n+1}^{x_1}$. This is best illustrated by the following equation

where double strands represent multiple parallel strands (the number of strands depending on n and a). We observe that the last two terms are in $A_n(m) \otimes T_n \cdots T_a \theta_a^{x_1}$ and in $A_n(m) \otimes T_n \cdots T_a \mathbb{Z}[x_a]$, respectively. Hence, only the first term is in $A_n(m) \otimes \theta_{n+1}^{x_1}$. Note that this remains true even if there is another element of $A_n(m)$ below. Indeed, by Lemma 3.10 it will decompose as a combination of elements without adding any T_n . This is important for the proof since we want to be able to ‘locally’ slide dots in the projection. Finally, since tight floating dots generates (with crossings and dots) all other floatings dots, we only need to show $\tilde{\omega}_1$ slides over $T_n \cdots T_1 \tilde{\omega}_1 T_1 \cdots T_n$. For this we observe using (18) and (16),

and the same for

Up to a sign corresponding with the parity of the projection morphism and of the floating dot, the last two terms in both equations coincide. Therefore, it only remains to show the first term in both equations are projected onto the same element (with a sign again). But this is a direct

consequence of Lemma 2.41, together with the relation $T_1\tilde{\omega}_1T_1\tilde{\omega}_1 = -\tilde{\omega}_1T_1\tilde{\omega}_1T_1$ as in the proof of Proposition 2.13,

and the observation made above about sliding of dots. $\square$

The second claim of the lemma above means the projection of an element in $A_{n+1}(m)$ onto the summands $A_n(m) \otimes \theta_{n+1}^{x_1}$ or $\theta_{n+1}^{x_1} \otimes A_n(m)$ when viewed as left or right $A_n(m)$ -module gives the same result.

Proof of Theorem 3.5. We define the morphism $s : q^{-2}A_n(m) \otimes_{n-1} A_n(m) \rightarrow A_{n+1}(m)$ as the one that adds a crossing to the right, that is $s(x \otimes_{n-1} y) = xT_n y$, or graphically

This is clearly a well-defined morphism of bimodules since T_n commutes with every element of $A_{n-1}(m)$. We define the projection morphisms as the projection on the left (equivalently right) $A_n(m)$ -submodules of $A_{n+1}(m)$ given by the summands $A_n(m) \otimes \mathbb{Z}[x_{n+1}]$ and $A_n(m) \otimes \theta_{n+1}^{x_1}$ from the decomposition in Lemma 3.10 (equivalently Lemma 3.11). By the second assertion in Lemma 3.11, this yields a well-defined bimodule morphism.

What is left to prove is that we have a short exact sequence of $A_n(m)$ -left modules (in fact, it is enough to prove we have a ‘short exact sequence’ of sets since we already know by the above that our maps respect the bimodule structure). Applying Lemma 3.10 we have

$$A_n(m) \otimes_{n-1} A_n(m) \cong \bigoplus_{a=1}^n A_n(m) \otimes T_{n-1} \cdots T_a(\mathbb{Z}[x_a] \oplus \theta_a^{x_1}),$$

and

$$s(A_n(m) \otimes T_{n-1} \cdots T_a(\mathbb{Z}[x_a] \oplus \theta_a^{x_1})) = A_n(m) \otimes T_n T_{n-1} \cdots T_a(\mathbb{Z}[x_a] \oplus \theta_a^{x_1}).$$

Therefore, each direct summand of $A_n(m) \otimes_{n-1} A_n(m)$ is sent to a different direct summand of $A_{n+1}(m)$, so that s is an injection. Moreover, the only remaining summands in $A_{n+1}(m)$ are $A_n(m) \otimes \mathbb{Z}[x_{n+1}]$ and $A_n(m) \otimes \theta_{n+1}^{x_1}$, which are exactly the right part of the sequence. We conclude that we have constructed a short exact sequence. $\square$

By the decomposition in Lemma 3.11, we know that $A_{n+1}(m)$ is bifree as a $(A_{n+1}(m), A_n(m))$ -bimodule (i.e. free as left module and free as right module, but not necessarily free as a bimodule). Therefore, we have:

Proposition 3.12. *The functors F_n and E_n are exact and send projectives to projectives.*

Applying recursively the short exact sequence from Theorem 3.5 also gives an interesting observation about the graded superdimension of $A_n(m)$ (i.e. graded rank of $A_n(m)$ seen as $\mathbb{Z}$ -module where we specialize the parity to -1).

Corollary 3.13. *We have*

$$\text{sdim } A_n(m) = (F^n m_0, F^n m_0)_{\lambda q^m},$$

where m_0 is the highest weight vector of the $U_q(\mathfrak{sl}_2)$ -Verma module of highest weight λq^m , and $(-, -)_{\lambda q^m}$ is the universal Shapovalov form (as in [23, § 2.3]).

3.2. The categorification theorem. We now consider $A_n(m)$ with coefficients in $\mathbb{Q}$. We write $\mathbb{Z}_\pi = \mathbb{Z}[\pi]/(\pi^2 - 1)$ and $\mathbb{Q}_\pi = \mathbb{Z}_\pi \otimes \mathbb{Q}$. Let $\mathbb{Z}((q, \lambda))$ be the ring of formal Laurent series in the variables q and λ , given by the order $0 < q < \lambda$, as explained in [23, §5.1] (see [1] for a general discussion about rings of formal Laurent series in several variables). It is given by formal series in variables $q^{\pm 1}$ and $\lambda^{\pm 1}$ for which the sum of the bidegrees of each term is contained in a cone compatible with the additive order $0 < q < \lambda$ on $\mathbb{Z}q \oplus \mathbb{Z}\lambda$. This ensures that we can multiply two formal Laurent series using only finite sums to determine each coefficient. Moreover, there is an inclusion $\mathbb{Z}(q, \lambda) \hookrightarrow \mathbb{Z}((q, \lambda))$. For example, we get $\frac{1}{q-q^{-1}} \mapsto -q(1 + q^2 + \dots)$.

Let $A_n(m)\text{-smod}_{\text{lf}} \subset A_n(m)\text{-smod}$ be the full subcategory of $A_n(m)$ -supermodules which are of cone bounded, locally finite dimension over $\mathbb{Q}$. By *cone bounded* we mean that their graded dimensions, which are elements of $\mathbb{Z}[[q, q^{-1}, \lambda, \lambda^{-1}]]$, are contained in a cone compatible with the additive order $0 < q < \lambda$ on $\mathbb{Z}q \oplus \mathbb{Z}\lambda$ and so, in particular, are elements of $\mathbb{Z}((q, \lambda))$.

Recall from §2.5.1 that $A_n(m)$ admits a unique indecomposable, graded projective module $P_{(n)}$, up to shift. This projective module is of locally finite dimension contained in a cone compatible with $<$. Therefore, following the results from [23, §5], $A_n(m)\text{-smod}_{\text{lf}}$ is cone complete and possesses the local Jordan–Hölder property (i.e. it admits all cone bounded, locally finite direct sums and any object admits an essentially unique (infinite) filtration with simple quotients). Thus, its (topological) Grothendieck group $\mathbf{G}_0(A_n(m)\text{-smod}_{\text{lf}})$ (i.e. the quotient of the usual Grothendieck group by the relations obtained through the infinite filtrations) is a $\mathbb{Z}_\pi((q, \lambda))$ -module freely generated by the unique simple module $S = R(m)/R_+^{S_n}(m)$, where $R_+^{S_n}(m) \subset R^{S_n}(m)$ is the maximal ideal. This simple module admits a projective cover given by $P_{(n)}$. Taking a projective resolution of S , it is not hard to see the Grothendieck group is also generated by the unique indecomposable projective module.

Theorem 3.14. *The functors $F = \bigoplus_{n \geq 0} F_n$ and $E = \bigoplus_{n \geq 0} E_n$ induce an action of quantum $\mathfrak{sl}_2$ on the topological Grothendieck group $\mathbf{G}_0(A(m)\text{-smod}_{\text{lf}})$. With this action there is an isomorphism*

$$\mathbf{G}_0(A(m)\text{-smod}_{\text{lf}}) \otimes_{\mathbb{Z}_\pi} \mathbb{Q}_\pi/(\pi + 1) \cong M(\lambda q^m)$$

of $U_q(\mathfrak{sl}_2)$ -modules. This isomorphism sends classes of projective indecomposables to the canonical basis elements and classes of simples to dual canonical elements (see [23, §2]).

Proof. By Corollary 3.6 and Proposition 3.12, we know that F and E induce an action of $U_q(\mathfrak{sl}_2)$ on $\mathbf{G}_0(A(m)\text{-smod}_{\text{lf}})$, which becomes a highest weight module induced by the action of F on the highest weight vector $[A_0(m)]$. The rest follows by direct comparison between the action

of $U_q(\mathfrak{sl}_2)$ on the canonical basis elements of $M(\lambda q^m)$ (resp. dual canonical) and on the image of the projectives $[P_{(n)}]$ (resp. the simples) in the Grothendieck group, as in [23, §6], using Proposition 2.33 and Lemma 3.10. $\square$

Remark 3.15. Alternatively, the full subcategory $A_n(m)\text{-psmod}_{\text{lf}} \subset A_n(m)\text{-smod}_{\text{lf}}$ of cone bounded, locally finitely generated projective modules is cone complete, locally Krull-Schmidt (see [23, §5] for details about these notions). Therefore, the (topological) split Grothendieck group $K_0(A_n(m)\text{-psmod}_{\text{lf}})$ is freely generated over $\mathbb{Z}_\pi((q, \lambda))$ by $P_{(n)}$. We conclude that $A(m)\text{-psmod}_{\text{lf}}$ also yields a categorification of the $U_q(\mathfrak{sl}_2)$ -module $M(\lambda q^m)$.

3.3. Recovering the irreducible finite-dimensional $U_q(\mathfrak{sl}_2)$ -module $V(N)$. For this section, we can again assume $A_n(m)$ to be defined over $\mathbb{Z}$, except when specified otherwise. We also upgrade the parity grading into a $\mathbb{Z}$ -grading by setting

$$\deg_h(x_i) = 0, \quad \deg_h(T_i) = 0, \quad \deg_h(\omega_i^a) = 1.$$

We can do this because all relations in $A_n(m)$ preserve the number of floating dots. We call this degree the *homological degree*, and we write $[1]$ for a shift up by one in it. Note that the parity of an element x is given by $\deg_h(x) \bmod 2$.

3.3.1. Turning $A_n(m)$ into a dg-algebra. As already explained in [23, §8], for each $N \in \mathbb{N}$ such that $m + N \geq 0$, we can turn $A_n(m)$ into a dg-algebra by defining the differential d_N via

$$d_N(x_i) = 0, \quad d_N(T_i) = 0, \quad d_N(\omega_i^a) = (-1)^{N+a-i} \hbar_{N+a-i}(\underline{x}_i),$$

where we recall that $\hbar_k(\underline{x}_i)$ is the k -th complete homogeneous symmetric polynomial in variables $(x_1, \dots, x_i)$, together with the graded Leibniz rule

$$d_N(xy) = d_N(x)y + (-1)^{\deg_h(x)} x d_N(y).$$

In particular, we get $d_N(\tilde{\omega}_1) = d_N(\omega_1^{m+1}) = (-1)^{m+N} x_1^{m+N}$. Using the isomorphism $A_n \cong (0, \infty)\text{-BA}_n / J_{\infty, \infty}$, we can express the differential d_N using bubbles, as a 'blow-up':

$$\begin{array}{ccccccc} \left| \begin{array}{c} \cdots \\ 1 \end{array} \right| & \left| \begin{array}{c} \\ i \end{array} \right| & \left| \begin{array}{c} \circ^a \\ i+1 \end{array} \right| & \left| \begin{array}{c} \cdots \\ n \end{array} \right| & \xrightarrow{d_N} & \left| \begin{array}{c} \cdots \\ 1 \end{array} \right| & \left| \begin{array}{c} \bigcirc^{Y_{N+a-i}} \\ i \\ i+1 \end{array} \right| & \left| \begin{array}{c} \cdots \\ n \end{array} \right| \end{array}$$

which shows d_N is well-defined since ω_i^a interacts like Y_{N+a-i} with crossings (24) and bubble slides (23), as explained in §2.8.2.

Lemma 3.16. *The homology $H(A_n(m), d_N)$ is concentrated in homological degree 0.*

Proof. By Proposition 2.30, we can construct a basis of $A_n(m)$ where all floating dots involved are concentrated in the right part of the diagrams and have labels between $m+1$ and $m+n$. Therefore, as $\mathbb{Z}$ -modules, we have

$$A_n(m) \cong \text{NH}_n \otimes \bigwedge^\bullet(\omega_n^{m+1}, \omega_n^{m+2}, \dots, \omega_n^{m+n}).$$

Also we have $d_N(\omega_n^{m+\ell}) \in Z_s(A_n(m)) \cong R^{S_n}(m)$. Hence we get a decomposition of $\mathbb{Z}$ -dg-modules

$$(A_n(m), d_N) \cong (R^{S_n}(m), d_N) \otimes (U_n, 0),$$

which means

$$H(A_n(m), d_N) \cong H(R^{S_n}(m), d_N) \otimes U_n,$$

as $\mathbb{Z}$ -modules. Moreover, it is clear that $(R^{S_n}(m), d_N)$ has trivial homology groups in non-zero homological degree and this concludes the proof. $\square$

Having in mind the basis (8) and Lemma 3.16, it is not hard to see that $H(A_n(m), d_N)$ is isomorphic to the cyclotomic quotient of the nilHecke algebra $\mathrm{NH}_n^{m+N} = \mathrm{NH}_n / (x_1^{m+N})$.

Proposition 3.17. *There is an isomorphism*

$$H(A_n(m), d_N) \cong \mathrm{NH}_n^{m+N}.$$

Moreover, $(A_n(m), d_N)$ is acyclic for $n > m + N$.

The second statement follows from $d_N(\omega_{m+N+1}^{m+1}) = 1$. Note also that $\deg_{q,\lambda}(d_N) = (2N, -2)$ and $H(A(m), d_N) \cong \mathrm{NH}^{m+N} = \bigoplus_{n \geq 0} \mathrm{NH}_n^{m+N}$.

We will now prove that the short exact sequence from Theorem 3.5 can be enhanced in terms of dg-bimodules for d_N . First observe that all arguments in §3.1 hold for the homological grading, so that we obtain a short exact sequence

$$\begin{aligned} 0 \rightarrow q^{-2} A_n(m) \otimes_{n-1} A_n(m) \rightarrow A_{n+1}(m) \\ \rightarrow (A_n(m) \otimes \mathbb{Z}[\xi]) \oplus (q^{2m-4n} \lambda^2 A_n(m)[1] \otimes \mathbb{Z}[\xi]) \rightarrow 0, \end{aligned}$$

where the parity Π in Theorem 3.5 becomes the homological shift $[1]$. The differential on $A_n(m)$ induce a differential on $A_n(m) \otimes_{n-1} A_n(m)$, also written d_N , with the Koszul sign rule

$$d_N(x \otimes_{n-1} y) = d_N(x) \otimes y + (-1)^{\deg_h(x)} x \otimes d_N(y).$$

Hence, both the left and the middle part of the short exact sequence can be enhanced into $((A_n(m), d_N), (A_n(m), d_N))$ -dg-bimodules (we will write this $(A_n(m), d_N)$ -bimodules for the sake of compactness). Then, it only remains to define a differential on the right part, compatible with d_N and the projection morphism. This can be achieved by taking a lift of $1 \otimes \xi^j \in (q^{2m-4n} \lambda^2 A_n(m)[1] \otimes \mathbb{Z}[\xi])$, for example $T_n \cdots T_1 x_1^j \tilde{\omega}_1 T_1 \cdots T_n$, then applying d_N on it, and finally computing its projection in $(A_n(m) \otimes \mathbb{Z}[\xi])$. Since $d_N(T_n \cdots T_1 x_1^j \tilde{\omega}_1 T_1 \cdots T_n) = (-1)^{N+m} T_n \cdots T_1 x_1^{j+N+m} T_1 \cdots T_n$, we first compute the projection of $T_n \cdots T_1 x_1^j T_1 \cdots T_n$ for any $j \geq 0$.

Lemma 3.18. *The projection*

$$\begin{array}{|c|} \hline \cdots \\ \hline n+1 \\ \hline \cdots \\ \hline \end{array} \rightarrow \bigoplus_{p \in \mathbb{N}} \begin{array}{|c|} \hline \cdots \\ \hline n \\ \hline \cdots \\ \hline \end{array} \begin{array}{l} p \\ \bullet \end{array}$$

from Theorem 3.5 sends

$$\begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } j \\ \vdots \\ \dots \end{array} \mapsto (-1)^n \sum_{p=n}^{j-n} \hbar_{p-n}(\underline{x}_n, \xi) \hbar_{j-n-p}(\underline{x}_n),$$

where we identify ξ with a dot on the rightmost strand.

Proof. We can suppose $n \geq 1$. By (15) and (17) we obtain

$$\begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } j \\ \vdots \\ \dots \end{array} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } i \\ \vdots \\ \dots \end{array} = \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } j-1 \\ \vdots \\ \dots \end{array} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } i+1 \\ \vdots \\ \dots \end{array} + \sum_{\substack{r+s \\ =n-1}} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } j-1 \\ \vdots \\ \dots \end{array} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } i \\ \vdots \\ \dots \end{array}$$

for all $i \geq 0$ and $j \geq 1$, where the $.r.$ means we take r such strands. We have

$$\begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } j \\ \vdots \\ \dots \end{array} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } i \\ \vdots \\ \dots \end{array} = \sum_{\substack{r+s \\ =n-1}} \sum_{i=0}^{j-1} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } j-i-1 \\ \vdots \\ \dots \end{array} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } i \\ \vdots \\ \dots \end{array}$$

By the nilHecke relation (17), we obtain

$$(28) \quad \begin{array}{c} \text{strand with dot } i \\ \text{strand with dot } i \end{array} = \begin{array}{c} \text{strand with dot } i \\ \text{strand with dot } i \end{array} - \sum_{\ell=0}^{i-1} \begin{array}{c} \text{strand with dot } \ell \\ \text{strand with dot } i-1-\ell \end{array}$$

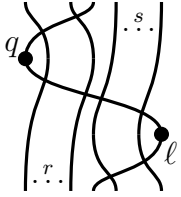
so that we get in the image of the projection

$$\begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } j \\ \vdots \\ \dots \end{array} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } i \\ \vdots \\ \dots \end{array} \mapsto - \sum_{\substack{r+s \\ =n-1}} \sum_{i=1}^{j-1} \sum_{\ell=0}^{i-1} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } j-i-1 \\ \vdots \\ \dots \end{array} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } i-1-\ell \\ \vdots \\ \dots \end{array}$$

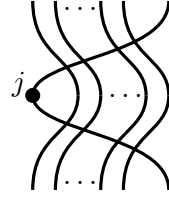
By a triangular change of variables, i.e. $p = i - 1 - \ell$ and $q = j - i - 1$, this can be rewritten as

$$(29) \quad - \sum_{p=0}^{j-2} \sum_{\substack{q+\ell \\ =j-2-p}} \sum_{\substack{r+s \\ =n-1}} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } q \\ \vdots \\ \dots \end{array} \begin{array}{c} \dots \\ \vdots \\ \text{strand with dot } p \\ \vdots \\ \dots \end{array}$$

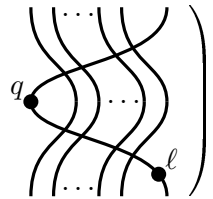
Now we claim that

$$(30) \quad \pi_t := - \sum_{\substack{q+\ell \\ =t}} \sum_{\substack{r+s \\ =n-1}} \left(\text{diagram} \right) = (-1)^n \sum_{q=n-1}^{t-n+1} \hbar_{q+1-n}(\underline{x}_n) \hbar_{t-n+1-q}(\underline{x}_n).$$


To prove it, first we observe that π_t must be in the supercenter $Z_s(A_n(m))$. Indeed, T_i commutes with



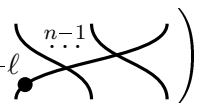
for $1 \leq i \leq n-1$ in $A_{n+1}(m)$, and thus $\pi_t \in A_n(m)$ has to commute with T_i in $A_n(m)$. Then by Proposition 2.15, it follows that $\pi_t \in Z_s(A_n(m))$. Therefore, since π_t has λ -degree 0, we know by Corollary 2.29 that it has to be a symmetric polynomial in $(x_1, \dots, x_n)$, and also it must be zero whenever $t < 2n-2$ (elements with λ -degree 0 in the supercenter are of non-negative q -degree). From the results in §2 we know that there is a faithful action of $A_n(m)$ on $\mathbb{Z}[\underline{x}_n] \otimes \wedge^\bullet(\underline{\omega}_n)$ and so we only need to check the image of 1 through the action of π_t . As ∂_i acts by zero on 1, we can restrict to $s = 0$ and we have

$$\pi_t(1) = - \sum_{q+\ell=t} \left(\text{diagram} \right) (1).$$


Using again the nilHecke relations (16) and (17), together with an generalization of (28) and the fact that ∂_i acts by zero on 1, we get

$$\left(\text{diagram} \right) (1) = \hbar_{i-k}(\underline{x}_{k+1}), \quad \left(\text{diagram} \right) (1) = (-1)^k \hbar_{i-k}(\underline{x}_{k+1}).$$


From this we can see that we must have $\ell \geq n-1$ in order not to get zero. Therefore, we have

$$\pi_t(1) = (-1)^{n-1} \sum_{\ell \geq n-1}^t \left(\text{diagram} \right) (\hbar_{\ell-n+1}(\underline{x}_{n-1})).$$


Since $\hbar_{\ell-n+1}(\underline{x}_n)$ lies in $Z_s(A_n(m))$, we get

$$\begin{aligned}\pi_t(1) &= (-1)^n \sum_{\ell \geq n-1}^t \hbar_{\ell-n+1}(\underline{x}_n) \left(\begin{array}{c} \text{Diagram: A crossing of two strands. The left strand is labeled } t-\ell \text{ and has a dot. The right strand is labeled } n-1. \end{array} \right) (1) \\ &= (-1)^n \sum_{\ell \geq n-1}^{t-n+1} \hbar_{\ell-n+1}(\underline{x}_n) \hbar_{t-\ell-n+1}(\underline{x}_n),\end{aligned}$$

which prove the claim. Finally, applying (30) on (29) gives

$$(-1)^n \sum_{p=0}^{j-2} \sum_{q=n-1}^{j-1-p-n} \hbar_{q+1-n}(\underline{x}_n) \hbar_{j-1-p-n-q}(\underline{x}_n) \xi^p,$$

which is equivalent to the statement of the lemma, after decomposing

$$(-1)^n \sum_{p'=n}^{j-n} \hbar_{p'-n}(\underline{x}_n, \xi) \hbar_{j-n-p'}(\underline{x}_n) = (-1)^n \sum_{p'=n}^{j-n} \sum_{q'=0}^{p'-n} \hbar_{p'-n-q'}(\underline{x}_n) \hbar_{j-n-p'}(\underline{x}_n) \xi^{q'},$$

and applying a triangular change of variables $p' = 1 + p + q$, $q' = p$ in the sums. $\square$

Now we can define a differential on $(A_n(m) \otimes \mathbb{Z}[\xi]) \oplus (q^{2m-4n} \lambda^2 A_n(m)[1] \otimes \mathbb{Z}[\xi])$, which we denote again by d_N . To do so, we only have to define it on each of the independant generators of the direct decomposition into free $A_n(m)$ -bimodules. Thus we set

$$d_N(0 \oplus \xi^i) = \left((-1)^{N+m+n} \sum_{p=n}^{N+m+i-n} \hbar_{p-n}(\underline{x}_n, \xi) \hbar_{N+m+i-n-p}(\underline{x}_n) \right) \oplus 0,$$

and $d_N(\xi^i \oplus 0) = 0$, for all $i \in \mathbb{N}_0$. The action of the differential in general is induced by the actions of $(A_n(m), d_N)$. Moreover, by Lemma 3.18, we know d_N commutes with the $(A_n(m), A_n(m))$ -bimodule maps from Theorem 3.5. This means we get a short exact sequence of dg-bimodules

$$\begin{aligned}0 \rightarrow (q^{-2} A_n(m) \otimes_{n-1} A_n(m), d_N) &\rightarrow (A_{n+1}(m), d_N) \\ &\rightarrow ((A_n(m) \oplus q^{2m-4n} \lambda^2 A_n(m)[1]) \otimes \mathbb{Z}[\xi], d_N) \rightarrow 0.\end{aligned}$$

By the snake lemma in the abelian category of complexes of graded bimodules, it descends to a long exact sequence

$$\begin{array}{ccc}
 & & \dots \\
 & \swarrow & \\
 H^1(A_{n+1}(m), d_N) & \xrightarrow{\quad} & H^1((A_n(m) \oplus q^{2m-4n} \lambda^2 A_n(m)[1]) \otimes \mathbb{Z}[\xi], d_N) \\
 & \searrow \delta & \\
 H^0(q^{-2} A_n(m) \otimes_{n-1} A_n(m), d_N) & \xrightarrow{\quad} & H^0(A_{n+1}(m), d_N) \\
 & \swarrow & \\
 H^0((A_n(m) \oplus q^{2m-4n} \lambda^2 A_n(m)[1]) \otimes \mathbb{Z}[\xi], d_N) & \xrightarrow{\delta} & H^{-1}(q^{-2} A_n(m) \otimes_{n-1} A_n(m), d_N) \\
 & \swarrow & \\
 \dots & &
 \end{array}$$

of $H(A_n(m), d_N)$ -bimodules. Then, since $H(A_n(m), d_N)$ is concentrated in homological degree 0 and isomorphic to NH_n^{m+N} , we obtain

$$\begin{array}{ccc}
 & & \dots \\
 & \swarrow & \\
 0 & \xrightarrow{\quad} & H^1((A_n(m) \oplus q^{2m-4n} \lambda^2 A_n(m)[1]) \otimes \mathbb{Z}[\xi], d_N) \\
 & \searrow \delta & \\
 q^{-2} \mathrm{NH}_n^{m+N} \otimes_{\mathrm{NH}_{n-1}^{m+N}} \mathrm{NH}_n^{n+m} & \xrightarrow{\quad} & \mathrm{NH}_{n+1}^{m+N} \\
 & \swarrow & \\
 H^0((A_n(m) \oplus q^{2m-4n} \lambda^2 A_n(m)[1]) \otimes \mathbb{Z}[\xi], d_N) & \xrightarrow{\delta} & 0 \\
 & \swarrow & \\
 \dots & &
 \end{array}$$

of NH_n^{m+N} -bimodules.

Lemma 3.19. *There are isomorphisms of NH_n^{m+N} -bimodules for $N + m - 2n \geq 0$,*

$$H^i((A_n(m) \oplus q^{2m-4n} \lambda^2 A_n(m)[1]) \otimes \mathbb{Z}[\xi], d_N) \cong \begin{cases} \bigoplus_{\{N+m-2n\}} \mathrm{NH}_n^{m+N}, & \text{if } i = 0, \\ 0, & \text{if } i \neq 0, \end{cases}$$

and for $N + m - 2n \leq 0$,

$$H^i((A_n(m) \oplus q^{2m-4n} \lambda^2 A_n(m)[1]) \otimes \mathbb{Z}[\xi], d_N) \cong \begin{cases} 0, & \text{if } i \neq 1, \\ \bigoplus_{\{2n-m-N\}} \lambda^2 q^{2m-4n} \mathrm{NH}_n^{m+N}, & \text{if } i = 1, \end{cases}$$

where $\bigoplus_{\{k\}} M = \bigoplus_{i=0}^{k-1} q^{2i} M$.

Proof. These homologies are easily computed by looking at the image of the independent generator elements $0 \oplus \xi^i$ and $\xi^j \oplus 0$, $i, j \geq 0$. We have $d_N(\xi^j \oplus 0) = 0$. For $N + m - 2n \leq 0$, we compute $d_N(0 \oplus \xi^i) = 0$ whenever $i < 2n - m - N$, and $d_N(0 \oplus \xi^{2n-m-N}) = 1 \oplus 0$. For $N + m - 2n \geq 0$, we observe that $d_N(0 \oplus \xi^i)$ gives a monic polynomial with respect to ξ , up to sign, and leading term given by $(-1)^{N+m+n} \xi^{N+m-2n+i}$. $\square$

After shifting by the degree of the connecting homomorphism, we recover the direct sum decompositions of the cyclotomic nilHecke algebra that categorify the $\mathfrak{sl}_2$ -commutator in the categorification of the irreducible, finite-dimensional $U_q(\mathfrak{sl}_2)$ -module $V(N + m)$ (see [13, Theorem 5.2]):

$$\begin{aligned} \mathrm{NH}_{n+1}^{m+N} &\cong (q^{-2} \mathrm{NH}_n^{m+N} \otimes_{n-1} \mathrm{NH}_n^{m+N}) \oplus_{\{N+m-2n\}} \mathrm{NH}_n^{m+N}, & \text{if } N + m - 2n \geq 0, \\ q^{-2} \mathrm{NH}_n^{m+N} \otimes_{n-1} \mathrm{NH}_n^{m+N} &\cong \mathrm{NH}_{n+1}^{m+N} \oplus_{\{2n-m-N\}} q^{2m-4n+2N} \mathrm{NH}_n^{m+N}, & \text{if } N + m - 2n \leq 0. \end{aligned}$$

3.3.2. Categorification of $V(N)$. Until the end of the section, we work with $\mathbb{Q}$ as ground ring.

For a dg-algebra (A, d) , let $\mathcal{D}^c(A, d)$ denote the triangulated, full subcategory of compact objects in the derived category $\mathcal{D}(A, d)$ of (A, d) , and $\mathcal{D}^f(A, d)$ be the triangulated, full subcategory of $\mathcal{D}(A, d)$ consisting in objects which are quasi-isomorphic to finite-dimensional (A, d) -modules (see [14] for explanations about those notions, see also [8] for a nice exposition of similar notions used in the context of categorification).

The dg-algebra $(A_n(m), d_N)$ is *formal*, i.e. it is quasi-isomorphic to its homology equipped with a trivial differential. Indeed, we can consider the surjective map $(A_n(m), d_N) \rightarrow (\mathrm{NH}_n^N, 0)$ which sends floating dots to zero and projects onto the quotient by the cyclotomic ideal. Let $Kom(\mathrm{NH}^{N+m}\text{-pmod}_{\mathrm{fg}})$ be the homotopy category of bounded complexes of finitely generated, projective left, NH^{N+m} -modules. Similarly, $Kom(\mathrm{NH}^{N+m}\text{-mod}_{\mathrm{fd}})$ is given by bounded complexes of finite-dimensional modules. By standard arguments (see [14]), we get the following:

Theorem 3.20. *There are equivalences of triangulated categories*

$$\mathcal{D}^c(A(m), d_N) \cong \mathcal{D}^c(\mathrm{NH}^{N+m}, 0) \cong Kom(\mathrm{NH}^{N+m}\text{-pmod}_{\mathrm{fg}}),$$

and equivalences of triangulated categories

$$\mathcal{D}^f(A(m), d_N) \cong \mathcal{D}^f(\mathrm{NH}^{N+m}, 0) \cong Kom(\mathrm{NH}^{N+m}\text{-mod}_{\mathrm{fd}}).$$

Recall from [13] that the category of finitely generated, projective left NH^{N+m} -modules (resp. category of finite-dimensional, left NH^{N+m} -modules), when working with ground ring $\mathbb{Q}$, categorify the irreducible highest weight module $V(N + m)$ (resp. the dual weight module $V(N + m)^*$) of Lutzig integral $U_{\mathbb{Z}[q, q^{-1}]}(\mathfrak{sl}_2)$ (see [21]). As a direct consequence of this fact together with Theorem 3.20 we have:

Corollary 3.21. *There are isomorphisms of $U_{\mathbb{Z}[q, q^{-1}]}(\mathfrak{sl}_2)$ -representations*

$$K_0(\mathcal{D}^c((A(m), d_N)\text{-mod})) \cong V(N + m),$$

and

$$K_0(\mathcal{D}^f((A(m), d_N)\text{-mod})) \cong V(N + m)^*.$$

Remark 3.22. In fact, the categorical action of $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{sl}_2)$ on $Kom(\mathrm{NH}^{N+m}\text{-pmod}_{\mathrm{fg}})$ (or equivalently on $Kom(\mathrm{NH}^{N+m}\text{-pmod}_{\mathrm{fg}})$) coincide with the one obtain by deriving the induction/restriction functors given by the inclusion $(A_n(m), d_N) \hookrightarrow (A_{n+1}(m), d_N)$ that adds a vertical strand at the right. Hence we can think of these as being ‘equivalent 2-representations up to homotopy’.

3.3.3. *Equivariant-cohomology.* As observed in [2, §4], there are other interesting differentials that can be defined on A_n . Let Σ denote the multiset of roots of the polynomial

$$P(x) = \sum_{j=0}^{N+m} \kappa_j x^{N+m-j},$$

where $\kappa_1, \dots, \kappa_{N+m}$ are generic parameters of degree $\deg(\kappa_i) = (2i, 0)$ and $\kappa_0 = 1$. The deformed differential is defined on $A_n^\Sigma(m) = A_n(m) \otimes \mathbb{Q}[\kappa_{N+m}]$ by

$$d_N^\Sigma(T_i) = 0, \quad d_N^\Sigma(x_i) = 0, \quad d_N^\Sigma(\omega_i^a) = (-1)^{N+a+i} \sum_{j=0}^{N+a-i} \kappa_j \hbar_{N+a+i-j}(x_1, \dots, x_i).$$

Proposition 3.23. ([2, Corollary 4.9]) *This defines a differential on $A_n^\Sigma(m)$ of (q, λ) -degree $(2N, -2)$.*

By [2, Theorem 4.10] there is a quasi-isomorphism between

$$(A_n^\Sigma(m), d_N^\Sigma) \xrightarrow{\cong} \mathrm{NH}_n^\Sigma = \mathrm{NH}_n \otimes \mathbb{Q}[\kappa_{N+m}] / I_{N+m}^\Sigma$$

where I_{N+m}^Σ is the two-sided ideal generated by $\sum_{j=0}^{N+m} \kappa_j x_1^{N+m-j}$.

Remark 3.24. In view of [10, Lecture 6], the center $Z(\mathrm{NH}_n^\Sigma)$ is isomorphic to the $GL(N)$ -equivariant cohomology $H_{GL(N)}^*(Gr(n, N))$.

Making use of Lemma 3.18, we define a differential d_N^Σ on the bimodule

$$(A_n^\Sigma(m) \otimes \mathbb{Q}[\xi]) \oplus (q^{2m-4n} \lambda^2 A_n^\Sigma(m)[1] \otimes \mathbb{Q}[\xi])$$

by setting $d_N^\Sigma(\xi^i \oplus 0) = 0$, and

$$d_N^\Sigma(0 \oplus \xi^i) = (-1)^{N+m+n} \sum_{j=0}^{N+m} \kappa_j \sum_{p=n}^{N+m+i-j-n} \hbar_{p-n}(\underline{x}_n, \xi) \hbar_{N+m+i-j-n-p}(\underline{x}_n).$$

This yields a short exact sequence of dg-bimodules

$$\begin{aligned} 0 \rightarrow (q^{-2} A_n^\Sigma(m) \otimes_{n-1} A_n^\Sigma(m), d_N^\Sigma) &\rightarrow (A_{n+1}^\Sigma(m), d_N^\Sigma) \\ &\rightarrow ((A_n^\Sigma(m) \otimes \mathbb{Z}[\xi]) \oplus (q^{2m-4n} \lambda^2 A_n^\Sigma(m)[1] \otimes \mathbb{Z}[\xi]), d_N^\Sigma) \rightarrow 0, \end{aligned}$$

which in turn implies direct sum decompositions

$$\begin{aligned} \mathrm{NH}_{n+1}^\Sigma &\cong (q^{-2} \mathrm{NH}_n^\Sigma \otimes_{n-1} \mathrm{NH}_n^\Sigma) \oplus_{\{N+m-2n\}} \mathrm{NH}_n^\Sigma, & \text{if } N+m-2n \geq 0, \\ q^{-2} \mathrm{NH}_n^\Sigma \otimes_{n-1} \mathrm{NH}_n^\Sigma &\cong \mathrm{NH}_{n+1}^\Sigma \oplus_{\{2n-m-N\}} q^{2m-4n+2N} \mathrm{NH}_n^\Sigma, & \text{if } N+m-2n \leq 0. \end{aligned}$$

Therefore, we get

Theorem 3.25. *There is an equivalence of triangulated categories*

$$\mathcal{D}^c(A^\Sigma(m), d_N^\Sigma) \cong \text{Kom}(\text{NH}^\Sigma\text{-pmod}_{\text{fg}})$$

and

$$K_0(\mathcal{D}^c(A^\Sigma(m), d_N^\Sigma)) \cong K_0(\text{NH}^\Sigma\text{-pmod}_{\text{fg}}) \cong V(N + m).$$

Remark 3.26. In [2] it is proved that d_N^Σ restricts to R^{S_n} (which is clear from the perspective of bubbled algebras) and $H(R^{S_n}, d_N^\Sigma) \cong H_{GL(N)}^*(G_{n;N}, \mathbb{Q})$ (see §2.8.3). We know by Proposition 2.44 that $R^{S_n} \cong \Omega_n$. Hence, we can induce a differential d_N^Σ on $\Omega_n \otimes \mathbb{Q}[\kappa_N]$. From the differential d_N^Σ on $A_n(-1)$, we get an induced differential d_N^Σ on $\Omega_{n,n+1}$, sending a floating dot to its corresponding bubble, so that $H(\Omega_{n,n+1}, d_N^\Sigma) \cong H_{GL(N)}^*(G_{n,n+1;N}, \mathbb{Q})$. This way we recover the categorification of $V(N)$ in [19] using equivariant cohomology from the (geometric) categorification of $M(\lambda q^{-1})$ constructed by the authors in [23].

3.4. Categorical $\mathfrak{sl}_2$ -action on $(0, \infty)$ -BA $_n$. The inclusion of algebras

$$(0, \infty)\text{-BA}_n \subset (0, \infty)\text{-BA}_{n+1}$$

yields induction and restriction functors allowing us to define

$$\begin{aligned} \tilde{F}_n &: (0, \infty)\text{-BA}_n(m)\text{-smod} \rightarrow (0, \infty)\text{-BA}_{n+1}(m) = \text{Ind}_n^{n+1}, \\ \tilde{E}_n &: (0, \infty)\text{-BA}_{n+1}(m)\text{-smod} \rightarrow (0, \infty)\text{-BA}_n(m) = q^{2n-m}\lambda^{-1} \text{Res}_n^{n+1}. \end{aligned}$$

Using similar arguments as above, one can show that this defines a categorical $\mathfrak{sl}_2$ -action on $(0, \infty)\text{-BA}(m)\text{-smod}$ (the main ingredient being that we can bring all Y 's and ϖ 's to the left, so that we can have analogs of Lemma 3.10 and Theorem 3.5).

Theorem 3.27. *There is a natural short exact sequence:*

$$0 \rightarrow \tilde{F}_{n-1}\tilde{E}_{n-1} \rightarrow \tilde{E}_n\tilde{F}_n \rightarrow q^{m-2n}\lambda Q_{n+1} \oplus q^{2n-m}\lambda^{-1}\Pi Q_{n+1} \rightarrow 0.$$

From this we deduce as in Remark 3.15:

Theorem 3.28. *The split (topological) Grothendieck group of $(0, \infty)\text{-BA}(m)\text{-psmod}_{\text{lf}}g$ is an $U_q(\mathfrak{sl}_2)$ -module and*

$$K_0((0, \infty)\text{-BA}(m)\text{-psmod}_{\text{lf}}g) \cong M(\lambda q^m).$$

With the aim of imitating §3.3, replacing $A_n(m)$ by $(0, \infty)\text{-BA}(m)$ and NH_n^{N+m} by NH_n^Σ , we observe the following:

Proposition 3.29. *There is an isomorphism $\text{NH}_n^\Sigma \cong (0, N + m)\text{-BNH}_n$ where κ_j is identified with $(-1)^j Y_{j,0}$.*

Proof. First, we observe that by (23) in BNH_n

$$Y_{N+m+k,0} = \sum_{r=0}^{N+m+k} e_r(\underline{x}_n) Y_{N+m+k-r,n}.$$

Since we kill all $Y_{i,n}$ for $i > N + m - n$ and $e_r(\underline{x}_n) = 0$ for $r > n$, we get $Y_{N+m+k,0} = 0$ if and only if $k > 0$. This means we can define a surjective map $(0, N + m)\text{-BNH}_n \rightarrow \text{NH}_n^\Sigma$ sending $(-1)^j Y_{j,0}$ to κ_j for $j \leq N + m$. Then, we observe that

$$\sum_{j=0}^{N+m} (-1)^j Y_{j,0} x_1^{N+m-j} = Y_{N+m,1} = \sum_{r=0}^{N+m} e_r(x_2, \dots, x_n) Y_{N+m-r,n} = 0,$$

since $e_r(x_2, \dots, x_n) = 0$ if $r > n - 1$ and $Y_{i,n} = 0$ if $i > N + m - n$. $\square$

As before, we define a differential $d_N : (0, \infty)\text{-BA}_n(m) \rightarrow (0, \infty)\text{-BA}_n(m)$ for $N + m \geq 0$ by

$$\begin{aligned} d_N(T_i) &= 0, & d_N(\omega_i^a) &= (-1)^{N+a+i} \sum_{j=0}^{N+a-i} (-1)^j Y_{j,0} \hbar_{N+a+i-j}(x_1, \dots, x_i) = Y_{N+a-i,i}, \\ d_N(x_i) &= 0, & d_N(\varpi_0^a) &= Y_{N+a,0}. \end{aligned}$$

This is well defined since, by the same arguments as in Proposition 2.30, we can bring all ϖ_i^a to the left, and both $Y_{N+a,0}$ and ϖ_0^a (super)commutes with everything.

Remark 3.30. We stress that in general $d_N(\varpi_i^a) \neq X_{N+a-i,i}$ and $d_N(\varpi_i^a) \neq Y_{N+a-i,i}$ but

$$\begin{aligned} d_N(\varpi_i^a) &= \sum_{\ell=0}^i e_\ell(x_1, \dots, x_i) d_N(\varpi_0^{i+a+\ell}) \\ &= \sum_{\ell=0}^i X_{\ell,i} Y_{N+a+\ell,0}. \end{aligned}$$

Moreover, since the minimal label for ϖ_0^a is $m + 1$ we have $d_N(\varpi_0^a) = Y_{N+a,0}$ with $a \geq m + 1$. Also, $d_N(\omega_1^{m+1}) = Y_{N+m,1} = \sum_{j=0}^{N+m} (-1)^j Y_{j,0} x_1^{N+m-j}$ does not involve any $Y_{i,0}$ for $i \geq N + m$, and thus we get

$$H((0, \infty)\text{-BA}_n(m), d_N) \cong (0, N + m)\text{-BNH}_n \cong \text{NH}_n^\Sigma.$$

Then, it yields a quasi-isomorphism

$$((0, \infty)\text{-BA}_n(m), d_N) \cong (\text{NH}_n^\Sigma, 0),$$

so that again

$$\mathcal{D}^c((0, \infty)\text{-BA}(m), d_N) \cong \text{Kom}(\text{NH}_n^\Sigma\text{-pmod}_{\text{fg}}),$$

and

$$K_0(\mathcal{D}^c((0, \infty)\text{-BA}(m), d_N)) \cong V(N + m).$$

Remark 3.31. It is also possible to first define two differentials d'_N and d_N^Σ on $(0, \infty)\text{-BA}_n(m)$, both sending all generators to zero except for $d'_N(\varpi_0^a) = Y_{N+a,0}$, and for $d_N^\Sigma(\omega_i^a) = Y_{N+a-i,i}$. This is well defined since ω_i^a behaves like $Y_{N+a-i,i}$. Then, $H((0, \infty)\text{-BA}_n(m), d'_N) \cong A_n^\Sigma(m)$ and $d_N = d'_N + d_N^\Sigma$. Since d'_N commutes with d_N^Σ , we get an induced differential d_N^Σ on $H((0, \infty)\text{-BA}_n(m), d'_N)$, coinciding with the one defined on $A_n^\Sigma(m)$ in §3.3.3. In particular, we

recover [2, Corollary 4.9]. Moreover, since $H((0, \infty)\text{-BA}_n(m), d'_N)$ is concentrated in homological degree zero with respect to the degree given by counting the number of ϖ_i^a 's, the spectral sequence going from $H((0, \infty)\text{-BA}_n(m), d'_N)$ to the total homology $H((0, \infty)\text{-BA}_n(m), d'_N + d_N^\Sigma)$ converges at the second page, so that

$$\begin{aligned} H(A_n^\Sigma(m), d_N^\Sigma) &\cong H(H((0, \infty)\text{-BA}_n(m), d'_N), d_N^\Sigma) \\ &\cong H((0, \infty)\text{-BA}_n(m), d'_N + d_N^\Sigma) \\ &\cong (0, N + m)\text{-BNH}_n \cong \text{NH}_n^\Sigma. \end{aligned}$$

and we recover [2, Theorem 4.10].

3.5. Cyclotomic quotients. For $N \in \mathbb{N}_0$, let A_n^N denote the quotient $A_n^N = A_n/(x_1^N)$ and put

$$A^N = \bigoplus_{n \geq 0} A_n^N,$$

which is a finite-dimensional, bigraded superalgebra thanks to [12, Proposition 2.8] (in particular $A_n^N \cong 0$ for $n > N$). In the following we let $K_0(B)$ denote the split Grothendieck group of the category $B\text{-pmod}_{\text{fg}}$ of finitely generated, projective, (super)modules over a finite-dimensional, bigraded (super)algebra B .

Let I_n be the minimal 2-sided ideal of A_n^N containing $\omega_1, \dots, \omega_n$ (it is generated by the ω_n^a 's cf. Proposition 2.30). Since the ω 's are “exterior elements”, I_n is nilpotent. Hence, the superalgebras A_n^N/I_n and NH_n^N are isomorphic (the latter seen as a superalgebra concentrated in parity 0), we have an isomorphism of $\mathbb{Z}$ -modules $K_0(A_n^N/I_n) \cong K_0(\text{NH}_n^N)$ for all n . However, the methods from §3.1, do not give a categorical $\mathfrak{sl}_2$ -action without a significative modification of the functors F and E .

3.6. Idempotent half 2-Kac-Moody algebra for $\mathfrak{sl}_2$. For $m \in \mathbb{Z}$ and for $(n_1, n_2, \dots, n_k) \in \mathbb{N}_0^k$ a composition of n let $(m_1, m_2, \dots, m_k) \in \mathbb{Z}^k$ be defined by the rule $m_1 = m$ and for $i > 1$, $m_{i+1} = m_i - n_i$. Define

$$A_{n_1, n_2, \dots, n_k}(m) = A_{n_1}(m_1) \otimes A_{n_2}(m_2) \otimes \cdots \otimes A_{n_k}(m_k).$$

We have inclusions of bigraded superalgebras

$$\psi_{\underline{n}, \underline{m}}: A_{n_1, n_2, \dots, n_k}(m) \rightarrow A_n(m).$$

Let $k = 2$ and define the functors

$$\text{Ind}_{(n_1, n_2), m}^{n_1+n_2, m}: A_{n_1, n_2}(m)\text{-smod} \rightarrow A_{n_1+n_2}(m)\text{-smod},$$

$$\text{Res}_{(n_1, n_2), m}^{n_1+n_2, m}: A_{n_1+n_2}(m)\text{-smod} \rightarrow A_{n_1, n_2}(m)\text{-smod}.$$

Similarly as before, we define functors of induction and restriction as sums of the above functors:

$$\text{Ind}_{n_1, n_2}^{n_1+n_2} = \bigoplus_{m \in \mathbb{Z}} \text{Ind}_{(n_1, n_2), m}^{n_1+n_2, m}: A_{n_1, n_2}\text{-smod} \rightarrow A_{n_1+n_2}\text{-smod},$$

$$\text{Res}_{n_1, n_2}^{n_1+n_2} = \bigoplus_{m \in \mathbb{Z}} \text{Res}_{(n_1, n_2), m}^{n_1+n_2, m}: A_{n_1+n_2}\text{-smod} \rightarrow A_{n_1, n_2}\text{-smod}.$$

As in the case of the nilHecke algebra, the functors of induction above define products on $K'_0(A)$ (as in §3.5) and the functors of restriction above define coproducts. These products and coproducts make $K'_0(A)$ into twisted bialgebra as in the case of the nilHecke algebras [15, Props. 3.1-3.2].

Let $U_{\mathbb{Z}[q,q^{-1}]}^-(\mathfrak{sl}_2)$ be an integral version of the half quantum group $\mathfrak{sl}_2$ corresponding with Beilinson–Lusztig–MacPherson’s [3] idempotent quantum $\mathfrak{sl}_2$, that is the subalgebra generated by the $1_{n-2}F1_n$ ’s in $U_{\mathbb{Z}[q,q^{-1}]}(\mathfrak{sl}_2)$.

Proposition 3.32. *The split Grothendieck groups $K'_0(A)$ is isomorphic with $U_{\mathbb{Z}[q,q^{-1}]}^-(\mathfrak{sl}_2)$.*

4. 2-VERMA MODULES

Let $\mathbb{k}$ be a field of characteristic 0. Let c be either an integer or a formal parameter and define ε_c to be zero if $c \in \mathbb{Z}$ and to be 1 otherwise. Let $\Lambda_{c,m} = c + m - 2\mathbb{N}$. Recall that a *Quillen exact category* [26, §2] is a full subcategory of an abelian category, closed under extensions. We can view it as an additive category equipped with a class of short exact sequences given by kernel-cokernel pairs respecting some axioms (e.g. direct sums yield short exact sequences). Moreover, we say an additive category is locally additive, cone complete [23, §5] if it is strictly bigraded (i.e. $q^a \lambda^b C \not\cong C$ for all $C \in \mathcal{C}$ and $(a, b) \in \mathbb{Z} \times \mathbb{Z}$) and if it admits all locally finite, cone bounded coproducts, and those coincide with the locally finite products. Now recall the definition of 2-Verma module for quantum $\mathfrak{sl}_2$ from [23, §6.3].

Definition 4.1. A 2-Verma module for $\mathfrak{sl}_2$ with highest weight $c + m$ consists of a bigraded, $\mathbb{k}$ -linear, idempotent complete (strict) 2-category $\mathfrak{M}$ admitting a parity 2-functor $\Pi : \mathfrak{M} \rightarrow \mathfrak{M}$, where:

- The objects of $\mathfrak{M}$ are indexed by weights $\mu \in c + \mathbb{Z}$.
- There are identity 1-morphisms $\mathbb{1}_\mu$ for each $\mu \in c + \mathbb{Z}$, as well as 1-morphisms $F\mathbb{1}_\mu : \mu \rightarrow \mu - 2$ in $\mathfrak{M}$ and their grading shift. We also assume that $F\mathbb{1}_\mu$ has a right adjoint and define the 1-morphism $E\mathbb{1}_\mu : \mu - 2 \rightarrow \mu$ as a grading shift of a right adjoint of $F\mathbb{1}_\mu$,

$$\mathbb{1}_\mu E = q^{c-\mu+2} \lambda^{-\varepsilon_c} (F\mathbb{1}_\mu)_R.$$

- The Hom-spaces between objects are locally additive, cone complete, Quillen exact categories.

On this data we impose the following conditions:

- (1) The identity 1-morphism $\mathbb{1}_\mu$ of the object μ is isomorphic to the zero 1-morphism if $\mu \notin \Lambda_{c,m}$.
- (2) The enriched 2-Hom, $\text{HOM}_{\mathfrak{M}}(\mathbb{1}_\mu, \mathbb{1}_\mu)$, is cone bounded for all μ .
- (3) There is an exact sequence

$$0 \longrightarrow FE\mathbb{1}_\mu \longrightarrow EF\mathbb{1}_\mu \longrightarrow q^{-c+\mu} \lambda^{\varepsilon_c} Q_\mu \oplus q^{c-\mu} \lambda^{-\varepsilon_c} \Pi Q_\mu \longrightarrow 0,$$

where $Q_\mu := \bigoplus_{k \geq 0} q^{2k+1} \Pi \mathbb{1}_\mu$.

- (4) For each $k \in \mathbb{N}_0$, $F^k \mathbb{1}_\mu$ carries a faithful action of the enlarged nilHecke algebra $A_k(\mu - c)$.

Following Rouquier [28], we now restrict to the case of 2-Verma modules which are sub-categories of the strict 2-categories of all bigraded, additive, $\mathbb{k}$ -linear supercategories, with 1-morphisms being (additive) $\mathbb{k}$ -linear functors, and 2-morphisms being grading preserving natural transformations. For such a 2-Verma module, we write $\mathcal{M} = \bigoplus_j \mathcal{M}_{c+j}$ with $\mathcal{M}_{c+j}$ being the category corresponding to the object $c + j$ in it. We will call this a $\mathbb{k}$ -linear 2-Verma module. When $\mathcal{M}$ is abelian, we will say it is an *abelian 2-Verma module*.

A nice property of $\mathbb{k}$ -linear 2-Verma module is that, since F and E are functors, the short exact sequence (3) yields a ker-coker pair of natural transformations. Therefore, evaluating the functors involved on an object of $\mathcal{M}$ gives a ker-coker pair of morphisms in $\mathcal{M}$. In particular, for a projective object P we obtain a short exact sequence of functors

$$\begin{aligned} 0 \longrightarrow \mathrm{Hom}(P, FE\mathbb{1}_\mu -) &\longrightarrow \mathrm{Hom}(P, EF\mathbb{1}_\mu -) \\ &\longrightarrow \mathrm{Hom}\left(P, \left(q^{-c+\mu}\lambda^{\varepsilon_c}Q_\mu \oplus q^{c-\mu}\lambda^{-\varepsilon_c}\Pi Q_\mu\right) -\right) \longrightarrow 0. \end{aligned}$$

Form the 2-category $\mathfrak{M}'(m)$ whose objects are the categories $A_k(m)$ - $\mathrm{smod}_{\mathrm{lf}}$, the 1-morphisms are cone bounded, locally finite direct sums of shifts of compositions and summands of functors from $\{E_k, F_k, Q_k, \mathrm{Id}_k\}$, and the 2-morphisms are (grading preserving) natural transformations of functors. We define $\mathfrak{M}(m)$ as the completion under extensions of $\mathfrak{M}'(m)$ in the abelian 2-category of abelian categories. Then, $\mathfrak{M}(m)$ is an example of abelian, $\mathbb{k}$ -linear 2-Verma module which categorify the Verma module $M(\lambda q^m)$. As a matter of fact, this 2-category is equivalent to the completion under extensions of the 2-category induced by the Ω_k^m - $\mathrm{smod}_{\mathrm{lf}}$ from [23] (this can be seen as a consequence of the Morita equivalence between Ω_k^m and $A_k(m)$).

Lemma 4.2. *Let $\mathcal{M}$ be a $\mathbb{k}$ -linear 2-Verma module with highest weight $c + m$. Suppose $\mathbb{M} \in \mathcal{M}_{c+m}$ is projective. Then we have*

$$\mathrm{gdim} \mathrm{HOM}_{c+m-2k}(F^k\mathbb{M}, F^k\mathbb{M}) = \mathrm{gdim}(A_k(m)) \mathrm{gdim}(\mathrm{END}_{c+m}(\mathbb{M})),$$

where gdim is the $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ -graded dimension over $\mathbb{k}$.

Proof. For the sake of simplicity we suppress the subscript $c + m - 2k$ from the equations. Using the adjunction hypothesis on E and F we get

$$\mathrm{HOM}(F^k\mathbb{M}, F^k\mathbb{M}) \cong \mathrm{HOM}(F^{k-1}\mathbb{M}, q^{m-2k}\lambda^{\varepsilon_c}EF^k\mathbb{M}),$$

and thus by applying it k times we get

$$\mathrm{HOM}(F^k\mathbb{M}, F^k\mathbb{M}) \cong \mathrm{HOM}(\mathbb{M}, q^{\sum_{\ell=0}^{k-1} m-2(k-\ell)}\lambda^{\varepsilon_c k}E^k F^k\mathbb{M}).$$

Since $\mathbb{M}$ is projective, the functor $\mathrm{HOM}(\mathbb{M}, -)$ is exact and the short exact sequence (3) yields a short exact sequence in the HOM 's. Recall also from [23, §5] that we have $\mathrm{Hom}(X, \bigoplus_i Y_i) \cong \prod_i \mathrm{Hom}(X, Y_i)$ in a locally additive category. Thus, we get that

$$\mathrm{gdim} \mathrm{HOM}(X, QY) = \pi q(1 + q^2 + \dots) \mathrm{gdim} \mathrm{HOM}(X, Y).$$

Also, we observe that

$$\begin{aligned} \mathrm{gdim} A_k(m) &= \mathrm{gdim} \mathrm{HOM}_{A(m)}(A_k(m), A_k(m)) \\ &= q^{\sum_{\ell=1}^k m-2\ell} \lambda^{\varepsilon_c k} \mathrm{gdim} \mathrm{HOM}_{A(m)}(A_0(m), E_0 \cdots E_{k-1} F_{k-1} \cdots F_0 A_0(m)), \end{aligned}$$

and that we can apply the short exact sequence from Corollary 3.6 on the right, since $A_0(m)$ is projective. Then the statement follows from the fact that the short exact sequences in (3) and in Corollary 3.6 are similar. $\square$

We will now show that the $\mathbb{k}$ -linear 2-Verma module constructed from the category of graded projective modules over A_n is the essentially unique $\mathbb{k}$ -linear 2-Verma module.

For an object $\mathbb{M}$ in a $\mathbb{k}$ -linear category and $\mathcal{C}$ another $\mathbb{k}$ -linear category, denote by $\mathcal{C} \otimes_{\mathbb{k}} \mathbb{M}$ the category having the same objects as $\mathcal{C}$ but with hom-spaces being $\text{Hom}_{\mathcal{C}}(-, -) \otimes_{\mathbb{k}} \text{End}(\mathbb{M})$.

For $\mathcal{M}$ a $\mathbb{k}$ -linear 2-Verma module with highest weight $c + m$ and $\mathbb{M} \in \mathcal{M}_{c+m}$, we define the coproduct preserving, $\mathbb{k}$ -linear functor

$$\left(\bigoplus_{k \geq 0} F^k \mathbb{M} \right) \otimes_{A(m)} (-) : A(m)\text{-psmod}_{\text{lfg}} \otimes_{\mathbb{k}} \mathbb{M} \rightarrow \mathcal{M},$$

which sends the free $A_k(m)$ -module $A_k(m)$ to $F^k \mathbb{M}$. Since $\mathfrak{M}$ is idempotent complete, the image of the idempotent $e_k \in \text{End}(F^k \mathbb{1}_{m+c})$ (see §2.5.1) gives a functor $F^{(k)} \mathbb{1}_{m+c}$. Then, the indecomposable projective $A_k(m)$ -module $P_{(k)}(m)$ is sent to $F^{(k)} \mathbb{M}$. A morphism $f \otimes_{\mathbb{k}} g \in \text{End}(A_k(m) \otimes_{\mathbb{k}} \mathbb{M}) \cong A_k(m) \otimes_{\mathbb{k}} \text{End}(\mathbb{M})$ is sent to $f \bullet F^k(g)$ where $f \bullet -$ is the action of $A_k(m)$ on $F^k \mathbb{1}_{c+m}$.

Proposition 4.3. *Let $\mathcal{M}$ be a $\mathbb{k}$ -linear 2-Verma module with highest weight $c + m$. For each projective object $\mathbb{M} \in \mathcal{M}_{c+m}$, the functor*

$$\left(\bigoplus_{k \geq 0} F^k \mathbb{M} \right) \otimes_{A(m)} (-) : A(m)\text{-psmod}_{\text{lfg}} \otimes_{\mathbb{k}} \mathbb{M} \rightarrow \mathcal{M},$$

is fully faithful.

Proof. The image of the functor is completely determined by the image of the indecomposable projectives $P_{(k)}(m)$ for various k , which are themselves determined by the image of the free modules $A_k(m)$, which have $F^k \mathbb{M}$ as images. Each of the $A_k(m)$'s and $F^k \mathbb{M}$'s live in a different weight space, so that the hom-spaces in $A(m)\text{-psmod}_{\text{lfg}} \otimes_{\mathbb{k}} \mathbb{M}$ and in the image of the functor are both determined by $\text{END}(A_k(m)) \cong A_k(m) \otimes_{\mathbb{k}} \text{END}(\mathbb{M})$ and $\text{END}(F^k \mathbb{M})$. Then, by Lemma 4.2, together with the faithful action of $A_k(n)$ on $F^k \mathbb{1}_{c+m}$, we get that $\text{END}_{c+m-2k}(F^k \mathbb{M}) \cong A_k(m) \otimes_k \text{END}(\mathbb{M})$. Therefore, the functor is fully faithful. $\square$

Clearly, if $\mathcal{M}$ is idempotent complete and all objects of $\mathcal{M}_{c+m-2k}$ are direct summands of $F^k \mathbb{M}$ for some $k \geq 0$, then the functor becomes an equivalence. Similarly, we get:

Proposition 4.4. *Let $\mathcal{M}$ be an abelian 2-Verma module with projective object $\mathbb{M} \in \mathcal{M}_{c+m}$. Then, there is a fully faithful functor*

$$\left(\bigoplus_{k \geq 0} F^k \mathbb{M} \right) \otimes_{A(m)} (-) : A(m)\text{-smod}_{\text{lfg}} \otimes_{\mathbb{k}} \mathbb{M} \rightarrow \mathcal{M}.$$

As before, if $\mathcal{M}_{c+m-2k}$ corresponds with the abelian category generated by $F^k(\mathbb{M})$, then the functor becomes an equivalence.

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