

On the parameterisation of momentum diffusion in depth-integrated models¹

Eric Deleersnijder, 26th January - 1st February 2018

Let t denote the time, and x and y represent horizontal, Cartesian coordinates. Let \mathcal{S} represent the domain of interest, with $\mathcal{S} \in \mathfrak{R}^2$, whilst its boundary is the curve denoted \mathcal{L} , which is rigid and impermeable. Assuming that there is no external forcing, the flow developing in \mathcal{S} is assumed to obey the shallow water equations

$$\frac{\partial \eta}{\partial t} + \nabla_h \bullet (H\mathbf{u}) = 0 \quad (1)$$

and

$$\frac{\partial (H\mathbf{u})}{\partial t} + f \mathbf{e}_z \times (H\mathbf{u}) = -gH \nabla_h \eta - \nabla_h \bullet [H\mathbf{u}\mathbf{u} - H\boldsymbol{\sigma} / \rho_\bullet] - C_D |\mathbf{u}| \mathbf{u}, \quad (2)$$

where usual notations are employed, except for the stress tensor $\boldsymbol{\sigma}$, which represents all of the unresolved processes. This tensor may be parameterised *à la Fourier-Fick*, leading to

$$\boldsymbol{\sigma} = \rho_\bullet v_h [\nabla_h \mathbf{u} + (\nabla_h \mathbf{u})^T] + \rho_\bullet (v'_h - v_h) (\nabla_h \bullet \mathbf{u}) \mathbf{I} \quad (3)$$

where v_h and v'_h (with $v_h, v'_h > 0$) are appropriate horizontal (kinematic) viscosities, and \mathbf{I} denotes the (two-dimensional) identity tensor. On the domain boundary, the velocity is prescribed to be zero (no-slip boundary condition):

$$[\mathbf{u}(t, x, y)]_{(x, y) \in \mathcal{L}} = 0. \quad (4)$$

The mechanical energy of the flow is readily seen to be

$$E = \underbrace{\frac{\rho_\bullet}{2} \int_{\mathcal{S}} H |\mathbf{u}|^2 d\mathcal{S}}_{\text{kinetic energy}} + \underbrace{g \rho_\bullet \int_{\mathcal{S}-h}^{\eta} (z - z_0) dz d\mathcal{S}}_{\text{potential (gravitational) energy}}, \quad (5)$$

where z is the vertical coordinate (increasing upward) and z_0 is a constant, whose value will be seen to be unimportant in the present problem.

As expected, the mechanical energy decreases according to the following expression

$$\begin{aligned} \frac{dE}{dt} = & - \frac{\rho_\bullet}{2} \int_{\mathcal{S}} H v_h [\nabla_h \mathbf{u} + (\nabla_h \mathbf{u})^T - (\nabla_h \bullet \mathbf{u}) \mathbf{I}] : [\nabla_h \mathbf{u} + (\nabla_h \mathbf{u})^T - (\nabla_h \bullet \mathbf{u}) \mathbf{I}] d\mathcal{S} \\ & - \rho_\bullet \int_{\mathcal{S}} H v'_h (\nabla_h \bullet \mathbf{u})^2 d\mathcal{S} - \rho_\bullet \int_{\mathcal{S}} C_D |\mathbf{u}|^3 d\mathcal{S} \end{aligned} \quad (6)$$

This is because all of the integrals in the right hand side of this relation are positive definite. Clearly, horizontal viscosities v_h and v'_h must be positive to guarantee that mechanical energy will be progressively dissipated. Their values may be determined independently of one another. The first coefficient is associated with the rate of shear of the flow, whilst the second is related to the rate of expansion.

Formulation (3) may be deemed by some to be unnecessarily complex. This is why simpler

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parameterisations may be worth considering. For instance, one may want to settle for the following expression:

$$\boldsymbol{\sigma} = \rho \cdot v_h [\nabla_h \mathbf{u} + (\nabla_h \mathbf{u})^T] , \quad (7)$$

which can be obtained by setting $v'_h = v_h$. In the context of the present working note, no indisputable objection can be raised against this formula.

Other plausible candidates are

$$\boldsymbol{\sigma} = \rho \cdot v_h \nabla_h \mathbf{u} , \quad (8)^2$$

and

$$\boldsymbol{\sigma} = \rho \cdot v_h (\nabla_h \mathbf{u})^T . \quad (9)^3$$

These tensors are not symmetric. Presumably, this flies in the face of a well-known result of continuum mechanics according to which any stress tensor must be symmetric. Whether or not this applies to a depth-integrated momentum equation is not entirely clear. I suspect, however, that this constraint must be satisfied no matter what.

Another argument in favour of the use of a symmetric tensor is as follows. The stress tensor is essentially meant to parameterise the integral over the height of the water column of $-\rho \cdot \hat{\mathbf{u}} \hat{\mathbf{u}}$ (i.e. shear dispersion of momentum), where $\hat{\mathbf{u}}$ denotes the deviation of the horizontal velocity with respect to its depth average. Tensor $-\rho \cdot \hat{\mathbf{u}} \hat{\mathbf{u}}$ is symmetric, which is the reason why it should be parameterised by means of a symmetric tensor. Thus, as far this criterion is concerned, expressions (8) and (9) are unacceptable.

Parameterisation (8) leads to

$$\frac{dE}{dt} = -\rho \cdot \int_S H v_h [\nabla_h \mathbf{u} : \nabla_h \mathbf{u}] dS - \rho \cdot \int_S C_D |\mathbf{u}|^3 dS . \quad (10)$$

The integral in the right hand side of (10) is positive definite. Thus, it is guaranteed that the mechanical energy will always decrease. On the other hand, expression (9) yields the following mechanical energy budget:

$$\frac{dE}{dt} = -\rho \cdot \int_S H v_h [\nabla_h \mathbf{u} : (\nabla_h \mathbf{u})^T] dS - \rho \cdot \int_S C_D |\mathbf{u}|^3 dS . \quad (11)$$

The integral in the right hand side of (11) is not positive definite. As a consequence, it is not guaranteed that the mechanical energy will always decrease.

If the parameterisation of stress tensor $\boldsymbol{\sigma}$ is such that the mechanical energy always decreases, then the following limit holds valid

$$\lim_{t \rightarrow \infty} \eta(t, x, y) = \frac{\int_S \eta(0, x, y) dS}{\int_S dS} \quad (12)$$

and

$$\lim_{t \rightarrow \infty} \mathbf{u}(t, x, y) = 0 . \quad (13)$$

² This parameterisation leads to $[\nabla_h \bullet (H \boldsymbol{\sigma} / \rho)]_{i,j} = \partial_j (H v_h \partial_j u_i)$.

³ This parameterisation leads to $[\nabla_h \bullet (H \boldsymbol{\sigma} / \rho)]_{i,j} = \partial_j (H v_h \partial_i u_j)$.

Clearly, formula (12) stems from the fact that the water volume present in the domain of interest remains constant, i.e.

$$\int_S \eta(t,x,y) dS = \int_S \eta(0,x,y) dS . \quad (14)$$

This result is readily obtained by integrating continuity equation (1) over the domain of interest.
