

Vintage capital and the dynamics of the AK model^a

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Abstract

This paper analyzes the equilibrium dynamics of an AK-type endogenous growth model with vintage capital. The inclusion of vintage capital leads to oscillatory dynamics governed by replacement echoes, which additionally influence the intercept of the balanced growth path. These features, which are in sharp contrast to those from the standard AK model, can contribute to explaining the short-run deviations observed between investment and growth rates time series. To characterize the convergence properties and the dynamics of the model we develop analytical and numerical methods that should be of interest for the general resolution of endogenous growth models with vintage capital.

Key words: Endogenous growth, Vintage capital, AK model, Difference-differential equations

JEL classification numbers: E22, E32, O40

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1 Introduction

This paper focuses on the equilibrium dynamics of an AK-type endogenous growth model with vintage capital and non-linear utility. Vintage capital has become a key feature to be incorporated into growth models toward a satisfactory account of the postwar growth experience of the United States.¹ However, existing endogenous growth models with vintage capital [e.g. Aghion and Howitt (1994), Parente (1994), Jovanovic and Rob (1998), Gort, Greenwood and Rupert (1999)] restrict the analysis to balanced growth paths. In our opinion, the main reason underlying this circumstance is not the lack of interest in the off-balanced growth path properties of this type of models, but rather a lack of tools to completely characterize their dynamics. These difficulties arise because dynamic general equilibrium models with vintage technology often collapse into a mixed delay differential equation system, which cannot in general be solved either mathematically or numerically.²

One aim of our paper is to propose a first attempt towards the complete resolution of endogenous growth models with vintage capital. In doing so we incorporate a simple depreciation rule into the simplest approach to endogenous growth, namely the AK model. More precisely, by assuming that machines have a finite lifetime, the one-hoss shay depreciation assumption, we add to the AK model the minimum structure needed to make the vintage capital technology economically relevant. This small departure from the standard model of exponential depreciation modifies dramatically the dynamics of the standard AK class of models. Indeed, convergence to the balanced growth path is no longer monotonic and the initial reaction to a shock affects the position of the balanced growth path.

The finding of persistent oscillations in investment is somewhat an expected result once non-exponential depreciation structures are incorporated into growth models. Indeed, the possibility of cyclical growth in the presence of vintage capital was pointed out by the earlier studies such as Johansen (1959). However, the literature in the 1960's dealt with descriptive growth models under neoclassical production technology and constant saving rates. Recognition that persistent and robust oscillations in investment can occur in models of vintage capital due to the effects of variable depreciation rates was first made by Benhabib and Rustichini (1991). We consider this theoretical paper to be an important advance in the understanding of

¹For a recent review see Greenwood and Jovanovic (1998). These authors stress the embodied nature of technical progress implicit in the permanent decline in equipment prices as well as the productivity slowdown, among other facts.

²For this reason, most of the theoretical literature on this ground has concentrated in some particular vintage technologies. First of all, Arrow (1962) proposes a vintage capital model in which learning-by-doing depends on cumulative past investment. Thus, integration with respect to time is substituted by integration with respect to knowledge and explicit results can be brought out. A second example is provided by Solow (1960) in a neoclassical framework where each vintage technology has a Cobb-Douglas specification. Under this assumption it is possible to derive an aggregate Cobb-Douglas technology, with a well defined aggregator for capital.

the effect of variable depreciation rates on the dynamics of investment and growth. Clearly, though, a complete model specification is needed to precisely characterize how the endogenous growth rate is affected by the determinants of the vintage structure of capital as well as to analyze the role of replacement echoes for the short-run dynamics.

To achieve these results it turns out to be useful to proceed in two stages. We start by specifying a Solow-Swan version of the model where explicit results can be brought about. Then, we incorporate our technology assumptions into an otherwise standard optimal growth framework. There are important insights we get from the Solow-Swan version of the model that we apply and extend in characterizing the dynamics in the optimal growth version. In solving for the Solow-Swan version of the model we are close to the strategy proposed by Benhabib and Rustichini (1991) since the vintage capital structure can be reduced to delayed differential equations with constant delays. However, the optimal growth version of the model requires an alternative strategy since the dynamic system augments to a mixed delayed-differential equation system. We have the advantage that some stability results can be proved in our setting though. In light of these results we are able to overcome the simultaneous occurrence of leads and lags by operating directly on the optimization problem without using the optimal conditions. We develop a numerical procedure that allows us in addition to deal with the important issue of the indeterminacy in levels that arises in an endogenous growth framework. Consequently, the analytical and numerical methods we present should be of interest in related applications.

Besides the methodological contribution there are some features we can learn from the AK vintage capital growth model, notwithstanding its simplicity as a theory of endogenous growth.³ On the empirical side, Jones (1995) uses the lack of large, persistent upward movements in growth rates in the post-World War II era for OECD economies to suggest apparent empirical rejection of endogenous growth theories, because during that period rates of investment have increased significantly, especially for equipment. On the basis of this statistical evidence Jones conclude that the early AK-style growth models, as well as subsequent models focusing more explicitly on endogenous technological change are confronted with a strong restriction: the rejection of “rate-of-growth” effects. However, McGrattan (1998), by using historical data going back to the 19th century, shows that the patterns Jones (1995) points to were short-lived and that the longer time series show evidence that periods of high investment rates roughly coincide with periods of high growth rates, just as AK models predict. She suggests variants of AK-style models in which changes in policy

³ The AK class of models has been criticized as having little empirical support its main assumption: the absence of diminishing returns. This critique vanishes once technological knowledge is assumed to be part of an aggregate of different sorts of capital goods. Furthermore, as stressed by Kocherlakota and Yi (1995), if exogenous technological shocks are introduced even an AK model may satisfy the convergence hypothesis claimed by the neoclassical growth theory. As stated below, more serious critiques [e.g. Jones (1995), Kocherlakota and Yi (1997), among others] analyze the testable predictions of this type of models.

variables directly affecting capital/output ratios and the labor/leisure trade-off can be consistent with the long-run evidence she finds and the short-lived evidence Jones found.

Therefore, the evidence on short-run deviations in trends of investment rates and growth rates could not be an appropriate criterion to distinguish exogenous from endogenous growth. We shall illustrate below that the vintage version of an endogenous growth model we discuss gives some implications for this controversy through comparison with its Benhabib and Rustichini's (1991) exogenous growth vintage counterpart. Also, even though growth rate and level of income and investment exhibit cyclical behaviors on the converging path towards the balanced-growth equilibrium it goes without saying our specification cannot be seen as a model of the business cycle. Instead, our model specification allows us to analyze the relative independence between the volatility of investment and the growth rate as well as their interaction with the length of duration of capital. Likewise, we would like to emphasize that we can build a case in favor of AK theory as far as deviations in trends of investment rates and growth rates are consistent with the patterns in postwar data, a testable prediction of our model specification of a different nature than those suggested in McGrattan (1998).

The paper is organized as follows. We first specify in Section 2 the AK one-hoss shay depreciation technology. In Section 3, we solve for the constant saving rate growth model, we characterize the balanced growth path and we prove non-monotonic convergence. An example is provided to explain the main economic properties of this type of model. In Section 4, the same type of analyses is carried out in the context of an optimal growth model. Based on the results presented in the previous section, some potentially interesting empirical implications of the model are suggested in Section 5. In particular, some ways to recast the model with decreasing returns to capital and embodied technological progress are discussed. Finally, in Section 6 some concluding remarks are made.

2 Technology

We propose a very simple AK technology with vintage capital:

$$y(t) = A \int_{t-T}^t i(z) dz; \quad (1)$$

where $y(t)$ represents production at time t and $i(z)$ represents investment at time z , which corresponds to the vintage z . As in the AK model, the productivity of capital A is constant and strictly positive, and only capital goods are required to produce. Machines depreciate suddenly after $T > 0$ units of time, the one-hoss shay

depreciation assumption. As we show below, the introduction of an exogenous life time for machines changes dramatically the behavior of the AK model.

Technology (1) has some interesting properties. First, let us denote by $k(t)$ the integral in the right hand side of (1). It can be interpreted as the stock of capital. Differentiating with respect to time, we have

$$\dot{k}(t) = i(t) - \delta(t)k(t);$$

where $\delta(t) = \frac{i(t-T)}{k(t)}$: In the standard AK model, the depreciation rate is assumed to be constant. However, in the one-hoss shay version, the depreciation rate depends on delayed investment, which shows the vintage capital nature of the model. Indeed, non-exponential depreciation schemes should be seen as a generalization of the classical view of capital. This view is related to the standard model of exponential depreciation and dramatically reduces the possible dynamics that an optimal growth model can describe.

Secondly, this specification of the production function does not introduce any type of technological progress. However, as in the standard AK model, the fact that returns to capital are constant results in sustained growth. Consequently, we have an endogenous growth model of vintage capital without (embodied) technical change. Notice that, even if vintage capital is a natural technological environment for the analyses of embodied technical progress these are two distinct concepts. Section 5 provides an interpretation of equation (1) in terms of human capital accumulation, that gives place to some type of embodied technological progress.

3 Constant saving rate

Let us start by analyzing an economy of the Solow-Swan type, where the saving rate, $0 < s < 1$, is supposed to be constant. The equilibrium for this economy can be written as a delayed integral equation on $i(t)$, i.e., $\forall t \geq 0$;

$$i(t) = sA \int_{t-T}^t i(z) dz \quad (2)$$

with initial conditions $i(t) = i_0(t) \geq 0$ for all $t \in [-T; 0]$. By differentiating (2), we can rewrite the equilibrium of this economy as a delayed differential equation (DDE) on $i(t)$, $\forall t \geq 0$;

$$\dot{i}(t) = sA (i(t) - i(t-T)) \quad (3)$$

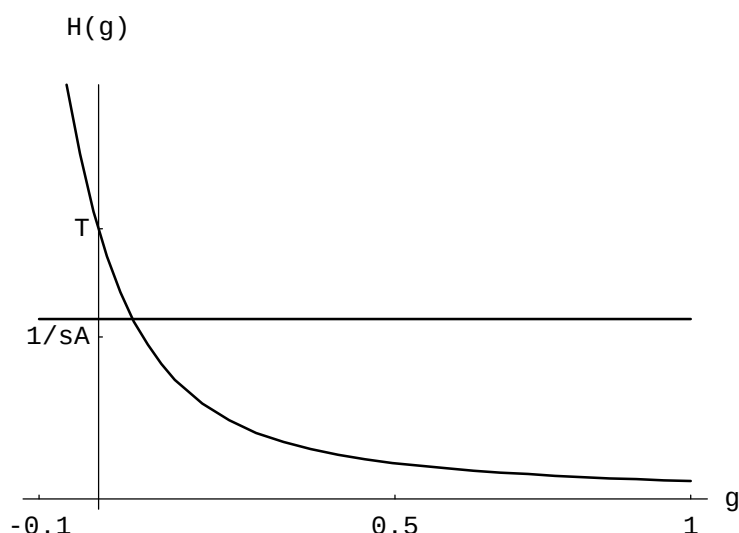


Figure 1: Determination of the growth rate on the BGP

with $i(t) = i_0(t)$, 0 for all $t \in [T; 0]$ and

$$i(0) = sA \int_{i(T)}^{\mathbf{z}_0} i_0(z) dz: \quad (4)$$

From the definition of technology in (1), we know that changes in output depend linearly on the difference between creation (current investment) and destruction (delayed investment). Since investment is a constant fraction of total output, changes in investment are also a linear function of creation minus destruction, as specified in equation (3). This type of dynamics are expected to be non monotonic and to be governed by echo effects.

3.1 Balanced growth path

A balanced growth path (hereafter BGP) solution for equation (2) is a constant growth rate $g \in \mathbb{R}$, such that

$$g = sA \int_{i(T)}^{\mathbf{z}_0} i_0(z) dz: \quad (5)$$

In what follows, $g = g(T)$ refers to the implicit BGP relation, in (5), between g and T , for given values of s and A .

Proposition 1 $g > 0$ exists and is unique iff $T > \frac{1}{sA}$

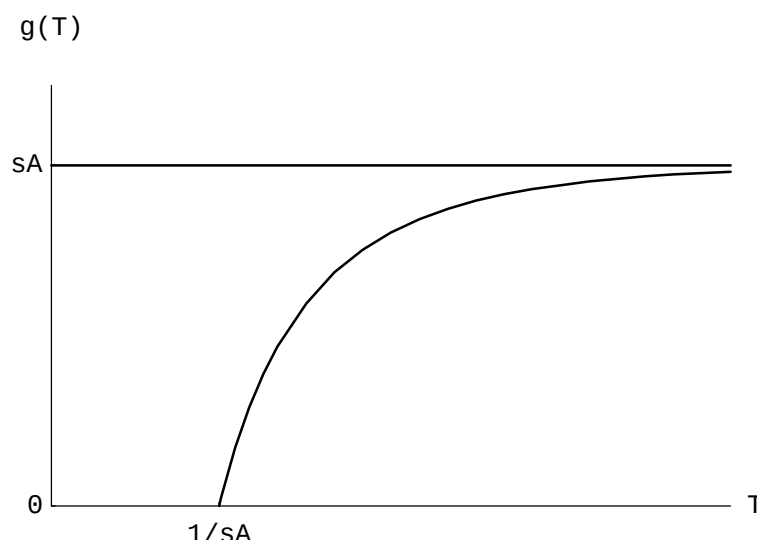


Figure 2: The BGP growth rate

Proof. From (5), we can write for $g > 0$

$$H(g) = \frac{1}{sA};$$

where $H(g) = \frac{1 - e^{-gT}}{g}$. By l'Hôpital rule, we can prove that $\lim_{g \rightarrow 0^+} H(g) = T$. Moreover, $\lim_{g \rightarrow 1} H(g) = 0$. Additionally, $H'(g) = \frac{(1+gT)e^{-gT} - 1}{g^2} < 0$, because the numerator $h(g) = (1+gT)e^{-gT} - 1$ is such that $h(0) = 0$ and $h'(g) = -gT^2 e^{-gT} < 0$ iff $g > 0$. Consequently, as it can be seen in Figure 1, if $T > \frac{1}{sA}$ there exists a unique $g > 0$ satisfying (5). ■

In what follows, we impose the restriction on parameters $T > \frac{1}{sA}$. Notice that a machine produces AT units of output during all its productive life and, given individuals' saving behavior, produces sAT units of capital. To have positive growth each machine must produce more than the one unit of good needed to produce it, i.e., sAT should be greater than one.

Proposition 2 Under $T > \frac{1}{sA}$, $\frac{\partial g}{\partial s}$, $\frac{\partial g}{\partial A}$ and $\frac{\partial g}{\partial T}$ are all positive

Proof. As we can see in Figure 1, the two first results are immediate. Notice that for any $g > 0$, $\frac{1 - e^{-gT}}{g} > \frac{1 - e^{-gT^0}}{g}$ if $T > T^0$. Then, we can still use Figure 1 to see that a proof for $\frac{\partial g}{\partial T} > 0$ is immediate. ■

Therefore, as it is shown in Figure 2, there is a positive relation between the lifetime of machines and the growth rate. Since machines from all generations are equally productive, an increase on T is equivalent to a decrease in the depreciation rate in the AK model, which is positive for growth. Indeed, as T goes to infinity, $g(T)$ is bounded above by sA which is the limit case for the AK model with zero depreciation rate: (5) reduces to $g = sA$. It turns out to be the case that property $\frac{\partial g}{\partial T} > 0$ is crucial for the statement of the stability results below. Finally, the positive effect on growth of both the saving rate and the productivity of capital are obvious and they are present in the AK model as well.

With respect to the average age of capital, let us define it as:

$$m(t) = \frac{\int_{t-T}^t (t-z) R_t \frac{i(z)}{\int_{t-T}^t i(z) dz} dz;$$

that is, a weighted average of the ages of active vintages, the weights being equal to the relative participations of the successive active vintages in the total operating capital.

Under the BGP assumption that $i(t)$ grows at the rate g , we can easily compute the BGP value for the average age:

$$m = \frac{1}{g} \left(1 - \frac{T e^{igT}}{1 - e^{igT}} \right); \quad (6)$$

and show that, for a given T , the average age of capital is negatively related to the growth rate. Notice that when $T = 1$, (6) reduces to the AK model with zero depreciation rate, where $m = \frac{1}{g}$. In this case, the average age of capital is negatively related to the growth rate. The reason is straightforward: given T and for a greater growth rate, the weight of new machines is larger and then the average age of capital is smaller. More in general, in the standard optimal growth model, if investment is growing at a constant rate on the BGP, there should be a negative relation between the average age of capital and the growth rate.⁴

3.2 Investment and output dynamics

3.2.1 Theoretical results on stability

In analyzing the stability properties of the DDE equation (3) we make use of a result in Hayes (1950).⁵ Let us define detrended investment as $\{t\} = i(t) e^{igt}$.

⁴Consequently, the Denison (1964) claim on the unimportance of the embodied question is per-se irrelevant.

⁵The basic Hayes theorem (see Theorem 13.8 in Bellman and Cooke, 1963) is a set of two necessary and sufficient conditions for the real parts of all the roots of the characteristic equation

From equations (3) and (5), we can show that

$$\dot{I}(t) = (sA - g) [I(t) - I(t - T)] : \quad (7)$$

Proposition 3 For $g > 0$ all the nonzero roots of (7) are stable

Proof. The characteristic equation associated to (7) is

$$z - (sA - g) + (sA - g)e^{-zT} = 0:$$

By defining $z = ze^z$ we obtain Hayes form: $pe^z - p - ze^z = 0$, with $p = (sA - g)T$. Consequently, in our case as in Benhabib and Rustichini (1991, example 4), $z = 0$ is a root. For the remaining roots to have strictly negative real parts, we must prove $p < 1$. From (5), it can be easily shown that $(sA - g)T = sAT e^{igT}$. Moreover, the first derivative of the implicit function $g(T)$ in (5) is

$$g'(T) = \frac{sgTe^{igT}}{1 - sATe^{igT}};$$

which is strictly positive by Proposition 2. $g'(T) > 0$ implies $p < 1$, which completes the proof. ■

Given that the characteristic equation has only $z = 0$ as a real root, the economy converges to the long-run growth trend by oscillations.⁶

3.2.2 Numerical resolution of the dynamics

The DDE (7) can be solved using the method of steps described in Bellman and Cooke (1963, p. 45). To this end, we now single out a numerical exercise by choosing parameter values as reported in Table 1. In the BGP, the growth rate is equal to 0.0296. Concerning initial conditions, we have assumed $i_0(t) = e^{g_0 t}$ for all $t < 0$, $g_0 = 0.0282$. Exponential initial conditions are consistent with the economy being in a different BGP before $t = 0$. In this sense, this exercise is equivalent to a permanent shock in s , A or T , which increases the BGP growth rate in a 5%. The nature of the shock has no effect on the solution, but it associates to $i_0(t)$ different output histories. Figures 3 and 4 show the solution for detrended output and the growth rate. It is worth to remark that alternative specifications of initial conditions should have consequences for the transitional dynamics.

to be strictly negative. See also Hale (1977, p. 109) for a complete bifurcation diagram for scalar one delay DDEs.

⁶Note that $z = -g$ is also a root of the characteristic function of the DDE describing detrended investment dynamics. It corresponds to constant solution paths for $i(t)$. Since under Proposition 1, $g > 0$, the latter solution paths are incompatible with the structural integral equation (2), so that we have to disregard this root.

Table 1: Parameter values

s	A	T	i_0	g_0	g
0.2751	0.30	15	1	0.0282	0.0296

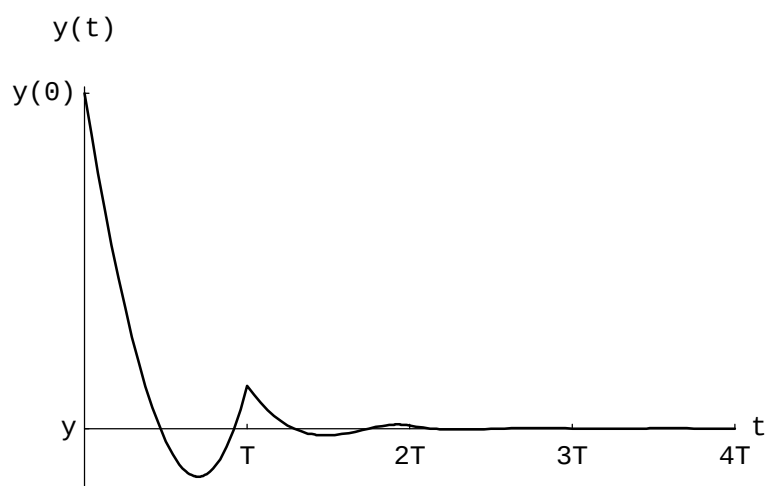


Figure 3: Constant saving rate: Detrended output

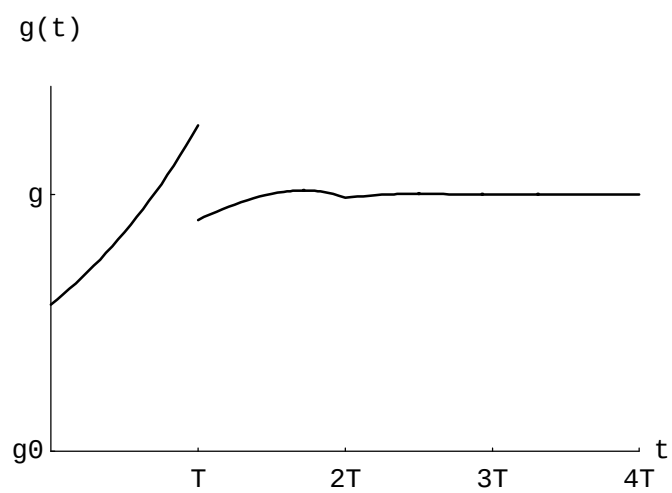


Figure 4: Constant saving rate: The growth rate

A first important observation, from Figure 4, is that the growth rate is non constant from $t = 0$, as it is in the standard AK model. It jumps at $t = 0$, is initially smaller than the BGP solution, increases monotonically over the first interval of length T and has a discontinuity in $t = T$. After this point the growth rate converges to its BGP value by oscillations. The behavior of the growth rate in the interval $[0; T[$, observed in Figure 4, is mathematically established in the following proposition:

Proposition 4 If $g_0 < g$, then

- (a) $g_0 < g(0) < g$
- (b) $g'(t) > 0$ for all $t \in [0; T[$
- (c) $g(t)$ is discontinuous at $t = T$
- (d) $g - g(0)$ is increasing in g

The Proposition is proved in the Appendix.

A permanent shock in A or in T makes output to jump at $t = 0$, thus investment also jumps. A permanent shock in s does affect investment directly. We have an equivalent jump in the AK model: under the same initial conditions but $T = 1$, $g_0 < g$ iff $s_0 A_0 < sA$, then $i(0) = \frac{sA}{g_0} > \frac{s_0 A_0}{g_0} = 1 = i_0$. Investment jumps in order to allow the growth rate of the capital stock to jump at $t = 0$.

Output at $t = 0$ is totally determined by initial conditions for investment. Moreover, the level of the new BGP solution depends crucially on the initial level of output. Since the adjustment is not instantaneous, the evolution of output on the adjustment period also influences the output level on the BGP as we can observe in Figure 3.

Finally, we have performed numerical exercises for different values of the parameters. They indicate that the profile of both detrended output and the growth rate do not depend on g_0 (of course, if $g_0 > g$ the solution profile is inverted but symmetric) or on s , A or T , provided that condition $T > \frac{1}{sA}$ holds. Only the initial jump on the growth rate, the BGP level of detrended output and the amplitude of fluctuations depend on these parameters. As stated in part (d) of Proposition 4, the greater is g with respect to g_0 the larger the distance between $g(0)$ and g . When the permanent shock is important, the economy starts relatively far from the BGP growth rate and this initial distance reduces the level of the BGP. Consequently, the greater is a positive shock, the larger is the slope of the BGP but the smaller is the intercept.

4 Optimal growth

In the previous section, we have fully characterized the dynamics of the one-hoss shay AK model under the assumption of a constant saving rate. Under the same technological assumptions, in this section we generalize these results for an optimal growth model. Let a planner solve the following problem:

$$\text{Max} \int_0^T \frac{c(t)^{1-\frac{3}{4}}}{1-\frac{3}{4}} e^{-\frac{1}{2}t} dt \quad (8)$$

s.t.

$$y(t) = A \int_{t-T}^t i(z) dz \quad (1)$$

$$c(t) + i(t) = y(t) \quad (9)$$

$$0 \leq i(t) \leq y(t)$$

and given $i(t) = i_0(t) \geq 0$ for all $t \in [t-T; 0]$; with parameters $\frac{1}{2} > 0$ and $\frac{3}{4} > 0$, $\frac{3}{4} \leq 1$. $c(t)$ represents consumption. The optimal conditions for this problem are:

$$(y(t) - i(t))^{-\frac{3}{4}} = \dot{\lambda}(t) \quad (10)$$

$$\dot{\lambda}(t) e^{-\frac{1}{2}t} = A \int_t^{t+T} \dot{\lambda}(z) e^{-\frac{1}{2}z} dz; \quad (11)$$

where $\dot{\lambda}(t)$ is the Lagrangian multiplier associated to the feasibility constraint.

Equation (11) says that at the optimum the cost of investment should be equal to its discounted flow of benefits, both evaluated at the marginal value of consumption.

4.1 Balanced growth path

From the previous equations, and assuming that $y(t) = y e^{gt}$ and $i(t) = i e^{gt}$, $y > 0$ and $i > 0$, we obtain:

$$\frac{3}{4}g + \frac{1}{2} = A \frac{i}{1-i} e^{-i(\frac{3}{4}g + \frac{1}{2})T} \quad (12)$$

$$g = \frac{i}{y} A \frac{i}{1-i} e^{-i g T} \quad (13)$$

Notice that equation (13) is equivalent to (5) if $\frac{i}{y} = s$. However, g is determined in equation (12), given the parameters $\frac{3}{4}$, $\frac{1}{2}$, A and T , and (13) determines the ratio $\frac{i}{y}$. In what follows, we still use the notation $g = g(T)$ to refer to the equilibrium relation between g and T implicit now in equation (12).

Proposition 5 If $H(\frac{1}{2}) > \frac{1}{A}$, then $g > 0$

Proof. Using the function $H(x) = \frac{(1 - e^{-xT})}{x}$, whose properties were analyzed in the proof of Proposition 1, we can easily show that this proposition is true ■

From equation (13), we know that if $\frac{3}{4}$ and $\frac{1}{2}$ are such that $\frac{i}{y} = s$ in the BGP, for s defined in the previous section, the BGP of the optimal growth model is identical to the BGP of the constant saving rate model. Moreover, as a direct consequence of Proposition 2, it can be easily checked that $g^0(T) > 0$, as in the Solow-Swan version of the model.

The condition $(1 - \frac{3}{4}g) < \frac{1}{2}$ is needed for utility to be bounded along the BGP. Under this condition, it can be shown that $\frac{i}{y} < 1$. Along the BGP the saving rate should be strictly smaller than one.

4.2 Investment and output dynamics

4.2.1 Theoretical results on stability

Notice that condition (11) only depends on the Lagrangian multiplier $\hat{A}(t)$, which grows at the rate $-\frac{3}{4}g$ on the BGP. Let us define $x(t) = \hat{A}(t) e^{\frac{3}{4}gt}$ and rewrite (11) as

$$x(t) e^{i(\frac{3}{4}g + \frac{1}{2})t} = A \int_t^{t+T} x(z) e^{i(\frac{3}{4}g + \frac{1}{2})z} dz; \quad (14)$$

This advanced integral equation is forward looking and forms a top block of the system, implying that the detrended marginal value of consumption, $x(t)$, can be solved first. By differentiating (14), we get the following advanced differential equation (ADE):

$$x^0(t) = -(x(t+T) - x(t)); \quad (15)$$

where $- = A - \frac{3}{4}g - \frac{1}{2}$, strictly positive for (12). In analyzing the stability of the ADE (15) we build upon similar arguments as in Section 3.2.1.

Proposition 6 $x(t) = x$ constant, for all $t \geq 0$, is the only stable solution of (15)

Proof. The characteristic equation is $z - \beta e^{zT} + \beta = 0$ and defining $z = \beta z^T$ we can easily obtain Hayes' form with $p = -T$ and $q = 1$. This implies a stability condition $-T < 1$ which it can be easily checked it is equivalent to $g^0(T) > 0$. Note this result is obtained for βz so that all the roots but $z = 0$ have strictly positive real parts. ■

Moreover, since $x(t)$ has to converge to $(y - i)^{1/4}$, Proposition 6 implies $x(t) = (y - i)^{1/4}$ for all $t \geq 0$. Detrended consumption is also constant and equal to $c(t) = c = x^{1-1/4}$. The value of c is determined by the initial conditions. The optimality of this result is straightforward. The block recursive structure of the problem allows the planner to choose detrended consumption without any restriction other than (14). It seems obvious that, from concavity of the utility function, he must prefer a constant detrended consumption path. Observe that, irrespective of the value of the intertemporal elasticity of substitution, the planner always chooses a constant detrended consumption, as it does in the standard AK model. However, in our model he needs to let the saving rate to fluctuate to compensate for fluctuations in output due to echo effects.

To analyze the transitional dynamics of detrended production and investment, we need to solve equations (1) and (10) jointly with the definition of $x(t)$ and the solution $x(t) = x$. Before doing that, let us define $\hat{f}(t) = i(t) e^{igt}$. By combining (1) and (10), the definition of $x(t)$ and Proposition 6, we can show that the dynamics of detrended investment are given by:

$$\hat{f}(t) = \beta g c + (A - \beta g) \hat{f}(t) - A e^{igt} \hat{f}(t - T) \quad (16)$$

with initial conditions $\hat{f}(t) = i_0(t) e^{igt}$ for all $t \in [-T; 0]$ and $\hat{f}(0) = y(0) - c$, where

$$y(0) = A \int_{-T}^0 i_0(z) dz$$

Since the constant $\beta g c$ adds only constant partial solutions, the stability of detrended investment depends upon the homogeneous part of equation (16).

Proposition 7 Any stable solution of the DDE (16) has the form:

$$\hat{f}(t) = \hat{f} + \sum_{r \in E_s} c_r e^{s_r t};$$

where $\hat{f} = \frac{gc}{A - \beta g - A e^{igt}}$, E_s is the set of stable roots of the characteristic function of the homogenous part of the DDE (16), and c_r are constant terms determined by the initial conditions.

The general solution form stated above is merely an application of the superposition principle to the non-homogenous DDE (16). $\hat{f}$ is a constant solution of the

DDE and s_r are the roots of $f(z) = z - (A - g) + A e^{igT} e^{izT}$, which turns out to be the characteristic function of the homogenous part of the DDE (16). The expansion representation of the stable solutions of the homogenous part of (16) is an application of Theorem 3.4 in Bellman and Cooke (1963). Note that the expansion involves constant terms c_r because the roots of $f(z)$ are all simple. Indeed, a multiple root arises if and only if $f(z) = f'(z) = 0$. It is trivial to show that this situation cannot occur in our case. On the other hand, one can put the characteristic function $f(z)$ into the form of Hayes with $p = (A - g)T$ and $q = ATe^{igT}$. Since equation (13) can be rewritten as $(A - \frac{gY}{T})T = ATe^{igT} = q$, it turns out that $p > q$ as far as the long run saving rate is strictly lower than one. Hence, one of the two necessary and sufficient conditions of Hayes theorem does not hold and the characteristic function admits generally both stable and unstable roots. For stability requirements (of detrended investment), we rule out the unstable roots. The constant terms c_r and the consumption term c cannot be fully determined if no initial function $\{b(t), t \in [-T; 0]$ is specified. But even if the latter function was specified, we would not be able to compute analytically the solution paths since this would require the computation of the entire set of the stable roots of function $f(z)$, which is typically infinite. So we resort to numerical resolution.

4.2.2 Numerical resolution of the dynamics

The computational procedure that we use to find the equilibrium paths of the optimal growth model is of the cyclic coordinate descent type (see Luenberger (1973), p. 158) and operates directly on the optimization problem. It is an extension of the algorithm proposed by Boucekkine, Germain, Licandro and Magnus (1999). The Appendix contains a description of the algorithm used to compute the optimal solution. Roughly, it consists of finding a fixed point vector $i(t)$ by sequentially maximizing the objective with respect to coordinate variables at time t . This methodological approach is of particular interest when both continuous time and discrete time phenomena are to be considered. This covers for instance optimal discrete time models with state variables evolving in continuous time as in certain optimal replacement investment problems.⁷ It is also useful to deal with the class of continuous time optimal growth models with Kaleckian lags (e.g. Asea and Zak (1999)).

We perform a comparable experiment to that of the Solow-Swan version of the model and parameter values are chosen correspondingly. This implies parameter values as those reported in Table 2.

We set $\frac{3}{4}$ and $\frac{1}{2}$ that correspond at the BGP value for s (0.2751) used in Section

⁷See Benhabib and Rustichini (1991) and Boucekkine et al. (1997). More recently, Whelan (2000) argues that the working of the information technologies is better captured in continuous time (flow of services in real time) while it certainly involves some crucial discrete timing variables as the scrapping of computers and softwares and the time length of the patent protection of new products.

Table 2: Parameter values

$\frac{3}{4}$	$\frac{1}{2}$	A	T	i_0	g_0	g
8.0	0.06	0.30	15	1	0.0282	0.0296

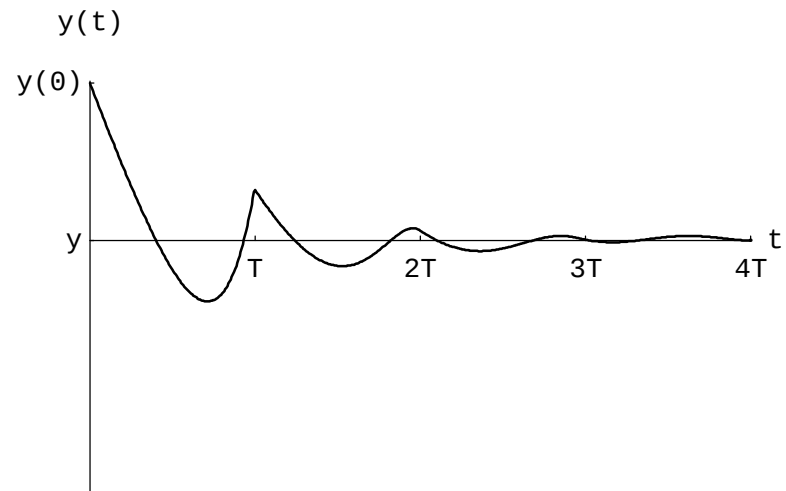


Figure 5: Optimal growth model: Detrended output

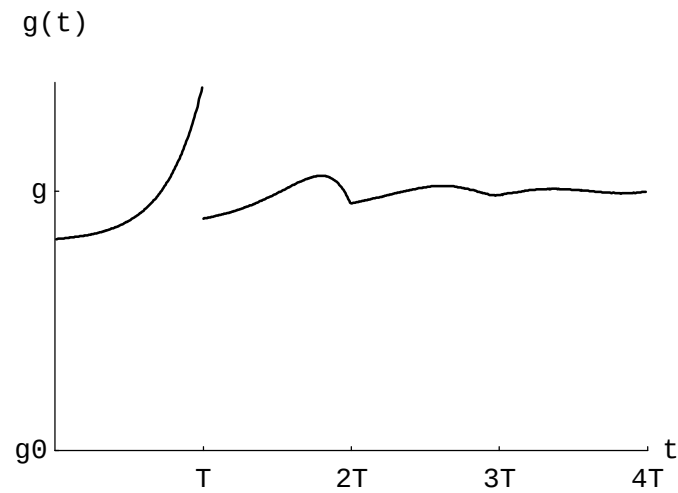


Figure 6: Optimal growth model: The growth rate

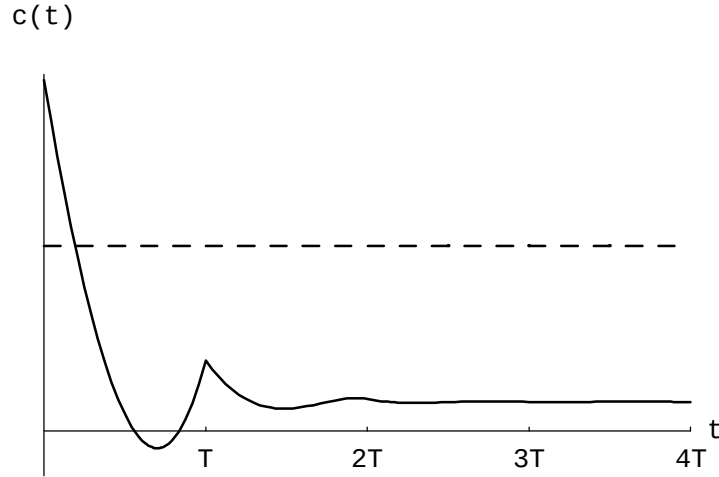


Figure 7: Consumption: optimal growth vs constant saving rate

3. Notice that the implied value of $\frac{3}{4}$ is relatively high. This quantitative peculiarity comes from the AK structure of our model: if we let $T = 1$ and we introduce a depreciation rate of about $\frac{1}{15}$ (to be consistent with a mean life time of 15 years), we need $\frac{3}{4} = 5:9$ to generate an endogenous growth rate of around 0.0296.

Figures 5 and 6 are plotted in the same scale as Figures 3 and 4 above, respectively. They depict the solution path for output and the growth rate, which behave very similar as in the constant saving rate model. From Proposition 6, we know that the planner optimally chooses to have a constant detrended consumption. For this reason, the saving rate rises at the beginning, increasing the growth rate (with respect to the Solow-Swan case) and therefore allowing output to converge to a higher long-run level. As a consequence, the planner generates longer lasting fluctuations than those that were obtained in the constant saving rate model. Indeed, in the optimal growth model it is the saving rate that bears most of the adjustment to the BGP. As stated in Proposition 6, detrended consumption should be constant from $t = 0$, but its level should be determined by initial conditions. Figure 7 compares the numerical solution obtained for detrended consumption in both models, the dashed line corresponds to the optimal growth solution and the solid line to the constant saving rate model. In the optimal growth model our numerical procedure illustrates on the fact that the planner is optimally choosing the stable solution, and the algorithm succeeds in calculating the constant detrended consumption level. In order to have a constant detrended consumption, the saving rate must increase at the beginning and fluctuate around its BGP solution afterward, as it is shown in Figure 10. Alternatively, in the Solow-Swan version of the model detrended consumption is just a constant fraction of output and fluctuates likewise.

A further analysis on stability can be achieved by computing numerically a subset of the infinite roots of the homogeneous part of (16), those with a negative real part

near to zero. This analysis is related to work by Engelborghs and Roose (1999), which allows not only to detect Hopf bifurcations but also to estimate the subset of rightmost roots of a DDE. We have found that this subset is non empty and therefore supports the convergence by oscillations result in Figures 5 and 6. For the optimal growth model and the parameter values in Table 2, Figure 8 shows the real parts in the x axe and the imaginary parts in the y axe. Figure 9 does the same for the constant saving rate model and parameters in Table 1. We can evaluate the convergence speed of the economy using the computed roots: the closer to zero is the smallest real part of the nonzero computed eigenvalues, the slower is convergence. These figures confirm that the Solow-Swan version of the model converges more rapidly.

5 Implications of the model

The introduction of vintage capital into an otherwise standard AK-type optimal growth model leads to three main conclusions. First, persistent oscillations in investment can occur with concave utility when we allow for some non-smooth depreciation scheme. Second, since investment involves creation and destruction as separate activities, those oscillations are the result of replacement echoes. Third, there is a trade-off between rapid expansion and hence rapid net investment and longer lasting fluctuations; thus changes in the rate of growth will have the same qualitative effects as when the saving rate is exogenous, but these effects will be more persistent although quantitatively smaller. We now proceed to a more formal analysis of these three conclusions.

5.1 Investment and growth

The dynamic properties of the vintage AK model are very different from those of the early AK-style growth models. The question remains of whether our model can do better than the standard model in explaining some features of the empirical data. In particular, can the vintage AK model contribute to explaining deviations in trends of investment rates and growth rates consistent with the patterns in data? Jones (1995) finds in a sample of OECD countries for the 1950-1989 period that investment rate increases do not coincide with increases in GDP growth rates. In fact, for some countries the investment rate increases coincide with decreases in GDP growth rates. McGrattan (1998) argues that using only postwar data for countries at similar stages of development is likely to emphasize temporary movements in the data and so hide trends, not reveal them. By using historical data she finds that Jones' deviations from investment and growth trends are relatively short-lived, and long-lived periods of high investment rates roughly do coincide with periods of high growth. Furthermore, by looking at cross-country data in a wider range of

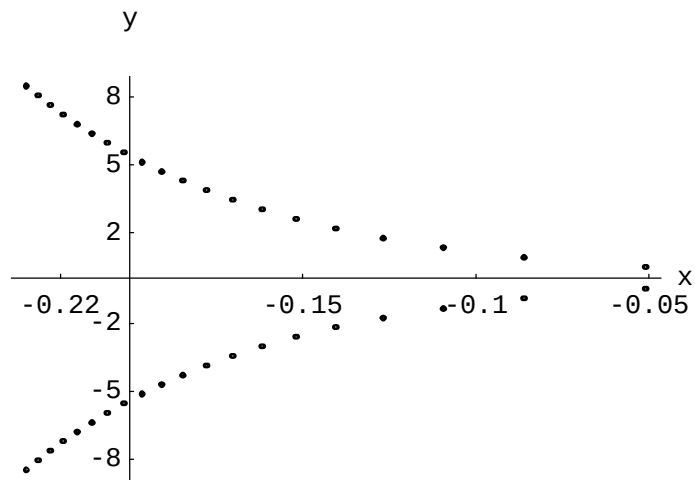


Figure 8: Eigenvalues of the optimal growth model

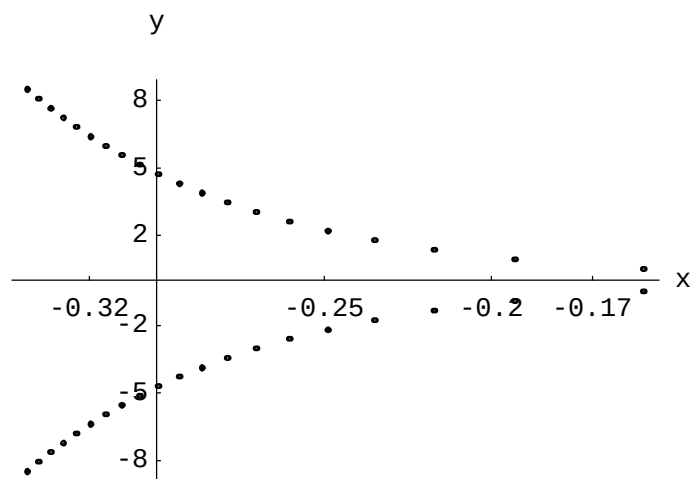


Figure 9: Eigenvalues of the constant saving rate model

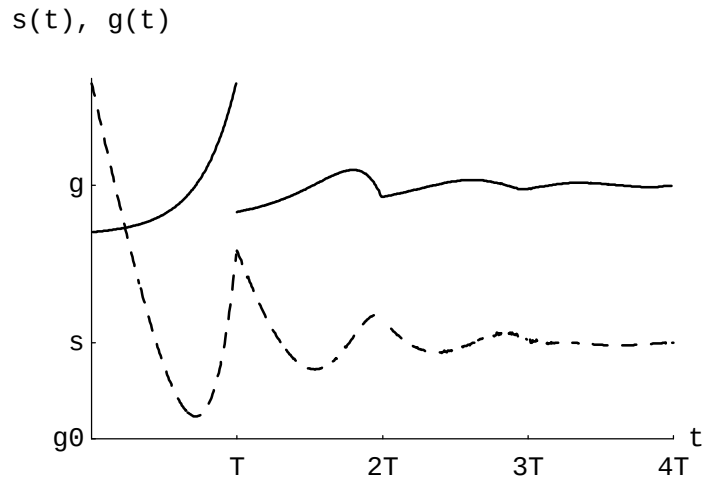


Figure 10: The growth and the saving rates

development experiences than that in the relatively advanced OECD countries she finds evidence consistent with long-run common trends.

Figure 10 summarizes the short-run dynamics of the investment share (dashed line) and the growth rate (solid line) in our model. Indeed, investment rates do not move in lock step with growth rates. The intuition is straightforward. In the standard AK model, the depreciation rate is constant and there is a linear relation between the growth and the investment rates: $g(t) = A \frac{i(t)}{y(t)}$. Consequently, both rates move in the same direction in the long and in the short run. However, in the vintage AK model this relation is non-linear:

$$g(t) = A \frac{i(t)}{y(t)} \pm \pm(t);$$

$\pm(t)$ being $A \frac{i(t - T)}{y(t)}$. In the long run the relation between both rates is positive, but in the short run the growth rate depends also upon delayed investment. Consider for instance a permanent increase in A at $t = 0$, and let us analyze the behavior of both the investment and the growth rates in the transition from a BGP to another. Initially, there is a shortage of capital that makes more profitable to save and invest: $s(0) > s$. As the capital stock increases, the incentives to save and the investment rate decrease. Concerning the growth rate, for $t \in [0; T[$ creation is larger than destruction, which makes the capital stock to increase at a rate larger than g_0 . This reduces the depreciation rate and increases the growth rate.

The observed pattern in growth and investment rates is consistent with the dynamics of the vintage AK model. As Jones (1995) points out, across OECD countries over relatively short periods investment rates are rising while growth rates are not. Of course, over long horizons, the vintage AK model predicts that higher investment rates coincide with higher growth rates. This is the evidence McGrattan

presents on trends using data for 1870-1989.

It should be stressed that the sort of fluctuations the model generates is not merely a mathematical property but derives testable implications for the vintage AK theory. Interestingly, only technological reasons are in action here. It is the echo effect due to the non-exponential depreciation assumption that explains the short-run deviations between saving rates and growth rates. In contrast, the argument suggested by McGrattan (1998) in explaining these deviations relies on fiscal policy changes affecting the capital-output ratio. In our model the output-capital ratio A remains constant by construction. Consequently, the prediction of our model is of a very different nature than the one she proposes.

Finally, our model can be seen as a limit case of the sort of specification that Benhabib and Rustichini (1991) have analyzed under the assumption of decreasing returns to capital and one-hoss shay depreciation. Recognition that persistent and robust oscillations in investment can occur in models of vintage capital due to echo effects was first made by these authors. Indeed, when returns to capital are close to unity, a one-hoss shay depreciation scheme will generate a similar behavior to our vintage AK model in the short-medium run. Consequently, as McCallum (1996) has emphasized for constant depreciation rate technologies, there is no such a quantitative difference between the one-hoss shay exogenous growth model and our AK model.⁸ However, when decreasing returns to capital are far from unity, the long run behavior of the model of Benhabib and Rustichini implies that increases in the saving rate are not associated with a long run increase in the growth rate, which is exogenously given by definition.

5.2 Physical and human capital

The vintage AK model can also be seen as a reduced form of a more general economy with both physical and human capital. This result is obtained in a one sector model using a constant returns to scale technology in both types of capital, in which output is allocated on a one-for-one basis to consumption, investment in physical capital and human capital accumulation.

As in section 2, vintages aged less than T are operative. Technology of a vintage $z \in [t - T; T]$ is given by

$$y(z) = B i(z)^{1-\theta} h(z)^{\theta}; \quad (17)$$

⁸With $y(t) = Ak(t)^{\theta}$ and $\theta = 0.99$, under parameter values as in Table 1, the behavior of the growth rate is very similar to that depicted in Figure 4 and it takes many periods to observe the convergence of the growth rate to zero. The main reason is that the steady state is well above initial conditions; thus the economy needs to grow for a long time to reach it. For small returns to capital and sufficiently low initial conditions, the growth rate is initially very high and converges monotonically (with a discontinuity at $t=T$) and very fast to zero.

where $B > 0$ and $0 < \theta < 1$. $h(z)$ represents human capital associated to vintage z . Let us assume that both physical and human capital are vintage specific and have the same lifetime $T > 0$. Machines use specific human capital, which is destroyed when machines are scrapped.

Given that both forms of capital face the same user cost, it is very easy to show that the optimal ratio of physical to human capital is $\frac{1-\theta}{\theta}$, the same for all vintages. Substituting it in (17), and aggregating over all operative plants at time t , we get (1) as the aggregate technology, where $A' = B \frac{\theta}{1-\theta}$.

We can now interpret our vintage AK model in terms of embodied technological progress. On a BGP, human capital is growing at the positive rate g . Consequently, labor associated to the representative plant of vintage z has $h(z)$ as human capital, which is larger than the human capital of all previous vintages. Under this interpretation, technical progress is embodied in new plants. Moreover, the life time of capital, T , can be interpreted as capturing some smoothing in adoption. More precisely, T introduces a lag in the diffusion of new technologies through variable depreciation. Even though it is optimal to increase the saving rate in order to profit from a rapid embodied technical progress, new technologies are only adopted by a small fraction of firms and the destruction of old technologies takes time.

When the economy faces a positive shock in A , to invest in human capital becomes more profitable, which increases the rate of technological progress and the incentives to save. It makes both the saving rate and the growth rate jump at the time of the shock. Afterwards, the saving rate decreases and converges by oscillations to its balance growth path value. The growth rate is however affected by the diffusion process of new technologies, through the simultaneous occurrence of creation and destruction. Since the capital stock is initially growing faster than during the time previous to the shock, the destruction process implies a decrease in the depreciation rate which makes the growth rate to increase even if the saving rate is decreasing.

6 Conclusions

Recent discussions on growth theory emphasize the ability of vintage capital models to explain growth facts. However, there is a small number of contributions endogenizing growth in vintage models, and most of them focus on the analysis of balanced growth paths. The model analyzed here goes part way toward developing the methods for a complete resolution of endogenous growth models with vintage capital. For analytical convenience it is limited to a case in which the engine of growth is simple: returns to capital are bounded below. However, the basic properties of the model are common to most endogenous growth models. Our framework represents a minimal departure from the standard model with linear technology: we impose

a constant lifetime for machines. Under this assumption we show that some key properties of the AK model change dramatically. In particular, convergence to the BGP is no more instantaneous. Instead, convergence is non monotonic due to the existence of replacement echoes. As a consequence, investment rates do not move in lock step with growth rates.

These findings indicate that there is much to be learned from the explicit modeling of variable depreciation rates. An obvious immediate extension of this line of research is to include an endogenous decision for the scrapping time. This is so since our numerical algorithm can be used to deal with time dependent and state dependent leads and lags. Also, a lot of our procedures should be at work when reducing the level of aggregation by thinking more carefully about the economics of technology and knowledge. Yet a model economy that includes both of these features would provide a significantly better framework for useful policy analysis. The findings obtained here should constitute an important first step toward the understanding and resolution of these more elaborate models.

Appendix

In this appendix we prove Proposition 4 and we present an outline of the algorithm used to compute equilibrium paths of the optimal growth model.

Proof of Proposition 4

(a) From (2) we can show that

$$g(0) = sA \cdot \frac{g_0 e^{i g_0 T}}{1 - e^{i g_0 T}} \quad (A1)$$

From (5), we can show that

$$g = sA \cdot \frac{g e^{i g T}}{1 - e^{i g T}} \quad (A2)$$

Since $G(g) \equiv \frac{g e^{i g T}}{1 - e^{i g T}}$ is such that $G'(g) < 0$, then $g(0) < g$. Finally, from Proposition 2, we know that the relation between g and s , implicit in (5), is decreasing. Consequently, there exists a $\bar{s} < sA$, such that

$$g_0 = a(1 - e^{i g_0 T}) = a \cdot \frac{g_0 e^{i g_0 T}}{1 - e^{i g_0 T}} < g(0):$$

(b) From (3)

$$g(t) \equiv \frac{i'(t)}{i(t)} = sA \cdot \frac{i(t + T)}{i(t)}:$$

Differentiating with respect to time gives, for all $t \in [0; T[$

$$g'(t) = g(t) - g_0$$

Since $g(0) > g_0$, $g'(t) > 0 \forall t \in [0; T[$.

(c) Given that $H'(g) < 0$ and $g_0 < g$, from (4) and (5), $i(0) > \lim_{t \rightarrow 0^+} i_0(t) = 1$. From (3), $i'(t)$ has a discontinuity at $t = T$.

(d) Combining (A1) and (A2), we get

$$g - g(0) = G(g_0) - G(g) > 0$$

At given g_0 , an increase in g rises $g - g(0)$ since $G'(g) < 0$. $\square$

Algorithm

The planner's problem can be reformulated in terms of variables for which its long-run is known.

Let define $\bar{i}(t) = \frac{i(t)}{i_0(T)}$ and $z(t) = \frac{y(t)}{i(t)}$, then (8) reads:

$$\max_{\bar{i}} \int_0^T \frac{[z(t) - 1]^{1-\frac{1}{4}}}{1 - \frac{1}{4}} \bar{i}(t)^{1-\frac{1}{4}} e^{-\frac{1}{2}t} dt$$

subject to

$$z(t) = A \int_{t-T}^t \frac{\bar{i}(z)}{\bar{i}(t)} dz \quad (A3)$$

$$\frac{\bar{i}'(t)}{\bar{i}(t)} = g(t) \quad (A4)$$

given initial conditions $\bar{i}(t) = i_0(t) = \frac{i_0(t)}{i_0(T)}$, 0 for all $t < 0$

The numerical procedure operates on this transformation of the problem and the optimization relies upon the objective. In line with the cyclic coordinate descent algorithm proposed by Boucekkine, Germain, Licandro and Magnus (1999), the unknowns are replaced by piecewise constants on intervals $(0; \Phi)$, $(\Phi; 2\Phi)$, ..., and iterations are performed to find a fixed-point $g(t)$ (and/or state variable $i(t); y(t)$) vector up to tolerance parameter 'Tol'. An outline of the algorithm used to compute an approximate solution of problem above is the following:

Step 1: Initialize $g^0(t)$, the base of the relaxation, with dimension K sufficiently large. For $t \in [K; N]$, $N > K$ and large enough, set $g(t) = g$ (the BGP solution). Notice that knowing $g(t)$ we can compute $\bar{i}(t)$ and $z(t)$ using (A3) and (A4).

Step 2: Maximization step by step:

- ² Step 2.0: maximize with respect to coordinate g_0 keeping unchanged coordinates g_i , $i > 0$
- ² Step 2.k: maximize with respect to coordinate g_k keeping unchanged coordinates g_i , $i > k$, with coordinates g_i , $0 \leq i \leq k-1$ updated
- ² Step 2.K: last $k < K$ step, get $g^1(t)$

Note that at each k step states must be updated.

Step 3: If $g^1(t) = g^0(t)$, we are done. Else update $g^0(t)$ and go to Step 2.

Table 3: Algorithm parameters

N	K	ϕ	Tol
10 T	4 T	0:1	10^{-5}

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