Chapter 3: Numerical resolution

The systems of equations proposed to model geomorphological flow have been described in chapter 2. Owing to their mathematical complexity, those differential systems cannot be solved analytically. Then, a finite-volume approach is adopted to discretise the Saint-Venant – Exner and the two-layer model equations described in chapter 2, which can be written under the general formulation

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}+\mathbf{H}(\mathbf{U})\,\frac{\partial \mathbf{G}(\mathbf{U})}{\partial x}=\mathbf{S}(\mathbf{U}) \tag{3.1}$$

The system is solved in two steps, the first solving the hyperbolic part of the system and the second one the source terms. The initial value of the main unknowns, at time t, is $\mathbf{U}_1$. The solution $\mathbf{U}_2$, obtained from the hyperbolic resolution (step 1), is used as starting point for the source terms treatment (second step), giving the final solution $\mathbf{U}_3$ at time $t + dt$:

$$\left(\frac{\partial \mathbf{U}}{\partial t}\right)_{1\to 2}+\frac{\partial \mathbf{F}(\mathbf{U}_1)}{\partial x}+\mathbf{H}(\mathbf{U}_1)\,\frac{\partial \mathbf{G}(\mathbf{U}_1)}{\partial x}=0 \tag{3.2}$$

$$\left(\frac{\partial \mathbf{U}}{\partial t}\right)_{2\to 3}=\mathbf{S}\left(\mathbf{U}_{2\to 3}\right) \tag{3.3}$$

First, the hyperbolic left-hand side of (3.1) is integrated using the Godunov scheme developed by Harten, Lax and van Leer (1983). This method allows to solve the conservative part of the system (3.2), it was extended by Fraccarollo et al. (2003) to take into account the non-conservative products $\mathbf{H}(\mathbf{U})\,\partial\mathbf{G}(\mathbf{U})/\partial x$ (pressure on the sloped interface) as additional terms (called lateralised terms) of the fluxes. This is distinct from some other methods, which include them in the source terms.

Then, the source treatment (3.3) is exposed for both models. Once more, as far as the two-layer model is concerned, a splitting strategy has to be adopted (Capart, 2004 and Spinewine 2005). The contents of this chapter are not new theory, most of the following developments can be found in the literature. Nevertheless, the HLL solver, its extensions and the source treatment are exposed in a unified way, in an attempt to be as clear as possible.

3.1. Hyperbolic solver

Neglecting the non-conservative product $\mathbf{H}(\mathbf{U})\,\partial\mathbf{G}(\mathbf{U})/\partial x$ and the source terms $\mathbf{S}(\mathbf{U})$, the two layer equations (3.1) are reduced to a conservative form

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}=0 \tag{3.4}$$

As demonstrated previously by other authors (see for example Soares, 2002), an integrate form of the reduced conservative system (3.4) can be obtained by using Stokes theorem, which states that for every vector $\mathbf{V}$

$$\int_{\Gamma} \mathbf{V} d\mathbf{\Gamma} = \int_{\Omega} \mathrm{rot}\mathbf{V} d\mathbf{\Omega} \tag{3.5}$$

where $\mathbf{\Omega}$ is the integration domain and $\mathbf{\Gamma}$ its contour.

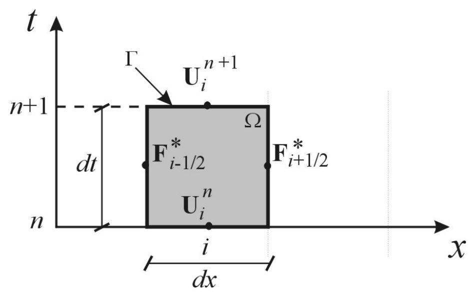

Figure 3.1. Computational cell, integration domain

The computational domain is discretised in finite volumes (Fig.3.1), in such a way that the general variables appearing in (3.5) are in the present case

$$\mathbf{V} = \begin{pmatrix} \mathbf{U} \\ -\mathbf{F} \\ 0 \end{pmatrix} \qquad d\mathbf{\Gamma} = \begin{pmatrix} dx \\ dt \\ 0 \end{pmatrix} \qquad d\mathbf{\Omega} = \begin{pmatrix} 0 \\ 0 \\ dxdt \end{pmatrix}$$

$$\mathrm{rot}\mathbf{V} = \begin{pmatrix} 0 \\ 0 \\ -\dfrac{\partial \mathbf{F}}{\partial x} - \dfrac{\partial \mathbf{U}}{\partial t} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \tag{3.6}$$

The system (3.4) is integrated along the space-time contour of a computational cell.

$$\int_{\Gamma} \left(\mathbf{U}\, dx - \mathbf{F}\, dt \right) = 0 \tag{3.7}$$

In a first-order approach, it is assumed that, for a fixed time *n*, each component of the vector **U**, containing the main variables, has a constant value in each cell *i*. The average value of the fluxes crossing an interface during a time interval *dt* is denoted by the superscript * (Fig 3.1).

$$\mathbf{U}_i^n\, dx - \mathbf{U}_i^{n+1} dx - \mathbf{F}_{i+1/2}^{*}\, dt + \mathbf{F}_{i-1/2}^{*}\, dt = 0 \tag{3.8}$$

This relation allows to calculate the solutions of the unknowns $\mathbf{U}_\mathrm{i}$ at time n+1 from the values of $\mathbf{U}_\mathrm{i}$ at time *n* and the fluxes **F*** across the interfaces neighbouring the cell *i*.

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{dt}{dx}\left(\mathbf{F}_{i-1/2}^{*} - \mathbf{F}_{i+1/2}^{*} \right) \tag{3.9}$$

To solve the general expression of the finite-volume scheme (3.9), the numerical fluxes **F*** at each computational cell have to be calculated.

3.1.1. Flux prediction HLL

Various methods are proposed in the literature to predict the fluxes at the interface of the computational cells (for example the predictors proposed by Roe (1981) or Braschi and Gallati (1992)). The HLL solver detailed here was proposed by Harten, Lax and van Leer (1983) (see also Harten, 1983). The problem to solve is discretised in computational cells (Fig. 3.2). The variable vector **U** has a constant value in each cell, the constant states are separated by interfaces crossed by a flux **F***.

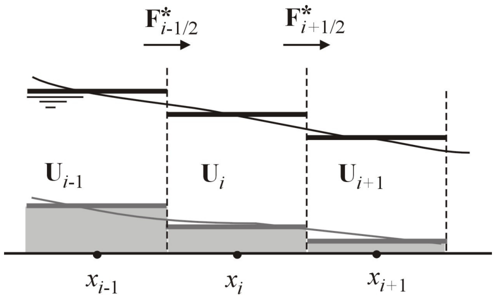

Figure 3.2. Computational cells

At each cell interface, there is a Riemann problem (Fig 3.3): a discontinuity between two constant states $\mathbf{U}_R$ (in the right cell) and $\mathbf{U}_L$ (in the left cell).

Information concerning the discontinuity propagates along the characteristic curves in the *x*-*t* plane. Along these curves, the partial differential equations (3.4) become ordinary differential equations called the compatibility relations. If the time step is short enough, the characteristic curves can be approximated by straight lines with constant slope λ_i, which are the n eigenvalues of the system (3.4). The eigenvalues λ_i are ranked from the smallest one $i = 1$ to the largest one $i = n$. The area disturbed by the discontinuity after a time dt (at time $n + 1$) is determined by the extreme characteristics, with the minimum slope upstream

$$S_L = \min(\lambda_1, 0) \tag{3.10}$$

and with the maximum slope downstream

$$S_R = \max(\lambda_n, 0) \tag{3.11}$$

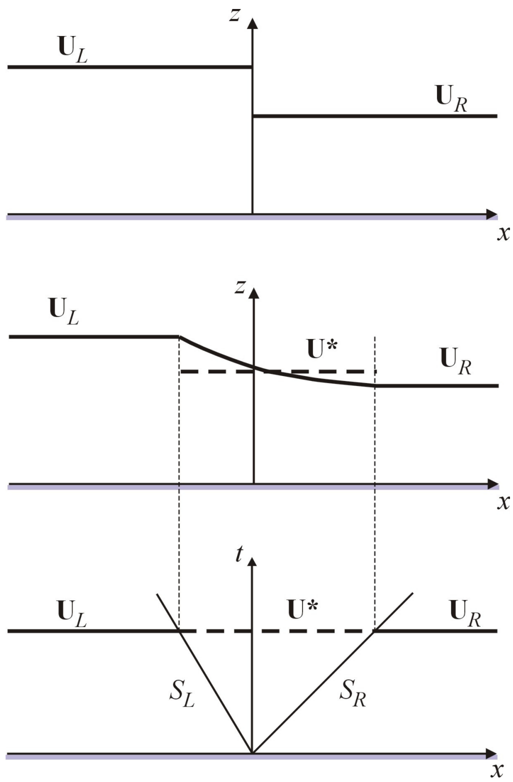

Figure 3.3. Riemann problem

The variation of **U** in this transition area, in time n+1, is a continuous curve linking the two constant states $\mathbf{U}_R$ and $\mathbf{U}_L$, it is approximated by a constant intermediate state **U***.

$$\mathbf{U}^* = \frac{1}{(S_R - S_L)dt} \int_{x_L}^{x_R} \mathbf{U}(x, t_{n+1})\, dx \qquad (3.12)$$

where $x_L = S_L\, dt$ and $x_R = S_R\, dt$ are the limits in space of the constant state **U*** (Fig. 3.4). This approximation, proposed in the HLL method, results in considering three constant states at time $n+1$ separated by two discontinuities corresponding to the extreme characteristics.

$$\mathbf{U}(x,t_{n+1}) = \begin{cases} \mathbf{U}_L & \text{if} \quad \dfrac{x}{dt} \le S_L \\ \mathbf{U}^* & \text{if} \quad S_L \le \dfrac{x}{dt} \le S_R \\ \mathbf{U}_R & \text{if} \quad \dfrac{x}{dt} \ge S_R \end{cases} \tag{3.13}$$

The time and space steps *dt* and *dx* have to satisfy the classical CFL (Courant-Friedrichs-Lewy) condition (Hirsch, 1997)

$$dt < \frac{dx}{\max\left(S_R, \mathrm{abs}(S_L)\right)} \tag{3.14}$$

It implies the size of the space step *dx* has to be greater than the transition area after a time *dt* in the upstream cell ($x_L = S_L dt$) and in the downstream cell ($x_R = S_R\, dt$).

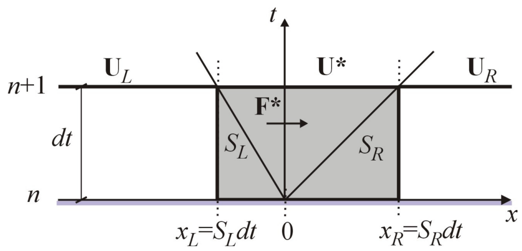

Figure 3.4. Integration domain

The system (3.4) is integrated, along the space-time contour $[x_L, x_R]$, $[t_n, t_{n+1}]$ bounding the grey surface sketched on Figure (3.4), using (3.7).

$$\int_{x_L}^{x_R} \mathbf{U}(x,t_{n+1})dx = \int_{x_L}^{x_R} \mathbf{U}(x,t_n)dx + \int_{t_n}^{t_{n+1}} \mathbf{F}\left(\mathbf{U}(x_L,t)\right)dt - \int_{t_n}^{t_{n+1}} \mathbf{F}\left(\mathbf{U}(x_R,t)\right)dt \tag{3.15}$$

If the definitions $\mathbf{F}_L = \mathbf{F}(\mathbf{U}_L)$ and $\mathbf{F}_R = \mathbf{F}(\mathbf{U}_R)$, with $\mathbf{U}_L = \mathbf{U}(x_L, t_n)$ and $\mathbf{U}_R = \mathbf{U}(x_R, t_n)$, it becomes

$$\int_{x_L}^{x_R} \mathbf{U}(x,t_{n+1})dx = x_R\mathbf{U}_R - x_L\mathbf{U}_L + dt\left(\mathbf{F}_L - \mathbf{F}_R\right). \tag{3.16}$$

The left hand-side member of (3.16) can also be written

$$\int_{x_L}^{x_R} \mathbf{U}(x, t_{n+1})\,dx = (S_R - S_L)\,dt\,\mathbf{U}^* \tag{3.17}$$

Comparing (3.16) to (3.17), an expression of the constant intermediate state **U*** is obtained

$$\mathbf{U}^* = \frac{S_R \mathbf{U}_R - S_L \mathbf{U}_L + \mathbf{F}_L - \mathbf{F}_R}{S_R - S_L} \tag{3.18}$$

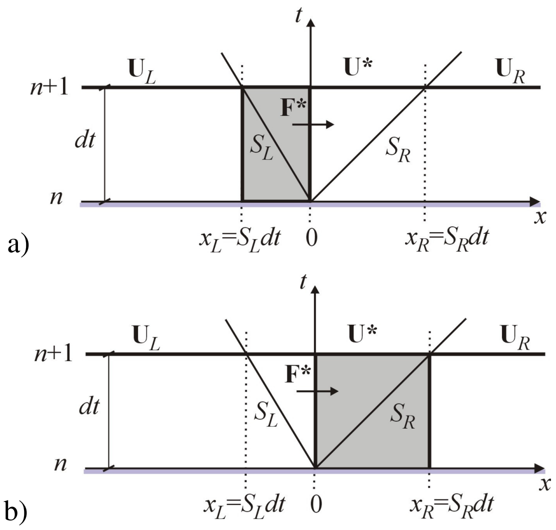

Figure 3.5. Integration domain

The flux at the interface **F***, calculated as an average value on dt, is deduced from the integral expression (3.7) along the space-time contour $[x_L, 0]$, $[t_n, t_{n+1}]$ (Fig. 3.4).

$$\mathbf{F}^* = \mathbf{F}_L - S_L \mathbf{U}_L - \frac{1}{dt}\int_{x_L}^{0} \mathbf{U}(x, t_{n+1})\,dx \tag{3.19}$$

Knowing that

$$\frac{1}{dt}\int_{x_L}^{0} \mathbf{U}(x, t_{n+1})\,dx = -\frac{S_L\,dt\,\mathbf{U}^*}{dt}, \tag{3.20}$$

expression (3.19) becomes

$$\mathbf{F}^* = \mathbf{F}_L + S_L\left(\mathbf{U}^* - \mathbf{U}_L\right) \tag{3.21}$$

and, using (3.18)

$$\mathbf{F}^* = \frac{S_R\mathbf{F}_L - S_L\mathbf{F}_R + S_L S_R\left(\mathbf{U}_R - \mathbf{U}_L\right)}{S_R - S_L} \tag{3.22}$$

The flux at the interface **F*** could also be identically deduced from the integral expression (3.7) along the space-time contour [0, x_R], [t_n, t_{n+1}] (Fig. 3.4). This general expression (3.22) can be simplified, depending on the signs of the extreme characteristic slopes S_L and S_R.

$$\mathbf{F}^* = \begin{cases} \mathbf{F}_L & \text{if} \quad S_L \geq 0 \\ \dfrac{S_R\mathbf{F}_L - S_L\mathbf{F}_R + S_L S_R\left(\mathbf{U}_R - \mathbf{U}_L\right)}{S_R - S_L} & \text{if} \quad S_L \leq 0 \leq S_R \\ \mathbf{F}_R & \text{if} \quad S_R \leq 0 \end{cases} \tag{3.23}$$

It can be checked that for pure hydrodynamic Saint-Venant equations, the expression of the interface flux **F*** (3.23) is identical to the one proposed by the Pavia scheme (Braschi and Gallati, 1992). In this particular case the extreme eigenvalues are $\lambda_{\min} = u - c$ and $\lambda_{\max} = u + c$ inducing

$$S_L = \min\left(\lambda_{\min}, 0\right) = \min\left(u - c,\, 0\right)$$

$$S_R = \max\left(\lambda_{\max}, 0\right) = \max\left(u + c,\, 0\right) \tag{3.24}$$

As sediments are added, the eigenstructure of the equations becomes more elaborate, with multiple eigenvalues.

When these eigenvalues have an analytical solution, as it is the case for the two-layer one-velocity model (2.106-2.109), the expressions of the extreme eigenvalues are directly used to find S_L and S_R.

$$S_L = \min\left(u_m - \sqrt{g\,\frac{h_m^2 + r\,h_s^2}{h_m + r\,h_s}}\,,\,0 \right)$$

$$S_R = \max\left(u_m + \sqrt{g\,\frac{h_m^2 + r\,h_s^2}{h_m + r\,h_s}}\,,\,0 \right) \tag{3.25}$$

When no simple analytical expression is available, as it is the case for the Saint-Venant – Exner (2.21-2.22) or the two-layer two-velocity model (2.73-2.77), S_L and S_R have to be calculated on the basis of simplified expressions or numerical evaluations of the eigenvalues.

In the case of the Saint-Venant – Exner system, several alternatives are proposed. The first approach consists in neglecting the effect of sediments into the information propagation. As exposed previously (chapter 2, section 2.2.2), for all flow regimes the maximum eigenvalue could be approximated by the hydrodynamic maximum eigenvalue $u + c$. The minimum eigenvalue could be approximated by the hydrodynamic minimum eigenvalue $u - c$ for subcritical flows ($\mathrm{Fr}_w < 1$) and by zero (as it is very small) for supercritical flows. It leads to the expressions of S_L and S_R (3.25) in the hydrodynamic case. A second possibility is to use the more precise approximated analytical expression (2.139) taking into account the contribution of sediments and giving

$$S_L = \lambda_{\min} = \frac{1}{2}\left(u - c - \sqrt{(u-c)^2 + 4\,\frac{1}{(u+c)}\,\frac{g}{(1-\varepsilon_0)}\left(u\,\frac{\partial q_s}{\partial u} - h\,\frac{\partial q_s}{\partial h} \right)} \right)$$

$$S_R = \lambda_{\max} = u + c \tag{3.26}$$

As the extreme eigenvalues $\lambda_{\min}$ and $\lambda_{\max}$ are respectively always negative and always positive, there is no need to compare them with zero. Another solution, more expensive in computational time, consists in evaluating the extreme eigenvalues by the numerical resolution of the characteristic polynomial at each computational cell and at each time step. The choice of the expression of S_L and S_R may have an important influence on the results. As illustrated on a practical example in chapter 7 section 7.2.2.4, the simplified approach based on the hydrodynamic case may generate instabilities. Indeed, the coupling introduced by the use of the third eigenvalue is necessary.

In the case of the two-layer two-velocity model, the numerical calculation of the eigenvalues is too expensive in computational time due to the important sizes of the matrix to treat and approximations of the extreme eigenvalues are preferred.

Capart and Young (2002) proposed the following estimations for S_L and S_R.

$$S_L = \min\left(\min(u_w, u_s) - \sqrt{g\,(h_w + h_s)},\, 0\right)$$

$$S_R = \max\left(\max(u_w, u_s) + \sqrt{g\,(h_w + h_s)},\, 0\right) \tag{3.27}$$

The main advantage of the HLL solver is that only the extreme positive and negative celerities S_R (maximum speed of a wave propagating downstream) and S_L(maximum speed of a wave propagating upstream) are needed. In case of complex systems dealing with sediments, the eigenstructure is based on multiple eigenvalues, a part of the information is propagated along the intermediate characteristics. Considering only the extreme celerities thus constitutes an approximation aiming at simplifying the resolution. As proposed further for the case of a three-eigenvalue system, it would be possible to take the intermediate celerity into account.

3.1.2. Lateralised Flux, LHLL method

The HLL method was extended by Fraccarollo et al. (2003) to take into account the non-conservative products $\mathbf{H}(\mathbf{U})\,\partial\mathbf{G}(\mathbf{U})/\partial x$ (pressure on the sloping interface) of equations (3.1) as additional terms of the fluxes. This is distinct from some other methods, which include them in the source terms.

This extended hyperbolic solver allows to solve both the two-layer model and the Saint-Venant – Exner equations (neglecting the source terms) written under the formulation

$$\frac{\partial \mathbf{U}}{\partial t}+\frac{\partial \mathbf{F}(\mathbf{U})}{\partial x}+\mathbf{H}(\mathbf{U})\,\frac{\partial \mathbf{G}(\mathbf{U})}{\partial x}=0 \tag{3.28}$$

The non-conservative products $\mathbf{H}(\mathbf{U})\,\partial\mathbf{G}(\mathbf{U})/\partial x$ in the equation systems developed in chapter 2 are equal to zero for continuity equations and different from zero for momentum equations. The non-conservative products are taken into account as additional parts (called lateralised part) of the fluxes. Two computational cells separated by an interface are considered (Fig. 3.6), there is a constant state $\mathbf{U}_L$ in the cell on the left of the interface and a constant state $\mathbf{U}_R$ in the cell on the right.

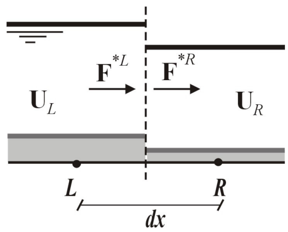

Figure 3.6. Lateralised flux, variables

The flux through the left side of this interface $\mathbf{F}^{*L}$ is artificially not equivalent to the flux through the right side of the interface $\mathbf{F}^{*R}$. In other words, the flux flowing out of the cell located upstream (left) at the interface is not identical to the one flowing into the cell located downstream (right). The non-conservative products $\mathbf{H}(\mathbf{U})\,\partial\mathbf{G}(\mathbf{U})/\partial x$, discretised on *dx*, are introduced as the artificial difference between $\mathbf{F}^{*L}$ and $\mathbf{F}^{*R}$.

$$\mathbf{F}^{*L}-\mathbf{F}^{*R}=\mathbf{H}^{*}\left(\mathbf{G}_R-\mathbf{G}_L\right) \tag{3.29}$$

where **H*** is the arithmetic average between the values of **H** in the neighbouring cells.

$$\mathbf{H}^{*}=\frac{\mathbf{H}_L+\mathbf{H}_R}{2} \tag{3.30}$$

where $\mathbf{H}_L = \mathbf{H}(\mathbf{U}_L)$, $\mathbf{H}_R = \mathbf{H}(\mathbf{U}_R)$, $\mathbf{G}_L = \mathbf{G}(\mathbf{U}_L)$ and $\mathbf{G}_R = \mathbf{G}(\mathbf{U}_R)$.

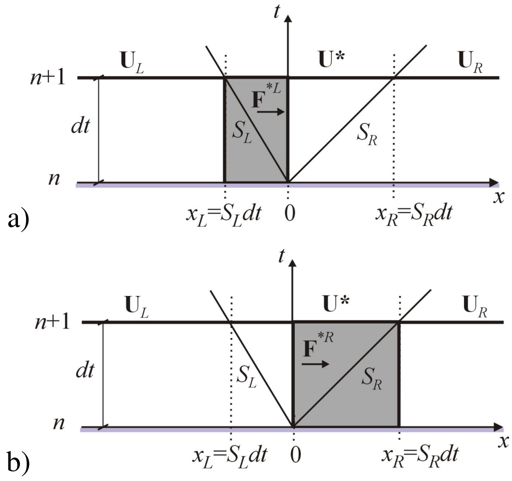

Figure 3.7. Integration domain

The flux at the left side of the interface $\mathbf{F}^{*L}$ is deduced from the integral expression (3.7) along the space-time contour $[x_L dt, 0]$, $[t_n, t_{n+1}]$ (Fig. 3.7a)

$$\mathbf{F}^{*L} = \mathbf{F}_L + S_L(\mathbf{U}^* - \mathbf{U}_L) \tag{3.31}$$

The flux at the right side of the interface $\mathbf{F}^{*R}$ is deduced from the integral expression (3.7) along the space-time contour $[0, x_R dt]$, $[t_n, t_{n+1}]$ (Fig. 3.7b)

$$\mathbf{F}^{*R} = \mathbf{F}_R + S_R(\mathbf{U}^* - \mathbf{U}_R) \tag{3.32}$$

If $\mathbf{U}^*$ is isolated from (3.31) and introduced in (3.32), an expression linking the fluxes corresponding to both sides of the interface $\mathbf{F}^{*L}$ and $\mathbf{F}^{*R}$ is obtained

$$\mathbf{F}^{*R} = \mathbf{F}_R + S_R(\mathbf{U}_L - \mathbf{U}_R) + \frac{S_R}{S_L}\left(\mathbf{F}^{*L} - \mathbf{F}_L\right) \tag{3.33}$$

The expression of the flux at the right side of the interface (3.33) is substituted in (3.29) to give the expression of the flux at the left side of the interface, only depending on the left and right cells states

$$\mathbf{F}^{*L} = \frac{S_R \mathbf{F}_L - S_L \mathbf{F}_R + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L} - \frac{S_L}{S_R - S_L} \frac{\mathbf{H}_L + \mathbf{H}_R}{2} (\mathbf{G}_R - \mathbf{G}_L) = \mathbf{F}^* - \frac{S_L}{S_R - S_L} \frac{\mathbf{H}_L + \mathbf{H}_R}{2} (\mathbf{G}_R - \mathbf{G}_L) \tag{3.34}$$

By the same way

$$\mathbf{F}^{*R} = \mathbf{F}^* - \frac{S_R}{S_R - S_L} \frac{\mathbf{H}_L + \mathbf{H}_R}{2} (\mathbf{G}_R - \mathbf{G}_L) \tag{3.35}$$

These general expressions can be simplified, depending on the signs of the extreme characteristic slopes, from which S_L (3.10) and S_R (3.11) (if $\lambda_1 > 0$ then $S_L = 0$ and if $\lambda_n < 0$ then $S_R = 0$):

$$\mathbf{F}^{*L} = \begin{cases} \mathbf{F}_L & \text{if} \quad \lambda_1 \geq 0 \\ \dfrac{S_R \mathbf{F}_L - S_L \mathbf{F}_R + S_L S_R (\mathbf{U}_R - \mathbf{U}_L)}{S_R - S_L} - \dfrac{S_L}{S_R - S_L} \dfrac{\mathbf{H}_L + \mathbf{H}_R}{2} (\mathbf{G}_R - \mathbf{G}_L) & \text{if} \quad \lambda_1 \leq 0 \leq \lambda_n \\ \mathbf{F}_R + \dfrac{\mathbf{H}_L + \mathbf{H}_R}{2} (\mathbf{G}_R - \mathbf{G}_L) & \text{if} \quad \lambda_n \leq 0 \end{cases} \tag{3.36}$$

$$\mathbf{F}^{*R} = \begin{cases} \mathbf{F}_L - \dfrac{\mathbf{H}_L + \mathbf{H}_R}{2}\left(\mathbf{G}_R - \mathbf{G}_L\right) & \text{if} \quad \lambda_1 \geq 0 \\[2ex] \dfrac{S_R \mathbf{F}_L - S_L \mathbf{F}_R + S_L S_R \left(\mathbf{U}_R - \mathbf{U}_L\right)}{S_R - S_L} & \\ \qquad - \dfrac{S_R}{S_R - S_L} \dfrac{\mathbf{H}_L + \mathbf{H}_R}{2}\left(\mathbf{G}_R - \mathbf{G}_L\right) & \text{if} \quad \lambda_1 \leq 0 \leq \lambda_n \\[2ex] \mathbf{F}_R & \text{if} \quad \lambda_n \leq 0 \end{cases} \tag{3.37}$$

If the non-conservative product $\mathbf{H}(\mathbf{U})\,\partial\mathbf{G}(\mathbf{U})/\partial x$ is zero, the expression of the fluxes (3.36) and (3.37) is reduced to the simpler expression (3.23) obtained previously for the conservative form. To take the eventual non-conservative product into account by the way of the lateralised fluxes, the general expression of the finite-volume (3.9) has to be modified to become

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^n + \frac{dt}{dx}\left(\mathbf{F}_{i-1/2}^{*R} - \mathbf{F}_{i+1/2}^{*L}\right) \tag{3.38}$$

3.1.3. Intermediate eigenvalue, HLLC method

In the HLL solver, only the extreme celerities are taken into account, the effect of the intermediate celerities is neglected. If the equation system has more than two eigenvalues, different constant states delimited by the characteristic lines should be taken into account (Fig. 3.8). The way to take into account the intermediate eigenvalue is exposed by Toro (1997) in the particular case of a system with three eigenvalues. In this case two constant states $\mathbf{U}_1$* and $\mathbf{U}_2$* and the intermediate eigenvalue S* have to be considered (Fig. 3.9).

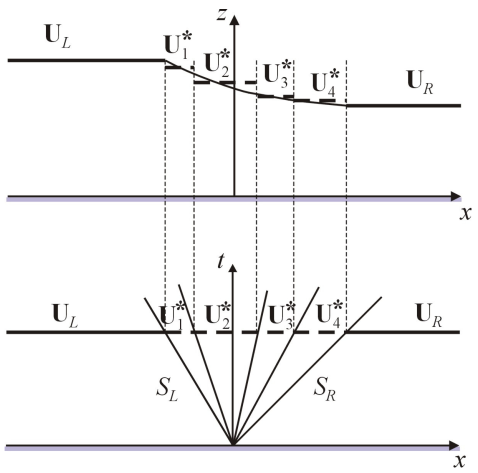

Figure 3.8. Riemann problem, multiple eigenvalues

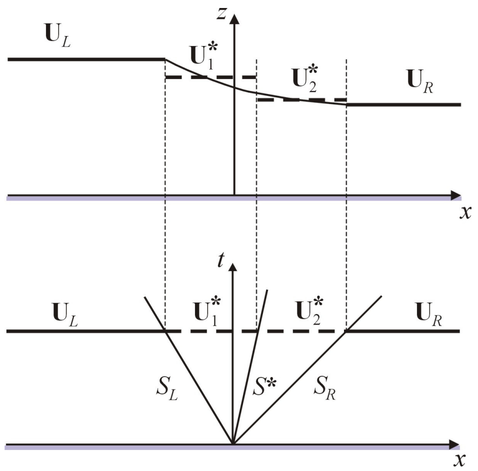

Figure 3.9. Riemann problem, 3 eigenvalues

Identically to the unique constant state **U*** in the common HLL approach (3.12), the two constant states $\mathbf{U}_1$* and $\mathbf{U}_2$* can be defined

$$\mathbf{U}_1^* = \frac{1}{(S^* - S_L)dt} \int_{x_L}^{x^*} \mathbf{U}(x, t_{n+1})dx \tag{3.39}$$

$$\mathbf{U}_2^* = \frac{1}{(S_R - S^*)dt} \int_{x^*}^{x_R} \mathbf{U}(x, t_{n+1})dx \tag{3.40}$$

where $x_L = S_L dt$, $x_R = S_R\,dt$ and $x^* = S^*dt$ are the limits in space of the constant states $\mathbf{U}_1$* and $\mathbf{U}_2$* (Fig. 3.9). The variation of **U** around the discontinuity is simplified as

$$\mathbf{U}(x,t_{n+1}) = \begin{cases} \mathbf{U}_L & \text{if} \quad \dfrac{x}{dt} \le S_L \\ \mathbf{U}_1^* & \text{if} \quad S_L \le \dfrac{x}{dt} \le S^* \\ \mathbf{U}_2^* & \text{if} \quad S^* \le \dfrac{x}{dt} \le S_R \\ \mathbf{U}_R & \text{if} \quad \dfrac{x}{dt} \ge S_R \end{cases} \tag{3.41}$$

In the expression of the single constant state **U*** (3.12), the left-hand side integral can be split

$$\begin{aligned} \mathbf{U}^* &= \frac{1}{(S_R - S_L)dt} \int_{x_L}^{x_R} \mathbf{U}(x,t_{n+1})dx \\ &= \frac{1}{(S_R - S_L)dt} \left(\int_{x_L}^{x^*} \mathbf{U}(x,t_{n+1})dx + \int_{x^*}^{x_R} \mathbf{U}(x,t_{n+1})dx \right) \end{aligned} \tag{3.42}$$

A relation between the different constant states is obtained by introducing (3.39) and (3.40) in (3.42)

$$\mathbf{U}^* = \frac{(S^* - S_L)\mathbf{U}_1^* + (S_R - S^*)\mathbf{U}_2^*}{(S_R - S_L)} \tag{3.43}$$

If the intermediate celerity is positive, the flux at the interface **F*** (Fig.3.10a) is deduced from the integral expression (3.7) along the space-time contour $[x_L, 0]$, $[t_n, t_{n+1}]$.

$$\mathbf{F}^* = \mathbf{F}_L + S_L\left(\mathbf{U}_1^* - \mathbf{U}_L\right) \tag{3.44}$$

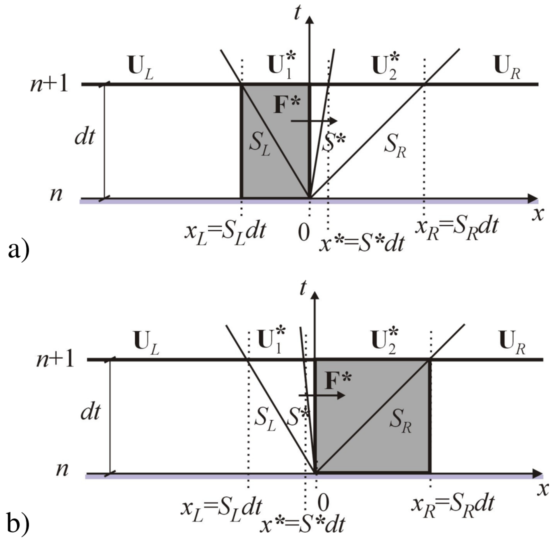

Figure3.10. Integration domain

If the intermediate celerity is negative, the flux at the interface **F*** (Fig. 3.10b) is deduced from the integral expression (3.7) along the space-time contour $[0, x_R]$, $[t_n, t_{n+1}]$.

$$\mathbf{F}^* = \mathbf{F}_R + S_R\left(\mathbf{U}_2^* - \mathbf{U}_R\right) \tag{3.45}$$

Independently of the sign of the intermediate celerity, two equations (3.43) and (3.44) or (3.45) are available to find the three unknowns **F*** $\mathbf{U}_1$* and $\mathbf{U}_2$*. An additional relation has to be introduced, for example an assumption on the constant states $\mathbf{U}_1$* or $\mathbf{U}_2$*. Inspired from this reasoning, Goutière et al. (submitted) proposed a formulation, taking the intermediate eigenvalue into account, for the sediment flux of the Saint-Venant – Exner model, this formulation will be presented in chapter 7, section 7.3.2.3.

More recently, Rosatti and Fraccarollo (2006) proposed a new solver, allowing to take the intermediate eigenvalue into account. They apply this new solver to the two-layer model assuming that the velocities are identical in both layers. In the present thesis, for the two-layer model, only the extreme characteristics are taken into account.

3.2. Source terms

Once the hyperbolic part of the system is solved , the effects of the source terms have to be taken into account and the following system (3.3) has to be solved.

$$\left(\frac{\partial \mathbf{U}}{\partial t}\right)_{2\to 3} = \mathbf{S}\left(\mathbf{U}_{2\to 3}\right) \tag{3.46}$$

The initial state $\mathbf{U}_2$ corresponds to the solution obtained for the hyperbolic part of the problem, state 2 corresponds to time t and state 3 corresponds to time $t + dt$. The source terms treatment is quite easy in the case of simple equation systems like Saint-Venant – Exner. This resolution becomes more elaborate in the case of the two layer models. Capart (2000) proposed to split the problem in several appropriate components, Spinewine (2005) exposed this method in detail in the case of the two-velocity / two-concentration model.

3.2.1. Saint-Venant – Exner

The system to solve (3.46) is expressed in the case of the Saint-Venant – Exner equations (2.21-2.22)

$$\frac{\partial}{\partial t}\begin{pmatrix} h \\ u \\ z_b \end{pmatrix} = \begin{pmatrix} 0 \\ \dfrac{-\tau}{\rho} \\ 0 \end{pmatrix} \tag{3.47}$$

where the friction shear stress τ (2.12) is based, for example, on the Manning formulation (2.13), which gives in the case of a unit-width flow

$$\tau = \rho_m \, h S_f = \frac{\rho_m n^2 u^2}{h^{1/3}} \tag{3.48}$$

where the variables h, u and the volumetric mass ρ_m correspond the mixture (sediment and water). The two equations of the system (3.47) corresponding to $\partial h/\partial t = 0$ and $\partial z_b/\partial t = 0$ indicates that the total depth h and the bed level z_b are not varying during the source terms treatment. The erosion and deposition process based on a transport formula take place during the hyperbolic resolution. The only variable changing with the source terms treatment is the velocity u, it is obtained by solving the remaining equation.

$$\frac{\partial u}{\partial t} = \frac{-\tau}{\rho_m} = -\frac{n^2 u^2}{h^{1/3}} \tag{3.49}$$

The initial state $\mathbf{U}_2$ corresponds to the known solution resulting from the hyperbolic treatment (state 2), the solution after the source treatment corresponds to the final state $\mathbf{U}_3$ (state 3). The system (3.47) can be discretised in order to find the solution at the final state $\mathbf{U}_3$. Knowing the intermediate state $\mathbf{U}_t$, the system can be solved easily using the explicit Euler scheme

$$h_{t+dt} = h_t \qquad u_{t+dt} = u_t + dt\,\frac{n^2 u_t^2}{h_t^{1/3}} \qquad z_{b,\,t+dt} = z_{b,\,t} \tag{3.50}$$

or using an implicit formulation, resolving a second-degree equation to find u_{t+dt}

$$h_{t+dt} = h_t \qquad u_{t+dt} = u_t + dt\,\frac{n^2 u_{t+dt}^2}{h_t^{1/3}} \qquad z_{b,\,t+dt} = z_{b,\,t} \tag{3.51}$$

3.2.2. Two-layer models

3.2.2.1. Two-layer source terms

The two-layer model equations were developed under various assumptions in chapter 2, the system to solve is depending on these assumptions. In the more complete case, taking into account different velocities in the water and in the transport layer and considering a change in concentration when sediment are in movement (Spinewine, 2005), according to (2.83-2.85) the system (3.46) to solve becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} h_w \\ h_s \\ z_b \\ h_w u_w \\ h_s u_s \end{pmatrix} = \begin{pmatrix} e_s \\ e_b - e_s \\ -e_b \\ u_w e_s - \tau_{ws}/\rho_w \\ -(\rho_w/\rho_s')\, u_w e_s + (\tau_{ws} - \tau_b)/\rho_s' \end{pmatrix} \tag{3.52}$$

Under more restrictive assumptions on the sediment concentration, considering the sediment concentration in the bed and in the transport layer is identical (Capart and Young, 2002) (see 2.88-2.92), e_s is equal to zero in (3.52). When, in addition, the velocity is assumed to be identical in both layers (Fraccarollo and Capart, 2002) (see 2.106-2.109), the system (3.46) to solve becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} h_m \\ h_s \\ z_b \\ u_m \end{pmatrix} = \begin{pmatrix} e_b \\ e_b \\ -e_b \\ -\tau_m/(\rho_w (h_m + r\, h_s)) \end{pmatrix} \tag{3.53}$$

When the ratio between those velocities is constant (2.100-2.102), the system (3.46) to solve becomes

$$\frac{\partial}{\partial t}\begin{pmatrix} h_w \\ h_s \\ z_b \\ (\rho_w h_w + \alpha \rho_s' h_s) u_w \end{pmatrix} = \begin{pmatrix} e_s \\ e_b - e_s \\ -e_b \\ -\tau_b \end{pmatrix} \tag{3.54}$$

3.2.2.2. Checking energy dissipation

The complex expression of the source terms does not allow to solve each of these systems in a single step. As exposed by Capart (2004) and Spinewine (2005), a splitting strategy has to be adopted. The system has to be replaced by the succession of several sub-systems easier to solve. A special care must be given to the choice of these sub-systems. In order to obey a physical behaviour, each of these systems has to ensure a dissipation or at least a conservation with time of the mechanical energy (kinetic and potential energy) contained within the system.

$$\frac{\partial\left(E_k + E_p \right)}{\partial t} \leq 0 \tag{3.55}$$

where the kinetic energy E_k and the potential energy E_p per unit surface in horizontal plane are respectively

$$E_k = \int_0^{z_w} \tfrac{1}{2} m(z) u^2(z) dz \tag{3.56}$$

$$E_p = \int_0^{z_w} m(z) g\, z\, dz \tag{3.57}$$

Capart (2004) developed the expression of the energy variation with time in the case of the two-velocity model assuming identical concentrations in the transport layer and in the bed (see 2.88-2.92).

$$\frac{\partial\left(E_k + E_p\right)}{\partial t} = \frac{\partial\left(\frac{1}{2}\rho_w h_w u_w^2 + \frac{1}{2}\rho_s' h_s u_s^2\right)}{\partial t} + \frac{\partial\left(\frac{1}{2}\rho_s' g z_s^2 + \frac{1}{2}\rho_w g h_w^2\right)}{\partial t} \tag{3.58}$$

According to (3.52), where $e_s = 0$ ($C_s = C_b$), $\partial z_s / \partial t = \partial z_b / \partial t + \partial h_s / \partial t = 0$ and $\partial h_w / \partial t = 0$, and then the energy variation becomes

$$\frac{\partial\left(E_k + E_p\right)}{\partial t} = \frac{\partial\left(\frac{1}{2}\rho_w h_w u_w^2 + \frac{1}{2}\rho_s' h_s u_s^2\right)}{\partial t} \tag{3.59}$$

The derivative is developed

$$\begin{aligned} \frac{\partial\left(E_k + E_p\right)}{\partial t} &= \frac{u_w \rho_w}{2}\left(u_w \frac{\partial h_w}{\partial t} + h_w \frac{\partial u_w}{\partial t}\right) + \frac{h_w u_w \rho_w}{2}\frac{\partial u_w}{\partial t} \\ &+ \frac{u_s \rho_s'}{2}\left(u_s \frac{\partial h_s}{\partial t} + h_s \frac{\partial u_s}{\partial t}\right) + \frac{h_s u_s \rho_s'}{2}\frac{\partial u_s}{\partial t} \end{aligned} \tag{3.60}$$

Once more according to (3.52), where $e_s = 0$ ($C_s = C_b$),

$$\frac{\partial h_w}{\partial t} = 0 \quad \text{and} \quad h_s \frac{\partial u_s}{\partial t} = \frac{\partial h_s u_s}{\partial t} - u_s \frac{\partial h_s}{\partial t} = \frac{\partial h_s u_s}{\partial t} - u_s e_b \tag{3.61}$$

if introduced in (3.60), the energy variation with time becomes

$$\frac{\partial \left(E_k + E_p\right)}{\partial t} = \frac{u_w \rho_w}{2}\left(\underbrace{\frac{\partial h_w u_w}{\partial t}}_{-\frac{\tau_w}{\rho_w}} + h_w \underbrace{\frac{\partial u_w}{\partial t}}_{-\frac{\tau_w}{\rho_w}} \right) + \frac{u_s \rho_s'}{2}\left(\underbrace{\frac{\partial h_s u_s}{\partial t}}_{\frac{\tau_w - \tau_b}{\rho_s'}} + h_s \underbrace{\frac{\partial u_s}{\partial t}}_{\frac{\tau_w - \tau_b}{\rho_s'} - u_s e_b} \right) \tag{3.62}$$

where the derivative are replaced by expressions depending on the shear stresses and the erosion rate using equations (2.91) (2.92) where the simplifications (3.61) have been introduced.

Substituting the expression of e_b depending on the shear stresses τ_s and τ_b (2.35)

$$e_b = -\frac{\partial z_b}{\partial t} = \frac{\tau_s - \tau_b}{\rho_b' u_s}, \tag{3.63}$$

the expression of the energy variation (3.62) is simplified

$$\frac{\partial \left(E_k + E_p\right)}{\partial t} = -\frac{1}{2}\left(\tau_s + \tau_b\right) u_s - \tau_w \left(u_w - u_s\right) \tag{3.64}$$

The energy dissipation is verified as all the right-hand terms of expression (3.64) are negative. In order to introduce the erosion rate e_b in (3.64), τ_s is replaced by a function of e_b and τ_b, using the expression linking e_b to the shear stresses τ_s and τ_b (3.63). Thus (3.64) becomes

$$\frac{\partial \left(E_k + E_p\right)}{\partial t} = -\frac{1}{2}\left(2\tau_b + e_b \rho_s' u_s\right) u_s - \tau_w \left(u_w - u_s\right) \tag{3.65}$$

If, this time, τ_b is replaced by a function of e_b and τ_s, using (3.63), (3.64) becomes

$$\frac{\partial\left(E_k+E_p\right)}{\partial t}=-\frac{1}{2}\left(2\tau_s-e_b\rho_s'u_s\right)u_s-\tau_w\left(u_w-u_s\right) \qquad (3.66)$$

Even if all the dissipative phenomena appearing in the source terms are closely linked, under these formulations (3.65) and (3.66), an attempt to divide the problem is made. The term containing τ_w corresponds to the loss of energy due to the friction between the water layer and the transport layer, τ_b or τ_s corresponds to the loss of energy due to the friction between the transport layer and the bed. Energy dissipation due to the mass and momentum exchanges occurring during the erosion or deposition process corresponds to the term containing e_b. In erosion ($e_b > 0$), in expression (3.65), each term, corresponding to a specific phenomena, dissipates energy. Identically, in deposition ($e_b < 0$) each right-hand term of (3.66) dissipates energy. We shall see the relevance of such considerations in the expression of the source terms (section 3.2.2.3).

The reasoning initially proposed by Capart (2004) and reproduced here, was extended by Spinewine (2005) to the less restrictive model allowing a change in concentration between the transport layer and the bed.

The variation of energy with time becomes

$$\frac{\partial\left(E_k+E_p\right)}{\partial t}=-\frac{1}{2}\left(\tau_s+\tau_b\right)u_s-\frac{1}{2}\left(\tau_{ws}+\tau_{sw}\right)\left(u_w-u_s\right)+\left(\rho_b'-\rho_s'\right)gh_se_b \qquad (3.67)$$

This expression is similar to (3.64) with an additional term taking into account the change in concentration. In the case of deposition ($e_b < 0$), all the terms are negative and the dissipation of energy is trivially verified, but in the case of erosion ($e_b > 0$), the additional term induces a rise of energy. This rise of energy is linked to the swelling of the transport layer due to the fact that the concentration in sediment is lower when they are in movement. According to Spinewine (2005), the dissipative effect of the first and the second terms of (3.67) largely counterbalance the energy gain due to the additional term. He has shown through different test cases, where erosion occurs, that the energy dissipates.

As proposed by Capart (2004) when the concentration in the transport layer is assumed to be the same than in the bed, (3.65) and (3.66), the erosion rate e_b and now, in addition, e_s can be introduced in the expression of the variation of energy with time in the case of distinct concentrations (3.67). Relation (2.39) expressing the link between e_s and the shear stresses τ_{ws} and τ_{sw}, is reminded

$$e_s = -\frac{\partial z_s}{\partial t} = \frac{\tau_{ws} - \tau_{sw}}{\rho_w (u_w - u_s)} \tag{3.68}$$

Using this expression (3.68), τ_{sw} is replaced in (3.67) by a function of e_s and τ_{ws}, and using (3.63), τ_s is replaced by a function of e_b and τ_b. The energy variation (3.67) becomes

$$\begin{aligned} \frac{\partial (E_k + E_p)}{\partial t} &= -\frac{1}{2}(2\tau_b + e_b \rho_s ' u_s) u_s \\ &\quad -\frac{1}{2}(2\tau_{ws} - \rho_w (u_w - u_s) e_s)(u_w - u_s) + (\rho_b ' - \rho_s ') g h_s e_b \end{aligned} \tag{3.69}$$

For erosion ($e_b > 0$, $e_s < 0$), each term of (3.69) induces an energy dissipation, except the last one as mentioned previously. Nevertheless, the sum of the terms linked to geomorphic changes (those containing e_s and e_b) should be negative.

$$-\frac{1}{2} e_b \rho_s ' u_s^2 + \frac{1}{2} \rho_w (u_w - u_s)^2 e_s + (\rho_b ' - \rho_s ') g h_s e_b \leq 0 \tag{3.70}$$

All the flows treated in the present work respect this condition. Only flows corresponding to very low sediment Froude number Fr_s (important h_s corresponding to low u_s) could violate this condition. In this case, numerical instabilities may be more susceptible to appear.

Using expression (3.68), τ_{ws} is replaced in (3.67) by a function of e_s and τ_{sw}, and using (3.63), τ_b is replaced by a function of e_b and τ_s.

The energy variation (3.67) becomes

$$\begin{aligned}\frac{\partial(E_k + E_p)}{\partial t} &= -\frac{1}{2}(2\tau_s - e_b\rho_b' u_s)u_s \\ &\quad -\frac{1}{2}(2\tau_{sw} + \rho_w(u_w - u_s)e_s)(u_w - u_s) + (\rho_b' - \rho_s')gh_s e_b\end{aligned} \tag{3.71}$$

In deposition ($e_b < 0$, $e_s > 0$), all the terms of (3.71) are negative. Then, under this formulation, the problem can be divided without any risks of numerical instabilities. Nevertheless, a problem remains as the friction between the two layers is expressed through τ_{sw}, for which, as mentioned in chapter 2, section 2.1.2.1, no expression is available. To avoid the problem, Spinewine (2005) made an assumption considering that τ_{sw} can be calculated the same way than τ_{ws} (2.80) , which is not the case when e_s is not equal to zero (see (3.68)). An alternative possibility should be to write expression (3.67) as a function of τ_{ws}. Using expression (3.68), τ_{sw} is replaced in (3.67) by a function of e_s and τ_{ws}, and, using (3.63), τ_b is replaced by a function of e_b and τ_s. The energy variation (3.67) becomes

$$\begin{aligned}\frac{\partial(E_k + E_p)}{\partial t} &= -\frac{1}{2}(2\tau_s - e_b\rho_s' u_s)u_s \\ &\quad -\frac{1}{2}(2\tau_{ws} - \rho_w(u_w - u_s)e_s)(u_w - u_s) + (\rho_b' - \rho_s')gh_s e_b\end{aligned} \tag{3.72}$$

In this expression, not all the terms induce an energy dissipation. In this case, the sum of the terms linked to geomorphic changes (those containing e_s and e_b) should be negative.

$$\frac{1}{2}e_b\rho_s' u_s^2 + \frac{1}{2}\rho_w(u_w - u_s)^2 e_s + (\rho_b' - \rho_s')gh_s e_b \le 0 \tag{3.73}$$

Condition (3.73), under deposition ($e_b < 0$, $e_s > 0$), is not respected for a wide range of flows, making the source treatment more sensitive to numerical instabilities. Then, as far as deposition is concerned, none of the previous formulations (3.71) with τ_{sw} or (3.72) with τ_{ws} is completely satisfactory. These energy formulations have an incidence on the source terms expression. For the test cases treated in this thesis, both expressions of the source terms deduced from both proposals (3.71) and (3.72) are tested and compared. Both give similar results without leading to numerical instabilities. Nevertheless, an analysis should be done on more cases before a generalisation can be made. In the next section, the source terms in deposition are expressed on the basis of (3.72).

3.2.2.3. Splitting of source terms

In order to simplify the resolution, the systems to solve during the source treatment (3.46) are divided into several sub-systems, the source terms are decomposed into three components

$$\mathbf{S}(\mathbf{U}) = \mathbf{S}_{fw}(\mathbf{U}) + \mathbf{S}_{fs}(\mathbf{U}) + \mathbf{S}_{g}(\mathbf{U}) \tag{3.74}$$

Each of these component is representative of a physical phenomenon dissipating energy. $\mathbf{S}_{fw}(\mathbf{U})$ is associated to friction between the water layer and the transport layer, $\mathbf{S}_{fs}(\mathbf{U})$ is associated to friction between the transport layer and the fixed bed and $\mathbf{S}_{g}(\mathbf{U})$ is associated to the mass and momentum exchanges between the bed and the transport layer. Each step is solved by an implicit or an explicit equation, using the solution of the previous system:

$$\left(\frac{\partial \mathbf{U}}{\partial t}\right)^{(2\to2')} = \mathbf{S}_{fw}(\mathbf{U})^{(2')} \text{ or } = \mathbf{S}_{fw}(\mathbf{U})^{(2)} \tag{3.75}$$

$$\left(\frac{\partial \mathbf{U}}{\partial t}\right)^{(2'\to2'')} = \mathbf{S}_{fs}(\mathbf{U})^{(2'')} \text{ or } = \mathbf{S}_{fs}(\mathbf{U})^{(2')} \tag{3.76}$$

$$\left(\frac{\partial \mathbf{U}}{\partial t}\right)^{(2''\to3)} = \mathbf{S}_{g}(\mathbf{U})^{(3)} \text{ or } = \mathbf{S}_{g}(\mathbf{U})^{(2'')} \tag{3.77}$$

where state 2 is the solution obtained by the hyperbolic solver, states 2’ and 2” represent intermediate steps between state 2 and final state 3 at time $t + dt$. These three source operators depend on the assumptions of the model and are expressed on a different way either there is an erosion or a deposition. In order to be numerically stable, each source resolution step has to dissipate energy. Then, expressions (3.69) and (3.72) are closely linked to the source operators, which are different in erosion and in deposition. Indeed, to ensure the physical principle of energy dissipation of each component, the source terms should be expressed as functions of e_b, e_s τ_b and τ_{ws} To ensure energy dissipation is induced by each of the source operator, Spinewine (2005) proposed to express the source terms as functions of e_b, e_s τ_s and τ_{sw} on the basis of (3.71), making the assumption that $\tau_{sw} = \tau_{ws}$. To avoid this assumption, in this thesis, the source terms in deposition are expressed as functions of e_b, e_s τ_s and τ_{ws}. Under this formulation, the energy dissipation of the geomorphic source operator is not ensured; nevertheless, no numerical instability is induced.

Table (3.1) recapitulates the system to solve for each source operator in erosion and deposition in the complete case of the two-velocity two-concentration model (Spinewine 2005). In deposition, τ_b has to be expressed as a function of e_b and τ_s using expression (3.63). As previously exposed, the only difference with Spinewine (2005) concerns the geomorphic operator in deposition.

Table (3.1) can be adapted to the two-velocity one-concentration model. The only differences concern the geomorphic operator. As there is no longer difference in sediment concentration between the bed and the sediment transport layer, the interface z_s does not move during the erosion and deposition processes inducing $e_s = 0$. Moreover, the volumetric mass in the bed ρ_b’ is identical to the volumetric mass in the transport layer ρ_s’.

When, in addition, the velocities in both layers are identical, the two friction operators reduce to one as there is not any more friction between the two layers. The friction and geomorphic operators (Fraccarollo and Capart, 2002) are given in case of erosion and deposition in Table (3.2). When a constant ratio is imposed between the velocity in each layer, even if the friction between the two layers is maintained, only two source operators are required (Table 3.3).

	Erosion $\lvert\tau_s\rvert>\lvert\tau_b\rvert$ $e_b>0$	**Deposition** $\lvert\tau_s\rvert<\lvert\tau_b\rvert$ $e_b<0$
Water friction operator S_{fw}	$\mathbf{S}_{fw}=\begin{pmatrix}0\\0\\0\\-\dfrac{\tau_{ws}}{\rho_w}\\\dfrac{\tau_{ws}}{\rho_s'}\end{pmatrix}$	$\mathbf{S}_{fw}=\begin{pmatrix}0\\0\\0\\0\\-\dfrac{\tau_b}{\rho_s'}\end{pmatrix}$
Sediment friction operator S_{fs}	$\mathbf{S}_{fs}=\begin{pmatrix}0\\0\\0\\-\dfrac{\tau_{ws}}{\rho_w}\\\dfrac{\tau_{ws}}{\rho_s'}\end{pmatrix}$	$\mathbf{S}_{fs}=\begin{pmatrix}0\\0\\0\\0\\-\dfrac{\tau_s}{\rho_s'}\end{pmatrix}$
Geomorphic operator S_g	$\mathbf{S}_g=\begin{pmatrix}e_s\\e_b-e_s\\-e_b\\u_we_s\\-\dfrac{\rho_w}{\rho_s'}u_we_s\end{pmatrix}$	$\mathbf{S}_g=\begin{pmatrix}e_s\\e_b-e_s\\-e_b\\u_we_s\\-\dfrac{\rho_w}{\rho_s'}u_we_s+\dfrac{\rho_b'}{\rho_s'}u_se_b\end{pmatrix}$

Table 3.1. Source operators two-velocity two-concentration model (inspired from Spinewine, 2005)

	Erosion $\lvert\tau_m\rvert > \lvert\tau_b\rvert$ $e_b>0$	**Deposition** $\lvert\tau_m\rvert < \lvert\tau_b\rvert$ $e_b<0$
Friction operator S_f	$\mathbf{S}_f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{\tau_b}{\rho_w(h_m + rh_s)} \end{pmatrix}$	$\mathbf{S}_f = \begin{pmatrix} e_b \\ -e_b \\ e_b \\ \rho_w(1+r)u_m e_b \end{pmatrix}$
Geomorphic operator S_g	$\mathbf{S}_g = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\dfrac{\tau_m}{\rho_w(h_m + rh_s)} \end{pmatrix}$	$\mathbf{S}_g = \begin{pmatrix} e_b \\ -e_b \\ e_b \\ 0 \end{pmatrix}$

Table 3.2. Source operators one-velocity one-concentration model (Fraccarollo and Capart, 2002)

	Erosion $\lvert\tau_s\rvert > \lvert\tau_b\rvert$ $e_b>0$	**Deposition** $\lvert\tau_s\rvert < \lvert\tau_b\rvert$ $e_b<0$
Friction operator S_f	$\mathbf{S}_f = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\tau_b \end{pmatrix}$	$\mathbf{S}_f = \begin{pmatrix} e_s \\ e_b - e_s \\ e_b \\ 0 \end{pmatrix}$
Geomorphic operator S_g	$\mathbf{S}_g = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -\tau_s \end{pmatrix}$	$\mathbf{S}_g = \begin{pmatrix} e_s \\ e_b - e_s \\ e_b \\ \alpha\rho_s' u_w e_b \end{pmatrix}$

Table 3.3. Source operators constant velocity ratio model

Each sub-system can be solved on an explicit or an implicit way. In case of an explicit resolution, the source operators are simply evaluated at the initial time and considered to be invariant during the time step. In the case of an implicit resolution the source operators are expressed as functions of the main unknowns **U** at the final time. None of these two approaches is exact, the source operators should be evaluated at an intermediate time corresponding to the integral average value in time of the main unknowns **U**. As the time steps are small due to the CFL conditions, no significant difference is observed between the two methods. The case of the implicit resolution is more complicated to implement than the explicit one. During each source operator treatment, some variables or functions of the variables are invariants with time, when the components of the source operators are equal to zero, they are useful to simplify the expressions in order to obtain a solution. The resolution proposed by Capart (2004) is adapted to the more complete case of the two-velocity two-concentration model previously detailed by Spinewine (2005).

Water friction operator

The water friction operator is the same in erosion and deposition (Table 3.1). The system to solve is

$$\left(\frac{\partial h_w}{\partial t}\right)^{(2\to 2')} = 0 \qquad \left(\frac{\partial h_s}{\partial t}\right)^{(2\to 2')} = 0 \qquad \left(\frac{\partial z_b}{\partial t}\right)^{(2\to 2')} = 0$$

$$\left(\frac{\partial h_w u_w}{\partial t}\right)^{(2\to 2')} = -\left(\frac{\tau_{ws}}{\rho_w}\right)^{(2')} \qquad \left(\frac{\partial h_s u_s}{\partial t}\right)^{(2\to 2')} = \left(\frac{\tau_{ws}}{\rho_s'}\right)^{(2')} \qquad (3.78a\text{-}e)$$

where state 2 is the solution of the hyperbolic solver and state 2' is the solution obtained by the treatment of the water friction source operator. In the following reasoning, state 2 corresponds to time t and state 2' corresponds to time $t + dt$.

According to (3.78a-c), h_w, h_s and z_b are invariant in time during the resolution of the water friction source operator.

$$h_{w,\,t+dt} = h_{w,\,t} \qquad h_{s,\,t+dt} = h_{s,\,t} \qquad z_{b,\,t+dt} = z_{b,\,t} \tag{3.79}$$

Removing the time invariant h_w and h_s from the partial derivatives (3.78d-e) and subtracting (3.78d) to (3.78e) gives

$$\frac{\partial u_w}{\partial t} - \frac{\partial u_s}{\partial t} = -\frac{\tau_{ws}}{\rho_w h_w} - \frac{\tau_{ws}}{\rho_s h_s} \tag{3.80}$$

if $\tau_{ws} = \rho_w C_{fw} (u_w - u_s)^2$ is substituted, it can be written

$$\frac{\partial (u_w - u_s)}{\partial t} = -\frac{C_{fw}}{h_w} \left(\frac{h_w \rho_w}{h_s \rho_s} + 1 \right) (u_w - u_s)^2 \tag{3.81}$$

Taking into account that h_w and h_s do not vary with time, the relative velocity $(u_w - u_s)_{t+dt}$ is implicitly calculated by the resolution of the following second order equation.

$$(u_w - u_s)_{t+dt} = (u_w - u_s)_t - dt \frac{C_{fw}}{h_{w,t}} \left(\frac{h_{w,t} \rho_w}{h_{s,t} \rho_s} + 1 \right) (u_w - u_s)^2_{t+dt} \tag{3.82}$$

Moreover, adding (3.78d) to (3.78e) multiplied by the ratio ρ_s'/ρ_w gives

$$\frac{\partial h_w u_w}{\partial t} + \frac{\rho_s'}{\rho_w} \frac{\partial h_s u_s}{\partial t} = 0\,, \tag{3.83}$$

which implies $h_w u_w + \rho_s'/\rho_w \; h_s u_s$ is an invariant with time for the water friction operator.

$$\left(h_w u_w + \frac{\rho_s'}{\rho_w} h_s u_s \right)_{t+dt} = \left(h_w u_w + \frac{\rho_s'}{\rho_w} h_s u_s \right)_t \tag{3.84}$$

The velocities $u_{w,t+dt}$ and $u_{s,t+dt}$ are deduced from a redistribution of the relative velocity $(u_w - u_s)_{t+dt}$ around the mean velocity, using the relative velocity $(u_w - u_s)_{t+dt}$ obtained from (3.82) and the invariant expression

(3.84).The state $t + dt$ is equal to the state 2', which is the departure values of the following step (sediment friction operator) in the source treatment.

Sediment friction operator

The system to solve is different in case of erosion and deposition (Table3.1). Four of the five equations of the sediment friction operator are common to erosion and deposition.

$$\left(\frac{\partial h_w}{\partial t}\right)^{(2'\to 2'')} = 0 \qquad \left(\frac{\partial h_s}{\partial t}\right)^{(2'\to 2'')} = 0 \qquad \left(\frac{\partial z_b}{\partial t}\right)^{(2'\to 2'')} = 0$$

$$\left(\frac{\partial h_w u_w}{\partial t}\right)^{(2'\to 2'')} = 0 \tag{3.85a-d}$$

The fifth equation is in case of erosion

$$\left(\frac{\partial h_s u_s}{\partial t}\right)^{(2'\to 2'')} = -\left(\frac{\tau_b}{\rho_s'}\right)^{(2'')} \tag{3.86e}$$

and in case of deposition

$$\left(\frac{\partial h_s u_s}{\partial t}\right)^{2'\to 2''} = -\left(\frac{\tau_s}{\rho_s'}\right)^{2''} \tag{3.87f}$$

where state 2' is the solution of the treatment of the water friction source operator and state 2'' is the solution obtained after the treatment of the sediment friction source operator (respectively corresponding, in the following reasoning, to time t and $t+dt$). According to (3.85a-d), h_w, h_s, z_b and $h_w u_w$ are invariant in time during the resolution of the sediment friction source operator. Taking into account the invariance of h_s with time, in the erosion equation (3.86e) can be written

$$\frac{\partial h_s u_s}{\partial t} = h_s \frac{\partial u_s}{\partial t} = -\frac{\tau_b}{\rho_s'} \tag{3.88}$$

with $\tau_b = \tau_{crit} + tg\varphi(\rho_s' - \rho_w)gh_s$ not varying with time, $u_{s,t+dt}$ is directly deduced in case of erosion

$$u_{s,t+dt} = u_{s,t} - dt \frac{\left(\tau_{crit} + tg\varphi(\rho_s' - \rho_w)gh_{s,t}\right)}{h_{s,t}\,\rho_s'} \tag{3.89}$$

By the same way, in deposition, equation (3.86f) can be written

$$\frac{\partial h_s u_s}{\partial t} = h_s \frac{\partial u_s}{\partial t} = -\frac{\tau_s}{\rho_s'} \tag{3.90}$$

where $\tau_s = \rho_s' C_{fs} \left| u_s \right| u_s$, then $u_{s,t+dt}$ can be deduced, in case of deposition, as the solution of the second order equation

$$u_{s,t+dt} = u_{s,t} - dt \frac{C_{fs}\, u_{s,t+dt}^2}{h_{s,t}} \tag{3.91}$$

The state $t + dt$ is equal to the state 2'', which is the departure state of the following step (geomorphic operator) in the source treatment.

Geomorphic operator in erosion

The system to solve is different in case of erosion and deposition. In the case of erosion it writes

$$\left(\frac{\partial h_w}{\partial t}\right)^{(2''\to 3)} = \left(e_s\right)^{(3)} \qquad \left(\frac{\partial h_s}{\partial t}\right)^{(2''\to 3)} = \left(e_b - e_s\right)^{(3)}$$

$$\left(\frac{\partial z_b}{\partial t}\right)^{(2''\to 3)} = \left(-e_b\right)^3 \qquad \left(\frac{\partial h_w u_w}{\partial t}\right)^{(2''\to 3)} = \left(u_w e_s\right)^{(3)}$$

$$\left(\frac{\partial h_s u_s}{\partial t}\right)^{(2''\to 3)} = \left(-\frac{\rho_w}{\rho_s'} u_w e_s\right)^{(3)} \tag{3.92a-e}$$

where state 2’’ is the solution of the treatment of the sediment friction source operator and state 3 is the solution obtained by treatment of the geomorphic source operator, ending the source terms treatment (respectively corresponding, in the following reasoning, to time t and $t + dt$). Subtracting (3.92a) multiplied by u_w from (3.92d) gives

$$\frac{\partial h_w u_w}{\partial t} - u_w \frac{\partial h_w}{\partial t} = h_w \frac{\partial u_w}{\partial t} = 0 \tag{3.93}$$

It implies the velocity in the water layer u_w is invariant with time during the geomorphic operator treatment in erosion. Using the relation between e_s and e_b (2.29), an expression of e_s can be deduced from (3.92b)

$$e_s = -\frac{\rho_b' - \rho_s'}{\rho_b' - \rho_w} \frac{\partial h_s}{\partial t} \tag{3.94}$$

Substituting (3.94) in (3.92e) and putting all the terms in the left hand side yields to the definition of a second invariant with time

$$\frac{\partial}{\partial t}\left(h_s \left(u_s - \frac{\rho_w}{\rho_s'} \frac{\rho_b' - \rho_s'}{\rho_b' - \rho_w} u_w \right)\right) = 0 \tag{3.95}$$

Using the relation between e_s and e_b (2.29), (3.92b) can be written

$$\frac{\partial h_s}{\partial t} = e_b - e_s = \frac{\rho_b' - \rho_w}{\rho_s' - \rho_w} e_b \tag{3.96}$$

Using the definition of the erosion rate e_b (3.63), (3.96) is expressed

$$h_{s,t+dt} = h_{s,t} + dt \frac{\rho_b - \rho_w}{\rho_s' - \rho_w} \frac{1}{\rho_b u_{s,t+dt}} \left(\tau_{s,t+dt} - \tau_{b,t+dt} \right) \tag{3.97}$$

where $\tau_{s,t+dt}$ (2.81) depends on $u_{s,t+dt}$ and $\tau_{b,t+dt}$ (2.82) on $h_{s,t+dt}$, the relation (3.97) linking $h_{s,t+dt}$ and $u_{s,t+dt}$ combined with invariant (3.95) also linking $h_{s,t+dt}$ and $u_{s,t+dt}$ allows to find a three order relation depending on $h_{s,t+dt}$

$$a h_{s,t+dt}^3 + b h_{s,t+dt}^2 + c h_{s,t+dt} + d = 0 \tag{3.98}$$

where the coefficients a, b, c and d depending of $h_{s,t}$, $u_{w,t}$ and $u_{s,t}$. The sediment transport depth $h_{s,t+dt}$ can be deduced either from the implicit third degree equation (3.98) or by the simpler explicit relation (3.99) based on the explicit Euler scheme.

$$h_{s,t+dt} = h_{s,t} + dt \frac{\rho_b' - \rho_w}{\rho_s' - \rho_w} \frac{1}{\rho_b u_{s,t}} \left(\tau_{s,t} - \tau_{b,t} \right)$$

$$= h_{s,t} + dt \frac{\rho_b' - \rho_w}{\rho_s' - \rho_w} \frac{1}{\rho_b u_{s,t}} \left(\rho_s' C_{fs} \left| u_{s,t} \right| u_{s,t} - \left(\tau_{crit} + \left(\rho_s' - \rho_w \right) g \, tg\varphi \, h_{s,t} \right) \right) \tag{3.99}$$

The invariant (3.95) linking $u_{s,t+dt}$ to $h_{s,t+dt}$, allows to calculate the velocity in the sediment transport layer $u_{s,t+dt}$ (using the results obtained for $h_{s,t+dt}$).

$$u_{s,t+dt} = \frac{h_{s,t}}{h_{s,t+dt}} \left(u_{s,t} - u_{w,t} \frac{\rho_w}{\rho_s'} \frac{\rho_b' - \rho_s'}{\rho_b' - \rho_w} \right) + u_{w,t} \frac{\rho_w}{\rho_s'} \frac{\rho_b' - \rho_s'}{\rho_b' - \rho_w} \tag{3.100}$$

Substituting e_s from (3.94) in (3.92a) gives

$$\frac{\partial h_w}{\partial t} = -\frac{\rho_b' - \rho_s'}{\rho_b' - \rho_w} \frac{\partial h_s}{\partial t}, \tag{3.101}$$

which allows to calculate the updated water depth $h_{w,t+dt}$

$$h_{w,t+dt} = h_{w,t} - \frac{\rho_b' - \rho_s'}{\rho_b' - \rho_w} \left(h_{s,t+dt} - h_{s,t} \right) \tag{3.102}$$

Adding the relations (3.92b) multiplied by $\left(\rho_s' - \rho_w \right) / \left(\rho_b' - \rho_w \right)$ and (3.92c), a new invariant with time is obtained.

$$\frac{\partial}{\partial t} \left(\frac{\rho_s' - \rho_w}{\rho_b' - \rho_w} h_s + z_b \right) = 0 \tag{3.103}$$

The bed level after the erosion process $z_{b,t+dt}$, can be calculated introducing the previously calculated sediment transport depth $h_{s,t+dt}$ in the invariant expression (3.103).

$$z_{b,t+dt} = z_{b,t} - \frac{\rho_s' - \rho_w}{\rho_b' - \rho_w}\left(h_{s,t+dt} - h_{s,t}\right) \tag{3.104}$$

When the transport layer is too thin (h_s very small), the shear stress applied on the bed is not τ_s (2.81) applied by the transport layer, but τ_{ws} (2.80) applied by the water layer. In this case the velocity in the thin transport layer is supposed to be the same than the velocity of water, it implies the expression of $h_{s,t+dt}$ (3.98) or (3.99) changes. The values obtained at $t + dt$ correspond to the final state 3.

Geomorphic operator in deposition

The system to solve in this case is

$$\left(\frac{\partial h_w}{\partial t}\right)^{(2"\to 3)} = \left(e_s\right)^{(3)} \qquad \left(\frac{\partial h_s}{\partial t}\right)^{(2"\to 3)} = \left(e_b - e_s\right)^{(3)}$$

$$\left(\frac{\partial z_b}{\partial t}\right)^{(2"\to 3)} = \left(-e_b\right)^{(3)} \qquad \left(\frac{\partial h_w u_w}{\partial t}\right)^{(2"\to 3)} = \left(u_w e_s\right)^{(3)}$$

$$\left(\frac{\partial h_s u_s}{\partial t}\right)^{(2"\to 3)} = \left(-\frac{\rho_w}{\rho_s'} u_w e_s + \frac{\rho_b'}{\rho_s'} u_s e_b\right)^{(3)} \tag{3.105a-e}$$

As it is the case for erosion (3.93), the water velocity u_w is an invariant for the geomorphic source operator in deposition. Contrarily, the invariant (3.95) during the erosion process is no longer available in deposition due to the additional term depending on u_s in equation (3.105e).

The transport layer depth $h_{s,t+dt}$ is calculated the same way than in erosion (3.99). Only the explicit expression can be used because $u_{s,t+dt}$, appearing in the implicit one, is not known as the invariant (3.95) does not exist anymore.

Once $h_{s,t+dt}$ is calculated, the water depth $h_{w,t+dt}$ and the bed level $z_{b,t+dt}$ are evaluated the same way than in erosion, using (3.102) and (3.104).

The major change between erosion and deposition consists in the calculation of the velocity in the transport layer $u_{s,t+dt}$. Expressing e_b as a function of e_s (2.29) and then replacing e_s by $\partial h_w/\partial t$ (3.105a) in (3.105e) gives

$$\frac{\partial h_s u_s}{\partial t} = \left(-\frac{\rho_w}{\rho_s'} u_w - \frac{\rho_b'(\rho_s'-\rho_w)}{\rho_s'(\rho_b'-\rho_s')} u_s \right) \frac{\partial h_w}{\partial t}, \tag{3.106}$$

which allows to calculate $u_{s,t+dt}$

$$u_{s,t+dt} = \frac{\left(h_{s,t} u_{s,t} + \left(-\frac{\rho_w}{\rho_s'} u_{w,t} - \frac{\rho_b'(\rho_s'-\rho_w)}{\rho_s'(\rho_b'-\rho_s')} u_{s,t+dt} \right) \left(h_{w,t+dt} - h_{w,t} \right) \right)}{h_{s,t+dt}} \tag{3.107}$$

The values obtained at $t + dt$ correspond to the final state 3.

These results slightly differ from Spinewine (2005), because the assumptions made to express the source terms are not the same.

3.3. Merging of both hyperbolic and source terms resolutions

At each time step, the system to solve is divided in two parts. The hyperbolic part is treated before the source terms.

$$\left(\frac{\partial \mathbf{U}}{\partial t} \right)^{(1\to 2)} + \frac{\partial \mathbf{F}(\mathbf{U})^{(1)}}{\partial x} + \mathbf{H}(\mathbf{U})^{(1)} \frac{\partial \mathbf{G}(\mathbf{U})^{(1)}}{\partial x} = 0 \tag{3.108}$$

$$\left(\frac{\partial \mathbf{U}}{\partial t} \right)^{(2\to 3)} = \mathbf{S}(\mathbf{U})^{(3)} \tag{3.109}$$

$\mathbf{U}^{(1)}$ corresponds to the initial time t, $\mathbf{U}^{(2)}$ corresponds to the intermediate state obtained after the resolution of the hyperbolic system and $\mathbf{U}^{(3)}$ is the final solution at time $t+dt$. The problem is discretised in time (time step dt) and in space (space step dx), the solver is applied to the cell numbered i (Fig. 3.11) between time t ($\mathbf{U}^{(1)}$) and $t+dt$ ($\mathbf{U}^{(3)}$).

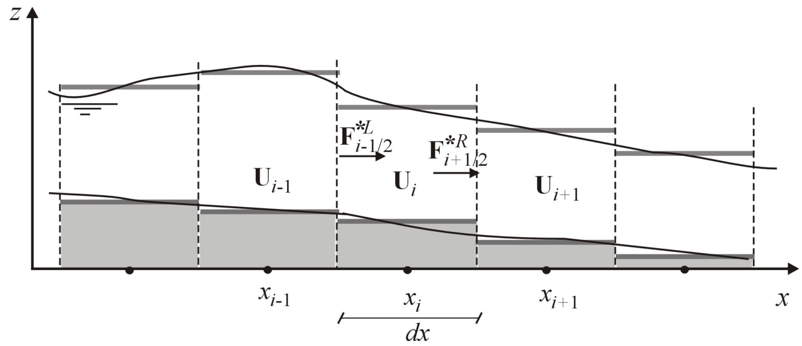

Figure 3.11. First order in space discretisation

First, the hyperbolic part is solved:

$$\mathbf{U}_i^{(2)} = \mathbf{U}_i^{(1)} + \frac{dt}{dx}\left(\mathbf{F}^{*R}_{i-1/2} - \mathbf{F}^{*L}_{i+1/2}\right)^{(1)} \tag{3.110}$$

where $\mathbf{F}^{*R}_{i-1/2}$ is the flux entering the cell i and $\mathbf{F}^{*L}_{i+1/2}$ is the flux exiting the cell i. These fluxes are calculated with (3.36) and (3.37), which take into account the non-conservative products generally related to the bed-slope. As constant states are assumed in each computational cell, the variation of the bed level along the channel is concentred at the cell interfaces.

Secondly, the source terms are calculated

$$\mathbf{U}_i^{(3)} = \mathbf{U}_i^{(2)} + \mathbf{S}(\mathbf{U}_i)^{(3)}\, dt \tag{3.111}$$

As exposed previously, the resolution of the source terms is divided in several steps.

3.4. Second order accuracy

The second-order accuracy in space is based on the MUSCL approach proposed by van Leer (1979). It consists in assuming linear states within each of the computational cells instead of constant states (Fig. 3.12).

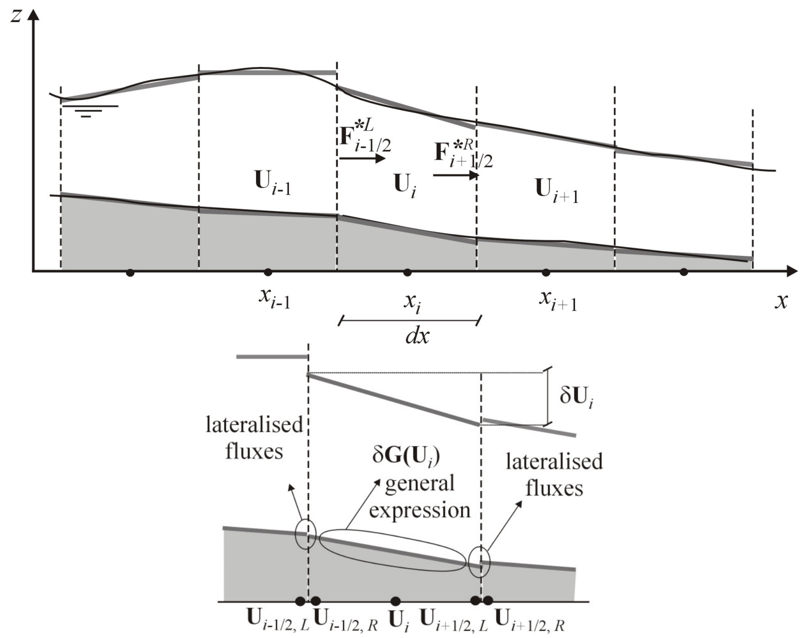

Figure 3.12. Second order in space discretisation

The state $\mathbf{U}_i$ corresponds to the middle of the cell i. Using the gradient $\partial\mathbf{U}_i$ corresponding to the slope of the linear state in the computational cell i, the states $\mathbf{U}_{L,\,i+1/2}$ and $\mathbf{U}_{R,\,i-1/2}$ are respectively defined on the left side of interface i+1/2 and on the right side of interface i-1/2.

$$\mathbf{U}_{L,i+1/2} = \mathbf{U}_i - \frac{1}{2}\partial\mathbf{U}_i$$

$$\mathbf{U}_{R,i-1/2} = \mathbf{U}_i + \frac{1}{2}\partial\mathbf{U}_i \qquad (3.112\text{a-b})$$

where the gradient $\partial\mathbf{U}_i$ is defined using the slope limiter operator minmod

$$\partial\mathbf{U}_i = \min\operatorname{mod}\left(\mathbf{U}_{i+1} - \mathbf{U}_i, \mathbf{U}_i - \mathbf{U}_{i-1}\right) \qquad (3.113)$$

with $$\min\operatorname{mod}(a,b) = \begin{cases} a & \text{if} \quad |a| \le |b| \quad \text{and} \quad ab > 0 \\ b & \text{if} \quad |a| > |b| \quad \text{and} \quad ab > 0 \\ 0 & \text{if} \quad ab \le 0 \end{cases}$$

The general expression of the finite-volume (3.110) becomes

$$\mathbf{U}_i^{(2)} = \mathbf{U}_i^{(1)} + \frac{dt}{dx}\left(\mathbf{F}_{i-1/2}^{*R} - \mathbf{F}_{i+1/2}^{*L}\right)^{(1)} - \frac{dt}{dx}\mathbf{H}(\mathbf{U}_i)^{(1)}\,\partial\mathbf{G}(\mathbf{U}_i)^{(1)} \qquad (3.114)$$

where $$\partial\mathbf{G}(\mathbf{U}_i) = \mathbf{G}\left(\mathbf{U}_{R,i-1/2}\right) - \mathbf{G}\left(\mathbf{U}_{L,i+1/2}\right) \qquad (3.115)$$

To calculate the fluxes through the interface i+1/2, the states on the left $\mathbf{U}_{L,\,i+1/2}$ and on the right $\mathbf{U}_{R,\,i+1/2}$ of the interface are taken into account in (3.36) and (3.37) instead of the states $\mathbf{U}_i$ and $\mathbf{U}_{i+1}$ respectively at the middle of the left and right cells. Contrarily to the first order method, only a part of the non-conservative product related to the slope is taken into account in the lateralised fluxes (discontinuity at the interface Fig. 3.12). The complementary part of non-conservative product, due to the variation of the bed level in the cell $\partial\mathbf{G}(\mathbf{U}_i)$ (Fig. 3.12) has to be introduced in the finite-volume expression (3.114). The minus sign before this term is linked to the definition (3.115) of the variation $\partial\mathbf{G}(\mathbf{U}_i)$ in the cell. There is no change in the source terms treatment between the first and second order in space methods.

The second-order accuracy in time is obtained by the decomposition of the time step into four steps using a predictor-corrector process (Fraccarollo et al., 2003).

First the source terms are integrated on half a time step

$$\mathbf{U}_i^{(1)} = \mathbf{U}_i^n + \mathbf{S}(\mathbf{U}_i)^n\,\frac{dt}{2} \qquad (3.116)$$

The second step is a predictor step for the hyperbolic operator

$$\mathbf{U}_i^{(2)} = \mathbf{U}_i^{(1)} + \frac{dt}{2\,dx}\left(\mathbf{F}_{i-1/2}^{*R} - \mathbf{F}_{i+1/2}^{*L}\right)^{(1)} - \frac{dt}{2\,dx}\left(\mathbf{H}_i^{(1)}\partial\mathbf{G}_i^{(1)}\right) \qquad (3.117)$$

The third step is a corrector of the previous step for the hyperbolic operator

$$\mathbf{U}_i^{(3)} = \mathbf{U}_i^{(2)} + \frac{dt}{dx}\left(\mathbf{F}_{i-1/2}^{*R} - \mathbf{F}_{i+1/2}^{*L}\right)^{(2)} - \frac{dt}{dx}\left(\mathbf{H}_i^{(2)}\partial\mathbf{G}_i^{(1)}\right) \qquad (3.118)$$

The last step is the second part (completing the first step) of the source terms treatment

$$\mathbf{U}_i^{n+1} = \mathbf{U}_i^{(4)} = \mathbf{U}_i^{(3)} + \mathbf{S}(\mathbf{U}_i)^{n+1} \frac{dt}{2} \tag{3.119}$$

In the previous expression, a second-order precision in space is added to a second-order precision in time. In the case of a first-order precision in space, the terms containing the bed level variation $\partial\mathbf{G}(\mathbf{U}_i)$ are equal to zero.