On the representability of actions of non-associative algebras

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Actions / Split extensions of groups

- Let B, X be groups. An *action* of B on X is a map

$$\xi\colon B \times X \to X\colon (b,x) \mapsto b \cdot x$$

  such that $(bb') \cdot x = b \cdot (b' \cdot x)$, $1_B \cdot x = x$ and $b \cdot (xx') = (b \cdot x)(b \cdot x')$.
- The *semi-direct product* $B \ltimes X$ w.r.t. ξ is $B \times X$ with multiplication

$$(b,x)(b',x') = (bb', x(b \cdot x')),$$

  where $1_{B \ltimes X} = (1_B, 1_X)$ and $(b,x)^{-1} = (b^{-1}, b^{-1} \cdot x^{-1})$.
- We can associate with ξ the split extension

$$\begin{tikzcd} 1 \arrow[r] & X \arrow[r, "i_2"] & B \ltimes X \arrow[r, shift left, "\pi_1"] & B \arrow[l, shift left, "i_1"] \arrow[r] & 1. \end{tikzcd}$$

- Conversely, given a split extension of groups

$$1 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 1$$

  one can define the action

$$B \times X \to X \colon (b, x) \mapsto \beta(b)k(x)\beta(b)^{-1}.$$

- There is a bijection $\mathrm{Act}(B, X) \cong \mathrm{SplExt}(B, X)$.
- In **Grp**, actions are represented by automorphisms: given a split extension of groups as above, one can define the homomorphism

$$\varphi \colon B \to \mathrm{Aut}(X) \colon b \mapsto \varphi_b,$$

  where $\varphi_b(x) = \beta(b)k(x)\beta(b)^{-1}$.
- Conversely, any homomorphism $\varphi \colon B \to \mathrm{Aut}(X)$ defines a semi-direct product $B \ltimes X$ with multiplication

$$(b, x)(b', x') = (bb', x\varphi_b(x'))$$

  and thus a split extension of B by X.

The semi-abelian context

- Semi-abelian categories were introduced in order to provide a categorical setting which would capture algebraic properties of groups, rings and algebras.
- Let $\mathcal{C}$ be a semi-abelian category (i.e. $\mathcal{C}$ is pointed, Barr-exact, Bourn-protomodular with finite coproducts).
- Let B, X be objects of $\mathcal{C}$. A *split extension* of B by X is a diagram

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

in $\mathcal{C}$ such that $\alpha \circ \beta = 1_B$ and (X, k) is a kernel of α.

- For any object X in $\mathcal{C}$, we consider the functor

$$\mathrm{SplExt}(-,X)\colon \mathcal{C}^{\mathrm{op}} \to \mathbf{Set}$$

where, for every object B in $\mathcal{C}$, $\mathrm{SplExt}(B,X)$ is the set of isomorphism classes of split extensions of B by X.
- There is an equivalence between split extensions and actions

$$\mathrm{SplExt}(-,X) \cong \mathrm{Act}(-,X).$$

Varieties of non-associative algebras

- A *non-associative algebra* A is a vector space A over a field $\mathbb{F}$ equipped with a bilinear operation $A \times A \to A \colon (x, y) \mapsto xy$ called the *multiplication*.
- We denote by **Alg** the category of non-associative algebras, where morphisms are the linear maps which preserve the multiplication.
- An *identity* of an algebra A is a non-associative polynomial $\varphi(x_1, \dots, x_n)$ such that $\varphi(a_1, \dots, a_n) = 0$, for every $a_1, \dots, a_n \in A$.
- Let I be a set of identities. The *variety of non-associative algebras* determined by I is the class $\mathcal{V}$ of all algebras which satisfy all the identities in I.
- Any variety of non-associative algebras $\mathcal{V}$ can be seen as a full subcategory of **Alg**. Any such variety is a semi-abelian category.

Examples

- **AbAlg** is the variety of *abelian* algebras: $xy = 0$.
- **Assoc** is the variety of *associative* algebras: $x(yz) - (xy)z = 0$.
- **CAlg** is the variety of *commutative* algebras: $xy - yx = 0$.
- **ACAlg** is the variety of *anti-commutative* algebras: $xy + yx = 0$.
- **CAssoc** is the variety of *commutative associative algebras*.
- **Lie** is the variety of *Lie algebras*, which is determined by anti-commutativity and the *Jacobi identity*: $xy + yx = 0$ and $x(yz) + y(zx) + z(xy) = 0$.
- **JJord** is the variety of *Jacobi–Jordan algebras*, which is determined by commutativity and the Jacobi identity.
- **Leib** is the variety of *(right) Leibniz algebras*, which is determined by the *(right) Leibniz identity*: $(xy)z - (xz)y - x(yz) = 0$.
- **Alt** is the variety of *alternative algebras*, which is determined by $(yx)x - y(xx) = 0$ and $x(xy) - (xx)y = 0$.

Action representable categories

Definition (F. Borceux, G. Janelidze and G. M. Kelly, 2005)

A semi-abelian category $\mathcal{C}$ is *action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ is representable. This means that there exists an object $[X]$ of $\mathcal{C}$, called the *actor* of X, and a natural isomorphism

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathcal{C}}(-, [X]).$$

- Prototype examples: **Grp**, **Lie** and any abelian category.
- In **Grp**, the actor of X is $\mathrm{Aut}(X)$.
- In **Lie**, the actor of X is $\mathrm{Der}(X)$.
- In any abelian category, the actor of X is 0.

- The notion of action representable category has proven to be quite restrictive.

Theorem (X. García Martínez, M. Tsishyn, T. Van der Linden and C. Vienne, 2021)

Let $\mathbb{F}$ *be an infinite field with* $\mathrm{char}(\mathbb{F}) \neq 2$ *and let* $\mathcal{V}$ *be a variety of non-associative algebras over* $\mathbb{F}$*. If* $\mathcal{V}$ *is action representable, then* $\mathcal{V} = \mathbf{AbAlg}$ *or* $\mathcal{V} = \mathbf{Lie}$*.*

- More recently, G. Janelidze introduced the notion of *weakly action representable category*, which includes a wider class of semi-abelian categories.

Weakly action representable categories

Definition (G. Janelidze, 2022)

A semi-abelian category $\mathcal{C}$ is *weakly action representable* if for every object X in $\mathcal{C}$, the functor $\mathrm{SplExt}(-, X)$ admits a weak representation. This means there exist an object T of $\mathcal{C}$ and a monomorphism of functors

$$\tau\colon \mathrm{SplExt}(-, X) \rightarrowtail \mathrm{Hom}_{\mathcal{C}}(-, T).$$

An object T as above is called *weak actor* of X, the pair (T, τ) is called *weak representation* of $\mathrm{SplExt}(-, X)$ and $(\varphi\colon B \to T) \in \mathrm{Im}(\tau_B)$ is called *acting morphism*.

Example

Every action representable category $\mathcal{C}$ is weakly action representable. In this case $T = [X]$ is the actor of X and τ is a natural isomorphism.

- **G. Janelidze, 2022.** The category **Assoc** is weakly action representable. A weak actor of X is the associative algebra of *bimultipliers* of X.

$$\mathrm{Bim}(X) = \{(f*-, -*f) \in \mathrm{End}(X)\times\mathrm{End}(X)^{\mathrm{op}} \mid f*(xy) = (f*x)y,$$
$$(xy)*f = x(y*f), x(f*y) = (x*f)y,\ \forall x, y \in X\}.$$

- **A. S. Cigoli, M. M., G. Metere, 2023.** The category **Leib** is weakly action representable. A weak actor of X is the Leibniz algebra of *biderivations* of X.

$$\mathrm{Bider}(X) = \{(d, D) \in \mathrm{End}(X)^2 \mid d(xy) = d(x)y + xd(y),$$
$$D(xy) = D(x)y - D(y)x,\ xd(y) = xD(y),\ \forall x, y \in X\}.$$

A counterexample: Jordan algebras

Definition

A *Jordan algebra* over a field $\mathbb{F}$ is a commutative algebra $(X, \cdot)$ over $\mathbb{F}$ which satisfies the *Jordan identity*

$$(xy)(xx) = x(y(xx)), \ \ \forall x, y \in X.$$

Theorem (G. Janelidze, 2022)

Every weakly action representable category is action accessible.

Remark (A. S. Cigoli and S. Mantovani, 2012)

The category **Jord** of Jordan algebras over $\mathbb{F}$ is not action accessible.

Conclusion: **Jord** is not weakly action representable.

Actions in varieties of algebras

- Let $\mathbb{F}$ be a field with $\mathrm{char}(\mathbb{F}) \neq 2$.
- Let $\mathcal{V}$ be a variety of non-associative algebras over $\mathbb{F}$.
- We suppose that $\mathcal{V}$ is *operadic*, i.e. all the identities of $\mathcal{V}$ are *multilinear*. This happens, for instance, when $\mathrm{char}(\mathbb{F}) = 0$.
- We also suppose that $\mathcal{V}$ is *action accessible*, or equivalently that $\mathcal{V}$ is an *Orzech category of interest*.
- This is equivalent to saying that the λ/μ *rules* hold, i.e. there exist $\lambda_1, \dots \lambda_8, \mu_1, \dots \mu_8 \in \mathbb{F}$ such that

$$\begin{aligned} x(yz) =& \lambda_1(xy)z + \lambda_2(yx)z + \lambda_3 z(xy) + \lambda_4 z(yx) + \\ &+ \lambda_5(xz)y + \lambda_6(zx)y + \lambda_7 y(xz) + \lambda_8 y(zx), \end{aligned}$$

and

$$\begin{aligned} (yz)x =& \mu_1(xy)z + \mu_2(yx)z + \mu_3 z(xy) + \mu_4 z(yx) + \\ &+ \mu_5(xz)y + \mu_6(zx)y + \mu_7 y(xz) + \mu_8 y(zx) \end{aligned}$$

are identities in $\mathcal{V}$.

The commutative case

- Let $\mathcal{V}$ be an operadic and action accessible variety of commutative algebras over $\mathbb{F}$. The λ/μ rules reduce to

$$x(yz) = \alpha(xy)z + \beta(xz)y,$$

for some $\alpha, \beta \in \mathbb{F}$.

Proposition

$\mathcal{V}$ is associative or satisfies the Jacobi identity:

$$x(yz) + y(zx) + z(xy) = 0.$$

Corollary

Let $\operatorname{char}(\mathbb{F}) \neq 3$*. If $\mathcal{V}$ is quadratic, then* $\mathcal{V} = \mathbf{CAssoc}$, $\mathcal{V} = \mathbf{JJord}$ *or* $\mathcal{V} = \mathbf{Nil}_2(\mathrm{Com})$.

Commutative associative algebras

Remark

Let X be a commutative associative algebra. Then

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(-), \mathrm{M}(X)),$$

where $\mathrm{M}(X) = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y\}$ is the associative algebra of *multipliers* of X and $U\colon \mathbf{CAssoc} \to \mathbf{Assoc}$ denotes the forgetful functor.

Theorem (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

Let X be a commutative associative algebra. Then $\mathrm{M}(X)$ is a commutative algebra if and only if the functor $\mathrm{SplExt}(-, X)$ is representable.

Example

If $\mathrm{Ann}(X)$ is trivial or $X^2 = X$, then $\mathrm{M}(X)$ is a commutative associative algebra.

Example

If $X = \mathbb{F}^2$ is the abelian two-dimensional algebra, then $\mathrm{M}(X) = \mathrm{End}(X)$ is not commutative.

Corollary (F. Borceux, G. Janelidze, G. M. Kelly, 2005)

The variety **CAssoc** *of commutative associative algebras over* $\mathbb{F}$ *is not action representable.*

Jacobi-Jordan algebras

Definition

A commutative algebra X is called a *Jacobi-Jordan* algebra if it satisfies the *Jacobi identity*

$$x(yz) + y(zx) + z(xy) = 0, \quad \forall x, y, z \in X.$$

Definition

An *anti-derivation* is a linear map $d\colon X \to X$ such that

$$d(xy) = -d(x)y - d(y)x, \quad \forall x, y \in X.$$

Example

The (left) multiplications $L_x = x \cdot -$, $x \in X$, are particular anti-derivations, called *inner anti-derivations*.

Split extensions

- Let

$$\begin{tikzcd} 0 \arrow[r] & X \arrow[r, "k"] & A \arrow[r, shift left, "\alpha"] & B \arrow[l, shift left, "\beta"] \arrow[r] & 0 \end{tikzcd}$$

be a split extension of Jacobi-Jordan algebras.
- We have that

$$A \cong B \ltimes X = (B \oplus X, \cdot)$$

and the Jacobi-Jordan algebra structure is given by

$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + b' * x).$$

- The action of B on X is defined by the bilinear map

$$b * x = \beta(b) \cdot_A k(x), \quad \forall b \in B, \, \forall x \in X.$$

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ using the formula above, we have the following characterization.

Proposition

$(B \oplus X, \cdot)$ is a Jacobi-Jordan algebra if and only if

1. $b * (xx') = -(b * x)x' - (b * x) * x'$;
2. $(bb') * x = -b * (b' * x) - b' * (b' * x)$;

for every $b, b' \in B$ and $x, x' \in X$.

In an equivalent way, a split extension of B by X (or an action of B on X) can be described as a linear map

$$B \to \mathrm{ADer}(X) \colon b \mapsto b * -$$

such that

$$(bb') * - = \{b * -, b' * -\}, \quad \forall b, b' \in B,$$

where $\{f, f'\} = -f \circ f' - f' \circ f$ denotes the *anti-commutator* between two linear endomorphisms of X.

Remark

The space $\mathrm{ADer}(X)$ endowed with the anti-commutator $\{-,-\}$ is not in general a (Jacobi-Jordan) algebra.

Example

Let $X = \mathbb{F}$ be the abelian one-dimensional algebra. Then $\mathrm{ADer}(X) = \mathrm{End}(X) \cong \mathbb{F}$ does not satisfy the Jacoby identity.

Definition

A *partial algebra* is a vector space X endowed with a *bilinear partial operation*

$$\Omega \subseteq X \times X \to X.$$

A homomorphism of partial algebras is a linear map $f\colon X \to Y$ such that $f(xy) = f(x)f(y)$, for every $x, y \in X$ such that xy and $f(x)f(y)$ are defined.

Definition

We denote by **PAlg** the category whose objects are the partial algebras over $\mathbb{F}$ and morphisms are the homomorphisms of partial algebras.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

Let X be a Jacobi-Jordan algebra. Then

(i) $\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{PAlg}}(U(-), \mathrm{ADer}(X))$, *where* $U\colon \mathbf{JJord} \to \mathbf{PAlg}$ *denotes the forgetful functor.*

(ii) *if* $\mathrm{ADer}(X)$ *is a Jacobi-Jordan algebra, then it is the actor of* X*.*

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

- *The variety* $\mathbf{Nil}_2(\mathrm{Com})$ *is weakly action representable;*
- *A weak actor of an object* X *of* $\mathbf{Nil}_2(\mathrm{Com})$ *is the abelian algebra*

$$[X]_2 = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y = 0,\ \forall x \in X\}.$$

- *An arrow* $B \to [X]_2 \colon b \mapsto b * -$ *is an acting morphism if and only if*

$$b * (b' * x) = 0, \quad \forall b, b' \in B,\ \forall x \in X.$$

The non-commutative case

- Let $\mathcal{V}$ be an operadic and action accessible variety of non-associative algebras. We fix a set of multilinear identities

$$\Phi_{k,i}(x_1, \ldots, x_k) = 0, \quad k = \deg(\Phi_{k,i}),\ i = 1, \ldots, n,$$

which determine $\mathcal{V}$.

- Our goal is to find a vector space $\mathcal{E}(X) \leq \mathrm{End}(X)^2$ together with a bilinear *partial* operation

$$\langle -, - \rangle \colon \Omega \subseteq \mathcal{E}(X) \times \mathcal{E}(X) \to \mathcal{E}(X),$$

such that we can associate in a natural way a morphism of algebras $B \to \mathcal{E}(X)$, with every split extension of B by X in $\mathcal{V}$.

Split extensions

- Let
$$0 \longrightarrow X \xrightarrow{k} A \underset{\beta}{\overset{\alpha}{\rightleftarrows}} B \longrightarrow 0$$
be a split extension in $\mathcal{V}$.
- We have that
$$A \cong B \ltimes X = (B \oplus X, \cdot)$$
with
$$(b, x) \cdot (b', x') = (bb', xx' + b * x' + x * b').$$
- The action of B on X is defined by the pair of bilinear maps
$$l \colon B \times X \to X, \quad r \colon X \times B \to X,$$
with $b * x' = l(b, x') = \beta(b) \cdot_A k(x')$, $x * b' = r(x, b') = k(x) \cdot_A \beta(b')$.

Conversely, if we define the product $(\cdot)$ on $B \oplus X$ as above:

Proposition

Let X and B be two algebras in $\mathcal{V}$. Given a pair of bilinear maps

$$l\colon B \times X \to X, \quad r\colon X \times B \to X,$$

we construct $(B \oplus X, \cdot)$ as before. Then $(B \oplus X, \cdot)$ is an object of $\mathcal{V}$ if and only if

$$\Phi_{k,i}(\alpha_1, \ldots, \alpha_k) = 0, \quad \forall i = 1, \ldots, n,$$

where at least one of the α_1, $\ldots$, α_k is an element of of the form $(0, x)$, with $x \in X$, and the others are of the form $(b, 0)$, with $b \in B$.

- The resulting algebra is the *semi-direct product* $B \ltimes X$.
- We denote $b * x \coloneqq (b, 0) \cdot (0, x)$ and $x * b \coloneqq (0, x) \cdot (b, 0)$.

The partial algebra $\mathcal{E}(X)$

We define

$$\mathcal{E}(X) = \{f = (f * -, - * f) \in \mathrm{End}(X)^2 \mid \Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0\}$$

for each choice of $\alpha_j = f$ and $\alpha_t \in X$, where $t \neq j$, $fx \coloneqq f * x$, $xf \coloneqq x * f$, together with a bilinear map

$$\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2 \colon (f, g) \mapsto h = (h * -, - * h)$$

where

$$\begin{aligned} x * h =& \lambda_1 (x * f) * g + \lambda_2 (f * x) * g + \lambda_3 g * (x * f) + \lambda_4 g * (f * x) \\ &+ \lambda_5 (x * g) * f + \lambda_6 (g * x) * f + \lambda_7 f * (x * g) + \lambda_8 f * (g * x) \end{aligned}$$

and

$$\begin{aligned} h * x =& \mu_1 (x * f) * g + \mu_2 (f * x) * g + \mu_3 g * (x * f) + \mu_4 g * (f * x) \\ &+ \mu_5 (x * g) * f + \mu_6 (g * x) * f + \mu_7 f * (x * g) + \mu_8 f * (g * x). \end{aligned}$$

Example

- If $\mathcal{V} = \mathbf{Assoc}$, then $\mathcal{E}(X) \cong \mathrm{Bim}(X)$.
- If $\mathcal{V} = \mathbf{Leib}$, then $\mathcal{E}(X) \cong \mathrm{Bider}(X)$.

Remark

The bilinear map $\langle -, - \rangle \colon \mathcal{E}(X) \times \mathcal{E}(X) \to \mathrm{End}(X)^2$ defines a partial operation $\langle -, - \rangle \colon \Omega \to \mathcal{E}(X)$, where Ω is the preimage $\langle -, - \rangle^{-1}(\mathcal{E}(X))$ of the inclusion $\mathcal{E}(X) \hookrightarrow \mathrm{End}(X)^2$.

Alternative algebras

Let $\mathcal{V} = \mathbf{Alt}$ be the variety of alternative algebras. Then $\mathcal{E}(X)$ consists of the pairs $(f * -, - * f) \in \mathrm{End}(X)^2$ satisfying

$$f * (xy) = (x * f)y + (f * x)y - x(f * y),$$

$$(xy) * f = x(f * y) + x(y * f) - (x * f)y,$$

$$x(y * f) = (yx) * f + (xy) * f - y(x * f)$$

$$(f * x)y = f * (yx) + f * (xy) - (f * y)x$$

for any $x, y \in X$, and the bilinear map

$$\langle (f * -, - * f), (g * -, - * g) \rangle = (h * -, - * h)$$

is given by

$$h * x = -(f * x) * g + f * (g * x) + f * (x * g)$$

$$x * h = (x * f) * g + (f * x) * g - f * (x * g).$$

One can check that $\langle -, - \rangle$ does not define an algebra structure.

Unitary alternative algebras

If X is a *unitary* alternative algebra with $e = 1_X$, then

$$\begin{aligned} f * x =& (x * f)e + (f * x)e - x(f * e), \\ x * f =& e(f * x) + e(x * f) - (e * f)x, \\ x * f =& x * f + x * f - x(e * f), \\ f * x =& f * x + f * x - (f * e)x. \end{aligned}$$

An element of $\mathcal{E}(X)$ is determined by $\alpha = f * e = e * f \in X$, i.e.

$$\mathcal{E}(X) \cong \{(\alpha, \alpha) \mid \alpha \in X\} \cong X$$

is an object of **Alt**.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

Let $\mathcal{V}$ be a variety and let $U\colon \mathcal{V} \to \mathbf{PAlg}$ denote the forgetful functor.

- *Let X be an object of $\mathcal{V}$. There exists a monomorphism of functors*

$$\tau\colon \operatorname{SplExt}(-, X) \rightarrowtail \operatorname{Hom}_{\mathbf{PAlg}}(U(-), \mathcal{E}(X))$$

where τ_B sends an element of $\operatorname{SplExt}(B, X)$ to the corresponding partial algebra homomorphism

$$U(B) \to \mathcal{E}(X)\colon b \mapsto (b * -, - * b).$$

- *The morphism $U(B) \to \mathcal{E}(X)\colon b \mapsto (b * -, - * b)$ belongs to $\operatorname{Im}(\tau_B)$ if and only if $\Phi_{k,i}(\alpha_1, \dots, \alpha_k) = 0$, as before.*
- *If $(\mathcal{E}(X), \langle -, - \rangle)$ is an object of $\mathcal{V}$, then $(\mathcal{E}(X), \tau)$ is a weak representation of $\operatorname{SplExt}(-, X)$.*

Definition

The partial algebra $\mathcal{E}(X)$ is called *external weak actor* of X. When τ is a natural isomorphism, we say that $\mathcal{E}(X)$ is an *external actor* of X.

Example

We may check that

1. if $\mathcal{V} = \mathbf{AbAlg}$, then $\mathcal{E}(X) = 0$ is the actor of X;
2. if $\mathcal{V} = \mathbf{Lie}$, then $\mathcal{E}(X) \cong \mathrm{Der}(X)$ is the actor of X;
3. if $\mathcal{V} = \mathbf{CAssoc}$, then $\mathcal{E}(X) \cong \mathrm{M}(X)$ is an external actor of X;
4. if $\mathcal{V} = \mathbf{JJord}$, then $\mathcal{E}(X) \cong \mathrm{ADer}(X)$ is an external actor of X;
5. if $\mathcal{V} = \mathbf{Nil}_2(\mathbf{CAlg})$, then $\mathcal{E}(X) \cong [X]_2$ is a weak actor of X;
6. if $\mathcal{V} = \mathbf{Alt}$ over a field $\mathbb{F}$ with $\mathrm{char}(\mathbb{F}) \neq 2$ and X is unitary, then $\mathcal{E}(X) \cong X$ is an alternative algebra and

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Alt}}(-, X).$$

What happens if $\mathcal{E}(X)$ is not an object of $\mathcal{V}$?

Example

If $\mathcal{V} = \mathbf{Leib}$, then $\mathcal{E}(X) \cong \mathrm{Bider}(X)$ as vector spaces. Choosing the λ/μ rules as

$$x(yz) = (xy)z - (xz)y, \qquad (yz)x = (yx)z - y(xz),$$

we get the standard multiplication of $\mathrm{Bider}(X)$ that defines a weak actor in **Leib**. If we choose the λ/μ rules as

$$x(yz) = (xy)z - (xz)y, \qquad (yz)x = (yx)z + y(zx),$$

we get a non-associative multiplication which, in general, does not define a Leibniz algebra structure on $\mathrm{Bider}(X)$.

Remark

Let X be a commutative associative algebra. Then

$$\mathrm{SplExt}(-, X) \cong \mathrm{Hom}_{\mathbf{Assoc}}(U(-), \mathrm{M}(X)),$$

where $\mathrm{M}(X) = \{f \in \mathrm{End}(X) \mid f(xy) = f(x)y\}$ is the associative algebra of multipliers and $U\colon \mathbf{CAssoc} \to \mathbf{Assoc}$ denotes the forgetful functor.

Theorem (J. Brox, X. García Martínez, M. M., T. Van der Linden and C. Vienne, 2024)

The category **CAssoc** *is weakly action representable.*

- **Key:** *Amalgamation property.*

Open problems and future directions

- Does the converse of the implication

$$\textit{weakly action representable category} \Rightarrow \textit{action accessible category}$$

  hold in the context of varieties of algebras over a field?
- To generalize the construction of the external weak actor to action accessible varieties of algebras with two not necessarily associative bilinear operations (**Pois**, **CPois**,...).
- To study the representability of actions in the context of *ideally exact categories* (**Ring**, **MVAlg**, categories of unitary algebras,...).

Thanks for your attention!

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