CDS PRICING WITH FRACTIONAL HAWKES PROCESSES

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CDS Pricing with Fractional Hawkes Processes

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Abstract

We propose a fractional self-exciting model for the risk of corporate default. We study the properties of a time-changed version of an intensity based model. As a time-change, we use the inverse of an α-stable subordinator. Performing such a time-change allows to incorporate two particular features in the survival probability curves implied by the model. Firstly, it introduces random periods of time where the survival probability is frozen, thereby modeling periods of time where the viability of the company is not threatened. Secondly, the time-change implies possible sharp drops in the survival probability. This feature corresponds to the occurence of one-time events that threaten the creditworthiness of the company. We show that the joint probability density function and Laplace transform of the time-changed intensity and associate compensator are solutions of fractional Fokker-Planck equations. After a discussion regarding approximation of Caputo fractional derivatives, we describe a simple and fast numerical method to solve the Fokker-Planck equation of the Laplace transform. This Laplace transform is used to obtain the survival probabilities implied by our model. Finally, we use our results to calibrate the model to real market data and show that it leads to an improvement of the fit.

Keywords: Finance, Credit risk, Caputo derivatives, Self-exciting processes.

Introduction

The most common paradigms for credit risk modeling are structural and intensity based models. The first approach is called firm value model, or structural default model. In this approach, the balance sheet of the firm is explicitly modeled and the default is triggered when assets fall below liabilities. For example of such models, we refer the reader to Merton (1974), Black and Cox (1976), Geske (1977), Longstaff and Schwartz (1995), Hainaut and Colwell (2016), Ayadi et al. (2016) and Ballota et al. (2019). The second approach, the one that is used in this paper, is called the reduced-form or intensity-based credit model. This second

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approach was initiated by the work of Duffie and Singleton (1999) and is for example used in Mari and Renò (2005), Hainaut and Le Courtois (2014) and Brigo and Vrins (2018). Textbooks concerning this second kind of models are Bielecki and Rutkowski (2002), Duffie and Singleton (2003), Lando (2004), Jeanblanc et al. (2009) and Schoutens and Cariboni (2009). This article deals with a new type of time-changed intensity-based credit model.

The intensity of our model is a mean-reverting process, a kind of process that is often used in finance, for example in Ekvall et al. (1997) or Wong and Lo (2009). The type of credit model used in this paper has an additional feature: it takes into account the spillover between defaults, also called contagion. The importance of this phenomenon has been emphasized by the financial crisis of 2008. A recent endogenous way to model contagion is provided by self-exciting point processes. For these processes, the default arrival intensity at a given point in time (that is also the instantaneous probability to observe a jump) depends on the number of past defaults, as well as on the elapsed time since the occurence of these defaults. This approach finds its origin in Hawkes (1971a, b) and Hawkes and Oakes (1974). More recently, Errais et al. (2010) develop a self-exciting approach for modeling defaults in a credit portfolio. In the most common and simplest specification, the default arrival intensity process is persistent and suddenly increases when a default occurs in the portfolio. Ait-Sahalia et al. (2015) use a similar approach for modeling contagion between different markets. In Hainaut (2016 a,b), self-exciting jump-diffusions are used for modeling the term structure of interest rates.

This article deals with the modeling of two phenomena within an intensity based model. The first phenomenon is the presence of periods of time during which the survival of a company is not threathened. In order to properly model such periods of time, the survival probability of the company should remain constant. This contrasts with the usual intensity based model, where the survival probability strictly decreases as time passes. The second phenomenon we aim to model is the random occurrences of one-time events that threaten the short term viability of the company. The proper modeling of this second phenomena would imply the possibility to observe very sharp drops in the survival probability of a company. This is again a feature that is not observed in classic intensity-based credit models. Obtaining those two features involves a non-Markov time-change built with the inverse of an α-stable subordinator. This time-change approach is applied to diffusions in physics for describing the movement of heavy particles that can get immobilized (see, e.g. Metzler and Klafter 2004 or Eliazar, Klafter 2004). This type of time-changed Brownian motions, called sub-diffusions, are also popular in econophysics (see Scalas 2006) and applied to financial derivatives by Magdziarz (2009, a) for illiquid markets. As shown in, e.g. Magdziarz (2009, b), the probability density function (PDF) of sub-diffusions is solution of a fractional Fokker-Planck equation. This equation is proposed in Barkai et al. (2000) and Metzler et al. (1999). Articles of Leonenko et al. (2013, a) Leonenko et al. (2013, b) go a step further and explore fractional Pearson diffusions and their correlation.

This article is organized in 3 sections. The first section introduces our framework, without dealing with the time-change. It is also concerned with the derivation of some useful results regarding this framework. The second section deals with the first main contribution of this paper: a model that incorporates the two aforementioned features and a study of its properties. To build such a model, a non-Markov time-change is added to the setting introduced in Section 1. Results of Section 1 are extended to this new setting with the help of fractional calculus. The third section deals with our second main contribution: a very simple numerical method to compute probability of survival that works in both settings, with or without time-changing by the inverse of an α-stable subordinator. Finally, our last main contribution is to show that our numerical method can be used to calibrate the model to real data and obtain a better fit.

1. Non-Fractional Setting

In the intensity approach to credit risk, defaults of companies are modelled by jumps of point processes. A jump of a point process corresponds to the default of a company. In this paper, we are interested in the time of default of a single company. The time of default is the time of the first jump of the point process $(D_t)_{t\geqslant 0}$, that is the positive random variable $\inf\{t > 0 : D_t \geqslant 1\}$. We start by describing precisely how the process $(D_t)_{t\geqslant 0}$ is built.

Let $(N_t)_{t\geqslant 0}$ be a point process on a filtered probability space $(\Omega, \mathbb{F}, P)$ where $\mathbb{F} = (\mathcal{F}_t)_{t\geqslant 0}$ is a filtration to be defined more precisely later on. We denote the intensity of $(N_t)_{t\geqslant 0}$ by $(\lambda_t)_{t\geqslant 0}$, another stochastic process. The intensity represents the instantaneous probability to observe a jump in the process $(N_t)_{t\geqslant 0}$. Let $\xi_1, \xi_2, \dots$ be independent identically distributed (i.i.d.) exponential random variables with parameter $\rho > 0$. Consequently, if the distribution of $\xi_1, \xi_2, \dots$ is given by the measure ν on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$, we have

$$\mathrm{d}\nu(x) = \rho e^{-\rho x}\mathrm{d}x, \quad \rho > 0$$

for all $x \in \mathbb{R}^+$, and $\mathbb{E}[\xi_i] = \rho^{-1}$ for all i. We also define the process $(P_t)_{t\geqslant 0}$ as

$$P_t = \sum_{k=1}^{N_t} \xi_k.$$

It is a pure positive jump process whose jump intensity is the stochastic process $(\lambda_t)_{t\geqslant 0}$. The dynamics of this intensity process is assumed to satisfy the stochastic differential equation (SDE)

$$\mathrm{d}\lambda_t = \kappa(\theta - \lambda_t)\mathrm{d}t + \eta \mathrm{d}P_t \tag{1}$$

where $\kappa, \theta, \eta > 0$ and $\lambda_0 \in [\theta, +\infty[$. As a consequence of SDE (1), the intensity depends upon the history of the process $(P_t)_{t\geqslant 0}$. This allows the process $(P_t)_{t\geqslant 0}$ to exhibit a self-exciting behaviour: a jump of the process $(P_t)_{t\geqslant 0}$ induces a jump in the intensity $(\lambda_t)_{t\geqslant 0}$, which in turn implies an increase of the probability to observe

another jump in $(P_t)_{t\geqslant 0}$. This is illustrated in Figure 1, where each jump of $(\lambda_t)_{t\geqslant 0}$ correspond to a jump of $(P_t)_{t\geqslant 0}$ (that is not represented here). This figure is obtained with the parameters of Table 1 (fractional case), with λ_0 fixed to θ. The parameter θ represents a minimum level of instantaneous jump probability, as well

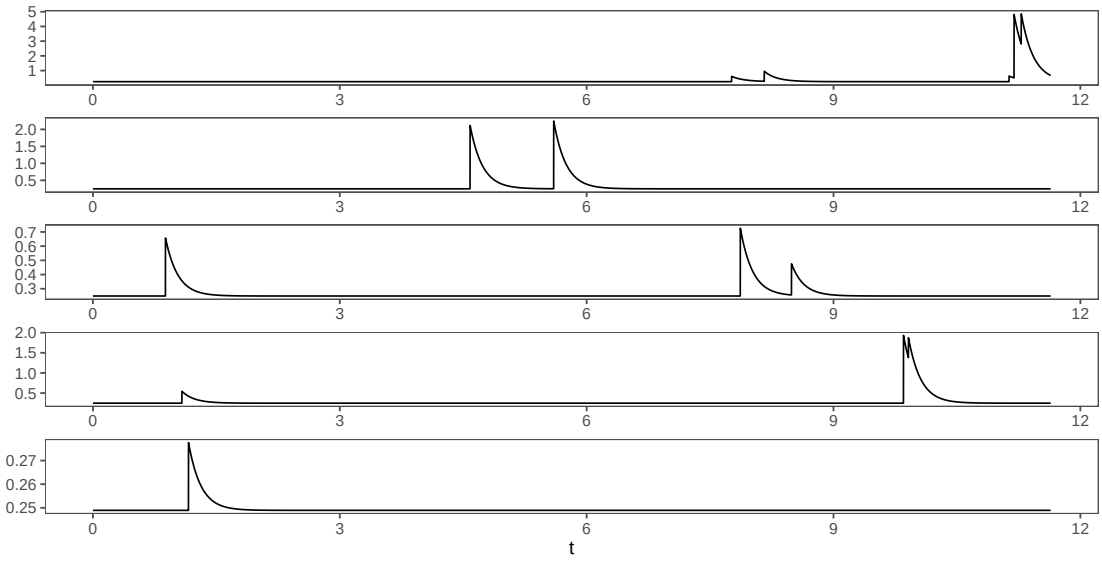


Figure 1: 5 Simulated Paths of $(\lambda_t)_{t\geqslant 0}$

as a level to which this instantaneous probability tends to revert. The fact that $\lambda_t \geqslant \theta$ almost surely for all t plays a role throughout this paper in the derivation of formulas, because it ensures that the PDF of λ_t equals 0 at the point 0. More precisely, if we denote by

$$p(t, x|s, y) = \frac{\partial}{\partial x} P(\lambda_t \leqslant x | \lambda_s = y)$$

the PDF of λ_t given λ_s, $t \geqslant s$, we always have $p(t, 0|s, y) = 0$. The parameter κ in Equation (1) represents the speed of the reversion towards θ.

From an economic point of view, $(\lambda_t)_{t\geqslant 0}$ represents the systemic risk in the economy. Large values of this process represent bad times for companies, increasing the odds to observe defaults. A jump of $(P_t)_{t\geqslant 0}$ represents a bad economic event and induces an increase of the default intensity. The self exciting behavior of $(\lambda_t, P_t)_{t\geqslant 0}$ aims to model the clustering of defaults observed in practice, see, e.g. Ait-Sahalia et al. (2015).

The time of default of the company is modeled as the first jump of the point process $(D_t)_{t\geqslant 0}$, whose intensity is assumed to be $(\lambda_t)_{t\geqslant 0}$. Despite the processes $(D_t)_{t\geqslant 0}$ and $(P_t)_{t\geqslant 0}$ both share the same intensity $(\lambda_t)_{t\geqslant 0}$, there is a very important difference between them. In order to clarify this difference, let us say a few words about how we decompose the filtration $\mathbb{F}$. We denote by $\mathbb{N} = (\mathcal{N}_t)_{t\geqslant 0}$ the natural filtration of $(N_t)_{t\geqslant 0}$. The filtration $\mathbb{H} = (\mathcal{H}_t)_{t\geqslant 0}$ is the one generated by the triplet $(N_t, P_t, \lambda_t)_{t\geqslant 0}$. A first important assumption

is that the sigma-algebra generated by all the random jump sizes $\xi_1, \xi_2, \ldots$ is independent of $\mathcal{N}_t$, for all t. Finally, the filtration $\mathbb{D} = (\mathcal{D}_t)_{t\geqslant 0}$ is the natural filtration of $(D_t)_{t\geqslant 0}$. The filtration $\mathbb{F}$ is then defined as $\mathbb{F} = \mathbb{H} \vee \mathbb{N} \vee \mathbb{D}$. We end this discussion about filtrations with a second important assumption: conditional on $\mathbb{H}$, $(D_t)_{t\geqslant 0}$ is a non-homogeneous Poisson process, in particular for any $0 \leqslant s \leqslant t$ and $k \in \mathbb{N}$,

$$P(D_s = k|\mathcal{H}_t) = \frac{\Lambda_s^k}{k!} e^{-\Lambda_s}$$

where $(\Lambda_t)_{t\geqslant 0} := (\int_0^t \lambda_u \mathrm{d}u)_{t\geqslant 0}$ is the compensator of $(D_t)_{t\geqslant 0}$. From this last assumption follows the fundamental difference between the processes $(D_t)_{t\geqslant 0}$ and $(N_t)_{t\geqslant 0}$: by definition of $(\lambda_t)_{t\geqslant 0}$, the jumps of $(N_t)_{t\geqslant 0}$ exactly correspond to the jumps of $(\lambda_t)_{t\geqslant 0}$, whereas it is absolutely not the case for $(D_t)_{t\geqslant 0}$. In other words, the jumps of $(N_t)_{t\geqslant 0}$ are observed in $(\lambda_t)_{t\geqslant 0}$, while the jumps of $(D_t)_{t\geqslant 0}$ are not. From an economic point of view, this means that the company whose default is modeled by the first jump of $(D_t)_{t\geqslant 0}$ is not systemic, but it is sensitive to defaults of systemic companies.

The first proposition is about the expectation and variance of the intensity $(\lambda_t)_{t\geqslant 0}$. It allows to derive some conditions on the parameters to ensure that the intensity process does not explode. The proof is in Appendix 1.

Proposition 1. *Assume that $\kappa > \frac{\eta}{\rho}$. For any $t \geqslant 0$, we have*

$$\mathbb{E}[\lambda_t] = e^{-a_1 t}\lambda_0 - \frac{\kappa\theta}{a_1}(e^{-a_1 t} - 1) \tag{2}$$

and

$$\mathbb{V}\mathrm{ar}(\lambda_t) = \frac{(1 - e^{-a_1 t})\left(2a_2\lambda_0 e^{-a_1 t} + a_3(1 - e^{-a_1 t})\right)}{2a_1} \tag{3}$$

where $a_1 := \kappa - \frac{\eta}{\rho}$, $a_2 := 2\left(\frac{\eta}{\rho}\right)^2$ and $a_3 := \kappa\theta\frac{a_2}{a_1}$.

From this result, we observe that the condition that ensures the stability of the model is $\kappa > \frac{\eta}{\rho}$. That is, the speed of the mean reversion must be sufficient to prevent the process from exploding. If this condition is satisfied, we have the following long-term values for the expected value and variance

$$\lim_{t\to+\infty} \mathbb{E}[\lambda_t] = \frac{\kappa\theta}{\kappa - \frac{\eta}{\rho}}$$

$$\lim_{t\to+\infty} \mathbb{V}\mathrm{ar}(\lambda_t) = \kappa\theta\left(\frac{\eta}{\eta - \rho\kappa}\right)^2.$$

In the following, we denote

$$p(t, x_1, x_2|s, y_1, y_2) = \frac{\partial^2 P(\lambda_t \leqslant x_1, \Lambda_t \leqslant x_2|\lambda_s = y_1, \Lambda_s = y_2)}{\partial x_1 \partial x_2} \tag{4}$$

and

$$\phi(t, z_1, z_2|s, y_1, y_2) = \mathbb{E}\left[\exp\left\{-z_1\lambda_t - z_2\Lambda_t\right\}|\lambda_s = y_1, \Lambda_s = y_2\right]$$

where $t \geqslant s \geqslant 0$, $y_1 \geqslant \theta$, $y_2 \geqslant 0$ and $\Lambda_t := \int_0^t \lambda_u \mathrm{d}u$. Note that p corresponds to the conditional PDF of (λ_t, Λ_t) given (λ_s, Λ_s) whereas ϕ is the corresponding Laplace transform. The conditional PDF p satisfies a partial differential equation (PDE).

Proposition 2. *The joint PDF $p(t, x_1, x_2|s, y_1, y_2)$ of (λ_t, Λ_t) is solution of the Fokker-Planck equation*

$$\begin{aligned}\frac{\partial p(t,x_1,x_2|s,y_1,y_2)}{\partial t} &= \kappa p(t,x_1,x_2|s,y_1,y_2) - \frac{\partial p(t,x_1,x_2|s,y_1,y_2)}{\partial x_2} x_1 \\ &- \frac{\partial p(t,x_1,x_2|s,y_1,y_2)}{\partial x_1}\kappa(\theta - x_1) - \eta\mathbb{E}[\xi p(t, x_1 - \eta\xi, x_2|s,y_1,y_2)] \\ &+ x_1\mathbb{E}[p(t,x_1-\eta\xi,x_2|s,y_1,y_2) - p(t,x_1,x_2|s,y_1,y_2)]\end{aligned} \tag{5}$$

with boundary condition $p(s, x_1, x_2|s, y_1, y_2) = \delta_{\{x_1-y_1, x_2-y_2\}}$, where δ stands for the Dirac measure located at $(0, 0) \in \mathbb{R}^2$.

The proof can be found in Appendix 1. The next result is the fact that the Laplace transform of $p(t, z_1, z_2|s, y_1, y_2)$ also satisfies a PDE. The proof has also been relegated to Appendix 1 and relies essentially on PDE (5).

Proposition 3. *The Laplace transform $\phi(t, z_1, z_2|s, y_1, y_2)$ of the joint PDF $p(t, x_1, x_2|s, y_1, y_2)$ is solution of the following forward Kolmogorov equation*

$$\frac{\partial \phi(t,z_1,z_2|s,y_1,y_2)}{\partial t} = -z_1\kappa\theta\phi(t,z_1,z_2|s,y_1,y_2) + \frac{\partial \phi(t,z_1,z_2|s,y_1,y_2)}{\partial z_1}\gamma(z_1,z_2)$$

where $\gamma(z_1, z_2) := 1-\kappa z_1-\mathbb{E}[e^{-z_1\eta\xi}]+z_2$. Moreover, the boundary conditions $\phi(s, z_1, z_2|s, y_1, y_2) = \exp\{-z_1y_1 - z_2y_2\}$ and $\phi(t, 0, 0|s, y_1, y_2) = 1$ hold.

The next section aims to extend the previous results to the time-changed framework.

2. Fractional Setting

This second section extends the model to the fractional case with the help of a change of time. This allows to obtain fractional Hawkes processes, as introduced in Hainaut (2020). We start this second section by describing the subordinator and some of its properties.

2.1. The Subordinator

We use a time-change with a subordinator that is the inverse of an α-stable Lévy process. Such processes are discussed in, e.g. the Chapter 3 of Sato (1999) and in Janicki and Weron (1994) We briefly introduce this class of processes and derive some properties of their inverses.

Definition 1. Let $(U_t)_{t\geqslant 0}$ be a stochastic process and $\alpha \in (0, 1)$. We say that the process $(U_t)_{t\geqslant 0}$ is an α-stable subordinator if it is an increasing Lévy process that satisfies

$$\mathbb{E}\left[e^{-uU_t}\right] = e^{-tu^\alpha}$$

for all $t \geqslant 0$.

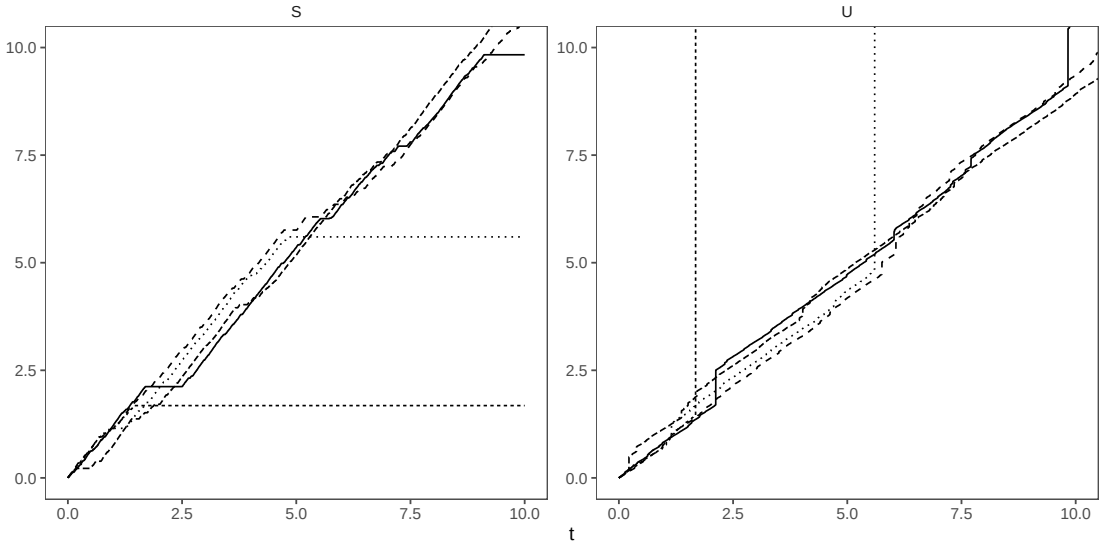


Figure 2: 5 Paths of $(U_t)_{t\geqslant 0}$ and Corresponding Paths of $(S_t)_{t\geqslant 0}$

Throughout all the upcoming developments, we assume that $(U_t)_{t\geqslant 0}$ is an α-stable subordinator. The inverse of $(U_t)_{t\geqslant 0}$ will be denoted by $(S_t)_{t\geqslant 0}$, and defined as

$$S_t := \inf\{s \geqslant 0 : U_s \geqslant t\}.$$

Figure 2 shows five simulated paths of $(U_t)_{t\geqslant 0}$ and the corresponding paths of $(S_t)_{t\geqslant 0}$. The algorithm used to obtain such simulations is from Magdziarz (2009, b) and is described in details in Appendix 2. The graphs are obtained with the fractional order $\alpha = 0.94151$ of Table 1 (fractional case). A property of interest in the paths of the process $(S_t)_{t\geqslant 0}$ is the presence of periods of time where these paths remain constant, as we can observe in Figure 2. The motionless periods correspond to the jumps of $(U_t)_{t\geqslant 0}$, which follows from the definition of $(S_t)_{t\geqslant 0}$. Figure 3 shows 50 simulations of survival probability curves[2]. The paths in the time-changed model are much more heterogeneous. In particular, the time-changed setting allows to model companies that do not encounter troubles during long periods of time, therefore having survival probabilities remaining constant during these periods. That property is absolutely not mimicable by a usual credit model, as we can see on the left graph of Figure 2. Indeed, the striclty positive intensity of these models necessarily imply decreasing survival probability curves. The PDF of U_t will be denoted by $p_U(t,\tau) = \frac{\partial}{\partial \tau} P(U_t \leqslant \tau)$ whereas the PDF of S_t will be denoted by $g(t,\tau) = \frac{\partial}{\partial \tau} P(S_t \leqslant \tau)$. The relations between the functions p_U and g are described in the following result. Proofs can be found in Hainaut (2020).

Proposition 4. *The following relations hold for all* t, τ

[2]We call these processes survival probability because in an intensity model, the probability of survival until time t is $\mathbb{E}[\exp\{-\Lambda_t\}]$.

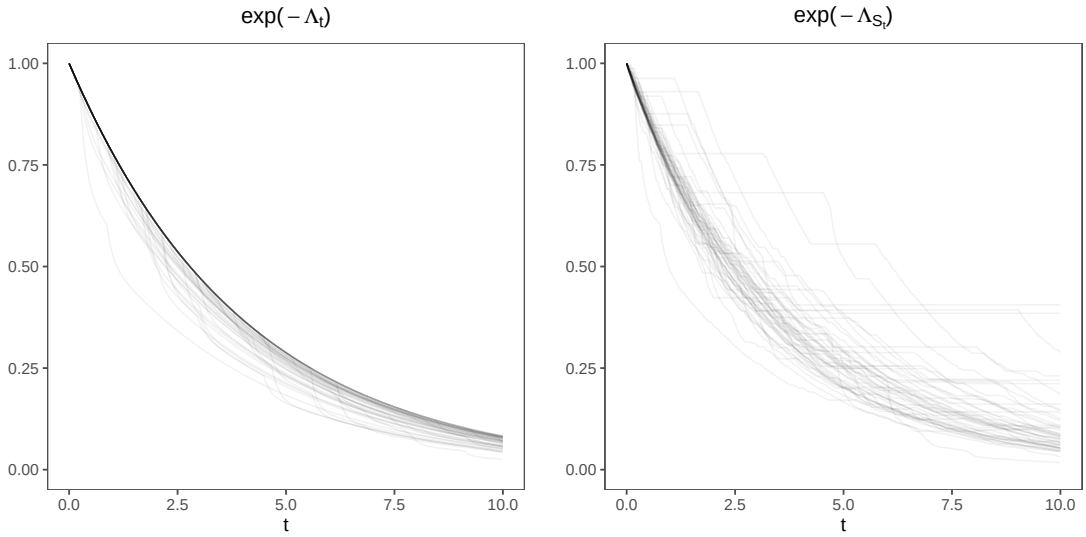


Figure 3: 50 Paths of $(\exp\{-\Lambda_t\})_{t\geqslant 0}$ and $(\exp\{-\Lambda_{S_t}\})_{t\geqslant 0}$

(1) $S_t =^d \left(\frac{t}{U_1}\right)^\alpha$*, i.e. both random variables share the same probability distribution;*

(2) $p_U(\tau,t) = \tau^{-\frac{1}{\alpha}} p_U(1,\tau^{-\frac{1}{\alpha}}t)$*;*

(3) $\tau g(t,\tau) = \frac{t}{\alpha} p_U(\tau,t)$*;*

(4) $g(t,\tau) = -\frac{\partial}{\partial\tau}\int_0^t p_U(\tau,u)\mathrm{d}u$*.*

We now state a result about the Laplace transform of $(S_t)_{t\geqslant 0}$. It will be used later as a boundary condition.

Proposition 5. *The Laplace transform of* $(S_t)_{t\geqslant 0}$ *is given by*

$$\begin{aligned}\mathbb{E}[e^{-\omega S_t}] &:= \int_0^{+\infty} e^{-\omega\tau} g(t,\tau)\mathrm{d}\tau \\ &= \mathrm{E}_\alpha(-\omega t^\alpha)\end{aligned}$$

where E_α *denotes the Mittag-Leffler function*

$$\mathrm{E}_\alpha(z) := \sum_{k=0}^{+\infty} \frac{z^k}{\Gamma(k\alpha+1)}.$$

Moreover, by differentiation of the Laplace transform of S_t*, we have for any* $n\in\mathbb{N}$

$$\mathbb{E}[S_t^n] = \frac{n! t^{n\alpha}}{\Gamma(n\alpha+1)}.$$

The reader is refered to Piryatinska et al. (2005) for a proof.

2.2. Caputo Derivatives

The main difference between the results obtained in the fractional setting (i.e. time-changed setting) compared to the non-fractional one is the replacement of the partial derivative with respect to time by a fractional derivative in the PDE's, turning them into fractional PDE's (FPDE). In the following, we present the needed material about Caputo fractional derivatives. Textbooks about fractional calculus are, e.g. Podlubny (1999) and Diethelm (2010). We start with the definition of Caputo derivative.

Definition 2. For a function $h : \mathbb{R} \to \mathbb{R}$, the Caputo derivative of h of order $\alpha \in (0, 1)$ is defined as

$$\frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}(t) := \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} h(s) \mathrm{d}s. \tag{6}$$

Caputo derivatives are also defined for any non-integer order $\alpha > 1$. We give the definition for the $\alpha \in (0, 1)$ case since it is the only one we need here. The interested reader may find a more geneal definition in the textbooks mentionned above. Note that all the values of the derivatives at times $s \in (0, t)$ are needed to compute the Caputo derivative at time t. The interpretation is that if we are interested in the change rate at time t, we need to take into account all the change rates at $s \in (0, t)$. From a modeling point of view, this introduces a memory effect. The following result will be useful regarding Caputo derivatives and Laplace transforms.

Proposition 6. *For any function $h : \mathbb{R} \to \mathbb{R}$, it holds that*

$$\widetilde{\frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}}(\omega) = \omega^\alpha \tilde{h}(\omega) - \omega^{\alpha-1} h(0), \tag{7}$$

where

$$\widetilde{\frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}}(\omega) := \int_0^{+\infty} \frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}(t) e^{-t\omega} \mathrm{d}t \text{ and } \tilde{h}(\omega) := \int_0^{+\infty} h(s) e^{-s\omega} \mathrm{d}s.$$

This is a well known result in fractional calculus. A proof can be found in the chapter 2 of Podlubny (1999). We can now turn to the main results regarding the fractional setting. Those are exposed in the next subsection.

2.3. Fractional Fokker-Planck Equation for the Time-Changed Process

This subsection is concerned with the main theoretical results of this paper, namely the FPDE for the joint PDF and joint Laplace transform of the time changed process $(\lambda_{S_t}, \Lambda_{S_t})_{t \geqslant 0}$. Before proving that the joint PDF and Laplace transform satisfy FPDE's, we need a preliminary result about Laplace transforms. We denote

$$p_\alpha(t, x_1, x_2|y) := \frac{\partial^2 P(\lambda_{S_t} \leqslant x_1, \Lambda_{S_t} \leqslant x_2 | \lambda_0 = y)}{\partial x_1 \partial x_2},$$

the joint PDF of $(\lambda_{S_t}, \Lambda_{S_t})_{t \geqslant 0}$. Also, recall that p is the joint PDF of $(\lambda_t, \Lambda_t)_{t \geqslant 0}$, as defined at Equation (4). The Laplace transform of this function satisfies the next lemma.

Lemma 7. *It holds that*

$$\tilde{p}_\alpha(\omega, x_1, x_2|y) = \omega^{\alpha-1}\tilde{p}(\omega^\alpha, x_1, x_2|y),$$

where

$$\tilde{p}_\alpha(\omega, x_1, x_2|y) := \int_0^{+\infty} e^{-\omega t} p_\alpha(t, x_1, x_2|y)\mathrm{d}t.$$

Proof. First note that

$$p_\alpha(t, x_1, x_2|y) = \int_0^{+\infty} p(\tau, x_1, x_2|0, y, 0)g(t, \tau)\mathrm{d}\tau,$$

where $g(t, \cdot)$ is the PDF of S_t. Therefore it follows that $\tilde{p}_\alpha$ satisfies

$$\begin{aligned}
\tilde{p}_\alpha(\omega, x_1, x_2|y) &:= \int_0^{+\infty} e^{-\omega t} p_\alpha(t, x_1, x_2|y)\mathrm{d}t \\
&= \int_0^{+\infty} e^{-t\omega} \int_0^{+\infty} p(\tau, x_1, x_2|0, y, 0)g(t, \tau)\mathrm{d}\tau\mathrm{d}t \\
&= \int_0^{+\infty} p(\tau, x_1, x_2|0, y, 0) \int_0^{+\infty} e^{-t\omega} g(t, \tau)\mathrm{d}t\mathrm{d}\tau \\
&= \omega^{\alpha-1} \int_0^{+\infty} e^{-\tau\omega^\alpha} p(\tau, x_1, x_2|0, y, 0)\mathrm{d}\tau = \omega^{\alpha-1}\tilde{p}(\omega^\alpha, x_1, x_2|0, y, 0),
\end{aligned}$$

as announced. □

We now have the tools to prove the first main theoretical result of this paper, namely that the joint PDF p_α satisfies a FPDE.

Proposition 8. *The joint PDF p_α satisfies the FPDE*

$$\begin{aligned}
\frac{\partial^\alpha p_\alpha(t, x_1, x_2|y)}{\partial t^\alpha} &= \kappa p_\alpha(t, x_1, x_2|y) - x_1 \frac{\partial}{\partial x_2} p_\alpha(t, x_1, x_2|y) - \kappa(\theta - x_1)\frac{\partial}{\partial x_1} p_\alpha(t, x_1, x_2|y) \\
&\quad - \eta\mathbb{E}[\xi p_\alpha(t, x_1 - \eta\xi, x_2|y)] + x_1\mathbb{E}[p_\alpha(t, x_1 - \eta\xi, x_2|y) - p_\alpha(t, x_1, x_2|y)]
\end{aligned}$$

with boundary condition $p_\alpha(s, x_1, x_2|y) = \delta_{\{x_1-y, x_2\}}$, where δ stands for the Dirac measure located at $(0, 0) \in \mathbb{R}^2$.

Proof. In the following, $p(t, x_1, x_2|y)$ is used as a shorthand for $p(t, x_1, x_2|0, y, 0)$. Firstly let us look at the integral

$$\int_0^{+\infty} \frac{\partial}{\partial t}\left(e^{-\omega t} p(t, x_1, x_2|y)\right)\mathrm{d}t. \tag{8}$$

From the fundamental theorem of calculus, it is clearly equal to $-p(0, x_1, x_2|y)$. However, by computing the derivative inside the integral, we obtain that the integral at Equation (8) is also equal to

$$-\omega\tilde{p}(\omega, x_1, x_2|y) + \int_0^{+\infty} e^{-\omega t}\frac{\partial}{\partial t} p(t, x_1, x_2|y)\mathrm{d}t$$

so that we infer

$$\omega\tilde{p}(\omega, x_1, x_2|y) - p(0, x_1, x_2|y) = \int_0^{+\infty} e^{-\omega t}\frac{\partial}{\partial t} p(t, x_1, x_2|y)\mathrm{d}t. \tag{9}$$

Secondly we use Proposition 2, i.e. the Fokker-Planck equation for p, on the right-hand side of Equation (9). After some simplifications, this gives

$$\begin{aligned}
&\omega\tilde{p}(\omega,x_1,x_2|y) - p(0,x_1,x_2|y)\\
&= -x_1\frac{\partial}{\partial x_2}\tilde{p}(\omega,x_1,x_2|y) - \kappa(\theta-x_1)\frac{\partial}{\partial x_1}\tilde{p}(\omega,x_1,x_2|y) + \kappa\tilde{p}(\omega,x_1,x_2|y)\\
&\quad - \eta\mathbb{E}\left[\xi\tilde{p}(\omega,x_1-\eta\xi,x_2|y)\right] + x_1\mathbb{E}\left[\tilde{p}(\omega,x_1-\eta\xi,x_2|y) - \tilde{p}(\omega,x_1,x_2|y)\right].
\end{aligned} \tag{10}$$

Since this is valid for any ω, we can replace ω by ω^α in Equation (10). Doing this, multiplying each side by $\omega^{\alpha-1}$, and using the fact that $p_\alpha(0,x_1,x_2|y) = p(0,x_1,x_2|y) = 0$ provides us with

$$\begin{aligned}
&\omega^\alpha(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|y)) - \omega^{\alpha-1}p_\alpha(0,x_1,x_2|y)\\
&= -x_1\frac{\partial}{\partial x_2}\left(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|y)\right) - \kappa(\theta-x_1)\frac{\partial}{\partial x_1}\left(\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|y)\right)\\
&\quad + \kappa\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|y) - \eta\mathbb{E}\left[\xi\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1-\eta\xi,x_2|y)\right]\\
&\quad + x_1\mathbb{E}\left[\omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1-\eta\xi,x_2|y) - \omega^{\alpha-1}\tilde{p}(\omega^\alpha,x_1,x_2|y)\right].
\end{aligned} \tag{11}$$

Using Lemma 7, we can rewrite Equation (11) as

$$\begin{aligned}
&\omega^\alpha\tilde{p}_\alpha(\omega,x_1,x_2|y) - \omega^{\alpha-1}p_\alpha(0,x_1,x_2|y)\\
&= -x_1\frac{\partial}{\partial x_2}\tilde{p}_\alpha(\omega,x_1,x_2|y) + -\kappa(\theta-x_1)\frac{\partial}{\partial x_1}\tilde{p}_\alpha(\omega,x_1,x_2|y)\\
&\quad + \kappa\tilde{p}_\alpha(\omega,x_1,x_2|y) - \eta\mathbb{E}\left[\xi\tilde{p}_\alpha(\omega,x_1-\eta\xi,x_2|y)\mathrm{d}t\right]\\
&\quad + x_1\mathbb{E}\left[\omega^{\alpha-1}\tilde{p}_\alpha(\omega,x_1-\eta\xi,x_2|y) - \tilde{p}_\alpha(\omega,x_1,x_2|y)\right].
\end{aligned} \tag{12}$$

An application of Proposition 6 to the function $t \in (0,+\infty) \mapsto p(t,x_1,x_2|y)$ allows to conclude that the left-hand side of Equation (12) equals $\widetilde{\frac{\partial^\alpha p}{\partial t^\alpha}}(\omega,x_1,x_2|y)$. □

We also obtained a FPDE for the Laplace transform, that is the second main theoretical result of this paper. In the following, the Laplace transform of $(S_t,\lambda_{S_t},\Lambda_{S_t})_{t\geqslant 0}$ is denoted by

$$\phi_\alpha(t,z_1,z_2,z_3|y) := \mathbb{E}\left[\exp\left\{-z_1S_t - z_2\lambda_{S_t} - z_3\Lambda_{S_t}\right\}|\lambda_0 = y\right].$$

Proposition 9. *The function ϕ_α satisfies the FPDE*

$$\frac{\partial^\alpha\phi_\alpha}{\partial t^\alpha}(t,z_1,z_2,z_3|y) = -z_1\phi_\alpha(t,z_1,z_2,z_3|y) - z_2\kappa\theta\phi_\alpha(t,z_1,z_2,z_3|y) + \gamma(z_2,z_3)\frac{\partial\phi_\alpha}{\partial z_2}(t,z_1,z_2,z_3|y) \tag{13}$$

where $\gamma(z_2,z_3) := 1 - \kappa z_2 - \mathbb{E}[e^{-z_2\eta\xi}] + z_3$. Moreover, the following boundary conditions hold:

$$\phi_\alpha(t,z_1,0,0|y) = \mathrm{E}_\alpha(-z_1t^\alpha)$$

$$\phi_\alpha(0,z_1,z_2,z_3|y) - e^{-z_2y} = \phi_\alpha(t,0,0,0|y) - 1 = 0.$$

Proof. The first boundary condition follows from Proposition 5, the second and the third are straightforward given the definition of ϕ_α. In order to derive Equation (13), we start by observing the joint PDF p_α of the triple $(S_t, \lambda_{S_t}, \Lambda_{S_t})$, that is

$$p_\alpha(t, x_1, x_2, x_3|y) := \frac{\partial^3}{\partial x_1 \partial x_2 \partial x_3} P\left(S_t \leqslant x_1, \lambda_{S_t} \leqslant x_2, \Lambda_{S_t} \leqslant x_3 | \lambda_0 = y\right).$$

From the definition of conditional PDF, we have

$$p_\alpha(t, x_1, x_2, x_3|y) = g(t, x_1) p(x_1, x_2, x_3|y). \tag{14}$$

See Equation (4) for the definition of p. From Equation (14), it follows that

$$\begin{aligned}
\tilde{p}_\alpha(\omega, x_1, x_2, x_3|y) &:= \int_0^{+\infty} e^{-\omega t} p_\alpha(t, x_1, x_2, x_3|y) \mathrm{d}t \\
&= p(x_1, x_2, x_3|y) \int_0^{+\infty} e^{-\omega t} g(t, x_1) \mathrm{d}t \\
&= \omega^{\alpha-1} e^{-x_1 \omega^\alpha} p(x_1, x_2, x_3|y).
\end{aligned}$$

Thanks to this result we can obtain

$$\begin{aligned}
&\tilde{\phi}_\alpha(\omega, z_1, z_2, z_3|y) \\
&:= \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-\omega t - z_1 x_1 - z_2 x_2 - z_3 x_3} p_\alpha(t, x_1, x_2, x_3|y) \mathrm{d}t \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\
&= \omega^{\alpha-1} \int_0^{+\infty} \int_0^{+\infty} \int_0^{+\infty} e^{-z_1 x_1 - z_2 x_2 - z_3 x_3} e^{-x_1 \omega^\alpha} p(x_1, x_2, x_3|y) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 \\
&= \omega^{\alpha-1} \int_0^{+\infty} e^{-(z_1 + \omega^\alpha) x_1} \phi(x_1, z_2, z_2|y) \mathrm{d}x_1,
\end{aligned} \tag{15}$$

where $\phi(x_1, z_2, z_3|y) = \int_0^{+\infty} \int_0^{+\infty} e^{-x_2 z_2 - x_3 z_3} p(x_1, x_2, x_3|y) \mathrm{d}x_2 \mathrm{d}x_3$. Next, an integration by parts yields

$$\tilde{\phi}_\alpha(\omega, z_1, z_2, z_3|y) = \frac{\omega^{\alpha-1}}{z_1 + \omega^\alpha} \left[\phi(0, z_2, z_3|y) + \int_0^{+\infty} e^{-(z_1 + \omega^\alpha) t} \frac{\partial \phi}{\partial t}(t, z_1, z_2|y) \mathrm{d}t \right]. \tag{16}$$

Proposition 3 gives an expression of $\frac{\partial \phi}{\partial t}(t, z_1, z_2|y)$ that we can use to obtain

$$\begin{aligned}
\int_0^{+\infty} e^{-(z_1 + \omega^\alpha) t} \frac{\partial \phi}{\partial t}(t, z_1, z_2|y) \mathrm{d}t = &- z_2 \kappa \theta \int_0^{+\infty} e^{-(z_1 + \omega^\alpha) t} \phi(t, z_2, z_3|y) \mathrm{d}t \\
&+ \gamma(z_2, z_3) \int_0^{+\infty} e^{-(z_1 + \omega^\alpha) t} \frac{\partial \phi}{\partial z_2}(t, z_2, z_3|y) \mathrm{d}t.
\end{aligned} \tag{17}$$

By this last equation, as well as Equation (15), Equation (16) is equivalent to

$$\begin{aligned}
(z_1 + \omega\alpha) \tilde{\phi}_\alpha(\omega, z_1, z_2, z_3|y) = &\omega^{\alpha-1} \phi(0, z_1, z_2|y) - z_2 \kappa \theta \tilde{\phi}_\alpha(\omega, z_1, z_2, z_3|y) \\
&+ \gamma(z_2, z_3) \frac{\partial \tilde{\phi}_\alpha}{\partial z_2}(\omega, z_1, z_2, z_3|y).
\end{aligned} \tag{18}$$

Observe that $\phi(0, z_2, z_3|y) = e^{-z_2 y} = \phi_\alpha(0, z_1, z_2, z_3|y)$, so that we obtain

$$\begin{aligned}
&\omega^\alpha \tilde{\phi}_\alpha(\omega, z_1, z_2, z_3|y) - \omega^{\alpha-1} \phi_\alpha(0, z_1, z_2, z_3|y) \\
&= - z_1 \tilde{\phi}_\alpha(\omega, z_1, z_2, z_3|y) - z_2 \kappa \theta \tilde{\phi}_\alpha(\omega, z_1, z_2, z_3|y) + \gamma(z_2, z_3) \frac{\partial}{\partial z_2} \tilde{\phi}_\alpha(\omega, z_1, z_2, z_3|y).
\end{aligned}$$

The announced result then follows from Proposition 6. □

Corollary 10. *If we denote*

$$\phi_\alpha(t, z_1, z_2|y) := \mathbb{E}\left[e^{-z_1\lambda_{S_t} - z_2\int_0^{S_t}\lambda_u \mathrm{d}u}\middle|\lambda_0 = y\right],$$

then it holds that

$$\frac{\partial^\alpha \phi_\alpha}{\partial t^\alpha}(t, z_1, z_2|y) = \phi_\alpha(t, z_1, z_2|y)\beta(z_1) + \frac{\partial}{\partial z_1}\phi_\alpha(t, z_1, z_2|y)\gamma(z_1, z_2). \tag{19}$$

Moreover, the boundary conditions $\phi_\alpha(t, 0, 0|y) = 1$ *and* $\phi_\alpha(0, z_1, z_2|y) = e^{-z_1 y}$ *are satisfied.*

Proof. Apply Proposition 9 with $z_1 = 0$. □

Note that this last PDE is nearly the same as in the non-frational case, see Proposition 3. The only difference is that the partial derivative with respect to time is replaced by a fractional Caputo derivative.

3. Numerical Illustration

This section shows how we can use the FPDE of the Laplace transform ϕ_α to compute survival probabilities. Several methods have been developped recently to solve FPDE's numerically. We can cite, e.g. the variational iteration method in Odibat and Momani (2009), the homotopy perturbation method in Golbabai and Sayevand (2011), the Laplace transform method in Jafari et al. (2013), the Haar-wavelet method in Wang et al. (2014), the Adomian decomposition method in Jafari and Daftardar-Gejji (2006) and El-Sayed et al. (2010) and the spectral regularization method in Zheng and Wei (2010). In our case, the FPDE is solved numerically with the help of a finite difference method. Finite difference methods are of common use in finance for solving PDE's, see Zvan et al. (2000) and Junseok et al. (2016) for examples and Duffy (2006) for a textbook. They have also been successfully used to solve FPDE's in, e.g. Murio (2008), Ding and Zhang (2011) and in Song and Wang (2013), where it is used to price European and American options in a fractional version of the Black and Scholes model. The finite difference approximation we use for the Caputo derivative is the same as in Murio (2008) and Ding and Zhang (2011). However, our numerical method differs in two aspects. Firstly the equation we solve is not the same as in the mentioned references. Secondly our method is adapted to solve a FPDE with very few boundary conditions, as explained later.

In our model, the time of default τ is the first jump of the time changed point process $(D_{S_t})_{t\geqslant 0}$, i.e.

$$\tau := \inf\{t \geqslant 0 | D_{S_t} > 0\}.$$

The process $(D_t)_{t\geqslant 0}$ is assumed to have intensity $(\lambda_t)_{t\geqslant 0}$. It follows that

$$P(\tau \geqslant t|\mathcal{F}_0) = \mathbb{E}\left[\exp\left\{-\Lambda_{S_t}\right\}|\mathcal{F}_0\right] = \phi_\alpha(t, 0, 1|y).$$

From Corollary 10, we know that the function $(t, z) \mapsto \phi_\alpha(t, z, 1|y)$ satisfies the FPDE

$$\frac{\partial^\alpha \phi_\alpha}{\partial t^\alpha}(t, z, 1|y) = -z\kappa\theta\phi_\alpha(t, z, 1|y) + \frac{\partial}{\partial z}\phi_\alpha(t, z, 1|y)\gamma(z, 1)$$

with the boundary condition $\phi_\alpha(0,z,1|y) = e^{-zy}$. We start by solving the simple non-fractional case ($\alpha = 1$) and then extend the method to the fractional case with $\alpha \in (0,1)$.

3.1. Non-Fractional Case

Recall that

$$\phi(t,z_1,z_2|s,y_1,y_2) = \mathbb{E}\left[\exp\left\{-z_1\lambda_t - z_2\Lambda_t\right\}|\lambda_s = y_1, \Lambda_s = y_2\right].$$

For the sake of shorter notations, we will denote $\phi(t,z_1,z_2) = \phi(t,z_1,z_2|0,y,0)$, for a fixed $y \geqslant \theta$. Recall that $\gamma(z_1,z_2) = 1 - \kappa z_1 - \mathbb{E}[e^{-z_1\eta\xi}] + z_2$. The goal is to solve the PDE

$$\frac{\partial\phi}{\partial t}(t,z_1,z_2) = -z_1\kappa\theta\phi(t,z_1,z_2) + \frac{\partial\phi}{\partial z_1}\gamma(z_1,z_2) \tag{20}$$

for a fixed z_2. Then, setting $z_2 = 1$ allows to find the survival probabilities implied by the model. In order to solve numerically this equation, we will work on two grids. The first one is $\{t_0,t_1,\ldots,t_{n_t}\}$, where $t_k = k\Delta_t$. The second one is $\{z_{-n_z},z_{-n_z+1},\ldots,z_{-1},z_0,z_1,\ldots,z_{n_z}\}$, where $z_j = j\Delta_z$. We write respectively $\hat{\phi}(k,j;z_2)$, $\frac{\partial\hat{\phi}}{\partial z_1}(k,j;z_2)$ and $\frac{\partial\hat{\phi}}{\partial z_1}(k,j;z_2)$ for our approximations of $\phi(k\Delta_t,j\Delta_z,z_2)$, $\frac{\partial\phi}{\partial t}(k\Delta_t,j\Delta_z,z_2)$ and $\frac{\partial\phi}{\partial z_1}(k\Delta_t,j\Delta_z,z_2)$. The partial derivatives are approximated as

$$\frac{\partial\hat{\phi}}{\partial t}(k,j;z_2) := \frac{\hat{\phi}(k,j;z_2) - \hat{\phi}(k-1,j;z_2)}{\Delta_t} \tag{21}$$

$$\frac{\partial\hat{\phi}}{\partial z_1}(k,j;z_2) := \begin{cases} \frac{\hat{\phi}(k,-n_z+1;z_2)-\hat{\phi}(k,-n_z;z_2)}{\Delta_z} & j = -n_z \\ \frac{\hat{\phi}(k,n_z;z_2)-\hat{\phi}(k,n_z-1;z_2)}{\Delta_z} & j = n_z \\ \frac{\hat{\phi}(k,j+1;z_2)-\hat{\phi}(k,j-1;z_2)}{2\Delta_z} & \text{otherwise.} \end{cases} \tag{22}$$

The special aspect of our method lies in Equation (22). Approximating the partial derivative with respect to z_1 in such a way, i.e. changing the approximation when $j \in \{-n_z,n_z\}$, allows us to obtain a solution even if the only boundary condition we know is $\phi(0,z_1,z_2|y) = e^{-z_1y}$. Using these notations, PDE (20) translates into

$$\frac{\partial\hat{\phi}}{\partial t}(k,j;z_2) = \hat{\phi}(k,j;z_2)\beta_j + \frac{\partial\hat{\phi}}{\partial z_1}(k,j;z_2)\gamma_j(z_2), \tag{23}$$

where $\beta_j = -j\Delta_z\kappa\theta$ and $\gamma_j(z_2) = \gamma(j\Delta_z,z_2)$. Using Equations (21) and (22), we find

$$\hat{\phi}(k-1,j;z_2) = \begin{cases} \left.\begin{array}{l}\left[1 - \Delta_t\beta_{-n_z} + \frac{\Delta_t}{\Delta_z}\gamma_{-n_z}(z_2)\right]\hat{\phi}(k,-n_z;z_2) \\ -\frac{\Delta_t}{\Delta_z}\gamma_{-n_z}(z_2)\hat{\phi}(k,-n_z+1)\end{array}\right\} & j = -n_z \\ \left.\begin{array}{l}\left[1 - \Delta_t\beta_{n_z} - \frac{\Delta_t}{\Delta_z}\gamma_{n_z}(z_2)\right]\hat{\phi}(k,n_z;z_2) \\ +\frac{\Delta_t}{\Delta_z}\gamma_{n_z}(z_2)\hat{\phi}(k,n_z-1+1)\end{array}\right\} & j = n_z \\ \left.\begin{array}{l}\frac{\Delta_t}{2\Delta_z}\gamma_j(z_2)(\hat{\phi}(k,j-1;z_2) - \hat{\phi}(k,j+1;z_2)) \\ +\left[1 - \Delta_t\beta_j\right]\hat{\phi}(k,j;z_2)\end{array}\right\} & j \notin \{-n_z,n_z\} \end{cases} \tag{24}$$

Equation (24) is then nothing more than a linear system of equations. Let us denote

$$\hat{\mathbf{\Phi}}_k(z_2) := \begin{pmatrix} \hat{\phi}(k, -n_z; z_2) \\ \hat{\phi}(k, -n_z + 1; z_2) \\ \vdots \\ \hat{\phi}(k, n_z; z_2) \end{pmatrix} \in \mathbb{R}^{2n_z+1}$$

and

$$\mathbf{A} := \begin{pmatrix} b_{-n_z} + a_{-n_z} & -a_{-n_z} & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \frac{a_{-n_z+1}}{2} & b_{-n_z+1} & \frac{-a_{-n_z+1}}{2} & 0 & 0 & \dots & 0 & 0 & 0 \\ 0 & \frac{a_{-n_z+2}}{2} & b_{-n_z+2} & \frac{-a_{-n_z+2}}{2} & 0 & \dots & 0 & 0 & 0 \\ 0 & 0 & \frac{a_{-n_z+3}}{2} & b_{-n_z+3} & \frac{-a_{-n_z+3}}{2} & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \dots & \frac{a_{-n_z-1}}{2} & b_{n_z-1} & \frac{-a_{n_z-1}}{2} \\ 0 & 0 & 0 & 0 & 0 & \dots & 0 & a_{n_z} & b_{n_z} - a_{n_z} \end{pmatrix}$$

where[3] $a_j := (\Delta_t/\Delta_z)\gamma_j(z_2)$ and $b_j := 1 - \Delta_t\beta_j$. Note that $\mathbf{A}(z_2) \in \mathbb{R}^{(2n_z+1)\times(2n_z+1)}$ for any z_2. Equation (24) then becomes

$$\hat{\mathbf{\Phi}}_{k-1}(z_2) = \mathbf{A}(z_2)\hat{\mathbf{\Phi}}_k(z_2),$$

so that given the (approximated) values $\hat{\mathbf{\Phi}}_{k-1}(z_2)$ of the function ϕ at time $(k-1)\Delta_t$, our problem reduces to inverting the matrix $\mathbf{A}(z_2)$.

From the boundary conditions of the PDE in Proposition 3, we know that for any z_1, z_2, $\phi(0, z_1, z_2|0, y_1, 0) = \exp\{-z_1 y_1\}$. It follows that, starting from the initial condition $\hat{\phi}(0, j; z_2) = \exp\{-j\Delta_z y_1\}$, i.e. the values in the vector $\hat{\mathbf{\Phi}}_0(z_2)$, we can determine all the vectors $\hat{\mathbf{\Phi}}_k(z_2)$ recursively using $\hat{\mathbf{\Phi}}_k(z_2) = (\mathbf{A}(z_2))^{-1}\hat{\mathbf{\Phi}}_{k-1}(z_2)$.

3.2. Approximating Caputo Fractional Derivatives

This subsection discusses the numerical approximation of the Caputo derivative of a function. Let us consider the grid $\{t_0, t_1, \dots, t_{n_t}\}$ of the previous section, that is $t_k = k\Delta_t$ for all k. The approximation formula is based on Equation (6) and the following development

$$\begin{aligned} \frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}(t_k) &= \frac{1}{\Gamma(1-\alpha)} \int_0^{t_k} (t_k - s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} h(s) \mathrm{d}s \\ &= \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{k-1} \int_{t_i}^{t_{i+1}} (t_k - s)^{-\alpha} \frac{\mathrm{d}}{\mathrm{d}s} h(s) \mathrm{d}s \\ &\approx \frac{1}{\Gamma(1-\alpha)} \sum_{i=0}^{k-1} \left(\frac{h(t_{i+1}) - h(t_i)}{\Delta_t} \right) \int_{t_i}^{t_{i+1}} (t_k - s)^{-\alpha} \mathrm{d}s \\ &= \frac{-\Delta_t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{k-1} (h(t_{i+1}) - h(t_i)) \left[(k-i-1)^{1-\alpha} - (k-i)^{1-\alpha} \right]. \end{aligned}$$

[3] Obviously we should write $a_j(z_2)$ and $\mathbf{A}(z_2)$ but the notations needed to be shortened here.

Therefore the approximation formula for the Caputo fractional derivative is

$$\frac{\mathrm{d}^\alpha h}{\mathrm{d}t^\alpha}(t_k) \approx \frac{-\Delta_t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{k-1} \left(h(t_{i+1}) - h(t_i)\right) \left[(k-i-1)^{1-\alpha} - (k-i)^{1-\alpha}\right]. \tag{25}$$

In the next subsection, we use this approximation to solve numerically the FPDE (19) of Corollary 10.

3.3. Fractional Case

We describe a numerical scheme to solve the FPDE of Corollary 10. The method is basically the same as in the non-fractional case, although it is a bit complicated by the fractional derivative with respect to time. We work with the same grids as in Section 3.1. Our approximations of $\phi_\alpha(k\Delta_t, j\Delta_z, z_2)$, $\frac{\partial \phi_\alpha}{\partial z_1}(k\Delta_t, j\Delta_z, z_2)$, and $\frac{\partial^\alpha \phi_\alpha}{\partial t^\alpha}(k\Delta_t, j\Delta_z, z_2)$ are respectively denoted by $\hat{\phi}_\alpha(k, j; z_2)$, $\frac{\partial \hat{\phi}_\alpha}{\partial z_1}(k, j; z_2)$ and $\frac{\partial^\alpha \hat{\phi}_\alpha}{\partial t^\alpha}(k, j; z_2)$. The partial derivative with respect to z_1 is approximated as in Section 3.1, that is

$$\frac{\partial \hat{\phi}_\alpha}{\partial z_1}(k, j; z_2) := \begin{cases} \frac{\hat{\phi}_\alpha(k, -n_z+1; z_2) - \hat{\phi}_\alpha(k, -n_z; z_2)}{\Delta_z} & j = -n_z \\ \frac{\hat{\phi}_\alpha(k, n_z; z_2) - \hat{\phi}_\alpha(k, n_z-1; z_2)}{\Delta_z} & j = n_z \\ \frac{\hat{\phi}_\alpha(k, j+1; z_2) - \hat{\phi}_\alpha(k, j-1; z_2)}{2\Delta_z} & \text{otherwise.} \end{cases} \tag{26}$$

As discussed in Section 3.2, the Caputo derivative is approximated by

$$\begin{aligned} &\frac{\partial^\alpha \hat{\phi}_\alpha}{\partial t^\alpha}(k, j; z_2) \\ &\quad := \frac{-\Delta_t^{-\alpha}}{\Gamma(2-\alpha)} \sum_{i=0}^{k-1} [(k-i-1)^{1-\alpha} - (k-i)^{1-\alpha}](\hat{\phi}_\alpha(i+1, j; z_2) - \hat{\phi}_\alpha(i, j; z_2)). \end{aligned} \tag{27}$$

Having introduced these notations, the FPDE in Corollary 10 translates into

$$\frac{\partial^\alpha \hat{\phi}_\alpha}{\partial t^\alpha}(k, j; z_2) = \hat{\phi}_\alpha(k, j; z_2)\beta_j + \frac{\partial \hat{\phi}_\alpha}{\partial z_1}(k, j; z_2)\gamma_j(z_2) \tag{28}$$

where β_j and γ_j are the same as in Equation (23). Starting from Equation (28), we can use approximations of Equations (27) and (26) to obtain

$$\begin{aligned} &\frac{\Delta_t^{-\alpha}}{\Gamma(2-\alpha)} \Bigg[\left((k-1)^{1-\alpha} - k^{1-\alpha}\right) \hat{\phi}_\alpha(0, j; z_2) \\ &- \sum_{i=1}^{k-1} \hat{\phi}_\alpha(i, j; z_2) \left(2(k-i)^{1-\alpha} - (k-i+1)^{1-\alpha} - (k-i-1)^{1-\alpha}\right) \Bigg] \\ &= \begin{cases} \hat{\phi}_\alpha(k, -n_z+1; z_2) \frac{\gamma_{-n_z}(z_2)}{\Delta_z} \\ + \hat{\phi}_\alpha(k, -n_z; z_2) \left[\frac{-\Delta_t^{-\alpha}}{\Gamma(2-\alpha)} + \beta_{-n_z} - \frac{\gamma_{-n_z}(z_2)}{\Delta_z}\right] \end{cases} & j = -n_z \\ \begin{cases} \hat{\phi}_\alpha(k, n_z-1; z_2) \frac{-\gamma_{n_z}(z_2)}{\Delta_z} \\ + \hat{\phi}_\alpha(k, n_z; z_2) \left[\frac{-\Delta_t^{-\alpha}}{\Gamma(2-\alpha)} + \beta_{n_z} + \frac{\gamma_{n_z}(z_2)}{\Delta_z}\right] \end{cases} & j = n_z \\ \begin{cases} \hat{\phi}_\alpha(k, j-1; z_2) \frac{-\gamma_j(z_2)}{2\Delta_z} + \hat{\phi}_\alpha(k, j+1; z_2) \frac{\gamma_j(z_2)}{2\Delta_z} \\ + \hat{\phi}_\alpha(k, j; z_2) \left[\frac{-\Delta_t^{-\alpha}}{\Gamma(2-\alpha)} + \beta_j\right] \end{cases} & \text{otherwise.} \end{cases} \end{aligned}$$

This is again a linear system of equations. Using vector and matrix notations as in Section 3.1, we have

$$\begin{aligned}
&\frac{\Delta_t^{-\alpha}}{\Gamma(2-\alpha)}\Bigg[\left((k-1)^{1-\alpha}-k^{1-\alpha}\right)\hat{\mathbf{\Phi}}_\alpha(0;z_2)\\
&-\sum_{i=1}^{k-1}\hat{\mathbf{\Phi}}_\alpha(i;z_2)\left(2(k-i)^{1-\alpha}-(k-i+1)^{1-\alpha}-(k-i-1)^{1-\alpha}\right)\Bigg]\\
&=\mathbf{A}_\alpha(z_2)\hat{\mathbf{\Phi}}_\alpha(k;z_2)
\end{aligned}$$

where[4]

$$\mathbf{A}_\alpha:=\begin{pmatrix}
b_{-n_z}-a_{-n_z} & a_{-n_z} & 0 & 0 & 0 & \cdots & 0 & 0 & 0\\
\frac{-a_{-n_z+1}}{2} & b_{-n_z+1} & \frac{a_{-n_z+1}}{2} & 0 & 0 & \ldots & 0 & 0 & 0\\
0 & \frac{-a_{-n_z+2}}{2} & b_{-n_z+2} & \frac{a_{-n_z+2}}{2} & 0 & \ldots & 0 & 0 & 0\\
0 & 0 & \frac{-a_{-n_z+3}}{2} & b_{-n_z+3} & \frac{a_{-n_z+3}}{2} & \ldots & 0 & 0 & 0\\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots\\
0 & 0 & 0 & 0 & 0 & \ldots & \frac{-a_{n_z-1}}{2} & b_{n_z-1} & \frac{a_{n_z-1}}{2}\\
0 & 0 & 0 & 0 & 0 & \ldots & 0 & -a_{n_z} & b_{n_z}+a_{n_z}
\end{pmatrix}$$

and

$$a_j(z_2):=\frac{\gamma_j(z_2)}{\Delta_z}\qquad b_j:=\frac{-\Delta_t^{-\alpha}}{\Gamma(2-\alpha)}+\beta_j.$$

As in the non-fractional case of Section 3.1, starting from the boundary condition $\hat{\phi}_\alpha(0,j;z_2)=\exp\{-j\Delta_z y_1\}$, we can obtain all the values on the grid using the recursion

$$\begin{aligned}
\hat{\mathbf{\Phi}}_\alpha(k;z_2)=(\mathbf{A}_\alpha(z_2))^{-1}\frac{\Delta_t^{-\alpha}}{\Gamma(2-\alpha)}&\Bigg[\left((k-1)^{1-\alpha}-k^{1-\alpha}\right)\hat{\mathbf{\Phi}}_\alpha(0;z_2)\\
&-\sum_{i=1}^{k-1}\hat{\mathbf{\Phi}}_\alpha(i;z_2)\left(2(k-i)^{1-\alpha}-(k-i+1)^{1-\alpha}-(k-i-1)^{1-\alpha}\right)\Bigg].
\end{aligned}$$

In the next section, we find lower and upper bounds on the survival probabilities. This allows us to ensure the precision of our numerical procedure.

3.4. Bounds on the Survival Probabilities

The bounds on the probabilities in the non-fractional model are given in the next proposition.

Proposition 11. *Let $q:\mathbb{R}^+\to[0,1], t\mapsto q(t):=\phi(t,0,1|0,y,0)$ be the survival probability function implied by the non-fractional model (see Proposition 3) and $f_1,f_2:\mathbb{R}^+\to[0,1]$ be defined as*

$$f_1(t):=\exp\left\{-\left(\left(y-\frac{\kappa\theta}{\kappa-\eta\rho^{-1}}\right)\frac{1-e^{-(\kappa-\eta\rho^{-1})t}}{\kappa-\eta\rho^{-1}}+\frac{\kappa\theta t}{\kappa-\eta\rho^{-1}}\right)\right\}$$

and $f_2(t):=e^{-\theta t}$. Then it holds that $f_1\leqslant q\leqslant f_2$.

[4] Again we should write $a_j(z_2)$ and $\mathbf{A}_\alpha(z_2)$ but shorter notations are needed.

Proof. The inequality $q \leqslant f_2$ follows from $\lambda_t \geqslant \theta$. For the other inequality, note that the convexity of the function $x \mapsto e^{-x}$ implies, by Jensen's inequality

$$q(t) = \mathbb{E}[\exp\{-\Lambda_t\}|\lambda_0 = y] \geqslant \exp\{-\mathbb{E}[\Lambda_t|\lambda_0 = y]\}.$$

The expression of f_1 is then obtained by integrating the expected value of λ_t, see Equation (2). □

We also have bounds in the fractional case.

Proposition 12. *Let $q_\alpha : \mathbb{R}^+ \to [0,1], t \mapsto q_\alpha(t) := \phi_\alpha(t, 0, 1|y)$ be the survival probability function implied by the fractional model (see Corollary 10) and $f_{\alpha,1}, f_{\alpha,2} : \mathbb{R}^+ \to [0,1]$ be defined as*

$$f_{\alpha,1}(t) := \exp\left\{-\left(\left(y - \frac{\kappa\theta}{\kappa - \eta\rho^{-1}}\right)\frac{1 - \mathrm{E}_\alpha(-(\kappa - \eta\rho^{-1})t^\alpha)}{\kappa - \eta\rho^{-1}} + \frac{\kappa\theta t^\alpha}{(\kappa - \eta\rho^{-1})\Gamma(\alpha+1)}\right)\right\}$$

and $f_{\alpha,2}(t) := E_\alpha(-\theta t^\alpha)$, where E_α is the Mittag-Leffler function. Then it holds that $f_{\alpha,1} \leqslant q_\alpha \leqslant f_{\alpha,2}$.

Proof. The inequality $q_\alpha \leqslant f_{\alpha,2}$ follows from $\Lambda_{S_t} \geqslant \theta S_t$ and Proposition 5. The other inequality follows from Jensen's inequality

$$\begin{aligned} q_\alpha(t) &\geqslant \exp\{-\mathbb{E}[\Lambda_{S_t}|\lambda_0 = y]\} \\ &= \exp\left\{-\mathbb{E}\left[\left(y - \frac{\kappa\theta}{\kappa - \eta\rho^{-1}}\right)\frac{1 - e^{-(\kappa - \eta\rho^{-1})S_t}}{\kappa - \eta\rho^{-1}} + \frac{\kappa\theta S_t}{\kappa - \eta\rho^{-1}}\right]\right\} \end{aligned}$$

where we used Equation 2. The proof is then easily completed with the help of Proposition 5. □

These two Propositions will be useful in the next section to show the accuracy of the numerical method.

3.5. Numerical Results

This section is devoted to the calibrations of both models (fractional and non-fractional) to real market data. The data have been taken on the 25th of January 2021 from Bloomberg. They consist of survival probabilities for American Airlines. These data are displayed in the *Market* column of Table 2. The model is calibrated by choosing the set of parameters that minimizes the quantity

$$\sum_{T=1}^{10} (Prob_T^{\text{Model}} - Prob_T^{\text{Market}})^2$$

where $Prob_T^{\text{Market}}$ denotes the market survival up to time T probability from Bloomberg and $Prob_T^{\text{Model}}$ denotes the corresponding survival probability implied by the model. Refering to the notations from above, the computations are made with $\Delta_t = 2 \times 10^{-3}$, $n_t = 5000$, $\Delta_z = 10^{-2}$, $n_z = 10$ and λ_0 was fixed at the target level θ. The parameters from the calibration are shown in Table 1. The graphical results of the calibrations are shown at Figure 4 and the numerical results are given in Table 2. In this table, we compare the market data to the survival probabilities implied by the model and computed by solving the (F)PDE with the method described above. As we can observe on Figure 4, the non-fractional model is unable to properly fit the market

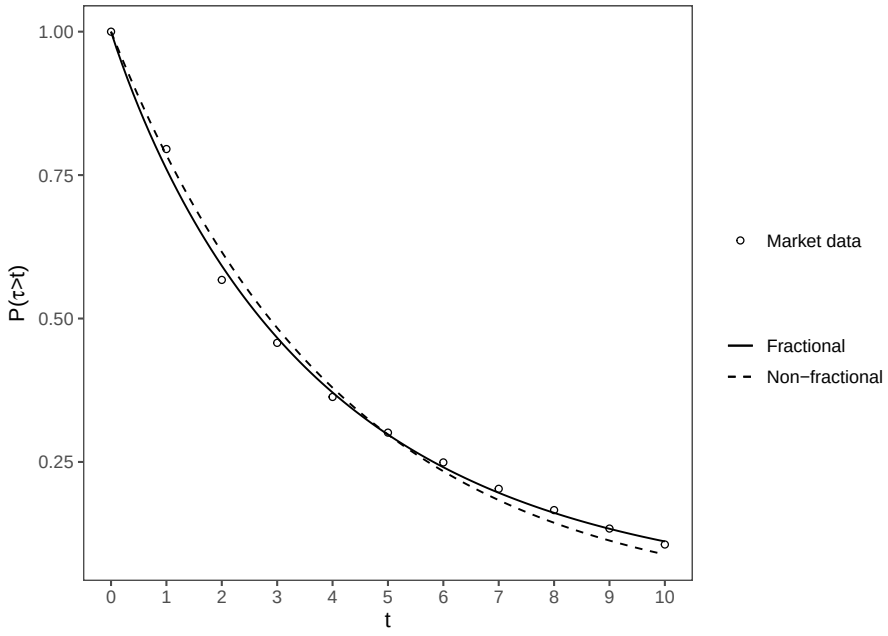


Figure 4: Market Survival Probabilities and Calibrated Survival Probabilities

data. Moving to the fractional model allows to introduce a greater degree of convexity and thus a better fit. To illustrate numerically the improvement of the fit, the absolute differences with respect to the market survival probabilities can also be found in Table 2. These differences between probabilities sum to 0.200165 in the non-fractional case whereas they sum to only 0.105476 in the fractional case. Thus the fractional model performs almost twice better according to the sum of absolute values criterion. Table 2 displays the lower

Parameter	θ	κ	η	ρ	α
Non-fractional	0.2381828	37.6009	1.728399	2.5861	1
Fractional	0.2489265	6.683165	1.728354	2.58613	0.9415109

Table 1: Parameters of the Calibrated Model

and upper bounds on the survival probabilities given at Propositions 11 and 12. Recall that f_1 and f_2 are respectively the lower and upper bounds in the non-fractional case and $f_{\alpha,1}$ and $f_{\alpha,2}$ are respectively the lower and upper bounds in the fractional case. Since the numerical method converges to values that are always between the lower and the upper bounds, we can deduce bounds on the error of the numerical results.

Table 3 shows that the results obtained from our numerical method are congruent with Monte-Carlo simulations of the process $(\exp\{-\Lambda_{S_t}\})_{t\geqslant 0}$. The algorithm we use to obtain such simulations is described in details in Appendix 2. Table 3 is based on 1000 Monte-Carlo simulations and displays the market survival

		Non-fractional				Fractional			
Time	Market	f_1	q	f_2	$\lvert\text{Market} - q\rvert$	$f_{\alpha,1}$	q_α	$f_{\alpha,2}$	$\lvert\text{Market} - q_\alpha\rvert$
1	0.7954	0.784761	0.784938	0.788059	0.010462	0.756789	0.760422	0.776569	0.034978
2	0.5673	0.615778	0.616056	0.621036	0.048756	0.583148	0.591944	0.617624	0.024644
3	0.4576	0.483182	0.483510	0.489413	0.025910	0.452881	0.466446	0.496494	0.008846
4	0.3633	0.379138	0.379481	0.385686	0.016181	0.353467	0.371079	0.402421	0.007779
5	0.3011	0.297498	0.297834	0.303943	0.003266	0.276881	0.297728	0.328537	0.003372
6	0.2492	0.233437	0.233754	0.239525	0.015446	0.217509	0.240802	0.270032	0.008398
7	0.2032	0.183171	0.183461	0.188760	0.019739	0.171271	0.196293	0.223397	0.006907
8	0.1661	0.143729	0.143989	0.148754	0.022111	0.135132	0.161266	0.186008	0.004834
9	0.1339	0.112780	0.113009	0.117227	0.020891	0.106803	0.133534	0.155875	0.000366
10	0.1061	0.088495	0.088695	0.092382	0.017405	0.084541	0.111452	0.131472	0.005352
Total					0.200165				0.105476

Table 2: Numerical Results of the Calibration and Bounds on the Computed Probabilities

probabilities (Market), the survival probabilities obtained from solving the FPDE numerically (FPDE), a 99% confidence interval based on the standard deviation (CI 99% and St. Dev.) of the simulations and the survival probabilities computed from the simulations (Average). The survival probabilities and their standard deviations are respectively computed as the empirical means and standard deviations of the generated paths of $(\exp\{-\Lambda_{S_t}\})_{t\geqslant 0}$. Note that the numerical results from the FPDE are very close to the Monte-Carlo simulations and always lie in the 99% confidence interval.

Figure 5 shows an example of the different shapes we can obtain through the choice of the fractional order α. To obtain the survival probabilities of Figure 5, we used the parameters of the calibrated fractional model (see Table 1), with the exception of α that ranges between 0.1 and 1 by steps of 0.1. The main observable feature is the degree of convexity of the curves. With a low fractional order α, the survival probability drops sharply at the beginning, but decreases much slower afterwards. This provides a good tool for modeling companies that have financial troubles in the short term, but have a good long term viability if they overcome their troubles.

Conclusions

This article proposes an application of fractional Hawkes jump processes to credit risk modeling. These processes present several interesting properties. Firstly, the intensity may be seen as a systematic factor driving the probability of failure of firms. The mechanism of self-excitation in this intensity ensures that the

			Monte-Carlo			
Time	Market	FPDE	CI 99%	Average	CI 99%	St. Dev.
1	0.7954	0.760422	0.755463	0.761501	0.767539	0.074130
2	0.5673	0.591944	0.584553	0.592788	0.601023	0.101101
3	0.4576	0.466446	0.455680	0.465248	0.474817	0.117471
4	0.3633	0.371079	0.359527	0.369701	0.379876	0.124911
5	0.3011	0.297728	0.285911	0.296311	0.306710	0.127669
6	0.2492	0.240802	0.229599	0.240198	0.250797	0.130120
7	0.2032	0.196293	0.186093	0.196737	0.207380	0.130668
8	0.1661	0.161266	0.152117	0.162516	0.172914	0.127660
9	0.1339	0.133534	0.124725	0.134715	0.144706	0.122647
10	0.1061	0.111452	0.103153	0.112737	0.122320	0.117655

Table 3: Numerical Solving of the FPDE and Monte-Carlo Simulations

model explains periods during which we observe a persistent increase of default probabilities. Secondly, the fractional model allows to obtain desirable features that differ from usual intensity models. Indeed, we have seen that obtaining survival probability curves that are constant during long periods of time is not possible with classical intensity models. Thirdly, the model is numerically tractable as it is possible to perform a calibration on real market data. As shown by this calibration, the fractional model fits better. We have seen that survival probabilities can be obtained by solving a fractional Fokker-Planck Equation. In this equation, the derivative with respect to time is replaced by the Caputo derivative. After a discussion about the proper approximation of Caputo derivatives, we have seen that the fractional model can explain a wide variety of term structures of survival probabilities.

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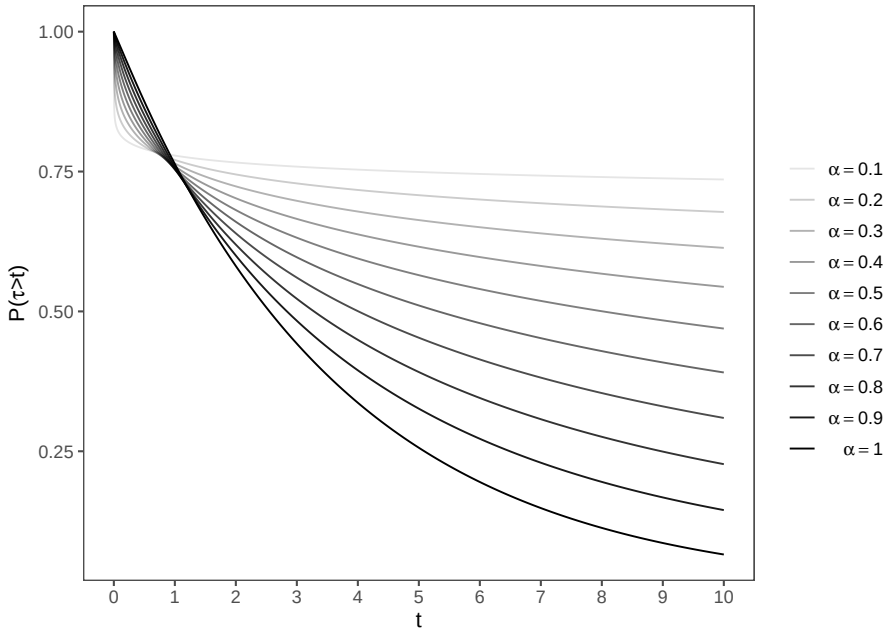


Figure 5: Survival Probability Curves in Function of α

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Appendices

Appendix 1: Proofs of the Results of Section 1

Proof of Proposition 1. First note that SDE (1) implies that

$$\lambda_t = \theta + e^{-\kappa t}(\lambda_0 - \theta) + \eta \int_0^t e^{-\kappa(t-s)} \mathrm{d}P_s. \tag{29}$$

By taking the expectation and deriving with respect to t, we obtain the ordinary differential equation (ODE)

$$\frac{\partial \mathbb{E}[\lambda_t]}{\partial t} = \kappa(\theta - \mathbb{E}[\lambda_t]) + \frac{\eta}{\rho}\mathbb{E}[\lambda_t] \tag{30}$$

whose solution is well given by Equation (2). For the variance, noting that $\mathbb{E}[\xi^2] = \rho^{-2}$ and using Ito's lemma yields

$$\frac{\partial \mathbb{E}[\lambda_t^2]}{\partial t} = \mathbb{E}[\lambda_t]\left(2\kappa\theta + 2\left(\frac{\eta}{\rho}\right)^2\right) + 2\mathbb{E}[\lambda_t^2]\left(\frac{\eta}{\rho} - \kappa\right). \tag{31}$$

By the chain rule, we find

$$\frac{\partial}{\partial t}(\mathbb{E}[\lambda_t])^2 = 2\kappa\theta\mathbb{E}[\lambda_t] + 2(\mathbb{E}[\lambda_t])^2\left(-\kappa + \frac{\eta}{\rho}\right). \tag{32}$$

Putting together Equations (31) and (32), we obtain $\frac{\partial}{\partial t}\mathbb{V}\mathrm{ar}(\lambda_t)$. Then, replacing $\mathbb{E}[\lambda_t]$ by its expression at Equation (2) and performing some simplifications leads an ODE for the variance. The solution of this ODE is well given by Equation (3) □

Proof of Proposition 2. We use the bivariate Kramers-Moyal expansion on the function p. This expansion says that the bivariate PDF p of (λ_t, Λ_t) satisfies

$$\begin{aligned} &p(t+\Delta, x_1, x_2|s, y_1, y_2) - p(t, x_1, x_2|s, y_1, y_2) \\ &\quad = \sum_{n=1}^{+\infty}\sum_{j=0}^{n} \frac{(-1)^n}{j!(n-j)!}\frac{\partial^j}{\partial x_1^j}\frac{\partial^{n-j}}{\partial x_2^{n-j}}\left(M(j, n-j, \Delta|t, x_1, x_2)p(t, x_1, x_2|s, y_1, y_2)\right) \end{aligned} \tag{33}$$

where $M(j, n-j, \Delta|t, x_1, x_2) = \mathbb{E}[(\lambda_{t+\Delta} - \lambda_t)^j(\Lambda_{t+\Delta} - \Lambda_t)^{n-j}|\lambda_t = x_1, \Lambda_t = x_2]$. A proof of this result can be found in Hainaut (2019). We need to determine the values of

$$M(j, n-j, \Delta|x_1, x_2) = \mathbb{E}[(\lambda_{t+\Delta} - \lambda_t)^j(\Lambda_{t+\Delta} - \Lambda_t)^{n-j}|\lambda_t = x_1, \Lambda_t = x_2]. \tag{34}$$

From Ito's Lemma applied to the function $h(x, y) = x^j y^{n-j}$ we find

$$\begin{aligned} &h(\lambda_{t+\Delta} - \lambda_t, \Lambda_{t+\Delta} - \Lambda_t) \\ &\quad = j\int_{s=0}^{\Delta}(\lambda_{t+s} - \lambda_t)^{j-1}(\Lambda_{t+s} - \Lambda_t)^{n-j}\kappa(\theta - \lambda_{t+s})\mathrm{d}s \\ &\quad + (n-j)\int_{s=0}^{\Delta}(\lambda_{t+s} - \lambda_t)^{j}(\Lambda_{t+s} - \Lambda_t)^{n-j-1}\lambda_{t+s}\mathrm{d}s \\ &\quad + \int_{s=0}^{\Delta}(\Lambda_{t+s} - \Lambda_t)^{n-j}\Big((\lambda_{t+s} - \lambda_t)^j - (\lambda_{t+s-} - \lambda_t)^j\Big)\mathrm{d}N_{t+s}. \end{aligned} \tag{35}$$

Now looking at the expectation of each term in Equation (35), we get

$$j\mathbb{E}\left[\int_{s=0}^{\Delta}(\lambda_{t+s}-\lambda_t)^{j-1}(\Lambda_{t+s}-\Lambda_t)^{n-j}\kappa(\theta-\lambda_{t+s})\mathrm{d}s\right] = \begin{cases}\kappa(\theta-\lambda_t)\Delta+O(\Delta^2) & \text{if } j=1 \text{ and } n=1\\ O(\Delta^2) & \text{otherwise,}\end{cases} \tag{36}$$

$$(n-j)\mathbb{E}\left[\int_{s=0}^{\Delta}(\lambda_{t+s}-\lambda_t)^{j}(\Lambda_{t+s}-\Lambda_t)^{n-j-1}\lambda_{t+s}\mathrm{d}s\right] = \begin{cases}\lambda_t\Delta+O(\Delta^2) & \text{if } j=0 \text{ and } n=1\\ O(\Delta^2) & \text{otherwise,}\end{cases} \tag{37}$$

$$\mathbb{E}\left[\int_{s=0}^{\Delta}(\Lambda_{t+s}-\Lambda_t)^{n-j}\Big((\lambda_{t+s}-\lambda_t)^j-(\lambda_{t+s-}-\lambda_t)^j\Big)\mathrm{d}N_{t+s}\right] = \begin{cases}\eta^j\mathbb{E}[\xi^j]\Delta\lambda_t+O(\Delta^2) & \text{if } n=j\\ O(\Delta^2) & \text{otherwise.}\end{cases} \tag{38}$$

Then dividing Equation (33) by Δ and replacing the terms by what we just found leads to

$$\begin{aligned}&\frac{p(t+\Delta,x_1,x_2|s,y_1,y_2)-p(t,x_1,x_2|s,y_1,y_2)}{\Delta}\\ &=-\kappa\Big(-p(t,x_1,x_2|s,y_1,y_2)+(\theta-x_1)\frac{\partial p(t,x_1,x_2|s,y_1,y_2)}{\partial x_1}\Big)\\ &+\mathbb{E}\left[\sum_{j=1}^{+\infty}\frac{(-\eta\xi)^j}{j!}\frac{\partial^j}{\partial x_1^j}(x_1p(t,x_1,x_2|s,y_1,y_2))\right]-x_1\frac{\partial p(t,x_1,x_2|s,y_1,y_2)}{\partial x_2}+O(\Delta).\end{aligned} \tag{39}$$

Recall that

$$\frac{\partial^j}{\partial x_1^j}(x_1p(t,x_1,x_2|s,y_1,y_2))=j\frac{\partial^{j-1}p(t,x_1,x_2|s,y_1,y_2)}{\partial x_1^{j-1}}+x_1\frac{\partial^j p(t,x_1,x_2|s,y_1,y_2)}{\partial x_1^j} \tag{40}$$

which we can use to write

$$\begin{aligned}&\mathbb{E}\left[\sum_{j=1}^{+\infty}\frac{(-\eta\xi)^j}{j!}\frac{\partial^j}{\partial x_1^j}(x_1p(t,x_1,x_2|s,y_1,y_2))\right]\\ &=\mathbb{E}\left[-\eta\xi\sum_{j=0}^{+\infty}\frac{(-\eta\xi)^j}{j!}\left(\frac{\partial^j}{\partial x_1^j}p(t,x_1,x_2|s,y_1,y_2)\right)\right]\\ &\quad+\mathbb{E}\left[\sum_{j=1}^{+\infty}\frac{(-\eta\xi)^j}{j!}x_1\frac{\partial^j p(t,x_1,x_2|s,y_1,y_2)}{\partial x_1^j}\right]\\ &=x_1\mathbb{E}[p(t,x_1-\eta\xi,x_2|s,y_1,y_2)-p(t,x_1,x_2|s,y_1,y_2)]\\ &\quad-\eta\mathbb{E}[p(t,x_1-\eta\xi,x_2|s,y_1,y_2)].\end{aligned} \tag{41}$$

It follows that letting Δ tend to zero in Equation (39) leads to the announced result. □

Proof of Proposition 3. We start by writing

$$\begin{aligned}\frac{\partial\phi(t,z_1,z_2|s,y_1,y_2)}{\partial t} &= \lim_{\Delta\to 0}\frac{\phi(t+\Delta,z_1,z_2|s,y_1,y_2)-\phi(t,z_1,z_2|s,y_1,y_2)}{\Delta}\\ &= \int_0^{+\infty}\int_0^{+\infty} e^{-z_1x_1-z_2x_2}\frac{\partial p(t,x_1,x_2|s,y_1,y_2)}{\partial t}\mathrm{d}x_1\mathrm{d}x_2.\end{aligned} \tag{42}$$

This allows us to use the Fokker-Planck Equation (5), which leads to

$$\begin{aligned}\frac{\partial\phi(t,z_1,z_2|s,y_1,y_2)}{\partial t} &= \kappa\int_0^{+\infty}\int_0^{+\infty} e^{-z_1x_1-z_2x_2}p(t,x_1,x_2|s,y_1,y_2)\mathrm{d}x_1\mathrm{d}x_2\\ &-\int_0^{+\infty}\int_0^{+\infty} e^{-z_1x_1-z_2x_2}\kappa(\theta-x_1)\frac{\partial p(t,x_1,x_2|s,y_1,y_2)}{\partial x_1}\mathrm{d}x_1\mathrm{d}x_1\\ &-\int_0^{+\infty}\int_0^{+\infty} e^{-z_1x_1-z_2x_2}x_1\frac{\partial p(t,x_1,x_2|s,y_1,y_2)}{\partial x_2}\mathrm{d}x_1\mathrm{d}x_2\\ &-\eta\int_0^{+\infty}\int_0^{+\infty} e^{-z_1x_1-z_2x_2}\mathbb{E}[\xi p(t,x_1-\eta\xi,x_1|s,y_1,y_2)]\mathrm{d}x_1\mathrm{d}x_2\\ &+\int_0^{+\infty}\int_0^{+\infty} x_1e^{-z_1x_1-z_2x_2}\mathbb{E}[p(t,x_1-\eta\xi,x_2|s,y_1,y_2)]\mathrm{d}x_1\mathrm{d}x_2\\ &-\int_0^{+\infty}\int_0^{+\infty} x_1e^{-z_1x_1-z_2x_2}\mathbb{E}[p(t,x_1,x_2|s,y_1,y_2)]\mathrm{d}x_1\mathrm{d}x_2.\end{aligned} \tag{43}$$

The proof is then completed by computing all the terms in Equation (43). This can be done with integrations by parts. □

Appendix 2: Monte-Carlo Simulations

The Monte-Carlo simulations of the processes $(\lambda_t)_{t\geqslant 0}$ are performed with a classic Euler discretization scheme. Let $\delta>0$ be the discretization step size. This step size was set to 10^{-3} in our computations. In order to simulate a discretized path $(\lambda^\delta_{n\delta})_{n\geqslant 0}$ of $(\lambda_t)_{t\geqslant 0}$, we set $\lambda^\delta_0:=\lambda_0\in\mathbb{R}$ with $\lambda_0\geqslant\theta$ and for $n\geqslant 0$, we set

$$\lambda^\delta_{(n+1)\delta}:=\lambda^\delta_{n\delta}+\kappa(\theta-\lambda^\delta_{n\delta})\delta+\eta\xi_nB_n$$

where ξ_n is exponentially distributed with mean ρ^{-1} and B_n is Poisson distributed with parameter $\delta\lambda^\delta_{n\delta}$. All the random variables ξ_n and B_n are independent. The integral $(\Lambda_t)_{t\geqslant 0}$ of $(\lambda_t)_{t\geqslant 0}$ is approximated by $(\Lambda^\delta_{n\delta})_{n\geqslant 0}$. The latter is computed with Riemann sums, that is $\Lambda^\delta_0:=0$ and

$$\Lambda^\delta_{(n+1)\delta}:=\Lambda^\delta_{n\delta}+\delta\lambda^\delta_{n\delta}.$$

Simulating the inverse α-stable process $(S_t)_{t\geqslant 0}$ is done with the algorithm proposed in Magdziarz (2009, b) which we describe now. We still work with the same discretization step $\delta=10^{-3}$. In order to simulate the discretized process $(S^\delta_{n\delta})_{n\geqslant 0}$, we start by simulating the discretized α-stable process $(U^\delta_{n\delta})_{n\geqslant 0}$. To do so, we set $U^\delta_0:=0$ and

$$U^\delta_{(n+1)\delta}:=U^\delta_{n\delta}+\delta^{1/\alpha}X_n$$

where X_n are i.i.d. totally skewed positive α-stable random variables. Realizations of such random variables are obtained with

$$X_n := \frac{\sin(\alpha(V_n + \frac{\pi}{2}))}{(\cos(V_n))^{1/\alpha}} \left(\frac{\cos(V_n - \alpha(V_n + \frac{\pi}{2})}{W_n} \right)^{\frac{1-\alpha}{\alpha}}$$

where V_n are i.i.d. random variables uniformly distributed on $(-\pi/2, \pi/2)$ and W_n are i.i.d. exponentially distributed random variables with mean 1. For more about simulations of α-stable random variables, see the Chapter 3 of Janicki and Weron (1994). It is then possible to approximate $(S_t)_{t \geqslant 0}$ with

$$S_t^\delta := (\min\{n \in \mathbb{N} : U_{n\delta}^\delta > t\} - 1)\delta.$$

By Theorem 2 in Magdziarz (2009, b), $\sup_{0 \leqslant s \leqslant T} |S_s - S_s^\delta| \leqslant \delta$ a.s. for all $T > 0$, so that the error in the approximation of $(S_t)_{t \geqslant 0}$ is always bounded by $\delta = 10^{-3}$. Given that $S_t^\delta \in \{\delta n : n \in \mathbb{N}\} \cup \{0\}$ for all $t \geqslant 0$, it is easy to combine $(\Lambda_{n\delta}^\delta)_{n \geqslant 0}$ and $(S_t^\delta)_{t \geqslant 0}$ to obtain a simulated path of $(\Lambda_{S_t})_{t \geqslant 0}$ by setting $\Lambda_{S_t}^\delta := \Lambda_{S_t^\delta}^\delta$.