The Axiom of Choice: Mathematics or Theology?

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Abstract

Through centuries, philosophers and theologians had tried to prove in many ways that God exists. Kurt Gödel gives once his own proof of existence of God based on modal logic S5 (since the modal formula $\Diamond\Box A \rightarrow \Box A$ grounds the proof) and some assumptions about existence as a perfect property. Ultimately, Gödel's ontological proof was a reforging of Leibniz's modal proof. Mimicking Gödel, Robert K. Meyer, a distinguished logician of the last century, had reworked and updated Aquinas's First Three Ways (namely the versions of the 'Cosmological Argument' that respectively focus on the notions of change, causation and contingence) by using some mathematical tools, namely set-theoretical ones as the Axiom of Choice and Zorn's Lemma. Unfortunately, this proof is not as well-known as it merits to be, so I shall show how such a set-theoretical demonstration of the existence of God works, and what are its assumptions. To do so, I will split my paper in two parts: first, a short discussion and, then, a proof of the Axiom of Choice in naïve-set-theory; second, a presentation and evaluation of the Mathematical Proofs (one cosmological due to Meyer, another ontological forged by Fisher) of the Existence of God based on the Axiom-of-Choice.

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Through centuries, philosophers and theologians had tried to prove in many ways that God exists (for a summary of Medieval 'classical' proofs, see Th. Aquinas, *Sum. Theol.*, 1a, q.2 and F. Suárez, *Disput. Met.* XXIX). Famously, Kurt Gödel gives once his own proof of existence of God based on modal logic S5 (since the formula $\Diamond\Box A \rightarrow \Box A$ grounds

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the proof) and some assumptions about existence as a perfect property (Gödel 1995b). Ultimately, Gödel's ontological proof was a reforging of Leibniz's modal proof (A II.1 585–591 (390–393 of the first edition), VI.3 571–579, 582–583, VI.4 18–19, 997–998, 1390–1391, 1566–1567, 1617, 1636–1637, VI.6 437–438, GP IV 401–406, etc., Leibniz's proof was itself a reworking of Descartes' ontological proof from the Fifth Metaphysical Meditation, see AT IX–1 50–56, 128–129, IX–2 31–32, which was itself heavily inspired by Anselm's *Proslogion*). Mimicking Gödel, Robert K. Meyer, a distinguished logician of the last century, had reworked and updated Aquinas's First Three Ways (namely the three versions of the 'Cosmological Argument' that respectively focus on the notions of change, causation and contingence, see Th. Aquinas, *Sum. Theol.*, 1a, q.2, art.3) by using some mathematical tools, namely set-theoretical ones as the Axiom of Choice and Zorn's Lemma (Meyer 1987). Unfortunately, this proof is not as well-known as it merits to be,[1] so I shall show how such a set-theoretical demonstration of the existence of God works, and what are its assumptions. To do so, I will split my paper in two parts: first, a short discussion and, then, a proof of the Axiom of Choice in naive set theory; second, a presentation and evaluation of the Mathematical Proofs (one cosmological due to Meyer 1987, another ontological forged by Fisher 1970) of the Existence of God based on the Axiom of Choice.

Step 1: Proving the Axiom of Choice in Naive Set Theory

Meyer 1987's Mathematical Proof of the existence of God (based on a suggestion of Hilary Putnam, see Routley 1980: 132, Meyer 1987: 361 n.4 and Putnam 1997: 488 n.1) is based on Zorn's Lemma,[2] that is one of the many equivalents of the Axiom of Choice (for a study of the many equivalent versions of the Axiom of Choice (*Auswahlpostulat*), see Rubin and Rubin 1963 and Moore 1982; on the equivalence between the Axiom of Choice and Zorn's Lemma, see Rosser 1953: 493–509, Rubin and Rubin 1963: 10–36, Sierpiński 1965: 430–433, Moore 1982: 220–227 and Bell 2021: §3). Roughly, an axiom of *n*-choice, where *n* is an assigned—either finite or transfinite—cardinality, states that at any point one may make *n* choices. Of course, axioms of finitary choice are not troublemaking (they only say that one can make a finite number of choices), but the situation is somewhat more questionable in the case of axioms of infinitary choices, i.e., where n is a transfinite cardinality (either denumerable or not). To borrow the usual terminology, I shall reserve the label 'Axiom of Choice' for the generalized axiom of choice which holds whatever the cardinality is.

> *Axiom of Choice (no choice function formulation)*: for every set x whose elements are sets y, non-empty and having no common elements, there is at least one set z having one and only one element from each of the sets y belonging to x.
>
> *Axiom of Choice (choice function formulation)*: for every set x whose elements are non-empty sets y, there is at least one choice function f that is defined on x and maps each set of x to an element of that set, i.e.,
>
> $\forall x$ [if $\emptyset \notin x$ then $\exists f\colon x \to \bigcup_{y \in x} y$ such that $\forall y(y \in x \to f(y) \in y)$]

Note that the Axiom of Choice does not imply that there must be an effective procedure for *constructing* the set z or the mapping f. In other words, the Axiom of Choice delivers the existence of sets which no mathematician can specify. How? Only *God knows how*!

This very anti-intuitionistic[3] point is nicely (and somewhat ironically) illustrated by Russell 1919: 125–127: for an infinite set of pairs of socks, given that right sock and left sock are notoriously indistinguishable (contrary to left shoe and right shoe), there is no obvious way to successfully select the left sock from every pair; however, as an almost magical trick (or as a *divine miracle*), once we have the Axiom of Choice, it easily allows us to pick out the left sock from each pair to obtain a new set of left-socks. In short, the Axiom of Choice is needed to choose one sock from each of infinitely many pairs of socks, but for shoes the Axiom is not required at all. In doing so, the mathematical usefulness of the Axiom is exhibited for it provides formerly unavailable tools for proving a battery of hitherto unattainable theorems.

The Axiom of Choice—albeit widely accepted nowadays—has aroused bitter controversy amongst mathematicians (on this controversy, see Russell 1992: 50–52, Rosser 1953: 490–493, Sierpiński 1965: 92–95 and Moore 1982). Some claim that this axiom is as obvious and as well-grounded as other set-theoretical axioms (Zermelo, who was the first to formulate the Axiom, Fraenkel, and Gödel

1990c: 177 n.2 for instance); other that anyway mathematical practice requires to assume it (Lewis 2020: 665, letter to Phillip Bricker, 6 April 1992); other that its usefulness suffices for accepting it (indeed, many theorems belonging to different mathematical disciplines have been drawn from it,[4] and its acceptance considerably simplifies the formulation of set theory); other that it is a dubious mathematical axiom, especially because it implies both that every set can be well-ordered (Well-Ordering Theorem, see Russell 1992: 50–52 and Rosser 1953: 493–494, 540–541 and Specker 1953 for the falsity of the Well-Ordering Theorem and, therefore, of the Axiom of Choice in Quine 1981's ML) and the so-called Banach-Tarski Paradox (Banach and Tarski 1924 and Krantz 2002: 125, see also similar paradoxes like Hausdorff Paradox and Von Neumann Paradox), in such a way that it is preferable to dispense with it.

The dispensability of the Axiom of Choice is a consequence of the fact that, as demonstrated by Gödel 1990a and Cohen 1963 (using his method of forcing), such an axiom is independent from the other axioms of Zermelo-Fraenkel Set Theory (note that, as points out by Specker 1953, the Axiom of Choice is false in Quine 1981's ML, Quine seems not to be aware of it however, see Quine 1981: 164). One can choose ZF rather than ZFC (ZF + the Axiom of Choice) depending of her opinion about the Axiom of Choice. But, once a naive set theory is provided, it readily follows that the Axiom of Choice, at least one variant of it, is provable (Routley 1980: 924–925; Priest, Routley and Norman 1989: 144, 373–374, 440; Berto 2007: 251; see also Weber 2009: 148–151, 188–189, 2012: 273–276, 2013: 327, 2021: 185 for another proof). Thus, although dispensable in sophisticated hierarchical set theories, the Axiom of Choice is a constituent principle of naive set theory.

The point is that there is a bundle of reasons to prefer a naive set theory rather than its sophisticated hierarchical or stratified reformulations as Zermelo-Fraenkel hierarchical theory (ZF henceforth), Russell's type-theory (RT), Neumann-Bernays-Gödel's class-theory (NBG), and Quine's New Foundations for Mathematical Logic (ML, see Quine 1981). To take just one example, hierarchizing and stratifying are always arbitrary and despairing theoretical moves. That is quite obvious while looking at the hierarchical and stratified solutions of the famous and venerable (at least 2400 years old, *viva Eubulides!*) semantic paradox called 'the Liar'—viz. the self-referential sentence 'this very sentence is not true'—defended by Russell, Ramsey, Tarski, Carnap, Quine and Cie (they all answer the paradox by ordering many different levels of language),[5] since there is always some strengthened Liar-paradox that the hierarchical solution fails to meet (on the unsuccessfulness of the alethic parametrization strategy in the case of self-referential paradoxes like the Liar, see Priest 1995: 149–154, 166–171, 2006: 18–25). For a defence of a naive set theory against usual classical hierarchical reforging, I refer the reader to Routley 1980: 911–935, Priest 1995: 123–194, esp. 172–186, 2006: 28–38, 141–143, Berto 2007: 63–92, 241–258 and Weber 2009: 148–151, 2021: 28–40. Thereafter, I shall take for granted the inadequacy (mostly for reason of artificiality and *ad-hocness*) of the hierarchical set-theories, and the urge to recapture the Cantorian naive insight.

Unlike classical complicated and hierarchical reforging of set theory, a naive set theory as the 'Dialectical Set Theory' (DST henceforth)[6] takes the set-theoretical paradoxes (Burali-Forti's, Cantor's, Russell's, etc.) at face value and commits itself to their teratological conclusions, for instance it accepts that the inconsistent Russell's Set—the set of the sets that do not belong to themselves (which has the property of belonging to itself and not belonging to itself)—is an inhabitant of the mathematical realm.

Russell's Set $\mathfrak{R} = \{x : x \notin x\}$

(and its paradoxical feature: $\mathfrak{R} \in \mathfrak{R}$ & $\mathfrak{R} \notin \mathfrak{R}$)

Indeed, to challenge the set-theoretical paradoxes, the classical strategy (shared by RT, ZF, NBG and ML) usually consists to *restrict* the comprehension axiom scheme in order to banish the teratological sets outside the mathematical realm (for instance, ZF limits the size of the sets that are acceptable, some alleged sets being 'too big', e.g., the universal set, i.e., the set of all sets, $\mathfrak{V}$ and the set of all ordinals Ω).[7] DST takes the opposite strategy (Routley 1980: 911-913), that is, the strategy of absolutely unrestricting the comprehension axiom and accepting all teratological sets since 'the paradox arguments are taken as proofs':

> "A dialectical set theory is one which accepts the paradoxes of set theory as part of the theory; it is a theory on which the underdeterminacy and overdeterminacy induced by the paradox-generating items of the set theoretical paradoxes is simply admitted and the paradox arguments are taken as proofs; it is a theory according to which the Russell class, for example, the class of all those classes which are not self-membered, both does and also does not belong to itself, and thus is perforce an inconsistent theory. But the hope (which can be vindicated under certain conditions) is that it is not a trivial theory on which not just the Russell paradox, but everything, holds." (Routley 1980: 911)

So,—as every genuine naive set theory should be—DST needs to be paraconsistent, that is, needs to be a theory in which sets and their members could have *contradictory* or *inconsistent* properties without trivializing the theory itself (on paraconsistency, its rational, and its defence, see the essays reunited in Priest, Routley and Norman 1989). Naive set theory has to drop out *horror contradictionis* (perhaps after having fought some commonsense 'incredulous stare'[8]) to regain Cantor's paradise that a lot of mathematicians mistakenly believe to be lost.

To be a genuine naive set theory, DST also has to capture Cantor's and Frege's intuitive set-theoretical ideas. To do so, DST is based on an absolutely *unrestricted* comprehension axiom (GCA) that generalizes the naive set comprehension axiom scheme (Routley 1980: 915–916),[9] namely: for every well-formed expression $\varphi(x)$ there is a set y such that $x \in y$ iff $\varphi(x)$, i.e., in a formal setting (where '$\leftrightarrow$' is taken to be an intensional biconditional, that is, for x to be φ *means* that x is in the set of φs):

$$\text{GCA} \qquad \exists y \forall x (x \in y \leftrightarrow \varphi(x))$$

Then, DST is defined by two nonlogical axioms, the first of which GCA is the existential generalization: *Abstraction* and *Extensionality* (Priest 2006: 141 and Weber 2021: 151–152). These two axioms analytically characterize the notion of set, i.e., they encapsulate what is to be a set (on what is a set/class in naive set theory—since there is no artificial and *ad-hoc* filter (see Priest 1995: 172–182, 2006: 34–36 and Weber 2021: 38) that discriminates genuine *sets* from mere *classes* as it the case in ZF and NBG –, namely a set is both an *extensional* collection and an *intensional* property; the set-membership ∈ having been coined to capture the idea of property instantiation or the predicative 'is', see Weber 2021: 33–35).

Abstraction	$x \in \{z : \varphi(z)\} \leftrightarrow \varphi(x)$
Extensionality	$\forall x(x \in y \leftrightarrow x \in z) \leftrightarrow y = z$

It immediately follows that, in DST, every predicate φ determines a set (i.e., for any predicate φ, there is a set y such that $x \in y$ iff $\varphi(x)$).

So, given GCA, there are sets that are defined by reference to themselves. Such an anti-hierarchical move is, of course, prohibited in classical set theories as ZF, NBG and ML and usually considered as foolish. However, sets defined by reference to themselves, whether inconsistent—for instance Routley's Set $\mathcal{Z}$ of all x that are not in (Routley 980: 915–916, the label comes from Weber 2021: 167–168) which has the inconsistent property to both have and do not have every object as a member, i.e.,

$$\textit{Routley's Set} \;\; \mathcal{Z} = \{x : x \notin \mathcal{Z}\}$$

– or not, are not as bizarre as one may believe. In fact, they appear to not be so teratological once they are compared to other mathematical weirdos, for instance to the nilpotent infinitesimals of Smooth Infinitesimal Analysis (an infinitesimal magnitude $\varepsilon \neq 0$ is nilpotent if $\varepsilon^k = 0$ for some $k \in \mathcal{N}$, on this alternative to Standard Analysis, see Lavendhomme 1998 and Bell 2008, 2019: 233–271). DST also includes non-well-founded sets (Weber 2021: 181; for a study on non-well-founded set theories, see Aczel 1988), that is another advantage of DST over ZF (standardly equipped with the Axiom of Foundation) in which all sets of the cumulative hierarchy necessarily are well-founded (a set is well-founded if all its non-empty parts have ∈-least members, i.e., if it has no infinitely descending set-membership chain).

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In DST, there is a straightforward proof of a strong version of the Axiom of Choice, namely the *generalized* Axiom of Choice. The proof is easy, since, in fact, the Axiom of Choice is directly derivable from GCA (Routley 1980: 924–925; Priest, Routley and Norman 1989:

373–374; and Weber 2009: 148–151, 188–189, 2012: 273–276, 2013: 327, 2021: 185 for an alternative proof that is more sophisticated but also is based on other disputable ideas).

First, let's define what is a *function*. A mapping $f: x \rightarrow y$ is a function from x to y iff f assigns to each element of x exactly one element of y. In other words, a set f is a function iff f is either a null relation or a univalent one, i.e.,

- either f has no members, i.e., there are no $u \in x$ and no $v \in y$ such that $\langle u, v\rangle \in f$
- or else f is univalent, i.e., for all $z \in x$ and all $u, v \in y$, if $\langle z, u\rangle \in f$ and $\langle z, v\rangle \in f$ then $u = v$

Usually, the first disjunct is not mentioned. But, if f is empty, that *classically* involves that every member of f satisfies any defining condition, so the ones of f and the conditions of functionality, but also the ones of any arbitrary set: admittedly, all *f*s are *f*s, but they are "purple unicorns" (cf. Weber 2021: 218) and round squares too.[10] The worry, as pointed out by Weber 2009: 148 (Routley 1980: 925 himself acknowledges this fact however), is that this is only correct within the framework of classical logic in which the empty set ∅ can be seen as a well full of absurdities (since—by *ex falso quodlibet*, i.e., from falsehood anything follows—$\forall x(x \in \emptyset \rightarrow \varphi(x))$ for any φ is a 'vacuous truth'), whereas DST is based on the weak relevant and paraconsistent logic DKQ (Routley 1980: 917–919; for the record, the variant of DST elaborated in Brady 2006: 171–274 is based on DJ^dQ) in which the *ex falso quodlibet* fails. Hence, the null relation disjunct is not allowed in DST. In other words, there is a patent incompatibility between the classical account of functionality used in Routley's proof and the logic underlying DST: the first needs a classical logic that the second aims to pillory. However, Routley's proof of the Axiom forces the classicist to accept that if GCA holds then so does the Axiom. And, arguably, that is already a significant result.

So, formally (where '*fnc*(*f*)' means '*f* is a function'):

1. $fnc(f)$ iff $\forall x \forall y \langle x, y\rangle \notin f \lor \forall x \forall y \forall z[(\langle x, y\rangle \in f \land \langle x, z\rangle \in f) \rightarrow y = z]$

Remind that a choice function on a set x of non-empty sets is a function f such that for every non-empty set y that belongs to x, $f(y)$ is an element of y, i.e.,

2. $choic\langle f, x\rangle$ iff $fnc(f) \wedge \forall y((y \in x \wedge y \neq \emptyset)) \rightarrow f(y) \in y)$

Hence the Axiom of Choice (AC) can be reformulated as follows: for all sets x, there is a function f which maps, for any non-empty set y in x, that set to an element of itself, i.e.,

AC. $\forall x \exists f\, [fnc(f) \wedge \forall y((y \in x \wedge y \neq \emptyset)) \rightarrow f(y) \in y)]$
AC. $\forall x \exists f\, choic\langle f, x\rangle$

Routley aims to prove the following variant of AC:

$$\exists f(fnc(f) \wedge \forall x \exists y[y \in x \wedge t) \rightarrow f(x) \in x])$$

that is,

$$\exists f(fnc(f) \wedge \forall x[x \neq \emptyset \wedge t) \rightarrow f(x) \in x])$$

To do so, Routley's proof of the Axiom of Choice is based on the construction of an impredicative (i.e., self-referential) set whose existence is guaranteed by GCA,

$$f = \{\langle x, y\rangle : x \text{ is a non-empty set, } y \text{ is in } x\text{, and } f \text{ is a function}\}$$

which is formally dressed as follows:

3. $\exists f \forall x \forall y\, [\langle x, y\rangle \in f \leftrightarrow (y \in x \wedge fnc(f))]$

The rest of the proof is designed to show that such a set, whose existence is ensured by GCA, does indeed describe, first, a *function*, and second, a *choice* function.

The first remaining part is straightforward: by the law Excluded-Middle (LEM), either there is some x and y such that $\langle x, y\rangle \in f$ or there is not.

4. $\exists x \exists y \langle x, y\rangle \in f \vee \forall x \forall y \langle x, y\rangle \notin f$ LEM

But, for such a f, both

5. $\forall x \forall y \langle x, y \rangle \notin f \rightarrow fnc(f)$ by 1

and,

6. $\exists x \exists y \langle x, y \rangle \in f \rightarrow fnc(f)$ by 3

hold (since, first, if f is empty f is a function by definition 1, and, second, if f has any members, then its defining conditions 3 are met, especially the one that says that f is a function). So either way, $fnc(f)$, i.e., f is a function (by the inferential pattern $A \vee B, A \rightarrow \Gamma, B \rightarrow \Gamma \vdash \Gamma$).

7. $fnc(f)$ by 4, 5 and 6

Then, the final part of the proof, which establishes that f is indeed a choice function. $f(x)$ can be characterized as follows:

8. $f(x) \leftrightarrow (fnc(f) \wedge \exists y(y \in x)) \rightarrow \langle x, f(x) \rangle \in f$
9. $(\exists y(y \in x) \wedge t) \rightarrow (\langle x, f(x) \rangle \in f \wedge t)$ by 8

where t is a constant that allows to dress up 8 in an enthymematic form (Routley 1980: 923). Hence

10. $(\langle x, y \rangle \in f \wedge t) \rightarrow y \in x$ from 3
11. $(\langle x, f(x) \rangle \in f \wedge t) \rightarrow f(x) \in x$ 10, $y/f(x)$
12. $\forall x \exists y[y \in x \wedge t) \rightarrow f(x) \in x]$ by 9 and 11

There is a function f such that for every non-empty set x, $f(x)$ belongs to x.

13. $\exists f(fnc(f) \wedge \forall x \exists y[y \in x \wedge t) \rightarrow f(x) \in x])$ by 7 and 12

Thus f is a choice function on x: for all $y \in x$, $f(y) \in y$. Take x to be $\mathfrak{V}$, f is a *global* choice function. So, the Axiom of Choice is a theorem of DST. Q.E.D.

14. $\exists f \forall x \forall y\ [(\langle x, y \rangle \in f \leftrightarrow (y \in x \wedge choic\langle f, x \rangle)]$

To sum up, Routley's proof runs as follows:

1. $fnc(f) \leftrightarrow \forall x \forall y \langle x, y \rangle \notin f \vee \forall x \forall y \forall z[(\langle x, y \rangle \in f \wedge \langle x, z \rangle \in f) \rightarrow y = z]$ *Def. fnc*
2. $choic\langle f, x \rangle \leftrightarrow fnc(f) \wedge \forall y((y \in x \wedge y \neq \emptyset)) \rightarrow f(y) \in y)$ *Def. choic*
3. $\exists f \forall x \forall y\ [\langle x, y \rangle \in f \leftrightarrow (y \in x \wedge fnc(f))]$ *Hyp.*

4.	$\exists x \exists y \langle x, y\rangle \in f \vee \forall x \forall y \langle x, y\rangle \notin f$	LEM
5.	$\forall x \forall y \langle x, y\rangle \notin f \rightarrow fnc(f)$	by 1
6.	$\exists x \exists y \langle x, y\rangle \in f \rightarrow fnc(f)$	by 3
7.	$fnc(f)$	by 4, 5 and 6
8.	$f(x) \leftrightarrow (fnc(f) \wedge \exists y(y \in x)) \rightarrow \langle x, f(x)\rangle \in f$	*Def. f*(*x*)
9.	$(\exists y(y \in x) \wedge t) \rightarrow ((\langle x, f(x)\rangle \in f \wedge t)$	
10.	$((\langle x, y\rangle \in f \wedge t) \rightarrow y \in x$	by 3
11.	$((\langle x, f(x)\rangle \in f \wedge t) \rightarrow f(x) \in x$	10, $y/f(x)$
12.	$\forall x \exists y[y \in x \wedge t) \rightarrow f(x) \in x]$	by 9 and 11
13.	$\exists f(fnc(f) \wedge \forall x \exists y[y \in x \wedge t) \rightarrow f(x) \in x])$	by 7 and 12
14.	$\exists f \forall x \forall y\ [((\langle x, y\rangle \in f \leftrightarrow (y \in x \wedge choic\langle f, x\rangle))]$	3, 13

Q.E.D.

Step 2: Zorn's Lemma and the Mathematical Proof of the Existence of God

The Axiom of Choice implies Zorn's Lemma—in fact, they are *mathematically equivalent* (on the equivalence between the Axiom of Choice and Zorn's Lemma, see Rosser 1953: 493–509, Rubin and Rubin 1963: 10–36, Sierpiński 1965: 430–433 and Bell 2021: §3). Of course, the equivalence between the two is a bit puzzling, since the Axiom of Choice is far more intuitive than Zorn's Lemma, i.e., although they are *mathematically* equivalent, they do not have the same *epistemic* obviousness or intuitiveness.[11] This point is nicely emphasized in the well-known joke of Jerry Bona quoted, for instance, in Krantz 2002: 121: "the Axiom of Choice is obviously true, the well-ordering principle obviously false, and who can tell about Zorn's Lemma?", the three being mathematically equivalent.

To get the joke, remind that the Axiom of Choice says that for every set *x* of pairwise disjunct and non-empty sets, there is at least one set *z* that has exactly one element in common with each of the sets in *x*. Its equivalent, Zorn's Lemma, is a maximal principle (alike to Hausdorff's one) that says that every non-empty partially ordered set in which every chain (= every totally ordered subset) has an upper bound has at least one maximal element. As for the Well-Ordering Theorem, it states that every set can be well-ordered, that is, for every set *x*, one can find a total ordering such that every non-empty subset of *x* has a least element.

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The Cosmological Argument takes three forms in Thomas Aquinas (there are the three first ways to prove God's existence; on Aquinas' Five Ways, see Kenny 1969 and Weingartner 2010, on the Cosmological Argument see also Sobel 2004: 168–199 and Oppy 2006: 97–173, 2009),[12] the difference being the kind of ordered and asymmetric relation <*x*, *y*> involved, respectively: the <moved, mover> pair, the <effect, cause> pair, and the <contingent, necessary> pair. The key idea, however, is the same in all three: on pain of alleged absurdity, there could not be an infinite regress of <*x*, *y*>, so there must be a being that is only a second member *y* of the pair, respectively: a being that moves without being moved (a First Unmoved Mover), a being that causes without itself being caused (a First Uncaused Cause), and a being whose existence is necessary. I shall only quote the Second Way (*Sum. Theol.* Ia, q.2, art.3, trans. Timothy McDermott):

> The Second Way is based on the nature of causation. In the observable world causes are found to be ordered in series; we never observe, nor ever could, something causing itself, for this would mean it preceded itself, and this is not possible. Such a series of causes must however stop somewhere; for in it an earlier member causes an intermediate and the intermediate a last (whether the intermediate be one or many). Now if you eliminate a cause you also eliminate its effects, so that you cannot have a last cause, nor an intermediate one, unless you have the first. Given therefore no stop in the series of causes, and hence no first cause, there would be no intermediate causes either, and no last effect, and this would be an open mistake. One is therefore forced to suppose some first cause, to which everyone gives the name "God."

The basic idea of Meyer 1987's reforging of the Cosmological Argument[13] is that if the universe has a partial ordering on the set of all existent items (taken, for instance, to capture the idea of a chain of causes with the assumption that the causal relation is transitive, irreflexive and asymmetric[14]), then, by Zorn's Lemma, this set has a maximal element which can be called 'God' since it is the First Cause. In a nutshell and a bit informally, Meyer 1987's proof runs as follows, once a few definitions are provided, namely: $\mathfrak{E}$ is the set of all entities that exist, whether they are concrete individuals, events, states of affairs, or else; a *god* is defined as an element g of the set of all existent entities $\mathfrak{E}$ having the property that if x is another element of $\mathfrak{E}$ such that $g > x$, where '$x > y$' symbolizes the strict partial ordering relation 'y is the cause of x',[15] then $g = x$; a *divine creator* of an element x in $\mathfrak{E}$ is a god g such that $x > g$.

1. The set $\mathfrak{E}$ is a set in a universe of set $\mathfrak{V}$ satisfying the Axiom of Choice.
2. The set $\mathfrak{E}$ is inductively ordered.
3. If 1 and 2 hold, there is a g in $\mathfrak{E}$ and any element of $\mathfrak{E}$ has a divine creator.

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4. There is a g in $\mathfrak{E}$ and any element of $\mathfrak{E}$ has a divine creator. (by *modus ponens*)

The same inferential pattern holds for the two other versions of the Cosmological Argument, i.e., when '$x > y$' is taken to be the partially ordered relation 'y is the mover of the moved x' or 'x is changed by y' (Aquinas' first way) and 'x owes its necessity to y' (Aquinas' third way).

As it has already been shown, Premiss 1 is true once DST is accepted (that is, once the One True Set Theory is taken to be a naive set theory, but, of course, Premiss 1 is also true in ZFC).

Premiss 3 is a deductive consequence of 1 and 2 obtained by Zorn's Lemma: given that $\langle \mathfrak{E}, > \rangle$ is partially ordered and all its chains (viz. a *chain* of $\mathfrak{E}$ is a totally ordered subset of , i.e., a subset of $\mathfrak{E}$ such that for any distinct elements x and y of this subset, either $x > y$ or $y > x$) have an upper bound (viz. an element x of the chain such that $y > x$ for all y of that chain), therefore $\langle \mathfrak{E}, > \rangle$ has a maximal element, viz. a First Cause g.

At first sight, all the burden of Meyer's proof is carried by Premiss 2: is it true that the set $\mathfrak{E}$ is inductively ordered (i.e., that every chain in $\mathfrak{E}$ has an upper bound), that it is structured as a causal nexus?[16] Indeed, some philosophers had argued for the idea that causation (usually taken to be a *grounding* relation) might be circular, make loops, or be infinitely descending, that is, they had argued for the idea that causation might be neither irreflexive, nor transitive, nor antisymmetric (see Bliss and Priest 2018: 7–31 for an overview of the conceptual alternatives and their possible defences, and Marion 2022: 31–40 for some examples in classical Chinese philosophy).[17] Premiss 2 is based on the very disputable idea that everything—apart the First *uncaused* Cause (provided that the causal relation is taken to be irreflexive; otherwise, the First Cause is self-caused, i.e. *causa sui*, rather than uncaused)—has a cause other than itself (the so-called 'Causal Principle'), more precisely on the idea that in every causal chain each element has a logically anterior cause *plus* the assumption that there is an element of $\mathcal{E}$ which causes all the other items of the chain (viz. by transitivity, the first cause of this causal chain).[18] Such an idea forms a *strong* Causal Principle (on different versions of this principle, see Pruss 2006: 41–71), since it amounts not only to asserting that everything apart the First uncaused Cause is caused, but also to claiming that for every causal chain, there is an element that is the cause of every other item in that chain (viz. the cause of the whole chain). It is not a coincidence that a greater part of Meyer 1987 is devoted to a discussion of the Causal Principle and, consequently, to a defence of Premiss 2 (namely Meyer 1987: 346–348, 355–356) rather than to the other premisses.

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The Cosmological Argument forms the first three Aquinas' ways of proving God's existence,[19] so these three ways can be formulated as involving the Axiom of Choice in guise of Zorn's Lemma. Is it the same for the fourth way, viz. the argument from degree? Yes. For this proof is also based on the idea that God is a maximal element. Hence the same inferential pattern and, therefore, the same conclusion: God exists (shortly: if there is a property x such that there is an arbitrary y that has x at an arbitrary degree z, there must be a g such that g is maximally x, i.e., such that g has x at the maximal degree u where $u \geq z$ for all z; by Zorn's Lemma, there is such a g). For the record, here Aquinas' text (*Sum. Theol.* Ia, q.2, art.3, trans. Timothy McDermott):

> The Fourth Way is based on the gradation observed in things. Some things are found to be more good, more true, more noble, and so on, and other things less. But such comparative terms describe varying degrees of approximation to a superlative; for example, things are hotter and hotter the nearer they approach what is hottest. Something therefore is the truest and best and most noble of things, and hence the most fully in being; for Aristotle says that the truest things are the things most fully in being [*Met.* α.1 993b30–31]. Now 'when many things possess some property in common, the one most fully

> possessing it causes it in the others: fire', to use Aristotle's example, 'the hottest of all things, causes all other things to be hot' [*Met.* α.1 993b24–26]. There is something therefore which causes in all other things their being, their goodness, and whatever other perfection they have. And this we call "God."

In fine, despite Aquinas's own beliefs (*Sum. Theol.* 1a, q.2, art.1), Aquinas' fourth way is very alike to Anselm's Ontological Argument since the fact that God is taken to be a maximal element grounds the proof (to show that Aquinas' fourth way is similar to Anselm's proof, it suffices to replace Aquinas' comparative *is truer than* or *has more being than* by Anselm's one *can be conceived as greater than*, plus the further assumption that, for all *x*, existent-*x* can be conceived as greater than non-existent *x*).

Contrary to the first four ways, Aquinas' fifth way, viz. the teleological argument from finality and design, does not implicitly appeal to the Axiom of Choice.

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The Axiom of Choice not only plays a key-role in Meyer's reforging of the Cosmological Argument, but has also been used to reconstruct Anselm's Ontological Argument. Even more unknown than Meyer 1987,[20] Fisher 1970 is a 2-page paper published in a mathematical (rather humoristic) magazine of the University of Warwick (the editor was Ian Stewart) that aims to mathematically dress up the Ontological Argument (first devised by Anselm of Canterbury in his *Proslogion*, then remodelled by Descartes, Leibniz and Gödel; on the many versions of the Ontological Argument, see Plantinga 1965, Barnes 1972, Oppy 1995, 2006: 49–96, Sobel 2004: 29–167, and the essays reunited in Oppy 2018). To do this, Fisher 1970 uses Zorn's Lemma. Without further ado, here Fisher 1970: 48's text:

> *Theorem* (St. Anselm, Aquinas,[21] and others.)
> The axiom of choice is equivalent to the existence of ~~a unique~~[22] God.
>
> *Proof* <of the first entailment underlying the Theorem: if the Axiom of Choice holds, then God exists>
>
> (Assuming the equivalence of the Axiom of Choice and Zorn's Lemma) Partially order the set of subsets of the set of all properties

of objects by inclusion. This set has maximal elements. God is by definition (due to Anselm) a maximal element in this set.

$$\text{God} \subseteq \text{God} \cup \{\text{existence}\}, \text{ so God} = \text{God} \cup \{\text{existence}\}$$
$$\therefore \text{God exists.}$$

After this proof, Vox Fisher provides, first, a question-begging 'demonstration' of the uniqueness of God based on Aquinas' Cosmological Argument (to be fair, Fisher 1970: 49 is well-aware that one can easily object to Aquinas' claim of God's uniqueness), and, then, a short statement about the nature of God: God is an *omnipotent* choice-maker whose existence makes true the Axiom of Choice (this is the proof of the second entailment underlying the Theorem: if God exists, then the Axiom of Choice holds).[23]

So, it appears that the Axiom of Choice in guise of Zorn's Lemma underlies two of the most celebrated proofs of the existence of God, namely Anselm's Ontological Argument and Aquinas' Cosmological One. Hence, under the insights of Gödel, Meyer and Fisher, Euler's dreams of mathematically proving God's existence finally came true.

Of course, although the old arguments for God's existence can be mathematically dressed up, it is quite obvious that the burden of these arguments do not lie in their mathematical features at all: the conclusiveness of the Cosmological Argument depends on the Causal Principle and some formal features attributed to the relation of causation (transitivity, irreflexivity, asymmetry), and the best-known versions of the Ontological Argument are based either on the idea that existence, as a perfection, is a genuine characterizing property in the same way as redness, roundness, etc. (an idea criticized both by Quine 1948's quantificationalist meta-ontology, and by the Meinongian one in which existence is often regarded as an extra-nuclear or non-characterizing property, e.g. in Routley 1980: 45–52, 180–187) or on Anselm's idea that God can be *defined* as 'that than which nothing greater can be conceived' (see the criticism of Anselm's Argument in Aquinas, *Sum. Theol.* 1a, q.2, art.1). That is the reason why the objections to these arguments usually are extra-mathematical, viz. they focus on Premiss 2 of Meyer 1987's Proof and either on the meta-ontological status of *existence* or on the definition of God as a *maximal* element for the Ontological Argument.[24] It is nonetheless worth noting that, while one might believe that because of this there is little to be gained by dressing up the old proofs of God's existence in mathematical clothes, in fact such a move has some significant advantages: on the one hand, it establishes the logical validity of these arguments once some assumptions are provided; and, on the other hand, it specifies the mathematical structure imposed on reality in order for their premisses to be true; in doing so, the mathematical reforging non-trivially improves and updates the venerable proofs (see Meyer 1987: 353–354, 359–361 and Oppy 2006: 124–125 for similar acknowledgements). Furthermore, such a mathematical dressing-up tells us that, ontologically speaking, the most distinctive property of God is her *maximality*, in whatever way such a property is understood. So, theology is the science that studies the maximal item.

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Ultimately, as Rosser 1953: 490–491 points out, while stating the Axiom of Choice, the mathematician is not doing mathematics but, rather, theology; a claim emphasized by Routley 1980: 133 as follows:

> Given the argument the proposition that God exists is, in fact, equivalent to the Axiom of Choice, using expected connections. For the Axiom of Choice materially implies Zorn's lemma, which materially implies that God exists, given that God is a First Cause. And conversely, that God exists materially implies that a supreme choice maker exists (given such expected properties of God as omnipotence), so the Axiom of Choice is guaranteed.

As pointed out in this quotation of Routley, there are two different but theologically related logical implications here: first, if there is an omnipotent God (i.e., an entity who is able to instantly make an *indenumerable* number of choices), then the Axiom of Choice is true (Rosser 1953: 490–491 emphasizes only this first implication, cf. Fisher 1970: 48 and Meyer 1987: 353); second, if the Axiom of Choice is true, then there is a God as Perfect Being or First Cause (respectively by Fisher 1970's and Meyer 1987's Proofs). But the Axiom is true, provided the alleged One True Set Theory (naive set theory in guise of DST). Therefore, by *modus ponens*, there is a First Cause, namely God.

It should be noted in passing that these two implications are theologically underdetermined, given that they are perfectly compatible with both monotheism and polytheism, and so cannot be used to support the existence of The Only One God of Abrahamic religions (despite Fisher 1970's and Hamdy 2023's attempts). Indeed, on the one hand, there could be *many* omnipotent godly choice-makers who secure the truth of the Axiom of Choice. On the other hand, Zorn's Lemma only concludes that, if the relevant conditions are satisfied, then there is *at least one* maximal element. Hence Zorn's Lemma does not preclude that there are *many* maximal elements, and, therefore, *many* gods *qua* first causes (by Meyer 1987's Proof, see Meyer 1987: 351–352 and Chaitin, da Costa and Doria 2011: 15 n.11). Of course, on the *further* assumption that $\langle \mathfrak{E}, > \rangle$ be an *upward directed* set—that is, be totally ordered rather than only partially ordered –, one can build a proof of the uniqueness of God, but this assumption is quite independent of the Causal Principle that grounds the Cosmological Argument (Meyer 1987: 355–357). In other words, polytheism and the Cosmological Argument are not incompatible as one may wrongly believe (for the fact that none of the first three Aquinas' ways entails the unicity of God, see Sobel 2004: 190–196 and Oppy 2006: 99, 103, 106).

Of course, Routley was a Meinongian (Routley 1980 is nothing other than a powerful well-argued 1000-page exploration of Meinong's Jungle) and, therefore, believe that all mathematical items, in fact, do not exist (there are objects that lack the property of existing, i.e., against Quine 1948's meta-ontology, Meinongian zealots do not believe that the ontology is unveiled by scrutinizing the extent of the domain of the quantifiers 'there is' and 'for all'). Consequently, Routley blocks the unwanted ontological commitments of mathematical theories, and debugs Meyer 1987's Proof by claiming that, albeit the truth of Zorn's Lemma involves that there is a God, such an intentional object can be existent as well as non-existent: mathematical theories are ontologically neutral, that is, they are ontologically unloaded theories (Routley 1980: 769–831). Perhaps, the situation is quite different for Robert K. Meyer himself who was once a Christian missionary in Japan (Osaka) in the late 1950s before becoming a high-minded logician (at Pittsburgh in the mid-1960s, there he met the American forerunners of relevant logic Anderson and Belnap, before moving to Antipodean non-classical paradise with Richard Routley/Sylvan and

Cie).[25] Perhaps, he was committed to his Mathematical Proof and considers it to be the only rigorous and definitive demonstration of the existence of God. I don't know his *psyché*.[26]

So, the issue ceases to be mathematical and theological for becoming deeply meta-ontological (true, this was already the case for the premisses of the Ontological Argument) and now also belongs to philosophy of science (the issue about mathematical realism *vs*. antirealism). Anyway, there is a Mathematical Proof, Atheists (and Dialetheist Atheists!) like me (I should confess) have to challenge it: could DST or another naive set theory be wrong? (consequently, a hierarchical theory as RT, ZF, NBG, or ML be right? Unbelievable!) Are mathematical existential proofs ontologically committing? (what is the right meta-ontology? Fregean? Quinean? Carnapian? Meinongian?) Etc.

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Appendix: Towards a Gödelian Theology

Throughout this paper, I often allude to some of Kurt Gödel's works on set theory and ontological proofs. Let us pursue such a Gödelian stance and, so, let us endorse Gödel's mathematical Platonism (Gödel 1990b: 128–131, 1995a): mathematical objects are as real as concrete spatiotemporal chairs and atoms are, i.e., mathematical items exist and are genuine parts of the mind-independent reality (for an overview of what 'reality' or 'existence' could mean, see Routley 1980: 697–768 and Marion 2024). Gödel does not only believe in mathematical realism that is a metaphysical widely accepted claim amongst philosophers of mathematics (at least since Frege's powerful defense of mathematical Platonism), but Gödel is also committed to a mathematical *theology*, for he writes in many private manuscripts that angels and demons inhabit in some way the godly mathematical realm. For instance, he tends to claim that his two incompleteness theorems implicitly show that evil is at work in mathematics, for incompleteness theorems indicate that our 'human' mathematics (built by mathematicians who live 'incarnated' in space and time) are in fact inadequate to grasp the divine mathematical truths which are beyond space and time—real mathematics in God's mind being necessary complete (Cassou-Noguès 2007: 157–163, 234–244, see also Lethen 2021).

Albeit Gödel remains a proponent of classical (hierarchical) set theory (in guise of NBG), reforming set theory in a naive and paraconsistent fashion, and therefore accepting DST (or amended versions of it), could have helped Gödel to rationally expound his mathematical theology, since God seems to be a *necessary* (given Meyer 1987's and Fisher 1970's Proofs) mathematical inhabitant of the set-theoretical universe drawn by DST.

Gödel's case shows that some requisites are needed to be a mathematical Theist (listed by acceptance in philosophical communities):

- Quinean 'quantificationalist' meta-ontology
- Mathematical realism
- Naive Set-Theory (so, DST or another paraconsistent set-theory)

A lot of philosophers are proponents of these three theses altogether. But a lot amongst them do not believe in God. Hence, they should reject at least one of these three theses. Most of them would ostracize the third, but it is the most well-grounded of the three for ZFC, RT, ML and NBG are notoriously highly counter-intuitive and philosophically puzzling. So, they must question either Quine's quantificationalism or mathematical realism. Their atheism becomes hardest to defend, but, of course, not undefendable. To give just one example, one atheistic strategy would be to follow the nominalist path taken by Field 1980 who, while agreeing with Quine 1948's idea that the ontological commitments of a theory are manifested by its translation into sentences that involve quantifiers, tries to show that modern physics does not involve sentences that quantify over mathematical entities. In so doing, Field 1980, on the one hand, accepts Quinean meta-ontology, but, on the other hand, rejects mathematical realism. Hence, Field 1980 can eschew mathematical theism.

Notes

1. The only discussions I found was a slightly different (but emphatic and not very rigorous) proof by Hamdy 2023, and a few summaries of Meyer 1987 in Routley 1980: 132–133, Oppy 2006: 123–125, Pruss 2006: 50–54, 2009: 87–88 (and http://alexanderpruss.com/papers/MeyerProof.html) and Weber 2009: 62. Oppy 2009: 42, Chaitin, da Costa and Doria 2011: 15–16 and Buzaglo 2019: 546 also briefly allude to Meyer's Proof.
2. Such a fact is interesting since one could believe that Aquinas' First Three Ways would be tied to another set-theoretical axiom, namely the Axiom of Foundation (indeed, the Non-Well-Founded Set Theories of Aczel 1988 that replace this axiom by another Axiom of Anti-Foundation invalidate the intuitive ideas behind the Cosmological Argument, see Marion 2022 for historical remarks on anti-foundationalist metaphysics).
3. The situation is in fact a little more subtle. On intuitionist constructive mathematics and the Axiom of Choice, see Dummett 1977: 37–39.
4. For a summary of great theorems of which the proofs use the Axiom of Choice, see Russell 1992: 48–50, Krantz 2002: 121–125 and Bell 2021: §4. For the (at least: pragmatical or temporary) indispensability of the Axiom of Choice in various mathematical subdomains (topology, algebra, analysis, measure-theory, etc.), see Rosser 1953: 510–512.
5. Russell 1906: 643–644, Russell and Whitehead 1910: 65, Ramsey 1926: 373–374, Tarski 1956, Carnap 1937: 214–217, Quine 1966: 9–11, etc.
6. On DST, see Routley 1980: 911–935, Brady 1989, Priest 2006: 141–143, Berto 2007: 63–103, 241–258, Weber 2009, 2013, 2021: 151–188 and Brady 2006: 171–274 for a variant of DST; and Mortensen 1995 for a general introduction to paraconsistent mathematics.
7. For a short more fine-grained overview of the various hierarchical and stratified answers to the set-theoretical paradoxes (ZF, RT and ML), see Rosser 1953: 200–207.
8. Lewis 2020: 416 (letter to Robert C. Stalnalker, 28 February 1996).
9. The Cantorian naive comprehension axiom was the following: for every well-formed expression $\varphi(x)$ there is a y such that $x \in y$ iff $\varphi(x)$, *provided that y is not free in* $\varphi(x)$.
10. On the unsatisfactory account of functionality in Routley 1980: 924–925 (amended in Priest, Routley and Norman 1989: 374), see also Weber 2013: 326.
11. Stephen Read emphasized this point once, in an interview for *The Reasoner* (vol. 5 n°12, December 2011): "And that seems odd—for instance, I am confident I could explain the Axiom of Choice to my students, but not confident I could explain Zorn's Lemma".
12. For the record, a sixth way to prove the existence of God appears recently, the proof is built as a criticism of classical logical ideas about the reference of empty terms, see Routley 1980: 133–137 and Marion 2022: 8–9.

13. Meyer 1987 is summarized in Routley 1980: 132–133, Oppy 2006: 123–125, Pruss 2006: 50–54, 2009: 87–88 (and http://alexanderpruss.com/papers/MeyerProof.html) and Weber 2009: 62.
14. Respectively: $\forall x \forall y \forall z[(y > x \wedge z > y) \rightarrow z > x]$, $\forall x \neg(x > x)$ and $\forall x \forall y[y > x \rightarrow \neg(x > y)]$. As points out by Meyer 1987: 350–351, the causal relation can also be a non-strict partial order ≥, that is, be transitive, reflexive and antisymmetric, namely $\forall x \forall y \forall z[(y \geq x \wedge z \geq y) \rightarrow z \geq x]$, $\forall x(x \geq x)$ and $\forall x \forall y[(y \geq x \wedge x \geq y) \rightarrow x = y)]$. The choice between ≥ and > depends on whether God is understood as a *self-caused* being (Cartesian way) or an *uncaused* one (Aristotelian and Thomistic way).
15. Of course, it is possible to construct the argument with '$x < y$' or '$x \leq y$' as the partial order relation 'x is the cause of y'. Then, Meyer's Proof uses a downward variant of Zorn's Lemma: every non-empty partially ordered set in which every chain has a lower bound has a minimal element, see Meyer 1987: 352*ss*, Oppy 2006: 123 and Weber 2009: 62.
16. Alexander R. Pruss (http://alexanderpruss.com/papers/MeyerProof.html), after focusing almost all his discussion on Premiss 2, ends his summary of Meyer 1987's Proof by asking: 'if you find a way of justifying Premiss 2, please *do* send me e-mail'. See also Pruss 2009: 88.
17. Meyer 1987: 351 (cf. 355) acknowledges that these cases are 'esoteric possibilities' which undermine Premiss 2, but immediately restricts the discussion to the cases in which Premiss 2 holds.
18. More formally: for every causal chain $\mathcal{C}$ of $\mathcal{E}$, there is some item x which is causally anterior to every item y in $\mathcal{C}$ (Meyer 1987: 347), that is, every causal chain $\langle \mathcal{C}, > \rangle$ has an upper bound (Meyer 1987: 352).
19. On this point Kenny 1969 and Oppy 2006: 98–107 disagree however. While Kenny 1969 argues that all of Aquinas' five ways share a similar structure, Oppy 2006 supports that the fourth and the fifth are not isomorphic to the first three ways.
20. As far as I know, only Pickover 2005: 236, 2009: 233, 238 (*postscript 1* on Gödel 1970's Mathematical Proof of the Existence of God) alludes to Fisher 1970.
21. *My footnote*. Vox Fisher does not know that Aquinas follows the Marmoutier monk Gaunilo in his rejection of Anselm's Ontological Proof, see *Sum. Theol.* 1a, q.2, art.1.
22. *My erasure*. "a unique" is written under erasure, because, in Fisher 1970, the proof of the existence of God is in fact independent of the proof of its uniqueness (note that it is also the case in Meyer 1987).
23. Such a claim (God's omnipotence guarantees the truth of the Axiom of Choice) is a commonplace of 'mathematical theology', see Rosser 1953: 490–491, Routley 1980: 133 and Meyer 1987: 353.
24. See Th. Aquinas, *Sum. Theol.* Ia, q.2. art.1, Hume, *DCNR*, part. 9, Kant, AK III 400–402, Frege 1884: §53 and, for an overview, Sobel 2004: 29–167; see also Pickover 2009: 227–242 for some opinions of mathematicians about Gödel 1970's Proof. For a recent defence of *modal* versions of the Ontological Argument, see Plantinga 1974: 197–221 (in which the reforging of Anselm's

Proof by Hasthorne is also discussed). For a famous objection against the modal reforging of Anselm's Argument, see Lewis 1983.

25. Hyde 2014: 69
26. For the record, Hamdy 2023 is very committed to see in this argument a genuine and unbeatable *proof* of the existence of God.

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