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Solvency measurement for defined benefits pension schemes

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SOLVENCY MEASUREMENT FOR DEFINED BENEFITS PENSION SCHEMES

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Abstract

Risk measurement as applicable for insurers (Solvency 2) or banks (Basel 2)can also be considered for pension fund liabilities. The purpose of this paper is to present various stochastic models in continuous time in order to estimate solvency capital for two important risks faced by pension funds: market risk and inflation risk. We address the situation of a Defined Benefit Pension Scheme (DB) with liabilities linked to final salary. We try to develop in this context a methodology coherent with IAS norms based on the so called projected unit credit cost method but including a risk measure approach. We also show that pension portability could be modelled using classical ruin theory. The models are developed first in a geometric Brownian motion environment and afterwards, using a Lévy process for the asset.

Key words : Inflation risk, market risk, solvency requirement, value at risk, probability of ruin, IAS norms

1 Introduction

Solvency measurement of financial institutions (banks, insurance companies, pension funds) has became a major topic in quantitative risk management these last years. Banks have first given the signal in the context of the successive Basel norms. These norms have especially asked banks to detain a minimum level of capital proportional to the real risks faced by the bank (risk based capital). This philosophy has then deeply influenced the insurance and pension regulation through the project Solvency 2 (substitution to the factor based approach of Solvency 1).
But even if all these institutions have clearly to play on the same financial international market, essential differences exist between them in terms of nature of the assets and of the liabilities. It should be then natural to take into account these particularities in their solvency measurement. One of the key aspects in this regard is for sure the time horizon of the risks to measure. Traditionally, banks are confronted to short time uncertainties. The influence of this short term can be seen in the Basel norms but has also inspired the solvency 2 projects by using as risk measure a value at risk on one year. But clearly life insurance companies and even more pension funds are much more oriented to a long term vision. Therefore the development of adequate risk tools taking into account the time horizon of the liabilities is crucial. The recent discussions about long term guarantees for life insurance in the context of the Solvency 2 project have illustrated this point. This time aspect is off course also present for pension funds. Our aim is to develop a solvency model for Defined benefit pension schemes on a risk based approach and taking into account the horizon of the liabilities. In this paper we have chosen to work with a lump sum at retirement (and not an annuity). We have therefore only considered two main risk factors: the market risk and the inflation risk (and not the longevity risk). Different papers have already studied solvency and funding of DB pension plans (for instance [Samuelson P] and [Thorley S]) but using stochastic optimal control tools and asset allocation purposes and not risk

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measures in the spirit of Solvency 2. This paper extends the result of [Samuelson P] for DB pension plans , introducing the inflation risk. . We have also generalized the asset model by considering not only geometric Brownian motions but also Levy processes. Following the IAS norms, we use as funding technique the so called projected unit credit cost method in order to compute contributions and actuarial liabilities. The long term aspect of pension liability is taken systematically into account by analyzing the risks not only on a one year horizon (as in Solvency 2) but until maturity (retirement age). An ALM approach for the assets and liabilities of the scheme is proposed and various classical risk measures are applied to the surplus of the pension fund (probability of default, value at risk). The effect of the duration of the liability is clearly illustrated and is surely one of the key factors if we want to consider solvency measurement for pension. We develop also a model based on probability of ruin instead of considering solvency only at maturity.

The paper is organized as follows. In section 2 we describe the general framework of the model in terms of asset and liability structure of the DB scheme. Section 3 is based on a static risk measurement and analyses the influence of the time horizon on the probability of default at maturity. In section 4, we compute a solvency level using a classical value at risk approach and in section 5, we consider a continuous time measurement, motivated by a pension portability argument and based on probability of ruin during a whole time interval as in the classical actuarial ruin theory. Finally in section 6, we will try to extend the same calculations and risk measures in the case of an asset which follows a Lévy process, more precisely a Variance Gamma Process.

2 ALM Model for a DB pension scheme

We consider a Defined Benefit pension scheme based on final salary. At time $t = 0$ an affiliate aged x is entering the scheme with an initial salary S(0). At retirement age, at time $t = T$, a lump sum will be paid, expressed as a multiple of its last wage, for instance:

$$B = \frac{N}{40}\alpha S,$$

where:

N	:	Years of service credited by the scheme
S	:	Final Salary
B	:	Benefit to pay (lump sum)
α	:	Coefficient

In order to compute the contributions to the pension fund, we will use the IAS norms and in particular the projected unit credit cost method as funding technique. We need then the following assumptions:

1. A fixed discount rate: r (risk free rate)
2. A life table: we will assume here no mortality before retirement (no longevity risk is considered in this paper)
3. A salary scale: the salary at time t denoted by S(t) will follow a stochastic evolution given by:

$$dS_t = \mu S_t dt + \eta S_t dz_t,$$

where :

μ	:	Average salary increase
η	:	Volatility on salary evolution
z	:	Standard Brownian motion

Using a best estimate approach, the contribution for the first year of service (or the normal cost) is then given by (assuming that the normal cost is paid in advance):

$$NC_0 = \frac{1}{40}\alpha S(0)e^{(\mu-r)T}.$$

(Present value at the risk free rate of the average projected benefit at retirement age).

At time t ($t = 1, 2, .., T-1$), the normal cost will have the same form:

$$NC_t = \frac{1}{40}\alpha S(t)e^{(\mu-r)(T-t)}$$

We can also introduce a loading factor β ($\beta \geq$)on the contribution; the normal cost becomes then:

$$NC_t = \frac{1}{40}\alpha S(t)(1+\beta)e^{(\mu-r)(T-t)}$$

In pure Best Estimate approach, this coefficient β is equal to zero. We just introduce it to look at the effect of a safety margin.

The actuarial liability AL at time t is then given by ($t = 0, 1, .., T-1$):

$$AL_t = \frac{t+1}{40}\alpha S(t)(1+\beta)e^{(\mu-r)(T-t)} \tag{1}$$

On the asset side we assume first that each contribution is invested in a Geometric Brownian motion whose evolution is solution of:

$$dA_s = \delta A_s ds + \sigma A_s dw_s,$$

where

δ	:	Mean return of the invesment fund
σ	:	Volatility of the return
w	:	Standard Brownian motion

This model will be generalized in section 6 using a Lévy process.

The two sources of risk (inflation and market risks) are off course correlated:

$$corr(w_t, z_t) = \rho t$$

Considering the risk attached to the first year of contribution, we could take as initial condition:

$$A_0 = NC_0$$

Then the corresponding final asset is given by (projection between $t = 0$ and $t = T$):

$$A_0(T) = NC_0 e^{(\delta-\frac{\sigma^2}{2})T+\sigma w(T)} = \frac{1}{40}\alpha S_0(1+\beta)e^{(\mu+\delta-r-\frac{\sigma^2}{2})T+\sigma w(T)} \tag{2}$$

More generally we could consider the risk between time t and time T by computing the future evolution till maturity of the investment of the actuarial liability AL existing at time t in the reference asset A:

$$A_t(T) = AL_t e^{(\delta-\frac{\sigma^2}{2})(T-t)+\sigma(w(T)-w(t))} = \frac{t+1}{40}\alpha S_t(1+\beta)e^{(\mu+\delta-r-\frac{\sigma^2}{2})(T-t)+\sigma(w(T)-w(t))} \tag{3}$$

the initial condition being then: $A_t(t) = AL_t$

In a ALM approach, these asset values must be compared to their respective liability counterparts. We obtain successively:

- for the final liability corresponding to the first year contribution:

$$L_0(T) = \frac{1}{40}\alpha S_0 e^{(\mu-\frac{\eta^2}{2})(T-t)+\eta z(T)} \tag{4}$$

- for the final liability corresponding to the actuarial liability until time t:

$$L_t(T) = \frac{t+1}{40}\alpha S_t e^{(\mu-\frac{\eta^2}{2})(T-t)+\eta(z(T)-z(t))} \tag{5}$$

3 Probability of default

A first interesting question is to look at the probability of default at maturity without any extra resource (i.e. the risk to have not enough assets at maturity to pay the required pension benefit). In particular we can consider this probability as a function of the residual time $T-t$. This probability computed at time t ($t = 0, 1, .., T-1$) is given by:

$$\phi(t,T) = P(A_t(T) < L_t(T)) = P(Y(t,T) < M)$$

Where:

$$\begin{aligned} Y(t,T) &= \sigma(w(T)-w(t)) - \eta(z(T)-z(t)) = N(0, \bar{\sigma}^2(T-t)) \\ \overline{\sigma}^2 &= \sigma^2 + \eta^2 - 2\rho\sigma\eta \\ M &= (r - \delta + \tfrac{\sigma^2}{2} - \tfrac{\eta^2}{2})(T-t) - \ln(1+\beta) \end{aligned}$$

So finally the probability of default at maturity depends on the residual time and is given by:

$$\phi(t,T) = \Phi(a(T-t)) \tag{6}$$

with:

$$\begin{aligned} \Phi &= \text{distribution function } N(0,1) \\ a(s) &= \frac{(r-\delta+\frac{\sigma^2}{2}-\frac{\eta^2}{2})\sqrt{s} - \frac{\ln(1+\beta)}{\sqrt{s}}}{\bar{\sigma}} \end{aligned}$$

Example 1:

risk free rate :	r	$=$	2%
mean return of the fund :	δ	$=$	6%
volatility of the fund :	σ	$=$	10%
average increase of salary :	μ	$=$	5%
volatility of the salary:	η	$=$	5%
correlation :	ρ	$=$	50%
safety loading :	β	$=$	5%

Figure 1 shows the evolution of the probability of default as a function of the residual time $T-t$.

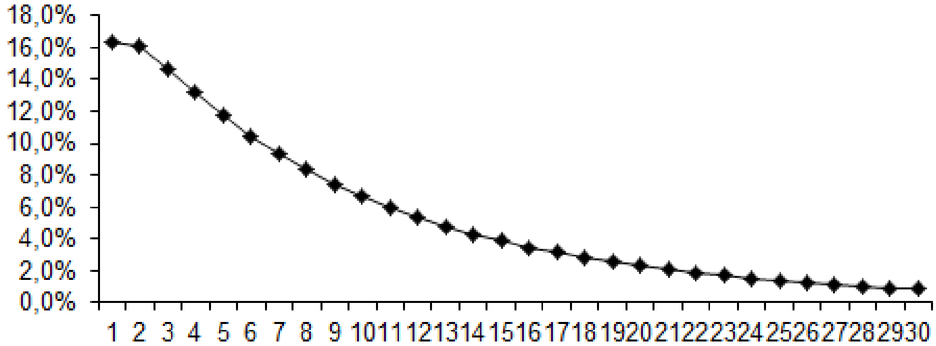

Figure 1: Probability of default

We can see clearly a time effect: for short residual time to maturity this probability is quite high but it decreases rapidly for long residual time.

4 Value at risk approach

In order to control this probability of default, we could as in Solvency 2 introduce a solvency level based on a value at risk approach. We will use the following notations:

SC	=	solvency capital using a value at risk methodology
VaR	=	value at risk
α_1	=	chosen safety level for a horizon of one year (for instance 99.5% in Solvency 2) .

For this safety level we can choice the following value based on yearly independent default probabilities (probability of default of $(1-\alpha_1)$ independently each year):

$$\alpha_N = (\alpha_1)^N$$

We will assume that the solvency capital is invested in the reference investment fund ; so we can define this solvency capital $SC(t,T)$ at time t ($t = 0, 1, .., T-1$) for the risk until time T as solution of:

$$P\{A_t(T) + SC(t,T)\frac{A_t(T)}{A_t(t)} < L_t(T)\} = 1 - \alpha_{T-t}$$

Using (3) and (5), this condition becomes:

$$P\left\{(\frac{t+1}{40}\alpha S(t)(1+\beta) + SC(t,T))e^{(\delta-\sigma^2/2)(T-t)+\sigma(w(T)-w(t))} < \frac{t+1}{40}\alpha S(t)e^{(\mu-\eta^2/2)(T-t)+\eta(z(T)-z(t))}\right\} = 1 - \alpha_{T-t}$$

After direct computation, we obtain the following value for the solvency capital:

$$SC(t,T) = \frac{t+1}{40}\alpha S(t)\left(e^{(\mu-\delta)(T-t)+z_{\alpha_{T-t}}\overline{\sigma}\sqrt{T-t}+(\sigma^2-\eta^2)(T-t)/2} - e^{(\mu-r)(T-t)}(1+\beta)\right)$$

where

$$z_k \quad = \quad k \text{ quantile of the normal distribution such that: } \Phi(z_k) = k$$

We can express the solvency capital as a percentage of the actuarial liability AL given by (1) (solvency level in percent):

$$SC^{\%}(t,T) = \frac{SC(t,T)}{AL_t} = \frac{1}{1+\beta}e^{-(\delta-r)(T-t)+z_{\alpha_{T-t}}\overline{\sigma}\sqrt{T-t}+(\sigma^2-\eta^2)(T-t)/2} - 1 \qquad (7)$$

This relative level does not depend on the salary increase. In particular if we look at a one year risk (as in Solvency 2), we get:

$$SC^{\%}(0,1) = \frac{SC(0,1)}{AL_0} = \frac{1}{1+\beta}e^{-(\delta-r)+z_{\alpha_1}\overline{\sigma}+(\sigma^2-\eta^2)/2} - 1$$

Example 2:

Same assumptions as in example 1

Safety level on one year :	α_1	=	99.5%
Safety level on N years :	α_N	=	$(\alpha_1)^N$

Figure 2 shows then the evolution of the solvency level in percent as a function of the residual time $T-t$. Negative values for the SCR correspond to cases where no additional solvency is needed.

We can also observe as in Figure 1 a time effect.

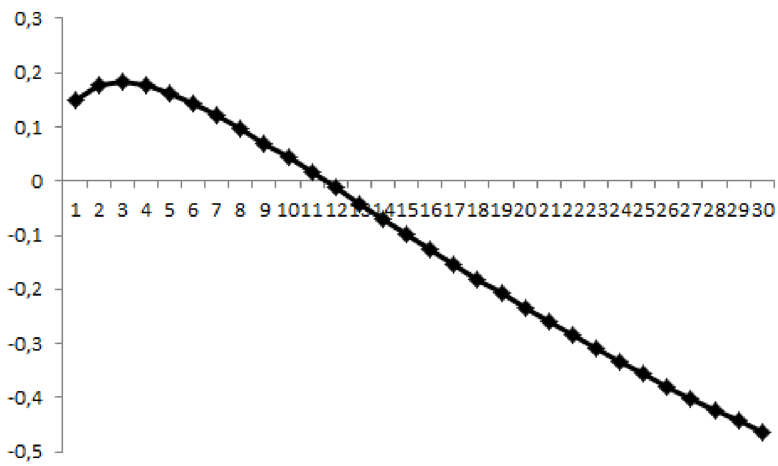

Figure 2: SCR in percent

5 Pension portability and ruin approach

The risk considered until now is only related to the payment of the pension benefit at retirement age (at time T). But sometimes, affiliates leave their pension scheme before retirement and a pension portability clause can be applied: the sponsor must then transfer the corresponding liability immediately (and generally without any possible penalty) to another scheme or another fund.

This means that the solvency should be checked not only at maturity of the contract but at any time. The purpose of this section is to analyze this philosophy directly inspired by the actuarial classical ruin theory.

5.1 Probability of ruin without capital

We first look at the probability of ruin without solvency capital between time t ($t = 0, \ldots.., T-1$) and maturity, given by:

$$\Psi(t,T) = 1 - P\left\{A_t(s) \geq L_t(s), \quad \forall\ s\ \in\ [t,T]\right\} \tag{8}$$

In order to compute this probability , we have to define the level of liability L corresponding to the years of service until time t , not only at maturity time T but also at any time s (between computation time t and maturity T). The asset process is simply defined by its market value generated by the investment at time t of the actuarial liability (cf. (3)):

$$\begin{aligned} A_t(s) &= AL_t e^{(\delta-\frac{\sigma^2}{2})(s-t)+\sigma(w(s)-w(t)} \\ &= \left(\frac{t+1}{40}\alpha S(t)(1+\beta)e^{(\mu-r)(T-t)}\right)e^{(\delta-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))} \end{aligned}$$

The corresponding liability process can be defined as present value at time s of the final benefit at retirement T based on the first $t+1$ years of service and incorporating inflation till time s of departure ($t < s < T$):

$$L_t(s) = \frac{t+1}{40}\alpha S(t)e^{(\mu-\eta^2/2)(s-t)+\eta(z(s)-z(t)}e^{-r(T-s)}$$

Then the probability (8) can be seen as a probability related to the minimum of a Geometric

Brownian motion along an interval:

$$
\begin{aligned}
& P\left\{A_t(s) \geq L_t(s), \quad \forall\ s\ \in\ [t,T]\right\} \\
= \ & P\left\{e^{(\delta-\mu-r+\eta^2/2-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))-\eta(z(s)-z(t)))} \geq \frac{e^{-\mu(T-t)}}{1+\beta}, \quad \forall\ s\ \in\ [t,T]\right\} \\
= \ & 1 - P\left\{\underset{t\leq s\leq T}{min}\left(e^{(\delta-\mu-r+\eta^2/2-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))-\eta(z(s)-z(t))}\right) \geq \frac{e^{-\mu(T-t)}}{1+\beta}\right\}
\end{aligned}
\tag{9}
$$

We can then obtain the following result:

Proposition 5.1. *The probability of ruin (8) is explicitly given by:*

$$
\begin{aligned}
\Psi(t,T) \ = \ & \Phi\left(\frac{-\ln(1+\beta)-(r-\delta-\eta^2/2+\sigma^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right) \\
+ \ & \left(\frac{e^{-\mu(T-t)}}{1+\beta}\right)^{\frac{2(\delta-\mu-r)+\eta^2-\sigma^2}{\delta^2}} \Phi\left(\frac{-\ln(1+\beta)-(r-\delta+2\mu-\eta^2/2+\sigma^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right)
\end{aligned}
\tag{10}
$$

Proof This is a direct consequence of the law of the minimum of a Geometric Brownian motion (see for instance [Back 1995]):

If the process S is given by:

$$
\begin{aligned}
S(t) &= e^{at+bw(t)} \\
z &= \underset{0\leq s\leq t}{min}\{S(s)\}
\end{aligned}
$$

and if: $0 < L \leq 1$ then

$$P(z \leq L) = \Phi(d_1) + L^{\frac{2a}{b^2}}\Phi(d_2)$$

with

$$d_{1,2} = \frac{\ln(L) \mp a.t}{\sigma\sqrt{t}}$$

Following (9), we can apply this result, using as parameters:

$$
\begin{aligned}
L &= \frac{e^{-\mu(T-t)}}{1+\beta} \\
a &= \delta - r - \mu + \eta^2/2 - \sigma^2/2 \\
b^2 &= \overline{\sigma}^2 = \sigma^2 + \eta^2 - 2\rho\sigma\eta
\end{aligned}
$$

Remark 5.1.

1. *The first term of (10) corresponds to the probability of default at maturity as given by (6). Then the second part can be seen as the additional probability to cover the ruin strictly before maturity:*

$$
\begin{aligned}
\Psi(t,T) \ = \ & \phi(t,T) \\
+ \ & \left(\frac{e^{-\mu(T-t)}}{1+\beta}\right)^{\frac{2(\delta-\mu-r)+\eta^2-\sigma^2}{\overline{\sigma}^2}} \Phi\left(\frac{-(r-\delta+2\mu-\eta^2/2+\sigma^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right)
\end{aligned}
$$

2. if $\beta = 0$ (no loading on the best estimate), then this probability becomes:

$$\begin{aligned}\Psi(t,T) &= \Phi\left(\frac{(r-\delta-\eta^2/2+\sigma^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right)\\ &+ \left(e^{-\mu(T-t)}\right)^{\frac{2(\delta-\mu-r)+\eta^2-\sigma^2}{\delta^2}}\Phi\left(\frac{-(r-\delta+2\mu-\eta^2/2+\sigma^2/2)}{\overline{\sigma}}\right)\end{aligned}$$

3. if $\beta = 0$ and $\mu = 0$ (no salary evolution) then the probability of ruin is equal to one.

5.2 Probability of ruin with risky capital

Assuming now that a solvency capital SC^* is invested at time t ($t = 0, 1, .., T-1$) in the reference investment fund A till maturity, we obtain for the probability of ruin:

$$\Psi(t,T,SC^*(t,T)) = 1 - P\left\{A_t(s) + SC^*(t,T)\frac{A_t(s)}{A_t(t)} \geq L_t(s), \quad \forall\ s\ \in\ [t,T]\right\}$$

As in (7), we can express this solvency level in percentage of the actuarial liability existing at computation time t.

$$SC^*_{\%}(t,T)) = \frac{SC^*(t,T))}{AL_t}$$

Then the probability of non ruin can be written as:

$$\begin{aligned}&P\left\{A_t(s) \geq L_t(s), \quad \forall\ s\ \in\ [t,T]\right\}\\ =\ &P\left\{(1+SC^*_{\%}(t,T)))e^{(\delta-\mu-r+\eta^2/2-\sigma^2/2)(s-t)+\sigma(w(s)-w(t))-\eta(z(s)-z(t))}) \geq \frac{e^{-\mu(T-t)}}{1+\beta}, \quad \forall\ s\ \in\ [t,T]\right\}\end{aligned}$$

We can obtain the following direct extension of Proposition 5.1 using the same basic result:

Proposition 5.2. *With a risky solvency capital SC^* (in percentage of the actuarial liability), the probability of ruin at time t becomes:*

$$\begin{aligned}\Psi(t,T,SC^*) &= \Phi\left(\frac{-\ln(1+\beta)-\ln(1+SC^*)+(r-\delta-\eta^2/2+\sigma^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right)\\ &+ \left(\frac{e^{-\mu(T-t)}}{(1+\beta)(1+SC^*)}\right)^{\frac{2(\delta-\mu-r)+\eta^2-\sigma^2}{\overline{\sigma}^2}}\Phi\left(\frac{-\ln(1+\beta)-\ln(1+SC^*)+(r-\delta+2\mu-\eta^2/2+\sigma^2/2)(T-t)}{\overline{\sigma}\sqrt{T-t}}\right)\end{aligned}$$

Remark 5.2.
This relation can be used to compute a well defined level of capital SC^* corresponding to a fixed level of safety. Then SC^* is solution of the implicit equation:

$$\Psi(t,T,SC^*) = 1-\alpha_1$$

6 Extension to Lévy Processes: Variance gamma process

6.1 The ALM model

A variance gamma process is a Lévy process with three parameters seen as a generalization of geometric Brownian motion. The idea introduced by this process is to model also the moments of trading in financial markets, the so-called business time, which are supposed random and following a gamma process, in order to evaluate a Brownian motion with drift at this time values.

The difference between this section and the previous ones, appears in the equation of the financial asset that is no longer a geometric Brownian motion but a variance gamma process.

Specifically, we consider a simple Brownian motion:

$$B_g(t) = \theta t + \sigma w_t \tag{11}$$

We replace the calendar time t by a gamma process $\gamma(t)$

The process $\gamma(t)$ has the following properties :

1) The increments $\gamma(t+h) - \gamma(t)$ are independent and following a gamma distribution $\Gamma(h/\nu;\nu)$
2) $E(\gamma(t)) = t$
3) $\text{var}(\gamma(t)) = \nu t$

where ν is a strictly positive integer

Instead of a simple Brownian motion : $B_g(t) = \theta t + \sigma w_t$, we will use in this section, a variance gamma process, to model the asset. We note by X_t^{VG}, the variance gamma process such as :

$$\begin{cases} X_0^{VG} &= 0 \\ X_t^{VG} &= \theta\gamma(t) + \sigma w_\gamma(t) \quad \forall\ t > 0 \end{cases} \tag{12}$$

Taking into account the parameters defined above, the density function of the variable X_t^{VG} is given by:

$$f_{X_t^{VG}}(x) = \int_0^\infty \frac{1}{\sigma\sqrt{2\pi g}} exp\left(-\frac{(x-\theta g)^2}{2\sigma^2 g}\right) \frac{g^{\frac{t}{\nu}-1} e^{-\frac{g}{\nu}}}{\nu^{\frac{t}{\nu}} \Gamma(\frac{t}{\nu})} dg$$

One advantage of the variance gamma process is that its characteristic function is given explicitly by the following formula:

$$\phi_{X_t^{VG}}(u) = \left(\frac{1}{1 - i\theta\nu u + (\frac{\sigma^2\nu}{2})u^2}\right)^{t/\nu} \tag{13}$$

In order to study the asset in DB pension scheme, we assume that each contribution is invested in a variance gamma process. We still use IAS norms context based on the Projected Unit Credit Cost method.
As seen above, the initial condition of the asset is :

$$A_0 = NC_0$$

To evaluate the final asset, we introduce the variable of variance gamma process such as :

$$A_0(T) = NC_0 e^{X_T^{VG}}$$

therefore :

$$\begin{cases} A_0 = \dfrac{1}{40}\alpha S(0)(1+\beta)e^{(\mu-r)T} \\ A_0(t) = A_0 e^{X_t^{VG}} = \dfrac{1}{40}\alpha S(0)(1+\beta)e^{(\mu-r)t}e^{\theta\gamma(t)+\sigma w_\gamma(t)} \quad \forall\ t > 0 \end{cases} \tag{14}$$

We take as a condition on the average return of the asset : $E(A_0(t)) = A_0 e^{\delta t}$
Then

Proposition 6.1.
The parameter θ can be written as a function of the parameters σ,δ and ν such as :

$$\theta = \frac{1 - \frac{\sigma^2\nu}{2} - e^{-\nu\delta}}{\nu} \qquad (\nu > 0)$$

Proof By using the condition on the average return of the asset : $E(A_0(t)) = A_0 e^{\delta t} \quad \forall t > 0$ and taking into account the form of the characteristic function of the variable X_t^{VG} , this expectation can be also expressed as :

$$\begin{aligned} E(A_0(t)) &= A_0 E(e^{X_t^{VG}}) \\ &= A_0 \phi_{X_t^{VG}}(\frac{1}{i}) \end{aligned}$$

where $\phi_{X_t^{VG}}$ is the characteristic function of the variable X_t^{VG}.

By using formula (13) we can write :

$$A_0 e^{\delta t} = A_0 \left(\frac{1}{1 - \theta\nu - \frac{\sigma^2\nu}{2}} \right)^{\frac{t}{\nu}}$$

Finally :

$$\theta = \frac{1 - \frac{\sigma^2\nu}{2} - e^{-\nu\delta}}{\nu}$$

■

More generally the future evolution of the asset between time t and the maturity T can be expressed by the following equation :

$$A_t(T) = AL_t e^{X_T^{VG} - X_t^{VG}}$$

$$\begin{aligned} A_t(T) &= AL_t e^{X_T^{VG} - X_t^{VG}} \\ &= \frac{t+1}{40}\alpha S(t)(1+\beta)e^{(\mu-r)(T-t)+\theta(\gamma(T)-\gamma(t))+\sigma(w(\gamma(T))-w(\gamma(t)))} \end{aligned}$$

With initial condition: $A_t(t) = AL_t$.

On the liability side we keep the assumption that wages evolve according to a geometric Brownian motion, and therefore we keep the same formulas seen previously :

$$\begin{aligned} L_0(T) &= \frac{1}{40}\alpha S(0)e^{(\mu-\frac{\eta^2}{2})T+\eta z(T)} && \text{Liability corresponding to the first year contribution} \\ L_t(T) &= \frac{t+1}{40}\alpha S(t)e^{(\mu-\frac{\eta^2}{2})(T-t)+\eta(z(T)-z(t))} && \text{Liability corresponding to the actuarial liability until time t} \end{aligned}$$

6.2 Probability of default

The probability of ruin at the time T, computed at time t(t=0,1,.....,T-1) is given by:

$$\varphi(t,T) = P(A_t(T) < L_t(T))$$

As X_t^{VG} is a Variance gamma process, $X_T^{VG} - X_t^{VG}$ has the same distribution as $X_{(T-t)}^{VG}$

$$\begin{aligned} \varphi(t,T) &= P(X_T^{VG} - X_t^{VG} < (r-\eta^2/2)(T-t) + \eta(z_T - z_t) - ln(1+\beta)) \\ &= P(X_{(T-t)}^{VG} < (r-\eta^2/2)(T-t) + \eta(z_T - z_t) - ln(1+\beta)) \\ &= \int_{g=0}^{\infty} P(\sigma w_g - \eta(z_T - z_t) < (r-\eta^2/2)(T-t) - \theta g - ln(1+\beta))P(\gamma(T-t) = g)dg. \end{aligned}$$

Note that : $z_T - z_t$ has the same law as $\sqrt{T-t}\epsilon_{T-t}$ and w_g has the same law as $\sqrt{g}\bar{\epsilon}_g$
where : ϵ_{T-t} and $\bar{\epsilon}_g$ are two standard normal variables.

We define $\bar{\rho}$, as a constant coefficient representing the correlation between ϵ_t and $\bar{\epsilon}_s$ $\forall s, t$
So, by introducing this coefficient of correlation, we can express the probability of default as :

$$\varphi(t,T) = \int_{g=0}^{\infty} \Phi\left(\frac{(r-\eta^2/2)(T-t) - \theta g - ln(1+\beta)}{\sqrt{\sigma^2 g + \eta^2(T-t) - 2\sigma\eta\bar{\rho}\sqrt{g(T-t)}}}\right) \frac{g^{\frac{T-t}{\nu}-1}e^{-\frac{g}{\nu}}}{\nu^{\frac{T-t}{\nu}}\Gamma(\frac{T-t}{\nu})}dg$$

Finally we can express the probability of default by this explicit formula :

$$\int_{g=0}^{\infty} \Phi\left(\frac{(r-\eta^2/2)(T-t) - \theta g - ln(1+\beta)}{\sqrt{\sigma^2 g + \eta^2(T-t) - 2\sigma\eta\bar{\rho}\sqrt{g(T-t)}}}\right) \frac{g^{\frac{T-t}{\nu}-1}e^{-\frac{g}{\nu}}}{\nu^{\frac{T-t}{\nu}}\Gamma(\frac{T-t}{\nu})}dg \tag{15}$$

To calculate the probability of default, we can for instance, use the Monte Carlo method for a large number of scenarios of the risk $A_t(T) - L_t(T)$ and calculate the number of negative scenarios, compared to the total number of scenarios generated. But more quickly and by using the formula (15) we can calculate this probability directly by the software R. We can see below the evolution of probability of default in function of number of years between current date t and date of retirement T, in other words, the probability of default in function of time horizon.Note that in practice, this period differs from affiliate to affiliate.

Example 3:

risk free rate :	r	=	2%
mean return of the fund :	δ	=	6%
volatility of the fund :	σ	=	10%
average increase of salary :	μ	=	5%
volatility of the salary:	η	=	5%
correlation :	$\bar{\rho}$	=	50%
safety loading :	β	=	5%
variance rate of the gamma time change :	ν	=	0.75

Figure 3 compares the probability of ruin, for an asset following geometric Brownian motion or a variance gamma process, function of residual time. The curves are given by explicit formulas (6) and (15).

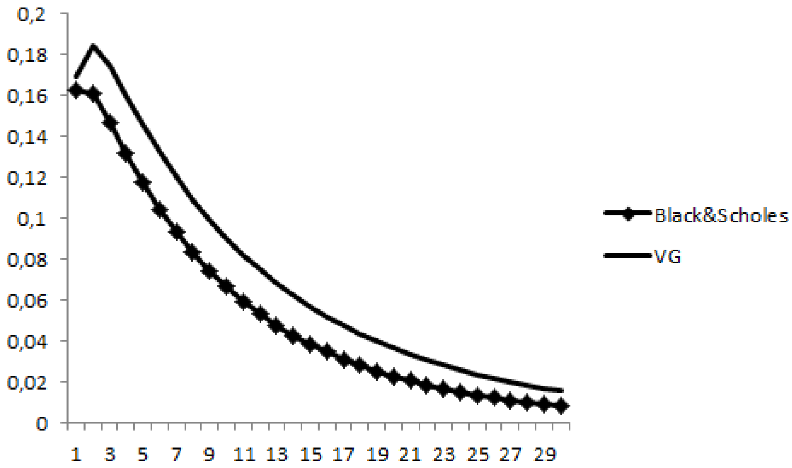

Figure 3: Probability of default BS vs VG

We note that if we invest the contributions in accordance with a variance gamma process, we treat more risk compared to an investment following a geometric Brownian motion which underestimates the financial risk, nevertheless, the curves of ruin probability obtained by the two models have the same shapes and are decreasing on the long term.

6.3 Solvency capital required

To calculate the Solvency Capital Required, we rely on a projection at time T. We suppose that the solvency capital is invested in a risky fund.

Proposition 6.2. *The solvency capital required SCR(t,T) at time t(t=1,2,....T-1) can be calculated by the following formula:*

$$SCR(t,T) = AL_t.(VaR_{1-\alpha_{T-t}}(Y_T^t) - 1)$$

where AL_t is defined by (1) and $Y_T^t = \frac{L_t(T)}{A_t(T)}$

PROOF The solvency capital required, invested in risky fund, can be seen as a solution of the following equation :

$$\begin{aligned} P\{A_t(T) + SCR(t,T)\frac{A_t(T)}{A_t(t)} < L_t(T)\} &= 1-\alpha_{T-t} \\ P\{A_t(T)(1+\frac{SCR(t,T)}{A_t(t)} < L_t(T)\}) &= 1-\alpha_{T-t} \\ P\{1+\frac{SCR(t,T)}{AL_t} < \frac{L_t(T)}{A_t(T)}) &= 1-\alpha_{T-t} \end{aligned}$$

So, this capital verifies :

$$1+\frac{SCR(t,T)}{AL_t(t)} = VaR_{1-\alpha_{T-t}}(Y_T^t)$$

where

$$Y_T^t = \frac{L_t(T)}{A_t(T)}$$

And finally :

$$SCR(t,T) = AL_t(VaR_{1-\alpha_{T-t}}(Y_T^t) - 1)$$

Here again, we can express the solvency capital required in percentage of the actuarial liability existing at computation time t. So this level in percent is given by :

$$SCR^{\%}(t,T) = VaR_{1-\alpha_{T-t}}(Y_T^t) - 1 \qquad (16)$$

In order to compare between the values of solvency capital required by each of process seen above,the figure 4 shows the evolution of solvency capital obtained by formula(16), by Monte Carlo method with a generation of 250.000 scenarios in parallel for each of the two processes (Variance gamma and geometric Brownian motion) :

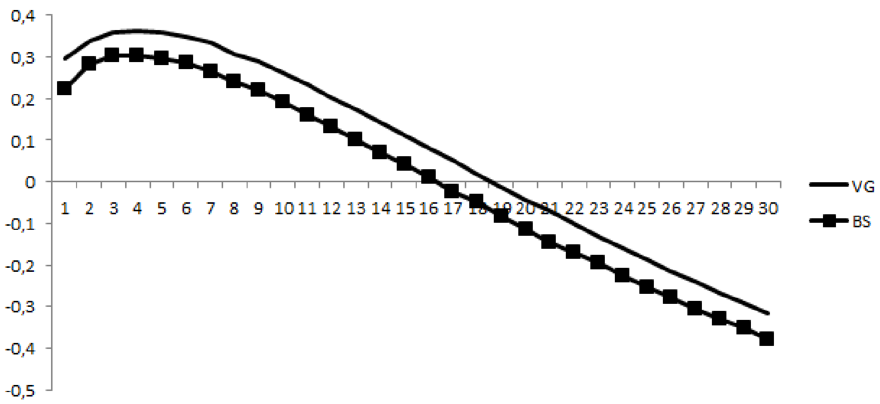

Figure 4: Solvency Capital Required BS vs VG

In terms of value-at-risk approach, we can see that the variance gamma process requires more capital than a simple geometric Brownian motion, this difference is due to the fact that this process takes into account the risks related to sudden crashes of financial markets.

7 Conclusion

This paper presents a new way to evaluate Defined benefits pension schemes, in coherence with IAS norms. The liabilities are subjected to disturbances due to the wage inflation, and the results are obtained by using the projected unit credit cost method.

This study proves that risk measures present similar shapes; in particular, a decreasing trend on long term can be observed.

To better estimate financial risks and in order to take into account brutal crashes of the financial markets, we have considered after a Brownian model, the case of an asset invested in a variance gamma process which is a pure jump process. The level of the solvency capital becomes higher in this case compared to an asset invested in a simple geometric Brownian motion.

In all the cases, the solvency capital is influenced by the duration of time horizon; a short residual time requires more capital.

We can conclude that the manager of a pension fund is more solvent with long-term contracts, what is for him an incentive to take more risk on these investments. This reality should be taken into account in future regulations of pension funds instead of applying as exepcted the traditional value at risk on one year. Asking too much capital for long term pension liability could induce on the long run poor returns and excessive conservative strategies.

References

[Artzner P., Delbaen F., Eber J.M., and Heath D] Coherent Measures of Risk *Mathematical Finance*, (9):203–228, 1999.

[Back 1995] A Course in Derivatives Securities *Springer*, 1995.

[Bodie 1995] On the Risks of Stocks in the Long Run *Financial Analysts Journal*, 51(3):68–76 1995.

[Bodie, Z. , Merton and Samuelson] Labor Supply Flexibility and Portfolio Choice in a Lifecycle Model. *Journal of Economic Dynamics and Control*, 16(3): 427-449, 1992.

[Briys E. and de Varenne F] On the Risk of Life Insurance Liabilities : Debunking Some Common Pitfalls. *Journal of Risk and Insurance*, 64(4): 673–694, 1997.

[Campbell J. and Viceira L] Strategic Asset Allocation ? Portfolio Choice for Long term Investors. *Oxford University Press* 2002.

[Devolder P.] Solvency requirement for long term guarantee: risk measure versus probability of ruin. *Insurance: Mathematics and Economices*, 1, 199–214 , 2011.

[Grosen A. and Jorgensen P] Life Insurance Liabilities at Market Value : An Analysis of Insolvency Risk, Bonus Policy and Regulatory Intervention Rules in a Barrier Option framework. *Journal of Risk and Insurance*, 69(1): 63–91, 2002.

[Hardy M. and Wirch J.L] The Iterated CTE : a Dynamic Risk Measure. *North American Actuarial Journal*, 8: 62–75, 2004.

[Josa-Fombellida R. and Rincon-Zapatero J.P.] Optimal risk management in defined stochastic pension funds. *Insurance: Mathematics and Economices*, 34, 489–503, 2004.

[Lee W] Diversification and Time ,Do Investment Horizons matter?. *Journal of Portfolio Management*, 4: 60–69, 1990.

[Madan D.P, Carr P.,Chang E.C 1998] The Variance Gamma Process and Option Pricing. *European Finance Review 2 : 79-105, 1998*

[Merton R] Continuous Time Finance. *Wiley*, 1992.

[Ngwira B. and Gerrard R.] Stochastic pension fund control in the presence of Poisson jumps. *Insurance: Mathematics and Economices*, 40,283–292,2007

[Pflug G. and Romisch W] Modeling, Measuring and Managing Risk, *World Scientific*, 2007.

[Samuelson P] The Long Term Case for Equities. *Journal of Portfolio Management*, 21(1): 15–24, 1994.

[Samuelson P] Risk and uncertainty : A fallacy of Large Numbers. *Scientia*, 98: 108–113, 1963.

[Thorley S] The Time Diversification Controversy. *Financial Analyst Journal*, 51(3): 18–22, 1995.

[Wirch J.L. and Hardy M.R] A synthesis of risk measures for capital adequacy. *European Actuarial Journal*, 25: 337–347, 1999.