PARTIAL INNER PRODUCT SPACES, A UNIFYING FRAMEWORK FOR QUANTUM MECHANICS

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1. Introduction

Many function spaces that play a central role in analysis come in the form of families, indexed by one or several parameters that characterize the behavior of functions (smoothness, behavior at infinity, ...). The typical structure is a *chain of Hilbert or (reflexive) Banach spaces*. Let us give two familiar examples.

(i) The Lebesgue L^p spaces on $[0,1]$, $\mathcal{I} = \{L^p([0,1], \mathrm{d}x), 1 \leqslant p \leqslant \infty\}$.
(ii) The scale of Hilbert spaces $\{\mathcal{H}_n\}_{n\in\mathbb{Z}}$ built on the powers of a positive self-adjoint operator $A \geq 1$ in a Hilbert space $\mathcal{H}_0$. Here $\mathcal{H}_n = D(A^n)$, with the graph norm $\|f\|_n = \|A^n f\|$, $f \in D(A^n)$, for $n \in \mathbb{N}$ or $n \in \mathbb{R}^+$, and $\mathcal{H}_{-n} = \mathcal{H}_n^\times$ (conjugate dual).

In both cases, the inner product of the central Hilbert space cannot be extended to the largest space of the chain, but only on *compatible* pairs. Thus one gets a *partial inner product space* (PIP-space). In this paper, we shall give a quick overview of PIP-spaces and operators on them. A complete information may be found in our monograph,[1] including references to the original work.

2. Partial inner product spaces

A formal way of introducing PIP-spaces is given by the idea of *linear compatibility*,[1] by which we mean a symmetric binary relation # on V which preserves linearity:

$$f\#g \Longleftrightarrow g\#f,\ \forall\, f,g \in V,$$
$$f\#g, f\#h \Longrightarrow f\#(\alpha g + \beta h),\ \forall\, f,g,h \in V,\ \forall\, \alpha,\beta \in \mathbb{C}.$$

Then, for any subset $S \subset V$, the set $S^{\#} = \{g \in V : g\#f, \forall\, f \in S\}$ is a vector subspace of V and one has $S^{\#\#} = (S^{\#})^{\#} \supseteq S$, $S^{\#\#\#} = S^{\#}$. Thus,

$$f\#g \iff f \in \{g\}^{\#} \iff \{f\}^{\#\#} \subseteq \{g\}^{\#}. \tag{1}$$

From now on, we will call *assaying subspace* of V a subspace S such that $S^{\#\#} = S$ and denote by $\mathcal{F}(V,\#)$ the family of all assaying subsets of V, ordered by inclusion. Assaying subsets will be denoted $V_r, V_q, \ldots$ and the index set by F. By definition, $q \leq r$ if and only if $V_q \subseteqq V_r$. We also write $V_{\overline{r}} = V_r^{\#}$, $r \in F$. Thus the relations (1) mean that $f\#g$ if and only if there is an index $r \in F$ such that $f \in V_r$, $g \in V_{\overline{r}}$. In other words, assaying subspaces are the building blocks of the whole structure.

Theorem 1. *The family $\mathcal{F}(V,\#) = \{V_r, r \in F\}$ of all assaying subspaces, ordered by inclusion, is a complete involutive lattice, under the following operations, arbitrarily iterated:*

$$\begin{array}{llll} . \textit{ involution:} & V_r & \leftrightarrow & V_{\overline{r}} := (V_r)^{\#}, \\ . \textit{ infimum:} & V_{p\wedge q} & := & V_p \wedge V_q = V_p \cap V_q, \qquad (p,q,r \in F) \\ . \textit{ supremum:} & V_{p\vee q} & := & V_p \vee V_q = (V_p + V_q)^{\#\#}. \end{array}$$

The smallest element of $\mathcal{F}(V,\#)$ is $V^{\#} = \bigcap_r V_r$ and the greatest element is $V = \bigcup_r V_r$. We also note the following relations:

$$(V_{p\wedge q})^{\#} = V_{\overline{p\wedge q}} = V_{\overline{p}\vee\overline{q}} = V_{\overline{p}} \vee V_{\overline{q}}.$$

A *partial inner product* on $(V, \#)$ is a hermitian form $\langle\cdot|\cdot\rangle$, not necessarily positive definite, defined exactly on compatible pairs of vectors. A *partial inner product space* (PIP-space) is a vector space V equipped with a linear compatibility and a partial inner product.

We require the PIP-space $(V, \#)$ to be *nondegenerate*, i.e., $\langle f|g\rangle = 0$ for all $f \in V^{\#}$ implies $g = 0$. Then $(V^{\#}, V)$ is a dual pair in the sense of topological vector spaces,[3] and so is every couple $(V_r, V_{\overline{r}})$, $r \in F$. In the sequel, we also assume that the partial inner product is positive definite.

Next we assume that every V_r carries its *Mackey topology* $\tau(V_r, V_{\overline{r}})$. As a consequence, if V_r is a Hilbert space or a reflexive Banach space, then $\tau(V_r, V_{\overline{r}})$ coincides with the norm topology. Next, $r < s$ implies $V_r \subset V_s$, and the embedding operator $E_{sr} : V_r \to V_s$ is continuous and has dense range. In particular, $V^{\#}$ is dense in every V_r.

Let us give two simple examples.

(1) $V = \omega$, the space of *all* complex sequences $x = (x_n)$, with the compatibility relation $x\#y \Leftrightarrow \sum_{n=1}^{\infty} |x_n\, y_n| < \infty$, and the partial inner product $\langle x|y\rangle = \sum_{n=1}^{\infty} \overline{x_n}\, y_n$, for $x\#y$.
(2) $V = L^1_{\rm loc}(\mathbb{R}, \mathrm{d}x)$, the space of Lebesgue measurable, locally integrable functions, with the compatibility relation $f\#g \Leftrightarrow \int_{\mathbb{R}} |f(x)g(x)|\, \mathrm{d}x < \infty$ and the partial inner product $\langle f|g\rangle = \int_{\mathbb{R}} \overline{f(x)}g(x)\, \mathrm{d}x$, for $f\#g$.

The previous examples show that $\mathcal{F}(V, \#)$ is a huge lattice (it is complete!) and that assaying subspaces may be complicated, such as Fréchet spaces, non metrizable spaces, etc. This situation suggests to choose an involutive sublattice $\mathcal{I}$ of $\mathcal{F}$, indexed by I, such that
(i) $\mathcal{I}$ is *generating*, that is, $f\#g$ iff $\exists\, r \in I$ such that $f \in V_r, g \in V_{\overline{r}}$;
(ii) every $V_r, r \in I$, is a Hilbert space or a reflexive Banach space;
(iii) $\mathcal{I}$ contains a unique self-dual Hilbert space $V_o = V_{\overline{o}}$.
In that case, the structure $V_I := (V, \mathcal{I}, \langle\cdot|\cdot\rangle)$ is called, respectively, a *lattice of Hilbert spaces* (LHS) or a *lattice of Banach spaces* (LBS). Note that $V^{\#}, V$ themselves usually do *not* belong to the family $\{V_r,\, r \in I\}$, but they can be recovered as $V^{\#} = \bigcap_{r\in I} V_r$, $V = \sum_{r\in I} V_r$.

In the LBS case, the lattice structure takes the following form

- $V_{p\wedge q} = V_p \cap V_q$, with the *projective* norm $\|f\|_{p\wedge q} = \|f\|_p + \|f\|_q$;
- $V_{p\vee q} = V_p + V_q$, with the *inductive* norm

$$\|f\|_{p\vee q} = \inf_{f=g+h} \left(\|g\|_p + \|h\|_q\right), g \in V_p, f \in V_q.$$

These norms are usual in interpolation theory.[2] In the LHS case, one takes similar definitions with squared norms, in order to get Hilbert norms.

We list a series of concrete examples of LBS/LHS (in one dimension, for simplicity). The first two are those described in the Introduction.

(i) Scales of Hilbert or Banach spaces
(a) The Lebesgue L^p spaces on $[0, 1]$, $\mathcal{I} = \{L^p([0, 1], \mathrm{d}x),\ 1 \leqslant p \leqslant \infty\}$.
(b) The scale of Hilbert spaces built on the powers of a positive self-adjoint operator $A \geq 1$ in a Hilbert space $\mathcal{H}_0$. For instance, in $\mathcal{H}_0 = L^2(\mathbb{R}, \mathrm{d}x)$:

- $(A_{\rm m} f)(x) = (1 - \frac{d^2}{dx^2})^{1/2} f(x)$;
- $(A_{\rm osc} f)(x) = (1 + x^2 - \frac{d^2}{dx^2}) f(x)$.

In the case of $A_{\rm m}$, the intermediate spaces are the Sobolev spaces $H^s(\mathbb{R})$, $s \in \mathbb{Z}$ or $\mathbb{R}$. Note that $\mathcal{H}_\infty(A_{\rm osc})$ coincides with the Schwartz space $\mathcal{S}(\mathbb{R})$ of smooth functions of fast decay, and $\mathcal{H}_{-\infty}(A_{\rm osc})$ with the space $\mathcal{S}^\times(\mathbb{R})$ of tempered distributions.

(ii) Sequence spaces
In ω, we may take the lattice $\mathcal{I} = \{\ell^2(r)\}$ of the weighted Hilbert spaces $\ell^2(r) = \{(x_n) : \sum_{n=1}^{\infty} \; |x_n|^2 \, r_n^{-1} < \infty\}$, $r = (r_n)$, $r_n > 0$.

(iii) Spaces of locally integrable functions
In $L^1_{\rm loc}(\mathbb{R}, \mathrm{d}x)$, we may take the lattice $\mathcal{I} = \{L^2(r)\}$ of the weighted Hilbert spaces $L^2(r) = \{f \in L^1_{\rm loc}(\mathbb{R}, dx) : \int_{\mathbb{R}} |f(x)|^2 \, r(x)^{-1} \; \mathrm{d}x < \infty\}$, with $r, r^{-1} \in L^2_{\rm loc}(\mathbb{R}, dx), r(x) > 0$ a.e.

(iv) The spaces $L^p(\mathbb{R}, \mathrm{d}x)$, $1 < p < \infty$
These spaces do not constitute a scale, since one has only the inclusions $L^p \cap L^q \subset L^s$, $p < s < q$. Thus one has to consider the lattice they generate, which is a genuine lattice of Banach spaces, $L^p \cap L^q$ and $L^p + L^q$, reflexive for $1 < p, q < \infty$.

3. Operators on pip-spaces

According to the general philosophy, an operator on a PIP-space should be defined in terms of assaying subspaces V_r only. Thus we state:

Given a LHS or LBS $V_I = \{V_r, \, r \in I\}$, an *operator* on V_I is a map A from a subset $\mathcal{D} \subseteq V$ into V, where

(i) $\mathcal{D}$ is a nonempty union of assaying subsets of V_I;

(ii) for every assaying subset V_q contained in $\mathcal{D}$, there exists a $p \in I$ such that the restriction A_{pq} of A to V_q is linear and continuous into V_p;

(iii) A has no proper extension satisfying (i) and (ii), i.e., it is maximal.

The linear bounded operator $A_{pq} : V_q \to V_p$ is called a *representative* of A. In terms of the latter, the operator A may be characterized by the set $\mathsf{j}(A) := \{(q, p) \in I \times I : A_{pq}$ exists$\}$. Thus the operator A may be identified with the (coherent) collection of its representatives,

$$A \simeq \{A_{pq} : V_q \to V_p : (q, p) \in \mathsf{j}(A)\}.$$

We denote by $\mathrm{Op}(V_I)$ the set of all operators on V_I. A similar definition may be given for operators $A : V_I \to Y_K$ between two LHSs or LBSs.

Algebraic operations may be defined for operators on a PIP-space:

(i) *Adjoint $A^\times$:* given $A \in \mathrm{Op}(V_I)$ with $(r, s) \in \mathsf{j}(A)$, it has a unique adjoint $A^\times \in \mathrm{Op}(V_I)$ defined by

$$\langle A^\times x | y \rangle = \langle x | A y \rangle, \text{ for } y \in V_r \text{ and } \; x \in V_{\overline{s}}\,,$$

that is, $(A^\times)_{\overline{r}\overline{s}} = (A_{sr})^*$ (usual Hilbert/Banach space adjoint).

It follows that $A^{\times\times} = A$, for every $A \in \mathrm{Op}(V_I)$: no extension is allowed, by the maximality condition (iii) of the definition.

(ii) *Partial multiplication:* AB is defined whenever there is a continuous factorization through some V_q :

$$V_r \xrightarrow{B} V_q \xrightarrow{A} V_s, \quad \text{i.e.,} \quad (AB)_{sr} = A_{sq} B_{qr} \, .$$

The concept of PIP-space operator allows to treat on the same footing all kinds of operators, from bounded ones to very singular ones. Take, for instance,

$$V_r \subset V_o \simeq V_{\overline{o}} \subset V_s \quad (V_o = \text{Hilbert space}).$$

Then three cases may arise:

- if A_{oo} exists, then A corresponds to a *bounded* operator $V_o \to V_o$;
- if A_{oo} does not exist, but only $A_{or} : V_r \to V_o, r < o$, then A corresponds to an *unbounded* operator, with Hilbert space domain containing V_r;
- if no A_{or} exists, but only $A_{sr} : V_r \to V_s$, $r < o < s$, then A corresponds to a *singular* operator, with domain in V_o possibly $\{0\}$.

A nice application is a rigorous analysis of singular quantum Hamiltonians, e.g., the Kronig–Penney crystal model or δ interactions (Ref. 1, Sec.7.1.3).

4. Special classes of operators on PIP-spaces

(1) Homomorphisms

An operator $A \in \mathrm{Op}(V_I, Y_K)$ is called a *homomorphism* if

(i) $\forall\, r \in I$ there exists $u \in K$ such that both A_{ur} and $A_{\overline{ur}}$ exist;

(ii) $\forall\, u \in K$, there exists $r \in I$ such that both A_{ur} and $A_{\overline{ur}}$ exist.

We denote by $\mathrm{Hom}(V_I, Y_K)$ the set of all homomorphisms between the two LHS V_I, Y_K. Clearly $A \in \mathrm{Hom}(V_I, Y_K)$ if and only if $A^\times \in \mathrm{Hom}(Y_K, V_I)$.

A homomorphism $A \in \mathrm{Hom}(V_I, Y_K)$ is an *isomorphism* if there exists $B \in \mathrm{Hom}(Y_K, V_I)$ such that $BA = \mathit{1}_V, AB = \mathit{1}_Y$ (identity operators).

An operator U is *unitary* if $U^\times U$ and $UU^\times$ are defined and $U^\times U = \mathit{1}_V, UU^\times = \mathit{1}_Y$. However, unitary operators need *not* be homomorphisms! This implies that the natural setting for group representations in a LHS is that of *unitary isomorphisms* (Ref. 1, Sec.3.3.4; see also (4) below) .

(2) Symmetric operators

An operator A is *symmetric* if $A^\times = A$. Symmetric operators satisfy a generalized KLMN theorem, stating when a symmetric operator has a self-adjoint restriction to the central Hilbert space V_o (Ref. 1, Sec.3.3.5).

(3) Orthogonal projections

An *orthogonal projection* on a nondegenerate PIP-space V is a homomorphism that satisfies the relations $P^2 = P = P^\times$. On the other hand, there is a natural notion of PIP-subspace, called *orthocomplemented*, which guarantees that such a subspace W of V is again a nondegenerate PIP-space with the induced compatibility relation and the restriction of the partial inner product. Then the basic theorem about projections states that a PIP-subspace W of V is orthocomplemented if and only if W is the range of an orthogonal projection $P : W = PV$. Then $V = W \oplus Z$, where $Z = W^{\perp}$ is another orthocomplemented PIP-subspace.

(4) Representations of Lie groups and Lie algebras

Given a strongly continuous unitary representation U_{00} of a Lie group G in a Hilbert space $\mathcal{H}_0$, define the Gårding domain $\mathcal{H}_0^G$, which consists of C^∞ vectors for U_{00}. The domain $\mathcal{H}_0^G$ is dense in $\mathcal{H}_0$ and is stable under $U_{00}(g)$, $\forall\, g \in G$, and also under the representatives of all elements of the Lie algebra $\mathfrak{g}$ of G. Define the Nelson operator $\Delta = \sum_{j=1}^{n} X_j^2$, where $\{X_j, j = 1, \ldots, n\}$ are the representatives under U_{00} of a basis of $\mathfrak{g}$. The operator Δ is essentially self-adjoint on $\mathcal{H}_0^G$ and $\overline{\Delta}$ is self-adjoint and positive. Define the canonical scale of Hilbert spaces $V_I := \{\mathcal{H}_n, n \in \mathbb{Z}\}$ generated by the powers of $(\overline{\Delta} + 1)$:

$$V^{\#} := \mathcal{D}^\infty(\overline{\Delta}) \subset \ldots \subset \mathcal{H}_n \subset \ldots \subset \mathcal{H}_0 \subset \ldots \subset \mathcal{H}_{-n} \subset \ldots \subset V := \mathcal{D}_{\overline{\infty}}(\overline{\Delta}).$$

Then $D^\infty(\overline{\Delta}) = \bigcap_{n=1}^{\infty} D(\overline{\Delta}^n) = \mathcal{H}_0^\infty = \{C^\infty\text{-vectors of } U_{00}\}$. For every $g \in G$ and every $n = 0, 1, 2, \ldots$, $U_{00}(g) : \mathcal{H}_n \to \mathcal{H}_n$ continuously. By transposition, the same holds for $U_{nn}(g^{-1}) : \mathcal{H}_{-n} \to \mathcal{H}_{-n}$, $n = 1, 2, \ldots$. Thus U_{00} extends to a unitary, totally regular representation U in the LHS V_I (Ref. 1, Sec.7.7.4). Therefore, the scale V_I is the natural tool for the representation of $\mathfrak{g}$ or $\mathfrak{U}(\mathfrak{g})$:

(i) Given $x \in \mathfrak{g}$ or $L \in \mathfrak{U}(\mathfrak{g})$, the representative $U(x)$, resp. $U(L)$, originally defined on $\mathcal{H}_0^G$, extends to a regular operator on V_I;

(ii) $\mathcal{H}_0^\infty = D^\infty(\overline{\Delta})$ is a domain of essential self-adjointness for many operators of the form $U(L), L \in \mathfrak{U}(\mathfrak{g})$, e.g. the generators $\{X_j, j = 1, \ldots, n\}$ or elliptic operators, such as the Nelson operator Δ of G or of $H \subset G$.

Thus, if G is the symmetry group of a quantum system, this analysis yields many candidates for physically meaningful observables.

References

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