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Artículos

ON SUNSPOT EQUILIBRIA IN OVERLAPPING GENERATIONS ECONOMIES

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Abstract We consider the existence of finite Stationary Sunspot Equilibria (k-SSE) in the simple Overlapping Generations (OLG) model for the ''negative outside money'' case. Necessary and sufficient conditions found are so weak that a large class of economies has k-SSE under standard assumptions on preferences.

ACERCA DE LOS EQUILIBRIOS «SUNSPOT" EN ECONOMÍAS DE GENERACIONES SOLAPADAS

Resumen En este artículo se estudia la existencia de Equilibrios con "Manchas Solares" Estacionarios de orden k (denominados k-SSE, del inglés *Stationary Sunspot Equilibria of order k)* en el modelo más sencillo de una economía de generaciones sucesivas cuando la "cantidad de dinero externo" es negativa. Las condiciones necesarias y suficientes para su existencia encontradas son tan débiles que una amplia clase de estas economías tiene k-SSE bajo hipótesis usuales sobre las preferencias.

1. INTRODUCTION

Most of the results of the Sunspot Equilibria literature using the framework of the simple Overlapping Generations (OLG) Model [Samuelson (1958), Gale (1973)] are concerned with the case where the ''stock of outside money'' is positive –or *Samuelsonian Case* in the terminology of Gale (1973)–, *i.e.* agents want to transfer wealth from the first period of their lives to the second one in the form of money savings.

Characterizations of the existence of finite Stationary Sunspot Equilibria (henceforth k-SSE) with positive money holdings in the simple 1-commodity OLG model with a two-period lived representative agent, are given in Azariadis (1981) and Spear (1984) (for 2-SSE), Azariadis and Guesnerie (1986) and Grandmont (1986) (for k-SSE in general), among others. The conditions given in this literature may be satisfied or not by a ''non pathologic'' economy of the class considered and, consequently, robust examples can be shown of simple OLG economies either with or without k-SSE.

In contrast with the previous results, we consider in this article the possibility of k-SSE in these simple OLG economies such that, in the resulting allocation of resources, agents

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want to transfer wealth from the second to the first period of their lives —like in the *Classical Case* of Gale (1973). This ''negative'' money case has been quite naturally considered in the deterministic OLG models, but very rarely studied in the sunspot equilibria literature. One can think of an extreme case in which agents have a non-zero endowment only in the second period of their lives but, nevertheless, we can suppose that some institutional mechanism —some kind of ''negative money'' or ''intergenerational debt'', like in Burnell (1989a)—, allows agents to attain non autarkic allocations of resources. In his article, Burnell states a result about the existence of SSE with "negative" money in a one-commodity OLG economy as a straightforward consequence of the sufficient characterization provided by Spear (1984). Furthermore, in Burnell (1989b) he uses an argument based on the Poincaré-Hopf Index Theorem to give a sufficient condition for the existence of SSE in the ''negative'' money case in an n-commodity OLG economy. Nevertheless, all these results consider only simple 2-SSE, *i.e.* Sunspot Equilibria fluctuating between just two states.

On the contrary, we give in this article a necessary and sufficient condition for the existence of k-SSE in the simple OLG model with "negative" money, whatever the number of states $k \geq 2$ between which the equilibrium fluctuates is. Moreover, we shall show that, whenever agents have a non-zero endowment only in their second period of life, the condition found is satisfied by any economy of this class with very general preferences for the representative agent.

This result will be established here by a straightforward transposition of the argument used in Grandmont (1986) into this case. Actually, the result does not depend upon some particular assumptions needed for this particular argument to work, like the additive separability of the utility function in the consumptions of different periods. It could be shown as a consequence of the more general invariant compact argument of Chiappori and Guesnerie (1991).

In section 2 we present the standard simple OLG model in its deterministic version. In section 3, its immediate generalization to a "world with sunspots" is considered. In section 4 we provide a characterization of the economies of this class with k-SSE and "negative" money and we interprete it in terms of the offer curve of the representative agent. An illustrative example of why k-SSE are more likely to appear in the *Classical Case* than in the *Samuelsonian Case* can be found there also. Some concluding remarks follow in the last section 5, and finally the Appendix collects proofs and some technicalities.

2. THE DETERMINISTIC ECONOMY

Consider the simple OLG model of a pure exchange economy: there is an infinite sequence of identical agents living for two consecutive periods each one, in such a way that at every period the current generation consists of an "old" agent and a "young" one. The decision-making problem of these agents, in a world without uncertainty, is to decide their consumption levels in their two periods of life, under the budgetary constraint imposed by the value of their endowments in the competitive market. This simple model of a sequential economy and its competitive equilibria can then be summarized by an infinite numerable sequence of optimization programs indexed by $t \in \mathbb{Z}$

$$\begin{aligned} &\max_{c_t^t\, c_{t+1}^t} U_0(c_t^t)+\mathrm{U}_1(c_{t+1}^t) \\ &\text{subject to } p_t(c_t^t - w_0)+p_{t+1}(c_{t+1}^t - w_1) \leq 0 \\ &\text{and } (c_t^t,\, c_{t+1}^t) \in \mathbb{R}_+^2 \end{aligned} \qquad [1]$$

the solutions of which, $c_t^t(p_t,p_{t+1})$ and $c_{t+1}^t(p_t,p_{t+1})$ –the optimal consumptions of the agent "born" at any period t– parameterized by a sequence of strictly positive prices $\{p_t\}_t$, $t \in \mathbb{Z}$, must satisfy the materials balance condition

$$\forall t \in \mathbb{Z},\ c_t^{t-1}(p_{t-1},p_t) + c_t^t(p_t,p_{t+1}) = w_0 + w_1 \qquad [2]$$

that expresses the feasibility of the allocation of resources.

The utility function is usually assumed to satisfy some properties that are sufficient for guaranteeing that these solutions $c_t^t(p_t,p_{t+1})$ and $c_{t+1}^t(p_t,p_{t+1})$ are well defined functions of the perfectly foreseen prices $\{p_t\}_t$, $t \in \mathbb{Z}$, *e.g.*

Hypothesis H.1. The utility function $U_0(c_t^t) + U_1(c_{t+1}^t)$ is such that

(1) $\forall i \in \{0,1\}$, $U_i(\cdot)$ *is a strictly monotonically increasing and strictly concave function* [1] *defined on* $\mathbb{R}_+$ *and* C^2 *on* $\mathbb{R}_{++}$, and

(2) *the marginal rate of substitution* $U_0'(c_t^t)/U_1'(c_{t+1}^t)$ *tends to 0 (respectively to* $+\infty$*) whenever the consumption bundle approaches to a point on the axis of abscissae (respectively the axis of ordinates) dictinct form the origin.*

A typical example of preferences satisfying the second condition in hypothesis H.1 is that of monotone preferences with indifference curves not intersecting the axes. That is to say, with such preferences, any consumption bundle with strictly positive consumptions *in both* periods is strictly preferred to a bundle on the axes. We shall speak of "interior indifference curves" to make reference to this situation [2]. An example of preferences fitting this case are those represented by a Cobb-Douglas utility function, which enter this framework by writing them as their monotone transformation given by the log-linear function

$$U(c_0,c_1) = \begin{cases} \ln c_0 + \ln c_1 & , \forall (c_0,c_1) \in \mathbb{R}_{++}^2 \\ -\infty & , \text{if } c_0 = 0 \text{ or } c_1 = 0. \end{cases}$$

This assumption, made also in Peck (1988) and Spear (1984), is quite natural in the context of an OLG model. It does not seem to be a too demanding assumption to claim that the agents prefer not to starve at any period because of a null consumption.

In fact, the second condition of hypothesis H.1 above will be satisfied by any utility function which verifies the Inada condition

$$\forall i \in \{0,1\}\ ,\ \lim_{x \to 0^+} U_i'(x) = +\infty$$

like, for instance,

$$U(c_0,c_1) = \sqrt{c_0} + \sqrt{c_1}.$$

Let us remark that, in this case, indifference curves are not "interior" in the sense we mentioned above, but they reach the axes –although in a tangent way. Nevertheless, we shall speak of preferences with "interior indifference curve" for short to make reference to pre-

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[1] Usually a real-valued function, but perhaps a function taking values in the extended real line $\mathbb{R} \cup \{-\infty\}$ –as in the example below.

[2] Actually, the continuity of preferences implies that the axes constitute by themselves the indifference "curve" associated to the lowest utility level.

ferences satisfying the second condition of hypothesis H.1. Actually, this assumption is unnecessarily strong –as we shall see later– but it will simplify the exposition quite a lot in what follows.

In the budget constraint, w_0 and w_1 are the endowments of the representative agent in his two periods of life. We shall assume that they are nonnegative in general, but both of them cannot be equal to zero simultaneously, *i.e.* $(w_0, w_1) \in \mathbb{R}^2_+ - \{0\}$. Usually, the budget constraint is written in an equivalent form

$$\begin{aligned} p_t c_t^t + m_t &\leq p_t w_0 \\ p_{t+1} c_{t+1}^t &\leq p_{t+1} w_1 + m_t \\ (c_t^t, c_{t+1}^t) &\in \mathbb{R}^2_+ \text{ and } m_t \in \mathbb{R} \end{aligned} \qquad [3]$$

which helps the monetary interpretation of the model. In this case m_t –the required transfer of wealth between periods in order to attain the consumption bundle (c_t^t, c_{t+1}^t)– is read, when positive, as the amount of "money" held by agents from their "young" to their "old" age. Let us assume that there exists in general a "monetary" mechanism allowing agents to transfer any desired amount of wealth across periods –in any direction in time– at a given rate determined by the relative prices of the commodities. In this sense, agents are allowed to hold "positive" as well as "negative" amounts of money.

It can be easily seen (*cf.* Appendix) that the perfect foresight equilibria of this economy without uncertainty result in allocations of resources $\{(c_t^{t*}, c_{t+1}^{t}{}^*)\}_t$, $t \in \mathbb{Z}$, which consist of a sequence of consumption bundles lying on the *offer curve* –the *locus* of points (c_t^t, c_{t+1}^t) satisfying the equation

$$W_0(c_t^t) + W_1(c_{t+1}^t) = 0 \qquad [4]$$

where $\forall i \in \{0,1\}$, $W_i(x) \equiv (x - w_i)U_i'(x)$– and verifying some "dynamics" imposed by the feasibility condition. This dynamics can be expressed by a backward difference equation in the equilibrium second period consumptions

$$c_t^{t\text{-}1*} = w_0 + w_1 - \phi(c_{t+1}^t{}^*) \qquad [5]$$

whenever at equilibrium agents hold a "positive" quantity of money, or by a forward difference equation in the equilibrium first period consumptions

$$c_{t+1}^{t+1*} = w_0 + w_1 - \psi(c_t^{t*}) \qquad [6]$$

if the agents want to be indebted in their first period of life at equilibrium (see the proof in the Appendix). The functions $\phi(\cdot)$ and $\psi(\cdot)$ are respectively $W_0^{-1}(-W_1(\cdot))$ and $W_1^{-1}(-W_0(\cdot))$, but these dynamics can actually be very easily visualized on the offer curve —see figures 1 and 2 below.

Figure 1

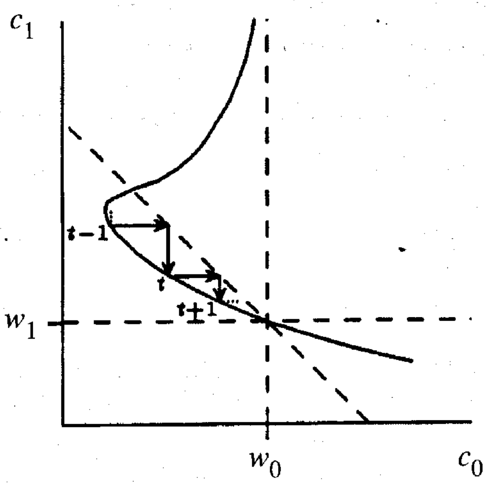
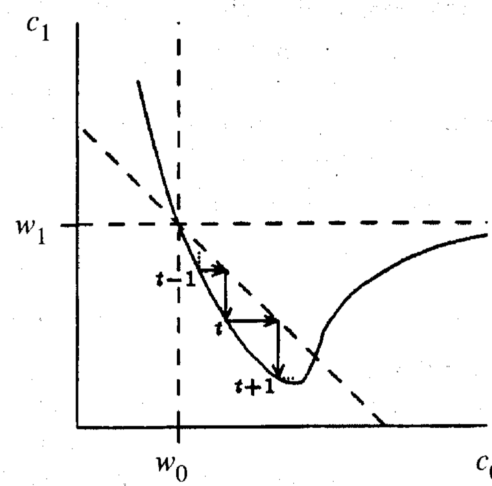


Figure 2



Remark. With the assumed hypotheses about the utility function of the representative agent, if the endowments are both strictly positive, then the *offer curve* is indeed a smooth curve that does not intersect the axes. It will just hit tangently one of the axes, precisely at the endowments point, whenever w_0 or w_1 –but not both– is equal to zero. This will be a usefull property in the characterization of economies with k-SSE given later.

3. THE ECONOMY WITH SUNSPOTS

Let us suppose now that there exists an observable extrinsic random signal[3] –let us call it a "sunspot"– taking at each period one out of k possible values $\omega^1, \omega^2, \ldots, \omega^k$, and following a k-state first order Markov chain that is known by the agents of the economy. All the agents also know that the fundamentals of the economy –namely, their endowments and preferences in this case– do not depend on the realizations of this random signal.

Nevertheless, if the agents believe that the equilibrium prices will be perfectly correlated to the observed sunspots –*i.e.*, according to their beliefs, the price p_t should be at equilibrium equal to some value p^i whenever the current observed sunspot is ω^i–, then they will make their consumption decisions rationally using the information about the future evolution of the economy revealed by the sunspot *via* the assumed perfect correlation between sunpots and prices. That is to say, agents will choose a (conditional) expected utility maximizing current consumption and plan of sunspot-contingent future consumptions, subject to the budgetary constraints imposed by the monetary mechanism. In particular, the optimization program faced by an agent "born" in a period in which the observed sunspot signal takes the value ω^i will be the following one, if their "sunspot theory" were true,

$$\max_{c_0^i, c_1^{i1}, \ldots, c_1^{ik}} U_0(c_0^i) + \sum_{j=1}^{k} \pi_{ij} U_1(c_1^{ij})$$

$$\text{subject to } \forall j \in \{1, 2, \ldots, k\}^2, \; p^i(c_0^i - w_0) + p^j(c_1^{ij} - w_1) \leq 0 \qquad [7]$$

[3] *i.e.*, fundamentals of the economy are the same for every possible state of the world (a complete history of realizations of the signal).

$$\text{and } (c_0^i, c_1^{i1}, \ldots, c_1^{ik}) \in \mathbb{R}_+ \times \mathbb{R}_+^k$$

where π_{ij} is the typical element of the Markov matrix Π governing the stochastic process followed by the sunspot signal, c_0^i is the first period consumption of the agent, and c_1^{ij} is his second period consumption contingent to observation the sunspot value ω^j in that period. Prices $p^1, p^2, \ldots, p^k$ are the sunspot-contingent foreseen prices, according to the agent beliefs.

A k-SSE consists then of k prices $p^1, p^2, \ldots, p^k$ such that the solutions of the previous optimization programs –when i takes all the values in $\{1, 2, \ldots, k\}$– satisfy the following condition about the feasibility of the allocation of resources, whatever the resolution of the extrinsic uncertainty is,

$$\forall h, l \in \{1, 2, \ldots, k\},\ c_0^{h*} + c_1^{lh*} = w_0 + w_1. \qquad [8]$$

It immediately follows from these feasibility conditions that the sunspot-contingent consumptions in the second period of a k-SSE do not depend on the value taken by the sunspot signal in the first period of life of the representative agent, *i.e.*

$$\forall h, l \in \{1, 2, \ldots, k\},\ c_1^{lh*} = c_1^{h*}. \qquad [9]$$

Let us see now a simple characterization of the economies of this class that have k-SSE in which the agents transfer wealth from their second period of life to the first one.

4. ECONOMIES WITH SUNSPOT EQUILIBRIA AND "NEGATIVE" MONEY

The solution of the optimization program of an agent believing in the perfect correlation between sunspots and prices, and "born" when the sunspot ω^i is observed, is charaterized by the condition (see the proof in the Appendix)

$$W_0(c_0^{i*}) + \sum_{j=1}^{k} \pi_{ij} W_1(c_1^{j*}) = 0 \qquad [10]$$

where the functions W_0 and W_1 are those defined in the section 2.

Then the allocation of resources of a k-SSE consists of a k-tuple of sunspot-contingent nonnegative consumptions in the first period $c_0^{1*}, c_0^{2*}, \ldots, c_0^{k*}$ and a k-tuple of consumptions in the second one $c_1^{1*}, c_1^{2*}, \ldots, c_1^{k*}$ such that

$$\forall i \in \{1, 2, \ldots, k\},\ W_0(c_0^{i*}) + \sum_{j=1}^{k} \pi_{ij} W_1(c_1^{j*}) = 0$$
$$\text{and } \forall h, \in \{1, 2, \ldots, k\},\ c_0^{h*} + c_1^{h*} = w_0 + w_1 \qquad [11]$$

since then

(1) each bundle $(c_0^{i*}, c_1^{1*}, c_1^{2*}, \ldots, c_1^{k*})$, for all possible $i \in \{1, 2, \ldots, k\}$, is the solution of the utility maximization problem of an agent "born" when the observed sunspot signal is ω^i and for a system of sunspot-contingent strictly positive prices $p^1, p^2, \ldots, p^k$ satisfying

$$\frac{p^i}{p^j} = -\frac{c_1^{j*} - w_1}{c_0^{i*} - w_0}, \qquad [12]$$

and

(2) the allocation of resources is feasible in any state of the world –a complete history of sunspot observations.

Let us now state the following proposition.

Proposition. *In a simple OLG economy satisfying hypothesis H.1 on preferences, there exist k-SSE in which agents are indebted in their first period of life if, and only if, the offer curve of the representative agent is such that there exit two positive values* c^m *and* c^M *satisfying*

$$\forall i \in \{\mathrm{m},\mathrm{M}\},\ \mathrm{c^m} < w_0 + w_1 - \psi(c^i) < c^M \qquad [13]$$

where $w_0 + w_1 - \psi(\cdot)$ *is the function characterizing the forward perfect foresight dynamics in the Classical Case.*

A corollary of this proposition is the following one:

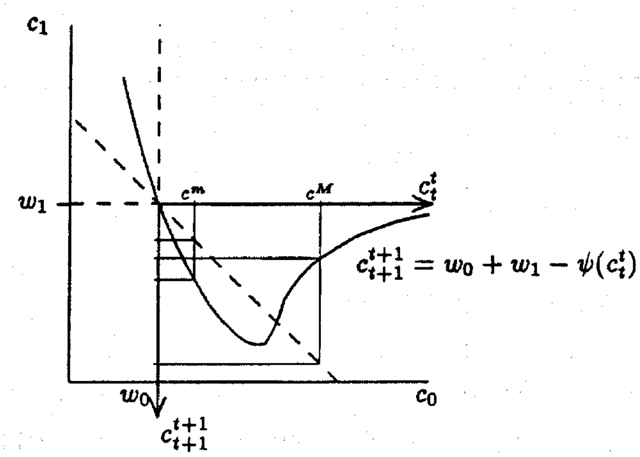
Corollary. *If moreover agents have a non zero endowment only in their second period of life – i.e.* $w_0 = 0$ *and* $w_1 > 0$*–, then all the economies of this class have k-SSE.*

The proof of this proposition and its corollary is given in the Appendix. The main point of the proposition –like in Grandmont (1986)– is that we just need to check a very simple geometrical condition on the offer curve of the representative agent to know something about the existence of k-SSE on this class of economies. In particular, for the case of k-SSE with indebted agents, we have to see if there exists an interval containing the images of its own extremes by the function implicitely defined by the offer curve between the first and second period consumptions. The characterization of k-SSE in the *Samuelsonian Case* in Grandmont (1986) led to a similarly simple condition, but there is nonetheless a difference worth to be noticed. While the value of the marginal rate of substitution between first and second period consumptions at the endowments point –the slope of the offer curve at (w_0, w_1)– did not suffice to determine the existence or not of k-SSE in the *Samuelsonian Case* (a sufficiently "backward bent" offer curve was also required), on the contrary, the value of this slope is enough by itself to determine the existence or not of k-SSE in the *Classical Case*.

In a simple OLG economy like the one described above –*i.e.* with "interior indifference curves"–, whenever the slope of the offer curve at the endowments point (w_0, w_1) is greater than one in absolute value there will exist k-SSE, and the corresponding allocations of resources are such that agents are borrowing in their first period of life against their future wealth. This can be easily seen (*cf.* Figure 3) from the fact that –with such preferences– the offer curve never reaches the axes, and consequently the interval (c^m, c^M) asked for in the proposition exists surely –just take its lower extreme greater than w_0 and the upper one smaller than $w_0 + w_1$ but as close to them as necessary respectively.

The corollary follows from the fact that, for economies with "interior indifference curves", if $w_0 = 0$ then the offer curve starts from the endowments point $(0, w_1)$ on the ordinate axis with vertical slope, and this implies immediately the fulfilment of the condition [13] given in the proposition. Hence, there is no "sunspot-proof" economy with such preferences and non-zero endowments only in the second period, in contrast with what happens in the symmetric case of non-zero endowments only in the first period. In that case, both "non-pathological" economies with and without k-SSE can be exhibited. Nevertheless, let us remark that in this case the "interior indifference curves" assumption is unnecesarily strong indeed, since any pattern of preferences giving an offer curve starting from the axis with a slope smaller than –1 and never reaching the abscissae axis will do for having k-SSE.

Figure 3



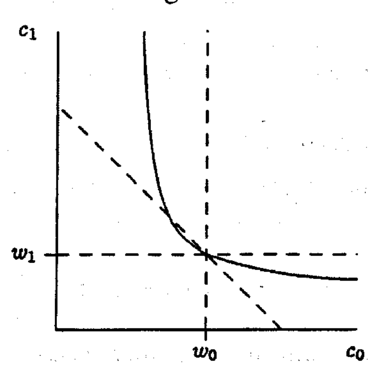
Example. Let us assume that preferences of the representative agent are described by the utility function

$$U(c_0,c_1) = \begin{cases} \ln c_0 + \ln c_1 & , \forall (c_0,c_1) \in \mathbb{R}^2_{++} \\ -\infty & , \text{if } c_0 = 0 \text{ or } c_1 = 0. \end{cases}$$

The corresponding offer curve is in this case easy to characterize, since the marginal rate of substitution between present and future consumptions at any point (c_0,c_1) is given by the ratio $-c_1/c_0$. It can be shown that, for any given endowments (w_0, w_1), the offer curve is a monotonous decreasing convex smooth curve going through (w_0, w_1) with slope $-w_1/w_0$ and the coordinate axis as asymptotes.

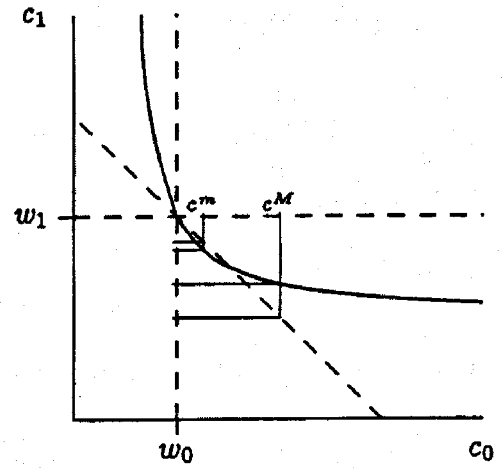
An immediate statement we can make about this economy is that it will not have k-SSE if $w_0>w_1$, but on the contrary there will exist k-SSE whenever $w_0<w_1$. This follows directly from the characterization of the existence of k-SSE in terms of the geometry of the offer curve . If $w_0>w_1$, neither the necessary and sufficient condition given in Grandmont (1986) nor the one given here is satisfied –the offer curve is not "sufficiently backward-bending", indeed it is not bent backwards at all (see figure 4).

Figure 4



On the contrary, if $w_0 < w_1$, the necessary and sufficient condition [13] given in the proposition above is fulfilled, as it can be seen in figure 5.

Figure 5



5. CONCLUDING REMARKS

In this article we have shown that in very general simple OLG economies, Stationary Sunspot Equilibria with finite support (k-SEE) –*i.e.* Rational Expectations Equilibria perfectly correlated to some extrinsic random signal fluctuating between a finite number of states– such that agents get indebted when "young", do exist much more "easily" than in the *Samuelsonian Case* with "positive money".

As it was noticed in the Introduction, this result is in fact a consequence of the Invariant Compact argument of Chiappori and Guesnerie (1991) which guarantees the existence of k-SSE in stochastic dynamical systems like

$$\tilde{Z}(x_t, \mu_{t+1}) = 0$$

–where μ_{t+1} is a probability distribution representing the expectations on x_{t+1}– whenever, for the associated deterministic system

$$Z(x_t, x_{t+1}) = 0$$

–like the dynamics [6] $c_{t+1}^{t+1*} = w_0 + w_1 - \psi(c_t^{t*})$ above– there exists a compact interval of values x_t in the state space of the system such that the values x_{t+1} for which the deterministic dynamics holds belong to the same interval. Notice that the existence of such a compact interval is equivalent to the geometrical condition imposed on the offer curve by the proposition given above. Nevertheless, the constructive strategy we have used to show the existence of k-SSE with "negative" money follows more closely from the economic nature of the model, and makes aware of the possibility of this phenomenon in a simpler way than the more abstract –and hence more general– argument of Chiappori and Guesnerie (1991).

Another interesting conclusion we can draw from the existence of k-SSE with "negative" money is the following one. Since these k-SSE are by no means related to the existence of an indeterminate steady state nor to the existence of deterministic cycles, then all the reasoning developped by Grandmont (1986) for stabilizing competitive business cycles –and hence k-SSE also, in the *Samuelsonian Case* that he considers– ceases to apply to this case, and the fiscal and monetary policies he designs would fail to eliminate the k-SSE in the *Classical Case*.

APPENDIX

The offer curve and the optimal consumptions

The consumption bundle which solves the representative agent's optimization program in the economy without uncertainty can also be characterized by the solution of the equivalent program

$$\max_{c_t^t} U_0(c_t^t)+U_1(w_1-\frac{p_t}{p_{t+1}}(c_t^t-w_0))$$
$$\text{subject to } 0<c_t^t<w_0+\frac{p_{t+1}}{p_t}w_1 \qquad \text{[A1]}$$

since the strict monotonicity of the utility function implies that the budgetary constraint will hold with equality at the solution, and the indifference curves contained in the strictly positive orthant $\mathbb{R}^2_{++}$ guarantee an interior solution. Hence the first period consumption c_t^{t*} solving this equivalent program is characterized by the condition

$$U_0'(c_t^{t*}) - \frac{p_t}{p_{t+1}}U_1'(w_1-\frac{p_t}{p_{t+1}}(c_t^{t*}-w_0))=0 \qquad \text{[A2]}$$

and the corresponding optimal second period consumption is

$$c_{t+1}^{t}{}^*=w_1-\frac{p_t}{p_{t+1}}(c_t^{t*}-w_0). \qquad \text{[A3]}$$

The conditions [A2] and [A3] can be rewritten as follows respectively, after rearranging terms and multiplying by p_{t+1},

$$p_{t+1}U_0'(c_t^{t*})-p_tU_1'(w_1-\frac{p_t}{p_{t+1}}(c_t^{t*}-w_0))=0 \qquad \text{[A4]}$$

$$p_{t+1}(c_{t+1}^{t}{}^*-w_1)=-p_t(c_t^{t*}-w_0). \qquad \text{[A5]}$$

Consequently, the values c_t^{t*} distinct from w_0 satisfying the condition [A4] are the same that those satisfying the following condition

$$p_{t+1}(c_t^{t*}-w_0)U_0'(c_t^{t*})-p_t(c_t^{t*}-w_0)U_1'(w_1-\frac{p_t}{p_{t+1}}(c_t^{t*}-w_0))=0 \qquad \text{[A6]}$$

what can be rewritten as follows, according to the condition [A5],

$$(c_t^{t*}-w_0)U_0'(c_t^{t*})-(c_{t+1}^{t}{}^*-w_1)U_1'(c_{t+1}^{t}{}^*)=0. \qquad \text{[A7]}$$

This is exactly the expression [4] of the main body the article

$$W_0(c_t^{t*})+W_1(c_{t+1}^{t}{}^*)=0 \qquad \text{[4]}$$

by the definition of the functions $W_i(\cdot)$.

Similarly, the (expected) utility maximizing current consumption and contingent second period consumptions of an agent "born" when the observed sunspot is ω^i can also be characterized by the solution of the equivalent program

$$\max_{c_0^i} U_0(c_0^i)+\sum_{j=1}^{k}\pi_{ij}U_1(w_1\frac{p^i}{p^j}(c_0^i-w_0))$$
$$\text{subject to } 0<c_0^i<\min_{j\in\{1,2,\dots,k\}}\left\{w_0+\frac{p^j}{p^i}w_1\right\} \qquad \text{[A8]}$$

because of the strict monotonicity of the utility function and the fact that indifference curves are contained in the strictly positive orthant $\mathbb{R}^2_{++}$ also. The previous reasoning can be redone step by step to arrive to the expression

$$(c_0^{i*} - w_0) U_0'(c_0^{i*}) - \sum_{j=1}^{k} \pi_{ij} (c_1^{j*} - w_1) U_1'(c_1^{j*}) = 0 \qquad \text{[A9]}$$

what is once more exactly equivalent to the expression [10] of the main body of the article

$$W_0(c_t^{t*}) + \sum_{j=1}^{k} \pi_{ij} W_1(c_{t+1}^{t}{}^*) = 0 \qquad [10]$$

according to the definition of the functions $W_i(\cdot)$.

The functions $W_i(\cdot)$.

In main part of the article, we have made use of the functions defined as follows

$$\forall i \in \{0,1\}\ ,\ W_i(x) \equiv (x - w_i) U_1'(x).$$

Just one property of this function is used in the reasoning hereafter: $W_i(\cdot)$ is strictly increasing –and hence its inverse does exist– in the interval $(0, w_i)$, *i.e.*

$$\forall x \in (0, w_i)\ ,\ W_i'(x) = U_i'(x) + (x - w_i) U_1''(x) > 0. \qquad \text{[A10]}$$

The Proof.

Proof of the proposition. Assume that, for a simple OLG economy like the one described in section 2, a k-SSE in which agents are indebted in the first period exists. Its allocation of resources is then characterized by a k-tuple of sunspot-contingent consumptions in the first first period $c_0^{1*}, c_0^{2*},\dots, c_0^{k*}$ and a k-tuple of consumptions in the second one $c_1^{1*}, c_1^{2*},\dots, c_1^{k*}$ such that

$$\forall i \in \{1, 2,\dots, k\},\ W_0(c_0^{i*}) + \sum_{j=1}^{k} \pi_{ij} W_1(c_1^{j*}) = 0 \qquad \text{[A11]}$$

$$\text{and } \forall h \in \{1, 2,\dots, k\}\ ,\ c_0^{h*} + c_1^{h*} = w_0 + w_1 \qquad \text{[A12]}$$

as we have just seen at the beginning of this Appendix.

Hence there exist two distinct values c_1^{r*} and c_1^{s*} among the sunspot-contingent second period consumptions $c_1^{1*}, c_1^{2*},\dots, c_1^{k*}$ such that

$$\forall i \in \{1, 2,\dots, k\}\ ,\ W_1(c_1^{r*}) < -\mathrm{W}_0(c_0^{i*}) < W_1(c_1^{s*}). \qquad \text{[A13]}$$

But since the second period consumptions are those of a k-SSE in which agents get indebted –*i.e.* they consume more than their endowment in the first period–, then they must satisfy all

$$\forall i \in \{1, 2,\dots, k\}\ ,\ 0 < c_1^{i*} < w_1$$

what implies, from the strictly increasing behaviour of the function $W_i(\cdot)$ in the interval $(0, w_1)$ –see [A10] above–, that for all the members of the inequalities [A13] above the pre-image $W_1^{-1}(\cdot)$ is well defined and the following inequalities then hold

$$\forall i \in \{1, 2,\dots, k\}\ ,\ c_1^{r*} < \psi(c_0^{i*}) < c_1^{s*}\ ,$$

where $\psi(\cdot)$ is $W_1^{-1}(W_0(\cdot))$.

Taking into account the feasibility conditions [A12] on the allocation of resources, these inequalities can be rewritten as

$$\forall i \in \{1, 2,\dots, k\}\ ,\ c_0^{s*} < w_0 + w_1 - \psi(c_0^{i}*) < c_0^{r*}\ ,$$

which implies the condition [13] we were looking for if c_0^{s*} and c_0^{r*} are renamed as c^m and c^M respectively.

Conversely, if there exist two positive values c^m and c^M in the interval $(w_0\, ,\ w_0 + w_1)$ satisfying

$$\forall i \in \{m, M\}\ ,\ c^m < w_0 + w_1 - \psi(c^i) < c^M$$

then, by continuity of the function $\psi(\cdot)$, we can choose an arbitrary finite number k of distinct "first period consumptions" $c_0^{1}*$, $c_0^{2}*$,..., $c_0^{k}*$ clustered around both c^m and c^M and close enough to them [4] for the following condition to hold for some c_0^{r*} and c_0^{s*} among them,

$$\forall i \in \{1, 2,\dots, k\}\ ,\ c_0^{s*} < w_0 + w_1 - \psi(c_0^{i}*) < c_0^{r*}\,.$$

Hence, besides these first period consumptions $c_0^{1}*$, $c_0^{2}*$,..., $c_0^{k}*$, there exist also a k-tuple of second period consumptions $c_1^{1}*$, $c_1^{2}*$,..., $c_1^{k}*$ such that

$$\forall h \in \{1, 2,\dots, k\}\ ,\ c_0^{h*} + c_1^{h*} = w_0 + w_1\,.$$

and

$$\forall i \in \{1, 2,\dots, k\}\ ,\ W_1(c_1^{r*}) < -W_0(c_0^{i}*) < W_1(c_1^{s*}).$$

after rearranging terms and composition with function $W_1(\cdot)$. It is then immediate the existence of a Markov matrix Π such that the conditions

$$\forall i \in \{1, 2,\dots, k\}\ ,\ W_0(c_0^{i}*) + \sum_{j=1}^{k} \pi_{ij} W_1(c_1^{j}*) = 0$$

$$\text{and } \forall h \in \{1, 2,\dots, k\}\ ,\ c_0^{h*} + c_1^{h*} = w_0 + w_1\,.$$

characterizing a k-SSE are satisfied. □

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[4] *i.e.*, any c_0^{i*} is within an arbitrary small distance from c^m and c^M, and not all of them are gathered around only one of these points; just consider k slight perturbations of c^m and c^M.

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