A MUTUALLY EXCITING ROUGH JUMP DIFFUSION FOR FINANCIAL MODELLING

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A mutually exciting rough jump diffusion for financial modelling

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This article introduces a new class of diffusive processes with rough mutually exciting jumps for modelling financial asset return. The novel feature is that the memory of positive and negative jump processes is defined by the product of a dampening factor and a kernel involved in the construction of the rough Brownian motion. The jump processes are nearly unstable because their intensity diverges to $+\infty$ for a brief duration after a shock. We first infer the stability conditions and explore the features of the dampened rough (DR) kernel, which defines a fractional operator, similar to the Riemann-Liouville integral. We next reformulate intensities as infinite dimensional Markov processes. Approximating these processes by discretization and then considering the limit allows us to retrieve the Laplace transform of asset log-return. We show that this transform depends on the solution of a particular fractional integro-differential equation. We also define a family of changes of measure that preserves the features of the process under a risk neutral measure. We next develop an econometric estimation procedure based on the peak over threshold (POT) method. To illustrate this work, we fit the mutually exciting rough jump diffusion to time-series of Bitcoin log-returns and compare the goodness of fit to its non-rough equivalent. Finally, we analyze the influence of roughness on option prices.

Keywords: self-exciting process, Epidemic Type Aftershock Sequence (ETAS), jump-diffusion, fractional Brownian motion, Riemann-Liouville fractional integral.

1 Introduction

The propensity of financial price jumps to cluster is abundantly documented in the literature. The phenomenon is specifically studied and evidenced by Yu [34] and Maheu and McCurty who respectively consider DJIA and individual stocks returns. More recently, Aït-Sahalia et al. [1] explore international equity market indices on a daily basis and conclude that Jump clustering in time is a strong effect for equity market indices.

This jump clustering phenomenon questions the relevance of classical jump-diffusion dynamics for modelling asset prices. With no surprise, a recent strand of the literature makes significant efforts to develop quantitative methods to model the clustering of jumps and to investigate implications of it for asset or option pricing. A natural way to capture clustering of jumps consists to use self-exciting point processes for which the jump arrival intensity depends on the number

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and sometimes the size of previous shocks on asset log-returns as in Hainaut and Moraux [18]. This approach is linked to the Hawkes self-exciting processes (see Hawkes [23, 24]). This type of point processes is also commonly used for modelling high-frequency data. Readers interested by high-frequency applications may consult Giot [13], Bowsher [4], Chavez-Demoulin and McGill [5], Bacry et al. [2], Da Fonseca and Zaatour [8] or Hainaut and Goutte [17], for more recent contributions. The literature on financial applications of self-exciting processes is vast and we refer to Hawkes [26] for a detailled review.

In a standard self-exciting model, the jump intensity increases after a shock and revert next to a baseline level. The speed of reversion is determined by a memory kernel. This category of model is also called "Epidemic Type Aftershock Sequence" (ETAS) and is used for modelling earthquakes as in Hawkes and Oakes [25]. In the most common specification, the memory kernel is exponential decreasing. In this particular case, the jump process is Markov and we can rely on Itô's calculus to find an analytical expression of its Laplace transform. In a general setting, we lose the analytical tractability offered by stochastic calculus, apart from moments or asymptotic properties. For instance, Muzy et al. [29] find the moments of stationary processes and their limit behaviour. Stabile and Torrisi [32] study the asymptotic behavior of non-stationary Hawkes process. Cristofaro et al. [7] propose a fractional differential equation that governs the intensity rate of self-exciting process by using the Caputo fractional derivative. Hainaut [19] finds the Laplace transform of self-exciting claim processes for memory functions with a spectral representation.

Recently, Jaisson and Rosenbaum [27] have observed that nearly unstable Hawkes processes fit well high-frequency financial time series. Under certain conditions, this unstable process asymptotically behaves as a Brownian Volterra process with a kernel, $k(u) = u^{\alpha-1}E_{\alpha,\alpha}(-u^{\alpha})$, where $\alpha \in (0, 1]$ and $E_{\alpha,\alpha}$ is the 2 parameters Mittag-Leffler function. Based on this, Chen et al. [6] and Habyarimana et al. [16] use this kernel for defining the fractional Hawkes process. This kernel diverges at zero but the jump process remains stable. In this paper, we combine to a diffusion, an alternative mutually exciting process with a kernel diverging at zero, directly inspired from the literature on fractional/rough Brownian motions and from Hainaut et al. [21].

The fractional Brownian motion (fBm) has dependent increments contrary to the Brownian motion (Bm). This dependence is measured by the Hurst index, noted $H \in (0, 1)$. A value of H greater (resp. lower) than 1/2 corresponds to positive (resp. negative) correlation between increments. For H =1/2, we retrieve the Bm with independent increments (for details, see e.g. Hainaut [20], chapter 6). fBm's with $H < 1/2$ display highly volatile sample paths and are called rough for this reason. At time $t > 0$, the rough fBm admits an integral representation with respect to a Bm: $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dW_s$ where $\alpha \in (0, 1]$ and W_t is a Bm. This stochastic integral is well-defined even if the rough kernel $\frac{u^{\alpha-1}}{\Gamma(\alpha)}$, diverges when $u \to 0$. FBm's and Rough processes recently received a great deal of attention in the literature. Guo et al. [14] study analytically and numerically the fractional Langevin equation driven by the fractional Brownian motion. Zeng et al. [35] deal with the stability problem of the fractional order Black-Scholes model driven by fractional Brownian motion. Gatheral et al. [12] proposed a model in which the variance of log-prices is driven by a rough Brownian motion, the rough Heston. Xu and Zhou [33] evaluate perpetual American put options when the underlying asset price follows a sub-mixed fractional Brownian motion. The properties of the rough Heston model are studied by El Euch and Rosenbaum [9] and [10], who find the characteristic function of the rough process based on fractional Riccati equations.

In this work, we combine a diffusion and a bivariate jump process for positive and negative shocks with mutual excitation. The novel feature is that kernels of jump processes are rough as for a fBm, but dampened by an exponential decaying function to ensure their stability. This new process, called "mutually exciting rough jump diffusion" (MERJD), presents several interesting features for modelling prices at high frequency or highly volatile assets such as cryptocurrencies. We first show in Section 2 that dampened rough (DR) kernel is a Sonine function and therefore admits a conjugate kernel. This property allows us to obtain analytical expressions of expected intensities and to find the stability conditions of the bivariate jump process. The MERJD admits an equivalent infinite dimensional Markov representation, presented in Section 3. We can approximate the Laplace transform of the MERJD by discretizing this representation. Considering the limit of the finite dimensional approximation leads to the Laplace transform of the MERJD in Section 4. In a similar manner to El Euch and Rosenbaum [10], this transform depends on the solution of a fractional differential equation (FDE). This FDE involves an operator based on the DR kernel and similar to the left fractional Riemann-Liouville (RL) integral. As detailed in Section 5, there exists a family of changes of measure preserving the features of the process under a risk neutral measure. This makes the MERJD adapted for pricing financial derivatives. In Section 6, we show that the log-likelihood of rough mutually exciting jump processes admits a closed form expression. This is combined to a POT method for estimating MERJD parameters from the time series of Bitcoin hourly returns. We conclude by an analysis of the influence of roughness on option prices.

2 A dampened rough kernel

We consider an asset price process, denoted by $(S_t)_{t\geq 0}$ defined on a probability space Ω endowed with the natural filtration $(\mathcal{F}_t)_{t\geq 0}$ of all processes involved in the dynamic of S_t and with a probability measure $\mathbb{P}$. The log-return $X_t = \ln \frac{S_t}{S_0}$ is ruled by a mutually exciting rough jump diffusion (MERJD) that is defined as follows

$$X_t = \left(\mu - \frac{\sigma^2}{2}\right) t + \sigma W_t + \sum_{j=1}^{2} \left(L_t^{(j)} - \mu_j \int_0^t \lambda_s^{(j)} ds \right) , \tag{1}$$

where $\left(L_t^{(1)}\right)_{t\geq 0}$ and $\left(L_t^{(2)}\right)_{t\geq 0}$ are respectively positive and negative point processes, with intensities $\left(\lambda_t^{(1)}\right)_{t\geq 0}$ and $\left(\lambda_t^{(2)}\right)_{t\geq 0}$. $(W_t)_{t\geq 0}$ is a Brownian motion and $\mu, \sigma \in \mathbb{R}^2$. $\mu_1 \in \mathbb{R}^+$ and $\mu_2 \in \mathbb{R}^-$ are respectively the expected jumps of $L_t^{(1)}$ and $L_2^{(2)}$. These point processes are the sum of random increments, noted $J_k^{(1)}$ and $J_k^{(2)}$,

$$L_t^{(j)} = \sum_{k=1}^{N_t^{(j)}} J_k^{(j)} , \; j = 1, 2 .$$

where $\left(N_t^{(j)}\right)_{t\geq 0}$ is the number of shocks observed up to time t. The statistical distributions of $J_k^{(j)} \sim J^{(j)}$ for $j = 1, 2$, are denoted by $m^{(j)}(.)$. $J^{(1)}$ and $J^{(2)}$ are respectively defined on $(\mathbb{R}^+, \mathcal{B}(\mathbb{R}^+))$ and $(\mathbb{R}^-, \mathcal{B}(\mathbb{R}^-))$. The jump expectations are $\mu_j = \mathbb{E}\left(J^{(j)}\right)$ and the moment generating function are $\mathcal{J}_j(\omega) = \mathbb{E}\left(e^{\omega J^{(j)}}\right)$ for $j = 1, 2$.

In the numerical illustration, we consider MERJD with positive and negative exponentially distributed jumps. The probability density functions (pdf's) of $J^{(1)}$ and $J^{(2)}$ are respectively equal to

$$m^{(1)}(z) = \rho_1 e^{-\rho_1 z} 1_{\{z\geq 0\}} , \, m^{(2)}(z) = -\rho_2 e^{-\rho_2 z} 1_{\{z\leq 0\}} . \tag{2}$$

where $\rho_1 \in \mathbb{R}^+$ and $\rho_2 \in \mathbb{R}^-$. In this particular case, $\mu_j = \frac{1}{\rho_j}$ and $\mathcal{J}_j(\omega) = \frac{\rho_j}{\rho_j - \omega}$ for $j = 1, 2$, with $\omega \leq \rho_1$ and $\omega \geq \rho_2$.

The intensities depend upon the past realizations of counting processes $\left(N_t^{(j)}\right)_{t\geq 0}$ for $j = 1, 2$ in the following way:

$$\begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \int_0^{t-} e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s^{(1)} \\ \int_0^{t-} e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s^{(2)} \end{pmatrix} \tag{3}$$

where $\alpha \in (0, 1]$, β, $\eta_{i,j} \in \mathbb{R}^+$ for $i, j \in \{1, 2\}$. We will discuss later the conditions ensuring the stability of these processes. The functions $k(u) = e^{-\beta u} \frac{u^{\alpha-1}}{\Gamma(\alpha)}$ is called the memory kernel. It is the product of a dampening term, $e^{-\beta u}$, and of the rough kernel, $\frac{u^{\alpha-1}}{\Gamma(\alpha)}$. This kernel is called "rough" because its is involved in the construction of the rough Brownian motion (rBm). This rBm is defined as an integral $\int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} dW_s$ where $(W_s)_{s\geq 0}$ is a Brownian motion. As shown in next paragraphs, the dampening factor is needed to avoid the divergence of intensities. When $\alpha = 1$, we retrieve a standard bivariate Hawkes process, with an exponential kernel. We assume that the dampening and roughness parameters (β and α) are common to positive and negative jumps. This assumption may appear restrictive but is necessary to maintain the analytical tractability in later developments.

Before further studying the properties of the MERJD, we focus on the features of the dampened rough kernels. They belong to the class of Sonine functions [31] and they define operators similar to left fractional Riemann-Liouville integrals (defined in Equation 60 of Appendix A).

Definition 1. The kernel $k(u) \in L^1_{loc}(\mathbb{R}^+)$ is a Sonine function if there exists a conjugate kernel $l(u) \in L^1_{loc}(\mathbb{R}^+)$ such that

$$\int_0^t l(t-u)\, k(u)\, du = 1 \,, \; \forall t \geq 0. \tag{4}$$

Let $\phi \in L^1(\mathbb{R}^+)$, the Sonine operators associated to $k(u)$ and $l(u)$ are defined as

$$\begin{aligned} (K\phi)(t) &= \int_0^t k\,(t-u)\, \phi\,(u)\, du \,, \; \forall t \geq 0, \\ (L\phi)(t) &= \int_0^t l\,(t-u)\, \phi\,(u)\, du \,, \; \forall t \geq 0. \end{aligned} \tag{5}$$

We observe a similarity between the Riemann-Liouville integral and $K\phi = \int_0^t e^{-\beta(t-s)} \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds$

$$I_{0+}^{\alpha} \phi = \int_0^t \frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)} \phi(s) ds \,, \tag{6}$$

The operator K is called in the rest of the paper, the dampened Riemann-Liouville (RL) integral. Remark that kernels $k(u)$ and $l(u)$ are necessary unbounded as $u \to 0$. We have then to find conditions on the kernel to guarantee the existence of the integral of L_p functions. The kernel $k(u)$ has the form

$$k(u) = \frac{g(u)}{u^{1-\alpha}} \quad , \quad x > 0 \,, \sup_{u \geq 0} |g(u)| < \infty \,,$$

where $g(u) = \frac{1}{\Gamma(\alpha)} e^{-\beta\, u}$ is a bounded function and $\alpha \in [0, 1)$. From the Hardy-Littlewood [22] Sobolev inequality, a sufficient condition is $\alpha < 1/p$. In this case, the operator acts from $L_p(\mathbb{R})$, $1 < p < 1/\alpha$ into $L_q(\mathbb{R})$ where $1/q = 1/p - \alpha$. In the rest of this article, we consider L_1-integrands which guarantees that $K\phi$ is well defined for $\alpha \in [0,1)$. We refer to Samko and Cardoso [30] for the necessary conditions of existence of other integrals with general Sonine kernels.

We denote by $(\mathcal{L}\phi)(z) = \int_0^\infty e^{-zu}\phi(u)\,du$, the Laplace transform of a function, $\phi \in L^1(\mathbb{R}^+)$. We infer by direct calculation the Laplace transform of the kernel:

$$(\mathcal{L}k)(z) \quad = \quad \frac{1}{(\beta+z)^\alpha}. \tag{7}$$

By parts integration, we find that the Laplace transform of the dampened RL integral of a function $\phi(t)$ is

$$(\mathcal{L}(K\phi))(s) = \frac{\mathcal{L}(\phi)(s)}{(z+\beta)^\alpha}. \tag{8}$$

Furthermore, the Sonine condition (4) is rewritten in terms of Laplace transforms of $k(.)$ and $l(.)$:

$$(\mathcal{L}k)(z)(\mathcal{L}l)(z) \quad = \quad \frac{1}{z}. \tag{9}$$

This last relation allows us to prove the next result.

Proposition 1. *The conjugate kernel $l(.)$ of $k(.)$ satisfying condition (4), is given by*

$$l(u) = \beta^\alpha + \frac{\alpha}{\Gamma(1-\alpha)}\int_u^\infty \frac{e^{-\beta s}}{s^{1+\alpha}}ds\,. \tag{10}$$

Proof. We check that the Laplace transform of $l(.)$ satisfies the condition (9). We first integrate by parts the integral in Eq. (10):

$$\alpha\int_u^\infty \frac{e^{-\beta s}}{s^{1+\alpha}}ds \quad = \quad e^{-\beta u}u^{-\alpha} - \beta\int_u^\infty e^{-\beta s}s^{-\alpha}\,du\,.$$

$(\mathcal{L}l)(z)$ is next rewritten as the sum:

$$\begin{aligned}(\mathcal{L}l)(z) = \beta^\alpha\int_0^\infty e^{-zu}du + \frac{1}{\Gamma(1-\alpha)}\int_0^\infty e^{-(z+\beta)u}u^{-\alpha}du \\ -\frac{\beta}{\Gamma(1-\alpha)}\int_0^\infty\int_u^\infty e^{-\beta s}e^{-zu}s^{-\alpha}\,ds\,du\,.\end{aligned} \tag{11}$$

After a change of variable $v = (z+\beta)u$, we obtain that

$$\int_0^\infty e^{-(z+\beta)u}u^{-\alpha}du = (z+\beta)^{\alpha-1}\Gamma(1-\alpha)\,.$$

Using the Fubini's theorem, we change the order of integration in the last integral of Equation (11). We deduce that

$$\begin{aligned}&\int_0^\infty\int_u^\infty e^{-\beta s}e^{-zu}s^{-\alpha}\,ds\,du \\ &\quad= \int_0^\infty e^{-\beta s}s^{-\alpha}\,ds - \frac{1}{z}\int_0^\infty e^{-(\beta+z)s}s^{-\alpha}\,ds \\ &\quad= \frac{\beta^{\alpha-1}}{z}\Gamma(1-\alpha) - \frac{(\beta+z)^{\alpha-1}}{z}\Gamma(1-\alpha)\,.\end{aligned}$$

Combining previous results allows us to find that the Laplace transform of the conjugate kernel of $k(.)$ is $(\mathcal{L}l)(z) = \frac{(\beta+z)^\alpha}{z}$, which satisfies the condition (9). $\square$

The dampened RL integral $(K\phi)(t)$ admits an inverse operator, presented in the following proposition and comparable to a fractional derivative.

Proposition 2. *The inverse operator of the dampened RL integral K, is the derivative of its conjugate kernel. For $\phi \in L^1(\mathbb{R}^+)$, it is equal to*

$$\begin{aligned} \left(K^{-1}\phi\right)(t) &= \frac{d}{dt}\left(L\phi\right)(t) \\ &= \frac{d}{dt}\int_0^t l\left(t-u\right)\phi\left(u\right)du\,. \end{aligned} \tag{12}$$

This inverse operator is called the dampened RL derivative.

Proof. This result is a consequence of the Sonine condition. We apply the operator L to $K\phi$ and permut the order of integration. If we do the change of variable $v = s - u$, we have that

$$\begin{aligned} \left(LK\phi\right)(t) &= \int_0^t l(t-s)\int_0^s k(s-u)\phi(u)\,du\,ds \\ &= \int_0^t \phi(u)\int_0^{t-u} l(t-s)k(s-u)\,ds\,du \\ &= \int_0^t \phi(u)\int_0^{t-u} l(t-u-v)k(v)\,dv\,du \end{aligned}$$

From the Sonine condition (4), the inner integral is equal to 1. Deriving both sides with respect to t, leads to the conclusion, $K^{-1}\phi = \frac{d}{dt}\left(L\phi\right)$. □

The dampened RL integral and derivative will be latter involved in the construction of the Laplace transform of the MERJD, in Section 4. Before we determine the expected intensities from which we infer conditions of stability. The next result implies the Mittag-Leffler function with one and two parameters. We refer to Appendix A for a summary of its properties.

Proposition 3. *Expected intensities at time $t \geq 0$ conditionally to the filtration $\mathcal{F}_0$, are given by*

$$\mathbb{E}_0\begin{pmatrix}\lambda_t^{(1)}\\ \lambda_t^{(2)}\end{pmatrix} = \begin{pmatrix}\lambda_0^{(1)}\\ \lambda_0^{(2)}\end{pmatrix} + \begin{pmatrix}\eta_{11} & \eta_{12}\\ \eta_{21} & \eta_{22}\end{pmatrix}\begin{pmatrix}\left(Ku_1\right)(t)\\ \left(Ku_2\right)(t)\end{pmatrix}.$$

where $\left(Ku_1\right)(t)$ and $\left(Ku_2\right)(t)$ are equal to

$$\begin{cases} \left(Ku_1\right)(t) &= \int_0^t \left(\lambda_0^{(1)} + \eta_{12}\left(Ku_2\right)(s)\right)e^{-\beta(t-s)}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\eta_{11}(t-s)^{\alpha}\right)\,ds\,, \\ \left(Ku_2\right)(t) &= \int_0^t \left(\lambda_0^{(2)} + \eta_{21}\left(Ku_1\right)(s)\right)e^{-\beta(t-s)}(t-s)^{\alpha-1}E_{\alpha,\alpha}\left(\eta_{22}(t-s)^{\alpha}\right)\,ds\,. \end{cases} \tag{13}$$

Proof. Let us denote $u_j(t) = \mathbb{E}_0\left(\lambda_t^{(j)}\right)$ for $j = 1, 2$. From Eq (3) and by definition of the dampened RL integral, we have

$$\begin{pmatrix}u_1(t)\\ u_2(t)\end{pmatrix} = \begin{pmatrix}\lambda_0^{(1)}\\ \lambda_0^{(2)}\end{pmatrix} + \begin{pmatrix}\eta_{11} & \eta_{12}\\ \eta_{21} & \eta_{22}\end{pmatrix}\begin{pmatrix}\left(Ku_1\right)(t)\\ \left(Ku_2\right)(t)\end{pmatrix}. \tag{14}$$

From Eq. (8), the Laplace transform of Ku_j , for $j = 1, 2$, is equal to $(s+\beta)^{-\alpha}\,U_j(s)$ where $U_j(s) = \int_0^{\infty} e^{-s\,t}u_j(t)\,dt$. Therefore, the Laplace transform of the system (14) is equal to

$$\begin{cases} U_1(s) &= \frac{\lambda_0^{(1)}}{s} + \eta_{11}\left(s+\beta\right)^{-\alpha}U_1(s) + \eta_{12}\left(s+\beta\right)^{-\alpha}U_2(s)\,, \\ U_2(s) &= \frac{\lambda_0^{(2)}}{s} + \eta_{21}\left(s+\beta\right)^{-\alpha}U_1(s) + \eta_{22}\left(s+\beta\right)^{-\alpha}U_2(s)\,. \end{cases}$$

Rearranging terms, allows us to express $U_1(s)$ and $U_2(s)$ as follows

$$\begin{cases} \frac{U_1(s)}{(s+\beta)^\alpha} & = \frac{\lambda_0^{(1)}}{s}\frac{1}{((s+\beta)^\alpha-\eta_{11})} + \frac{\eta_{12}}{((s+\beta)^\alpha-\eta_{11})}\frac{U_2(s)}{(s+\beta)^\alpha}\,, \\ \frac{U_2(s)}{(s+\beta)^\alpha} & = \frac{\lambda_0^{(2)}}{s}\frac{1}{((s+\beta)^\alpha-\eta_{22})} + \frac{\eta_{21}}{((s+\beta)^\alpha-\eta_{22})}\frac{U_1(s)}{(s+\beta)^\alpha}\,. \end{cases} \tag{15}$$

Given that $\frac{U_j(s)}{(s+\beta)^\alpha} = \mathcal{L}\left(Ku_j\right)(s)$ and $\mathcal{L}\left(e^{-\beta t}t^{\alpha-1}E_{\alpha,\alpha}(\pm\eta t^\alpha)\right)(s) = \frac{1}{(s+\beta)^\alpha\mp\eta}$ we obtain the system (13). ☐

The intensities are by construction unstable since the DR kernel diverges at origin. The next proposition presents the conditions that ensure the existence of counting processes.

Proposition 4. *If the parameters defining $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$ fulfill the following three conditions*

$$\begin{array}{c} \beta^\alpha \geq \eta_{11} \quad , \quad \beta^\alpha \geq \eta_{22}\,, \\ (\beta^\alpha-\eta_{11})(\beta^\alpha-\eta_{22}) \geq \eta_{12}\eta_{21}\,, \end{array} \tag{16}$$

the expected intensities admit a limit when $t\to\infty$, that are:

$$\begin{pmatrix} \lambda_\infty^{(1)} \\ \lambda_\infty^{(2)} \end{pmatrix} = \lim_{t\to\infty}\mathbb{E}_0\begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \frac{\lambda_0^{(1)}(\beta^\alpha-\eta_{22})\beta^\alpha+\eta_{12}\lambda_0^{(2)}\beta^\alpha}{(\beta^\alpha-\eta_{11})(\beta^\alpha-\eta_{22})-\eta_{12}\eta_{21}} \\ \frac{\lambda_0^{(2)}(\beta^\alpha-\eta_{11})\beta^\alpha+\eta_{21}\lambda_0^{(1)}\beta^\alpha}{(\beta^\alpha-\eta_{11})(\beta^\alpha-\eta_{22})-\eta_{12}\eta_{21}} \end{pmatrix}. \tag{17}$$

Proof. From Equations (15), we deduce that the Laplace transform $U_1(s)$ of $\mathbb{E}_0\left(\lambda_t^{(1)}\right)$ is equal to

$$\begin{aligned} \frac{U_1(s)}{(s+\beta)^\alpha} &= \frac{\lambda_0^{(1)}}{s\left((s+\beta)^\alpha-\eta_{11}\right)} + \frac{\eta_{12}\lambda_0^{(2)}}{s\left((s+\beta)^\alpha-\eta_{11}\right)\left((s+\beta)^\alpha-\eta_{22}\right)} \\ &\quad + \frac{\eta_{12}\eta_{21}U_1(s)}{\left((s+\beta)^\alpha-\eta_{11}\right)\left((s+\beta)^\alpha-\eta_{22}\right)(s+\beta)^\alpha}\,. \end{aligned}$$

If we isolate $U_1(s)$, the Laplace transform of $\mathbb{E}_0\left(\lambda_t^{(1)}\right)$ is

$$U_1(s) = \frac{\lambda_0^{(1)}\left((s+\beta)^\alpha-\eta_{22}\right)(s+\beta)^\alpha+\eta_{12}\lambda_0^{(2)}(s+\beta)^\alpha}{s\left(\left((s+\beta)^\alpha-\eta_{11}\right)\left((s+\beta)^\alpha-\eta_{22}\right)-\eta_{12}\eta_{21}\right)}\,.$$

Using the following property of Laplace transform allows us to retrieve $\lambda_\infty^{(1)}$:

$$\begin{aligned} \lim_{t\to\infty}u_1(t) &= \lim_{s\to0^+}sU_1(s) \\ &= \frac{\lambda_0^{(1)}\left(\beta^\alpha-\eta_{22}\right)\beta^\alpha+\eta_{12}\lambda_0^{(2)}\beta^\alpha}{\left(\beta^\alpha-\eta_{11}\right)\left(\beta^\alpha-\eta_{22}\right)-\eta_{12}\eta_{21}}\,. \end{aligned}$$

$\lambda_\infty^{(2)}$ is obtained in the same manner. By definition, these expectations must be positive and bounded and therefore conditions (16) must be fulfilled. ☐

By construction, the MERJD is not a Markov process. Nevertheless, we can rewrite the model as an infinite dimensional Markov process because the power $x^{\alpha-1}$ has an integral representation. This step is detailed in the next proposition.

Proposition 5. *For $j=1,2$, let us consider a family of auxiliary jump processes $Z_t^{(j,\xi)}$, indexed by $\xi\in\mathbb{R}^+$:*

$$Z_t^{(j,\xi)} = \int_0^t e^{-(\beta+\xi)(t-s)}dN_s^{(j)}\,. \tag{18}$$

Le us denote $\gamma(d\xi) := \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi$ for $\xi \geq 0$. The intensities $\lambda_t^{(j)}$ are expressed as integrals of $Z_t^{(j,\xi)}$ with respect to $\gamma(d\xi)$:

$$\begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} = \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(1,\xi)} \gamma(d\xi) \\ \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(2,\xi)} \gamma(d\xi) \end{pmatrix}. \tag{19}$$

Proof. We check by direct integration that $x^{\alpha-1}$ is the following integral

$$x^{\alpha-1} = \int_0^\infty e^{-x\xi} \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi \,.$$

By construction, the process $Z_t^{(j,\xi)}$ is an Ornstein-Uhlenbeck jump process reverting toward 0, driven by the dynamic

$$dZ_t^{(j,\xi)} = -(\beta+\xi) Z_t^{(j,\xi)} dt + dN_t^{(j)} \,. \tag{20}$$

This process has a finite expectation for all $\xi \in \mathbb{R}^+$ and we can use the Fubini's theorem to rewrite the intensities as follows:

$$\begin{aligned} \begin{pmatrix} \lambda_t^{(1)} \\ \lambda_t^{(2)} \end{pmatrix} &= \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \int_0^{t-} \int_0^\infty e^{-(\xi+\beta)(t-s)} \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi \, dN_s^{(1)} \\ \int_0^{t-} \int_0^\infty e^{-(\xi+\beta)(t-s)} \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi \, dN_s^{(2)} \end{pmatrix} \\ &= \begin{pmatrix} \lambda_0^{(1)} \\ \lambda_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(1,\xi)} \gamma(d\xi) \\ \frac{1}{\Gamma(\alpha)} \int_0^\infty Z_t^{(2,\xi)} \gamma(d\xi) \end{pmatrix}. \end{aligned}$$

□

By construction, the multivariate processes $\left(\lambda_t^{(1)}, \lambda_t^{(2)}, L_t^{(1)}, L_t^{(2)}, \left(Z_t^{(j,\xi)}\right)_{j\in\{1,2\},\,\xi\in\mathbb{R}^+}\right)$ is Markov. A similar rewriting of a non-Markov Hawkes process is used in Hainaut [19] for processes with kernels having a spectral representation. The main differences with this article are that considered kernels do not diverge at the origin and that equivalent processes to $Z_t^{(j,\xi)}$ are defined in the complex plane. We will also check that the divergence at the origin of the dampened rough kernel prevent us to express the Laplace transform of the MERJD in terms of backward ordinary equations.

3 Finite dimensional approximation

In this section, we approximate the integral in (19) on a finite grid of processes $Z_t^{(j,\xi)}$. This method allows us to find the Laplace transform of the log-return $(X_t)_{t\geq 0}$, using Itô's calculus. For this purpose, we approach $\gamma(.)$ by a discrete measure on a finite numbers of atoms. We consider a partition $\mathcal{E}^{(n)} := \{0 < \xi_0^{(n)} < \xi_1^{(n)} < ... < \xi_n^{(n)} < \infty\}$. The mid point of the interval $(\xi_l^{(n)}, \xi_{l+1}^{(n)})$ is

$$b_l = \frac{\xi_l^{(n)} + \xi_{l+1}^{(n)}}{2} \,, l \in \{0, ..., n-1\} \tag{21}$$

The mass of atoms at b_l, is defined as the integral of $\gamma(.)$ over the interval :

$$w_l = \int_{\xi_l^{(n)}}^{\xi_{l+1}^{(n)}} \gamma(dz) \,, l \in \{0, ..., n-1\} \,. \tag{22}$$

If δ_{b_l} is the Dirac measure located at point b_l, the discrete measure $\gamma_n(.)$ approximating $\gamma(.)$ is defined as

$$\gamma_n(d\xi) = \sum_{l=0}^{n-1} w_l \delta_{b_l}(\xi)\, d\xi.$$

In a similar manner to [3], the partition $\mathcal{E}^{(n)}$ satisfies three conditions:

1. $\lim_{n\to\infty} \xi_0^{(n)} = 0$ and $\lim_{n\to\infty} \xi_n^{(n)} = \infty$,
2. $\lim_{n\to\infty} \max |\xi_{i+1}^{(n)} - \xi_i^{(n)}| = 0$,
3. $\mathcal{E}^{(n)} \subset \mathcal{E}^{(n+1)}$.

Proposition 6. *Under assumptions (1)-(2)-(3), a non-negative and convex function $g(.)$, $\gamma-$integrable, is such that*

$$\lim_{n\to\infty} \int_0^\infty g(\xi)\gamma_n(d\xi) = \int_0^\infty g(\xi)\gamma(d\xi)\,,$$

and the convergence is monotone increasing.

The sketch of the proof is provided e.g. in [3]. we note $\tilde{Z}_t^{(j,l)} := Z_t^{(j,b_l)}$ for $j = 1,2$ and $l = 1,...n-1$. Each $\tilde{Z}_t^{(j,l)}$ is mean reverting and ruled by the SDE

$$d\tilde{Z}_t^{(j,l)} = \left(-(\beta + b_l)\, \tilde{Z}_t^{(j,l)}\right) dt + d\tilde{N}_t^{(j)} \quad j \in \{1,2\},$$

where $\tilde{N}_t^{(j)}$ is the counting process in the finite dimensional model. Its intensity, noted $\tilde{\lambda}_t^{(j)}$, is the sum of $\tilde{Z}_t^{(j,l)}$, weighted by the mass of atoms:

$$\begin{pmatrix} \tilde{\lambda}_t^{(1)} \\ \tilde{\lambda}_t^{(2)} \end{pmatrix} = \begin{pmatrix} \tilde{\lambda}_0^{(1)} \\ \tilde{\lambda}_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \begin{pmatrix} \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \tilde{Z}_t^{(1,l)} \\ \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \tilde{Z}_t^{(2,l)} \end{pmatrix}.$$

The next corollary states the convergence of $\tilde{\lambda}_t^{(j)}$ to $\lambda_t^{(j)}$.

Corollary 1. *Under the assumptions of Proposition 6, it holds for $t \geq 0$ that $\lim_{n\to\infty} \tilde{\lambda}_t^{(j)}(\omega) = \lambda_t^{(j)}(\omega)$, almost surely.*

Proof. Let us consider fixed sample paths $\tilde{N}_s^{(j)}(\omega)$ for $s \in [0,t]$ where $\omega \in \Omega$. Thus,

$$\tilde{Z}_t^{(j,\xi)}(\omega) = \int_0^t e^{-(\beta+\xi)(t-s)}\, d\tilde{N}_s^{(j)}(\omega)$$

is a positive function of ξ and is decreasing and convex since

$$\frac{\partial}{\partial \xi} \tilde{Z}_t^{(j,\xi)}(\omega) \leq 0\,, \quad \frac{\partial^2}{\partial \xi^2} \tilde{Z}_t^{(j,\xi)}(\omega) \geq 0\,.$$

Therefore, from Proposition 6,

$$\int_0^\infty \tilde{Z}_t^{(j,\xi)}(\omega)\gamma_n(d\xi) \to \int_0^\infty Z_t^{(j,\xi)}(\omega)\gamma(d\xi)\,,$$

with $N_s^{(j)}(\omega) = \tilde{N}_s^{(j)}(\omega)$. ∎

The infinitesimal dynamics of jump intensities are then equal to

$$\begin{pmatrix} d\tilde{\lambda}_t^{(1)} \\ d\tilde{\lambda}_t^{(2)} \end{pmatrix} = \begin{pmatrix} \eta_{11} & \eta_{12} \\ \eta_{21} & \eta_{22} \end{pmatrix} \left(-\sum_{l=0}^{n-1} \frac{w_l\,(\beta+b_l)}{\Gamma(\alpha)} \begin{pmatrix} \tilde{Z}_t^{(1,l)} \\ \tilde{Z}_t^{(2,l)} \end{pmatrix} dt + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \begin{pmatrix} d\tilde{N}_t^{(1)} \\ d\tilde{N}_t^{(2)} \end{pmatrix} \right) \tag{23}$$

In the discrete framework, the approximated log-return, denoted $\left(\tilde{X}_t\right)_{\geq 0}$ is driven by the following SDE:

$$d\tilde{X}_t = \left(\mu - \frac{\sigma^2}{2}\right) dt + \sigma dW_t + \sum_{j=1}^{2} \left(d\tilde{L}_t^{(j)} - \mu_j \tilde{\lambda}_t^{(j)} dt \right). \tag{24}$$

Applying the Itô's lemma to $\tilde{S}_t = S_0 e^{\tilde{X}_t}$, allows us to show that the approximated asset price follows a geometric dynamic:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} = \mu dt + \sigma dW_t + \sum_{j=1}^{2} \left(\left(e^{J^{(j)}} - 1\right) d\tilde{N}_t^{(j)} - \mu_k \tilde{\lambda}_t^{(j)} dt \right). \tag{25}$$

The next proposition provides the Laplace transform of the approached MERJD conditionally to the filtration of $\mathcal{F}_t$, in terms of backward ordinary differential equations (ODE's). To lighten developments, we adopt bold symbols for all vectors, i.e.

$$\tilde{\boldsymbol{\lambda}}_t = \begin{pmatrix} \tilde{\lambda}_t^{(1)} \\ \tilde{\lambda}_t^{(2)} \end{pmatrix}, \; \tilde{\boldsymbol{Z}}_t^{(l)} = \begin{pmatrix} \tilde{Z}_t^{(1,l)} \\ \tilde{Z}_t^{(2,l)} \end{pmatrix}, \; \boldsymbol{\eta}_{.,j} = \begin{pmatrix} \eta_{1j} \\ \eta_{2j} \end{pmatrix}.$$

Proposition 7. *Let $\omega \in \mathbb{R}^+$. TheLaplace function of the MERJD log-return $\tilde{X}_t$ at time t, conditionally to $\mathcal{F}_t$, is given by the following expression*

$$\mathbb{E}\left(e^{-\omega \tilde{X}_s} \,|\, \mathcal{F}_t\right) = \exp\left(q_0(t,s) + \boldsymbol{q}_\lambda(t,s)^\top \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \boldsymbol{q}_l(t,s)^\top \tilde{\boldsymbol{Z}}_t^{(l)} - \omega \tilde{X}_t \right) \tag{26}$$

where

$$q_0(t,s) = -\omega\left(\mu - \frac{\sigma^2}{2}\right)(s-t) + \omega^2 \frac{\sigma^2}{2}(s-t), \tag{27}$$

and vector of functions $\boldsymbol{q}_\lambda(t,s), \boldsymbol{q}_l(t,s) : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^2$, solve ODE's:

$$\begin{cases} \partial_t q_\lambda^{(j)}(t,s) = -\omega\mu_j - \left(\mathcal{J}_j(-\omega) \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(t,s) + q_l^{(j)}(t,s) \right) \right) - 1 \right), \\ \partial_t q_l^{(j)}(t,s) = (\beta + b_l)\left(q_l^{(j)}(t,s) + \boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(t,s) \right). \end{cases} \tag{28}$$

with the terminal conditions $q_\lambda^{(j)}(s,s) = 0$ and $q_l^{(j)}(s,s) = 0$ for $j \in \{1,2\}$ and $l = 0, ..., n-1$.

Proof. We denote by $f\left(t, \tilde{X}_t, \tilde{\boldsymbol{\lambda}}_t, \left(\tilde{\boldsymbol{Z}}_t^{(l)}\right)_{l=0,\dots,n-1}, \tilde{\boldsymbol{L}}_t\right)$, the Laplace transform $\mathbb{E}\left(e^{-\omega \tilde{X}_s} | \mathcal{F}_t\right)$. $f(.)$ being a conditional expectation, is also a martingale and $\mathbb{E}\left(df \,|\, \mathcal{F}_t\right) = 0$. From the Itô's lemma, we infer that $f(.)$ satisfies a stochastic differential equation (SDE):

$$\begin{aligned} 0 = {} & \partial_t f + \left(\mu - \frac{\sigma^2}{2} - \mu_1 \tilde{\lambda}_t^{(1)} - \mu_2 \tilde{\lambda}_t^{(2)}\right) \partial_{\tilde{X}} f + \frac{\sigma^2}{2} \partial_{\tilde{X}\tilde{X}} f \\ & - \sum_{l=0}^{n-1} (\beta + b_l) \left(\tilde{Z}_t^{(1,l)} \partial_{\tilde{Z}_t^{(1,l)}} f + \tilde{Z}_t^{(2,l)} \partial_{\tilde{Z}_t^{(2,l)}} f \right) \\ & - \sum_{l=0}^{n-1} \frac{w_l\,(\beta+b_l)}{\Gamma(\alpha)} \sum_{j=1}^{2} \left(\eta_{j1} \tilde{Z}_t^{(1,l)} + \eta_{j2} \tilde{Z}_t^{(2,l)} \right) \partial_{\tilde{\lambda}^{(j)}} f + \\ & \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \int_0^\infty f\left(t, \tilde{\boldsymbol{\lambda}}_t + \frac{\sum_{l=0}^{n-1} w_l}{\Gamma(\alpha)} \begin{pmatrix} \eta_{1j} \\ \eta_{2j} \end{pmatrix}, \left(\tilde{Z}_t^{(j,l)} + 1\right)_l, \tilde{X}_t + z \right) - f(.)\, m^{(j)}(dz) \end{aligned} \tag{29}$$

We do the assumption that $f(.)$ is an exponential affine function of the type

$$f(.) = \exp\left(q_0(t,s) + \boldsymbol{q}_\lambda(t,s)^\top \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \boldsymbol{q}_l(t,s)^\top \tilde{\boldsymbol{Z}}_t^{(l)} - \omega \tilde{X}_t\right) . \tag{30}$$

The partial derivatives of $f(.)$ with respect to state variables are given by

$$\begin{aligned}
\partial_{\tilde{Z}^{(j,l)}} f &= f \, \frac{w_l}{\Gamma(\alpha)} q_l^{(j)}(t,s) \,, \\
\partial_{\tilde{\lambda}_j} f &= f \, q_\lambda^{(j)}(t,s) \,, \\
\partial_{\tilde{X}} f &= -\omega f \,,\; \partial_{\tilde{X}\tilde{X}} f = \omega^2 f \,.
\end{aligned}$$

Whereas, the derivative of f with respect to time is equal to

$$\partial_t f(.) \;=\; f(.) \left(\partial_t q_0(t,s) + \partial_t \boldsymbol{q}_\lambda(t,s)^\top \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \partial_t \boldsymbol{q}_l(t,s)^\top \tilde{\boldsymbol{Z}}_t^{(l)}\right) .$$

Under assumption (30), the jump terms in Equation (29) are developed in the following way:

$$\begin{aligned}
& f\left(t, \tilde{\boldsymbol{\lambda}}_t + \frac{\sum_{l=0}^{n-1} w_l}{\Gamma(\alpha)} \begin{pmatrix} \eta_{1j} \\ \eta_{2j} \end{pmatrix}, \left(\tilde{Z}_t^{(j,l)} + 1\right)_l, \tilde{X}_t + z\right) - f(.) = \\
& f(.) \left(\exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(t,s) + q_l^{(j)}(t,s)\right) - \omega z\right) - 1\right)
\end{aligned}$$

Combining the previous equations allows us to show that $q_0(t,s)$, $\boldsymbol{q}_\lambda(t,s)$ and $\boldsymbol{q}_l(t,s)$ solve the equation

$$\begin{aligned}
0 &= \partial_t q_0(t,s) + \partial_t \boldsymbol{q}_\lambda(t,s)^\top \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \partial_t \boldsymbol{q}_l(t,s)^\top \tilde{\boldsymbol{Z}}_t^{(l)} - \omega \left(\mu - \frac{\sigma^2}{2}\right) \\
& + \frac{\omega^2 \sigma^2}{2} - \sum_{l=0}^{n-1} \frac{(\beta + b_l) \, w_l}{\Gamma(\alpha)} \left(\tilde{Z}_t^{(1,l)} q_l^{(1)}(t,s) + \tilde{Z}_t^{(2,l)} q_l^{(2)}(t,s)\right) \\
& - \sum_{l=0}^{n-1} \frac{(\beta + b_l) \, w_l}{\Gamma(\alpha)} \sum_{j=1}^{2} \left(\eta_{j1} \tilde{Z}_t^{(1,l)} + \eta_{j2} \tilde{Z}_t^{(2,l)}\right) q_\lambda^{(j)}(t,s) + \omega \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \mu_j + \\
& \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \left(\int_0^\infty \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(t,s) + q_l^{(j)}(t,s)\right) - \omega z\right) - 1 \, m^{(j)}(dz)\right) .
\end{aligned}$$

Grouping and cancelling terms independent of state variables leads to the expression (27) for $q_0(t,s)$. Cancelling terms multiplying $\tilde{Z}_t^{(j,l)}$ give us for $k = 1, 2$,

$$\begin{aligned}
0 = & \frac{w_l}{\Gamma(\alpha)} \partial_t q_l^{(j)}(t,s) - \frac{w_l \, (\beta + b_l)}{\Gamma(\alpha)} \\
& \times \left(\eta_{1j} q_\lambda^{(1)}(t,s) + \eta_{2j} q_\lambda^{(2)}(t,s) + q_l^{(j)}(t,s)\right) ,
\end{aligned}$$

that is the second equation in (28). The first equation in (28) is obtained by cancelling terms multiplying $\tilde{\lambda}_t^{(j)}$. □

The next corollary states that functions $\boldsymbol{q}_l(t,s)$ admit an integral representation.

Corollary 2. *The function $q_l^{(j)}(t,s)$ solving the ODE's in Equation (28) are equal to*

$$q_l^{(1)}(t,s) = -\int_t^s (\beta+b_l)\, e^{-(\beta+b_l)(u-t)} \left(\eta_{11} q_\lambda^{(1)}(t,s) + \eta_{21} q_\lambda^{(2)}(t,s)\right) du\,. \tag{31}$$
$$q_l^{(2)}(t,s) = -\int_t^s (\beta+b_l)\, e^{-(\beta+b_l)(u-t)} \left(\eta_{12} q_\lambda^{(1)}(t,s) + \eta_{22} q_\lambda^{(2)}(t,s)\right) du\,.$$

This result is checked by deriving the expression of $q_l^{(j)}(t,s)$ with respect to t. We find by this way the first ODE in (28). In Hainaut ([20], Chapter 5), the characteristic function of a non-Markov Hawkes process is retrieved by considering the limit of the partition $\mathcal{E}^{(n)}$. We cannot apply the same approach for the dampened rough kernel. The reason is that the limit of the sum of w_l, involved in Equation (28), is not defined when $n \to \infty$:

$$\lim_{n\to\infty} \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} = \frac{1}{\Gamma(\alpha)} \int_0^\infty \frac{\xi^{-\alpha}}{\Gamma(1-\alpha)} d\xi = \infty\,.$$

Nevertheless, the backward ODE's (28) are convertible into forward ones, which admit a limit when $n \to \infty$. To establish these forward ODE's, we need the next result.

Corollary 3. *For any $s, s' \in \mathbb{R}^+$ and $v \in \mathbb{R}^+$ such that $v \leq \min(s, s^{'})$, the following equality holds*

$$q_\lambda^{(j)}(s-v,s) = q_\lambda^{(j)}(s'-v,s')\,,\; j=1,2. \tag{32}$$

Proof. By definition, the equality (32) is true for $v=0$, $q_\lambda^{(j)}(s,s) = q_\lambda^{(j)}(s',s') = 0$. Let us assume that the property (32) holds for all $r \in [0,v]$ where $0 < v \leq s$. From previous equations, we notice that

$$\begin{aligned}\frac{\partial q_\lambda^{(j)}(s-v,s)}{\partial v} &= -\left.\frac{\partial q_\lambda^{(j)}(t,s)}{\partial t}\right|_{t=s-v} = \\ &\omega\mu_j + \mathbb{E}\left(e^{-\omega J^{(j)}}\right) \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(q_\lambda^{(j)}(s-v,s) + \boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_l(s-v,s)\right)\right) - 1\end{aligned}$$

where

$$q_l^{(j)}(s-v,s) = -(\beta+b_l)\, \boldsymbol{\eta}_{.,j}^\top \int_{s-v}^s e^{-(\beta+b_l)(u-(s-v))} \boldsymbol{q}_\lambda(s-v,s)\, du\,.$$

We rewrite the integrals in the previous equation using the change of variable $u = s - r$,

$$\int_{s-v}^s e^{-(\beta+b_l)(u-(s-v))} \boldsymbol{q}_\lambda(u,s)\, du = \int_0^v e^{-(\beta+b_l)(v-r)} \boldsymbol{q}_\lambda(s-r,s)\, dr$$

Since $\boldsymbol{q}_\lambda(s-r,s) = \boldsymbol{q}_\lambda(s'-r,s')$ for all $r \in [0,v]$, we infer that

$$\begin{aligned}\int_0^v e^{-(\beta+b_l)(v-r)} \boldsymbol{q}_\lambda(s-r,s)\, dr &= \int_0^v e^{-(\beta+b_l)(v-r)} \boldsymbol{q}_\lambda(s'-r,s')\, dr \\ &= \int_{s'-v}^{s'} e^{-(\beta+b_l)(u-(s'-v))} \boldsymbol{q}_\lambda(u,s')\, du\end{aligned}$$

and $q_l^{(j)}(s-v,s) = q_l^{(j)}(s'-v,s')$. Therefore we have well

$$\frac{\partial q_\lambda^{(j)}(s-v,s)}{\partial v} = \frac{\partial q_\lambda^{(j)}(s'-v,s')}{\partial v}$$

and conclude that the equality (32) also holds for $v + dv$. □

A direct consequence of this last corollary is that $q_{\lambda}^{(j)}(t,s)=q_{\lambda}^{(j)}(s-t)$ and $q_{l}^{(j)}(t,s)=q_{l}^{(j)}(s-t)$. Then, we infer that

$$
\begin{aligned}
\partial_t q_{\lambda}^{(j)}(t,s) &= -\partial_s q_{\lambda}^{(j)}(t,s)\,,\\
\partial_t q_{l}^{(j)}(t,s) &= -\partial_s q_{l}^{(j)}(t,s)\,.
\end{aligned}
$$

This allows to rewrite the Laplace transform in terms of a forward differential equation ruling $q_{\lambda}(t,s)$.

Proposition 8. *Let $\omega\in\mathbb{R}^{+}$. The Laplace transform of $\tilde{X}_s$ at time s, conditionally to $\mathcal{F}_t$, is given by the following expression*

$$
\mathbb{E}\left(e^{-\omega\tilde{X}_s}\,|\,\mathcal{F}_t\right) = \exp\left(q_0(t,s)+\boldsymbol{q}_{\lambda}(t,s)^{\top}\tilde{\boldsymbol{\lambda}}_t+\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\boldsymbol{q}_l(t,s)^{\top}\tilde{\boldsymbol{Z}}_t^{(l)}-\omega\tilde{X}_t\right) \tag{33}
$$

where $q_0(t,s)$ is defined in Eq. (27).

$$
\begin{cases}
\partial_s q_{\lambda}^{(j)}(t,s)=\omega\mu_j+\left(\mathcal{J}_j\left(-\omega\right)\exp\left(\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\left(\boldsymbol{\eta}_{.,j}^{\top}\boldsymbol{q}_{\lambda}(t,s)+q_l^{(j)}(t,s)\right)\right)-1\right)\,,\\
\partial_s q_l^{(j)}(t,s)=-\partial_s\int_t^s\,\left(\beta+b_l\right)e^{-\left(\beta+b_l\right)\left(u-t\right)}\boldsymbol{\eta}_{.,j}^{\top}\boldsymbol{q}_{\lambda}(t,s)\,du\,.
\end{cases} \tag{34}
$$

with the initial condition $q_{\lambda}^{(j)}(t,t)=0$ and $q_l^{(j)}(t,t)=0$ for $j=1,2$ and $l=0,...,n-1$.

Proof. From equation (28) and as $\partial_t q_l^{(j)}(t,s)=-\partial_s q_l^{(j)}(t,s)$, we have that

$$
q_l^{(j)}(t,s)+\boldsymbol{\eta}_{.,j}^{\top}\boldsymbol{q}_{\lambda}(t,s)=\frac{1}{\beta+b_l}\partial_t q_l^{(j)}(t,s)=-\frac{1}{\beta+b_l}\partial_s q_l^{(j)}(t,s)\,.
$$

On the other hand, the backward ODE ruling $q_{\lambda}^{(j)}(.,.)$ is rewritten as follows

$$
\partial_t q_{\lambda}^{(j)}(t,s) = -\omega\mu_j-\left(\mathbb{E}\left(e^{-\omega J^{(j)}}\right)\exp\left(\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)\left(\beta+b_l\right)}\partial_t q_l^{(j)}(t,s)\right)-1\right)\,.
$$

As $\partial_t q_{\lambda}^{(j)}(t,s)=-\partial_s q_{\lambda}^{(j)}(t,s)$ and $\partial_t q_l^{(j)}(t,s)=-\partial_s q_l^{(j)}(t,s)$, this is equivalent to

$$
\partial_s q_{\lambda}^{(j)}(t,s) = \omega\mu_j+\left(\mathbb{E}\left(e^{-\omega J^{(j)}}\right)\exp\left(-\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)\left(\beta+b_l\right)}\partial_s q_l^{(j)}(t,s)\right)-1\right)\,.
$$

□

This last proposition allows us to find the Laplace transform of the MERJD by considering the limit, in the next section.

4 Laplace transform of the MERJD log-return

We dispose of necessary tools for retrieving the Laplace transform of the MERJD process, X_t. The next proposition states that its initial value depends on a function solving a fractional differential equation involving the dampened RL integral and derivative such as defined in Section 2.

Proposition 9. *The Laplace transform of the log-return $(X_s)_{s\geq 0}$, conditionally to $\mathcal{F}_0$, for $\omega \in \mathbb{R}^+$, is equal to*

$$\Upsilon_s(-\omega) := \mathbb{E}\left(e^{-\omega X_s} \,|\, \mathcal{F}_0\right) \quad = \quad \exp\left(-\left(\omega\left(\mu - \frac{\sigma^2}{2}\right) - \frac{\omega^2\sigma^2}{2}\right) s + \boldsymbol{q}_\lambda(s)^\top \boldsymbol{\lambda}_t\right) \,, \tag{35}$$

where $q_\lambda^{(j)}(s)$ for $j = 1, 2$ solves a forward ODE:

$$\frac{dq_\lambda^{(j)}(s)}{ds} \quad = \quad \omega\mu_j + \mathcal{J}_j(-\omega)\exp\left(\boldsymbol{\eta}_{.,j}^\top\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s)\right) - 1 \tag{36}$$

with the initial condition $q_\lambda^{(j)}(0) = 0$ and where $K\frac{dq_\lambda^{(j)}}{ds}$ is the dampened Riemann-Liouville integral of $\frac{dq_\lambda^{(j)}}{ds}$:

$$\left(K\frac{dq_\lambda^{(j)}}{ds}\right)(s) = \frac{1}{\Gamma(\alpha)}\int_0^s \frac{e^{-\beta(s-u)}}{(s-u)^{1-\alpha}}\frac{dq_\lambda^{(j)}(u)}{du}\,du.$$

An equivalent representation is obtained by defining $\psi^{(j)}(s) := \left(K\frac{dq_\lambda^{(j)}}{ds}\right)(s)$. this function solves the fractional differential equation

$$\left(K^{-1}\psi^{(j)}\right)(s) \quad = \quad \omega\mu_j + \mathcal{J}_j(-\omega)\exp\left(\boldsymbol{\eta}_{.,k}^\top\,\boldsymbol{\psi}(s)\right) - 1\,, \tag{37}$$

where $\left(K^{-1}\psi^{(j)}\right)(s) = \frac{dq_\lambda^{(j)}}{ds}(s)$ is the dampened Riemann-Liouville derivative of $\psi^{(j)}(s)$:

$$\left(K^{-1}\psi^{(j)}\right)(s) = \frac{d}{ds}\int_0^s \left(\beta^\alpha + \frac{\alpha}{\Gamma(1-\alpha)}\int_{s-u}^\infty \frac{e^{-\beta v}}{v^{1+\alpha}}dv\right)\psi^{(j)}(u)\,du\,.$$

Proof. As , $q_\lambda^{(j)}(s,s) = 0$, we develop the forward derivative of $q_l^{(j)}(t,s)$ provided in Equation (34), as follows

$$\begin{aligned}
\partial_s q_l^{(j)}(t,s) &= -\partial_s \int_t^s (\beta + b_l)\, e^{-(\beta+b_l)(u-t)}\boldsymbol{\eta}_{.,j}^\top \boldsymbol{q}_\lambda(u,s)\,du \\
&= -\int_t^s (\beta + b_l)\, e^{-(\beta+b_l)(u-t)}\boldsymbol{\eta}_{.,j}^\top \partial_s\boldsymbol{q}_\lambda(u,s)\,du\,.
\end{aligned}$$

We have proven in the previous section that $\boldsymbol{q}_\lambda(u,s)$ is in fact a function of $s-u$: $\boldsymbol{q}_\lambda(s-u)$, if we perform the change of variable $v = s-u$, the derivative $\partial_s q_l^{(j)}(t,s)$ can be rewritten as

$$\partial_s q_l^{(j)}(t,s) = -\int_0^{s-t} (\beta + b_l)\, e^{-(\beta+b_l)(s-t-v)}\boldsymbol{\eta}_{.,j}^\top \frac{d\boldsymbol{q}_\lambda(v)}{dv}\,dv\,.$$

Furthermore, we have seen that $\partial_s q_\lambda^{(j)}(t,s) = \frac{dq_\lambda^{(j)}(s-t)}{ds}$. Then, we reformulate the forward ODE (34) as:

$$\begin{aligned}
\frac{dq_\lambda^{(j)}(s-t)}{ds} &= \omega\mu_j + \\
&\mathbb{E}\left(e^{-\omega J^{(j)}}\right)\exp\left(\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\int_0^{s-t} e^{-(\beta+b_l)(s-t-v)}\boldsymbol{\eta}_{.,j}^\top\frac{d\boldsymbol{q}_\lambda(v)}{dv}\,dv\right) - 1\,.
\end{aligned} \tag{38}$$

We next consider the limit of the term in the exponential when the size of the partition $\mathcal{E}^{(n)}$ tends to infinity. By construction, the following limit is well defined:

$$\begin{aligned}
&\lim_{n\to\infty}\sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\int_0^{s-t} e^{-(\beta+b_l)(s-t-v)}\boldsymbol{\eta}_{.,j}^\top\frac{d\boldsymbol{q}_\lambda(v)}{dv}\,dv\\
&=\frac{1}{\Gamma(\alpha)}\int_0^{s-t}\int_0^\infty e^{-\xi(s-t-v)}\gamma(d\xi)e^{-\beta(s-t-v)}\boldsymbol{\eta}_{.,j}^\top\frac{d\boldsymbol{q}_\lambda(v)}{dv}\,dv\\
&=\boldsymbol{\eta}_{.,j}^\top\int_0^{s-t}\frac{e^{-\beta(s-t-v)}}{\Gamma(\alpha)(s-t-v)^{1-\alpha}}\frac{d\boldsymbol{q}_\lambda(v)}{dv}\,dv\,.
\end{aligned}$$

At time $t=0$, i.e. conditionally to $\mathcal{F}_0$, we recognize the the dampened Riemann-Liouville of $\partial_s q_\lambda$:

$$\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s) \;=\; \int_0^s\frac{e^{-\beta(s-u)}}{\Gamma(\alpha)(s-u)^{1-\alpha}}\frac{d\boldsymbol{q}_\lambda(u)}{du}\,du\,, \tag{39}$$

and combining Equations (38) and (39) leads to the fractional equation (36).

$$\frac{dq_\lambda^{(j)}(s)}{ds} \;=\; \omega\mu_j+\mathbb{E}\left(e^{-\omega J^{(j)}}\right)\exp\left(\boldsymbol{\eta}_{.,j}^\top\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s)\right)-1$$

Given that $K^{-1}K\phi=\phi$, We immediately infer that $\boldsymbol{\psi}(s)=\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s)$ and Equation (63). $\square$

When $\alpha\to 1$, the rough jump process converges toward a standard Hawkes process with an exponential kernel. In this case, the dampened RL integral of $\frac{d\boldsymbol{q}_\lambda}{ds}$ converges toward the following integral

$$\lim_{\alpha\to 1}\left(K\frac{d\boldsymbol{q}_\lambda}{ds}\right)(s) \;=\; \int_0^s e^{-\beta(s-u)}\frac{d\boldsymbol{q}_\lambda(u)}{du}\,du\,,$$

and from Equation (38), $\frac{dq_\lambda^{(j)}}{ds}$ solves the integro-differential equation:

$$\frac{dq_\lambda^{(j)}(s)}{ds} \;=\; \omega\mu_j+\mathcal{J}_j(-\omega)\exp\left(\boldsymbol{\eta}_{.,j}^\top\int_0^s e^{-\beta(s-u)}\frac{d\boldsymbol{q}_\lambda(u)}{du}\,du\right)-1\,. \tag{40}$$

In practice, we solve numerically Equation (36). We divide $[0,s]$ in n subintervals $[s_k,s_{k+1}]$ of length Δ, for $k=0,...,n-1$. We denote by $\boldsymbol{g}(k):=\left.\frac{d\boldsymbol{q}_\lambda(s)}{ds}\right|_{s=s_k}$, the differential of $\boldsymbol{q}_\lambda$ at time s_k and we next use an explicit approximation of the dampened RL fractional integral :

$$g^{(j)}(k)=\omega\mu_j+\mathbb{E}\left(e^{-\omega J^{(j)}}\right)\exp\left(\frac{\boldsymbol{\eta}_{.,j}^\top}{\Gamma(\alpha)}\sum_{u=0}^{k-1}\frac{e^{-\beta(s_k-s_u)}}{(s_k-s_u)^{1-\alpha}}\boldsymbol{g}(u)\,\Delta\right)-1\,. \tag{41}$$

The recursion is initialized by setting $g^{(j)}(0)=\omega\mu_j+\mathbb{E}\left(e^{-\omega J^{(j)}}\right)-1$. We can exploit previous results to compute the probability density function of the log-return $(X_t)_{t\geq 0}$. Our approach is based on a discrete fast Fourier transform (DFFT). It consists to invert the characteristic function of the process, that is the Laplace transform (35) valued on the imaginary axis. Let us denote the characteristic function of X_s by $\Upsilon_s(i\omega)=\mathbb{E}\left(e^{i\omega X_s}\,|\,\mathcal{F}_0\right)$ for $\omega\in\mathbb{R}$. This is also the inverse Fourier transform of the probability density function (pdf) $f_s^X(x)$ of $X_s|\mathcal{F}_0$. Therefore, this density can be retrieved by computing the following integral (the Fourier transform, $\mathcal{F}[.]$, of $\Upsilon_s(.)$):

$$\begin{aligned} f_s^X(x) &= \frac{1}{2\pi}\mathcal{F}[\Upsilon_s(i\omega)](x) \\ &= \frac{1}{2\pi}\int_{-\infty}^{+\infty}\Upsilon_s(i\omega)e^{-i\omega x}d\omega\,. \end{aligned} \tag{42}$$

This integral is approximated by discretization with the DFFT algorithm recalled in Appendix B.

5 Change of measure

To avoid arbitrages, the valuation of derivatives is performed under an equivalent measure, called risk-neutral. Under this measure, discounted asset prices are martingales. Nevertheless, the market in our model, is incomplete because there are multiple non-traded risk factors. A consequence of this incompleteness is the existence of several equivalent measures that are potential candidates for the definition of a risk neutral one.

We first consider changes of measure in the finite dimensional approximation of Section 3. In a similar manner to Section 4, the dynamic of the MERJD under the changed measure will be obtained by taking the limit of the partition, $\lim_{n\to\infty}\mathcal{E}^{(n)}$. We focus on a family of changes of measure that are induced by exponential martingales of the form:

$$\begin{aligned} \tilde{M}_t &= \exp\left(-\frac{1}{2}\int_0^t\varphi(s)^2ds-\int_0^t\varphi(s)dW_s\right)\times \\ &\quad \exp\left(\sum_{j=1}^{2}\left[\zeta_j\tilde{L}_t^{(j)}+(1-\mathcal{J}_j(\zeta_j))\int_0^t\tilde{\lambda}_s^{(j)}ds\right]\right)\,, \end{aligned} \tag{43}$$

where $\varphi(t)$ is a $\mathcal{F}_t$-adapted process such that $\int_0^t|\varphi(s)|^2\,ds<\infty$ and $\zeta_j\in\mathbb{R}$ are such that $\mathcal{J}_j(\zeta_j)<\infty$ for $j=1,2$. Let us recall that the moment generating function of jumps is denoted by $\mathcal{J}_j(\omega)=\mathbb{E}\left(e^{\omega J^{(j)}}\right)$ for $j=1,2$. We can check that $\tilde{M}_t$ is a local martingale. Its differential is indeed equal to

$$\begin{aligned} d\tilde{M}_t = -\tilde{M}_t\varphi(t)dW_t+\tilde{M}_t\sum_{j=1}^{2}\left(1-\mathcal{J}_j(\zeta_j)\right)\tilde{\lambda}_t^{(j)}dt \\ +\tilde{M}_t\sum_{j=1}^{2}\left(\exp\left(\zeta_jJ^{(j)}\right)-1\right)dN_t^{(j)}\,. \end{aligned}$$

The expectation is null, $\mathbb{E}(d\tilde{M}_t)=0$ and the integral of $d\tilde{M}_t$ is a local martingale. We denote by $\mathbb{Q}$ the measure defined by the change of measure (43). This new measure preserves the structure of the MERJD as demonstrated in the next proposition.

Proposition 10. *For $j=1,2$,let us denote by $\tilde{N}_t^{Q(j)}$ the counting processes of intensity*

$$\tilde{\lambda}_t^{Q(j)} = \mathcal{J}_j(\zeta_j)\tilde{\lambda}_t^{(j)}\,, \tag{44}$$

We also define random variables $J^{Q(j)}$, through their moment generating functions under the measure $\mathbb{Q}$:

$$\mathcal{J}_j^Q(\omega)=\mathbb{E}^{\mathbb{Q}}\left(e^{\omega J^{Q(j)}}\right) = \frac{\mathcal{J}_j(\omega+\zeta_j)}{\mathcal{J}_j(\zeta_j)}\,,\ j=1,2\,, \tag{45}$$

and processes $\tilde{L}_t^Q = \sum_{k=1}^{\tilde{N}_t^{Q(j)}} J_k^{Q(j)}$. *Under the measure* $\mathbb{Q}$, *the dynamic of the log-return is ruled by the following SDE*

$$d\tilde{X}_t = \left(\mu - \frac{\sigma^2}{2} - \sigma\varphi(t)\right) dt + \sigma dW_t^Q + \sum_{j=1}^{2}\left(d\tilde{L}_t^{Q(j)} - \frac{\mu_j}{\mathcal{J}_j(\zeta_j)}\int_0^t \tilde{\lambda}_s^{Q(j)} ds\right), \tag{46}$$

where $dW_t^Q = dW_t + \sigma\varphi(t)dt$. *Furthermore, the intensities* $\tilde{\lambda}_t^{Q(j)}$ *are such that*

$$\begin{pmatrix} \tilde{\lambda}_t^{Q(1)} \\ \tilde{\lambda}_t^{Q(2)} \end{pmatrix} = \begin{pmatrix} \mathcal{J}_1(\zeta_1)\tilde{\lambda}_0^{(1)} \\ \mathcal{J}_2(\zeta_2)\tilde{\lambda}_0^{(2)} \end{pmatrix} + \begin{pmatrix} \eta_{11}\mathcal{J}_1(\zeta_1) & \eta_{12}\mathcal{J}_1(\zeta_1) \\ \eta_{21}\mathcal{J}_2(\zeta_2) & \eta_{22}\mathcal{J}_2(\zeta_2) \end{pmatrix} \begin{pmatrix} \sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\tilde{Z}_t^{(1,l)} \\ \sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\tilde{Z}_t^{(2,l)} \end{pmatrix}.$$

Proof. $\tilde{M}_t$ is product of a Brownian and jump changes of measure. The impact of the Brownian change of measure on the drift of $\tilde{X}_t$ being standard, we set $\varphi(t) = 0$ and consider a change of measure $\tilde{M}_t = \exp(Y_t)$ where

$$dY_t = \sum_{j=1}^{2}(1 - \mathcal{J}_j(\zeta_j))\,\tilde{\lambda}_t^{(j)} dt + \sum_{j=1}^{2}\zeta_j\, d\tilde{L}_t^{(j)}.$$

Under the $\mathbb{Q}$-measure, the Laplace transform of $\tilde{X}_s$ conditionally to $\mathcal{F}_t$ is equal to

$$\mathbb{E}^{\mathbb{Q}}\left(e^{-\omega\tilde{X}_s} \,|\, \mathcal{F}_t\right) = e^{-Y_t}\mathbb{E}\left(e^{-\omega\tilde{X}_s + Y_s} \,|\, \mathcal{F}_t\right).$$

We denote $\mathbb{E}\left(e^{-\omega\tilde{X}_s+Y_s} \,|\, \mathcal{F}_t\right)$ by $f\left(t, \tilde{X}_t, \tilde{\boldsymbol{\lambda}}_t, \left(\tilde{\boldsymbol{Z}}_t^{(l)}\right)_{l=0,\dots,n-1}, \tilde{\boldsymbol{L}}_t, Y_t\right)$. From the Itô's lemma, $f(.)$ satisfies the following stochastic differential equation (SDE):

$$\begin{aligned} 0 = {} & \partial_t f + \left(\mu - \frac{\sigma^2}{2} - \mu_1\tilde{\lambda}_t^{(1)} - \mu_2\tilde{\lambda}_t^{(2)}\right)\partial_{\tilde{X}} f + \frac{\sigma^2}{2}\partial_{\tilde{X}\tilde{X}} f \\ & + \partial_Y f \sum_{j=1}^{2}(1 - \mathcal{J}_j(\zeta_j))\,\tilde{\lambda}_t^{(j)} - \sum_{l=0}^{n-1}(\beta + b_l)\sum_{j=1}^{2}\tilde{Z}_t^{(j,l)}\partial_{\tilde{Z}_t^{(j,l)}} f \\ & - \sum_{l=0}^{n-1}\frac{w_l(\beta + b_l)}{\Gamma(\alpha)}\sum_{j=1}^{2}\left(\eta_{j1}\tilde{Z}_t^{(1,l)} + \eta_{j2}\tilde{Z}_t^{(2,l)}\right)\partial_{\tilde{\lambda}^{(j)}} f + \sum_{j=1}^{2}\tilde{\lambda}_t^{(j)} \times \\ & \int_0^\infty f\left(t, \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1}\frac{w_l\mathcal{J}_k(\zeta_k)}{\Gamma(\alpha)}\boldsymbol{\eta}_{.,j}, \left(\tilde{Z}_t^{(j,l)} + 1\right)_l, \tilde{X}_t + z, Y_t + \zeta_1 z\right) - f(.)\, m^{(j)}(dz) \end{aligned} \tag{47}$$

We do the ansatz that $f(.)$ is an exponential affine function of risk factors:

$$f(.) = \exp\left(q_0(t,s) + \boldsymbol{q}_\lambda(t,s)^\top\left(\tilde{\boldsymbol{\lambda}}_t \odot \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta})\right) + \sum_{l=0}^{n-1}\frac{w_l}{\Gamma(\alpha)}\boldsymbol{q}_l(t,s)^\top\tilde{\boldsymbol{Z}}_t^{(l)} - \omega\tilde{X}_t + q_Y(t,s)Y_t\right),$$

where $\tilde{\boldsymbol{\lambda}}_t \odot \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta})$ is the dot product of $\tilde{\boldsymbol{\lambda}}_t$ and $\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) = (\mathcal{J}_1(\zeta_1), \mathcal{J}_2(\zeta_2))$. The partial derivatives of $f(.)$ with respect to state variables are given by

$$\begin{aligned} &\partial_{\tilde{Z}^{(j,l)}} f = f\,\frac{w_l}{\Gamma(\alpha)}q_l^{(j)}(t,s), \\ &\partial_{\tilde{\lambda}_j} f = \mathcal{J}_j(\zeta_j)\, f\, q_\lambda^{(j)}(t,s), \\ &\partial_{\tilde{X}} f = -\omega f, \ \partial_{\tilde{X}\tilde{X}} f = \omega^2 f, \\ &\partial_Y f = f q_U(t,s). \end{aligned}$$

The derivative of f with respect to time is equal to

$$\partial_t f \quad = \quad f\left(\partial_t q_0(t,s) + \partial_t \boldsymbol{q}_\lambda(t,s)^\top \tilde{\boldsymbol{\lambda}}_t \odot \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \partial_t \boldsymbol{q}_l(t,s)^\top \tilde{\boldsymbol{Z}}_t^{(l)} + Y_t \partial_t q_Y(t,s)\right) .$$

The jump terms in Equation (47) become for $j = 1, 2$:

$$f\left(t, \tilde{\boldsymbol{\lambda}}_t + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\,, \left(\tilde{Z}_t^{(j,l)} + 1\right)_l, \tilde{X}_t +\ z, Y_t + \zeta_1 z\right) - f(.) =$$
$$f(.)\left(\exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(t,s) + q_l^{(j)}(t,s)\right) - \omega z + \zeta_j z\, q_Y(t,s)\right) - 1\right) .$$

Combining the previous equations in Equation (47) allows us to infer that $\partial_t q_Y(t,s) = 0$ and $q_Y(t,s) = 1$. Therefore, $q_0(t,s)$, $\boldsymbol{q}_\lambda(t,s)$ and $\boldsymbol{q}_l(t,s)$ satisfy the relation

$$\begin{aligned}
0 &= \partial_t q_0(t,s) + \partial_t \boldsymbol{q}_\lambda(t,s)^\top \tilde{\boldsymbol{\lambda}}_t \odot \boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) + \sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \partial_t \boldsymbol{q}_l(t,s)^\top \tilde{\boldsymbol{Z}}_t^{(l)} \\
&\quad + Y_t \partial_t q_Y(t,s) - \omega\left(\mu - \frac{\sigma^2}{2} - \mu_1 \tilde{\lambda}_t^{(1)} - \mu_2 \tilde{\lambda}_t^{(2)}\right) + \omega^2 \frac{\sigma^2}{2} \\
&\quad + \sum_{j=1}^{2} \left(1 - \mathcal{J}_j(\zeta_j)\right) \tilde{\lambda}_t^{(j)} - \sum_{l=0}^{n-1} \frac{(\beta + b_l)\, w_l}{\Gamma(\alpha)} \left(\tilde{Z}_t^{(1,l)} q_l^{(1)}(t,s) + \tilde{Z}_t^{(2,l)} q_l^{(2)}(t,s)\right) \\
&\quad - \sum_{l=0}^{n-1} \frac{(\beta + b_l)\, w_l}{\Gamma(\alpha)} \sum_{j=1}^{2} \mathcal{J}_j(\zeta_j) \left(\eta_{j1} \tilde{Z}_t^{(1,l)} + \eta_{j2} \tilde{Z}_t^{(2,l)}\right) q_\lambda^{(j)}(t,s) + \sum_{j=1}^{2} \tilde{\lambda}_t^{(j)} \times \\
&\int_0^\infty \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(t,s) + q_l^{(j)}(t,s)\right) - \omega z + \zeta_j z\, q_Y(t,s)\right) - 1\, m^{(j)}(dz)
\end{aligned}$$

Grouping terms allows us to infer that $q_0(t,s)$ is equal to Equation (27). Cancelling terms multiplying $\tilde{Z}_t^{(j,l)}$ give us for $j = 1, 2$,

$$0 = \frac{w_l}{\Gamma(\alpha)} \partial_t q_l^{(j)}(t,s) - \frac{w_l\,(\beta + b_l)}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(t,s) + q_l^{(j)}(t,s)\right),$$

which corresponds to the second Equation (28) with parameters $\boldsymbol{\eta}_{.,k}^\top$ multiplied elementwise by $\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta})$. Cancelling terms multiplying $\tilde{\lambda}_t^{(j)}$ leads to

$$\partial_t q_\lambda^{(j)}(t,s) \quad = \quad -\omega \mu_j - \left(\frac{\mathcal{J}_j(\zeta_j - \omega)}{\mathcal{J}_j(\zeta_j)} \exp\left(\sum_{l=0}^{n-1} \frac{w_l}{\Gamma(\alpha)} \left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \boldsymbol{q}_\lambda(t,s) + q_l^{(j)}(t,s)\right)\right) - 1\right),$$

similar to the first Equation in (28). The Laplace transform $\mathbb{E}^{\mathbb{Q}}\left(e^{-\omega \tilde{X}_s} \,|\, \mathcal{F}_t\right)$ has therefore the same form as the one of $\mathbb{E}\left(e^{-\omega \tilde{X}_s} \,|\, \mathcal{F}_t\right)$, provided in Proposition 7 with intensities (44), jumps (45) for $j = 1, 2$. $\square$

When jumps are exponential random variables with pdf's of Equation (2), the distribution of jumps under $\mathbb{Q}$ remains exponential as stated in the next corollary.

Corollary 4. ***Corollary*** *If $J^{(j)} \sim expo(\rho_j)$ then for $\zeta_1 \in (-\rho_1, +\infty)$ and $\zeta_2 \in (-\infty, -\rho_2)$, $J^{Q(j)}$ are distributed as exponential random variable with parameters $\rho_j^Q = \rho_j - \zeta_j$, under $\mathbb{Q}$, for $j = 1, 2$.*

This result is a direct consequence of Equation (45) and combined with $\mathcal{J}_j(\omega) = \frac{\rho_j}{\rho_j - \omega}$ for $j = 1, 2$, with $\omega \leq \rho_1$ and $\omega \geq \rho_2$. Another direct consequence of Proposition 10 is the possibility to identify the familly of change of measure defining a risk neutral measure, i.e. a measure under which the discounted asset price is a martingale. We consider a constant discount rate, denoted by $r \in \mathbb{R}^+$.

Corollary 5. *The equivalent measures* $\mathbb{Q}$ *defined by the change of measure* (43) *are risk neutral if*

$$\varphi(t) \quad = \quad \frac{\mu - r}{\sigma} + \sum_{j=1}^{2} \frac{\tilde{\lambda}_t^{(j)} \mathcal{J}_j(\zeta_j) \left(\mathbb{E}\left(e^{J^{Q(j)}}\right) - 1 \right) - \mu_j \int_0^t \tilde{\lambda}_s^{(j)} ds}{\sigma} , \tag{48}$$

where r *is the discount rate.*

Proof. Replacing the expression (48) of $\varphi(t)$ into the dynamic (46) of $\tilde{X}_t$ under $\mathbb{Q}$, allows us to infer that

$$d\tilde{X}_t = \left(r - \frac{\sigma^2}{2}\right) dt + \sigma dW_t^Q + \sum_{j=1}^{2} \left(d\tilde{L}_t^{Q(j)} - \tilde{\lambda}_t^{Q(j)} \mathbb{E}\left(e^{J^{Q(j)}} - 1\right) dt \right) . \tag{49}$$

If we remember that the asset price is $\tilde{S}_t = \tilde{S}_0 e^{\tilde{X}_t}$, using the Itô's lemma leads to the following SDE:

$$\frac{d\tilde{S}_t}{\tilde{S}_t} \quad = \quad rdt + \sigma dW_t^Q + \sum_{k=1}^{2} \left[\left(e^{J^{Q(k)}} - 1\right) d\tilde{N}_t^{Q(k)} - \mathbb{E}\left(e^{J^{Q(k)}} - 1\right) \tilde{\lambda}_t^{Q(k)} dt \right] . \tag{50}$$

Since $\mathbb{E}^{\mathbb{Q}}\left(\frac{d\tilde{S}_t}{\tilde{S}_t}\right) = rdt$, the discounted price $e^{-rt}\tilde{S}_t$ is a well martingale under $\mathbb{Q}$ and $\mathbb{Q}$ is risk neutral. □

By construction, when the size n of the partition $\mathcal{E}^{(n)}$ tends to $+\infty$, the process $\tilde{X}_t$ converges toward X_t. From Equation (49), we deduce that the MERJD, X_t, under the risk neutral measure is the following sum:

$$X_t \quad = \quad \left(r - \frac{\sigma^2}{2}\right) t + \sigma W_t^Q + \sum_{j=1}^{2} \left(L_t^{Q(j)} - \mu_j^Q \int_0^t \lambda_s^{Q(j)} ds \right) , \tag{51}$$

where $\mu_j^Q = \left(\mathbb{E}\left(e^{J^{Q(j)}}\right) - 1\right)$ for $j = 1, 2$. The structure of X_t under $\mathbb{Q}$ and $\mathbb{P}$ are similar. Therefore, the Laplace transform of the log-return under $\mathbb{Q}$ is provided by Proposition 9 with updated parameters:

$$\Upsilon_s^Q(\omega) := \mathbb{E}^{\mathbb{Q}}\left(e^{-\omega X_s} \,|\, \mathcal{F}_0\right) \quad = \quad \exp\left(- \left(\omega \left(r - \frac{\sigma^2}{2} \right) - \frac{\omega^2 \sigma^2}{2} \right) s + \boldsymbol{q}_\lambda^Q(s)^\top \boldsymbol{\lambda}_t \right) , \tag{52}$$

where $q_\lambda^{Q(j)}(s)$ for $j = 1, 2$ solves a forward ODE:

$$\frac{dq_\lambda^{Q(j)}(s)}{ds} \quad = \quad \omega \mu_j^Q + \mathcal{J}_j^Q(-\omega) \exp\left(\left(\boldsymbol{\mathcal{J}}(\boldsymbol{\zeta}) \odot \boldsymbol{\eta}_{.,j}\right)^\top \left(K \frac{d\boldsymbol{q}_\lambda^Q}{ds} \right)(s) \right) - 1 \tag{53}$$

with the initial condition $q_\lambda^{Q(j)}(0) = 0$. Inverting by DFFT the characteristic function as explained in Section 4, permit us to retrieve the pdf of X_t under the risk neutral measure and to price European options on the corresponding asset. The influence of roughness parameters, α and β, on call prices is studied in Section 7.

6 Econometric estimation

Estimating the parameters of a MERJD is a challenging task since jumps are not directly observable. For this reason, we adopt a peak-over-threshold approach for detecting jumps, similar to that in [11].

The discrete record of p asset log-returns, equally spaced with a lag Δ, is noted $\{x_1, x_1, x_2, ..., x_p\}$. The times of observation are $\{s_0, s_1, ..., s_p\}$. We assume that a jump occurs when the log-return is above or below some thresholds. These thresholds are denoted by $g(\alpha_1)$ and $g(\alpha_2)$ depend on two confidence levels, α_1 and α_2. To define thresholds, we first fit a pure Gaussian process: $x_k \sim \mu_g \Delta + \sigma_g W_\Delta$ to time-series. The unbiased estimators of μ_g and σ_g are:

$$\widehat{\mu}_g = \frac{1}{p\Delta} \sum_{j=1}^{p} x_j \qquad \widehat{\sigma}_g^2 = \frac{1}{(p-1)\,\Delta} \sum_{j=1}^{p} (x_j - \widehat{\mu}_g)^2 \,. \tag{54}$$

If $\Phi(.)$ denotes the cdf of a standard normal, $g(\alpha_1)$, $g(\alpha_2)$ are set to the α_1 and α_2 percentiles of the Brownian motion: $g(\alpha_i) = \widehat{\mu}_g \Delta + \widehat{\sigma}_g \sqrt{\Delta} \Phi^{-1}(\alpha_i)$ for $i = 1, 2$. The times of the k^{th} jump of $L_t^{(1)}$ and $L_t^{(2)}$ are :

$$\begin{aligned}
\tau_k^{(1)} &= \min\{s_j \in \{s_1, ..., s_p\} \,|\, x_j \geq g(\alpha_1)\,,\, s_j \geq \tau_{k-1}^{(1)}\}\,, \\
\tau_k^{(2)} &= \min\{s_j \in \{s_1, ..., s_p\} \,|\, x_j \leq g(\alpha_2)\,,\, s_j \geq \tau_{k-1}^{(2)}\}\,.
\end{aligned}$$

The sample path of $\left(N_t^{(j)}\right)_{t\geq 0}$ for $j = 1, 2$ are approached by the following time series:

$$N^{(j)}(s_i) = \max\{k \in \mathbb{N} \,|\, \tau_k^{(j)} \leq s_i\}\,,\ i = 1, ..., n.$$

The levels of confidence, α_1 and α_2, are optimized such that the skewness and kurtosis of x_i for periods without jumps are close to those of a normal distribution. If the sets of times without jump is noted $\mathcal{T}$, parameter estimates of μ and σ are

$$\begin{cases}
\widehat{\sigma} = \frac{1}{(|\mathcal{T}|-1)\Delta} \sum_{s_j \in \mathcal{T}} \left(x_j - \frac{1}{|\mathcal{T}|\Delta} \sum_{s_i \in \mathcal{T}} x_i \right)^2 \,, \\
\widehat{\mu} = \frac{1}{|\mathcal{T}|\Delta} \sum_{s_j \in \mathcal{T}} x_j + \frac{\widehat{\sigma}}{2}\,.
\end{cases}$$

We next estimate the bivariate rough Hawkes process by log-likelihood maximization. We detail the calculation of this log-likelihood. From Equation (3), the sample intensities is equal to

$$\lambda_{t-}^{(j)} = \lambda_0^{(j)} + \sum_{k=1}^{2} \left(\frac{\eta_{jk}}{\Gamma(\alpha)} \sum_{\tau_u^{(k)} < t} e^{-\beta(t-\tau_u^{(k)})} (t - \tau_u^{(k)})^{\alpha - 1} \right) \quad j = 1, 2\,.$$

These realized intensities are involved in the calculation of the log-likelihood.

Proposition 11. *We denote the Gamma incomplete function by* $\Gamma(\alpha, x) = \int_x^\infty e^{-u} u^{\alpha-1} du$. *The log-likelihood of a sample of observations over* $[0, \mathcal{S}]$ *is defined as:*

$$\ln \mathcal{L} = \sum_{j=1}^{2} \left(-\int_0^{\mathcal{S}} \lambda_s^{(j)} ds + \sum_{k=1}^{N_{\mathcal{S}}^{(j)}} \log\left(\lambda_{\tau_k -}^{(j)}\right) \right)\,, \tag{55}$$

where the integral of the intensity is equal to

$$\int_0^{\mathcal{S}} \lambda_s^{(j)} ds = \lambda_0^{(j)} \mathcal{S} + \sum_{k=1}^{2} \frac{\eta_{jk}}{\beta^\alpha} \sum_{u=1}^{N_{\mathcal{S}}^{(k)}} \left(1 - \frac{\Gamma\left(\alpha, \beta\left(\mathcal{S} - \tau_u^{(k)}\right)\right)}{\Gamma(\alpha)} \right)\,. \tag{56}$$

Proof. From e.g. Embrechts et al. [11], the log-likelihood of the sample is given by Equation (55). Using the expression (3) of λ_t and changing the order of integration, we develop the integral of the intensity as follows:

$$\begin{aligned}\int_0^{\mathcal{S}} \lambda_u^{(j)} du &= \lambda_0^{(j)}\mathcal{S} + \sum_{k=1}^{2}\eta_{jk}\int_0^{\mathcal{S}}\int_0^{u-} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} dN_s^{(j)}\, du \qquad (57)\\ &= \lambda_0^{(j)}\mathcal{S} + \sum_{k=1}^{2}\eta_{jk}\int_0^{\mathcal{S}}\int_s^{\mathcal{S}} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)} du\, dN_s^{(j)} .\end{aligned}$$

The inner integrals are reformulated in terms of Gamma functions by a change of variable $v = \beta(u-s)$

$$\begin{aligned}&\int_s^{\mathcal{S}} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\, du \qquad (58)\\ &= \int_s^{\infty} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\, du - \int_{\mathcal{S}}^{\infty} e^{-\beta(u-s)}\frac{(u-s)^{\alpha-1}}{\Gamma(\alpha)}\, du\\ &= \frac{\beta^{-\alpha}}{\Gamma(\alpha)}\int_0^{\infty} e^{-v}v^{\alpha-1}\, dv - \frac{\beta^{-\alpha}}{\Gamma(\alpha)}\int_{\beta(\mathcal{S}-s)}^{\infty} e^{-v}v^{\alpha-1}\, dv\\ &= \beta^{-\alpha}\left(1-\frac{\Gamma(\alpha,\beta(\mathcal{S}-s))}{\Gamma(\alpha)}\right)\end{aligned}$$

Combining Equations (57) and (58) leads to the expression (56). □

For comparison, we consider a bivariate Hawkes process with an exponential memory kernel

$$\lambda_t^{(j)} = \lambda_0^{(j)} + \sum_{k=1}^{2}\eta_{jk}\left(\int_0^{t-} e^{-\beta(t-s)} dN_s^{(k)}\right).$$

The log-likelihood in this case has the same form as Equation (55) with

$$\int_0^{\mathcal{S}} \lambda_u^{(j)} du = \lambda_0^{(j)}\mathcal{S} + \sum_{k=1}^{2}\eta_{jk}\sum_{u=1}^{N_{\mathcal{S}}^{(k)}}\left(1-e^{-\beta\left(\mathcal{S}-\tau_u^{(k)}\right)}\right). \qquad (59)$$

We recall that this corresponds to the rough model with $\alpha = 1$. If we denote by Θ_N, the set of parameters of intensities, their estimates are obtained by maximization of the log-likelihood (55):

$$\widehat{\Theta}_N = \arg\max_{\Theta_N} \ln\mathcal{L}(\Theta_N) .$$

The distribution $m^{(j)}(.)$ of jumps is fitted independently of counting processes. In absence of jumps, the log-return has a normal distribution. If a single jump occurs over Δ, From Chapter 1 of Hainaut [20] Proposition 1.3, the pdf of the sum $\sigma W_\Delta + J^{(j)}$ is equal to:

$$\begin{aligned}h^{(1)}(z|\sigma,\rho_1) &= 1_{\{z\geq 0\}}\rho_1\exp\left(\frac{1}{2}(\rho_1\sigma)^2\Delta-\rho_1 z\right)\Phi\left(\frac{z-\rho_1\sigma^2\Delta}{\sqrt{\Delta}\sigma}\right),\\ h^{(2)}(z|\sigma,\rho_2) &= -1_{\{z\leq 0\}}\rho_2\exp\left(\frac{1}{2}(\rho_2\sigma)^2\Delta-\rho_2 z\right)\left(1-\Phi\left(\frac{z-\rho_2\sigma^2\Delta}{\sqrt{\Delta}\sigma}\right)\right).\end{aligned}$$

where $\Phi(.)$ is the cdf of a standard normal random variable. If $\{J_1^{(j)},...,J_{N_t^{(j)}}^{(j)}\}$ for $j = 1,2$ and Θ_J are respectively the sample of jumps and the set of parameters of $m^{(j)}(.)$, estimates are found by log-likelihood maximization:

$$\widehat{\Theta}_J = \arg\max_{\Theta_J}\sum_{j=1}^{2}\sum_{k=1}^{N_t^{(j)}} h^{(j)}\left(J_k^{(j)}|\widehat{\sigma},\rho_j\right).$$

7 Numerical illustration

To illustrate this article, we fit the MERJD to time-series of hourly Bitcoin returns from the 9/2/2018 to 9/2/2023, traded in USD on the platform Gemini. The upper graph of Figure 1 shows hourly returns and the lower plot displays Bitcoin prices. The bitcoin is traded 24h/24h and the time interval between two successive observation is $\Delta = 1/8760$ year.

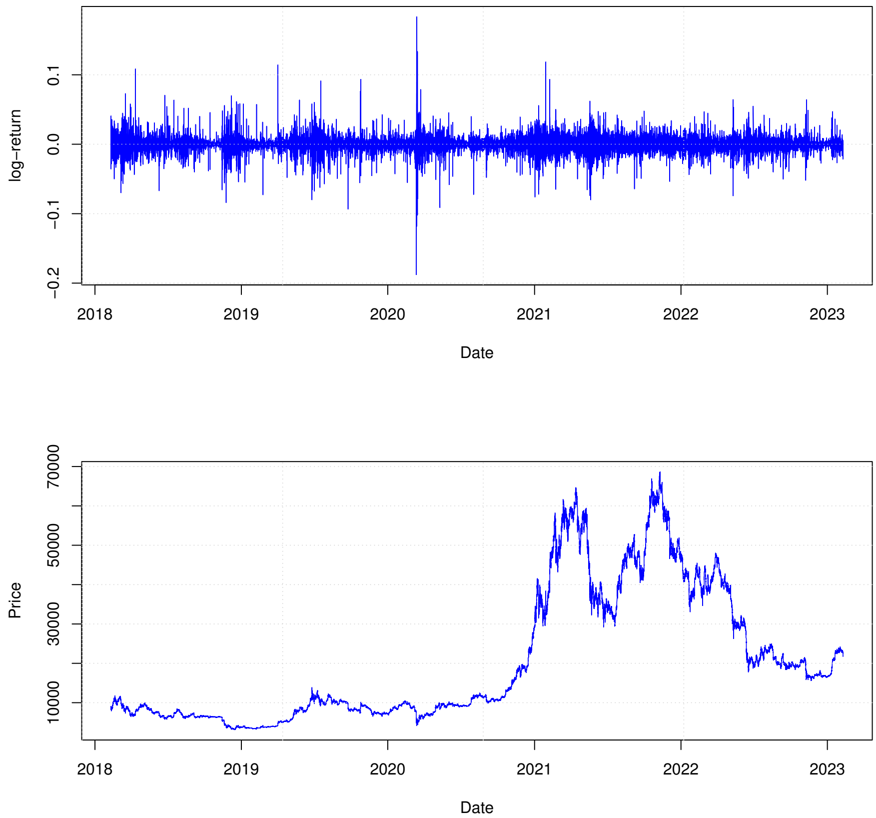

Figure 1: Hourly log-returns of Bitcoin.

Table 1 reports the average and standard deviation of hourly log-returns, such as defined in Equation (54). Jumps are detected with the POT method described in Section 6. The skewness and kurtosis of log-returns for observations without jump are respectively equal to -9.19e-5 and 3.0002. The upper and lower thresholds are close to 1% in absolute values. The estimated Brownian volatility is 0.38% per hour which is huge but relevant for cryptocurrencies. Figure 2 shows the sample paths of intensities, reconstructed from jumps detected by the POT method. Peaks of intensities correspond to large fluctuations of Bitcoin.

Parameters	Values	Parameters	Values
$g(\alpha_1)$	-0.9752%	$g(\alpha_2)$	1.0001%
$\widehat{\mu}_g\Delta$	0.0021%	$\widehat{\sigma}_g\sqrt{\Delta}$	0.7974%
$\widehat{\mu}\Delta$	0.0082%	$\widehat{\sigma}\sqrt{\Delta}$	0.3830%

Table 1: Mean and standard deviation of hourly returns. Thresholds for the POT method.

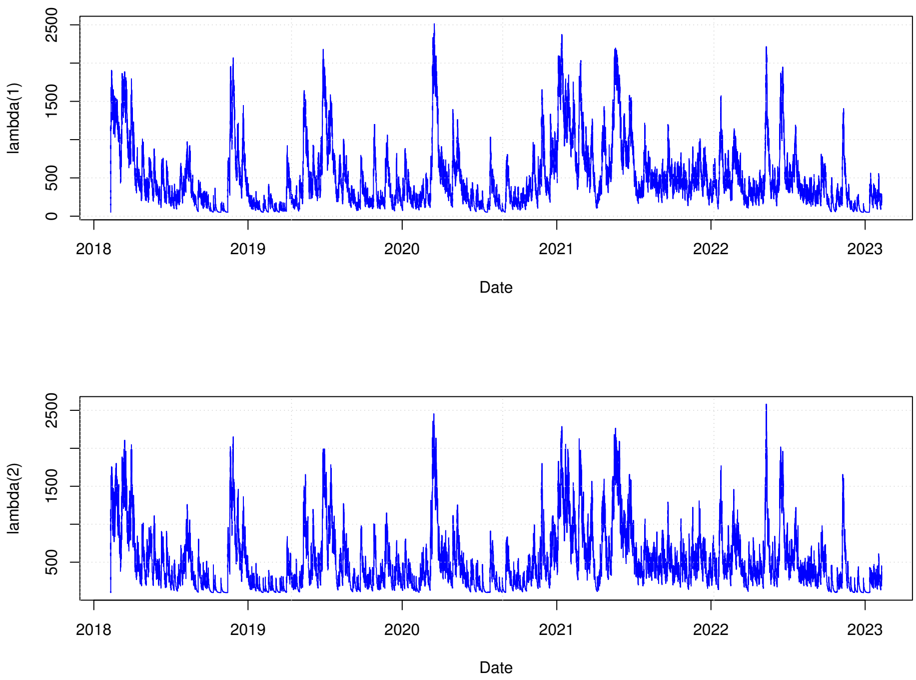

Figure 2: Sample paths of intensities $\lambda_t^{(1)}$ and $\lambda_t^{(2)}$.

The parameter estimates of the bivariate rough jump process are listed in Table 2. The α determines the level of roughness and is around 0.90. The dampening parameter, β is close to 182. The matrix of $\eta_{i,j}$ provides interesting information about self and mutual excitations between shocks. Negative jumps are self-excited ($\widehat{\eta}_{22} = 87.13$) but nearly not influenced by positive jumps as $\widehat{\eta}_{21}$ is close to zero. On the contrary, positive shocks tend to be triggered by negative jumps ($\widehat{\eta}_{11} = 48.68$). In practice, this contagion is explainable by bounce trading strategies. The level of self-excitation is also nearly twice lower for positive jumps than for negative ones ($\widehat{\eta}_{11}$=48.68 whereas $\widehat{\eta}_{22}$= 87.13). The baseline intensity of negative jumps, $\widehat{\lambda}_0^{(2)}$, is twice higher than the one of positive shocks, $\widehat{\lambda}_0^{(1)}$. Nevertheless, asymptotic expected intensities are similar and close to 500 positive and negative jumps per year. In absolute values, the average sizes of positive and negative jumps are comparable and close to $\pm 1.7\%$.

Parameters	Values	Parameters	Values
$\widehat{\alpha}$	0.9061	$\widehat{\beta}$	181.7853
$\widehat{\eta}_{11}$	48.6850	$\widehat{\eta}_{12}$	48.9050
$\widehat{\eta}_{21}$	2.0365	$\widehat{\eta}_{22}$	87.1365
$\widehat{\lambda}_0^{(1)}$	53.8714	$\widehat{\lambda}_0^{(2)}$	101.8721
$\lambda_\infty^{(1)}$	488.9064	$\lambda_\infty^{(2)}$	505.8213
Log-lik. rough Hawkes process, $\mathcal{L}(\widehat{\Theta_N})$: 28046.24			
$\widehat{\rho}_1$	59.4260	$\widehat{\rho}_2$	-57.8427
Log-lik. jumps : 15 458.44			

Table 2: Parameter estimates, rough jump process.

We compare the bivariate rough jump process to its non-rough version ($\alpha = 1$), whose parameters are reported in Table 3. The dampening factor, β and levels of self-excitation, η_{11} and η_{22}, are higher in this last model. As for the rough process, we observe a one way contagion of negative jumps on positive ones. Baseline intensities and asymptotic expected intensities are comparable to those of the rough jump process. In order to check the relevance of the rough jump model, we perform a log-likelihood ratio test. Under the assumption that $\alpha = 1$, the log of the squared ratio of likelihoods,

$$2\left(\ln \mathcal{L}(\widehat{\Theta}_N) - \ln \mathcal{L}(\widehat{\Theta^h_N})\right) \sim \chi^2_1$$

is (asymptotically) a chi-square random variable with one degree of freedom. Using figures reported in Tables 2 and 3, we calculate a p-value of 0.0382%. This confirms that the rough jump model achieves a better goodness of fit than its non-rough version.

Parameters	Values	Parameters	Values
$\widehat{\alpha}$	1.0000	$\widehat{\beta}$	221.0378
$\widehat{\eta}_{11}$	107.089	$\widehat{\eta}_{12}$	90.60763
$\widehat{\eta}_{21}$	0.0416	$\widehat{\eta}_{22}$	174.0245
$\widehat{\lambda}_0^{(1)}$	38.0073	$\widehat{\lambda}_0^{(2)}$	106.9566
$\lambda_\infty^{(1)}$	473.9214	$\lambda_\infty^{(2)}$	503.2868
Log-lik. rough Hawkes process, $\mathcal{L}(\widehat{\Theta^h_N})$: 28 039.94			

Table 3: Parameter estimates, bivariate (non-rough) Hawkes process.

To evaluate the contribution of jump components to the total volatility of log-return, we set σ to zero. To estimate a potential bias induced by the DFFT, we also cancel the drift, $\mu = 0$. Table 4 presents expectations, standard deviations, 5%-95% percentiles of X_t under these assumptions for various maturities (all other parameters being equal to their estimate, Table 2). Calculations are done with the DFFT algorithm of Appendix B, with $x_{max} = 2.2$ and $M = 2^{10}$. We observe a small bias of 0.20% for the log-return, which has a theoretical null expectation. Increasing M reduces the bias but raises the computation time and requires adjusting x_{max} to keep Δ_ω small enough. The standard deviation of X_t climbs from 18% for one month up to 41.2% for a half year. The 90% confidence interval of X_t is large and grows up to $[-70\%, 65\%]$ for a time horizon of 6 months. These statistics confirm that the jump part of X_t is a significant driver of the total log-return volatility.

Days	Expectation	Standard deviation	percentiles 5%	percentiles 95%
30	0.002	0.180	-0.32	0.269
60	0.002	0.262	-0.458	0.398
90	0.002	0.315	-0.548	0.484
180	0.002	0.412	-0.703	0.652

Table 4: Statistics of X_t, without Brownian activity and no drift ($\sigma = 0$ and $\mu = 0$). DFFT with $M = 2^{10}$ components (and $x_{max} = 2.2$).

To understand the influence of rough jumps on derivatives, we evaluate European call options by DFFT and compute their implied volatility by inverting the Black & Scholes formula. We again set the Brownian volatility and the drift to zero to focus on the jump components of log-return. The risk free rate is also null. Options are valued with the same parameters of Table 2. We assume that $S_0 = 100$, we consider strikes K from 50 to 150 and expiry dates from 1 to 6 months.

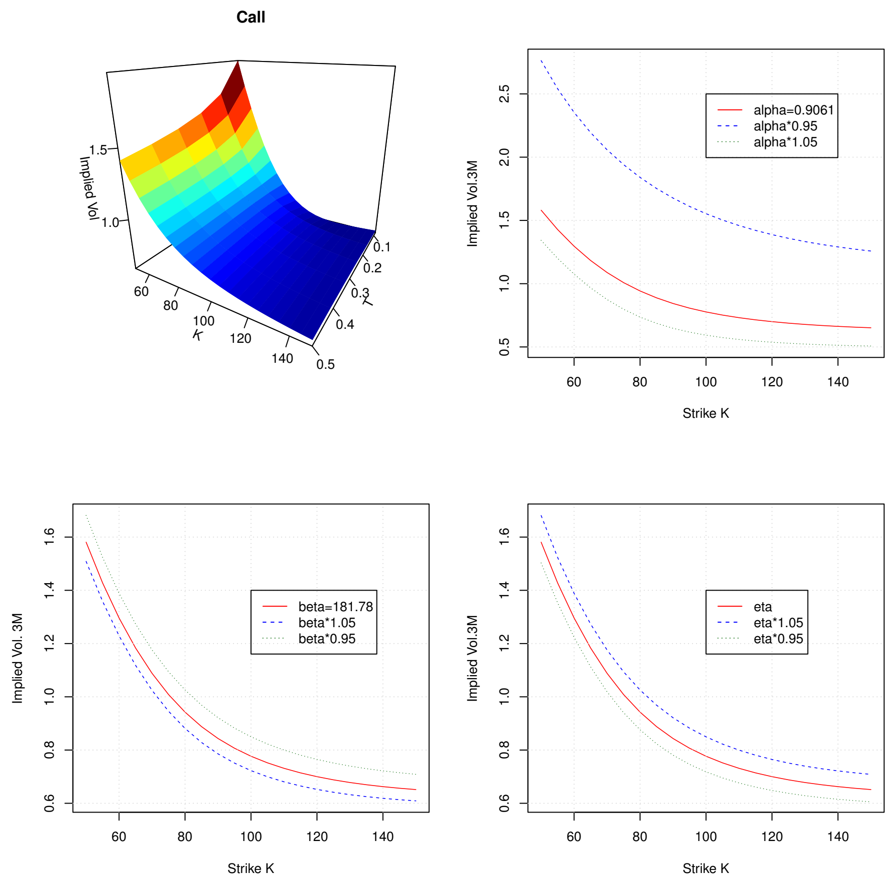

Figure 3: Implied volatility surfaces of European call options ($\sigma = 0$ and $\mu = 0$).

The upper left plot of Figure 3 shows the surface of implied volatilities. They ranges from 61.39% up to 195.58% for deep out and in the money options. The implied volatility generated by jumps with $\mathbb{P}$ parameters, seems large but it is relevant with market data. For instance, the BitVol index, which measures the expected 30-day implied volatility in Bitcoin, evolves between 60% and 100% with a peak up to 168% on the 17/3/2020. The upper right plot of Figure 3 displays 3 months implied volatilities for an α increased and decreased by 5%. Reducing this parameter raises the roughness of the bivariate jump process and therefore implied volatilities. On the contrary, increasing the dampening parameter, β, speeds up the reversion of intensities to their baseline values and then reduces implied volatilities as shown in the left lower plot of Figure 3. The last plot reveals that increasing parameters of self and mutual excitations, drives up implied volatilities.

8 Conclusions

This paper proposes a new type of jump diffusion, the MERJD, with mutual excitation between positive and negative jumps, ruled by dampened rough (DR) memory kernel. This process presents multiple interesting properties for the modelling of asset return with a high volatility such as cryptocurrencies.

Even if the memory kernel diverges at zero, the process remains stable under mild conditions. The MERJD admits an infinite dimensional Markov representation. Considering the limit of a finite approximation of this process allows us to retrieve the Laplace transform of the MERJD. The DR kernel being a Sonine function, we can define a fractional operator close to the Riemann-

Liouville derivative. The Laplace transform of the MERJD may then rewritten in term of a solution of a fractional differential equation (FDE) using this new operator. This FDE can be solved numerically.

The MERJD is perfectly adapted for pricing derivatives. We have indeed defined a family of changes of measure preserving the features of the process and found the conditions under which the new measure is risk neutral. The sensitivity analysis of option prices to parameters reveals that the level of roughness, set by α, has a major impact on implied volatilities.

The MERJD can also be used for risk management purposes since parameters under the real measure may be estimated by a peak over threshold method. This is feasible because the log-likelihood of the bivariate rough jump process admits an analytical expression. The numerical illustration emphasizes the ability of the MERJD to explain price jumps of volatile assets such as the Bitcoin.

Appendix A. Mittag Leffler function

The Mittag-Leffler functions with one and two parameters are respectively defined by

$$\begin{aligned} E_{\alpha}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+1)} \,, \\ E_{\alpha,\beta}(x) &= \sum_{k=0}^{\infty} \frac{x^k}{\Gamma(\alpha k+\beta)} \,, \end{aligned}$$

where $\alpha > 0$ and $\beta \in \mathbb{C}$. The function $u(x) = E_{\alpha}(\eta x^{\alpha})$ is closely related to fractional calculus when $\alpha \in (0,1)$. We denote by $I_{0+}^{\alpha} u$ is the following Riemann-Liouville fractional integral

$$\left(I_{0+}^{\alpha} u\right)(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{u(s)}{(t-s)^{1-\alpha}} ds \,. \tag{60}$$

The left Riemann-Liouville derivative, denoted by $D_{0+}^{\alpha} u(t)$, is the derivative of $I_{0+}^{1-\alpha} u(t)$:

$$\left(D_{0+}^{\alpha} u\right)(t) = \frac{d\left(I_{0+}^{1-\alpha} u\right)(t)}{dt} = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_0^t \frac{u(s)}{(t-s)^{\alpha}} \, ds \,,$$

and is such that $D_{0+}^{\alpha} I_{0+}^{\alpha} u(t) = u(t)$. The solution of the fractional integral/differential equations

$$\left(D_{0+}^{\alpha} u\right)(t) = \eta \, u(t)$$

is precisely the function $u(x) = E_{\alpha}(\eta x^{\alpha})$. In this article we also use the relation

$$\frac{dE_{\alpha}(\,\eta x^{\alpha})}{dx} = \eta x^{\alpha-1} E_{\alpha,\alpha}(\,\eta x^{\alpha}) \,. \tag{61}$$

From the Laplace's transform of $E_{\alpha}(\pm x^{\alpha})$,

$$\mathcal{L}\left(E_{\alpha}(\pm x^{\alpha})\right) := \int_0^{\infty} e^{-zx} E_{\alpha}(\pm x^{\alpha}) \, dx = \frac{z^{\alpha-1}}{z^{\alpha} \mp 1} \,, \tag{62}$$

(see Gorenflo et al. [15], pages 40 and 41), we infer that

$$\mathcal{L}\left(E_{\alpha}(\pm \eta x^{\alpha})\right) = \frac{z^{\alpha-1}}{z^{\alpha} \mp \eta} \,. \tag{63}$$

Appendix B. Fast Fourier's transform

Let M be the number of steps used in the Discrete Fast Fourier's Transform (DFFT) and $\Delta_x = \frac{2x_{max}}{M-1}$, be a step of discretization. Let us denote $\Delta_\omega = \frac{2\pi}{M\,\Delta_x}$ and $\omega_j = (j-1)\Delta_\omega$ for $j = 1, ..., M+1$. The pdf of X_s at points $x_k = -\frac{M}{2}\Delta_x + (k-1)\Delta_x$ for $k = 1, ..., M$ are approached by

$$\begin{aligned} f_s^X(x_k) &= \frac{2}{M\,\Delta_x}\sum_{j=1}^{M}\delta_j \Upsilon_s\left(i\omega_j\right)\exp\left(i\left((k-1)\pi\right)\right) \\ &\quad \times \exp\left(-i(k-1)(j-1)\frac{2\pi}{M}\right), \end{aligned} \tag{64}$$

where$\delta_j = \left(\frac{1}{2}\right)^{1_{\{j_1=1\}}} + 1_{\{j\neq 1\}}$ and $\Upsilon_s(\omega) = \mathbb{E}\left(e^{\omega X_s} \,|\, \mathcal{F}_0\right)$.

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