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Sunspot fluctuations in dynamics with predetermined variables★

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Summary. In this paper, I study the existence of Sunspot Equilibria in a general framework whose dynamics allow for the presence of predetermined variables in the system. The main motivation for this research comes from the fact that previous studies did not allow for such predetermined variables which, nevertheless, appear quite naturally in economic models. I show, for a non-negligible subset of dynamics with predetermined variables verifying usual assumptions, the existence of Stationary Sunspot Equilibria fluctuating between an arbitrary finite number of states arbitrarily close to a steady state.

JEL Classification Numbers: E32, D84, C62.

1 Introduction

The concept of Sunspot Equilibrium (see [4]) has already been used to give some insight about the role played by expectations in the persistent fluctuations and instability of a dynamic economy (see for instance [1], [2], [3], [5], [10], [13], [15]). It is a well-known fact that Rational Expectations Equilibria exhibiting unpredictable but repetitive fluctuations in prices and the allocation of resources may appear in economies with "stable" fundamentals, *i.e.* economies in which the evolution of the fundamental data is perfectly foreseen at every period. Hence, in such economies the unpredictability of the outcome cannot be the consequence of random shocks on the fundamentals. These Rational Expectations Equilibria are traditionally referred to as Sunspot Equilibria in the literature.

In a Sunspot Equilibrium, prices and the allocation of resources are perfectly correlated to some random signal, a "sunspot", by definition

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without influence upon the fundamentals of the economy, just because agents expect them to do so (correctly indeed, since it is a Rational Expectations Equilibrium) and make their decisions accordingly dependent upon such expectations. Moreover, this phenomenon happens even if the agents are aware of the irrelevance of the sunspot for the determination of the fundamentals, *i.e.* endowments, preferences and technology. The belief that the outcome of the economy depends on the sunspot is self-fulfilling.

A class of particularly simple Sunspot Equilibria is that of markovian Stationary Sunspot Equilibria which fluctuate between a finite number k of states following some Markov chain (k-SSE henceforth, standing for Stationary Sunspot Equilibrium of order k). This class of equilibria has been extensively studied in most of the previous references. In [6], [7], [8] and [11] conditions are provided for their existence as solutions to dynamics like $\tilde{Z}(x_t, \mu_{t+1}) = 0$, where x_t is a vector of state variables at period t and μ_{t+1} is the probability distribution of x_{t+1} conditional to x_t representing agents' expectations. Nevertheless, several economic models do not fit this framework. For instance, the dynamics of the economies studied in [18] and [14] take the form of a stochastic difference equation with predetermined variables like $\tilde{Z}(x_{t-1}, x_t, \mu_{t+1}) = 0$.

It is pointed in [7] that the wider range of economic models encompassed by the dynamics with predetermined variables is offset by "clear drop in the stock of existing results. The only clear cut conclusion that has been established so far is a generalization of Woodford's 'conjecture' ". The "conjecture" links the existence of so-called "local" SSE around a steady state with the indeterminacy of such steady state in the perfect foresight dynamics. Specifically, [17] shows that the indeterminacy of the steady state in the perfect foresight dynamics is necessary and sufficient for the existence of local SSE around this steady state. Nonetheless, the SSE shown in [17] depend on the whole infinite past history of the observed sunspots. Spontaneous coordination of agents on one of these equilibria seems to be somewhat problematic since it assumes implicitly economic agents which are able to record and process infinite sets of information and have extremely sophisticated "sunspot theories". On the contrary, finite Sunspot Equilibria require the agents just to be able to distinguish between a finite number of sunspot values and make their decisions according to an extremely simple "sunspot theory". Therefore, it is interesting to know if, alternatively, simple markovian Stationary Sunspot Equilibria with finite support still exist for such systems.

A result related to the one presented in this paper can be found in [16], where the authors study the existence of continua of stationary rational expectations equilibria (SREE) in overlapping generations models with multiple goods and distinct agents per period. They find continua of high-order markovian SREE which may be Sunspot Equilibria for generic economies of this class. The approach followed in the present paper differs substantially from that of [16], since they do not discuss finite-state equilibria and here the analysis does not focuses on an explicit OLG framework as they do, but rather starts from an abstract reduced form of the equilibrium conditions (as in [8]).

Specifically, in this paper I show the existence of *finite* Stationary Sunspot Equilibria with support arbitrarily close to some regular steady state in models with predetermined variables when the state variable is one-dimensional. I provide a sufficient characterization of the systems with k-SSE showing that these are not negligible among those satisfying usual hypotheses. Indeed the characterization becomes necessary too for linear systems, but not for general nonlinear systems as a counterexample will show.

In section 2, I shall present the framework considered. For an example of economies fitting it, see [14] and [18]. The results are provided in section 3. Section 4 gives conclusions and the appendix collects proofs and auxiliary lemmas.

2 The model

One approach to the study of Sunspot Equilibria in dynamical models is developed in [8], where dynamics of the form

$$\tilde{Z}(x_t, \mu_{t+1}) = 0 \tag{1}$$

are studied. In [8] is provided a complete characterization of dynamics of the form (1) which exhibit finite Stationary Sunspot Equilibria following a first order Markov chain, and a close relationship is shown to exist between the existence of these equilibria arbitrarily close to an isolated steady state and the indeterminacy of this steady state in the perfect foresight dynamics. In this paper, I test the robustness of these results with respect to the introduction of predetermined variables, as in the dynamics given by

$$\tilde{Z}(x_{t-1}, x_t, \mu_{t+1}) = 0 \tag{2}$$

I shall give conditions for the existence of *local* (*i.e.* with support arbitrarily close to a steady state) Stationary Sunspot Equilibria of equation (2) fluctuating between any number k of states or, more specifically, k-state second order[2] Markov chains solving (2), when the state variable is one-dimensional.

A k-SSE of (2) can be identified with a k-tuple $\mathbf{x} = (x^1, x^2, \ldots, x^k)$ of states within an open interval X of $\mathbb{R}$ with all of them distinct[3] and k^2 vectors in the interior of the $(k-1)$-dimensional unit simplex, $m_{ij} \in S^{k-1}$, one for each $1 \leq i, j \leq k$, such that

$$\tilde{Z}^k(x^i, x^j, \mathbf{x}, m_{ij}) = 0 \tag{3}$$

for all $i, j, \tilde{Z}^k$ being the restriction of the map $\tilde{Z}$ to probability measures with finite support of cardinal not bigger than k, *i.e.*

[2] The greater length of the Markov chain is needed to obtain non-degenerate existence of k-SSE.

[3] If some of the states coincide, but two of them at least are distinct, the corresponding concept will be that of markovian Stationary Sunspot Equilibrium of *cardinal* k (denoted $\hat{k}$-SSE), that can be seen as a k'-SSE for some $k' < k$.

$$
\begin{aligned}
\tilde{Z}^k : X \times X \times X^k \times S^{k-1} &\longrightarrow \mathbb{R} \\
(x_{-1}, x_0, \mathbf{x}_1, m) &\longrightarrow \tilde{Z}(x_{-1}, x_0, \{\mathbf{x}_1, m\})
\end{aligned} \tag{4}
$$

where $\{\mathbf{x}_1, m\}$ denotes the distribution that associates to the h-th state "tomorrow", x_1^h, a probability equal to the h-th component of m, m^h.

In (4), the arguments of $\tilde{Z}$ are to be interpreted as follows: let x_0 be "the state of a system" at a given date; x_{-1} is then the state at the previous date and $\{\mathbf{x}_1, m\}$ is a finite support probability distribution of the state of the system at the following date. The zeros of $\tilde{Z}$ are interpreted as the equilibria of a sequential economy in which, at each date, agents have some expectations about the state of the system at the following date, possibly conditioned to the information they have about the history of the system, which in this case is the observed current state and the recorded state at the previous date.

An immediate consequence of the definition of the maps $\tilde{Z}^k$ is that if, for any pair of integers $k < k'$, both $\{\mathbf{x}_1, m\} \in X^k \times S^{k-1}$ and $\{\mathbf{y}_1, n\} \in X^{k'} \times S^{k'-1}$ represent the same probability measure on X[4], then for any x_{-1}, x_0 in X

$$
\tilde{Z}^k(x_{-1}, x_0, \mathbf{x}_1, m) = \tilde{Z}^{k'}(x_{-1}, x_0, \mathbf{y}_1, n)\,. \tag{5}
$$

This property conveys the idea that what determines an equilibrium is the actual probability distribution representing the expectations about the state tomorrow, and not the way of writing it. Finally, I shall denote by means of Z the map $\tilde{Z}^1$, that is to say, the restriction of $\tilde{Z}$, to measures of Dirac associating a probability 1 to a given state.

Let us assume the three following hypotheses:

(S) Smoothness of $\tilde{Z}$. *$\tilde{Z}$ is such that, for any integer k, the map $\tilde{Z}^k$ defined in (4) is C^r, for r high enough.*

(R) Existence of a regular steady state. *There exists $\bar{x}$ in X such that $Z(\bar{x}, \bar{x}, \bar{x}) = 0$, $Z_{-1} + Z_0 + Z_1 \neq 0$ and $Z_1 \neq 0$, where, for $h = -1, 0, 1$, Z_h denotes the partial derivative of the map Z with respect to x_h, evaluated at $(\bar{x}, \bar{x}, \bar{x})$. The steady state $\bar{x}$ will be supposed to be the origin without loss of generality.*

(C) Consistency of derivatives. *For any integer $k \geq 2$, if $(x_{-1}, x_0, \mathbf{x}_1, m) \in X \times X \times X^k \times S^{k-1}$ is such that $\mathbf{x}_1$ is in Δ, the diagonal of $\mathbb{R}^k$ (i.e. every state x_1^i is equal to a given value $\bar{x}_1 \in X$) then for any $i = 1, 2, \ldots, k$,*

$$
\tilde{Z}^k_{1,i}(x_{-1}, x_0, \mathbf{x}_1, m) = m^i Z_1(x_{-1}, x_0, \bar{x}_1) \tag{6}
$$

where, for $i = 1, 2, \ldots, k$, $\tilde{Z}^k_{1,i}$ denote the partial derivatives of the map $\tilde{Z}^k$ with respect to x_1^i respectively.

Hypothesis (R) just asks for the existence of some isolated steady state of the system. In the OLG models, hypothesis (C) is a consequence of the

[4] *e.g.* $\{(1,2),(1/3,2/3)\}$ and $\{(1,2,2),(1/3,1/3,1/3)\}$ represent actually the same finite probability measure.

expected utility maximizing behaviour assumption. For a more detailed discussion on the rationale of this assumption see [6], [7] and [11].

For a given integer k, integers $1 \leq i, j \leq k$ and a given vector of the simplex $m \in S^{k-1}$, I shall denote by means of $V^k_{i,j}(m)$ the subset of the space of supports X^k

$$V^k_{ij}(m) = \left\{ \mathbf{x} \in X^k \mid \tilde{Z}^k(x^i, x^j, \mathbf{x}, m) = 0 \right\}. \tag{7}$$

According to (7), the existence of $\hat{k}$-SSE, Stationary Sunspot Equilibria of cardinal[5] k (*i.e.* of order at least 2 but at most k) becomes equivalent to the existence of k^2 vectors of the unit simplex $m_{ij} \in S^{k-1}$, one for each couple $1 \leq i, j \leq k$, such that the off-diagonal intersection of all the sets $V^k_{ij}(m)$ is not empty. Moreover, if for some point in this intersection all the k states are distinct, then that point will be in fact a k-SSE. Finally, if it is possible to find k-SSE's with support arbitrarily close to the steady state, we shall say that there exists local k-SSE around it.

In what follows, I shall make a local study of the sets $V^k_{ij}(m)$ around the steady state $\bar{\mathbf{x}} = (\bar{x}, \bar{x}, \ldots, \bar{x})$, for arbitrary probabilities of transition between states m_{ij}. Afterwards, I shall characterize a "big" set of maps $\tilde{Z}$, verifying the assumptions, for which uncountably many local k-SSE of any order exist. This will suffice to show that sunspot fluctuations between a finite number of states is a possible and likely rational expectations equilibrium outcome for a large family of systems fitting this framework.

3 Results

It follows immediately from the previous definitions the following property.

Proposition 1. *If hypothesis* (R) *holds, then, for any integer $k \geq 2$, and for any $m_{ij} \in S^{k-1}$ (one for each $1 \leq i, j \leq k$), $\bar{\mathbf{x}}$ is in the intersection of all $V^k_{ij}(m_{ij})$.*

That is to say, for any integer k and for any probabilities of transition, the support corresponding to the steady state $\bar{\mathbf{x}}$ satisfies all the equilibrium conditions. Thus the steady state is a "degenerate k-SSE" with all its states identical and correlated to any Markov chain. But what is actually interesting is the appearance of "true" k-SSE (or $\hat{k}$-SSE in general), *i.e.* the existence of elements of the intersection of all $V^k_{ij}(m_{ij})$ which are not in the diagonal Δ of $\mathbb{R}^k$, for some adequate choice of the probabilities m_{ij}. In order to cope with this question I shall make use of the following property.

Proposition 2. *If hypotheses* (S), (R) *and* (C) *hold, then, for any integer $k \geq 2$, for all $1 \leq i, j \leq k$ and for any $m_{ij} \in S^{k-1}$, the set $V^k_{ij}(m_{ij})$ is locally a smooth manifold of dimension $k-1$, in a neighbourhood of the support of the steady state $\bar{\mathbf{x}}$. Moreover, its tangent space at this point is the orthogonal complement of the space spanned by the vector $m_{ij} - (Ae^i + Ce^j)$, where e^i is the i-th vector of the canonical basis in $\mathbb{R}^k$, $A = -Z_{-1}/Z_1$, $C = -Z_0/Z_1$, with Z_{-1}, Z_0 and Z_1*

[5]See footnote 1.

being the partial derivatives of the map Z with respect to x_{-1}, x_0 and x_1 evaluated at $(\bar{x}, \bar{x}, \bar{x})$.

Proposition 2 allows to provide a simple characterization of the local structure of the sets $V_{ij}^k(m_{ij})$ around the steady state, for any given $m_{ij} \in S^{k-1}$. In effect, the tangent space of $V_{ij}^k(m_{ij})$ at $\bar{\mathbf{x}}$ is characterized by its orthogonal complement, *i.e.* the space spanned by the gradient $Z_{-1}e^i + Z_0 e^j + Z_1 m_{ij}$. Hence, only the ratios A and C are needed for characterizing completely the local structure of all the manifolds $V_{ij}^k(m_{ij})$ around the steady state.

Notice that the manifolds $V_{ij}^k(m_{ij})$ depend continuous and injectively on the vectors of the simplex m_{ij}, as their tangent spaces do. Notice as well that, as a consequence of the hypothesis (R), the diagonal Δ of $\mathbb{R}^k$ never lies in a tangent space of a manifold $V_{ij}^k(m_{ij})$ at the steady state.

The next proposition establishes a sufficient condition for off-diagonal intersections of the manifolds $V_{ij}^k(m_{ij})$ arbitrarily close to the steady state to exist.

Proposition 3. *Assume that $\tilde{Z}$ verifies hypotheses* (S) (R) *and* (C). *If $\tilde{Z}$ is such that (condition (*)), for some integer $k \geq 2$, there exists vectors $m_{ij}^* \in S^{k-1}$, one for each couple $1 \leq i, j \leq k$, such that the tangent spaces of all the manifolds $V_{ij}^k(m_{ij}^*)$ at the steady state $\bar{\mathbf{x}}$ have a non-trivial intersection, then there exists a neighbourhood of the steady state in this intersection subspace, such that any support in this neighbourhood distinct from the steady state is in all the manifolds $V_{ij}^k(m_{ij})$ for some vectors m_{ij} in the simplex S^{k-1}.*

Notice that all the supports in this neighbourhood will correspond to $\hat{k}$-SSE, since as it was remarked above, the diagonal Δ of $\mathbb{R}^k$ cannot be contained in any of the tangent spaces, nor *a fortiori* in their intersection. Moreover, when the dimension of the intersection subspace is $k-1$ (*i.e.* all the manifolds have the same tangent space at the steady state) then "most"[6] of these $\hat{k}$-SSE are in fact k-SSE. But what is important to know is whether the maps $\tilde{Z}$ satisfying hypotheses (S), (R) and (C) for which the condition (*) in Proposition 3 holds are "general enough", in order to make this proposition relevant for characterizing systems exhibiting k-SSE. I shall give, for any $k \geq 2$, a characterization of the set of maps $\tilde{Z}$ such that (*) holds and it will turn out to be "non-negligible" among those verifying the assumptions. To do this, let us first characterize the maps $\tilde{Z}$ which verify the condition (*) for the special case where the intersection subspace has the maximal dimension $k-1$; that is to say, the case in which all the manifolds $V_{ij}^k(m)$ share the same tangent space at the steady state.

Proposition 4. *For any integer $k \geq 2$, a map $\tilde{Z}$ satisfying hypotheses* (S), (R) *and* (C) *is such that the condition (*) is satisfied with maximal dimension $k-1$ if, and only if, (A, C) is in $S(k)$, defined as follows*

[6]An open and dense subset of them.

$$S(k) = \left\{ (A, C) \in \mathbb{R}^2 \middle| -\frac{1}{k-1} < A + C < 1 \right.$$
$$\left. \text{and } \frac{1}{k-1} A - \frac{1}{k-1} < C < (k-1)\, A + 1 \right\}. \tag{8}$$

Notice that if $k = 2$, whenever $(^*)$ is satisfied, then it will be so with maximal dimension necessarily. Thus in this case, $S(2)$ characterizes completely the maps $\tilde{Z}$ verifying the condition $(^*)$. On the contrary, if $k > 2$, $S(k)$ characterizes a proper subset of the maps $\tilde{Z}$ verifying the condition $(^*)$. Nevertheless, actually $S(2)$ characterizes completely the maps $\tilde{Z}$ verifying the condition $(^*)$ for any integer $k \geq 2$ too, as it established in the following proposition.

Proposition 5. *For any integer $k \geq 2$, a map $\tilde{Z}$ satisfying hypotheses* (S), (R) *and* (C) *is such that the condition* $(^*)$ *is verified if, and only if,* (A, C) *is in* $S(2)$, *defined as follows*

$$S(2) = \left\{ (A, C) \in \mathbb{R}^2 \middle| -1 < A + C < 1 \textit{ and } -1 < A - C < 1 \right\}. \tag{9}$$

Since $S(2)$ is a full-dimension subset of the parameters (A, C) space $\mathbb{R}^2$ (it is the unit open ball of $\mathbb{R}^2$ with the norm $||\cdot||_1$), according to Proposition 5, Proposition 3 is verified by a non negligible family of dynamics (2) verifying the assumptions (S), (R) and (C). As a corollary of the previous propositions, we have the following result about the existence of k-SSE for dynamics of the class (2).

Theorem 1. *If the map $\tilde{Z}$ of* (2) *verifies hypotheses* (S), (R) *and* (C), *and* (A, C) *is in* $S(2)$, *then the equation* (2) *has local k-SSE of any order* $k \geq 2$.

This result follows from the existence of local $\hat{k}$-SSE under the conditions of the theorem, which is an immediate consequence of propositions 3 and 5. The fact that "most" of these $\hat{k}$-SSE can be chosen to be actual k-SSE is shown in the proof of proposition 5.

As it can be seen in the Appendix, the result depends on the fact that every point in a tangent space of a manifold close enough to the steady state can be attained by some other manifold of the same family perturbing the probabilities m. Thus if for some probabilities all the tangent space have a non-trivial intersection, all the manifolds can be made intersect at any of its points close to the steady state. Clearly, if Z is a linear mapping, the condition $(^*)$ is not only sufficient but necessary too for this procedure to work, since in this case manifolds and tangent spaces are the same objects. Unfortunately, when Z is not linear the condition $(^*)$ is not necessary in general as the following counterexample shows.

Consider in $\mathbb{R}^2$ two families, parameterized by $\alpha \in (0, 1)$, of arcs of circle going through $(0, 0)$ and $(\alpha, 0)$ and one with centre at $(\frac{\alpha}{2}, \ln(\frac{\alpha}{2} + 1))$, the other with centre at $(\frac{\alpha}{2}, -\ln(\frac{\alpha}{2} + 1))$. Clearly any point $(x, 0)$ with $0 < x < 1$ is in both curves if $x = \alpha$, while there is no α for which the corresponding curves share the same tangent.

4 Conclusions

In this paper it has been shown that Stationary Sunspot Equilibria of the class k-SSE exist arbitrarily close to a regular steady state for general one-dimensional one-step forward looking dynamical systems depending on predetermined variables under usual assumptions. Such systems appear naturally in economic models that do not fit the frameworks previously studied in the literature. The characterization provided in this paper of the dynamics with predetermined variables with k-SSE shows that, actually, they are not negligible.

Notice that the systems characterized by Theorem 1 have both parameters A and C "close" to 0. Therefore, the perfect foresight dynamics associated to them converges very fast to the steady state in a neighbourhood of it. Actually, the region $S(2)$ of parameters A and C is a proper subset of the region

$$W = \left\{(A, C) \in \mathbb{R}^2 | C < 1 - A, C > A - 1 \text{ et } A > -1\right\} \tag{10}$$

characterizing the systems for which the steady state is indeterminate in the perfect foresight dynamics[7]. Hence, according to the results shown in [17], local Stationary Sunspot Equilibria exist for any dynamics of the class (2) belonging to this bigger set. Nevertheless, the SSE shown in [17] depend on the whole infinite past history of the system at any period t, in contrast with the simple k-SSE shown in this paper. More interestingly, the sufficient characterization of the dynamics (2) with k-SSE provided by Theorem 1 becomes necessary too if the map Z is linear. Since [17] shows the existence of local SSE for dynamics (2) not satisfying this necessary characterization, for the linear case such SSE cannot be of the class k-SSE. This shows that the existence of SSE *à la Woodford* and of k-SSE close to a steady state are distinct questions in general.

Finally, the results provided here can be generalized to the case of a system with predetermined variables up to any finite number of periods in the past, as in

$$\tilde{Z}(x_{t-p}, \ldots, x_{t-1}, x_t, \mu_{t+1}) = 0 . \tag{11}$$

Straightforward restatements of the assumptions of smoothness, existence and regularity of a steady state, and consistence of derivatives suffice to show the existence of local k-SSE of any order for such an equation. The only condition needed to be fulfilled is a strongly enough convergent perfect foresight dynamics around the steady state (for a detailed development of this question see [9]).

[7] This follows from the characterization of the systems with all the eigenvalues of their linearization around the steady state within the unit disk in the complex plane.

Appendix

Proof of Proposition 1. Immediate from the definitions.
Proof of Proposition 2. For any $k \geq 2$, $1 \leq i, j \leq k$ and $m \in S^{k-1}$ consider the following C^r map:

$$\begin{aligned} F_m^{ij} &: X^k \longrightarrow \mathbb{R} \\ &\mathbf{x} \longrightarrow \tilde{Z}^k(x^i, x^j, \mathbf{x}, m) \end{aligned} \tag{12}$$

Then, $V_{ij}^k(m)$ is the preimage of 0 by this map, $(F_m^{ij})^{-1}(0)$. By hypothesis (C), the gradient of F_m^{ij} at the steady state $\bar{\mathbf{x}}$, $DF_m^{ij}(\bar{\mathbf{x}})$, is $Z_{-1}e^i + Z_0 e^j + Z_1 m$, with a straightforward notation for partial derivatives evaluated at $(\bar{x}, \bar{x}, \bar{x})$. But then, for any $m \in S^{k-1}$, $\langle DF_m^{ij}(\bar{\mathbf{x}}), \mathbf{1}\rangle \neq 0$ (where $\mathbf{1} = (1, \ldots, 1) \in \mathbb{R}^k$), since

$$\begin{aligned} \langle DF_m^{ij}(\bar{\mathbf{x}}), \mathbf{1}\rangle &= Z_{-1} + Z_0 + Z_1 \sum_{h=1}^{k} m^h \\ &= Z_{-1} + Z_0 + Z_1 \neq 0 \end{aligned} \tag{13}$$

by the hypothesis (R). Then, it is clear that $DF_m^{ij}(\bar{\mathbf{x}})$ is not the null vector. Hence, F_m^{ij} is a submersion in a neighbourhood U of $\bar{\mathbf{x}}$ in $\mathbb{R}^k$. Thus $0 \in \mathbb{R}$ is a regular value of F_m^{ij} and, by the Preimage Theorem (see [12]), its preimage $(F_m^{ij})^{-1}(0) \cap U$ is a smooth manifold of dimension $k-1$, with tangent space orthogonal to any vector in the space spanned by its gradient, namely orthogonal to $m - (Ae^i + Ce^j)$. □

The next two lemmas that will be used in the proof of the Proposition 3.

Lemma 1. *Let $f : \mathcal{O} \times (H \cap IntP) \to W$ be a C^r map, where $\mathcal{O}$ is an open interval of $\mathbb{R}$, H is a line in $\mathbb{R}^2$ not containing the origin, P a cone in $\mathbb{R}^2$ with non-empty interior such that $H \cap IntP \neq \emptyset$, and W is a neighbourhood of 0 in $\mathbb{R}^2$. If given any $d \in H \cap IntP$, the image $f(\mathcal{O}, d)$ is a differentiable curve C_d going through 0 in $\mathbb{R}^2$ and its tangent at this point, $T_0(C_d)$, is orthogonal to d, then for any $d \in H \cap IntP$, there exists a neighbourhood of 0 in $T_0(C_d)$, V_0, such that $\forall y \in V_0$, $\exists d' \in H \cap IntP \,|\, y \in C_{d'}$.*

Proof of Lemma 1. For any $d \in H \cap \mathrm{Int}P$, and for any neighbourhood W of 0 in $\mathbb{R}^2$, $T_0(C_d)$ splits W in two subsets, W^1 and W^2, such that $W^1 \cup W^2 = W$ and $W^1 \cap W^2 = T_0(C_d) \cap W$. Similarly, the line orthogonal to $T_0(C_d)$, D, spanned by d splits as well W in two other subsets, W^a and W^b, such that $W^a \cup W^b = W$ and $W^a \cap W^b = D \cap W$. Let us define $W^{ij} = W^i \cap W^j$, $\forall (i, j) \in \{1, 2\} \times \{a, b\}$. Then, necessarily, it is the case that either $C_d \cap W \subseteq T_0(C_d)$ or not. In the first case, $C_d \cap W$ is already the neighbourhood we are looking for and C_d itself is the curve that contains any point of it. In the second case, if there is no other neighbourhood W' of 0, $W' \subset W$, such that the first case applies, then C_d intersects the interior of at least one of the W^{ij}, say W^{1a} without loss of generality. Then, for W small enough, C_d does not intersect the interior of W^{2a} in the other halfspace defined by the tangent space, *i.e.* $C_d \cap \mathrm{Int}W^{1a} \neq \emptyset$ and $C_d \cap \mathrm{Int}W^{2a} = \emptyset$. Now, since

$d \in \mathrm{Int}P$, there exists $d' \in (H \cap \mathrm{Int}P)$ arbitrarily close to d, such that $T_0(C_{d'}) \cap \mathrm{Int}W^{2a} \neq \emptyset$, that is to say, such that $C_{d'} \cap \mathrm{Int}W^{2a} \neq \emptyset$ actually. But d' can be chosen sufficiently close to d for $C_{d'} \cap \mathrm{Int}W^{1a} \neq \emptyset$ to hold, by the continuity of the map f. Hence, given that $C_{d'}$ is itself a continuous curve, there must exist a point $x \neq 0$ in $T_0(C_d) \cap W^a$ in which $C_{d'}$ and $T_0(C_d)$ intersect. This intersection point can be found arbitrarily close to 0, for d' close enough to d. Should this not be true, there would be some neighbourhood $W' \subset W$ such that the first case does apply. □

Lemma 2. *Let $F : \mathcal{O} \times (H \cap \mathrm{Int}P) \to W$ be a C^r map, where $\mathcal{O}$ is an open set of $\mathbb{R}^{k-1}$, H is a $(k-1)$-dimensional hyperplane in $\mathbb{R}^k$ not containing the origin, P a cone in $\mathbb{R}^k$ with non-empty interior such that $H \cap \mathrm{Int}P \neq \emptyset$, and W is a neighbourhood of 0 in $\mathbb{R}^k$. If given any $d \in H \cap \mathrm{Int}P$, the image $F(\mathcal{O}, d)$ is a smooth $(k-1)$-dimensional manifold, M_d, going through 0 and its tangent space at this point, $T_0(M_d)$, is orthogonal to d, then for any $d \in H \cap \mathrm{Int}P$, there exists a neighbourhood of 0 in $T_0(M_d)$, V_0, such that $\forall y \in V_0$, $\exists d' \in H \cap \mathrm{Int}P \,|\, y \in M_{d'}$.*

Proof of Lemma 2. For any $d \in H \cap \mathrm{Int}P$, let M_d be the associated manifold, x a point of $T_0(M_d)$ distinct from 0, $S_x \subseteq T_0(M_d)$ the one-dimensional linear subspace spanned by x, and S_x^d the two-dimensional linear subspace spanned by x and d. For any $d' \in (H \cap \mathrm{Int}P) \cap S_x^d$, $\tilde{M}_{d'} = M_{d'} \cap S_x^d$ is locally a one-dimensional manifold in a neighbourhood of 0 in R^k (*i.e.* a differentiable curve) contained in S_x^d, since S_x^d contains d', the orthogonal direction of $M_{d'}$ at 0, and S_x^d is then locally transversal to $M_{d'}$. Then, there exists a C^r map $f : \mathcal{O}_x \times (H_x \cap \mathrm{Int}P_x) \to W_x$ (where $\mathcal{O}_x$ is an open interval in $\mathbb{R}$, and H_x, P_x and W_x the intersections of H, P and W with S_x^d), such that Lemma 1 applies. Consequently, for $d \in (H \cap \mathrm{Int}P) \cap S_x^d$ there exists a neighbourhood V_0^x of 0 in S_x, such that $\forall y \in V_0^x$, $\exists d' \in H_x \cap \mathrm{Int}P_x \subseteq H \cap \mathrm{Int}P \,|\, y \in M_{d'}$. Since this argument does not depend on x in $T_0(M_d)$, there exists then a $V_0 \subseteq T_0(M_d)$ satisfying the conclusion of the lemma. □

Proposition 3 follows from Lemma 2.

Proof of Proposition 3. Assume that, for $k \geq 2$, there exists probability distributions $m_{ij}^* \in S^{k-1}$, $1 \leq i,j \leq k$, such that the intersection of the tangent spaces of all the manifolds $V_{ij}^k(m_{ij}^*)$ at the steady state, is a linear subspace of dimension $k' \in \{1, \ldots, k-1\}$. Define for each $1 \leq i,j \leq k$ a parallel shift of the simplex S^{k-1} parameterized by (A, C), d_{ij}^{AC}, associating to each m in S^{k-1} the vector $m - (Ae^i + Ce^j)$. Whenever $A + C \neq 1$, such a shift associates distinct directions $d_{ij}^{AC}(m)$ in $\mathbb{R}^k$ to different vectors m in S^{k-1}. For any $1 \leq i,j \leq k$, consider the following C^r map

$$\begin{aligned} G^{ij} : X^k \times S^{k-1} &\longrightarrow \mathbb{R} \\ (\mathbf{x}, m) &\longrightarrow \tilde{Z}^k(x^i, x^j, \mathbf{x}, m) \end{aligned} \tag{14}$$

and the equation $G^{ij}(\mathbf{x}, m) = 0$, verified in particular by $(\bar{\mathbf{x}}, m_{ij}^*)$. The jacobian of G^{ij} at $(\bar{\mathbf{x}}, m_{ij}^*)$ is $DG^{ij}(\bar{\mathbf{x}}, m_{ij}^*) = (D_x G^{ij}(\bar{\mathbf{x}}, m_{ij}^*), D_m G^{ij}(\bar{\mathbf{x}}, m_{ij}^*))$. But actually

$D_xG^{ij}(\bar{\mathbf{x}}, m^*_{ij})$ is $DF^{ij}_{m^*_{ij}}(\bar{\mathbf{x}})$ (where F^{ij}_m is the map defined in the proof of the Proposition 2), which is non-null by (13). Then, by the Implicit Function Theorem, $G^{ij}(\mathbf{x}, m_{ij}) = 0$ defines locally one of the coordinates of $\mathbf{x}$, say x^1 without loss of generality, as a function of the other $k-1$ coordinates (denoted by $\mathbf{x}^{-1} \in \mathbb{R}^{k-1}$) and the vector m_{ij}, in an open neighbourhood $\mathcal{O}^{ij} \times \mu^{ij}$ of $(\bar{\mathbf{x}}^{-1}, m^*_{ij})$ in $\mathbb{R}^{k-1} \times S^{k-1}$. This function $x^1(\bar{\mathbf{x}}^{-1}, m_{ij})$ is C^r too. Now, consider the map

$$\begin{aligned} \mathbf{\Phi}^{ij} : \mathcal{O}^{ij} \times d^{AC}_{ij}(\mu^{ij}) &\longrightarrow \mathbb{R}^k \\ (\mathbf{x}^{-1}, d^{AC}_{ij}(m_{ij})) &\longrightarrow (x^1(\mathbf{x}^{-1}, m_{ij}), \mathbf{x}^{-1}) . \end{aligned} \tag{15}$$

This map is well defined since d^{AC}_{ij} is a bijection onto its image and thus so is its restriction to μ^{ij}. Moreover, $\mathbf{\Phi}^{ij}$ is also C^r, $\mathcal{O}^{ij}$ is an open neighbourhood of $0 \in \mathbb{R}^{k-1}$, $d^{AC}_{ij}(\mu^{ij})$ is contained in the affine hyperplane $\langle \mathbf{x}, \mathbf{1} \rangle = 1 - (A + C)$ and $d^{AC}_{ij}(\mu^{ij})$ spans a cone in $\mathbb{R}^k$ with non-empty interior. Besides, for any $d^{AC}_{ij}(m_{ij}) \in d^{AC}_{ij}(\mu^{ij})$, and for $\mathcal{O}^{ij}$ small enough, the image $\mathbf{\Phi}^{ij}(\mathcal{O}^{ij}, d^{AC}_{ij}(m_{ij}))$ is a smooth manifold of dimension $k-1$ going through $0 \in \mathbb{R}^k$ and such that $d^{AC}_{ij}(m_{ij})$ is orthogonal to its tangent space. That is to say, the map $\mathbf{\Phi}^{ij}$ satisfies the hypotheses of the Lemma 2, and hence there exists a neighbourhood of the steady state $\bar{\mathbf{x}} = 0$ in the tangent space of the manifold $V^k_{ij}(m^*_{ij})$, such that any point of it is in some other $V^k_{ij}(m_{ij})$ for some $d^{AC}_{ij}(m_{ij})$, *i.e.* for some $m_{ij} \in S^{k-1}$. Applying this argument to any point distinct from the steady state $\bar{\mathbf{x}}$ and belonging to all these neighbourhoods, for $1 \le i, j \le k$, in the intersection of the tangent spaces, the conclusion of Proposition 3 follows. □

Proposition 4 is an immediate consequence of the following lemma:

Lemma 3. *For any integer $k \geq 2$, $\cap_{i,j} d^{AC}_{ij}(S^{k-1})$ is non-empty if, and only if $k \max\{0, -A, -C, -(A + C)\} < 1 - (A + C)$.*

Proof of Lemma 3. For any $1 \le i, j \le k$, from the definition of the map d^{AC}_{ij} it can be seen that $d^{AC}_{ij}(S^{k-1})$ is $\{\mathbf{x} \in \mathbb{R}^k | \, \mathbf{x} + Ae^i + Ce^j \in S^{k-1}\}$ or, what is the same thing, $\{\mathbf{x} \in \mathbb{R}^k | \, \langle \mathbf{x}, \mathbf{1} \rangle = 1 - (A + C)$ and $\mathbf{x} \gg o_{ij}\}$ where $o_{ij} = (\ldots, o^s_{ij}, \ldots)$ is $-(Ae^i + Ce^j)$. Then, the intersection for all i, j of $d^{AC}_{ij}(S^{k-1})$ is $\{\mathbf{x} \in \mathbb{R}^k | \, \langle \mathbf{x}, \mathbf{1} \rangle = 1 - (A + C)$ and $\mathbf{x} \gg o_{ij}, \forall i, j\}$. Now, $\{\mathbf{x} \in \mathbb{R}^k | \, \mathbf{x} \gg o_{ij}, \forall i, j\}$ is $\{\mathbf{x} \in \mathbb{R}^k | \, \mathbf{x} \gg \sum_{s=1}^k \max_{i,j}\{o^s_{ij}\} e^s\}$, but since $o^s_{ij} \in \{0, -A, -C, -(A + C)\}$ for any s, j, i and, for a given s all these values are taken by o^s_{ij} for some i,j, then $\max_{i,j}(o^s_{ij})$ does not depend on s, but is equal to $\max\{0, -A, -C, -(A + C)\}$. Thus, the intersection of all $d^{AC}_{ij}(S^{k-1})$ is all $\mathbf{x} \in \mathbb{R}^k$ such that $\langle \mathbf{x}, \mathbf{1} \rangle = 1 - (A + C)$ and $\mathbf{x} \gg \max\{0, -A, -C, -(A + C)\} \sum_{s=1}^k e^s$. Thus it is non-empty if, and only if, $k \max\{0, -A, -C, -(A + C)\} < 1 - (A + C)$. □

Let us now prove Proposition 4 with the aid of Lemma 3.

Proof of Proposition 4. For any $k \geq 2$, the map $\tilde{Z}$ satisfying hypotheses (S), (R) and (C) is such that the condition (*) is verified with maximal dimension

$k-1$ if, and only if, there exist m^*_{ij}, one for each $1 \le i,j \le k$, such that all the tangent spaces of the manifolds $V^k_{ij}(m^*_{ij})$ at the steady state have the same orthogonal complement, *i.e.* only if the normal vectors $d^{AC}_{ij}(m^*_{ij}) = m^*_{ij} - (Ae^i + Ce^j)$ are equal, since they belong to the same affine subspace. But this is only possible if $\cap_{i,j} d^{AC}_{ij}(S^{k-1})$ is non-empty, what happens, by Lemma 3, only if $k\ \max\{0, -A, -C, -(A+C)\} < 1-(A+C)$. It is immediate to verify that the values of (A, C) satisfying this condition are those of $S(k)$ in Proposition 4. □

Proof of Proposition 5. For $k = 2$, the result is already established in Proposition 4. Next notice that the condition (*) is equivalent to ask for the vectors $m^*_{ij} \in S^{k-1}$ to be such that the k^2 hyperplanes $\langle m^*_{ij} - (Ae^i + Ce^j), \mathbf{x}\rangle = 0$ have a non-trivial intersection, *i.e.* equivalent to ask for the existence of $\mathbf{x} \neq 0$ such that $Ax^i + Cx^j = \langle m^*_{ij}, \mathbf{x}\rangle$. This amounts to characterizing the (A, C) such that there is a $\mathbf{x} \neq 0$ with all $Ax^i + Cx^j$ in the interior of its convex hull in $\mathbb{R}$. Now, if $(A, C) \in S(2)$ then, by Proposition 4, the condition (*) is verified for $k = 2$, *i.e.* there exist vectors $m^*_{ij} \in S^1$, one for each $1 \le i,j \le 2$, and a support (x^1, x^2) such that $Ax^i + Cx^j = \langle m^*_{ij}, \mathbf{x}\rangle$ Now, let $\mathbf{y}$ be (x^1, x^1, x^2). Then, there exist vectors $n^*_{ij} \in S^2$ such that $Ay^i + Cy^j = \langle n^*_{ij}, \mathbf{y}\rangle$ holds for all $1 \le i,j \le 3$, since any $Ay^i + Cy^j$ is equal to some $Ax^r + Cx^s$, for adequate values $1 \le r,s \le 2$. Thus if (A, C) is in $S(2)$, the condition (*) is verified for $k = 3$. Moreover, for any support $\tilde{\mathbf{y}}$ close enough to $\mathbf{y}$ but with all its coordinates distinct, continuity guarantees that all the $A\tilde{y}^i + C\tilde{y}^j$ are still in the interior of the convex hull of $\tilde{\mathbf{y}}$ in $\mathbb{R}$. It can be shown by induction, following this continuity argument, that if condition (*) is verified for $k = 2$ (what is true only for the (A, C) in $S(2)$), then it is verified for any integer $k > 2$ also.

Conversely, if condition (*) is verified for some $k > 2$ then there exist some support $\mathbf{x}$ distinct from the steady state, such that all the $Ax^i + Cx^j$ are in the interior of its convex hull in $\mathbb{R}$. Let y^1 and y^2 be respectively the smallest and biggest coordinates of $\mathbf{x}$. Then, it is still true that, for any $1 \le i,j \le 2$, $Ay^i + Cy^j$ is in the interior of the convex hull of (y^1, y^2) in $\mathbb{R}$, so that for $k = 2$ condition (*) must be verified and this, by Proposition 4, happens only if $(A, C) \in S(2)$. □

Proof of Theorem 1. If the map $\tilde{Z}$ verifies hypotheses (S), (R) and (C), Proposition 3 provides, for any given integer $k \ge 2$, a sufficient condition (*) for the existence of probabilities of transition m_{ij} such that the manifolds $V^k_{ij}(m_{ij})$, intersect at supports arbitrarily close to the steady state. These supports correspond to $\hat{k}$-SSE and not to other steady states since the normal directions of the manifolds are not orthogonal to the diagonal Δ of $\mathbb{R}^k$ by (R). Hence, Δ cannot be in any of the tangent spaces, nor in their intersection. Now, by Proposition 5, the maps $\tilde{Z}$ satisfying that sufficient condition (*) are only those for which $(A, C) \in S(2)$ holds.

Moreover, as seen in the proof of Proposition 5, for any integer $k \ge 2$, supports arbitrarily close to the steady state with all the k states distinct can be attained by the manifolds $V^k_{ij}(m_{ij})$, thus corresponding to k-SSE. □

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