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Invariance with respect to Re-Evaluations of Coalitional Power

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Abstract

If x is a reasonable agreement in a game V , then so should it remain in the associated game V_x^* where the coalitions can buy up the cooperation of non-members by “paying” them according to x . This new stability property called “Invariance with respect to re-evaluations of coalitional power” (IRCP) allows to characterize the core as the largest solution specifying feasible allocations that are individually rational. In addition, a natural adaptation of IRCP allows to elegantly characterize the inner core for NTU games with convex and smooth feasible sets.

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Introduction

The reduced game property is a self-consistency requirement: the restriction of a payoff vector in the solution of a game to a subset of players is in the solution of the reduced game. This property allows to axiomatize most traditional solution concepts in game theory (cf. Thomson (1996) for a survey). The key is to adequately define the reduced games. Peleg (1985-1986) used the traditional Davis and Maschler (1965) reduction process in his axiomatization of the core. Let $x \in \mathbb{R}^N$ be a potential agreement for splitting the surplus of cooperation and let S be a subset of players. Then, the reduced game defined on S is obtained by considering that each strict subset of S *may* buy up the cooperation of players outside S , while S itself *has to* buy up the cooperation of all the players outside S , the price of the players being specified by x . Why such an asymmetry in the definition of the reduced games? In fact, many alternative definitions have already been considered in the literature, imposing for instance that each subset of S has to buy up the cooperation of players outside S (cf. Tadenuma (1992)) or that S itself can buy up the subset of non-members it prefers (cf. Serrano and Volij (1998) in the context of exchange economies). More fundamentally, we argue that, at least as far as the core is concerned, it is not as much the focus on subgames that matters as the idea of re-evaluating the power of coalitions at the solution. If x is a reasonable agreement in a game V , then so should it remain in the associated game V_x^* where the coalitions can buy up the cooperation of non-members by “paying” them according to x . We prefer to push forward this natural stability property called “invariance with respect to re-evaluations of coalitional power” (IRCP) instead of blurring it in a variable population framework. The core is indeed the largest solution satisfying IRCP and specifying feasible allocations that are individually rational.

In the context of games with convex feasible sets, the inner core is a natural refinement of the core. It is obtained by applying the fictitious-transfer procedure to the core defined for games with transferable utility (cf. Shapley (1969) and Myerson (1992)). The inner core has been characterized as the set of feasible allocations that are not objected even if coalitions are allowed to use some sophisticated “randomized blocking plans” (cf. Myerson (1991), Qin (1993) and de Clippel and Minelli (2002)). Basing ourselves on these results, we offer a beautiful axiomatic justification of the inner core for games with convex and smooth feasible sets. Indeed, the inner core is the largest solution satisfying the modified IRCP where V_x^* is replaced by its convex hull, and specifying feasible allocations that are individually rational. It is quite natural, in a cardinal context, to consider that coalitions can use lotteries for determining the subgroup whose cooperation they will buy up. In contrast to Peleg (1985) who proposed as an open question in section 7.6 to find a large class of convex-valued games where the reduced game property may be applied for characterizing the core, we find it more natural to axiomatize the inner core with a variant of our related property of IRCP on the class of games with convex and smooth feasible sets.

The paper is organized as follows. We start by introducing the IRCP property and our characterization of the core for games with transferable utility.

The interpretation of the result is straightforward on this class of games. We then extend the analysis to games with non transferable utility. Besides the interest of the result itself, it serves as a transition for introducing the last section where we show how natural is the inner core for games with convex and smooth feasible sets.

1 Games with Transferable Utility

Let n be a positive integer and let $N := \{1, \dots, n\}$ be the finite set of players. $P(N)$ denotes the set of *coalitions*, that is the set of non-empty subsets of N . Elements of $\mathbb{R}^N$ that represent a distribution of payoffs for the players are called *allocations*. If S is a coalition and x is an allocation, then $x(S)$ denotes the sum of the S -components of x , i.e. $x(S) := \sum_{i \in S} x_i$. A *game with transferable utility* (TU) is a function $v : P(N) \rightarrow \mathbb{R}$. The number $v(S)$ represents the maximal amount of money that its member can guarantee themselves if they cooperate. A *solution* specifies for each game a reasonable way to split the surplus generated when all the players cooperate. It is thus a correspondence $\sigma : \mathbb{R}^{P(N)} \rightarrow \mathbb{R}^N$ such that $\sum_{i \in N} x_i \leq v(N)$ for each $x \in \sigma(v)$ and each TU-game v . For instance, the *core* specifies for each TU-game v the set of feasible allocations against which no coalition can object, i.e. $c(v) := \{x \in \mathbb{R}^N \mid \sum_{i \in N} x_i \leq v(N) \wedge [(\forall S \in P(N)) : x(S) \geq v(S)]\}$.

Let us suppose that the players are about to agree on an allocation x for a TU-game v . Then, the coalitional worths could be re-evaluated as follows. The members of each coalition could buy up the cooperation of non-members at prices specified by x in order to obtain more surplus. This generates a new characteristic function v_x^* , where

$$v_x^*(S) := \max_{T \in P(N \setminus S) \cup \{\emptyset\}} [v(S \cup T) - x(T)]$$

for each $S \in P(N)$. As a stability requirement, we impose that x remains a reasonable agreement for v_x^* . We call this new property “Independence with respect to Re-Evaluations of Coalitional Power” (IRCP).

Axiom (IRCP) $(\forall v \in \mathbb{R}^{P(N)}) (\forall x \in \sigma(v)) : x \in \sigma(v_x^*)$.

The core satisfies IRCP as the following argument shows. Let v be a TU-game, let $x \in c(v)$, let $S \in P(N)$ and let $T \in P(N \setminus S) \cup \{\emptyset\}$. We have that $x(S \cup T) \geq v(S \cup T)$ and so $x(S) \geq v(S \cup T) - x(T)$. Hence, $x \in c(v_x^*)$. It can be checked that both the prenucleolus and the prekernel satisfy IRCP. On the contrary, the Shapley value violates IRCP, as the following example shows. Let $N = \{1, 2, 3\}$ and let v be the TU-game defined by $v(\{1\}) = v(\{2\}) = v(\{3\}) = v(\{2, 3\}) = 0$ and $v(\{1, 2\}) = v(\{1, 3\}) = v(\{1, 2, 3\}) = 6$. Then, $Sh(v) = (4, 1, 1)$. The associated TU-game $v_{Sh(v)}^*$ is defined by: $v_{Sh(v)}^*(\{1\}) = 5$, $v_{Sh(v)}^*(\{2\}) = v_{Sh(v)}^*(\{3\}) = v(\{2, 3\}) = 2$ and $v_{Sh(v)}^*(\{1, 2\}) = v_{Sh(v)}^*(\{1, 3\}) = v_{Sh(v)}^*(\{1, 2, 3\}) = 6$. Its Shapley value is $(13/3, 5/6, 5/6)$.

“Individual rationality” (IR) is another natural property that could be imposed on a solution. It requires that no individual prefers to refuse to cooperate.

Axiom (IR) $(\forall v \in \mathbb{R}^N)(\forall x \in \sigma(v))(\forall i \in N) : x_i \geq v(\{i\})$.

Proposition 1 *The core is the largest solution defined on the class of TU-games that satisfies IR and IRCP.*

Proof: We already know that the core satisfies the two axioms. Let then σ be a solution satisfying the two axioms, let v be a TU-game, let $x \in \sigma(v)$, let $S \in P(N)$ and let $i \in S$. Then, $S \setminus \{i\} \in P(N \setminus \{i\}) \cup \{\emptyset\}$ and we have by applying both IR and IRCP that $x_i \geq v(S) - x(S \setminus \{i\})$. So, $x(S) \geq v(S)$ and $x \in c(v)$.

□

The intuition is clear: objections against some $x \in \mathbb{R}^N$ in the original TU-game v can be seen as a violation of some individual rationality constraint in the enlarged game v_x^* . The fact that the prenucleolus and the prekernel both satisfy IRCP but are not always included in the core does not contradict the proposition, as they do not satisfy IR.

In order to obtain an exact characterization of the core on the class of balanced TU-games (i.e. with non-empty cores) with at least¹ three players, we might impose in addition the two conditions of “non-emptiness” (NE) and “super-additivity” as in Peleg (1986). Alternatively, we may impose NE together with a dual version of IRCP: each allocation x that verifies our stability criterion (i.e. $x \in \sigma(v_x^*)$) belongs to the solution (i.e. $x \in \sigma(v)$). Indeed, the core satisfies it, as $v \leq v_x^*$. On the other hand, let σ be a solution satisfying IR and the dual property, let v be a TU-game and let $x \in c(v)$. Then, $v_x^*(S) = x(S)$ for each $S \in P(N)$. As $x \in c(v_x^*)$, v_x^* is balanced. By IR and NE, $x \in \sigma(v_x^*)$. By the dual property, $x \in \sigma(v)$. Let us notice finally that the dual version of IRCP is implied by the “conditional decreasingness property”: if $v \leq v'$ and $v(N) = v'(N)$, then $\sigma(v') \subseteq \sigma(v)$. This property is quite natural for solutions specifying allocations that are not objected in some reasonable sense.

2 Games with Non-Transferable Utility

Our arguments can naturally be extended to games with non-transferable utility (NTU). Let S be a coalition. $\mathbb{R}^S$ is seen as a subset of $\mathbb{R}^N$ as follows: $\mathbb{R}^S := \{x \in \mathbb{R}^N | (\forall i \in N \setminus S) : x_i = 0\}$. An *NTU-game* is a function V that associates to each coalition S a nonempty and comprehensive subset $V(S)$ of $\mathbb{R}^S$. We assume that $V(S)$ is *nonlevel* in the sense that, for all $x \in V(S)$, if there exists $y \in V(S)$ such that $y >_S x$, then there exists $y' \in V(S)$ such that $y' >>_S x$. The set of all such NTU-games is denoted by $\mathcal{G}$. A *solution* is a correspondence $\Sigma : \mathcal{G} \rightarrow \mathbb{R}^N$ such

¹The equal split solution $ES(v) := (\frac{v(\{1,2\})+v(\{1\})-v(\{2\})}{2}, \frac{v(\{1,2\})+v(\{2\})-v(\{1\})}{2})$ satisfies the axioms when there are only two players.

that $\Sigma(V) \subseteq V(N)$ for each NTU-game $V \in \mathcal{G}$. For instance, the *core* specifies for each NTU-game V the set of feasible allocations against which no coalition can object, i.e. $C(V) := \{x \in \mathbb{R}^N \mid \neg[(\exists S \in P(N))(\exists y \in V(S)) : y \succ_S x]\}$.

Let $V \in \mathcal{G}$ and let $x \in \mathbb{R}^N$ be a reasonable agreement. Then, the natural way to re-evaluate what coalitions can do is to consider the NTU-game V_x^* defined by:

$$V_x^*(S) := \bigcup_{T \in P(N \setminus S) \cup \{\emptyset\}} \{y \in \mathbb{R}^S \mid y + x_T \in V(S \cup T)\}$$

for each coalition S . It is easy to check that $V_x^* \in \mathcal{G}$.

Axiom (IRCP) $(\forall V \in \mathcal{G})(\forall x \in \Sigma(V)) : x \in \Sigma(V_x^*)$

The core satisfies IRCP as the following argument shows. Let $V \in \mathcal{G}$, let $x \in C(V)$, let $S \in P(N)$, let $T \in P(N \setminus S) \cup \{\emptyset\}$ and let $y \in \mathbb{R}^S$ be such that $y + x_T \in V(S \cup T)$. If $y \succ_S x_S$, then $y + x_T \succ_{S \cup T} x_{S \cup T}$, which is impossible (strong and weak objections are equivalent given the nonlevelness assumptions).

Axiom (IR) $(\forall V \in \mathcal{G})(\forall x \in \Sigma(V))(\forall i \in N) : x_i \geq \sup_{y \in V(\{i\})} y_i$.

Proposition 2 *The core is the largest solution defined on the class of NTU-games that satisfies IR and IRCP.*

Proof: We already know that the core satisfies the two axioms. Let then Σ be a solution satisfying the two axioms, let V be an NTU-game, let $x \in \Sigma(V)$, let $S \in P(N)$ and let $y \in V(S)$ be such that $y \succ_S x$. Let $i \in S$. Then, $S \setminus \{i\} \in P(N \setminus \{i\}) \cup \{\emptyset\}$ and we have that the individual rationality constraint of player i is contradicted in V_x^* . This is absurd.

□

Peleg (1985) imposes the “converse reduced game property” in order to obtain an exact axiomatization of the core: if the restrictions of an allocation belong to the solution of all the reduced games with two players, then it is in the solution of the original game. We are unable to express such a property in our fixed-population framework. Nevertheless, if we restrict ourselves to games with nonempty cores, we may use NE together with a dual version of IRCP $[x \in \Sigma(V_x^*) \rightarrow x \in \Sigma(V)]$ or the conditional decreasingness property $[V \subseteq V' \rightarrow \Sigma(V') \cap V(N) \subseteq \Sigma(V)]$, as in the previous section.

3 Games with Convex and Smooth Feasible Sets

We refine our study in the case of NTU-games with convex feasible sets. It is indeed frequently assumed that $V(S)$ is convex for each $S \in P(N)$. For instance, players can use lotteries and evaluate them according to expected utility. The context is cardinal in contrast to the previous section where it was ordinal. We

impose in addition some smoothness assumption: for each² $S \in P(N)$, there is a unique supporting hyperplane at each point on the boundary of $V(S)$. Let $\mathcal{G}_{CS}$ be the class of all such NTU-games. The core is a relevant solution concept as in the previous section. Nevertheless, applying the fictitious-transfer procedure allows to refine it by taking into account the more specific framework. An allocation $x \in V(N)$ belongs to the inner core of V ($x \in IC(V)$) if there exists $\lambda \in \mathbb{R}_{++}^N$ such that $(\lambda_1 x_1, \dots, \lambda_n x_n) \in c(v^\lambda)$, where v^λ is the TU-game defined as follows:

$$v^\lambda(S) := \sup_{x \in V(S)} \lambda \cdot x$$

for each $S \in P(N)$.

The way we re-evaluated what coalitions can do in the last section is not relevant anymore as V_x^* is not necessarily a game with convex feasible sets. The obvious mathematical solution to this problem is to take a convex hull. It is also the most natural way to proceed given the interpretation we gave to the convexity of the feasible sets: lotteries should be used as well for re-evaluating the power of coalitions. It is easy to check that $coV_x^* \in \mathcal{G}_{CS}$.

Axiom (IRCP') $(\forall V \in \mathcal{G}_{CS})(\forall x \in \Sigma(V)) : x \in \Sigma(coV_x^*)$

The inner core satisfies IRCP' as the following argument shows. Let $V \in \mathcal{G}_{CS}$, let $x \in IC(V)$, let $\lambda \in \mathbb{R}_{++}^N$ be the unique normalized supporting vector and let $S \in P(N)$. If there exists $\alpha \in \Delta(P(N \setminus S) \cup \{\emptyset\})$ and $y : P(N \setminus S) \cup \{\emptyset\} \rightarrow V(S)$ such that $y(T) + x_T \in V(S \cup T)$ for each $T \in P(N \setminus S) \cup \{\emptyset\}$ and $\lambda \cdot \sum_{T \in P(N \setminus S) \cup \{\emptyset\}} \alpha(T) y(T) > \lambda \cdot x_S$, then $\sum_{T \in P(N \setminus S) \cup \{\emptyset\}} \alpha(T) \lambda \cdot y(T) > \lambda \cdot x_S$. So, there exists $T \in P(N \setminus S) \cup \{\emptyset\}$ such that $\lambda \cdot y(T) > \lambda \cdot x_S$. But then, $\lambda \cdot (y(T) + x_T) > \lambda \cdot x_{S \cup T}$ which is impossible.

Lemma Let $V \in \mathcal{G}_{CS}$ and let $x \in V(N)$. If $x \in C(coV_x^*)$, then $x \in IC(V)$.

Indeed, we already proved in de Clippel and Minelli (2002) the stronger result that if there does not exist $\alpha \in [0, 1]$, $y \in V(S)$ and $z \in \{x' \in \mathbb{R}^S | x' + x_{N \setminus S} \in V(N)\}$ such that $\alpha y + (1 - \alpha)z \succ_S x_S$, then $x \in IC(V)$. Example 3 in de Clippel and Minelli (2002) shows that the smoothness assumption on the games is necessary for the lemma to hold.

Proposition 3 *The inner core is the largest solution defined on the class of NTU-games with convex and smooth feasible sets that satisfies IR and IRCP'.*

Proof: We already know that the inner core satisfies the two axioms. Let then Σ be a solution satisfying the two axioms. By applying an argument similar to the one used to prove the previous proposition, we have that $\Sigma \subseteq C$. It is then easy to conclude by combining IRCP' with the lemma.

□

²In fact, the smoothness assumption is needed only for the feasible set of the grand coalition.

As before, if we further restrict ourselves to games with nonempty inner cores, we may use NE together with a dual version of IRCP' [$x \in \Sigma(\text{co}V_x^*) \rightarrow x \in \Sigma(V)$] or the conditional decreasingness property in order to exactly characterize the inner core. The result should be compared with de Clippel (2001). There, we used the conditional superadditivity axiom of Aumann (1985) together with the restriction that the solution should specify efficient allocations and coincide with the core on TU-games, instead of IRCP' and IR. We didn't need to impose NE.

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