On the properties of some time steppings for solving the linearised, one-dimensional, shallow water equations

Eric Deleersnijder, 9th February 2005

In a flat bottom domain, we consider the linearised, one-dimensional, shallow water equations:

$$(1)\qquad \begin{cases} \dfrac{\partial \eta}{\partial t} = -\dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial t} = -\dfrac{\partial \eta}{\partial x} \end{cases} ,$$

where t and x denote time and the space coordinate, respectively; the surface elevation is denoted $\eta(t,x)$ while the velocity reads $u(t,x)$. All these variables are dimensionless, and have been rendered so as in Beckers and Deleersnijder (1993). As is well known, the equations above can be combined to give either of the following wave equations:

$$(2)\qquad \frac{\partial^2 \eta}{\partial t^2} - \frac{\partial^2 \eta}{\partial x^2} = 0 \quad \text{or} \quad \frac{\partial^2 u}{\partial t^2} - \frac{\partial^2 u}{\partial x^2} = 0 .$$

Plane, progressive waves are part of the solutions that this problem admits. In other words, we can consider solutions of the following type:

$$(3)\qquad [\eta(t,x), u(t,x)] = \mathrm{Re}\left[(E,U)e^{\sigma t - i\omega t + i\kappa x}\right] ,$$

where E and U are complex constants; the real constants σ, ω and κ denote the growth rate of the solution, the angular frequency and the wavenumber, respectively. Substituting (3) into (1) or (2) yields

$$(4)\qquad \begin{cases} \sigma = 0 \\ \omega = \pm\kappa \end{cases} .$$

So, the solution is neither amplified nor damped — this is no surprise! — and the phase speed of the waves is

$$(5)\qquad c_p = \pm 1 .$$

We discretise equations (1) in space on the one-dimensional counterpart of Arakawa's C-grid[1]:

[1] The C-grid is equivalent to that of the MAC method, i.e. it is a staggered grid, in which the velocity components and the pressure, or elevation, are all calculated at different locations.

(6) $$\begin{cases} \dfrac{d\eta_m}{dt} = -\dfrac{u_{m+1} - u_m}{\Delta x} \\ \dfrac{du_m}{dt} = -\dfrac{\eta_m - \eta_{m-1}}{\Delta x} \end{cases},$$

with

(7) $$\begin{cases} \eta_m(t) = \eta[t,(m+1/2)\Delta x] \\ u_m(t) = u(t,m\Delta x) \end{cases},$$

where m is an integer, while Δt and Δx denote the time and space increments, respectively.

Keeping the same space discretisation, different time discretisations will be considered (see Table 1). To do so, it is convenient to introduce the following notations

(8) $$\begin{cases} \eta_m^n = \eta[n\Delta t,(m+1/2)\Delta x] \\ u_m^n = u(n\Delta t,m\Delta x) \end{cases},$$

where n is an integer. According to the well-known von Neumann method, we consider a wave solution similar to that of the continuous problem, i.e. we substitute into the numerical schemes under consideration a solution of the form

(9) $$[\eta_m^n, u_m^n] = \mathrm{Re}\left[(Ee^{i\kappa\Delta x/2}, U)\lambda^n e^{i\kappa m\Delta x}\right]$$

or, equivalently,

(10) $$[\eta_m^n, u_m^n] = \mathrm{Re}\left[(Ee^{i\kappa\Delta x/2}, U)e^{\sigma n\Delta t - i\omega n\Delta t + i\kappa m\Delta x}\right],$$

where $\lambda = \exp[(\sigma - i\omega)\Delta t]$ is the amplification factor. To compare the properties of the discrete approximate solutions with the exact one, we will estimate the growth rate and the phase speed of the former as

(11) $$\sigma = \frac{\log|\lambda|}{\Delta t},$$

(12) $$c_p = \frac{-\arg\lambda}{C\theta},$$

where C is the Courant number, $C = \Delta t/\Delta x$, and $\theta = \kappa\Delta x$ is a normalised wavenumber. According to the sampling theorem, the latter must obey inequalities $0 \le \theta \le \pi$. Finally, it is useful to introduce $\mu = C^2(1-\cos\theta)$. Table 2 displays the results of this analysis.

Table 1. The discrete equations associated with the seven numerical time steppings considered herein, the space discretisation being the same for all of them.

Scheme	***Discrete equations***
Euler Forward	$\begin{cases} \eta_m^{n+1} = \eta_m^n - C(u_{m+1}^n - u_m^n) \\ u_m^{n+1} = u_m^n - C(\eta_m^n - \eta_{m-1}^n) \end{cases}$
Euler Backward	$\begin{cases} \eta_m^{n+1} = \eta_m^n - C(u_{m+1}^{n+1} - u_m^{n+1}) \\ u_m^{n+1} = u_m^n - C(\eta_m^{n+1} - \eta_{m-1}^{n+1}) \end{cases}$
Forward-Backward	$\begin{cases} \eta_m^{n+1} = \eta_m^n - C(u_{m+1}^n - u_m^n) \\ u_m^{n+1} = u_m^n - C(\eta_m^{n+1} - \eta_{m-1}^{n+1}) \end{cases}$ or $\begin{cases} \eta_m^{n+1} = \eta_m^n - C(u_{m+1}^{n+1} - u_m^{n+1}) \\ u_m^{n+1} = u_m^n - C(\eta_m^n - \eta_{m-1}^n) \end{cases}$
Crank-Nicolson	$\begin{cases} \eta_m^{n+1} = \eta_m^n - \frac{C}{2}(u_{m+1}^{n+1} - u_m^{n+1}) - \frac{C}{2}(u_{m+1}^n - u_m^n) \\ u_m^{n+1} = u_m^n - \frac{C}{2}(\eta_m^{n+1} - \eta_{m-1}^{n+1}) - \frac{C}{2}(\eta_m^n - \eta_{m-1}^n) \end{cases}$
Leapfrog	$\begin{cases} \eta_m^{n+1} = \eta_m^{n-1} - 2C(u_{m+1}^n - u_m^n) \\ u_m^{n+1} = u_m^{n-1} - 2C(\eta_m^n - \eta_{m-1}^n) \end{cases}$
2nd-order Runge-Kutta	$\begin{cases} \eta_m^{n+1} = \eta_m^n - C(u_{m+1}^n - u_m^n) + \frac{C^2}{2}(\eta_{m+1}^n - 2\eta_m^n + \eta_{m-1}^n) \\ u_m^{n+1} = u_m^n - C(\eta_m^n - \eta_{m-1}^n) + \frac{C^2}{2}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) \end{cases}$
4th-order Runge-Kutta	$\begin{cases} \eta_m^{n+1} = \eta_m^n - C(u_{m+1}^n - u_m^n) + \frac{C^2}{2}(\eta_{m+1}^n - 2\eta_m^n + \eta_{m-1}^n) \\ \qquad - \frac{C^3}{6}(u_{m+2}^n - 3u_{m+1}^n + 3u_m^n - u_{m-1}^n) + \frac{C^4}{24}(\eta_{m+2}^n - 4\eta_{m+1}^n + 6\eta_m^n - 4\eta_{m-1}^n + \eta_{m-2}^n) \\ u_m^{n+1} = u_m^n - C(\eta_m^n - \eta_{m-1}^n) + \frac{C^2}{2}(u_{m+1}^n - 2u_m^n + u_{m-1}^n) \\ \qquad - \frac{C^3}{6}(\eta_{m+1}^n - 3\eta_m^n + 3\eta_{m-1}^n - \eta_{m-2}^n) + \frac{C^4}{24}(u_{m+2}^n - 4u_{m+1}^n + 6u_m^n - 4u_{m-1}^n + u_{m-2}^n) \end{cases}$

Table 2. Key properties of the numerical schemes listed in Table 1, which should be compared with their exact solution counterparts, i.e. $\lambda = \exp(\mp iC\theta)$, $|\lambda| = 1$, $\sigma = 0$ and $c_p = \pm 1$.

Scheme	*Amplification factor*	*Stability*	*Growth rate* σ as $\theta \to 0$	*Phase speed* c_p as $\theta \to 0$
Euler Forward	$\lambda = 1 \mp i\sqrt{2\mu}$ $\lvert\lambda\rvert = \sqrt{1+2\mu}$	unstable	$\sim \dfrac{C^2}{2\Delta t}\theta^2$	$\sim \pm 1 \mp \dfrac{1+8C^2}{24}\theta^2$
Euler Backward	$\lambda = \dfrac{1 \mp i\sqrt{2\mu}}{1+2\mu}$ $\lvert\lambda\rvert = \dfrac{1}{\sqrt{1+2\mu}}$	stable	$\sim -\dfrac{C^2}{2\Delta t}\theta^2$	$\sim \pm 1 \mp \dfrac{1+8C^2}{24}\theta^2$
Forward-Backward	$\lambda = 1 - \mu \mp i\sqrt{2\mu - \mu^2}$ $\lvert\lambda\rvert = 1$ if stable	stable if $C \le 1$	0	$\sim \pm 1 \mp \dfrac{1-C^2}{24}\theta^2$
Crank-Nicolson	$\lambda = \dfrac{2 - \mu \mp i\sqrt{8\mu}}{2+\mu}$ $\lvert\lambda\rvert = 1$	stable	0	$\sim \pm 1 \mp \dfrac{1+2C^2}{24}\theta^2$
Leapfrog	$\lambda = \sqrt{1 - 4\mu \mp i\sqrt{8\mu(1-2\mu)}}$ $\lvert\lambda\rvert = 1$ if stable	stable if $C \le \dfrac{1}{2}$	0	$\sim \pm 1 \mp \dfrac{1-4C^2}{24}\theta^2$
2nd-order Runge-Kutta	$\lambda = 1 - \mu \mp i\sqrt{2\mu}$ $\lvert\lambda\rvert = \sqrt{1+\mu^2}$	unstable	$\sim \dfrac{C^4}{8\Delta t}\theta^4$	$\sim \pm 1 \mp \dfrac{1-4C^2}{24}\theta^2$
4th-order Runge-Kutta	$\lambda = 1 - \mu(1-\mu/6) \mp i\sqrt{2\mu}(1-\mu/3)^2$ $\lvert\lambda\rvert = \sqrt{1 - \dfrac{\mu^3}{9} + \dfrac{\mu^4}{36}}$ if stable	stable if $C \le \sqrt{2}$	$\sim -\dfrac{C^6}{144\Delta t}\theta^6$	$\sim \pm 1 \mp \dfrac{1}{24}\theta^2$

In Table 2, the two artificial modes of the Leapfrog schemes are ignored. They have inappropriate properties. This is why these modes must be dampened. To do so, use can be made of the Asselin filter (Asselin 1972).

For given values of the time and space increments, a scheme is said to be unstable if there exists at least one value of the normalised wave number θ such that $\sigma > 0$, implying that the wave amplitude grows exponentially; otherwise, the scheme is considered to be stable. However, stability does not necessarily occur for all possible values Δt and Δx.

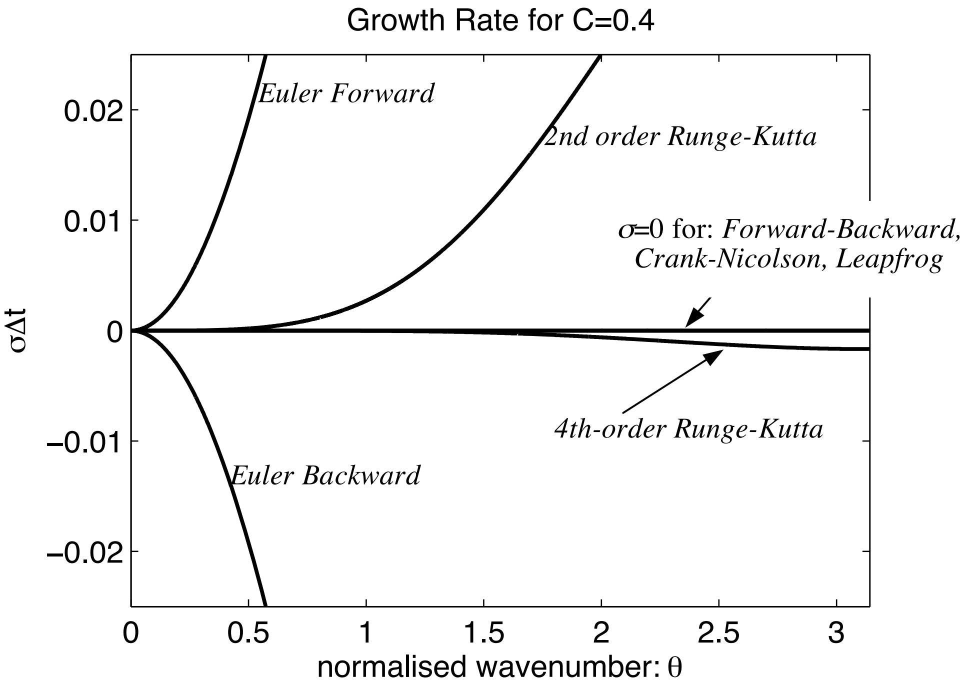

Figure 1. Plot of the normalised growth rate $\sigma\Delta t$ against the normalised wavenumber $\theta = \kappa\Delta x$ for the seven schemes considered herein. The Courant Number C is assumed to be equal to 0.4. The schemes for which σ>0 are unstable, while those for which σ<0 exhibit some dissipation. Three of the schemes (Forward-Backward, Crank-Nicolson and Leapfrog) have neither amplification nor dissipation, which is in agreement with the exact solution.

If the growth rate (Figure 1) is zero, the amplitude of the wave solution remains constant. This holds true for an infinite domain of interest, but, if the corresponding numerical scheme is applied in a finite domain with closed boundaries, the total — i.e. potential + kinetic — energy is not necessarily conserved. Strict conservation may be obtained for the Crank-Nicholson scheme, if the energy and the boundary conditions are dealt with in an appropriate manner (see Appendix). On the other hand, the Forward-Backward scheme does not strictly conserve the total energy — though it generally remains bounded. This issue should be addressed in depth.

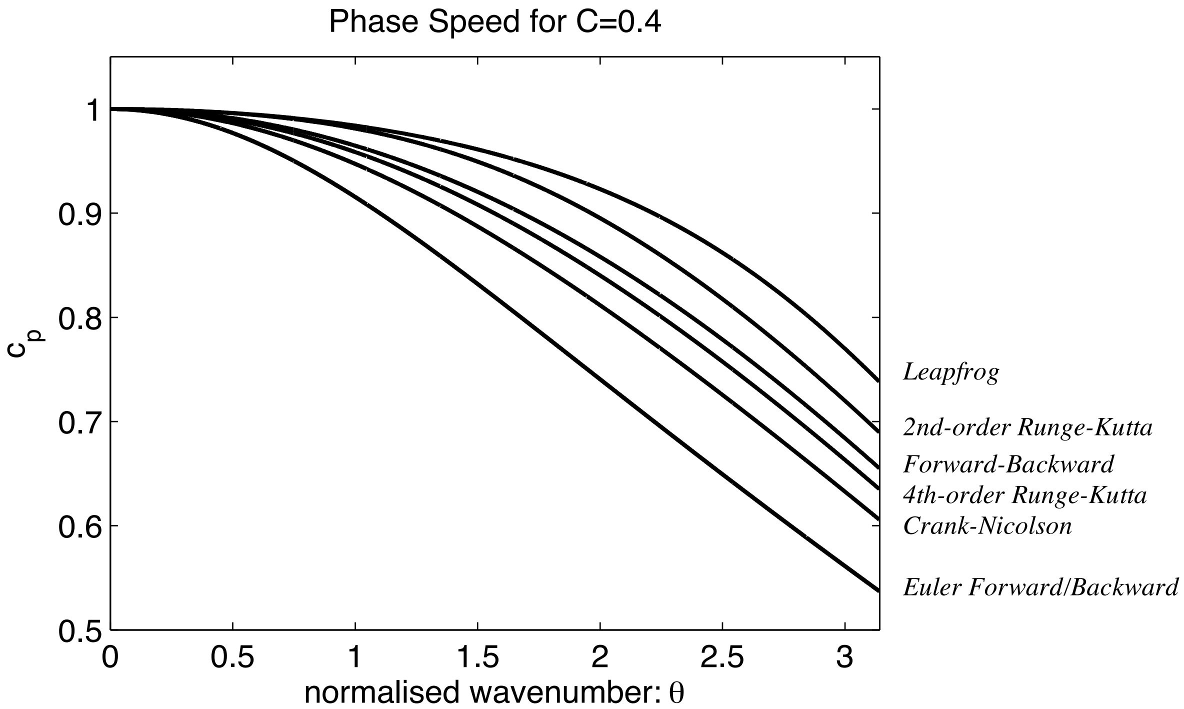

Figure 2. Plot of the phase speed as a function of the normalised wavenumber $\theta = \kappa \Delta x$ for all the schemes considered herein. The Courant number C is assumed to be equal to 0.4.

Phase speeds for all the schemes under study are displayed in Figure 2 for C=2. In this case, the phase speed of the Leapfrog scheme is that which is closest to unity, the exact value. However, this scheme also produces spurious — oscillating — modes that have to be dampened by having recourse to a filter, which might affect the phase speed of the physical mode too[2]. By manipulating the Forward-Backward discrete equations, we can obtain the following expressions

$$\text{(13)} \qquad \frac{\eta_m^{n+1} - 2\eta_m^n + \eta_m^{n-1}}{\Delta t^2} - \frac{\eta_{m+1}^n - 2\eta_m^n + \eta_{m-1}^n}{\Delta x^2} = 0 \quad \text{or} \quad \frac{u_m^{n+1} - 2u_m^n + u_m^{n-1}}{\Delta t^2} - \frac{u_{m+1}^n - 2u_m^n + u_{m-1}^n}{\Delta x^2} = 0 \,,$$

which are second-order accurate discretisations of the wave equations (2). Presumably, this is why this scheme exhibits the best propagation properties of all the stable schemes considered herein, save the Leapfrog scheme.

Clearly, increasing the accuracy of the time discretisation without increasing that of the space discretisation is unlikely to improve the propagation properties of discrete solutions of the shallow water equations. However, in a finite-size domain, increasing the accuracy of the space discretisation demands additional efforts in the treatment of the boundary conditions.

References

Asselin R., 1972, Frequency filter for time integrations, *Monthly Weather Review*, 100, 487-490

Beckers J.-M. and E. Deleersnijder, 1993, Stability of a FBTCS scheme applied to the propagation of shallow-water inertia-gravity waves on various space grids, *Journal of Computational Physics*, 108, 95-104

[2] I am not sure what is the exact impact of the Asselin filter on the physical modes. This needs to be checked.

Appendix

Conservation de l'énergie dans un système linéaire

Eric Deleersnijder, le 26 février 2002

On considère le système d'équations algébriques linéaires obtenu par discrétisation dans l'espace d'un problème différentiel dont les inconnues dépendent du temps t et de l'espace:

$$\frac{dx}{dt} = Dx \ , \tag{1}$$

où x est le vecteur d'inconnues et D est une matrice appropriée. On appelle “énergie” le produit scalaire du vecteur x par lui-même, $x^T x = |x|^2$. On suppose que le système conserve l'énergie: pour tout vecteur y le produit $y^T D y$, de sorte que

$$\frac{d}{dt}(x^T x) = 2x^T Dx = 0 \ . \tag{2}$$

Il faut que la matrice D soit antisymétrique. Pour le montrer, on suppose d'abord qu'elle est quelconque et on la décompose en sa partie symétrique, D_s, et sa partie antisymétrique, D_a, de sorte que $D = D_s + D_a$, avec $D_s^T = D_s$ et $D_a^T = -D_a$. Le produit $y^T D y$ qui doit être nul pour tout vecteur y s'écrit alors

$$y^T D y = y^T D_s y + y^T D_a y \ . \tag{3}$$

On vérifie sans peine que $y^T D_a y = (y^T D_a y)^T = y^T D_a^T y = -y^T D_a y$, ce qui implique que $y^T D_a y = 0$, pour tout vecteur y. La matrice D_s étant symétrique, on peut toujours l'écrire sous la forme d'un produit matriciel du type $D_s = S^T S$. Donc, $y^T D_s y = y^T S^T S y = (Sy)^T (Sy)$, une expression qui n'est nulle pour tout vecteur y qu'à la condition que toutes les composantes de S, et donc de D_s, soient nulles. La partie symétrique de la matrice D doit donc bien être nulle.

On discrétise dans le temps l'équation (1) par un schéma de type Euler dont le taux d'implicité est $\alpha \in [0,1]$:

$$\frac{x_{n+1} - x_n}{\Delta t} = (1-\alpha) D x_n + \alpha D x_{n+1} \ . \tag{4}$$

On peut ré-écrire cet algorithme comme suit:

$$\frac{x_{n+1} - x_n}{\Delta t} = \frac{1}{2} D(x_{n+1} + x_n) - \frac{1-2\alpha}{2} D(x_{n+1} - x_n) \ . \tag{5}$$

On multiplie (5) par $(x_{n+1} + x_n)^T$ et, après quelques manipulations, on obtient

$$\frac{|x_{n+1}|^2 - |x_n|^2}{\Delta t} = \frac{1-2\alpha}{2} (x_{n+1} - x_n)^T D (x_{n+1} + x_n) . \tag{6}$$

En combinant (5) et (6), il vient

$$\boxed{|x_{n+1}|^2 - |x_n|^2 = (1-2\alpha) |x_{n+1} - x_n|^2 \ .} \tag{7}$$

En conséquence, pour $0 \le \alpha < 1/2$, $\alpha = 1/2$, et $1/2 < \alpha \le 1$, l'énergie augmente, reste constante et diminue, respectivement. En d'autres termes, le schéma semi-implicite (α=1/2) conserve l'énergie, les schémas à dominante explicite “explosent”, tandis que les schémas à dominance implicite “implosent”.