The identification of preferences from the equilibrium prices of commodities and assets [1] [2]

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Abstract

The competitive equilibrium correspondence, which associates equilibrium prices of commodities and assets with allocations of endowments, identifies the preferences and beliefs of individuals; this is the case even if the asset market is incomplete.

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By watching the behavior of individuals throughout their lives, you should be able to reverse - engineer their utility functions.

R. Dawkins (1995), *River out of Eden,* Widenfeld and Nicholson.

1 Introduction

Explanation and prediction require the behavior of individuals, which is observable, to identify their characteristics, which are not.

In a market, it is competitive equilibria that are observable. The theory fails to specify out of equilibrium behavior and, as a consequence, demand at arbitrary prices and incomes is not observable. Experimental observations may be less restrictive.

The preferences of individuals are unobservable - beliefs are unobservable as well, even though they may not be exogenous and might vary with equilibrium prices or aggregate behavior.

Observations may involve different degrees of aggregation. At the most disaggregated level, one can observe the demand of individuals as prices and the allocation of endowments or revenue varies. More appropriately, with less disaggregation, observations are restricted to equilibrium prices as the allocation of endowments varies. The endowments of individuals may be in part unobservable, though redistributions of revenue are mostly observable; production possibilities may be observable or not.

Here, it is shown that the competitive equilibrium correspondence, which associates equilibrium prices of commodities and assets with allocations of endowments, identifies the preferences and beliefs of individuals; this is the case even if the asset market is incomplete.

It follows that generality in competitive equilibrium theory as put forward by Arrow and Debreu (1954) and McKenzie (1954), for abstract economies, and by Radner (1972), for economies with an operative asset market, allows for explanation and prediction.

Under certainty, the transfer paradox, generalized in Donsimoni and Polemarchakis (1994), makes it clear that knowledge of the utility functions is necessary in order to identify welfare effects of transfers.

Under uncertainty, with an incomplete asset market, the identification of preferences from observed behavior has strong positive as well as normative implications. A competitive equilibrium allocation is not optimal; more pertinently, according to Geanakoplos and Polemarchakis (1986), redistributions of portfolios of assets can result in a pareto improvement; alternatively, following Herings and Polemarchakis (1998), the regulation of prices and the imposition of rationing in order to attain market clearing can be pareto improving. The question of the informational requirements for the determination of improving redistributions of portfolios of assets or regulation of prices immediately arises.

The identification of preferences from the equilibrium correspondence allows possibly counterintuitive distributional effects of financial innovation, as in Hart (1975), to be predicted without ambiguity. Similarly, it allows for the determination of the investment decisions of firms as in Drèze (1974) without recourse to unobservable characteristics.

The argument for identification is developed for exchange economies. The introduction of production possibilities poses no problem if the production choices of firms are observable. However, "household production" prevents identification even if individual demand, which aggregates consumption and production decisions is observable.

The lack of available data and problems with econometric estimation procedures notwithstanding, it is an important theoretical question whether the necessary information concerning the unobservable characteristics of individuals can be identified from their observable, market behavior.

Under certainty or, equivalently, for economies with a complete market either in contingent commodities or in elementary securities, the identification of preferences from the equilibrium correspondence builds on earlier arguments of Brown and Matzkin (1990) and Chiappori and Ekeland (1998); the former argued that identification of income effects is possible from the equilibrium correspondence while the latter showed that identification of preferences is possible from the aggregate demand function.

The identification of the preference relation of an individual from his demand behavior as the prices of commodities and his income vary is evident, under standard regularity assumptions; sufficient conditions for identification were examined in detail in Mas - Colell (1977). The identification of the preference relations of individuals is less evident if only aggregate demand behavior is observable; it is not evident that it is impossible to disaggregate the observed behavior into different families of individual demand functions generated by different profiles of utilities; indeed, an example with quasi - linear preferences shows that this may be the case. Nevertheless, under a condition of non - vanishing income effects, the slutzky decomposition of the demand functions of individuals can be exploited to identify their preferences from aggregate behavior; this was the argument in Chiappori and Ekeland (1998). Indeed, the equilibrium correspondence, a priori less informative than aggregate demand, suffices for the identification of the preferences of individuals; which is surprising, since, on the equilibrium correspondence, prices and endowments do not vary independently, and there are, typically, finitely many prices associated with each profile of endowments. Identification from the equilibrium correspondence requires variation in the allocation of endowments. Nevertheless, observation of the distribution of income associated with alternative allocations of endowments and associated competitive equilibrium prices suffices for identification.

Under uncertainty, with an operative, possibly incomplete asset market, the identification of preferences and beliefs from the equilibrium correspondence for commodities and assets relies on the argument of Geanakoplos and Polemarchakis (1990) who considered the individual demand function for commodities and assets as observable. With an incomplete asset market, the identification of preferences even from individual demand behavior is problematic: the first order conditions for individual optimization do not determine, at least immediately, the marginal rate of substitution between consumption at different states of the world. With one commodity, restrictions on the structure of payoffs of assets, as in Dybvig and Polemarchakis (1981) and in Polemarchakis and Rose (1984), or on the utility function that represents the preferences, as in Green , Lau and Polemarchakis (1979) permit identification. With multiple commodities, the variation of relative price of commodities at each state of the world permits identification, as in Geanakoplos and Polemarchakis (1990). The argument extends to the more restrictive settings where only aggregate demand or the equilibrium correspondence are observable.

The identification of beliefs poses additional problems in the limiting case of one commodity as in Polemarchakis (1983). With multiple commodities the identification of beliefs is obtained without further complications.

The identification of unobservable characteristics is distinct from the integrability of demand functions or from the restrictions that apply to individual or aggregate demand or the equilibrium correspondence.

Integrability, introduced by Samuelson (1956) and further considered in Debreu (1972, 1976), concerns conditions for the existence of a complete and transitive preference relation that generates demand behavior; identification presupposes the existence of an underlying preference relation, often one that satisfies additional regularity conditions.

In principal, restrictions on behavior may fail to arise even if identification is possible. Nevertheless, in the context of aggregate excess demand where indeed restrictions vanish, as shown by Debreu (1974), Mantel (1974) and Sonnenschein (1973, 1974), identification is not possible; which may account for some confusion.

Recent work by Chiappori and Ekeland (1998) argues that no local restrictions apply on the equilibrium correspondence while according to our argument identification is possible - this illustrates the distinction between identification and a failure of restrictions. Nevertheless global restrictions remain, as in Brown and Matzkin (1996).

While recoverability and observable restrictions are two separate issue, there is an important caveat to the argument since identifications implies that restrictions on unobservables have observable implications. Indeed, recoverability implies that all additional restrictions on preferences or beliefs translate immediately to restrictions on the equilibrium correspondence.

The argument for recoverability is local. Given any profile of endowments with equilibrium prices, we uniquely recover the associated consumption allocation as well as preferences over consumption in a neighborhood of this allocation. This argument extends immediately to preferences over the whole consumption set if additional assumptions on preferences assure that the associated allocations are attained at some equilibrium.

2 Certainty

Individuals are $i \in \mathcal{I} = \{1, \ldots, I\}$, a finite, non - empty set.

Commodities are $l \in \mathcal{L} = \{1, \ldots, L\}$, a finite, non - empty set, and a bundle of commodities is [1] $x = (\ldots, x_l, \ldots)'$.

The preferences of an individual are represented by the utility function, u^i, with domain the consumption set.

Utility functions, u^i_1 and u^i_2, are ordinally equivalent if u^i_2 is a strictly monotonically increasing transformation of u^i_1.

The endowment of an individual is e^i, a bundle of commodities.

Assumption 1 *For every individual* [2],

1. *The consumption set is the set of non - negative bundles of commodities;*
2. *the utility function,* u^i, *is continuous and quasi - concave; in the interior of the consumption set, it is differentiably strictly monotonically increasing:* $Du^i(x) \gg 0$, *and strictly quasi - concave* [3]*:* $y \in [Du^i(x)]^\perp \setminus \{0\} \Rightarrow y' D^2 u^i(x) y < 0$; *for a sequence of strictly positive consumption bundles,* $(x_n : n = 1, \ldots)$, *and for* $\overline{x}$, *a non - zero consumption bundle on the boundary of the consumption set,* $(\lim_{n \to \infty} x_n = \overline{x}) \Rightarrow (\lim_{n \to \infty} (\|Du^i(x_n)\|)^{-1} Du^i(x_n) x_n = 0)$;
3. $e^i \gg 0$: *the endowment is a consumption bundle in the interior of the consumption set.*

The profile of utility functions is

$$u^{\mathcal{I}} = (\ldots, u^i, \ldots),$$

and the allocation of endowments is

$$e^{\mathcal{I}} = (\ldots, e^i, \ldots).$$

[1] "$'$ "denotes the transpose.

[2] "$\gg$, " "$>$ "and "$\geq$ "are vector inequalities; also, "$\ll$, " "$<$ "and "$\leq$. "

[3] "[ ] "denotes the span of a set of vectors or the column span of a matrix; "$\perp$ "denotes the orthogonal complement.

Profiles of utility functions, $u_1^{\mathcal{I}}$ and $u_2^{\mathcal{I}}$, are ordinally equivalent if, for every individual, the utility functions u_1^i and u_2^i are ordinally equivalent.

The profile of utility functions is fixed, while the allocation of endowments varies.

The aggregate endowment is $e^a = \sum_{i \in \mathcal{I}} e^i$.

Prices of commodities are $p = (\ldots, p_l, \ldots) \gg 0$.

The optimization problem of an individual is

$$\begin{aligned} \max \quad & u^i(x), \\ \text{s.t} \quad & px \leq pe^i. \end{aligned}$$

The solution to the individual optimization problem, $x^i(p, e^i)$, which exists, is unique and lies in the interior of the consumption set, defines x^i, the demand function of the individual.

For ordinally equivalent utility functions, the demand functions coincide.

Lemma 1 *The demand function identifies the utility function of an individual up to ordinal equivalence.*

Proof By the separating hyperplane theorem, the demand function is surjective.

Since, at a solution to the individual optimization problem, the gradient of utility function and the vector of prices are colinear, the demand function identifies the utility function up to a strictly monotonically increasing transformation. □

Remark In the presence of individual production possibilities described by the production function f, the optimization problem of an individual takes the form

$$\begin{aligned} \max \quad & u^i(x), \\ \text{s.t} \quad & px \leq p(e^i + y), \\ & f(y) = 0. \end{aligned}$$

This is equivalent to the problem

$$\begin{aligned} \max \quad & v^i(x), \\ \text{s.t} \quad & px \leq pe^i, \end{aligned}$$

where the function v^i is defined by

$$v^i(x) = \max\{u^i(x + y) : f(y) = 0).$$

From the previous argument, v^i is identified by the demand behavior of the individual. If the production function, f is known or, alternatively, if profit maximizing production decisions are observable, the utility function, u^i, can be identified; otherwise it cannot be.

The demand function is continuously differentiable; price effects are

$$D_p x^i = (\ldots, \frac{\partial x_l^i}{\partial p_k}, \ldots),$$

and income effects are

$$D_{e_1^i} x^i = (\ldots, \frac{\partial x_l^i}{\partial e_1^i}, \ldots).$$

Across individuals,

$$x^a(p, e^{\mathcal{I}}) = \sum_{i \in \mathcal{I}} x^i(p, e^i),$$

which defines x^a, the aggregate demand function.

For ordinally equivalent profiles of utility functions, the aggregate demand functions coincide.

Assumption 2 *For every individual,*

1. *the income effect for every commodity, $\partial x_l^i / \partial e_1^i$, is a twice differentiable function of revenue, e_1^i;*
2. *there exist commodities, m and n, other than the numeraire, such that*

$$\frac{\partial^2 x_m^i}{\partial (e_1^i)^2} \neq 0, \quad \text{and} \quad \frac{\partial^2 x_n^i}{\partial (e_1^i)^2} \neq 0,$$

and

$$\frac{\partial}{\partial e_1^i} (\ln \frac{\partial^2 x_m^i}{(\partial e_1^i)^2}) \neq \frac{\partial}{\partial e_1^i} (\ln \frac{\partial^2 x_n^i}{(\partial e_1^i)^2}).$$

Income effects do not vanish for any commodity, while there are two commodities for which the partial elasticities of the income effects with respect to revenue do not vanish.

It is reasonable to conjecture that appropriate perturbations of the utility functions of individuals guarantee that this is, generically, the case.

Lemma 2 (Chiappori and Ekeland (1998)) *The aggregate demand function identifies the profile of utility functions up to ordinal equivalence.*

Proof It suffices that the aggregate demand function identify the demand function of every individual.

The argument is developed in a sequence of steps.

The solution to the optimization problem of an individual is determined by the necessary and sufficient first order conditions

$$Du^i - \lambda^i p = 0,$$

$$px - pe^i = 0.$$

Differentiating the first order conditions and setting

$$\begin{pmatrix} K^i & -v^i \\ -v^{i\prime} & b^i \end{pmatrix} = \begin{pmatrix} D^2u^i & -p' \\ -p & 0 \end{pmatrix}^{-1},$$

and

$$S^i = \lambda^i K^i,$$

yields, by the implicit function theorem, that

$$D_p x^i = S^i - v^i(x^i - e^i)',$$

$$D_{e^i} x^i = v^i p,$$

and, as a consequence,

$$dx^i = (S^i - v^i(x^i - e^i)')dp + (v^i p)de^i = 0.$$

This is the slutzky decomposition [4] of the jacobian of the demand function of an individual; the matrix, S^i, of substitution effects, $s^i_{l,k} = (\partial x^i_l/\partial p_k)_{\overline{u}^i}$, is symmetric and negative semi - definite, it has rank $(L-1)$, and satisfies $pS^i = 0$, and the vector, v^i, of income effects, $v^i_l = \partial x^i_l/\partial e^i_1$, satisfies $pv^i = 1$.

For the aggregate demand function,

$$D_p x^a = \textstyle\sum_{i\in\mathcal{I}}(S^i - v^i(x^i - e^i)'),$$

$$D_{e^i} x^a = v^i p,$$

and, as a consequence,

$$dx^a = \sum_{i\in\mathcal{I}}(S^i - v^i(x^i - e^i)')dp + \sum_{i\in\mathcal{I}}(v^i p)de^i.$$

To simplify notation, without loss of generality, all derivatives are evaluated at prices of commodities p with $p_1 = 1$.

Step 1 Since $D_{e^i_1} x^a = v^i$, the aggregate demand function identifies the income effects for every individual.

[4] Antonelli (1886), Slutzky (1915)

Step 2 The functions

$$f_{j,k} = \frac{\partial x^a_j}{\partial p_k} - \frac{\partial x^a_k}{p_j} - \sum_{i \in \mathcal{I}} (v^i_j e^i_k - v^i_k e^i_j), \quad j, k \in \mathcal{L} \setminus \{1\}, j \neq k,$$

for pairs of distinct commodities other than the numeraire, are determined by the aggregate demand function.

By direct substitution and the symmetry of the matrices of substitution effects,

$$f_{j,k} = \sum_{i \in \mathcal{I}} (v^i_k x^i_j - v^i_j x^i_k).$$

Step 3 By direct computation,

$$v^{i\prime}_k x^i_j - v^{i\prime}_j x^i_k = \tfrac{\partial}{\partial e^i_1} f_{j,k},$$

$$v^{i\prime\prime}_k x^i_j - v^{i\prime\prime}_j x^i_k = \tfrac{\partial^2}{\partial (e^i_1)^2} f_{j,k} - (v^{i\prime}_k v^i_j - v^{i\prime}_j v^i_k),$$

where $v^{i\prime}_l = (\partial^2 v^i_l / \partial (e^i_1)^2)$ and $v^{i\prime\prime}_l = (\partial^2 v^i_l / \partial (e^i_1)^2)$. The aggregate demand function identifies the functions of income effects of individuals and, hence, their derivatives, while the endowments of individuals are observable; at prices of commodities and allocation of endowments $(p, e^{\mathcal{I}},)$ this is a system of $(L-1)(L-2)$ linear equations in the $(L-1)$ variables $x^i_2, \ldots, x^i_L$.

Step 4 In order to identify the demand of an individual, it suffices to select a subset of the equations and show that the associated matrix of coefficients has full column rank.

Without loss of generality, $m = 2$, and $n = 3$,

The subset of equations is

$$v^{i\prime}_k x^i_2 - v^{i\prime}_2 x^i_k = \tfrac{\partial f_{2,k}}{\partial e^i_1},$$

$$v^{i\prime\prime}_3 x^i_2 - v^{i\prime\prime}_2 x^i_3 = \tfrac{\partial^2 f_{2,3}}{\partial (e^i_1)^2} - (v^{i\prime}_3 x^i_2 - v^{i\prime}_2 x^i_3),$$

and the associated matrix of coefficients is

$$\begin{pmatrix} v_3^{i\prime} & -v_2^{i\prime} & \dots & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ v_k^{i\prime} & 0 & \dots & -v_2^{i\prime} & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \dots \\ v_L^{i\prime} & 0 & \dots & 0 & \dots & -v_2^{i\prime} \\ v_3^{i\prime\prime} & -v_2^{i\prime\prime} & \dots & 0 & \dots & 0 \end{pmatrix}.$$

Since $v_2^{i\prime} \neq 0$, the matrix of coefficients is invertible if and only if the matrix

$$\begin{pmatrix} v_3^{i\prime} & -v_2^{i\prime} \\ v_3^{i\prime\prime} & -v_2^{i\prime\prime} \end{pmatrix}$$

has full rank. Indeed, the determinant is

$$(-v_2^{i\prime} v_3^{i\prime})^{-1}(\tfrac{v_2^{i\prime\prime}}{v_2^{i\prime}} - \tfrac{v_3^{i\prime\prime}}{v_3^{i\prime}}) =$$

$$-((\tfrac{\partial^2 x_2^i}{\partial(e_1^i)^2})(\tfrac{\partial^2 x_3^i}{\partial(e_1^i)^2}))^{-1}(\tfrac{\partial}{\partial e_1^i}(\ln \tfrac{\partial^2 x_2^i}{(\partial e_1^i)^2}) - \tfrac{\partial}{\partial e_1^i}(\ln \tfrac{\partial^2 x_3^i}{(\partial e_1^i)^2}) \neq 0.$$

□

Remark Assumption 2 implies that there are at least three commodities: $L \geq 3$; a different argument is required for economies with two commodities, $L = 2$.

Remark If the utility functions of individuals are defined by

$$u^i(x) = x_1 + (\frac{1}{\alpha}) \sum_{l \in \mathcal{L} \setminus \{1\}} \delta_l^i x_l^\alpha, \quad \alpha < 1, \delta_l^i > 0, l \in \mathcal{L} \setminus \{1\},$$

the aggregate demand functions for commodities other than the numeraire is defined by

$$x_l^a(p) = \sum_{i \in \mathcal{I}} \min\{0, (\delta_l^i)^{\frac{1}{1-\alpha}} p_l^{\frac{1}{\alpha-1}}\}, \quad l \in \mathcal{L} \setminus \{1\}.$$

The utility functions of individuals fail the condition of non - vanishing income effects in assumption 2: the income effects on commodities other than the numeraire vanish; they also fail the boundary condition in assumption 1, but this is not important. The aggregate demand function fails to identify the utility functions of individuals.

Competitive equilibrium prices are such that $x^a(p, e^{\mathcal{I}}) - e^a = 0$.

The competitive equilibrium correspondence associates competitive equilibrium prices to profiles of endowments,

$$\omega(e^{\mathcal{I}}) = \{p : x^a(p, e^{\mathcal{I}}) - e^a = 0, \text{ and } p_1 = 1\}.$$

For ordinally equivalent profiles of utility functions, the competitive equilibrium correspondences coincide.

Proposition 1 *The competitive equilibrium correspondence, on an open set of endowments, identifies the profile of utility functions, up to ordinal equivalence, on the associated subsets of the consumption sets of individual.*

Proof It suffices that the competitive equilibrium correspondence identify the demand function of every individual.

The argument is developed in a sequence of steps.

The graph of the competitive equilibrium correspondence has the structure of a continuously differentiable manifold.

The tangent space to the competitive equilibrium manifold is defined by

$$dx^a = de^a,$$

and, as a consequence, by

$$\sum_{i \in \mathcal{I}} (S^i - v^i(x^i - e^i)')dp + \sum_{i \in \mathcal{I}} (v^i p - I)de^i = 0.$$

The competitive equilibrium correspondence determines the competitive e - quilibrium manifold and, consequently, everywhere, its tangent space.

Step 1 The tangent space of the equilibrium manifold identifies the income effects of every individual, v^i, and the matrix of aggregate price effects, $\sum_{i \in \mathcal{I}}(S^i - v^i(x^i - e^i)')$.

For every individual, the matrix $(v^i p - I)$ has rank $(L-1)$, since $p(v^i p - I) = 0$, while $y \in [p]^{\perp} \Rightarrow (v^i p - I)y = y$. Hence, there exists a subspace of dimension 1 — it suffices that the dimension be greater than 0 — of vectors that satisfy $(v^i p - I)y = 0$. This identifies v^i, since $p \gg 0$ — it suffices that $p \neq 0$ — while $pv^i = 1$.

The rank of the matrix of aggregate price effects is at most equal to $(L-1)$, since $\sum_{i \in \mathcal{I}}(S^i - v^i(x^i - e^i)')p = 0$. Since, for every individual, the matrix $(v^i p - I)$ has rank $(L-1)$ – it suffices that this is the case for one individual – the space $\sum_{i \in \mathcal{I}}[(v^i p - I)]$ has dimension $(L-1)$ which identifies the matrix of aggregate price effects, since

$$\sum_{i \in \mathcal{I}} (S^i - v^i(x^i - e^i)')dp = -\sum_{i \in \mathcal{I}} (v^i p - I)de^i.$$

Step 2 For competitive equilibrium prices of commodities and allocations of endowments, the jacobian of the aggregate demand function

$$D_p x^a = \sum_{i \in \mathcal{I}} (S^i - v^i (x^i - e^i)'),$$

$$D_{e^i} x^a = v^i p,$$

is identified.

The functions $f_{j,k}$ for pairs of distinct commodities other than the numeraire are defined as in the proof of lemma 2.

For the identification of the demand functions of individuals it is required that the first and second derivatives of the functions $f_{j,k}$ with respect to the revenue of every individual, e^i_1, be identified by the competitive equilibrium manifold. It suffices, then, that, for every individual, i, the projection of the set of competitive equilibria to prices of commodities and endowments of individual i, be surjective.

The competitive equilibrium correspondence identifies the profile of utility functions on the set of consumption bundles for each individual that obtain at prices of commodities and allocations of endowments on the graph of the competitive equilibrium correspondence. □

Remark If, for every individual and for every commodity, for any sequence, $((p_n, e^i_n) : n = 1, \ldots)$, of prices of commodities and endowments,

$$\lim_{n \to \infty} e^i_{1,n} = \infty \quad \Rightarrow \quad \lim_{n \to \infty} x^i_{l,n}(p_n, e^i_n) = \infty, \quad l \in \mathcal{L} :$$

for every individual, every commodity is normal, in a strong sense, then the competitive equilibrium correspondence identifies the utility functions of individuals on their entire domain of definition. The argument is as follows:
Given $(\overline{p}, \overline{e}^i)$, prices of commodities and an endowment for an individual, i, there exist endowments $\overline{e}^2, \ldots, \overline{e}^{i-1}, \overline{e}^{i+1}, \ldots, \overline{e}^L$, such that $x^a(\overline{p}, \overline{e}^{\mathcal{I}}) = \overline{e}^a$.

It suffices to set the endowment of an individual $h \neq i$; the endowments of individuals $g \neq h, i$ are set arbitrarily, at $\overline{e}^g$. The solution, $x^h(p, \tau^h)$, to the optimization problem of individual h with the value of his endowment replaced by revenue, $\tau^h > 0$, exists, it is unique and strictly positive, and it satisfies $px^h(p, \tau^h) = \tau^h$. For every commodity, $(\lim_{n \to \infty} \tau^h_n = \infty) \Rightarrow (\lim_{n \to \infty} x^h_{l,n}(p_n, \tau^h_n) = \infty)$. If $e^h(\tau^h) = \sum_{j \in \mathcal{I} \setminus \{h\}} x^j(\overline{p}, \overline{e}^j) + x^h(\overline{p}, \tau^h) - \sum_{j \in \mathcal{I} \setminus \{h\}} x^j$, then there exists $\overline{\tau}^h$, sufficiently large, such that $\overline{e}^h = e^h(\tau^h) \gg 0$. But, then, $\overline{p} \in \omega(\overline{e}^{\mathcal{I}})$.

3 Uncertainty

Individuals are $i \in \mathcal{I} = \{1, \ldots, I\}$, a finite, non - empty set.

States of the world, exhaustive and exclusive descriptions of the environment, are $s \in \mathcal{S} = \{1, \ldots, S\}$, a finite, non - empty set.

Commodities are $l \in \mathcal{L} = \{1, \ldots, L\}$, a finite, non - empty set.

At the state of the world s, commodities are $(l, s) \in \mathcal{L} \times \{s\}$, and a bundle of commodities is $x_s = (\ldots, x_{l,s}, \ldots)'$; across states of the world, commodities are $(l, s) \in \mathcal{L} \times \mathcal{S}$, and a bundle of commodities is $x = (\ldots, x'_s, \ldots)' = (\ldots, x_{l,s}, \ldots)'$.

Assets for the transfer of revenue across states of the world are $a \in \mathcal{A} = \{1, \ldots, A\}$, a finite set, and a portfolio of assets is $y = (\ldots, y_a, \ldots)'$. The payoff of asset a at the state of the world s is $r_{a,s}$; across states of the world, the payoffs of an asset are $r_a = (\ldots, r_{a,s}, \ldots)'$. The payoffs of assets at the state of the world s are $R_s = (\ldots, r_{a,s}, \ldots)$; across states of the world, the matrix of payoffs of assets is

$$R = (\ldots, r_a, \ldots) = (\ldots, R_s, \ldots)'.$$

The payoff of a portfolio of assets, y, at state of the world s is $r_s y$; across states of the world, the payoffs of a portfolio of assets are

$$Ry = (\ldots, r_s y, \ldots).$$

The column span of the matrix of payoffs of assets, $[R]$, is the subspace of attainable reallocations of revenue across states of the world.

Assumption 3 *The asset structure is such that*

1. *there are at least two assets,* $A \geq 2$.
2. *the matrix of payoffs of assets has full column rank;*
3. *the payoff of asset* $a = 1$ *is positive:* $r_1 > 0$.
4. *at every state of the world, the payoffs of assets do not vanish:* $R_s \neq 0$.

Redundant assets, whose payoffs are linear combination of the payoffs of other assets can be priced by arbitrage.

The asset market is either complete: $A = S$, or incomplete: $A < S$.

A portfolio of assets with positive payoffs serves to eliminate satiation over portfolios; that this portfolio consist of only one asset, $a = 1$, simplifies the exposition.

The preferences of an individual are described by the utility function, w^i, with domain the consumption set.

The preferences of the individual admit a representation that is additively separable across states of the world, $\{u^i_s : s \in \mathcal{S}\}$: the consumption set has a product structure; the cardinal utility function, at a state of the world, is u^i_s, and the utility function is $w^i = \sum_{s \in \mathcal{S}} u^i_s$. The preferences may, but need not admit a von Neumann - Morgenstern representation, (u^i, μ^i) : the consumption sets at different states of the world and the cardinal utility functions, u^i, coincide; $\mu^i = (\ldots, \mu^i_s, \ldots)$, is a subjective probability measure on the set of states of the world, and the utility function is [5] $w^i = \mathrm{E}_\mu u^i$.

[5] "E_μ "denotes the expectation with respect to the probability measure μ.

Utility functions, w_1^i and w_2^i, are cardinally equivalent if w_2^i, is a monotonically increasing, affine transformation of w_1^i.

The endowment of an individual in commodities is e^i, a bundle of commodities across states of the world; his endowment of assets is f^i, a portfolio of assets.

The effective endowment of the individual in commodities is [6] $\tilde{e}^i = (\ldots, \tilde{e}_s^i, \ldots)'$, where $\tilde{e}_s^i = e_s^i + \mathbf{1}_1^L R_s f^i$ is the effective endowment in commodities at a state of the world.

Assumption 4 *For every individual and for every state of the world,*

1. *the consumption set is the set of non - negative bundles of commodities;*
2. *the cardinal utility function, u_s^i, is continuous and concave; in the interior of the consumption set, the utility function is differentiably strictly monotonically increasing: $Dv_s^i(x) \gg 0$, and strictly concave: $y \neq 0 \Rightarrow y' D^2 u_s^i(x) y < 0$; for a sequence of strictly positive consumption bundles, $(x_{s,n} \gg 0 : n = 1, \ldots)$, and for $\overline{x}_s$, a consumption bundle on the boundary of the consumption set, $(\lim_{n\to\infty} x_{s,n} = x_s) \Rightarrow (\lim_{n\to\infty}((\|Du^i(x_{s,n})\|)^{-1} Du^i(x_{s,n})x_{s,n}) = 0)$, while $\lim_{n\to\infty} \|Du^i(x_{s,n})\| = \infty$;*
3. *$\tilde{e}^i \gg 0 :$ the effective endowment in commodities is a consumption bundle in the interior of the consumption set.*

The distinction between endowments in commodities and endowments in assets simplifies the exposition, but is not essential.

The profile of utilities functions is

$$u^{\mathcal{I}} = (\ldots, \{u_s^i : s \in \mathcal{S}\}, \ldots),$$

and the allocation of endowments is

$$(e^{\mathcal{I}}, f^{\mathcal{I}}) = (\ldots, (e^i, f^i), \ldots).$$

Profiles of utility functions, $u_1^{\mathcal{I}}$ and $u_2^{\mathcal{I}}$, are cardinally equivalent if, for every individual, the utility functions w_1^i and w_2^i are cardinally equivalent.

The profile of utility functions, $u^{\mathcal{I}}$, and the matrix of payoffs of assets, R, are fixed, while the allocation of endowments varies.

The aggregate endowment in commodities is $e^a = \sum_{i\in\mathcal{I}} e^i$, and the aggregate endowment in commodities is $f^a = \sum_{i\in\mathcal{I}} f^i$; the aggregate endowment is (e^a, f^a).

At a state of the world, prices of commodities are $p_s = (\ldots, p_{l,s}, \ldots) \gg 0$; Across states of the world, prices of commodities are $p = (\ldots, p_s, \ldots)$.

[6] "$\mathbf{1}_k^K$ "denotes the k – th unit vector of dimension K.

Across states of the world, the expenditures associated with bundle of commodities, x, at prices of commodities p are

$$p \otimes x = (\ldots, p_s x_s, \ldots).$$

Prices of assets are $q = (\ldots, q_a, \ldots)$.

Prices of assets do not allow for arbitrage if $Ry > 0 \Rightarrow qy > 0$; this is the case if and only if $q = \pi R$, for some $\pi = (\ldots, \pi_s, \ldots) \gg 0$.

Prices are a pair, (p, q), of prices of commodities and of prices of assets.

The optimization problem of an individual is

$$\begin{aligned} \max \quad & w^i(x), \\ \text{s.t} \quad & p \otimes x \leq p \otimes e^i + Ry, \\ & qy \leq qf^i. \end{aligned}$$

The solution of the individual optimization problem, $(x^i, y^i)(p, q, e^i)$, exists and is unique, and the consumption plan, $x^i(p, q, e^i)$, lies in the interior of the consumption set; it defines (x^i, y^i), the demand function of the individual for consumption plans and portfolios of assets.

The demand function is continuously differentiable; price effects are

$$D_{p_t} x^i_s = (\ldots, \frac{\partial x^i_{l,s}}{\partial p_{k,t}}, \ldots), D_q x^i_s = (\ldots, \frac{\partial x^i_{l,s}}{\partial q_a}, \ldots),$$

$$D_{p_{l,s}} y^i = (\ldots, \frac{\partial y^i_a}{\partial p_{l,s}}, \ldots), D_q y^i = (\ldots, \frac{\partial y^i_a}{\partial q_b}, \ldots);$$

income effects are

$$D_{e^i_{1,t}} x^i_s = (\ldots, \frac{\partial x^i_{l,s}}{\partial e^i_{1,t}}, \ldots), D_{f^i_1} x^i_s = (\ldots, \frac{\partial x^i_{l,s}}{\partial f^i_1}, \ldots), D_{e^i_{1,s}} y^i = (\ldots, \frac{\partial y^i_a}{\partial e^i_{1,s}}, \ldots),$$

$$D_{f^i_1} y^i = (\ldots, \frac{\partial y^i_a}{\partial f^i_1}, \ldots).$$

For cardinally equivalent utility functions, the demand functions coincide.

Associated with the individual optimization problem, at each state of the world, there is a conditional optimization problem

$$\begin{aligned} \max \quad & u^i_s(z_s + e^i_s), \\ \text{s.t} \quad & p_s z_s \leq p_s e^i_s + R_s y^i, \end{aligned}$$

where $y^i > 0$ is a fixed portfolio of assets, such that $p_s e^i_s + R_s y^i > 0$.

The solution of the auxiliary optimization problem, $z^i_s(p_s, e^i, y^i)$, exists, is unique, and lies in the interior of the consumption set; it defines z^i_s, the conditional demand function of the individual.

The conditional demand functions are continuously differentiable; price effects are

$$D_{p_s} z_s^i = (\ldots, \frac{\partial z_{l,s}^i}{\partial p_{k,s}}, \ldots);$$

income effects are

$$D_{e_{1,s}^i} z_s^i = (\ldots, \frac{\partial z_{l,s}^i}{\partial e_{1,s}^i}, \ldots),$$

and

$$D_{y^i} z_s^i = D_{e_{1,s}^i} z_s^i R_s.$$

Assumption 5 *For every individual and for every state of the world,*

1. *the vectors* $z_s^i = (\ldots, z_{l,s}^i, \ldots)$ *and* $D_{e_{1,s}^i} z_s^i = (\ldots, \partial z_{l,s}^i / \partial e_{1,s}^i, \ldots)$ *are linearly independent, and*
2. $(D_q y^i + D_{f_1^i} y^i (y^i - f^i)')R_s \neq 0,$

In the conditional demand function, the vector of income effects and the vector of demands are not colinear. This excludes homothetic utility functions. For this case identification is still possible but the argument has to be modified - the remark at the end of this section clarifies this point.

In the asset demand function, the sum of the matrix of price effects and the matrix formed by the product of the column vector of income effects and the transpose of the column vector which is the portfolio of assets does not vanish; this sum is the matrix of substitution effects in the demand for assets.

To simplify notation, all derivatives are evaluated at a price system which satisfies $p_{1,s} = 1$, for all $s \in \mathcal{S}$.

Lemma 3 (Geanakoplos and Polemarchakis (1990)) *The demand function for consumption plans and portfolios of assets identifies the utility function of the individual up to cardinal equivalence.*

Proof The argument is developed in a sequence of steps.
Step 1 The demand function for consumption plans and portfolios of assets, (x^i, y^i), determines the conditional demand function, z_s^i, at every state of the world. By the supporting hyperplane theorem, given prices of commodities and a portfolio of assets revenue, (p_s, y^i), there exist commodity prices, p_t, for $t \in \mathcal{S} \setminus \{s\}$, states of the world other than s, and prices of assets, q, such that $y^i(p, q) = y^i$. It follows that $x_s^i(p, q) = z_s(p_s, e_s^i, y^i)$.

The necessary and sufficient conditions for a solution of the conditional individual optimization problem at a state of the world are

$$Du_s^i - \lambda_s^i p_s = 0,$$

$$p_s z_s^i - p_s e_s^i - R_s y^i = 0.$$

Differentiating the first order conditions and setting

$$\begin{pmatrix} K_s^i & -v_s^i \\ -v_s^{i\prime} & b_s^i \end{pmatrix} = \begin{pmatrix} D^2 u_s^i & -p_s' \\ -p_s & 0 \end{pmatrix}^{-1},$$

and

$$S_s^i = \lambda_s^i K_s^i,$$

yields, by the implicit function theorem, that

$$D_{p_s} z_s^i = S_s^i - v_s^i (z_s^i - e_s^i)',$$

$$D_{e_{1,s}^i} z_s^i = v_s^i,$$

$$D_{y^i} z_s^i = v_s^i R_s,$$

$$D_{p_s} \lambda_s^i = -\lambda_s^i v_s^{i\prime} + b_s^i (z_s^i - e_s^i)',$$

$$D_{y^i} \lambda_s^i = -b_s^i R_s,$$

and, as a consequence

$$dz_s^i = (S_s^i - v_s^i (z_s^i - e_s^i)')dp + v_s^i (p_s de_s^i + R_s dy^i),$$

$$d\lambda_s^i = (-\lambda_s^i v_s^{i\prime} + b_s^i (z_s^i - e_s^i)'))dp_s - b_s^i v_s^i (p_s de_s^i + R_s dy^i).$$

The matrix, S_s^i, of substitution effects, $s_{l,k,s}^i = (\partial x_{l,s}^i / \partial p_{k,s})_{\overline{v}_s^i}$, is symmetric and negative semi - definite, it has rank $(L-1)$, and satisfies $p_s S_s^i = 0$, and the vector, v_s^i, of income effects, $v_{l,s}^i = \partial x_{l,s}^i / \partial t_s^i$, satisfies $p_s v_s^i = 1$. The marginal utility of revenue, $\lambda_s^i = \partial v_s^i / \partial t_s^i$, decreases with revenue: $\partial \lambda_s^i / \partial t_s^i = -b_s^i < 0$.

The necessary and sufficient first order conditions for a solution to the individual portfolio choice problem are

$$\lambda^i R = \mu^i q,$$

$$qy - qf^i = 0,$$

where, across states of the world, $\lambda^i = (\ldots, \lambda_s^i, \ldots)$ are the marginal utilities of revenue obtained from the conditional optimization with $y^i = y$.

Differentiating the first order conditions and setting

$$\begin{pmatrix} K^i & -v^i \\ -v^{i\prime} & b^i \end{pmatrix} = \begin{pmatrix} -\sum_{s \in \mathcal{S}} b_s^i R_s R_s' & -q \\ -q' & 0 \end{pmatrix}^{-1},$$

and

$$S^i = \lambda^i K^i,$$

yields by the implicit function theorem that

$$D_{p_s} y^i = S^i R_s (\lambda_s^i v_s^{i\prime} - b_s^i z_s^{i\prime}),$$

$$D_q y^i = S^i - v^i (y^i - f^i)'$$

$$D_{f_1^i} y^i = v^i,$$

where, for each state of the world, v_s^i and b_s^i are the income effects and the derivative of the marginal utility of revenue, respectively, obtained from the conditional optimization with $y^i = y$.

Step 2 The demand function for assets, y^i, and its derivatives with respect to revenue, $D_{f_1^i} y^i$, the prices of assets, $D_q y^i$, determine the vector of income effects v^i, and the S^i, matrix of substitution effects.

The conditional demand function for commodities at a state of the world, z^i, and its derivative with respect to revenue, $D_{e^i_{1,s}} z_s^i$, or, alternatively, $D_{y^i} z_s^i$, determine the vector of income effects, v_s^i — in the latter case, since $R_s \neq 0$.

The derivatives with respect to the prices of commodities of the demand for assets, $D_{p_s} y^i$, determines the marginal utility of revenue, λ_s^i and its first derivative b_s^i; this is the case, since, by assumption, the vectors $D_{e^i_{1,s}} z_s^i = v_s^i$ and z_s^i and linearly, while $S^i R_s = (D_q y^i + D_{f_1^i} y^i) R_s \neq 0$.

By the separating hyperplane theorem, the demand function for commodities is surjective. Since, at a solution of the individual optimization problem, the gradient of the utility function is colinear with the vector $\lambda^i \otimes p$, the demand function for consumption plans and portfolios of assets identifies the utility function up to a monotonically increasing transformation.

Since the utility function is additively separable across states of the world, of which there are, at least, two, the demand function for consumption plans and portfolios of assets identifies the utility function up to a monotonically increasing, affine transformation. □

Remark The argument for the identification of the utility function does not require variations in the endowments of the individual in commodities at each state of the world; variations in the endowment of assets suffice. It is shown below that a relaxation of assumption 5 is possible when endowments of individuals at each states vary.

Across individuals,

$$(x^a, y^a)(p, q, e^{\mathcal{I}}) = \sum_{i \in \mathcal{I}} (x^i, y^i)(p, q, e^i),$$

which defines (x^a, y^a), the aggregate demand function for consumption plans and portfolios of assets.

For cardinally equivalent profiles of utility functions, the aggregate demand functions coincide.

At each state of the world, for $y^{\mathcal{I}} = (\ldots, y^i, \ldots)$, a fixed allocation of portfolios of assets, such that $p_s e_s^i + R_s y^i > 0$, for every individual,

$$z_s^a(p_s, e_s^{\mathcal{I}}, y^{\mathcal{I}}) = \sum_{i \in \mathcal{I}} z_s^i(p_s, e_s^i, y^i),$$

which defines z_s^a, the aggregate, conditional demand function.

Assumption 6 *For every individual,*

1. *the income effect for every asset, $\partial y_a^i / \partial f_1^i$, is a twice differentiable function of revenue, f_1^i;*

2. *there exist assets, d and e, other than the numeraire, such that*

$$\frac{\partial^2 y_d^i}{\partial (f_1^i)^2} \neq 0, \quad and \quad \frac{\partial^2 y_e^i}{\partial (f_1^i)^2} \neq 0,$$

   and

$$\frac{\partial}{\partial f_1^i}(\ln \frac{\partial^2 y_d^i}{(\partial f_1^i)^2}) \neq \frac{\partial}{\partial f_1^i}(\ln \frac{\partial^2 y_e^i}{(\partial f_1^i)^2});$$

   for every state of the world,

3. *the income effect in the conditional demand for every commodity, $\partial z_{l,s}^i / \partial e_{1,s}^i$, is a twice differentiable function of revenue, $e_{1,s}^i$;*

4. *there exist commodities, m and n, other than the numeraire, such that*

$$\frac{\partial^2 z_{m,s}^i}{\partial (e_{1,s}^i)^2} \neq 0, \quad and \quad \frac{\partial^2 z_{n,s}^i}{\partial (e_{1,s}^i)^2} \neq 0,$$

   and

$$\frac{\partial}{\partial e_1^i}(\ln \frac{\partial^2 z_{m,s}^i}{(\partial e_{1,s}^i)^2}) \neq \frac{\partial}{\partial e_{1,s}^i}(\ln \frac{\partial^2 z_{n,s}^i}{(\partial e_{1,s}^i)^2}).$$

This is the analogue of the condition of non - vanishing income effects that was employed in the argument under certainty.

Lemma 4 *The aggregate demand function for consumption plans and portfolios of assets identifies the profile of utilities up to cardinal equivalence.*

Proof It suffices that the aggregate demand function identify, for every individual, the demand functions for portfolios of assets, y^i, and z^i_s, the conditional demand function for commodities, at every state of the world.

The argument is developed in a sequence of steps.

Step 1 The aggregate demand function for consumption plans and portfolios of assets, (x^a, y^a), determines the aggregate, conditional demand function, z^a_s, at every state of the world.

For the aggregate, conditional demand function,

$$D_{p_s} z^a_s = \textstyle\sum_{s \in \mathcal{S}} (S^i_s - v^i_s (z^i_s - e^i_s)'),$$

$$D_{e^i_{1,s}} z^a_s = v^i_s,$$

$$D_{y^i} z^a_s = v^i_s R_s,$$

and, as a consequence

$$dz^a_s = \sum_{s \in \mathcal{S}} (S^i_s - v^i_s (z^i_s - e^i_s)') dp + \sum_{s \in \mathcal{S}} v^i_s (p_s de^i_s + R_s dy^i).$$

Since $D_{e^i_{1,s}} z^a_s = v^i_s$, or, alternatively, $D_{y^i} z^i_s = v^i_s R_s$, the aggregate, conditional demand function identifies, v^i_s, the income effects of every individual — in the latter case, since $R_s \neq 0$.

The functions

$$f_{j,k,s} = \frac{\partial z^a_{j,s}}{\partial p_{k,s}} - \frac{\partial z^a_{k,s}}{\partial p_{j,s}} - \sum_{i \in \mathcal{I}} (v^i_{j,s} e^i_{k,s} - v^i_{k,s} e^i_{j,s}), \quad j, k \in \mathcal{L} \setminus \{1\}, j \neq k,$$

for pairs of distinct commodities other than the numeraire, are identified by the aggregate demand function.

By direct substitution and the symmetry of the matrices of substitution effects,

$$f_{j,k,s} = \sum_{i \in \mathcal{I}} (v^i_{k,s} e^i_{j,s} - v^i_{j,s} e^i_{k,s}).$$

As in the proof of lemma 2, the first and second derivatives of the functions $f_{j,k,s}$ with respect to revenue, $e^i_{1,s}$, or, alternatively, y_i, identify z^i_s, the conditional demand function of the individual.

For the aggregate demand function for portfolios of assets,

$$D_q y^a = \textstyle\sum_{i \in \mathcal{I}} (S^i - v^i (y^i - f^i)'),$$

$$D_{f^i_1} y^a = v^i,$$

and, as a consequence

$$dy^a = \sum_{i \in \mathcal{I}} (S^i - v^i (y^i - f^i)')dp + \sum_{i \in \mathcal{I}} v^i df_1^i.$$

Since $D_{f_1^i} y^a = v^i$, the aggregate demand function for portfolios of assets identifies v^i, the income effects of every individual.

The functions

$$f_{b,c} = \frac{\partial y_b^a}{\partial q_c} - \frac{\partial y_c^a}{q_b} - \sum_{i \in \mathcal{I}} (v_b^i f_c^i - v_c^i f_b^i), \quad b, c \in \mathcal{A} \setminus \{1\}, b \neq c,$$

for pairs of distinct assets other than the numeraire, is identified by the aggregate demand function.

By direct substitution and the symmetry of the matrices of substitution effects,

$$f_{b,c} = \sum_{i \in \mathcal{I}} (v_c^i f_b^i - v_b^i f_c^i).$$

As in the proof of lemma 2, the first and second derivatives of the functions $f_{b,c}$ with respect to revenue, f_1^i, identify y^i, the demand function of the individual for portfolios of assets. □

Competitive equilibrium prices are such that

$$(x^a(p, e^{\mathcal{I}}), y^a(p, e^{\mathcal{I}})) - (e^a, f^a) = 0.$$

The competitive equilibrium correspondence associates competitive equilibrium prices of commodities and assets to profiles of endowments,

$$\omega(e^{\mathcal{I}}, f^{\mathcal{I}}) =$$

$$\{(p, q) : (x^a(p, e^{\mathcal{I}}, y^a(p, e^{\mathcal{I}},) - (e^a, f^a) = 0, \text{ and } p_{1,s} = 1, s \in \mathcal{S}, q_1 = 1\}.$$

For cardinally equivalent profiles of utility functions, the competitive equilibrium correspondences coincide.

Proposition 2 *The competitive equilibrium correspondence, on an open set of endowments, identifies the the profile of utility functions, up to cardinal equivalence, on the associated subsets of the consumption sets of individuals.*

Proof It suffices that the competitive equilibrium correspondence identify the profile of demand functions.

The argument is developed in a sequence of steps.

Step 1 For an allocation of portfolios of assets, $y^{\mathcal{I}} = (\ldots, y^i, \ldots)$, the aggregate portfolio of assets is $y^a = \sum_{i \in \mathcal{I}} y^i$.

The graph of the competitive equilibrium correspondence determines the graph of the conditional competitive equilibrium correspondence at every state of the world, which assigns competitive equilibrium prices of commodities to allocations of endowments and portfolios of assets,

$$\omega_s(e_s^{\mathcal{I}}, y^i) = \{p_s : z_s^a(p_s, e_s^{\mathcal{I}}, y^{\mathcal{I}},) - e_s^a - \mathbf{1}_1^L R_s y^a = 0, \text{ and } p_{1,s} = 1\}.$$

The graph of the conditional competitive equilibrium correspondence has the structure of a continuously differentiable manifold.

The tangent space to the conditional competitive equilibrium manifold is defined by

$$dz_s^a = de_s^a + \mathbf{1}_\mathbf{1}^\mathbf{L} R_s dy^a,$$

and, as a consequence, by

$$\sum_{i\in\mathcal{I}} (S_s^i - v_s^i(z_s^i - e_s^i)')dp_s + \sum_{i\in\mathcal{I}} (v_s^i p - I)de_s^i + \sum_{i\in\mathcal{I}} (v_s^i - \mathbf{1}_\mathbf{1}^\mathbf{L})R_s dy^i = 0.$$

The conditional competitive equilibrium correspondence determines the competitive equilibrium manifold and, consequently, everywhere, its tangent space.

As in the proof of proposition 1, at every state of the world, the graph of the conditional competitive equilibrium correspondence identifies the conditional demand function of every individual.

Step 2 By substitution, the tangent space to the competitive equilibrium manifold satisfies

$$\sum_{s\in\mathcal{S}} (\sum_{i\in\mathcal{I}} S_s^i R_s(\lambda_s^i v_s^i - b_s^i z^{i\prime}))dp_s + \sum_{i\in\mathcal{I}} (S^i - v^i(y^i - f^i)')dq + \sum_{i\in\mathcal{I}} (v^i q - I)df^i = 0$$

As in the proof of proposition 1,the graph of the competitive equilibrium correspondence identifies the demand function for assets of every individual and therefore, with step 1, the entire demand function. □

Remark If for every individual, for every commodity and for every state of the world, for any sequence $((p_{s,n}, e_{s,n}^i, y_s^i) : n = 1, \ldots)$, of prices of commodities, endowments of commodities and portfolios of assets,

$$\lim_{n\to\infty} e_{1,s,n}^i + R_s y_s^i = \infty \quad \Rightarrow \quad \lim_{n\to\infty} z_{l,s,n}^i(p_{s,n}, e_{s,n}^i, y_s^i) = \infty, \quad l \in \mathcal{L} :$$

for every individual, every commodity is normal in a strong sense, then the competitive equilibrium correspondence identifies the utility functions of individuals on their entire domain of definition. The argument is the analogue of the argument under certainty.

Remark It is an open question whether identification under uncertainty and an incomplete asset market extends to non - separable preferences, as was done in Polemarchakis and Selden (1984), for individual demand functions.

Remark The identification result requires that there are at least 3 commodities. As is the case under certainty, the case of 2 commodities remains an open question. The case of 1 commodity, evidently vacuous under certainty, is indeed of interest under uncertainty. Though identification from individual demand requires additional assumptions as pointed out in the earlier literature, identification from the competitive equilibrium correspondence is not problematic. This is due to the freedom afforded by the equilibrium correspondence, namely the variation in the endowments of individuals across states of the world. The following argument clarifies this - a similar argument shows that identification is possible if cardinal utilities, u^i_s, are homothetic.

Step 2 of the proof of proposition 2 implies that — as long as part 1) and 2) of assumption 6 are satisfied — individual asset demand can be identified from the equilibrium manifold even if $L = 1$. In this framework, individual asset demand as a function of individual endowments at all states and of prices does identify the utility function: if $L = 1$, the individual asset demand function is a solution to

$$\begin{aligned} \max \quad & w^i(Ry), \\ \text{s.t} \quad & qy \leq qf^i. \end{aligned}$$

Differentiating the first order conditions and setting

$$\begin{pmatrix} K^i & -v^i \\ -v^{i\prime} & b^i \end{pmatrix} = \begin{pmatrix} -R_s' D^2 w^i R_s & -q \\ -q' & 0 \end{pmatrix}^{-1},$$

one obtains that $\partial^2 u^i_s$ can be recovered by variation of e^s_i after the identification of K^i and v^i from variation in prices and endowments in portfolios, because $D^2 w^i$ is a diagonal matrix.

4 Implications and extensions

In applied general equilibrium, the preferences of individuals and the technologies of firms are chosen within a small parametric class to match empirical properties of data. The restriction to a given parametric class is viewed as essential: failure of identification makes counterfactual policy analysis problematic: different models that match the data may produce different prediction — Hansen and Heckman (1996) discuss this in depth.

The argument here is that there is no failure of identification if aggregate consumptions, incomes and equilibrium prices are observable, along varying profiles of individual endowments. It implies that it is possible to estimate preferences econometrically, from data on prices and incomes, as in Brown and Matzkin (1990).

However, data on prices and incomes is likely to be in he form of time series. In a more applicable argument, one needs to take into account that for time series data, prices and incomes might be part of one, intertemporal equilibrium, and not points on an equilibrium correspondence. According to Ḱubler (1999), the assumption of time separable expected utility restores global in an intertemporal model. For identification, the assumption of separable utility is not enough, since sufficiently complete asset markets allow individuals to smooth their expenditure across dates and states of the world. Examining identification in an overlapping generations model without bequest is subject to further research.

An additional complication arises under uncertainty. The rational expectations hypothesis implies that agents take all prices and dividends into account when choosing their portfolio and consumption. However, observations can only consist of one sample path, and it seems unlikely that identification of preferences is possible without any knowledge of equilibrium prices at nodes that that do not lie on the sample path.

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