An attempt at defining a position-dependent flushing time

Eric Deleersnijder, 2 July 2020

Abstract. The concept of position-dependent residence time is revisited, eventually leading to the definition of another diagnostic timescale, tentatively called flushing time. As opposed to the residence time, this timescale is not prescribed to be zero on open boundaries (Dirichlet boundary conditions). Instead, Robin and Neumann boundary conditions are prescribed, with a different treatment of the inlet and the outlet of the domain of interest. In an idealised channel flow, the behaviour of the flushing time is similar to that obtained for a well-mixed reservoir (resp. plug flow) if advection (resp. diffusion) is negligible. The residence time behaves in a markedly different manner for small values of the Peclet number, which may be deemed to be somewhat counterintuitive.

Motivation

According to Deleersnijder (2019), most (if not all) of the diagnostic timescales looking into the future can be viewed as one of the (many) variants of the exposure time. The latter is defined as follows: **the exposure time of a particle is the time it will spend in the domain of interest**. Needless to say, the fate of a single particle is rather irrelevant: timescale should be evaluated for a sufficiently large number of particles (e.g. van Sebille et al. 2018, Deleersnijder 2019, Lucas and Deleersnijder 2020).

To actually calculate such a timescale, a key ingredient is the definition of the moment and/or location at which the particles under study will cease to be taken into consideration. For instance, for the **residence time**, a particle is no longer considered at moment it hits for the first time an open boundary of the domain. In an Eulerian model, this leads to a Dirichlet boundary condition: the residence time is prescribed to be zero on all the open boundaries of the domain of interest (e.g. Delhez and Deleersnijder 2006). **Other approaches** may be contemplated. Some of them require the modelling of the environment of the domain of interest (e.g. Delhez 2013) and some do not, as is exemplified below.

In this working note, the classical residence time and a variant of it (ensuing from Neumann or Robin boundary conditions prescribed on the open boundaries), tentatively called flushing time for want of a better expression, are derived for a steady-state, multi-dimensional flow. For a highly idealised one-dimensional flow, it is seen that the latter timescale tends to that obtained for a **plug flow** (resp. **well-mixed reservoir**) if the **Peclet number** is much **larger** (resp. **smaller**) than **unity**. The behaviour of the classical residence time is radically different for small values of the Peclet number, which may be regarded by some as a weakness of this approach.

Geometry and hydrodynamics

Let Ω, Γ and $\mathbf{n}$ denote the domain of interest, its boundary and the outward unit vector normal to the latter, respectively. The volume of the domain of interest is

$$V = \int_{\Omega} d\Omega \tag{1}$$

with $d\Omega = dx\,dy\,dz$, where $\mathbf{x} = (x,y,z)$ denotes the position vector, whilst x, y and z are Cartesian coordinates.

The three-dimensional velocity field $\mathbf{v}(\mathbf{x})$ is time-independent and divergenceless, i.e.

$$\nabla \bullet \mathbf{v} = 0 \tag{2}$$

The domain boundary is split into three sub-regions according to the sign of the normal velocity on them (Figure 1):

$$\Gamma^{imp}: \text{ impermeable boundary} \quad \Rightarrow \quad [\mathbf{v} \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^{imp}} = 0 \tag{3}$$

$$\Gamma^{in}: \text{ incoming boundary} \quad \Rightarrow \quad [\mathbf{v} \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^{in}} < 0 \tag{4}$$

$$\Gamma^{out}: \text{ outgoing boundary} \quad \Rightarrow \quad [\mathbf{v} \bullet \mathbf{n}]_{\mathbf{x} \in \Gamma^{in}} > 0 \tag{5}$$

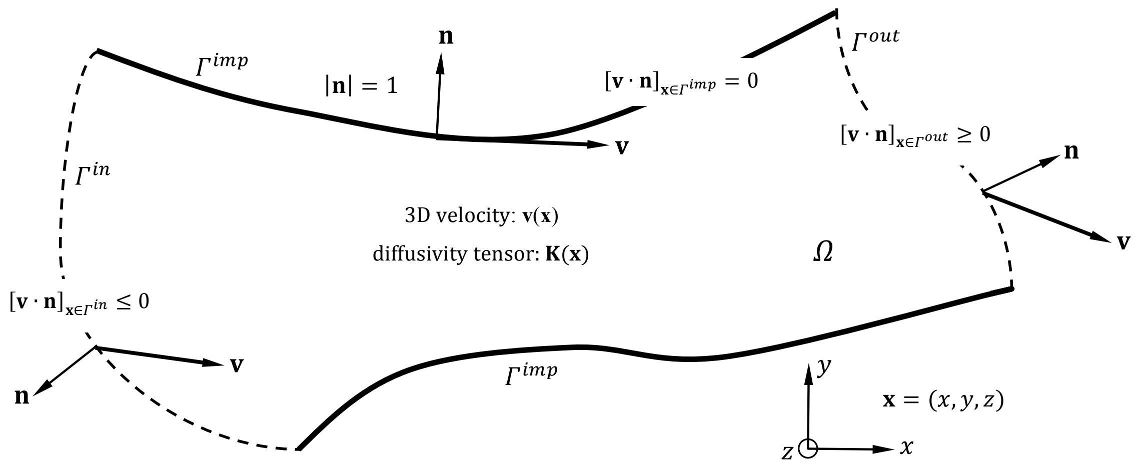

Figure 1. Schematic representation of the geometry of the domain of interest (Ω), its boundary ($\Gamma = \Gamma^{imp} \cup \Gamma^{in} \cup \Gamma^{out}$) and the behaviour of the water velocity on them.

Unresolved processes are parameterised by means of a Fourier-Fick formula involving diffusivity tensor $\mathbf{K}(\mathbf{x})$, which is symmetric ($\mathbf{K}^T = \mathbf{K}$) and positive-definite (e.g. Deleersnijder et al. 2001):

$$\forall\, \mathbf{y} \neq \mathbf{0}: \;\; \mathbf{y} \bullet \mathbf{K} \bullet \mathbf{y} > 0 \tag{6}$$

Residence time

The **residence time** $\theta(\mathbf{x})$ is the time needed to hit for the first time an open boundary, be it an incoming or an outgoing one (e.g. Delhez et al. 2004). It can be obtained from expression

$$\theta(\mathbf{x}) = \int_0^\infty \int_\Omega G(\mathbf{x};t',\mathbf{x}')\, d\Omega'\, dt' \tag{7}$$

where $d\Omega' = dx'\, dy'\, dz'$; Green's function $G(\mathbf{x};t',\mathbf{x}')$ is the solution of the following partial differential problem:

$$\frac{\partial G}{\partial t'} = -\nabla' \bullet (G\mathbf{v} - \mathbf{K} \bullet \nabla' G) \tag{8}$$

$$G(\mathbf{x};0,\mathbf{x}') = \delta(\mathbf{x} - \mathbf{x}') \tag{9}$$

$$\left[(G\mathbf{v} - \mathbf{K} \bullet \nabla' G) \bullet \mathbf{n}\right]_{\mathbf{x}' \in \Gamma^{imp}} = 0 \tag{10}$$

$$\left[G(\mathbf{x};t',\mathbf{x}')\right]_{\mathbf{x}' \in \Gamma^{in} \cup \Gamma^{out}} = 0 \tag{11}$$

Clearly, the physical dimension of the Green's function is length^{-3}.

The Green's function may be viewed as the “concentration” of particles of which a unit amount is located at point $\mathbf{x}' = \mathbf{x}$ at time $t' = 0$. Then, according to (7), residence time $\theta(\mathbf{x})$ is the mean time they take to hit for the first time an open (incoming or outgoing) boundary of the domain.

The negative part of the Green's function, $G^- = (G - |G|)/2$, satisfies

$$\frac{d}{dt'} \int_\Omega (G^-)^2\, d\Omega' = 2 \int_\Omega \underbrace{(-\nabla G^- \bullet \mathbf{K} \bullet \nabla G^-)}_{\leq 0}\, d\Omega' \leq 0 \tag{12}$$

Since G^- is zero at $t' = 0$, G^- will be zero at any time, implying that the Green's function is non-negative. Therefore, as it should be, the residence time is also non-negative:

$$\theta(\mathbf{x}) \geq 0 \tag{13}$$

The following (dimensionless) concentration may be derived from the Green's function:

$$C(t',\mathbf{x}') = \int_\Omega G(\mathbf{x};t',\mathbf{x}')\, d\Omega \tag{14}$$

By integrating (8)-(11) over Ω , the differential problem satisfied by concentration (14) is obtained:

$$\frac{\partial C}{\partial t'} = -\nabla' \bullet (C\mathbf{v} - \mathbf{K} \bullet \nabla' C) \tag{15}$$

$$C(0,\mathbf{x}') = 1 \tag{16}$$

$$\left[(C\mathbf{v} - \mathbf{K} \bullet \nabla' C) \bullet \mathbf{n}\right]_{\mathbf{x}' \in \Gamma^{imp}} = 0 \tag{17}$$

$$\left[C(t',\mathbf{x}')\right]_{\mathbf{x}' \in \Gamma^{in} \cup \Gamma^{out}} = 0 \tag{18}$$

Then, it is readily seen that the domain-averaged residence time is

$$\begin{aligned} \langle \theta \rangle &= \frac{1}{V} \int_\Omega \theta(\mathbf{x})\, d\Omega \\ &= \frac{1}{V} \int_\Omega \int_0^\infty \int_\Omega G(\mathbf{x};t',\mathbf{x}')\, d\Omega'\, dt'\, d\Omega = \frac{1}{V} \int_0^\infty \int_\Omega \underbrace{\int_\Omega G(\mathbf{x};t',\mathbf{x}')\, d\Omega}_{=C(t',\mathbf{x}')}\, d\Omega'\, dt' \end{aligned} \tag{19}$$

which simplifies to

$$\langle\theta\rangle = \frac{1}{V}\int_0^\infty\int_\Omega C(t',\mathbf{x}')\,d\Omega'\,dt' \tag{20}$$

In other words, the **average** over the domain of interest of pointwise residence time $\theta(\mathbf{x})$ is the **mean time** that particles of a **passive tracer** whose **initial concentration** is equal to **unity** will spend in the domain of interest.

Since the differential problems under consideration are linear, it is not necessary that the abovementioned tracer concentration be equal to unity. It is however essential that $C(0,\mathbf{x}')$ be uniform. Accordingly, if $C(0,\mathbf{x}') = C^0$, then the domain-averaged residence time is

$$\langle\theta\rangle = \frac{\int_0^\infty\int_\Omega C(t',\mathbf{x}')\,d\Omega'\,dt'}{\int_\Omega C(0,\mathbf{x}')\,d\Omega'} = \frac{1}{VC^0}\int_0^\infty\int_\Omega C(t',\mathbf{x}')\,d\Omega'\,dt' \tag{21}$$

Using the theoretical developments of Delhez et al. (2014), it may be seen that the residence time can also be obtained by solving the relevant adjoint problem, i.e.

$$\nabla\bullet(\mathbf{K}\bullet\nabla\theta+\theta\mathbf{v}) = -1 \tag{22}$$

$$\left[(\mathbf{K}\bullet\nabla\theta)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{imp}} = 0 \tag{23}$$

$$\left[\theta(\mathbf{x})\right]_{\mathbf{x}\in\Gamma^{in}\cup\Gamma^{out}} = 0 \tag{24}$$

It must be stressed that the boundary conditions cannot be chosen freely. They are deduced (through the mechanism of the adjoint problem) from boundary conditions (10)-(11), i.e. from the boundary conditions prescribed in the forward problem.

Flushing time

The residence time of a collection of particles as defined above is the mean time spent in the domain, given that the particles under study cease to be considered at the moment they hit for the first time an open boundary of the domain. The time spent in the domain may be calculated in many other ways, by modifying the boundary conditions.

It is suggested that pointwise **flushing time** $\tilde{\theta}(\mathbf{x})$ be evaluated as follows:

$$\tilde{\theta}(\mathbf{x}) = \int_0^\infty\int_\Omega \tilde{G}(\mathbf{x};t',\mathbf{x}')\,d\Omega'\,dt' \tag{25}$$

where

$$\frac{\partial\tilde{G}}{\partial t'} = -\nabla'\bullet(\tilde{G}\mathbf{v}-\mathbf{K}\bullet\nabla'\tilde{G}) \tag{26}$$

$$\tilde{G}(\mathbf{x};0,\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}') \tag{27}$$

$$\left[(\tilde{G}\mathbf{v}-\mathbf{K}\bullet\nabla'\tilde{G})\bullet\mathbf{n}\right]_{\mathbf{x}'\in\Gamma^{imp}\cup\Gamma^{in}} = 0 \tag{28}$$

$$\left[(-\mathbf{K}\bullet\nabla'\tilde{G})\bullet\mathbf{n}\right]_{\mathbf{x}'\in\Gamma^{out}} = 0 \tag{29}$$

There are no Dirichlet boundary conditions any more. On the incoming boundary, the (advective + diffusive) flux is prescribed to be zero, whilst only the diffusive flux is prescribed to be zero on the outgoing boundary. There is no modification of the boundary condition on Γ^{imp}.

The negative part of the Green's function, $\tilde{G}^{-} = (\tilde{G} - |\tilde{G}|)/2$, satisfies

$$\frac{d}{dt'} \int_{\Omega} (\tilde{G}^{-})^2 \, d\mathbf{x}' = \int_{\Gamma^{in}} \overbrace{(\tilde{G}^{-})^2}^{\geq 0} \overbrace{\mathbf{v} \bullet \mathbf{n}}^{\leq 0} \, d\Gamma' + \int_{\Gamma^{out}} \underbrace{(\tilde{G}^{-})^2}_{\geq 0} \underbrace{(-\mathbf{v} \bullet \mathbf{n})}_{\leq 0} \, d\Gamma' + 2 \int_{\Omega} \underbrace{(-\nabla \tilde{G}^{-} \bullet \mathbf{K} \bullet \nabla \tilde{G}^{-})}_{\leq 0} \, d\Omega' \leq 0 \tag{30}$$

Since $\tilde{G}^{-}$ is zero at $t' = 0$, $\tilde{G}^{-}$ will be zero at any time, implying that the Green's function is non-negative. Therefore, as it should be, the flushing time is also non-negative:

$$\tilde{\theta}(\mathbf{x}) \geq 0 \tag{31}$$

The following (dimensionless) concentration may be derived from the Green's function:

$$\tilde{C}(t', \mathbf{x}') = \int_{\Omega} \tilde{G}(\mathbf{x}; t', \mathbf{x}') \, d\Omega \tag{32}$$

By integrating (26)-(29) over Ω, the differential problem satisfied by concentration (32) is obtained:

$$\frac{\partial \tilde{C}}{\partial t'} = -\nabla' \bullet (\tilde{C}\mathbf{v} - \mathbf{K} \bullet \nabla' \tilde{C}) \tag{33}$$

$$\tilde{C}(0, \mathbf{x}') = 1 \tag{34}$$

$$\left[(\tilde{C}\mathbf{v} - \mathbf{K} \bullet \nabla' \tilde{C}) \bullet \mathbf{n} \right]_{\mathbf{x}' \in \Gamma^{imp} \cup \Gamma^{in}} = 0 \tag{35}$$

$$\left[(-\mathbf{K} \bullet \nabla' \tilde{C}) \bullet \mathbf{n} \right]_{\mathbf{x}' \in \Gamma^{out}} = 0 \tag{36}$$

Then, it is readily seen that the domain-averaged flushing time is

$$\left\langle \tilde{\theta} \right\rangle = \frac{1}{V} \int_{\Omega} \tilde{\theta}(\mathbf{x}) \, d\Omega = \frac{1}{V} \int_{\Omega} \int_{0}^{\infty} \int_{\Omega} \tilde{G}(\mathbf{x}; t', \mathbf{x}') \, d\Omega' \, dt' \, d\Omega = \frac{1}{V} \int_{0}^{\infty} \int_{\Omega} \underbrace{\int_{\Omega} \tilde{G}(\mathbf{x}; t', \mathbf{x}') \, d\Omega}_{= \tilde{C}(t', \mathbf{x}')} \, d\Omega' \, dt \tag{37}$$

which simplifies to

$$\left\langle \tilde{\theta} \right\rangle = \frac{1}{V} \int_{0}^{\infty} \int_{\Omega} \tilde{C}(t', \mathbf{x}') \, d\Omega' \, dt \tag{38}$$

In other words, the **average** over the domain of interest of pointwise flushing time $\tilde{\theta}(\mathbf{x})$ is the **mean time** that particles of a **passive tracer** whose **initial concentration** is equal to **unity** will spend in the domain of interest.

Using the theoretical developments of Delhez et al. (2014), it may be seen that residence time can also be obtained by solving the relevant adjoint problem, i.e.

$$\nabla \bullet (\mathbf{K} \bullet \nabla \tilde{\theta} + \tilde{\theta} \mathbf{v}) = -1 \tag{39}$$

$$\left[(\mathbf{K} \bullet \nabla\tilde{\theta}) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^{imp} \cup \Gamma^{in}} = 0 \tag{40}$$

$$\left[(\mathbf{K} \bullet \nabla\tilde{\theta} + \tilde{\theta}\mathbf{v}) \bullet \mathbf{n}\right]_{\mathbf{x} \in \Gamma^{out}} = 0 \tag{41}$$

As for the residence time, it must be stressed that the boundary conditions cannot be chosen freely. They are deduced (through the mechanism of the adjoint problem) from boundary conditions (28)-(29), i.e. from the boundary conditions prescribed in the forward problem.

Properties of the flushing time

Let $\bar{\tilde{\theta}}$ and $\hat{\tilde{\theta}}$ represent the average over the incoming boundary of the flushing time and the deviation with respect to the mean value, respectively. Then, integrating (39) over the domain of interest, using the divergence theorem and boundary conditions (40)-(41), one obtains

$$\bar{\tilde{\theta}} = \frac{V}{Q} + \frac{1}{Q} \int_{\Gamma^{in}} \hat{\tilde{\theta}} \mathbf{v} \bullet \mathbf{n} \, d\Gamma \tag{42}$$

where Q denotes the incoming volumetric flow rate ($\mathrm{m^3/s}$),

$$Q = - \int_{\Gamma^{in}} \mathbf{v} \bullet \mathbf{n} \, d\Gamma \tag{43}$$

which is equal to the outgoing one, for the flow is at a steady state. The second term in the right-hand side of (42) may be either positive or negative, and is likely to be rather small if the entrance is relatively narrow. This suggests that the **order of magnitude of the flushing time on the inlet is *V/Q*** (i.e. "volume over flux"), which is an expression typical of a well-mixed reservoir. No such property can be established for the residence time, since the latter is identically zero on the open boundaries.

Misguided physical intuition could suggest that the maximum of the flushing time is located on the inlet. This assumption does not necessarily hold valid. To address this issue, it is appropriate to first introduce the difference between the flushing time and its maximum value on the incoming boundary:

$$\delta\tilde{\theta}(\mathbf{x}) = \tilde{\theta}(\mathbf{x}) - \max_{\mathbf{x} \in \Gamma^{in}} \tilde{\theta}(\mathbf{x}) \tag{44}$$

The positive part of this variable is $\delta\tilde{\theta}^{+} = (\delta\tilde{\theta}^{+} + \left|\delta\tilde{\theta}^{+}\right|)/2$. Multiplying (39) by $\delta\tilde{\theta}^{+}$ and using boundary conditions (40)-(41), one obtains after lengthy calculations

$$\underbrace{2\int_{\Omega} \nabla(\delta\tilde{\theta}^{+}) \bullet \mathbf{K} \bullet \nabla(\delta\tilde{\theta}^{+}) \, d\Omega}_{\geq 0,\text{ since } \mathbf{K} \text{ is positive definite}} + \underbrace{\int_{\Gamma^{out}} \delta\tilde{\theta}^{+} \mathbf{v} \bullet \mathbf{n} \, d\Gamma}_{\geq 0,\text{ since } \delta\tilde{\theta}^{+} \text{ is positive definite and } \mathbf{v}\bullet\mathbf{n} \geq 0 \text{ on } \Gamma^{out}} + \underbrace{2\int_{\Omega} (-\delta\tilde{\theta}^{+}) \, d\Omega}_{\leq 0,\text{ since } (-\delta\tilde{\theta}^{+}) \text{ is negative definite}} = 0 \tag{45}$$

This does not imply that $\delta\tilde{\theta}^{+}$ is identically zero. Accordingly, the **maximum** of the flushing time can be located **inside the domain**.

Now consider the difference between the flushing time and the residence time,

$$\eta(\mathbf{x}) = \tilde{\theta}(\mathbf{x}) - \theta(\mathbf{x}) \tag{46}$$

From (22)-(24), (31) and (39)-(40), it is readily seen that this variable satisfies

$$\nabla \bullet (\mathbf{K} \bullet \nabla\eta + \eta\mathbf{v}) = 0 \tag{47}$$

$$\left[(\mathbf{K}\bullet\nabla\eta)\bullet\mathbf{n}\right]_{\mathbf{x}\in\Gamma^{imp}} = 0 \tag{48}$$

$$\left[\eta(\mathbf{x})\right]_{\mathbf{x}\in\Gamma^{in}\cup\Gamma^{out}} \geq 0 \tag{49}$$

Inequality (49) stems from the fact that the flushing time is non-negative and the residence time is prescribed to be zero on the open boundaries. Then, the negative part of $\eta(\mathbf{x})$, which is defined as $\eta^{-} = (\eta - |\eta|)/2$, obeys

$$-2\underbrace{\int_{\Omega}\nabla\eta^{-}\bullet\mathbf{K}\bullet\nabla\eta^{-}\,d\Omega}_{\geq 0\text{, since }\mathbf{K}\text{ is positive definite}} + \underbrace{\int_{\Gamma^{in}\cup\Gamma^{out}}\eta^{-}(2\mathbf{K}\bullet\nabla\eta^{-}+\eta^{-}\mathbf{v})\bullet\mathbf{n}\,d\Gamma}_{=0\text{, since }\eta\geq 0\text{ on }\Gamma^{in}\cup\Gamma^{out}\text{, implying that }\eta^{-}=0} = 0 \tag{50}$$

which implies that $\eta^{-}(\mathbf{x})$ is zero. As a result, the flushing time is greater than or equal to the residence time

$$\tilde{\theta}(\mathbf{x}) \geq \theta(\mathbf{x}) \tag{51}$$

This is unsurprising since both timescales obey the same partial differential equations, they obey a zero flux condition on the impermeable boundaries, the residence time is prescribed to be zero on the open boundaries, whilst the flushing time is non-negative on the same boundaries.

Channel flow: general solutions

In this section, we will consider domains whose length (distance from inlet to outlet) can be unambiguously defined. If, in addition, the width of the domain is much smaller than its length, then it may be deemed appropriate to have recourse to a one-dimensional approach, relying only on cross section-averaged variables. Clearly, the domain of interest may be termed a channel.

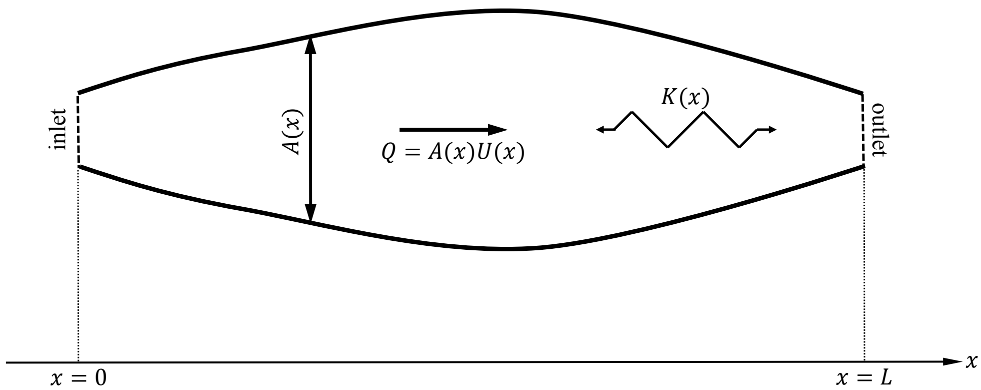

Figure 2. Schematic representation of the channel and the one-dimensional flow taking place in it. The longitudinal coordinate is denoted x, whilst $A(x)$, $U(x)$, and $K(x)$ represent the cross-sectional area, the velocity and the relevant along-flow diffusivity, respectively. The volumetric flow rate, $Q = A(x)U(x)$, is a constant.

Let x, L, $A(x)$ and $U(x)$ denote the longitudinal coordinate (distance to the incoming boundary), the length of the channel ($0 \le x \le L$), its cross-sectional area and the average of the longitudinal velocity over a cross section, respectively (Figure 2). Since the flow is at a steady state, the volumetric flow rate must be constant:

$$Q = A(x)U(x) \tag{52}$$

The unresolved processes are parameterised by means of longitudinal diffusivity $K(x)$, which is assumed to be positive.

It is convenient to introduce the following auxiliary variables:

$$\Xi(x) = \int_0^x A(x')\,dx' \tag{53}$$

$$\Pi(x) = \frac{U(x)L}{K(x)} = \frac{QL}{A(x)K(x)} \tag{54}$$

$$\lambda(x) = \exp\left[-\frac{1}{L}\int_0^x \Pi(x')\,dx'\right] \tag{55}$$

$$\zeta(x) = \int_0^x \frac{\Xi(x')\,dx'}{A(x')K(x')\lambda(x')} \tag{56}$$

Clearly, $\Pi(x)$ may be viewed as a local Peclet number. It is readily seen that $\lambda(0)=1$, $\zeta(0)=0$ and $[d\zeta/dx]_{x=0}=0$.

The residence time obeys (adjoint problem)

$$\begin{cases} \dfrac{d}{dx}\left(AK\dfrac{d\theta}{dx}+Q\theta\right) = -A \\ \theta(0)=0=\theta(L) \end{cases} \tag{57}$$

and, hence, reads

$$\theta(x) = \frac{\zeta(L)\lambda(L)[1-\lambda(x)] - [1-\lambda(L)]\zeta(x)\lambda(x)}{1-\lambda(L)} \tag{58}$$

On the other hand, the flushing time is the solution of the following adjoint differential problem:

$$\begin{cases} \dfrac{d}{dx}\left(AK\dfrac{d\tilde{\theta}}{dx}+Q\tilde{\theta}\right) = -A \\ \left[AK\dfrac{d\tilde{\theta}}{dx}\right]_{x=0}=0\,, \quad \left[AK\dfrac{d\tilde{\theta}}{dx}+Q\tilde{\theta}\right]_{x=L}=0 \end{cases} \tag{59}$$

This yields

$$\tilde{\theta}(x) = \frac{V}{Q} - \zeta(x)\lambda(x) \tag{60}$$

The flushing time is equal to V/Q at the entrance of the channel ($x=0$). This is in agreement with (42), for in a one-dimensional model $\hat{\tilde{\theta}}$ is identically zero.

If the channel is unbounded ($-\infty < x < \infty$), the exposure time, $\Theta(x)$, in domain of interest $0 \le x \le L$ satisfies

$$\begin{cases} \dfrac{d}{dx}\left(AK\dfrac{d\Theta}{dx} + Q\Theta \right) = \begin{cases} -A, \ x \in [0,L] \\ 0, \ x \notin [0,L] \end{cases} \\ \left[\dfrac{d\Theta}{dx} \right]_{x=-\infty} = 0 \ , \ \left[\Theta(x) \right]_{x=\infty} = 0 \end{cases} \tag{61}$$

The solution of this differential problem is

$$\Theta(x) = \begin{cases} \dfrac{V}{Q} \ , \ x < 0 \\ \tilde{\theta}(x) = \dfrac{V}{Q} - \zeta(x)\lambda(x) \ , \ 0 \le x \le L \\ \left[\dfrac{V}{Q} - \zeta(L)\lambda(L) \right] \dfrac{\lambda(x)}{\lambda(L)} \ , \ L < x \end{cases} \tag{62}$$

Remarkably, the **exposure time** and the **flushing time** are **equal** to each other in the domain of interest ($0 \le x \le L$). This property should prompt further investigations aimed at addressing the key issue related to the evaluation of the exposure time (or similar diagnostic timescales), i.e. the need to carry out calculations in a computational domain much larger then the domain of interest per se while preventing the exposure time from becoming unacceptably large (Delhez 2013, Liu et al. 2019).

Channel flow with constant parameters

Now assume that the cross-sectional area, A, and the along-channel diffusivity, K, are constant. Then, introducing dimensionless space coordinate $\xi = x/L$ and Peclet number

$$Pe = \frac{UL}{K} \tag{63}$$

auxiliary functions (53)-(56) simplify to

$$\Xi(x) = Ax \ , \ \Pi(x) = Pe \ , \ \lambda(x) = e^{-Pe\xi} \ , \ \zeta(x) = \frac{V}{Q}\frac{1 - e^{Pe\xi} + Pe\xi\, e^{Pe\xi}}{Pe} \tag{64}$$

so that the residence time, the flushing time and the exposure time transform to (Figure 3)

$$\theta(x) = \frac{V}{Q}\left[\frac{1 - e^{-Pe\xi}}{1 - e^{-Pe}} - \xi \right] \tag{65}$$

$$\tilde{\theta}(x) = \frac{V}{Q}\left[\frac{1 - e^{-Pe\xi}}{Pe} + 1 - \xi \right] \tag{66}$$

$$\Theta(\xi) = \begin{cases} \dfrac{V}{Q} \ , \ -\infty < \xi < 0 \\ \tilde{\theta}(\xi) = \dfrac{V}{Q}\left[\dfrac{1 - e^{-Pe\xi}}{Pe} + 1 - \xi \right], \ 0 \le \xi \le 1 \\ \dfrac{V}{Q}\left[\dfrac{e^{Pe} - 1}{Pe} e^{-Pe\xi} \right], \ 0 < \xi < 1 \end{cases} \tag{67}$$

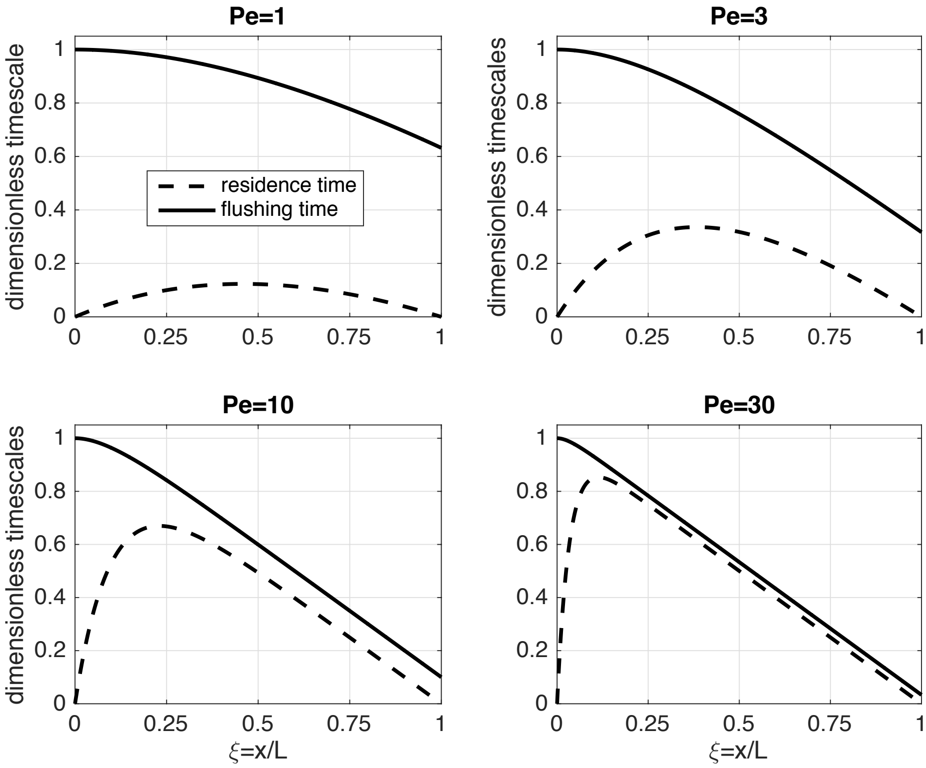

Figure 3. Dimensionless residence time ($Q\theta / V$) and flushing time ($Q\tilde{\theta} / V$) for various values of the Peclet number ($Pe = UL / K$) as a function of the dimensionless distance to the incoming boundary ($\xi = x / L$) for a channel flow with constant cross-sectional area and along-flow diffusivity. For a well-mixed reservoir (resp. plug flow) the dimensionless flushing time is equal to unity (resp. $1 - \xi$). For large values of the Peclet number, the residence time exhibits a boundary layer adjacent to the domain inlet ($\xi = 0$) (Delhez and Deleersnijder 2006).

If advection tends to be negligible, the flushing time admits asymptotic expansion

$$\tilde{\theta}(x) \sim \frac{V}{Q}\left(1 - \frac{\xi^2}{2} Pe\right) , \quad Pe \rightarrow 0 \tag{68}$$

The dominant term, V / Q, is **independent of the position** and, obviously, would be obtained in a zero-dimensional, **well-mixed reservoir** model (e.g. Bolin and Rodhe 1973, Deleersnijder 2019, Lucas and Deleersnijder 2020). If, on the other hand, diffusion tends to be negligible, the following asymptotic expansion holds valid

$$\tilde{\theta}(x) \sim \frac{V}{Q}\left(1 - \xi + \frac{1}{Pe}\right) , \quad Pe \rightarrow \infty \tag{69}$$

The leading order term, $(V / Q)(1 - \xi)$, is typical of a **plug flow** (e.g. Lucas and Deleersnijder 2020). The flushing time is such that $\partial\tilde{\theta} / \partial Pe < 0$. Moreover, the following inequalities are easily established (Figure 4)

$$\underbrace{\frac{V}{Q}(1-\xi)}_{\text{plug flow}} \leq \tilde{\theta}(x) \leq \underbrace{\frac{V}{Q}}_{\text{reservoir}} \tag{70}$$

It must be stressed that the residence exhibits a markedly different behaviour. The residence time is close to the flushing time for $Pe \to \infty$ (except in the boundary layer adjacent to the inlet), but tends to zero if diffusion is dominant, because of the Dirichlet boundary at both ends of the domain. Indeed, we have

$$\theta(x) \sim \frac{V}{Q}\frac{\xi(1-\xi)}{2}Pe \ , \quad Pe \to 0 \tag{71}$$

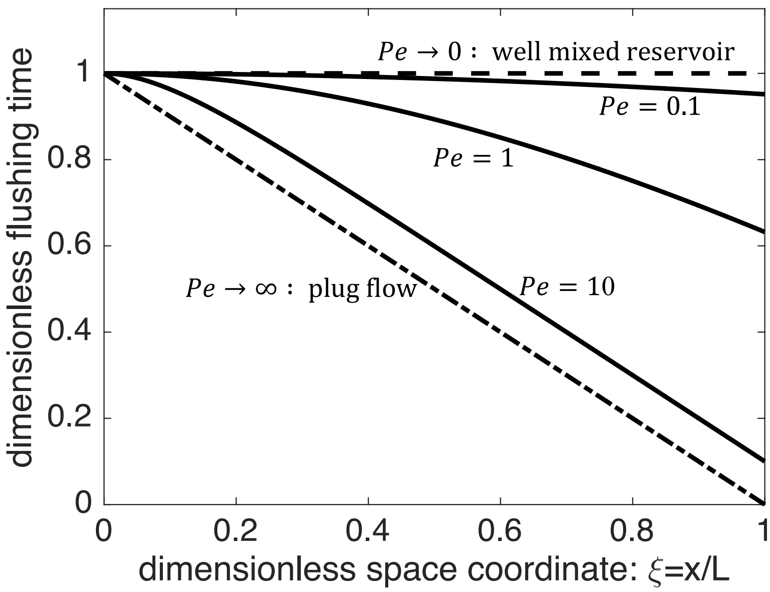

Figure 4. Dimensionless flushing time ($Q\tilde{\theta}/V$) for various values of the Peclet number ($Pe = UL/K$) as a function of the dimensionless distance to the incoming boundary ($\xi = x/L$) for a channel flow with constant cross-sectional area and along-flow diffusivity. The limit $Pe \to 0$ (resp. $Pe \to \infty$) corresponds to a well mixed reservoir (resp. plug flow). The fact that $\partial\tilde{\theta}/\partial Pe < 0$ is clearly illustrated.

The domain-averaged values of the residence and flushing times are (Figure 5)

$$\langle\theta\rangle = \frac{1}{L}\int_0^L \theta(x)\,dx = \frac{V}{Q}\frac{-2+Pe+(2+Pe)e^{-Pe}}{2Pe(1-e^{-Pe})} \sim \begin{cases} \frac{V}{Q}\left(\frac{1}{2}-\frac{1}{Pe}\right), & Pe \to \infty \\ \frac{V}{Q}\left(\frac{Pe}{12}-\frac{Pe^3}{720}\right), & Pe \to 0 \end{cases} \tag{72}$$

$$\langle\tilde{\theta}\rangle = \frac{1}{L}\int_0^L \tilde{\theta}(x)\,dx = \frac{V}{Q}\frac{2Pe+Pe^2-2(1-e^{-Pe})}{2Pe^2} \sim \begin{cases} \frac{V}{Q}\left(\frac{1}{2}+\frac{1}{Pe}\right), & Pe \to \infty \\ \frac{V}{Q}\left(1-\frac{Pe}{6}\right), & Pe \to 0 \end{cases} \tag{73}$$

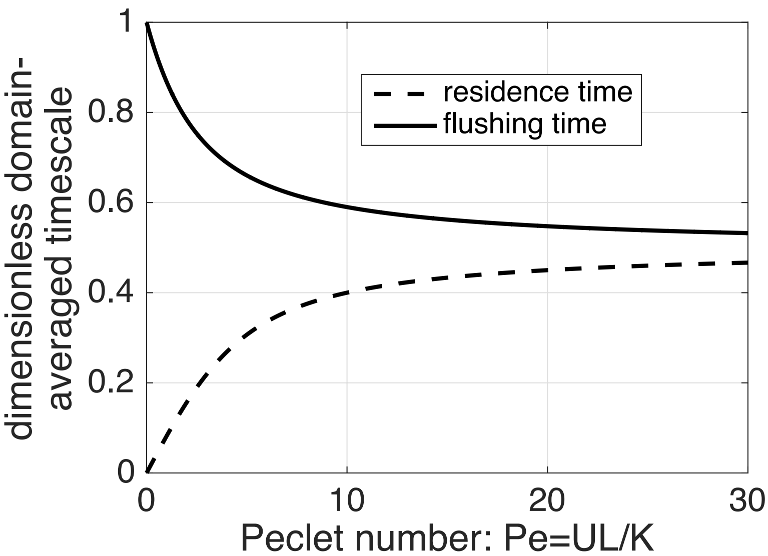

Figure 5. Dimensionless domain-averaged residence time ($Q\langle\theta\rangle/V$) and flushing time ($Q\langle\tilde{\theta}\rangle/V$) as a function of the Peclet number ($Pe=UL/K$) for a channel flow with constant cross-sectional area and along-flow diffusivity. For a well-mixed reservoir (resp. plug flow) the dimensionless domain-averaged flushing time is equal to unity (resp. 1/2). The behaviour of the domain-averaged flushing time is in agreement with such limits, which may be deemed to be in agreement with physical intuition.

Discussion

The boundary conditions leading to the so-called flushing time are similar to those implemented in Deleersnijder et al. (2006), which, however, addressed radically different issues. The boundary conditions of that article were physically grounded as opposed to those of the present study, which resorts to some form of reverse engineering. Indeed, the boundary conditions introduced above are those leading to a dependency of the new diagnostic timescale as a function of the Peclet number that is deemed to be in agreement with elementary physical intuition derived from the well established well-mixed reservoir and plug flow models. Clearly, it has yet to be examined in depth whether **physical justifications** may be built that would **support these boundary conditions**.

The **well-foundedness of the boundary conditions** for the flushing time must be assessed in the framework of the wider context of **water renewal** as was introduced by Gourgue et al. (2007).

The present working note is concerned only with steady state flows. Nonetheless, it must be stressed that the concept of flushing time should be applicable to **unsteady flows**. However, the impact of the time-dependency of the flow (e.g. tidal flow in an estuary) has yet to be investigated. This is obviously no minor issue, since the water velocity may not always cross open boundaries in the same direction. As a result, a given portion of an open boundary may alternatively be of an incoming and outgoing nature.

Owing to the nature of the open boundary conditions, it is **unlikely** that the **flushing time** will ever exhibit a **boundary layer adjacent to an open boundary** — as opposed to the residence time (Delhez and Deleersnijder 2006). Therefore, numerically solving the partial differential problem governing the flushing time may well prove to be significantly easier than dealing with that obeyed by the residence time (Blaise et al. 2010).

Building on Liu et al. (2012), Mouchet et al. (2016) generalised the concept of age to that of partial ages. Then, Lin and Liu (2019) developed the theory of partial residence times and applied it to various flows. It is believed that the flushing time may undergo a similar type of generalisation, which would lead to **partial flushing times**. Nonetheless, this has yet to be achieved.

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