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What could be learnt from Positronium for Quarkonium?

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Abstract

In order to fulfill Low's theorem requirements, a new lowest order basis for bound state decay computations is proposed, in which the binding energy is treated non-perturbatively. The properties of the method are sketched by reviewing standard positronium decay processes. Then, it is shown how applying the method to quarkonia sheds new light on some longstanding puzzles.

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1 Introduction

The properties of positronium provide some of the most precise tests of QED. Both the experimental and theoretical considerations have reached a very high level of precision, requiring for the latter the computations of many-loop diagrams[1]. In the present talk, we will address one particular aspect of the current QED bound state models, namely the factorization between the bound state dynamics and the annihilation process. The present study is motivated by the recurrent contradiction between factorized models and Low's theorem[2]. Our central result is a simple alternative method that allows an exact non-perturbative treatment of binding energy (BE) effects, and which produces bound state decay amplitudes with correct analytical behaviors.

Having in hand a formalism in which the BE is treated non-perturbatively is especially interesting in the context of quarkonium physics. Compared to positronium, quarkonium binding energies are much larger and in addition, they cannot be related to the coupling constant. Implications for a number of quarkonium puzzles are presented.

The present talk is based on a series of papers[3], to which we refer for details and references.

1.1 Basic formalism: factorization

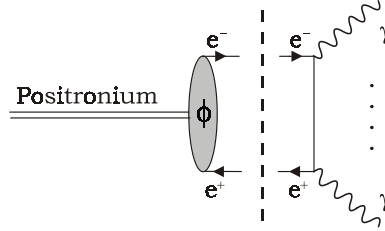


Figure 1 : Basic factorization of bound state decay amplitudes.

Historically, the first theoretical model designed in the forties to compute decay rates of para and orthopositronium ($J = 0$ and 1) states was (see Fig.1)

$$\Gamma(Ps(J) \rightarrow n\gamma) = \frac{1}{2J+1} |\phi_0|^2 (4v_{rel}\sigma_{scatt.}(e^+e^- \rightarrow n\gamma))_{v_{rel} \rightarrow 0} \quad (1)$$

which amount to replace the initial flux factor of the scattering cross section by the probability of e^+e^- contact inside the bound state, i.e. the Schrödinger wavefunction at zero separation $|\phi_0|^2 = \alpha^3 m^3 / 8\pi$ with m the electron mass. Using this naive factorization formula, it was computed

$$\Gamma(p-Ps \rightarrow \gamma\gamma) = \frac{\alpha^5 m}{2} \quad \text{Pirenne and Wheeler[4]} \quad (2)$$

$$\Gamma(o-Ps \rightarrow \gamma\gamma\gamma) = m\alpha^6 \frac{2(\pi^2 - 9)}{9\pi} \quad \text{Ore-Powell[5]}$$

Currently, a more developed type of factorization is used: the bound state decay amplitude is constructed as a three-dimensional convolution integral of the momentum-space wavefunction with the amplitude for the free constituents to scatter into the final

state (M is the positronium mass, γ the BE):

$$\mathcal{M}(Ps(J) \rightarrow n\gamma) = \sqrt{2M} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\phi(\mathbf{k})}{2E_{\mathbf{k}}} \mathcal{M}_{scatt.} \left(e_{\mathbf{k},\xi}^- e_{-\mathbf{k},\xi'}^+ \rightarrow n\gamma \right)_{\xi,\xi' \rightarrow J} \quad (3)$$

$$\text{with } \phi(\mathbf{k}) = \phi_0 \frac{8\pi\gamma}{(\mathbf{k}^2 + \gamma^2)^2}, \quad \gamma^2 = m^2 - M^2/4 \approx m^2\alpha^2/4$$

Using this basis, theoretical computations have reached a great level of precision:

$$\Gamma_{p-Ps} = \frac{\alpha^5 m}{2} (1 + \delta\Gamma_{p-Ps}), \quad \Gamma_{o-Ps} = m\alpha^6 \frac{2(\pi^2 - 9)}{9\pi} (1 + \delta\Gamma_{o-Ps}) \quad (4)$$

with the perturbative series (see [1] and references in [3])

$$\delta\Gamma_{p-Ps} = -A_p \frac{\alpha}{\pi} + 2\alpha^2 \ln \frac{1}{\alpha} + B_p \frac{\alpha^2}{\pi^2} - \frac{3\alpha^3}{2\pi} \ln^2 \frac{1}{\alpha} + C_p \frac{\alpha^3}{\pi} \ln \frac{1}{\alpha} + \delta_{4\gamma} \frac{\alpha^2}{\pi^2} \quad (5)$$

$$\delta\Gamma_{o-Ps} = -A_o \frac{\alpha}{\pi} - \frac{\alpha^2}{3} \ln \frac{1}{\alpha} + B_o \frac{\alpha^2}{\pi^2} - \frac{3\alpha^3}{2\pi} \ln^2 \frac{1}{\alpha} + C_o \frac{\alpha^3}{\pi} \ln \frac{1}{\alpha} + \delta_{5\gamma} \frac{\alpha^2}{\pi^2} \quad (6)$$

with

$$\begin{aligned} A_p &= 5 - \pi^2/4 \approx 2.5326 & A_o &= 10.286606 \text{ (10)} & \delta_{4\gamma} &= 0.274 \text{ (1)} \\ B_p &= 5.14 \text{ (30)} & B_o &= 44.52 \text{ (26)} & \delta_{5\gamma} &= 0.19 \text{ (1)} \\ C_p &= -7.919 \text{ (1)} & C_o &= 5.517 \text{ (1)} \end{aligned} \quad (7)$$

1.2 Analyticity? Why and how

From a quantum field theory perspective, the positronium has the quantum numbers of a neutral boson. As such, its radiation emissions are constrained to obey very general requirements. From QED gauge invariance and quantum field theory analyticity, the low-energy end of the photon energy spectrum is unambiguously predicted. This is the content of Low's theorem[2]. Note that the analyticity requirement invoked here has nothing to do with non-analytical terms ($\ln \alpha$) in the series expansions (5) and (6), but instead refers to a general property of decay amplitudes as functions of external photon energies, following from microcausality[6].

To state Low's theorem precisely, let us take the amplitude for orthopositronium to three photons

$$\mathcal{M}(o-Ps \rightarrow \gamma\gamma\gamma) = f(\omega_\gamma, \dots) \quad (8)$$

Gauge invariance and analyticity imply that the Laurent expansion of the amplitude for small photon energy has no singular and no constant term, and thus vanish linearly:

$$\mathcal{M}(o-Ps \rightarrow \gamma\gamma\gamma) = \mathcal{O}(\omega_\gamma) \text{ near } \omega_\gamma = 0 \quad (9)$$

As we now show, factorization-type approaches are in contradiction with this requirement. Using the naive factorization model (1), Ore and Powell computed the energy spectrum[5]

$$\frac{d\Gamma}{dx}(o-Ps \rightarrow \gamma\gamma\gamma) = \frac{2m\alpha^6}{9\pi} \Omega(x_1) \quad (10)$$

$$\Omega(x) = \frac{2(2-x)}{x} + \frac{2(1-x)x}{(2-x)^2} + 4 \left[\frac{(1-x)}{x^2} - \frac{(1-x)^2}{(2-x)^3} \right] \ln(1-x) \quad (11)$$

with $x_1 = 2\omega_\gamma/M$ the reduced photon energy. For very soft photon, this spectrum behaves as

$$\Omega(x) = \frac{5}{3}x + \mathcal{O}(x^2) \text{ near } x = 0 \quad (12)$$

while an $\mathcal{O}(x^3)$ behavior is required from (9). The problem originates in bremsstrahlung radiations contained in $\sigma_{scatt.}$ of (1), since its corresponding amplitude behaves as

$$\mathcal{M}_{scatt.}(e^+e^- \rightarrow n\gamma) = \mathcal{O}(\omega_\gamma^{-1}) + \mathcal{O}(\omega_\gamma^0) + \mathcal{O}(\omega_\gamma) + \dots \quad (13)$$

While the $\mathcal{O}(\omega_\gamma^{-1})$ term is cancelled in the limit $v_{rel} \rightarrow 0$ (selection rules), the constant one is not disposed of. Since it is a bremsstrahlung radiation, it is as unphysical as an IR divergence for positronium.

The convolution-type model (3) does not solve the problem, as can be shown using the language of dispersion relations. Starting from a four-dimensional loop model, and taking into account only the vertical cut (see Fig.2) in the imaginary part, the dispersion integral is precisely the convolution integral (3):

$$\begin{aligned} \mathcal{M}(o-Ps \rightarrow n\gamma) &= \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{s - M^2} [\text{Im } \mathcal{M}(o-Ps(s) \rightarrow n\gamma)]_{\text{vertical cut only}} \\ &= \sqrt{2M} \int \frac{d^3\mathbf{k}}{(2\pi)^3} \frac{\phi(\mathbf{k})}{2E_{\mathbf{k}}} \mathcal{M}_{scatt.}(e_{\mathbf{k},\xi}^- e_{-\mathbf{k},\xi'}^+ \rightarrow n\gamma)_{\xi,\xi' \rightarrow J} \end{aligned} \quad (14)$$

provided $F(s)$, the vertex form factor in $\text{Im } \mathcal{M}(o-Ps(s) \rightarrow n\gamma)$, is related to the Schrödinger momentum wavefunction as ($C = \sqrt{M}/m$)

$$F(s = 4(\mathbf{k}^2 + m^2)) = C\phi(\mathbf{k})(\mathbf{k}^2 + m^2) = C\phi_0 \frac{32\pi\gamma}{s - M^2} \quad (15)$$

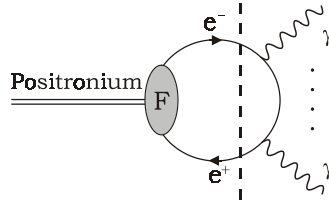


Figure 2 : The vertical cut of the convolution-type factorization.

Since only the vertical cut is taken into account to get (3), this last model obviously cannot be analytical in general: Cutkosky rules to get imaginary parts ask for all the possible cuts to be included. As we will see, it is precisely a cancellation between vertical and oblique cuts that enforces Low's theorem.

A third way to look at the problem is from the perspective of NRQED scaling rules. Typically, the bound state dynamics is soft scale (typical energy $< m_{e^-}$) while the annihilation process is hard scale (typical energy $> m_{e^-}$). However, in the corner of phase-space where the energy of the emitted photon is very soft, $\omega_\gamma^2 < \gamma^2 = m^2 - M^2/4$, this separation breaks down. Therefore, to get physical energy spectra, it is necessary to keep the BE as an arbitrary parameter, and not as a small expansion parameter, during the whole computation.

2 A new basis for perturbation theory

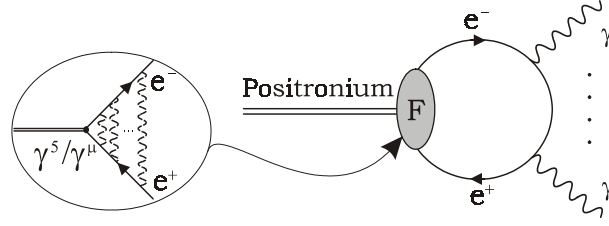


Figure 3a : The lowest order loop amplitude.

From the previous section, it appears that to get analytical amplitudes it is necessary to take a four-dimensional loop model (for oblique cuts) and to keep the BE as arbitrary. Our proposal for the lowest order amplitude on which perturbation theory is to be built is to revert to the Bethe-Salpeter loop amplitude depicted on Fig.3a, with $F(s)$ given in (15). This is a viable lowest order basis because it is easily computed from standard point-like QED loop amplitude through differentiation:

$$\text{Ps} \text{ (loop)} = \frac{\partial}{\partial P^3} \left[\text{Ps} \text{ (point-like)} \right]$$

Figure 3b : Coulomb bound state decay amplitudes from point-like QED ones.

To see this, it suffices to write the (unsubtracted) dispersion integral for the point-like and bound state loop decay amplitudes:

$$\begin{aligned} \text{Point-like case:} \quad \mathcal{I}_p(M^2) &= \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{s - M^2} \text{Im } \mathcal{I}_p(s) \\ \text{Bound state case:} \quad \mathcal{I}(M^2) &= \frac{1}{\pi} \int_{4m^2}^{\infty} \frac{ds}{s - M^2} F(s) \text{Im } \mathcal{I}_p(s) \end{aligned} \quad (16)$$

Thus, with (15)

$$\mathcal{I}(M^2) = (C\phi_0 32\pi\gamma) \frac{\partial}{\partial M^2} \mathcal{I}_p(M^2) \quad (17)$$

Since QED amplitudes are analytical and gauge invariant, so are our lowest order bound state decay amplitudes.

3 Application to QED bound states

3.1 $p\text{-Ps} \rightarrow \gamma\gamma$

The point-like amplitude shown on Fig.4 is written

$$\mathcal{M}_p = \frac{8m^2}{M^2} \varepsilon^{\mu\nu\rho\sigma} l_{1,\rho} l_{2,\sigma} \varepsilon_\mu^*(l_1) \varepsilon_\nu^*(l_2) \mathcal{I}_p \quad \text{with } \mathcal{I}_p = \frac{-2i}{16\pi^2} \arctan^2 \left(\frac{4m^2}{M^2} - 1 \right)^{-\frac{1}{2}} \quad (18)$$

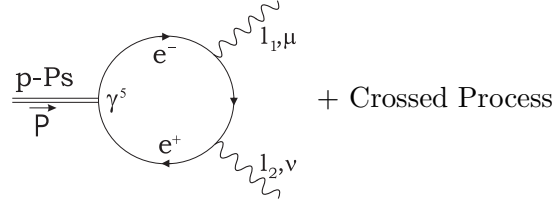


Figure 4 : The point-like amplitude for $p\text{-}Ps \rightarrow \gamma\gamma$

The corresponding parapositronium decay amplitude is obtained by replacing $\mathcal{I}_p$ by $\mathcal{I}$ obtained from (17). The decay rate is then

$$\Gamma(p\text{-}Ps \rightarrow \gamma\gamma) = \frac{m\alpha^5}{2} \frac{4m^2}{M^2} \left(\frac{2}{\pi} \arctan \frac{M}{2\gamma} \right)^2 \quad (19)$$

Note that this result is non-perturbative with respect to γ . Interestingly, the convergence of the perturbative series is greatly accelerated when BE effects are factored out. Writing

$$\Gamma_{p\text{-}Ps} = \frac{\alpha^5 m}{2} \frac{4m^2}{M^2} \left(\frac{2}{\pi} \arctan \frac{M}{2\gamma} \right)^2 (1 + \delta\Gamma'_{p\text{-}Ps}) \quad (20)$$

we get for $\delta\Gamma'_{p\text{-}Ps}$ the same expression (5) but with much reduced coefficients $A'_p \approx 0.5326$, $B'_p \approx 0.607$ and $C'_p \approx -3.919$.

3.2 $p\text{-}Dm \rightarrow \gamma e^+ e^-$

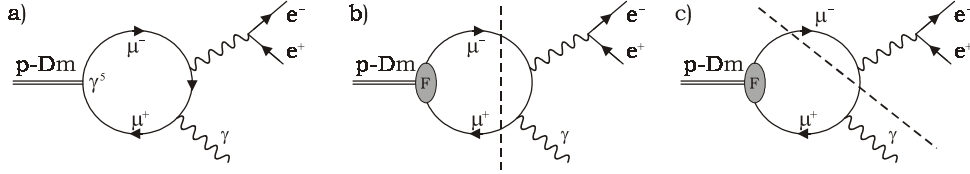


Figure 5 : (a) The point-like amplitude for $p\text{-}Dm \rightarrow \gamma e^+ e^-$, (b) the vertical cut and (c) the oblique cut (crossed process understood in each case).

The simplest process in which the implementation of Low's theorem can be analyzed is the annihilation of a paradimuonium ($^1S_0 (\mu^+ \mu^-)$ bound state) into a photon and a Dalitz pair (see Fig.5a). Following the same steps as above, we get the total rate as the phase-space integral over the photon energy spectrum:

$$\Gamma(p\text{-}Dm \rightarrow e^+ e^- \gamma) = \frac{m\alpha^6}{6\pi} \frac{4m^2}{M^2} \int_0^{1-a_e} dx \left| \frac{\mathcal{J}(x)}{x} \right|^2 \rho(x, a_e) \quad (21)$$

with

$$\begin{cases} x = 2\omega_\gamma/M, & a_e = 4m_e^2/M^2, & m = m_\mu \\ \rho(x, a_e) = \sqrt{1 - \frac{a_e}{1-x}} [2(1-x) + a_e] \frac{x^3}{(1-x)^2} \\ \frac{\mathcal{J}(x)}{x} = \frac{1}{x} \left(\frac{2}{\pi} \arctan \frac{M}{2\gamma} - \frac{4\gamma y}{\pi M} \arctan y \right) \\ y = \left(\frac{4m^2}{M^2(1-x)} - 1 \right)^{-1/2} \end{cases} \quad (22)$$

The first (second) term of $\mathcal{J}(x)/x$ can be traced to the vertical (oblique) cut, see Fig.5b(5c), respectively.

Low's theorem requirement is met since

$$\mathcal{M}(p\text{-}Dm \rightarrow e^+e^-\gamma) \stackrel{\omega_\gamma \rightarrow 0}{\simeq} \mathcal{O}(\omega_\gamma) \Rightarrow \frac{\mathcal{J}(x)}{x} \stackrel{x \rightarrow 0}{\simeq} \mathcal{O}(1) \quad (23)$$

To get this behavior, the loop is essential, since taken alone the vertical cut leads to $\mathcal{J}(x)/x \sim 1/x$. The non-perturbative treatment of γ is equally essential: the limits $\gamma \rightarrow 0$ and $x \rightarrow 0$ are mathematically incompatible

$$\frac{\mathcal{J}(x)}{x} \stackrel{x \rightarrow 0}{\simeq} \frac{M^2}{8\gamma^2} + \frac{1}{2} + \mathcal{O}(\gamma) \quad (24)$$

This incompatibility is also apparent on the plot of $\mathcal{J}(x)$ in Fig.6.

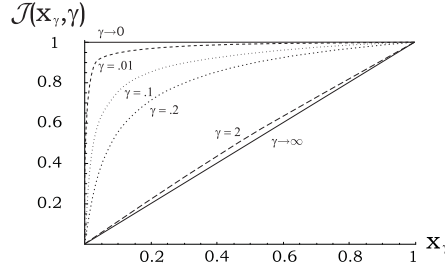


Figure 6 : The behavior of the Coulomb form factor $\mathcal{J}(x)$, normalized to $[\mathcal{J}(x)]_{\text{vert. cut}}$, as a function of $x = 2\omega_\gamma/M$ for various binding energies γ .

Finally, note that the total rate is well behaved as $\gamma \rightarrow 0$ (as is the surface in Fig.6). For total rate, oblique cuts are subleading.

3.3 $o\text{-}Dm \rightarrow e^+e^-$

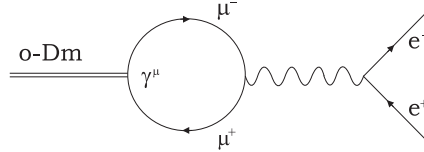


Figure 7 : The point-like amplitude for $o\text{-}Dm \rightarrow e^+e^-$.

Another application of interest as a laboratory for leptonic decays of quarkonium is $o\text{-}Dm \rightarrow e^+e^-$ as shown on Fig.7. Proceeding as before, we find

$$\Gamma(o\text{-}Dm \rightarrow e^+e^-) = \frac{\alpha M}{3} \left(1 + \frac{a_e}{2}\right) \sqrt{1 - a_e} |\Pi(M^2)|^2 \quad (25)$$

with $\Pi(M^2)$ obtained from the photon vacuum polarization function $\Pi_p(M^2)$ through (17). Again, we find that the bulk of radiative correction is accounted for at our lowest order

$$\begin{aligned} \Gamma(o\text{-}Dm \rightarrow e^+e^-) &= \frac{m\alpha^6}{6} \left(1 + \frac{a_e}{2}\right) \sqrt{1 - a_e} \left(1 - 4\frac{\alpha}{\pi} + \dots\right)_{NRQED} \\ &= \frac{m\alpha^6}{6} \left(1 + \frac{a_e}{2}\right) \sqrt{1 - a_e} \left(1 - 5.33\frac{\alpha}{\pi} + \dots\right)_{\text{Our lowest order}} \end{aligned} \quad (26)$$

This process is especially interesting because the derivative of the photon vacuum polarization function obey an anomalous Ward identity[7]

$$p^2 \frac{\partial}{\partial p^2} \Pi_p(p^2) = -\frac{1}{2} \Delta(p^2) - \frac{e^2}{12\pi^2} \quad (27)$$

$$\Delta(p^2) (p^\mu p^\nu - g^{\mu\nu} p^2) = \int dx dy e^{ipy} \langle 0 | T \{ \theta_\alpha^\alpha(x) J^\mu(y) J^\nu(0) \} | 0 \rangle$$

where θ_α^α is the trace of the improved energy-momentum tensor. The anomalous term $-e^2/12\pi^2$ account for 25% of the BE correction $-5.33\alpha/\pi$. Obvious open questions are whether this is a genuine contribution, and whether perturbative approaches like NRQED can catch it.

3.4 $o\text{-Ps} \rightarrow \gamma\gamma\gamma$

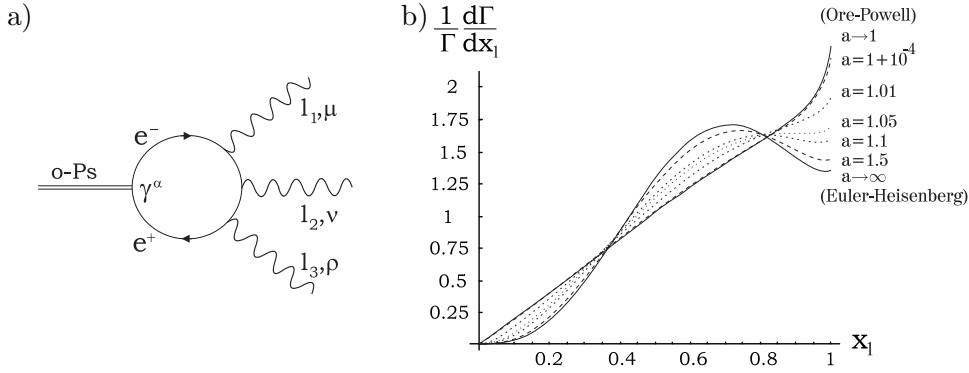


Figure 8 : (a) The point-like amplitude for $o\text{-Ps} \rightarrow \gamma\gamma\gamma$ and (b) the normalized photon energy spectrum as γ varies ($x_1 = 2\omega_\gamma/M$ and $a = 4\gamma^2/M^2 + 1$).

The next application is orthopositronium to three photons (Fig.8a). The computation is more involved, but proceeds as previously by taking the derivative of the light-by-light scattering amplitude. Details can be found in [3]. The behavior of the photon energy spectrum can now be studied as γ varies (see Fig.8b):

- $\gamma \neq 0$: $\frac{d\Gamma}{dx_1} \propto x_1^3 \left(\frac{M^2}{\gamma^2} + \dots \right)$ near $x_1 = 0$ (correct analytical behavior)
- $\gamma = 0$: If $\gamma \rightarrow 0$ is taken before $x_1 \rightarrow 0$: $\frac{d\Gamma}{dx_1} \propto x_1$ (Ore-Powell)
- $\gamma \rightarrow \infty$: Euler-Heisenberg limit: $\mathcal{L}_{E-H} = \frac{\alpha^2}{90m^4} ((F_{\mu\nu} F^{\mu\nu})^2 + \frac{7}{4} (F_{\mu\nu} \tilde{F}^{\mu\nu})^2)$

We observe the same mathematical incompatibility between the two limits $\omega_\gamma \rightarrow 0$ and $\gamma \rightarrow 0$ as in $p\text{-Dm} \rightarrow \gamma e^+ e^-$. However, note well that because of the high correlation in the three-photon phase-space, the Low's theorem suppression at low energy modifies the high end of the spectrum also.

For total rate, we write the BE corrections as

$$\Gamma_{o\text{-Ps}} = m\alpha^6 \frac{2(\pi^2 - 9)}{9\pi} B(\gamma/M)_{\text{Our lowest order}} \quad (28)$$

with, for small γ , the expansion (note well the slow convergence)

$$B(\gamma/M) = 1 - 15.4 \frac{\gamma}{M} + 122 \frac{\gamma^2}{M^2} - 889 \frac{\gamma^3}{M^3} + \dots = 1 - 12.1 \frac{\alpha}{\pi} + 80.2 \frac{\alpha^2}{\pi^2} - 502 \frac{\alpha^3}{\pi^3} + \dots \quad (29)$$

and we find again that the bulk of radiative corrections are accounted for as BE corrections:

$$\Gamma_{o-Ps} = m\alpha^6 \frac{2(\pi^2 - 9)}{9\pi} B(\gamma/M) (1 + \delta\Gamma'_{o-Ps}) \quad (30)$$

with the same series (6), but with much reduced coefficients $A'_o \approx -1.81$, $B'_o \approx -13.7$ and $C'_o \approx 1.48$. By the way, note that our method permits an estimation of the yet unknown non-logarithmic $\mathcal{O}(\alpha^3)$ term.

3.5 Other applications

Application of the method to spherically symmetric wavefunction is immediate. For example, radial excitation decay amplitudes can be obtained from the punctual QED amplitude as

$$\mathcal{I}_n(M_n^2) = (32\pi C\phi_{noo}) \left[{}_1F_1 \left(1 - n, 2, 16\gamma_n^2 \frac{\partial}{\partial M_n^2} \right) \right] \left[\gamma_n(M_n^2) \frac{\partial}{\partial M_n^2} \mathcal{I}_p(M_n^2) \right] \quad (31)$$

where the hypergeometric function is essentially the Laguerre polynomial.

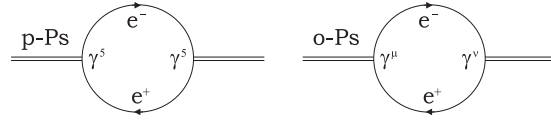


Figure 9 : The point-like mass renormalization graphs.

Hyperfine splitting can also be dealt with using the present method. From double derivatives of the punctual mass renormalization amplitudes depicted on Fig.9

$$\Pi_{para(ortho)}(M^2) = \frac{1}{2} (32\pi C\phi_o\gamma)^2 \frac{1}{2} \left(\frac{\partial}{\partial M^2} \right)^2 \Pi_{p,para(ortho)}(M^2) \quad (32)$$

and taking into account the one-photon annihilation graph, we find again that our lowest order $\Delta E_{hf} = (0.53m\alpha^4)_{\text{Our lowest order}}$ reproduces much of the NRQED correction[1] $\Delta E_{hf} = (0.5833m\alpha^4)_{NRQED}$. For further details, see [3].

4 Application to QCD bound states

The charmonium and bottomonium are the strong analogues of the positronium. Their binding energies are relatively small

$$\begin{array}{ll} \text{Positronium } (e^-e^+) & 4m_e^2/M^2 \approx 1 + 10^{-5} \quad (\approx 1 + \alpha^2/4) \\ \text{Charmonium } (c\bar{c}) & 4m_D^2/M_{J/\omega}^2 \approx 1.45 \\ \text{Bottomonium } (b\bar{b}) & 4m_B^2/M_Y^2 \approx 1.25 \end{array} \quad (33)$$

Their decay rates are computed using (1) adapted to the present case

$$\Gamma \left((Q\bar{Q})_{2S+1S_J} \rightarrow X \right) \propto |\phi_0|^2 (v_{rel} \sigma_{scatt.} (Q\bar{Q} \rightarrow X))_{v_{rel} \rightarrow 0} \quad (34)$$

and we have the analogies ($\mathcal{R} = |\phi_0|^2 / M^2$ and e_c the electric charge of the c):

Table I : Correspondence between QED and QCD applications.

QED	QCD	Decay rate[8]
$p\text{-}Ps \rightarrow \gamma\gamma$	$\eta_c \rightarrow \gamma\gamma$	$48\pi\alpha^2 e_c^4 \mathcal{R}$
	$\eta_c \rightarrow gg$	$\frac{32}{3}\pi\alpha_S^2 \mathcal{R}$
$o\text{-}Dm \rightarrow e^+e^-$	$J/\psi \rightarrow \gamma^* \rightarrow e^+e^-$	$16\pi e_c^2 \alpha^2 \mathcal{R}$
$o\text{-}Ps \rightarrow \gamma\gamma\gamma$	$J/\psi \rightarrow \gamma\gamma\gamma$	$\frac{64}{3}(\pi^2 - 9) e_c^6 \alpha^3 \mathcal{R}$
	$J/\psi \rightarrow \gamma gg$	$\frac{128}{9}(\pi^2 - 9) e_c^2 \alpha \alpha_S^2 \mathcal{R}$
	$J/\psi \rightarrow ggg$	$\frac{160}{81}(\pi^2 - 9) \alpha_S^3 \mathcal{R}$

We now review a number of observables in quarkonium physics that may be affected by BE effects. Since we do not know the precise form of the quarkonium wavefunctions, the following discussions are rather qualitative.

4.1 Prompt photon spectra

The vector quarkonium differential decay rates into three gauge bosons are:

$$\begin{aligned} \frac{d\Gamma}{dx} (^3S_1 \rightarrow \gamma\gamma\gamma) &= \frac{64}{3} e_Q^6 \alpha^3 \mathcal{R} \Omega(x) \\ \frac{d\Gamma}{dx} (^3S_1 \rightarrow \gamma gg) &= \frac{128}{9} e_Q^2 \alpha \alpha_S^2 \mathcal{R} \Omega(x) \\ \frac{d\Gamma}{dx} (^3S_1 \rightarrow ggg) &= \frac{160}{81} \alpha_S^3 \mathcal{R} \Omega(x) \end{aligned} \quad (35)$$

with the Ore-Powell spectrum function $\Omega(x)$ given in (11), which contradicts Low's theorem requirements. In Sec.3.4 it was shown that BE effects introduce a softening of the spectrum at high energies, and this is precisely what is observed for both $J/\psi \rightarrow \gamma + hadrons$ and $\Upsilon \rightarrow \gamma + hadrons$, compare Fig.8b and Fig.10 (where the low energy increase is due to bremsstrahlung processes[11]).

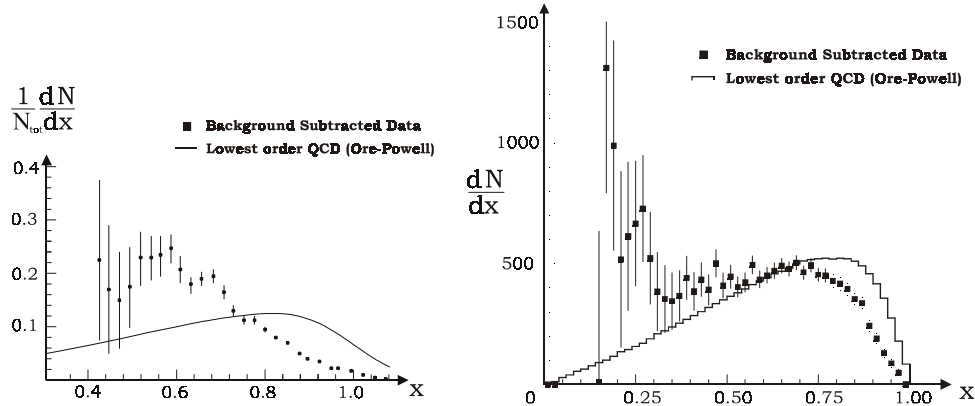


Figure 10: Photon spectrum in $J/\psi \rightarrow \gamma + had.$ [9] and in $\Upsilon \rightarrow \gamma + had.$ [10].

4.2 $\rho\pi$ puzzle

The decay rates in table I are valid for both J/ψ and $\psi(2S)$, the only change being in the wavefunction ϕ_0 and mass M . From them, one could expect that the ratio of the three-gluon decay rates of J/ψ and $\psi(2S)$ is the same as the one into e^+e^- . Going one step further, one could argue that the ratio into exclusive hadronic modes would also be constant[12]:

$$\frac{B(\psi(2S) \rightarrow X)}{B(J/\psi \rightarrow X)} \approx \frac{B(\psi(2S) \rightarrow e^+e^-)}{B(J/\psi \rightarrow e^+e^-)} \quad (36)$$

Experimentally, the ratio of leptonic modes is roughly 14%, while for hadronic modes, some are close to 14% and others much suppressed[13]:

$$\frac{B(\psi(2S) \rightarrow \omega\pi)}{B(J/\psi \rightarrow \omega\pi)} \approx 14\% \quad \frac{B(\psi(2S) \rightarrow \rho\pi)}{B(J/\psi \rightarrow \rho\pi)} < 0.2\% \quad (37)$$

The BE corrections to the one-virtual photon and three-gauge boson rates originate in the dynamics of the photon 2-point and 4-point functions, respectively. Those two are completely different: the 4-point function is much more sensitive to the ratio of the masses of the loop particle and initial particle. As a result, no matter the form of the wavefunction, three-gauge boson modes will be much more affected for non-negligible γ . Now, the isospin violating mode $\omega\pi$ proceeds through a virtual photon, as is the leptonic mode, and thus both receive the same correction. On the other hand, the 14% rule will be invalidated by large BE correction for the $\rho\pi$ mode, which proceeds from ggg .

4.3 Extraction of α_S and the perturbation series

The strong coupling constant can be extracted from ratios of inclusive hadronic modes and electromagnetic ones[13]. For instance, from η_c and J/ψ modes, one gets

$$\left. \begin{array}{l} \eta_c \rightarrow \gamma\gamma \\ \eta_c \rightarrow gg \end{array} \right\} \begin{array}{l} \rightarrow \alpha_S(m_c) \approx 0.31 \\ \Rightarrow \alpha_S(M_Z) \approx 0.12 \end{array} \quad \left. \begin{array}{l} J/\psi \rightarrow e^+e^- \\ J/\psi \rightarrow \gamma gg \\ J/\psi \rightarrow ggg \end{array} \right\} \begin{array}{l} \rightarrow \alpha_S(m_c) \approx 0.19 \\ \Rightarrow \alpha_S(M_Z) \approx 0.10 \end{array} \quad (38)$$

For η_c , both decay modes involve the same dynamics and their ratio is unaffected by BE corrections; α_S is in agreement with the world average[13] $\alpha_S(M_Z) \approx 0.117 \pm 0.002$. On the other hand, for J/ψ decays, the dynamics are very different and large BE corrections may arise.

As we have repeatedly seen in the case of positronium, the possibility of factoring out BE corrections improves the behavior of perturbative series. In the context of quarkonium, where first order corrections tend to be of the same order as the leading term, this may be crucial to get meaningful results. To take the worst case[8]

$$\Gamma(J/\psi \rightarrow \gamma\gamma\gamma) = \frac{64}{3} (\pi^2 - 9) e_Q^6 \alpha^3 \mathcal{R} \left(1 - 12.6 \frac{\alpha_S}{\pi} \right) \quad (39)$$

For $\alpha_S \approx 0.25$, the width vanishes. Now, the BE correction can be sizeable

$$\Gamma(J/\psi \rightarrow \gamma\gamma\gamma) = \frac{64}{3} (\pi^2 - 9) e_Q^6 \alpha^3 \mathcal{R} B(\gamma_{J/\psi}/M_{J/\psi}) \quad (40)$$

If the charmonium wavefunctions were Coulombic, $B(\gamma_{J/\psi}/M_{J/\psi}) \approx 0.01$, illustrating that really large suppression may indeed occur.

5 Conclusions

For QED bound states, a new basis for perturbation theory is proposed, in agreement with analyticity. Differential rates then behave as predicted by Low's theorem. Also, BE effects can be factored out of the perturbation series, thereby accelerating their convergence.

For QCD bound states, a perturbative expansion in γ like in NRQCD appears as very questionable, see the slow convergence of (19) or (29). BE effects have to be dealt with non-perturbatively, and can change the whole picture. In particular, prompt photon spectra in vector quarkonium decays are softened. Also, the difference in the dynamics of the photon 2-point and 4-point function can explain, at least in part, both the $\rho\pi$ puzzle and the smallness of α_S as extracted from vector quarkonium.

Work is in progress to extend the method to higher order computations, P -wave positronium and quarkonium decays, transitions among states, production rates,...

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