

ODD SYMMETRY OF LEAST ENERGY NODAL SOLUTIONS FOR THE CHOQUARD EQUATION

DAVID RUIZ AND JEAN VAN SCHAFTINGEN

ABSTRACT. We consider the Choquard equation (also known as stationary Hartree equation or Schrödinger–Newton equation)

$$-\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u.$$

Here I_α stands for the Riesz potential of order $\alpha \in (0, N)$, and $\frac{N-2}{N+\alpha} < \frac{1}{p} \leq \frac{1}{2}$. We prove that least energy nodal solutions have an odd symmetry with respect to a hyperplane when α is either close to 0 or close to N .

1. INTRODUCTION

In this paper we are interested in the Choquard equation (also known as stationary Hartree equation or Schrödinger–Newton equation)

$$(1.1) \quad -\Delta u + u = (I_\alpha * |u|^p)|u|^{p-2}u,$$

where $I_\alpha : \mathbb{R}^N \rightarrow \mathbb{R}$ denotes the Riesz potential, which is defined for each $x \in \mathbb{R}^N \setminus \{0\}$ by

$$(1.2) \quad I_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}}, \quad \text{where } A_\alpha = \frac{\Gamma(\frac{N-\alpha}{2})}{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}.$$

Problem (1.1) is the Euler–Lagrange equation of the *Choquard action functional* $J_\alpha : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ which is defined for each u in the Sobolev space $H^1(\mathbb{R}^N)$ by

$$(1.3) \quad J_\alpha(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p)|u|^p.$$

When $N = 3$, $\alpha = 2$ and $p = 2$, the equation (1.1) arises in Pekar’s model of the polaron [14, 23]. It has also appeared by introducing classical Newtonian gravitation in quantum physics [9, 12, 24]. The Choquard equation has been the object of many mathematical works (see [21]).

The existence of groundstate solutions (or least energy solutions) is quite well-known, see [14, 17–20, 25]. Those solutions are positive and radially symmetric. The uniqueness is known in some cases (see for instance [14]). It is also well-known that problem (1.1) admits sign-changing solutions with various symmetries [6–8; 27, Theorem 9.5].

Recently, other type of sign-changing solutions have been found for the Choquard equation. If $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$ there exist solutions which are odd with respect to a

D. R. was supported by the FEDER-MINECO Grant MTM2015-68210-P and by J. Andalucia (FQM-116). J.V.S. was supported by the Projet de Recherche (Fonds de la Recherche Scientifique-FNRS) T.1110.14 “Existence and asymptotic behavior of solutions to systems of semilinear elliptic partial differential equations”.

hyperplane of $\mathbb{R}^N$, see [11]. Those solutions have minimal energy among all solutions with that symmetry. Furthermore, there are also nodal solutions which minimize the energy in the so-called Nehari nodal set in the case $\frac{N-2}{N+\alpha} < \frac{1}{p} \leq \frac{1}{2}$ (see [10, 11]), which will be called *least energy nodal solutions*. We point out that, in both cases, those solutions do not have a counterpart in the framework of the usual stationary nonlinear Schrödinger equation.

At this point, it is quite reasonable to ask whether those solutions coincide; in other words, *whether the least energy nodal solutions are odd-symmetric with respect to a hyperplane*. The aim of this work is to give an affirmative answer to that question, if the order α of the Riesz potential is either close to 0 or close to N .

We first state the result for α close to 0.

Theorem 1. *If $1 - \frac{2}{N} < \frac{1}{p} \leq \frac{1}{2}$, then there exists $\alpha_\star \in (0, N)$ such that for any $\alpha \in (0, \alpha_\star)$, any least energy nodal solution $u_\alpha \in H^1(\mathbb{R}^N)$ of the Choquard equation (1.1) is odd with respect to a hyperplane of $\mathbb{R}^N$.*

By odd, we mean that there exists a reflection $R : \mathbb{R}^N \rightarrow \mathbb{R}^N$ of the Euclidean space $\mathbb{R}^N$ with respect to an affine hyperplane of $\mathbb{R}^N$ such that $u \circ R = -u$ in $\mathbb{R}^N$.

For the case where α is close to N our result is the following.

Theorem 2. *If $\frac{1}{2} - \frac{1}{N} < \frac{1}{p} < \frac{1}{2}$, then there exists $\alpha^\star \in (0, N)$ such that for any $\alpha \in (\alpha^\star, N)$, any least energy nodal solution $u_\alpha \in H^1(\mathbb{R}^N)$ of the Choquard equation (1.1) is odd with respect to a hyperplane.*

In general, the hypothesis $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$ is necessary by a Pohožaev-type inequality for the existence of sufficiently regular finite energy solutions to (1.1) [20, Theorem 2]. Since Theorems 1 and 2 are concerned with the case $\alpha \rightarrow 0$ and $\alpha \rightarrow N$ respectively, the restrictions on p imposed are the natural limit of these conditions.

The proofs use an argument by contradiction. We study the behavior of least energy nodal solutions u_α of the Choquard equation (1.1) when either $\alpha \rightarrow 0$ or $\alpha \rightarrow N$. This process leads us naturally to certain limit problems. If $\alpha \rightarrow 0$ the limit problem is just a usual stationary nonlinear Schrödinger equation, but in the case $\alpha \rightarrow N$ the equation includes an additional coefficient depending on the nonlocal quantity $\|u\|_{L^p}$.

A crucial ingredient of the proofs is the asymptotics of the Riesz potential energy $\int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p$. In the régime $\alpha \rightarrow 0$, the approximation is uniform on bounded sets of the Sobolev space $H^1(\mathbb{R}^N)$ (see § 3.1 below), which suits perfectly in our proofs. When $\alpha \rightarrow N$, the analysis is more delicate, because there is only a unilateral uniform approximation property on bounded sets (see § 3.2).

We point out that the family u_α *does not converge* to a solution to the limit problem, even up to translations in $\mathbb{R}^N$ and up to the extraction of a subsequence, an issue that makes our proof more involved. In the proof of Theorem 1 we show that the sequence of solutions actually forms a *Palais–Smale sequence* for the limit equation. As a consequence, our solutions behave asymptotically like differences of two positive solutions of the local problem moving away one from the other. With this in hand, we use the nondegeneracy of solutions to the local problem to conclude that the solution has an odd symmetry.

For Theorem 2 we need to describe more accurately the solutions, and we prove that the positive and negative parts of u_α converge to a groundstate of the corresponding limit problem. Moreover we also need to estimate the distance between the two bumps: it is going to infinity but slowly enough to preserve the interaction between the bumps as much as possible. In contrast with Theorem 1 which still holds for *low-energy nodal solutions* (see Proposition 4.1 below), the proof of Theorem 2 uses essentially the minimizing character of the nodal solutions.

The rest of the paper is organized as follows. In Section 2 we state some known results about groundstate solutions and nodal solutions of the Choquard equation. We also review properties of the limit problems that we encounter in the proofs. Section 3 is devoted to the study of the asymptotic behavior of the Riesz potential when α tends to 0 or N . Theorems 1 and 2 are proved in Sections 4 and 5, respectively.

2. PRELIMINARIES

2.1. Groundstates and least energy nodal solutions of the Choquard equations. For any $\alpha \in (0, N)$ and $p \in (1, \infty)$ such that $\frac{N-2}{N+\alpha} < \frac{1}{p} < \frac{N}{N+\alpha}$, the solutions in $H^1(\mathbb{R}^N)$ to the Choquard equation (1.1) correspond to critical points of the energy functional J_α defined on $H^1(\mathbb{R}^N)$ by (1.3). The Choquard equation (1.1) has a positive, radially symmetric groundstate solution $U_\alpha \in H^1(\mathbb{R}^N)$ [14, 17–20, 25] whose energy level will be denoted by $c_\alpha^{\text{gst}} = J_\alpha(U_\alpha)$.

The groundstate level c_α^{gst} has many different characterizations [20, §2.1]; it can be obtained as a minimum

$$c_\alpha^{\text{gst}} = \inf \{ J_\alpha(u) \mid u \in \mathcal{N}_\alpha \},$$

on the *Nehari manifold* which is defined by

$$\mathcal{N}_\alpha = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\} \mid \langle J'_\alpha(u), u \rangle = 0 \}.$$

The level c_α^{gst} can be equivalently characterized variationally as a minimax level:

$$(2.1) \quad c_\alpha^{\text{gst}} = \inf \left\{ \max_{t \geq 0} J_\alpha(tu) \mid u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}.$$

We now turn our attention to *nodal solutions for the Choquard equation*. As for local problems in bounded domains (see [3–5]) least energy nodal solutions can be constructed when $p \geq 2$ by minimizing the action functional on the Nehari nodal set [10, 11]:

$$c_\alpha^{\text{nod}} = \inf \{ J_\alpha(u) \mid u \in \mathcal{N}_\alpha^{\text{nod}} \},$$

where the *Nehari nodal set* $\mathcal{N}_\alpha^{\text{nod}}$ is defined by

$$\mathcal{N}_\alpha^{\text{nod}} = \{ u \in H^1(\mathbb{R}^N) \mid \langle J'_\alpha(u), u^+ \rangle = 0, \langle J'_\alpha(u), u^- \rangle = 0, u^+ \neq 0 \text{ and } u^- \neq 0 \}.$$

This level can also be characterized by

$$(2.2) \quad c_\alpha^{\text{nod}} = \inf \left\{ \max_{t, s \geq 0} J_\alpha(tu^+ + su^-) \mid u \in H^1(\mathbb{R}^N), u^+ \neq 0 \text{ and } u^- \neq 0 \right\}.$$

This can be seen as follows (see [11]): if $u \in H^1(\mathbb{R}^N)$, and if $u^+ \neq 0$ and $u^- \neq 0$, then for every $\sigma, \tau \in [0, \infty)$, we have

$$\begin{aligned} J_\alpha(\tau^{1/p}u^+ + \sigma^{1/p}u^-) \\ = \frac{\tau^{2/p}}{2} \int_{\mathbb{R}^N} |\nabla u^+|^2 + |u^+|^2 + \frac{\sigma^{2/p}}{2} \int_{\mathbb{R}^N} |\nabla u^-|^2 + |u^-|^2 \\ - \frac{1}{2p} \int_{\mathbb{R}^N} (I_\alpha * (\tau|u^+|^p + \sigma|u^-|^p))(\tau|u^+|^p + \sigma|u^-|^p); \end{aligned}$$

the right-hand side is a strictly concave function in the variables σ, τ (see [16, Theorem 9.8]) and achieves its maximum at a unique point $(\sigma, \tau) \in (0, \infty)^2$. If $u \in \mathcal{N}_\alpha^{\text{nod}}$, then $(\tau, \sigma) = (1, 1)$ is a critical point and the conclusion thus follows.

The level c_α^{nod} can be estimated by the groundstate level c_α^{gst} [11].

Proposition 2.1. *If $p \geq 2$ and $((N-2)p - N)_+ < \alpha < N$, then*

$$c_\alpha^{\text{nod}} < 2c_\alpha^{\text{gst}}.$$

2.2. Limiting problems. When $\alpha \rightarrow 0$, the Choquard equation (1.1) reduces at least formally to the nonlinear Schrödinger equation with exponent $q = 2p$

$$(2.3) \quad -\Delta u + u = |u|^{q-2}u.$$

The latter equation (2.3) is the Euler–Lagrange equation of the energy functional $\Phi_q : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined for each $u \in H^1(\mathbb{R}^N)$ by

$$(2.4) \quad \Phi_q(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{1}{q} \int_{\mathbb{R}^N} |u|^q.$$

Problem (2.3) has a positive groundstate U which is *radially symmetric, unique up to translations* and *nondegenerate*, that is, any solution $v \in H^1(\mathbb{R}^N)$ to the linearized problem

$$-\Delta v + v = (q-1)U^{q-2}v$$

is a directional derivative of the function U : it can be written $v = h \cdot \nabla U$, for some constant vector $h \in \mathbb{R}^N$ [13, 22, 26].

The *groundstate level* $\gamma_q = \Phi_q(U)$ has many different variational characterizations. We will be using the fact that the groundstate solution minimizes

$$(2.5) \quad \gamma_q = \inf \left\{ \max_{t \geq 0} \Phi_q(tu) \mid u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\}.$$

Indeed, it can be proved that the above infimum is attained by the groundstate U and $\max_{t \geq 0} \Phi_q(tU) = \Phi_q(U)$.

It is also well known that any other solution u of (2.3) must change sign and satisfies

$$(2.6) \quad \Phi_q(u) > 2\gamma_q.$$

Finally, the behavior of the Palais–Smale sequences of (2.3) has been fully described [1, Proposition II.1; 2; 28, Theorem 8.4]:

Lemma 2.2. *If the sequence $(u_n)_{n \in \mathbb{N}}$ in $H^1(\mathbb{R}^N)$ is a Palais–Smale sequence for the functional Φ_q , that is, if the sequence $(\Phi_q(u_n))_{n \in \mathbb{N}}$ is bounded in $\mathbb{R}$ and if the sequence $(\Phi'_q(u_n))_{n \in \mathbb{N}}$ converges to 0 in $H^{-1}(\mathbb{R}^N)$, then, there exists an integer $m \geq 0$, sequences $(a_n^i)_{n \in \mathbb{N}}$ in $\mathbb{R}^N$ for $i = 1, \dots, m$, and nonzero solutions U_i of (2.3) such that, as $n \rightarrow \infty$,*

- i) $u_n - \sum_{i=1}^m U_i(\cdot - a_n^i) \rightarrow 0$ strongly in $H^1(\mathbb{R}^N)$,
- ii) $\Phi_q(u_n) \rightarrow \sum_{i=1}^m \Phi_q(U_i)$,
- iii) $|a_n^i - a_n^j| \rightarrow +\infty$ if $i \neq j$.

In the study of the Choquard equation (1.1) for α close to N , we will encounter the following variant of the nonlinear Schrödinger equation:

$$(2.7) \quad -\Delta u + u = \mu \left(\int_{\mathbb{R}^N} |u|^p \right) |u|^{p-2} u,$$

for some parameter $\mu > 0$. This equation (2.7) is the Euler–Lagrange equation of the energy functional $\Psi_{p,\mu} : H^1(\mathbb{R}^N) \rightarrow \mathbb{R}$ defined for each $u \in H^1(\mathbb{R}^N)$ by

$$(2.8) \quad \Psi_{p,\mu}(u) = \frac{1}{2} \int_{\mathbb{R}^N} (|\nabla u|^2 + |u|^2) - \frac{\mu}{2p} \left(\int_{\mathbb{R}^N} |u|^p \right)^2.$$

The solutions of the problems (2.3) and (2.7) are related to each other. Indeed, if $u \in H^1(\mathbb{R}^N)$ is a solution of the equation (2.3) with $q = p$, we define

$$v = \frac{u}{\left(\mu \int_{\mathbb{R}^N} |u|^p \right)^{\frac{1}{2p-2}}}.$$

We observe that v is solution to problem (2.7) and that

$$(2.9) \quad \Psi_{p,\mu}(v) = \frac{\frac{1}{2} - \frac{1}{2p}}{\mu^{\frac{1}{p-1}}} \left(\frac{\Phi_p(u)}{\frac{1}{2} - \frac{1}{p}} \right)^{\frac{p-2}{p-1}}.$$

Conversely, if $v \in H^1(\mathbb{R}^N)$ is a solution of problem (2.7), then the function

$$u = \left(\mu \int_{\mathbb{R}^N} |v|^p \right)^{\frac{1}{p-2}} v$$

is a solution of equation (2.3) with $q = p$.

The groundstate V of problem (2.7) inherits the sign, uniqueness and symmetry properties of the groundstate U of problem (2.3). The groundstate levels are related as follows:

$$(2.10) \quad \kappa_{p,\mu} = \Psi_{p,\mu}(V) = \frac{\frac{1}{2} - \frac{1}{2p}}{\mu^{\frac{1}{p-1}}} \left(\frac{\Phi_p(U)}{\frac{1}{2} - \frac{1}{p}} \right)^{\frac{p-2}{p-1}} = \frac{\frac{1}{2} - \frac{1}{2p}}{\mu^{\frac{1}{p-1}}} \left(\frac{\gamma_p}{\frac{1}{2} - \frac{1}{p}} \right)^{\frac{p-2}{p-1}}.$$

The groundstate level $\kappa_{p,\mu}$ can be characterized variationally as

$$(2.11) \quad \kappa_{p,\mu} = \inf \left\{ \max_{t \geq 0} \Psi_{p,\mu}(tu) \mid u \in H^1(\mathbb{R}^N) \setminus \{0\} \right\} = \inf \{ \Psi_{p,\mu}(u) : u \in \mathcal{N}_{p,\mu} \},$$

where the Nehari manifold associated to (2.7) is defined by

$$(2.12) \quad \mathcal{N}_{p,\mu} = \{ u \in H^1(\mathbb{R}^N) \setminus \{0\}, \Psi'_{p,\mu}(u)(u) = 0 \}.$$

The following lemma will be needed later in the proofs, and basically states that minimizing sequences in $\mathcal{N}_{p,\mu}$ are convergent to the groundstate, up to translations. Its proof is standard and will be omitted.

Lemma 2.3. *Let $(u_k)_{k \in \mathbb{N}}$ be a sequence in $\mathcal{N}_{p,\mu}$. If $\Psi_{p,\mu}(u_k) \rightarrow \kappa_{p,\mu}$ as $k \rightarrow \infty$, then there exists sequences $(\xi_k)_{k \in \mathbb{N}}$ in $\mathbb{R}^N$ and $(\gamma_k)_{k \in \mathbb{N}}$ in $\{-1, 1\}$ such that*

$$u_k - \gamma_k V(\cdot - \xi_k) \rightarrow 0 \text{ in } H^1(\mathbb{R}^N),$$

where V is the groundstate of problem (2.7).

Finally, if u is a sign-changing solution of problem (2.7), then, in view of (2.6)

$$(2.13) \quad \Psi_{p,\mu}(u) > 2^{\frac{p-2}{p-1}} \kappa_{p,\mu} = 2\kappa_{p,2\mu}.$$

3. ASYMPTOTIC BEHAVIOR OF THE RIESZ POTENTIAL ENERGY

3.1. Concentrating Riesz potentials. In order to understand the asymptotic behavior of the Riesz potential energy as $\alpha \rightarrow 0$, we rely on the following L^2 estimate.

Lemma 3.1. *Let $s \in (0, N)$ and $\beta \in (0, \infty)$. For every $f, g \in L^2(\mathbb{R}^N)$ and every $\alpha \in (0, \beta]$ such that $I_\beta * f \in L^2(\mathbb{R}^N)$ and $(-\Delta)^{s/2} f \in L^2(\mathbb{R}^N)$, one has*

$$\left| \int_{\mathbb{R}^N} (I_\alpha * f)g - \int_{\mathbb{R}^N} fg \right| \leq \left(\frac{\alpha}{\beta} \|I_\beta * f\|_{L^2(\mathbb{R}^N)} + \frac{\alpha}{s} \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^N)} \right) \|g\|_{L^2(\mathbb{R}^N)}.$$

Proof. If $\hat{f}$ and $\hat{g}$ denote the Fourier transforms of the functions f and g , we have by the Plancherel theorem and by the formula for the Fourier transform of a Riesz potential

$$\int_{\mathbb{R}^N} (I_\alpha * f)g - \int_{\mathbb{R}^N} fg = \int_{\mathbb{R}^N} ((2\pi|\xi|)^{-\alpha} - 1) \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi.$$

We first observe by the Young inequality that if $2\pi|\xi| \leq 1$, then

$$1 \leq (2\pi|\xi|)^{-\alpha} \leq 1 - \frac{\alpha}{\beta} + \frac{\alpha}{\beta} (2\pi|\xi|)^{-\beta} \leq 1 + \frac{\alpha}{\beta} (2\pi|\xi|)^{-\beta}$$

and therefore

$$|(2\pi|\xi|)^{-\alpha} - 1| = (2\pi|\xi|)^{-\alpha} - 1 \leq \frac{\alpha}{\beta} (2\pi|\xi|)^{-\beta}.$$

It follows thus by the Cauchy–Schwarz inequality that

$$\begin{aligned} \left| \int_{B_{1/(2\pi)}} ((2\pi|\xi|)^{-\alpha} - 1) \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi \right| &\leq \frac{\alpha}{\beta} \left| \int_{B_{1/(2\pi)}} (2\pi|\xi|)^{-\beta} \hat{f}(\xi) \overline{\hat{g}(\xi)} d\xi \right| \\ &\leq \frac{\alpha}{\beta} \left(\int_{\mathbb{R}^N} (2\pi|\xi|)^{-2\beta} |\hat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\hat{g}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\ &= \frac{\alpha}{\beta} \|I_\beta * f\|_{L^2(\mathbb{R}^N)} \|g\|_{L^2(\mathbb{R}^N)}. \end{aligned}$$

On the other hand, if $2\pi|\xi| \geq 1$, we have, by Young's inequality again,

$$(2\pi|\xi|)^{-\alpha} \leq 1 \leq \frac{s}{\alpha+s} (2\pi|\xi|)^{-\alpha} + \frac{\alpha}{\alpha+s} (2\pi|\xi|)^s \leq (2\pi|\xi|)^{-\alpha} + \frac{\alpha}{s} (2\pi|\xi|)^s,$$

so that

$$|(2\pi|\xi|)^{-\alpha} - 1| = 1 - (2\pi|\xi|)^{-\alpha} \leq \frac{\alpha}{s} (2\pi|\xi|)^s.$$

Therefore,

$$\begin{aligned}
 \left| \int_{\mathbb{R}^N \setminus B_{1/(2\pi)}} ((2\pi|\xi|)^{-\alpha} - 1) \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi \right| &\leq \frac{\alpha}{s} \left| \int_{\mathbb{R}^N \setminus B_{1/(2\pi)}} (2\pi|\xi|)^{-s} \widehat{f}(\xi) \overline{\widehat{g}(\xi)} d\xi \right| \\
 &\leq \frac{\alpha}{s} \left(\int_{\mathbb{R}^N} (2\pi|\xi|)^{2s} |\widehat{f}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^N} |\widehat{g}(\xi)|^2 d\xi \right)^{\frac{1}{2}} \\
 &= \frac{\alpha}{s} \|(-\Delta)^{\frac{s}{2}} f\|_{L^2(\mathbb{R}^N)} \|g\|_{L^2(\mathbb{R}^N)}.
 \end{aligned}$$

This concludes the proof. $\square$

A variant of Lemma 3.1 can then be deduced, where the error is estimated in classical $L^q(\mathbb{R}^N)$ and Sobolev norms.

Lemma 3.2. *If $q > 2$, $\max\{\frac{2N}{N+2}, 1\} < r < 2$, and if*

$$0 < \alpha \leq \frac{N}{q} - \frac{N}{2},$$

then

$$\left| \int_{\mathbb{R}^N} (I_\alpha * f)g - \int_{\mathbb{R}^N} fg \right| \leq C\alpha (\|f\|_{L^q(\mathbb{R}^N)} + \|\nabla f\|_{L^r(\mathbb{R}^N)}) \|g\|_{L^2(\mathbb{R}^N)}.$$

Proof. We shall apply the estimate of Lemma 3.1. We take $\beta = N(\frac{1}{q} - \frac{1}{2})$, so that, by the classical Hardy–Littlewood–Sobolev inequality (see for example [16, theorem 4.3]),

$$\|I_\beta * f\|_{L^2(\mathbb{R}^N)} \leq C\|f\|_{L^q(\mathbb{R}^N)}.$$

We next take $s = 1 - N(\frac{1}{r} - \frac{1}{2})$ and we estimate by the Hardy–Littlewood–Sobolev inequality:

$$\begin{aligned}
 \|(-\Delta)^{\frac{s}{2}} f\|_{L^2} &= \|(2\pi|\xi|)^s \widehat{f}(\xi)\|_{L^2(\mathbb{R}^N, d\xi)} = (2\pi)^s \| |\xi|^{s-1} |\widehat{f}(\xi)| \|_{L^2(\mathbb{R}^N, d\xi)} \\
 &= \|I_{1-s} * |\nabla f|\|_{L^2(\mathbb{R}^N)} \leq C\|\nabla f\|_{L^r}.
 \end{aligned}$$

$\square$

Remark 3.1. The control in terms of $\|\nabla f\|_{L^r(\mathbb{R}^N)}$ might seem unnatural in Lemma 3.2, but it is actually necessary for Lemma 3.2 to hold. Indeed, if we choose a nonzero function $\psi \in C_0^\infty(\mathbb{R}^N)$ and define $f_n(x) = g_n(x) = e^{2\pi i n \eta \cdot x} \psi(x)$ for some fixed $\eta \in \mathbb{R}^N \setminus \{0\}$, clearly, $|f_n| = |\psi|$ in $\mathbb{R}^N$ and then the sequence $(f_n)_{n \in \mathbb{N}}$ is bounded in $L^q(\mathbb{R}^N)$ for every $q > 2$. By the translation properties of the Fourier transform, $\widehat{f}_n(\xi) = \widehat{\psi}(\xi - n\eta)$. Now, if $n^{\alpha_n} \rightarrow 0$ as $n \rightarrow \infty$, we have

$$\begin{aligned}
 \int_{\mathbb{R}^N} ((2\pi|\xi|)^{-\alpha_n} - 1) |\widehat{f}_n(\xi)|^2 d\xi &= \int_{\mathbb{R}^N} ((2\pi|\zeta + n\eta|)^{-\alpha_n} - 1) |\widehat{\psi}(\zeta)|^2 d\zeta \\
 &\rightarrow - \int_{\mathbb{R}^N} |\widehat{\psi}(\zeta)|^2 d\zeta < 0.
 \end{aligned}$$

Observe that in this case, the sequence $(|\nabla f_n|)_{n \in \mathbb{N}}$ is not bounded in any $L^r(\mathbb{R}^N)$ space.

Remark 3.2. Given $u \in H^1(\mathbb{R}^N)$, we set $f = g = |u|^p$. By the Sobolev embedding theorem and by the Hölder inequality, we have $|u|^p \in L^q(\mathbb{R}^N)$ for every $q > 1$ such that $\frac{1}{q} \geq p(\frac{1}{2} - \frac{1}{N})$ and

$$\nabla(|u|^p) = p|u|^{p-2}u \nabla u \in L^r(\mathbb{R}^N), \text{ for each } r \in [1, \infty) \text{ such that } \frac{1}{r} \geq \frac{p}{2} - \frac{p-1}{N},$$

by Hölder's inequality. Since $\frac{1}{p} > 1 - \frac{2}{N}$ we have

$$\frac{p}{2} - \frac{p-1}{N} < \frac{1}{N} + \frac{1}{2} = \frac{N+2}{2N},$$

and we are thus in the applicability range of Lemma 3.2, and we have

$$\left| \int_{\mathbb{R}^N} |u|^{2p} - \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \right| \leq C\alpha \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \right)^p.$$

3.2. Delocalizing Riesz potentials. In the régime $\alpha \rightarrow N$, we consider the potential $\tilde{I}_\alpha$ defined for $x \in \mathbb{R}^N \setminus \{0\}$ by

$$(3.1) \quad \tilde{I}_\alpha(x) = \frac{1}{|x|^{N-\alpha}};$$

this potential is related to the Riesz potential as follows

$$\tilde{I}_\alpha = \frac{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}{\Gamma(\frac{N-\alpha}{2})} I_\alpha,$$

and, as $\alpha \rightarrow N$,

$$\frac{\Gamma(\frac{\alpha}{2})\pi^{N/2}2^\alpha}{\Gamma(\frac{N-\alpha}{2})} = \Gamma(\frac{N}{2})\pi^{N/2}2^{N-1}(N-\alpha)(1+o(1)).$$

In the next lemma we give an upper bound for the Riesz potential energy:

Lemma 3.3. *Let $r \in (1, \infty)$. For every $\alpha \in (N/r, N)$, if $f \in L^1(\mathbb{R}^N) \cap L^r(\mathbb{R}^N)$ is nonnegative and $x \in \mathbb{R}^N$,*

$$(\tilde{I}_\alpha * f)(x) \leq \int_{\mathbb{R}^N} f + C \frac{N-\alpha}{(r\alpha - N)^{1-1/r}} \left(\int_{\mathbb{R}^N} f^r \right)^{\frac{1}{r}}.$$

In particular, if the function $g \in L^1(\mathbb{R}^N)$ is nonnegative, then

$$\int_{\mathbb{R}^N} (\tilde{I}_\alpha * f) g \leq \left(\int_{\mathbb{R}^N} f \right) \left(\int_{\mathbb{R}^N} g \right) + C \frac{N-\alpha}{(r\alpha - N)^{1-1/r}} \left(\int_{\mathbb{R}^N} f^r \right)^{\frac{1}{r}} \int_{\mathbb{R}^N} g.$$

Proof. Since the function f is summable and nonnegative, for each $x \in \mathbb{R}^N$,

$$(\tilde{I}_\alpha * f)(x) - \int_{\mathbb{R}^N} f = \int_{\mathbb{R}^N} \left(\frac{1}{|x-y|^{N-\alpha}} - 1 \right) f(y) dy \leq \int_{B_1(x)} \left(\frac{1}{|x-y|^{N-\alpha}} - 1 \right) f(y) dy.$$

Therefore, by the classical Hölder inequality, we have

$$(3.2) \quad (\tilde{I}_\alpha * f)(x) - \int_{\mathbb{R}^N} f \leq \left(\int_{B_1(0)} \left(\frac{1}{|z|^{N-\alpha}} - 1 \right)^{\frac{r}{r-1}} \right)^{1-\frac{1}{r}} \left(\int_{\mathbb{R}^N} f^r \right)^{\frac{1}{r}}.$$

In order to estimate the first integral, we first perform a radial integration:

$$(3.3) \quad \int_{B_1(0)} \left(\frac{1}{|z|^{N-\alpha}} - 1 \right)^{\frac{r}{r-1}} dz = |\partial B_1(0)| \int_0^1 \left(\frac{1-s^{N-\alpha}}{s^{N-\alpha}} \right)^{\frac{r}{r-1}} s^{N-1} ds.$$

On the one hand, the latter integral can be bounded by

$$(3.4) \quad \int_0^1 \left(\frac{1-s^{N-\alpha}}{s^{N-\alpha}} \right)^{\frac{r}{r-1}} s^{N-1} ds \leq \int_0^1 s^{N-(N-\alpha)\frac{r}{r-1}-1} ds = \frac{r-1}{N-r\alpha}.$$

On the other hand, by elementary convexity, we have for each $s \in (0, 1]$,

$$1 + (N - \alpha) \ln s \leq \exp((N - \alpha) \ln s) = s^{N - \alpha},$$

and therefore

$$(3.5) \quad \int_0^1 \left(\frac{1 - s^{N - \alpha}}{s^{N - \alpha}} \right)^{\frac{r}{r-1}} s^{N-1} ds \leq (N - \alpha)^{\frac{r}{r-1}} \int_0^1 \left(\ln \frac{1}{s} \right)^{\frac{r}{r-1}} s^{N-1 - \frac{r}{r-1}(N - \alpha)} ds.$$

The first inequality follows from (3.2), (3.3), (3.4) and (3.5).

The second inequality is obtained by multiplying the first one by $g(x)$ and integrating with respect to $x \in \mathbb{R}^N$. $\square$

Next lemma is concerned with the reversed inequality:

Lemma 3.4. *Let $r \in (1, \infty)$, $(\alpha_n)_{n \in \mathbb{N}}$ be a sequence in $(N/r, N)$ converging to N , $(\xi_n)_{n \in \mathbb{N}}$ be a sequence in $\mathbb{R}^N$ and let $(f_n)_{n \in \mathbb{N}}$ be a bounded sequence of functions in $L^r(\mathbb{R}^N)$. If $(f_n(\cdot - \xi_n))_{n \in \mathbb{N}}$ converges strongly to f in $L^1(\mathbb{R}^N)$ and if $(1/(1 + |\xi_n|)^{N - \alpha_n})_{n \in \mathbb{N}}$ converges to $\varrho \in [0, 1]$, then*

$$(3.6) \quad \lim_{n \rightarrow \infty} (\tilde{I}_{\alpha_n} * f_n)(x) = \varrho \int_{\mathbb{R}^N} f \quad \text{for any } x \in \mathbb{R}^N.$$

If moreover $(g_n)_{n \in \mathbb{N}}$ is a sequence of functions that converges to g strongly in $L^1(\mathbb{R}^N)$, then

$$(3.7) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (\tilde{I}_{\alpha_n} * f_n) g_n = \varrho \left(\int_{\mathbb{R}^N} f \right) \left(\int_{\mathbb{R}^N} g \right).$$

Lemma 3.4 gives a good idea of the validity of the reversed bound of Lemma 3.3. Indeed, $\tilde{I}_{\alpha_n} * f_n(x) \rightarrow \int_{\mathbb{R}^N} f$ for converging sequences f_n , but it can fail for sequences of functions given by translations in the x -variable. We shall prove later that the least energy nodal solutions behave as two signed bumps whose distance diverges. As we shall see, this makes our proofs more involved in the case α close to N .

Proof of Lemma 3.4. We can assume that $x = 0$, by making a suitable translation. We rewrite for each $n \in \mathbb{N}$ the quantities appearing in (3.6) in integral form

$$(\tilde{I}_{\alpha_n} * f_n)(0) - \varrho \int_{\mathbb{R}^N} f_n = \int_{\mathbb{R}^N} f_n(y) \left(\frac{1}{|y|^{N - \alpha_n}} - \varrho \right) dy.$$

Given $\delta \in (0, 1)$, by the Hölder inequality and by Lemma 3.3, we first have

$$\begin{aligned} \left| \int_{B_\delta} f_n(y) \left(\frac{1}{|y|^{N - \alpha_n}} - \varrho \right) dy \right| &\leq \int_{B_\delta} |f_n(y)| \left(\frac{1}{|y|^{N - \alpha_n}} + \varrho \right) dy \\ &\leq (1 + \varrho) |B_\delta|^{1 - \frac{1}{p}} \|f_n\|_{L^p(\mathbb{R}^N)} + C(N - \alpha_n) \|f_n\|_{L^r(\mathbb{R}^N)}, \end{aligned}$$

Next, we write

$$\int_{\mathbb{R}^N \setminus B_\delta} f_n(y) \left(\frac{1}{|y|^{N - \alpha_n}} - \varrho \right) dy = \int_{\mathbb{R}^N \setminus B_\delta(\xi_n)} f_n(z - \xi_n) \left(\frac{1}{|z - \xi_n|^{N - \alpha_n}} - \varrho \right) dy.$$

We observe that for every $z \in \mathbb{R}^N$,

$$(3.8) \quad \lim_{n \rightarrow \infty} \left(\frac{1}{|z - \xi_n|^{N - \alpha_n}} - \varrho \right) \chi_{B_\delta(\xi_n)} = 0.$$

Indeed, we have by the triangle inequality, on the one hand,

$$|z - \xi_n| \leq |z| + |\xi_n| \leq (1 + |z|)(1 + |\xi_n|)$$

and on the other hand, if $z \in \mathbb{R}^N \setminus B_\delta(\xi_n)$, we have

$$|z - \xi_n| \geq (1 - \lambda)\delta + \lambda(|\xi_n| - |z|) = \lambda(|\xi_n| + 1).$$

with $\lambda = \delta/(1 + |z| + \delta)$. Therefore it follows that, if $z \notin B_\delta(\xi_n)$

$$\left(\frac{1}{|z| + 1}\right)^{N-\alpha_n} \frac{1}{(1 + |\xi_n|)^{N-\alpha_n}} \leq \frac{1}{|z - \xi_n|^{N-\alpha_n}} \leq \left(\frac{1 + |z| + \delta}{\delta}\right)^{N-\alpha_n} \frac{1}{(1 + |\xi_n|)^{N-\alpha_n}},$$

and the limit (3.8) follows. Therefore, by Lebesgue's dominated convergence theorem,

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N \setminus B_\delta(\xi_n)} f_n(z - \xi_n) \left(\frac{1}{|z - \xi_n|^{N-\alpha_n}} - \rho \right) dy = 0.$$

Since $\delta > 0$ is arbitrary, the first conclusion (3.6) follows.

In order to prove (3.7), we pass to a subsequence so that $g_n \rightarrow g$ almost everywhere in $\mathbb{R}^N$ as $n \rightarrow \infty$ and for each $n \in \mathbb{N}$, $|g_n| \leq h$ in $\mathbb{R}^N$ for some $h \in L^1(\mathbb{R}^N)$. By (3.6), the sequence $(\tilde{I}_{\alpha_n} * f_n)_{n \in \mathbb{N}}$ converges to the constant $\varrho \int_{\mathbb{R}^N} f$ everywhere in $\mathbb{R}^N$. By Lemma 3.3, the sequence $(\tilde{I}_{\alpha_n} * f_n)_{n \in \mathbb{N}}$ is uniformly bounded over $\mathbb{R}^N$, so that Lebesgue's dominated convergence theorem applies and brings the conclusion. $\square$

4. PROOF OF THEOREM 1

This section is devoted to the proof of Theorem 1. As a first step, in the next proposition we show that least energy nodal solutions are asymptotically odd with respect to a hyperplane.

Proposition 4.1. *Let u_α be a family of solutions to (1.1) that changes sign and satisfying $I_\alpha(u_\alpha) \leq 2c_\alpha$, then*

$$\lim_{\alpha \rightarrow 0} \inf_{\xi^+, \xi^- \in \mathbb{R}^N} \|u_\alpha - (U(\cdot - \xi^+) - U(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)} = 0.$$

Moreover, for $\alpha \in (0, N)$ small enough there exist $\xi_\alpha^+, \xi_\alpha^-$ such that

$$\|u_\alpha - (U(\cdot - \xi_\alpha^+) - U(\cdot - \xi_\alpha^-))\|_{H^1(\mathbb{R}^N)} = \inf_{\xi^+, \xi^- \in \mathbb{R}^N} \|u_\alpha - (U(\cdot - \xi^+) - U(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)}$$

and

$$(4.1) \quad \lim_{\alpha \rightarrow 0} |\xi_\alpha^+ - \xi_\alpha^-| = +\infty.$$

By Proposition 2.1, if u_α is a least energy nodal solution we have $I_\alpha(u_\alpha) = c_\alpha^{\text{nod}} < 2c_\alpha$ and thus u_α satisfies the assumption of Proposition 4.1.

Proof of Proposition 4.1. The proof relies on some preliminary results which are stated in form of claims.

Claim 1. *One has*

$$\lim_{\alpha \rightarrow 0} c_\alpha^{\text{gst}} = \gamma_{2p}.$$

Proof of the claim. Given $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, by Lemma 3.2 we have, as $\alpha \rightarrow 0$,

$$\begin{aligned} c_\alpha^{\text{gst}} &\leq \max_{t>0} J_\alpha(tu) = \frac{t^2}{2} \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \right) - \frac{t^{2p}}{2p} \int_{\mathbb{R}^N} (I_\alpha * |u|^p) |u|^p \\ &\rightarrow \max_{t>0} \frac{t^2}{2} \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \right) - \frac{t^{2p}}{2p} \int_{\mathbb{R}^N} |u|^{2p} = \max_{t>0} \Phi_{2p}(tu). \end{aligned}$$

Taking the infimum with respect to $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we deduce that

$$(4.2) \quad \limsup_{\alpha \rightarrow 0} c_\alpha^{\text{gst}} \leq \gamma_{2p}.$$

Since

$$c_\alpha^{\text{gst}} = \left(\frac{1}{2} - \frac{1}{2p} \right) \left(\int_{\mathbb{R}^N} |\nabla U_\alpha|^2 + U_\alpha^2 \right),$$

the groundstate solution U_α remains bounded in $H^1(\mathbb{R}^N)$ as $\alpha \rightarrow 0$. By Lemma 3.2 again, we also have

$$c_\alpha^{\text{gst}} = J_\alpha(U_\alpha) = \max_{t>0} J_\alpha(tU_\alpha) = \max_{t>0} \Phi_{2p}(tU_\alpha) + o(\alpha),$$

as $\alpha \rightarrow 0$. This implies

$$(4.3) \quad \liminf_{\alpha \rightarrow 0} c_\alpha^{\text{gst}} \geq \gamma_{2p}.$$

The claims follows from the combination of the inequalities (4.2) and (4.3). $\diamond$

Claim 2. *The family $(u_\alpha)_{\alpha>0}$ is bounded in $H^1(\mathbb{R}^N)$ as $\alpha \rightarrow 0$.*

Proof of the claim. By assumption, we have for each $\alpha \in (0, N)$ $I_\alpha(u_\alpha) \leq 2c_\alpha$. Since for each $\alpha \in (0, N)$, we also have $\langle I'_\alpha(u_\alpha), u_\alpha \rangle = 0$, we deduce

$$\int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + |u_\alpha|^2 = \frac{2p}{p-1} I_\alpha(u_\alpha) \leq \frac{2p}{p-1} 2c_\alpha.$$

The claim follows then from Claim 1. $\diamond$

Claim 3. *One has*

$$\liminf_{\alpha \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_\alpha^\pm|^2 + |u_\alpha^\pm|^2 = \liminf_{\alpha \rightarrow 0} \int_{\mathbb{R}^N} (I_\alpha * |u_\alpha|^p) |u_\alpha^\pm|^p > 0.$$

Proof of the claim. We recall that the optimal Hardy–Littlewood–Sobolev inequality [15, Theorem 3.1; 16, Theorem 4.3] states that if $f, g \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, then

$$(4.4) \quad \int_{\mathbb{R}^N} (I_\alpha * f)g \leq C_{N,\alpha} \left(\int_{\mathbb{R}^N} |f|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |g|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}},$$

and the optimal constant $C_{N,\alpha}$ is given in terms of the gamma function Γ by

$$(4.5) \quad C_{N,\alpha} = \frac{\Gamma(\frac{N-\alpha}{2})}{2^\alpha \pi^{\alpha/2} \Gamma(\frac{N+\alpha}{2})} \left(\frac{\Gamma(\frac{N}{2})}{\Gamma(N)} \right)^{\frac{\alpha}{N}}.$$

If $p > 2$, we observe that, by the Hardy–Littlewood–Sobolev inequality (4.4) and by the classical Sobolev inequality

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u_\alpha^\pm|^2 + |u_\alpha^\pm|^2 &= \int_{\mathbb{R}^N} (I_\alpha * |u_\alpha|^p) |u_\alpha^\pm|^p \\ &\leq CC_{N,\alpha} \left(\int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + |u_\alpha|^2 \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^N} |\nabla u_\alpha^\pm|^2 + |u_\alpha^\pm|^2 \right)^{\frac{p}{2}}. \end{aligned}$$

We deduce therefrom that

$$1 \leq CC_{N,\alpha} \left(\int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + |u_\alpha|^2 \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^N} |\nabla u_\alpha^\pm|^2 + |u_\alpha^\pm|^2 \right)^{\frac{p-2}{2}}.$$

In view of (4.5), we have

$$\lim_{\alpha \rightarrow 0} C_{N,\alpha} = 1,$$

and the constant $C_{N,\alpha}$ remains thus bounded as $\alpha \rightarrow 0$. Since the family $(u_\alpha)_{\alpha \in (0,N)}$ is bounded in $H^1(\mathbb{R}^N)$ in view of Claim 2, we have, if $p > 2$,

$$\liminf_{\alpha \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_\alpha^\pm|^2 + |u_\alpha^\pm|^2 > 0.$$

The claim is thus proved in the case $p > 2$.

If $p = 2$ we adapt the strategy of [10]. Since for each $\alpha \in (0, N)$

$$\int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + |u_\alpha|^2 = \int_{\mathbb{R}^N} (I_\alpha * |u_\alpha|^2) |u_\alpha|^2 \leq CC_{N,\alpha} \left(\int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + |u_\alpha|^2 \right)^2,$$

the functions u_α stay away from 0 as $\alpha \rightarrow 0$:

$$\liminf_{\alpha \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_\alpha|^2 + |u_\alpha|^2 > 0.$$

Without loss of generality, we can assume that

$$\liminf_{\alpha \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_\alpha^+|^2 + |u_\alpha^+|^2 > 0.$$

We are going to prove that

$$\liminf_{\alpha \rightarrow 0} \int_{\mathbb{R}^N} |\nabla u_\alpha^-|^2 + |u_\alpha^-|^2 > 0.$$

Otherwise, there would exist a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $(0, N)$ converging to 0 such that

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |\nabla u_{\alpha_n}^-|^2 + |u_{\alpha_n}^-|^2 = 0.$$

We could then define for each $\alpha \in (0, N)$ the normalized negative part of u_α by

$$v_\alpha = \frac{u_\alpha^-}{\|u_\alpha^-\|_{H^1(\mathbb{R}^N)}}.$$

By the Hardy–Littlewood–Sobolev and by the Sobolev inequalities, for every $\alpha \in (0, N)$,

$$\int_{\mathbb{R}^N} (I_\alpha * |u_\alpha^-|^2) |u_\alpha^-|^2 \leq C \left(\int_{\mathbb{R}^N} |\nabla u_\alpha^-|^2 + |u_\alpha^-|^2 \right)^2,$$

and therefore, we would write

$$\begin{aligned} 1 &= \frac{\int_{\mathbb{R}^N} (I_{\alpha_n} * |u_{\alpha_n}|^2) |u_{\alpha_n}^-|^2}{\int_{\mathbb{R}^N} |\nabla u_{\alpha_n}^-|^2 + |u_{\alpha_n}^-|^2} = \lim_{n \rightarrow \infty} \frac{\int_{\mathbb{R}^N} (I_{\alpha_n} * |u_{\alpha_n}^+|^2) |u_{\alpha_n}^-|^2}{\int_{\mathbb{R}^N} |\nabla u_{\alpha_n}^-|^2 + |u_{\alpha_n}^-|^2} \\ &= \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} (I_{\alpha_n} * |u_{\alpha_n}^+|^2) |v_{\alpha_n}^-|. \end{aligned}$$

Now, by taking into account Remark 3.2, we would apply Lemma 3.2 with $g = |v_{\alpha_n}|^2$ to obtain

$$\lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}^N} (I_{\alpha_n} * |v_{\alpha_n}|^2) |u_{\alpha_n}^+|^2 - \int_{\mathbb{R}^N} |u_{\alpha_n}^+|^2 |v_{\alpha_n}|^2 \right) = 0.$$

But, by construction, $|u_{\alpha}^+|^2 |v_{\alpha}|^2 = 0$ almost everywhere in $\mathbb{R}^N$, and we would thus reach a contradiction. $\diamond$

With those results in hand, we are now in condition to prove Proposition 4.1. By Lemma 3.2 and Remark 3.2, if $\alpha \rightarrow 0$, then the family $(u_{\alpha})_{\alpha \in (0, N)}$ forms a Palais–Smale sequence for the limit problem (2.3). Moreover, by Claim 1 and by our assumption

$$\limsup_{\alpha \rightarrow 0} I_{\alpha}(u_{\alpha}) \leq \limsup_{\alpha \rightarrow 0} 2c_{\alpha}^{gst} \leq 2\gamma_{2p}.$$

In view of (2.6), Lemma 2.2 implies that $m \leq 2$ and $u_i = \pm U$. By Claim 3, we can assume without loss of generality that

$$(4.6) \quad u_{\alpha} = U(\cdot - \tilde{\xi}_{\alpha}^+) - U(\cdot - \tilde{\xi}_{\alpha}^-) + o(1), \quad |\tilde{\xi}_{\alpha}^+ - \tilde{\xi}_{\alpha}^-| \rightarrow +\infty.$$

We observe that, by Fatou's lemma, for each $\alpha \in (0, N)$,

$$\liminf_{|\xi^+| + |\xi^-| \rightarrow +\infty} \|u_{\alpha} - (U(\cdot - \xi^+) - U(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)} \geq \min\{\|u_{\alpha}\|_{H^1(\mathbb{R}^N)}, \|U\|_{H^1(\mathbb{R}^N)}\},$$

By Claim 3, the right-hand side stays away from 0 as $\alpha \rightarrow 0$. When $\alpha \in (0, N)$ is small enough, by the first part of the claim, the function

$$(\xi^+, \xi^-) \in \mathbb{R}^N \times \mathbb{R}^N \mapsto \|u_{\alpha} - (U(\cdot - \xi^+) - U(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)}$$

achieves thus its minimum at some pair of vectors $(\xi_{\alpha}^+, \xi_{\alpha}^-) \in \mathbb{R}^N \times \mathbb{R}^N$. By (4.6), that minimum goes to 0, that is,

$$u_{\alpha} = U(\cdot - \xi_{\alpha}^+) - U(\cdot - \xi_{\alpha}^-) + o(1).$$

We note that, again by (4.6), $|\tilde{\xi}_{\alpha}^{\pm} - \xi_{\alpha}^{\pm}| \rightarrow 0$. Finally, Lemma 2.2 implies that $|\xi_{\alpha}^+ - \xi_{\alpha}^-| \rightarrow +\infty$ as $\alpha \rightarrow 0$. $\square$

Proof of Theorem 1. Since the Choquard equation (1.1) is invariant under translations and rotations, we can assume that for each $\alpha \in (0, N)$ sufficiently close to 0, $\xi_{\alpha,+} = 0$ and $\xi_{\alpha,-} = \xi_{\alpha} = (m_{\alpha}, 0, \dots, 0)$, for some $m_{\alpha} \rightarrow +\infty$. We define then R_{α} to be the orthogonal reflection that sends 0 to ξ_{α} , that is, for each $x = (x_1, \dots, x_N) \in \mathbb{R}^N$,

$$R_{\alpha}(x) = (m_{\alpha} - x_1, x_2, \dots, x_N).$$

We set $v_{\alpha} = u_{\alpha} + \check{u}_{\alpha}$, where $\check{u}_{\alpha} = u_{\alpha} \circ R_{\alpha}$. We define also for every such $\alpha \in (0, N)$ the half-space

$$\Omega_{\alpha} = \left\{ x = (x_1, \dots, x_N) \in \mathbb{R}^N \mid x_1 < \frac{m_{\alpha}}{2} \right\}.$$

By construction, $v_\alpha \circ R_\alpha = v_\alpha$, and thus the function v_α is even with respect to $\partial\Omega_\alpha$.

Our purpose is to show that for $\alpha > 0$ small enough, $v_\alpha = 0$, from which Theorem 1 will follow immediately. The proof will rely on some preliminary results:

Step 1. For any direction $\zeta \in \mathbb{R}^N$,

$$(v_\alpha | \partial_\zeta U)_{H^1(\mathbb{R}^N)} = 0, \quad (v_\alpha | \partial_\zeta U)_{H^1(\Omega_\alpha)} = o(\|v_\alpha\|_{H^1(\Omega_\alpha)}),$$

as $\alpha \rightarrow 0$.

The function

$$(\xi^+, \xi^-) \in \mathbb{R}^N \times \mathbb{R}^N \mapsto \|u_\alpha - (U(\cdot - \xi^+) - U(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)}^2$$

attains a minimum at $(0, \xi_\alpha) \in \mathbb{R}^N \times \mathbb{R}^N$. Differentiating with respect to the variable ξ^+ in the direction $\zeta \in \mathbb{R}^N$, we obtain

$$(4.7) \quad (u_\alpha | \partial_\zeta U)_{H^1(\mathbb{R}^N)} = 0.$$

Reasoning in an analogous way on the variable ξ^- , we get

$$(u_\alpha | \partial_\zeta U(\cdot - \xi_\alpha))_{H^1(\mathbb{R}^N)} = 0.$$

We now observe that

- (1) if $\zeta = \xi_\alpha$, then $\partial_\zeta U \circ R_\alpha(x) = -\partial_\zeta U(\cdot - \xi_\alpha)$,
- (2) if $\zeta \cdot \xi_\alpha = 0$, then $\partial_\zeta U \circ R_\alpha(x) = \partial_\zeta U(\cdot - \xi_\alpha)$,

so that, in any case,

$$(u_\alpha | \partial_\zeta U(\cdot - \xi_\alpha))_{H^1(\mathbb{R}^N)} = \pm(\check{u}_\alpha | \partial_\zeta U)_{H^1(\mathbb{R}^N)} = 0.$$

This, together with (4.7), concludes the proof of the first assertion. The second follows since $\|\partial_\zeta U\|_{H^1(\mathbb{R}^N \setminus \Omega_\alpha)} = o(1)$.

Step 2. The function v_α satisfies the linear equation

$$(4.8) \quad \mathcal{L}_\alpha v_\alpha = 0,$$

where the linear operator $\mathcal{L}_\alpha$ is defined by

$$\mathcal{L}_\alpha v = -\Delta v + v - (I_\alpha * G_\alpha v)H_\alpha - (I_\alpha * K_\alpha)L_\alpha v,$$

with the functions G_α , H_α , K_α and L_α being given by

$$\begin{aligned} G_\alpha &= \begin{cases} \frac{|u_\alpha|^{p-2} - |\check{u}_\alpha|^{p-2}}{u_\alpha + \check{u}_\alpha} & \text{where } u_\alpha \neq -\check{u}_\alpha, \\ p|u_\alpha|^{p-2}u_\alpha & \text{elsewhere,} \end{cases} \\ H_\alpha &= \frac{1}{2}(|u_\alpha|^{p-2}u_\alpha - |\check{u}_\alpha|^{p-2}\check{u}_\alpha), \\ K_\alpha &= \frac{1}{2}(|u_\alpha|^p + |\check{u}_\alpha|^p), \\ L_\alpha &= \begin{cases} \frac{|u_\alpha|^{p-2}u_\alpha + |\check{u}_\alpha|^{p-2}\check{u}_\alpha}{u_\alpha + \check{u}_\alpha} & \text{where } u_\alpha \neq -\check{u}_\alpha, \\ (p-1)|u_\alpha|^{p-2} & \text{elsewhere.} \end{cases} \end{aligned}$$

By definition of v_α in terms of u_α and by the equation (1.1) satisfied by u_α , the function v_α obeys the equation

$$(4.9) \quad -\Delta v_\alpha + v_\alpha = (I_\alpha * |u_\alpha|^p)|u_\alpha|^{p-2}u_\alpha + (I_\alpha * |\check{u}_\alpha|^p)|\check{u}_\alpha|^{p-2}\check{u}_\alpha.$$

We observe that

$$\begin{aligned} & (I_\alpha * |u_\alpha|^p)|u_\alpha|^{p-2}u_\alpha + (I_\alpha * |\check{u}_\alpha|^p)|\check{u}_\alpha|^{p-2}\check{u}_\alpha \\ &= \frac{1}{2}(I_\alpha * (|u_\alpha|^p + |\check{u}_\alpha|^p))(|u_\alpha|^{p-2}u_\alpha + |\check{u}_\alpha|^{p-2}\check{u}_\alpha) \\ & \quad + \frac{1}{2}(I_\alpha * (|u_\alpha|^p - |\check{u}_\alpha|^p))(|u_\alpha|^{p-2}u_\alpha - |\check{u}_\alpha|^{p-2}\check{u}_\alpha). \end{aligned}$$

Step 3. *Conclusion of the proof of Theorem 1.*

As commented above, Theorem 1 follows if we show that $v_\alpha = 0$ for $\alpha > 0$ small enough. We assume by contradiction that there is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $(0, N)$ that converges to 0 such that for every $n \in \mathbb{N}$, $v_{\alpha_n} \neq 0$. For each $n \in \mathbb{N}$, we define the normalized sequence

$$w_n = \frac{v_{\alpha_n}}{\|v_{\alpha_n}\|_{H^1(\mathbb{R}^N)}}.$$

Without loss of generality, the sequence $(w_n)_{n \in \mathbb{N}}$ converges weakly in $H^1(\mathbb{R}^N)$ to some function $w \in H^1(\mathbb{R}^N)$.

By Proposition 4.1, we have the following convergences:

$$(4.10) \quad \begin{aligned} G_\alpha - p(U^{p-1} - U^{p-1}(\cdot - \xi_\alpha)) &\rightarrow 0 \quad \text{in } L^q(\mathbb{R}^N), \quad \text{if } \frac{1}{2} - \frac{1}{N} \leq \frac{p-1}{q} \leq \frac{1}{2}, \\ H_\alpha - (U^{p-1} - U^{p-1}(\cdot - \xi_\alpha)) &\rightarrow 0 \quad \text{in } L^q(\mathbb{R}^N), \quad \text{if } \frac{1}{2} - \frac{1}{N} \leq \frac{p-1}{q} \leq \frac{1}{2}, \\ K_\alpha - (U^p + U^p(\cdot - \xi_\alpha)) &\rightarrow 0 \quad \text{in } L^q(\mathbb{R}^N), \quad \text{if } \frac{1}{2} - \frac{1}{N} \leq \frac{p}{q} \leq \frac{1}{2}, \\ L_\alpha - (p-1)(U^{p-2} + U^{p-2}(\cdot - \xi_\alpha)) &\rightarrow 0 \quad \text{in } L^q(\mathbb{R}^N), \quad \text{if } \frac{1}{2} - \frac{1}{N} \leq \frac{p-2}{q} \leq \frac{1}{2}. \end{aligned}$$

If we test the equation $\mathcal{L}_\alpha w_\alpha$, against the function $\varphi \in C_0^\infty(\mathbb{R}^N)$, we have

$$\int_{\mathbb{R}^N} (\nabla w_\alpha \cdot \nabla \varphi + w_\alpha \varphi) = \int_{\mathbb{R}^N} (I_\alpha * G_\alpha w_\alpha) H_\alpha \varphi + (I_\alpha * K_\alpha) L_\alpha w_\alpha \varphi$$

We now apply Lemma 3.2 first to $f = G_\alpha w_\alpha$ and $g = H_\alpha \varphi$ and next to $f = K_\alpha$ and $g = L_\alpha w_\alpha \varphi$. As in Remark 3.2, the boundedness of ∇f in $L^r(\mathbb{R}^N)$ for some $r > 1$ follows from the Sobolev and Hölder inequalities. In combination with the asymptotic behavior of G_α , H_α , K_α , and L_α in (4.10), we deduce that w is a weak solution of the equation

$$-\Delta w + w = (2p-1)U^{2p-2}w.$$

By Step 1 and the nondegeneracy of the limiting problem (2.3) we have $w = 0$.

For each $n \in \mathbb{N}$, we now test the equation $\mathcal{L}_{\alpha_n} v_{\alpha_n} = 0$ against v_{α_n} and divide by $\|v_{\alpha_n}\|_{H^1(\mathbb{R}^N)}^2$, to obtain

$$(4.11) \quad \begin{aligned} 1 &= \int_{\mathbb{R}^N} |\nabla w_n|^2 + |w_n|^2 \\ &= \int_{\mathbb{R}^N} (I_{\alpha_n} * (G_{\alpha_n} w_n)) H_{\alpha_n} w_n + (I_{\alpha_n} * K_{\alpha_n}) L_{\alpha_n} |w_n|^2. \end{aligned}$$

By Lemma 3.2, on the other hand, we have as $n \rightarrow \infty$

$$\begin{aligned}
 (4.12) \quad & \int_{\mathbb{R}^N} (I_{\alpha_n} * (G_{\alpha_n} w_n)) H_{\alpha_n} w_n + (I_{\alpha_n} * K_{\alpha_n}) L_{\alpha_n} |w_n|^2 \\
 &= \int_{\mathbb{R}^N} (G_{\alpha_n} H_{\alpha_n} + K_{\alpha_n} L_{\alpha_n}) |w_n|^2 + o(1) \\
 &= 2 \int_{\Omega_{\alpha_n}} (G_{\alpha_n} H_{\alpha_n} + K_{\alpha_n} L_{\alpha_n}) |w_n|^2 + o(1) \\
 &= 2(2p-1) \int_{\mathbb{R}^N} |U|^{2p-2} |w_n|^2 + o(1),
 \end{aligned}$$

in view of (4.10). Since the sequence $(w_n)_{n \in \mathbb{N}}$ converges weakly to 0 in $H^1(\mathbb{R}^N)$, we have in view of Rellich's compactness theorem and the decay of U at infinity,

$$(4.13) \quad \lim_{n \rightarrow \infty} \int_{\mathbb{R}^N} |U|^{2p-2} |w_n|^2 = 0,$$

By taking into account (4.11), (4.12) and (4.13) are in contradiction. Hence $v_\alpha = 0$ for α close enough to 0. The proof of Theorem 1 is thus complete. $\square$

5. PROOF OF THEOREM 2

We now turn our attention to the proof of Theorem 2. The main difficulty with respect to Theorem 1 comes from the fact that the asymptotics of Riesz potential energy are not as accurate when $\alpha \rightarrow N$ as in the case $\alpha \rightarrow 0$. This requires additional steps in the proof.

To alleviate the notations, we define $\underline{\alpha} = \max\{0, (N-2)p - N\}$. For $\alpha \in (\underline{\alpha}, N)$, we first set

$$\tilde{u}_\alpha = (A_\alpha)^{\frac{1}{2p-2}} u_\alpha,$$

where A_α is the normalizing constant in the Riesz potential I_α coming from (1.2). The function $\tilde{u}_\alpha$ satisfies then the equation

$$(5.1) \quad -\Delta \tilde{u}_\alpha + \tilde{u}_\alpha = (\tilde{I}_\alpha * |\tilde{u}_\alpha|^p) |\tilde{u}_\alpha|^{p-2} \tilde{u}_\alpha,$$

with the unnormalized Riesz potential $\tilde{I}_\alpha$ that was defined in (3.1). We let $\tilde{J}_\alpha$, $\tilde{c}_\alpha^{\text{gst}}$ and $\tilde{c}_\alpha^{\text{nod}}$ denote the corresponding functional, groundstate and least energy nodal solution levels.

In the next proposition we prove an analogue to Proposition 4.1 of the previous section.

Proposition 5.1. *If $\tilde{u}_\alpha$ are least energy nodal solutions of (5.1), then*

$$\lim_{\alpha \rightarrow N} \inf_{\xi^+, \xi^- \in \mathbb{R}^N} \|\tilde{u}_\alpha - (V(\cdot - \xi^+) - V(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)} = 0,$$

where $V = V_2$ denotes a groundstate of (2.7) for $\mu = 2$. Moreover, for $\alpha \in (\underline{\alpha}, N)$ close enough to N there exists vectors $\xi_\alpha^+, \xi_\alpha^-$ such that

$$\|\tilde{u}_\alpha - (V(\cdot - \xi_\alpha^+) - V(\cdot - \xi_\alpha^-))\|_{H^1(\mathbb{R}^N)} = \inf_{\xi^+, \xi^- \in \mathbb{R}^N} \|\tilde{u}_\alpha - (V(\cdot - \xi^+) - V(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)};$$

moreover they satisfy the asymptotics

$$\lim_{\alpha \rightarrow N} |\xi_\alpha^+ - \xi_\alpha^-| = \infty, \quad \text{and} \quad \lim_{\alpha \rightarrow N} |\xi_\alpha^+ - \xi_\alpha^-|^{N-\alpha} = 1.$$

Proof. The proof of Proposition 5.1 requires some preliminaries, stated in the form of claims.

Claim 1. *One has*

$$\lim_{\alpha \rightarrow N} \tilde{c}_\alpha^{\text{gst}} = \kappa_{p,1}.$$

We recall that the quantity $\kappa_{p,1}$ is the groundstate level of the limiting problem (2.7) defined in (2.10).

Proof of the claim. Given $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, by Lemma 3.4 with $\xi_n = 0$ we have, as $\alpha \rightarrow N$,

$$\begin{aligned} \tilde{c}_\alpha^{\text{gst}} &\leq \max_{t>0} J_\alpha(tu) = \frac{t^2}{2} \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \right) - \frac{t^{2p}}{2p} \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |u|^p) |u|^p \\ &\rightarrow \max_{t>0} \frac{t^2}{2} \left(\int_{\mathbb{R}^N} |\nabla u|^2 + |u|^2 \right) - \frac{t^{2p}}{2p} \left(\int_{\mathbb{R}^N} |u|^p \right)^2 = \max_{t>0} \Psi_{p,1}(tu). \end{aligned}$$

Taking the infimum with respect to $u \in H^1(\mathbb{R}^N) \setminus \{0\}$, we get that $\limsup_{\alpha \rightarrow N} \tilde{c}_\alpha^{\text{gst}} \leq \kappa_{p,1}$. In particular, the groundstate solution $\tilde{U}_\alpha$ of (5.1) remains bounded in $H^1(\mathbb{R}^N)$ as $\alpha \rightarrow N$, since

$$\tilde{c}_\alpha^{\text{gst}} = \left(\frac{1}{2} - \frac{1}{2p} \right) \left(\int_{\mathbb{R}^N} |\nabla \tilde{U}_\alpha|^2 + |\tilde{U}_\alpha|^2 \right).$$

Lemma 3.3 yields

$$\liminf_{\alpha \rightarrow N} \tilde{c}_\alpha^{\text{gst}} = \liminf_{\alpha \rightarrow N} \max_{t>0} J_\alpha(tU_\alpha) \geq \liminf_{\alpha \rightarrow N} \max_{t>0} \Psi_{p,1}(tU_\alpha) \geq \kappa_{p,1},$$

from which the reversed inequality follows. $\diamond$

Claim 2. *The family $(\tilde{u}_\alpha)_{\alpha \in (\underline{\alpha}, N)}$ is bounded in $H^1(\mathbb{R}^N)$ as $\alpha \rightarrow N$.*

Proof of the claim. We observe that, by Proposition 2.1,

$$\left(\frac{1}{2} - \frac{1}{2p} \right) \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha|^2 + |\tilde{u}_\alpha|^2 = \tilde{J}_\alpha(\tilde{u}_\alpha) \leq 2\tilde{c}_\alpha^{\text{gst}},$$

and the conclusion follows then from Claim 1. $\diamond$

Claim 3. *One has*

$$\liminf_{\alpha \rightarrow N} \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^\pm|^2 + |\tilde{u}_\alpha^\pm|^2 > 0 \quad \text{and} \quad \liminf_{\alpha \rightarrow N} \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha^\pm|^p) |\tilde{u}_\alpha^\pm|^p > 0.$$

Proof of the claim. We recall that by the optimal Hardy–Littlewood–Sobolev inequality [15, Theorem 3.1; 16, Theorem 4.3] for all functions $f, g \in L^{\frac{2N}{N+\alpha}}(\mathbb{R}^N)$, we have

$$(5.2) \quad \int_{\mathbb{R}^N} (\tilde{I}_\alpha * f)g \leq \tilde{C}_{N,\alpha} \left(\int_{\mathbb{R}^N} |f|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}} \left(\int_{\mathbb{R}^N} |g|^{\frac{2N}{N+\alpha}} \right)^{\frac{N+\alpha}{2N}},$$

with an optimal constant $\tilde{C}_{N,\alpha}$ that can be expressed as

$$(5.3) \quad \tilde{C}_{N,\alpha} = \frac{\pi^{\frac{N-\alpha}{2}} \Gamma(\frac{\alpha}{2})}{\Gamma(\frac{N+\alpha}{2})} \left(\frac{\Gamma(N)}{\Gamma(\frac{N}{2})} \right)^{\frac{\alpha}{N}}.$$

By the Hardy–Littlewood–Sobolev inequality (5.2) and by the Sobolev inequality, we observe that

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^\pm|^2 + |\tilde{u}_\alpha^\pm|^2 &= \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha|^p) |\tilde{u}_\alpha^\pm|^p \\ &\leq C\tilde{C}_{N,\alpha} \left(\int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha|^2 + |\tilde{u}_\alpha|^2 \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^\pm|^2 + |\tilde{u}_\alpha^\pm|^2 \right)^{\frac{p}{2}}. \end{aligned}$$

so that, since $p > 2$, we have

$$1 \leq C\tilde{C}_{N,\alpha} \left(\int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha|^2 + |\tilde{u}_\alpha|^2 \right)^{\frac{p}{2}} \left(\int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^\pm|^2 + |\tilde{u}_\alpha^\pm|^2 \right)^{\frac{p-2}{2}}.$$

In view of (5.3), we have

$$\lim_{\alpha \rightarrow N} \tilde{C}_{N,\alpha} = 1,$$

so that, by Claim 2,

$$(5.4) \quad \liminf_{\alpha \rightarrow N} \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^\pm|^2 + |\tilde{u}_\alpha^\pm|^2 = \liminf_{\alpha \rightarrow N} \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha|^p) |\tilde{u}_\alpha^\pm|^p > 0$$

For the second estimate, we write, by the positive definiteness of the Riesz potential energy and the Cauchy–Schwarz inequality (see [16, Theorem 9.8]),

$$\begin{aligned} &\int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^+|^p \\ &= \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha|^p) |u_\alpha^+|^p - \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha^-|^p) |\tilde{u}_\alpha^+|^p \\ &\geq \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha|^p) |u_\alpha^+|^p - \left(\int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^+|^p \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha^-|^p) |\tilde{u}_\alpha^-|^p \right)^{\frac{1}{2}}. \end{aligned}$$

The conclusion follows then from the fact that $u_\alpha^+ \neq 0$, from the boundedness of the family u_α^- in $H^1(\mathbb{R}^N)$ and from the Hardy–Littlewood–Sobolev inequality (5.2). $\diamond$

Claim 4. *We have*

$$\lim_{\alpha \rightarrow N} \tilde{c}_\alpha^{\text{nod}} = 2\kappa_{p,2} = 2^{\frac{p-2}{p-1}} \kappa_{p,1}.$$

Moreover, define $t_\alpha, s_\alpha \in (0, \infty)$ such that

$$(5.5) \quad t_\alpha \tilde{u}_\alpha^+ \in \mathcal{N}_{p,2}, \quad \text{and} \quad s_\alpha \tilde{u}_\alpha^- \in \mathcal{N}_{p,2}$$

where $\mathcal{N}_{p,2}$ is the Nehari manifold associated to the functional $\Psi_{p,2}$ (see (2.12)). Then, t_α, s_α are bounded and they satisfy the following asymptotics as $\alpha \rightarrow 0$:

$$\begin{aligned} (5.6) \quad &t_\alpha^p s_\alpha^p \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^-|^p = \frac{t_\alpha^{2p}}{2} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^+|^p + \frac{s_\alpha^{2p}}{2} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^-|^p) |\tilde{u}_\alpha^-|^p + o(1), \\ (5.7) \quad &\Psi_{p,2}(t_\alpha \tilde{u}_\alpha^+) \rightarrow \kappa_{p,2}, \quad \Psi_{p,2}(s_\alpha \tilde{u}_\alpha^-) \rightarrow \kappa_{p,2}. \end{aligned}$$

Proof of the claim. We take a function $v \in C_c^\infty(\mathbb{R}^N)$ and we choose a vector $\xi \in \mathbb{R}^N$ such that $|\xi| > \text{diam}(\text{supp } v)$. We now define the function $u : \mathbb{R}^N \rightarrow \mathbb{R}$ for each $x \in \mathbb{R}^N$ by $u(x) = v(x) - v(x - \xi)$. In view of Lemma 3.4, we have

$$\int_{\mathbb{R}^N} (\tilde{I}_\alpha * |tu^+ + su^-|^p) |tu^+ + su^-|^p = \left(\int_{\mathbb{R}^N} |tu^+ + su^-|^p \right)^2 + o(1) (|t|^{2p} + |s|^{2p}).$$

It follows therefore that

$$(5.8) \quad \lim_{\alpha \rightarrow N} \max \{ \tilde{J}_\alpha(tu^+ + su^-) \mid t, s \in [0, \infty) \} \leq \max \{ \Psi_{p,1}(tu^+ + su^-) \mid t, s \in [0, \infty) \}.$$

Moreover, we have for every $s, t \in [0, \infty)$,

$$(5.9) \quad \begin{aligned} \Psi_{p,1}(tu^+ + su^-) &= \frac{t^2 + s^2}{2} \int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 - \frac{(t^p + s^p)^2}{2p} \left(\int_{\mathbb{R}^N} |v|^p \right)^2 \\ &\leq r^2 \int_{\mathbb{R}^N} |\nabla v|^2 + |v|^2 - \frac{2r^{2p}}{p} \left(\int_{\mathbb{R}^N} |v|^p \right)^2 = 2\Psi_{p,2}(rv), \end{aligned}$$

where $r = \sqrt{\frac{t^2 + s^2}{2}}$. By combining (5.8) and (5.9), we get, in view of the definition of the level $\tilde{c}_\alpha^{\text{nod}}$

$$\limsup_{\alpha \rightarrow N} \tilde{c}_\alpha^{\text{nod}} \leq 2\Psi_{p,2}(rv).$$

Since the latter inequality holds for every $v \in C_c^\infty(\mathbb{R}^N)$ and since the set $C_c^\infty(\mathbb{R}^N)$ is dense in the Sobolev space $H^1(\mathbb{R}^N)$, we have in view of the characterization (2.11) and of the identity (2.10),

$$\limsup_{\alpha \rightarrow N} \tilde{c}_\alpha^{\text{nod}} \leq 2\kappa_{p,2} = 2^{\frac{p-2}{p-1}} \kappa_{p,1}.$$

For the reversed inequality, first observe that

$$t_\alpha^{2(p-1)} = \frac{\int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^+|^2 + |\tilde{u}_\alpha^+|^2}{2 \left(\int_{\mathbb{R}^N} |\tilde{u}_\alpha^+|^p \right)^2} \quad \text{and} \quad s_\alpha^{2(p-1)} = \frac{\int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^-|^2 + |\tilde{u}_\alpha^-|^2}{2 \left(\int_{\mathbb{R}^N} |\tilde{u}_\alpha^-|^p \right)^2}.$$

Combining Lemma 3.3 and Claim 3, we conclude that t_α and s_α remain bounded and bounded away from 0 as $\alpha \rightarrow N$. Then,

$$(5.10) \quad \tilde{c}_\alpha^{\text{nod}} = \tilde{J}_\alpha(\tilde{u}_\alpha) \geq \tilde{J}_\alpha(t_\alpha \tilde{u}_\alpha^+ + s_\alpha \tilde{u}_\alpha^-).$$

By the positive definiteness of the Riesz potential energy and by the Cauchy–Schwarz inequality (see [16, Theorem 9.8]), we have

$$(5.11) \quad t_\alpha^p s_\alpha^p \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^-|^p \leq \frac{t_\alpha^{2p}}{2} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^+|^p + \frac{s_\alpha^{2p}}{2} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^-|^p) |\tilde{u}_\alpha^-|^p.$$

Therefore, in view of (5.10), we deduce that

$$(5.12) \quad \begin{aligned} \tilde{c}_\alpha^{\text{nod}} &\geq \frac{t_\alpha^2}{2} \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^+|^2 + |\tilde{u}_\alpha^+|^2 - \frac{t_\alpha^{2p}}{p} \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^+|^p \\ &\quad + \frac{s_\alpha^2}{2} \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^-|^2 + |\tilde{u}_\alpha^-|^2 - \frac{s_\alpha^{2p}}{p} \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha^-|^p) |\tilde{u}_\alpha^-|^p. \end{aligned}$$

By Lemma 3.3, we have, as $\alpha \rightarrow N$,

$$(5.13) \quad \int_{\mathbb{R}^N} (\tilde{I}_\alpha * |\tilde{u}_\alpha^\pm|^p) |\tilde{u}_\alpha^\pm|^p \leq \left(\int_{\mathbb{R}^N} |\tilde{u}_\alpha^\pm|^p \right)^2 + O(N - \alpha).$$

In view of (5.12) this leads us to

$$(5.14) \quad \tilde{c}_\alpha^{\text{nod}} \geq \Psi_{p,2}(t_\alpha \tilde{u}_\alpha^+) + \Psi_{p,2}(s_\alpha \tilde{u}_\alpha^-) + O(N - \alpha).$$

By the characterization (2.11) and by the identity (2.10), it follows that

$$\liminf_{\alpha \rightarrow N} \tilde{c}_\alpha^{\text{nod}} \geq 2\kappa_{p,2} = 2^{\frac{p-2}{p-1}} \kappa_{p,1},$$

which proves the first part of the claim.

As a byproduct, the inequalities (5.11) and (5.14) become equalities in the limit $\alpha \rightarrow N$; this gives (5.6) and (5.7). $\diamond$

We are now in conditions to prove Proposition 5.1. First, let us show that $t_\alpha \rightarrow 1$ and $s_\alpha \rightarrow 1$ as $\alpha \rightarrow N$. In view of Claim 2 and Lemma 3.3, and by using the positive definiteness of the Riesz potential energy and the Cauchy–Schwarz inequality (see [16, Theorem 9.8]) we have:

$$\begin{aligned} \tilde{c}_\alpha^{\text{nod}} &= \tilde{J}_\alpha(\tilde{u}_\alpha^+ + \tilde{u}_\alpha^-) \geq \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_\alpha^+|^2 + |u_\alpha^+|^2 - \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_\alpha^+|^p) |u_\alpha^+|^p \\ &\quad + \frac{1}{2} \int_{\mathbb{R}^N} |\nabla u_\alpha^-|^2 + |u_\alpha^-|^2 - \frac{1}{p} \int_{\mathbb{R}^N} (I_\alpha * |u_\alpha^-|^p) |u_\alpha^-|^p \\ &\geq \Psi_{p,2}(\tilde{u}_\alpha^+) + \Psi_{p,2}(\tilde{u}_\alpha^-) + o(1). \end{aligned}$$

By Claim 4, we have

$$\begin{aligned} 2\kappa_{p,2} &\geq \Psi_{p,2}(t_\alpha \tilde{u}_\alpha^+) + \Psi_{p,2}(s_\alpha \tilde{u}_\alpha^-) \\ &\quad + \Psi_{p,2}(\tilde{u}_\alpha^+) - \Psi_{p,2}(t_\alpha \tilde{u}_\alpha^+) + \Psi_{p,2}(\tilde{u}_\alpha^-) - \Psi_{p,2}(s_\alpha \tilde{u}_\alpha^-) + o(1) \\ &\geq 2\kappa_{p,2} + \frac{1}{2} \left(1 - \left(1 - \frac{1}{p} \right) t_\alpha^2 - \frac{1}{p t_\alpha^{2p-2}} \right) \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^+|^2 + |\tilde{u}_\alpha^+|^2 \\ &\quad + \frac{1}{2} \left(1 - \left(1 - \frac{1}{p} \right) s_\alpha^2 - \frac{1}{p s_\alpha^{2p-2}} \right) \int_{\mathbb{R}^N} |\nabla \tilde{u}_\alpha^-|^2 + |\tilde{u}_\alpha^-|^2 + o(1) \end{aligned}$$

Since the integrals on the right-hand side remain bounded away from 0 (Claim 3), we have

$$\lim_{n \rightarrow \infty} 1 - \left(1 - \frac{1}{p} \right) t_\alpha^2 - \frac{1}{p t_\alpha^{2p-2}} = 0, \quad \text{and} \quad \lim_{n \rightarrow \infty} 1 - \left(1 - \frac{1}{p} \right) s_\alpha^2 - \frac{1}{p s_\alpha^{2p-2}} = 0.$$

By Young's inequality, we have for each $\tau \in (0, \infty)$,

$$1 \leq \frac{1}{p} \frac{1}{\tau^{2p-2}} + \left(1 - \frac{1}{p} \right) \tau^2,$$

Therefore the function $\theta : (0, \infty) \rightarrow \mathbb{R}$ defined for every $\tau \in (0, \infty)$

$$\theta(\tau) = 1 - \left(1 - \frac{1}{p} \right) \tau^2 - \frac{1}{p \tau^{2p-2}},$$

is nonnegative and $\theta(\tau) = 0$ if and only if $\tau = 1$. Since we have $\lim_{\tau \rightarrow 0} \theta(\tau) = \infty$ and $\lim_{\tau \rightarrow \infty} \theta(\tau) = \infty$, we conclude that $t_\alpha \rightarrow 1$ and $s_\alpha \rightarrow 1$ as $\alpha \rightarrow N$.

By (5.7), the families $t_\alpha u_\alpha^+$, $s_\alpha u_\alpha^-$ minimize the functional $\Psi_{p,2}$ restricted to its Nehari manifold $\mathcal{N}_{p,2}$ (as $\alpha \rightarrow N$). Lemma 2.3 implies the existence of vectors $\tilde{\xi}_\alpha^+$, $\tilde{\xi}_\alpha^- \in \mathbb{R}^N$ such that

$$(5.15) \quad u_\alpha^+ - V(\cdot - \tilde{\xi}_\alpha^+) \rightarrow 0, \quad u_\alpha^- - V(\cdot - \tilde{\xi}_\alpha^-) \rightarrow 0 \quad \text{in } H^1(\mathbb{R}^N),$$

as $\alpha \rightarrow N$ where $V = V_2$ is the groundstate of problem (2.7) for $\mu = 2$. In particular,

$$u_\alpha - (V(\cdot - \tilde{\xi}_\alpha^+) - V(\cdot - \tilde{\xi}_\alpha^-)) \rightarrow 0 \quad \text{in } H^1(\mathbb{R}^N).$$

If the sequence $\tilde{\xi}_\alpha^+ - \tilde{\xi}_\alpha^-$ were bounded, taking positive and negative part of the above expression yields a contradiction with (5.15). Hence $|\tilde{\xi}_\alpha^+ - \tilde{\xi}_\alpha^-| \rightarrow +\infty$. Then,

$$\lim_{\alpha \rightarrow N} \inf_{\xi^+, \xi^- \in \mathbb{R}^N} \|u_\alpha - (V(\cdot - \xi^+) - V(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)} = 0.$$

Since by Fatou's lemma,

$$\liminf_{|\xi^+| + |\xi^-| \rightarrow +\infty} \|\tilde{u}_\alpha - (V(\cdot - \xi^+) - V(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)} \geq \min \{\|\tilde{u}_\alpha\|_{H^1(\mathbb{R}^N)}, \|V\|_{H^1(\mathbb{R}^N)}\},$$

the function

$$(\xi^+, \xi^-) \in \mathbb{R}^N \times \mathbb{R}^N \mapsto \|\tilde{u}_\alpha - (V(\cdot - \xi^+) - V(\cdot - \xi^-))\|_{H^1(\mathbb{R}^N)}$$

attains its infimum at some pair of vectors $(\xi_\alpha^+, \xi_\alpha^-) \in \mathbb{R}^N \times \mathbb{R}^N$ for α sufficiently close to N . As in Section 4 we can conclude that $|\xi_\alpha^\pm - \tilde{\xi}_\alpha^\pm| \rightarrow 0$; in particular, $|\xi_\alpha^+ - \xi_\alpha^-| \rightarrow +\infty$.

Finally, we prove that $|\xi_\alpha^+ - \xi_\alpha^-|^{N-\alpha} \rightarrow 1$ as $\alpha \rightarrow N$. By (5.6),

$$\int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^-|^p \rightarrow \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^+|^p + \frac{1}{2} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^-|^p) |\tilde{u}_\alpha^-|^p.$$

But,

$$\begin{aligned} \int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^+|^p) |\tilde{u}_\alpha^-|^p &= \int_{\mathbb{R}^N} (I_\alpha * V(\cdot - \xi_\alpha^+)^p) V(\cdot - \xi_\alpha^-)^p + o(1) \\ &= \int_{\mathbb{R}^N} (I_\alpha * V(\cdot - (\xi_\alpha^+ - \xi_\alpha^-)^p) V^p + o(1), \end{aligned}$$

where in the last equality we have just made a change of variables. Analogously,

$$\int_{\mathbb{R}^N} (I_\alpha * |\tilde{u}_\alpha^\pm|^p) |\tilde{u}_\alpha^\pm|^p = \int_{\mathbb{R}^N} (I_\alpha * V(\cdot - \xi_\alpha^\pm)^p) V(\cdot - \xi_\alpha^\pm)^p + o(1) = \int_{\mathbb{R}^N} (I_\alpha * V^p) V^p + o(1).$$

Lemma 3.4 implies that $|\xi_\alpha^+ - \xi_\alpha^-|^{N-\alpha} \rightarrow 1$, concluding the proof. $\square$

Proof of Theorem 2. With Proposition 5.1 in hand, we follow the same ideas used to prove Theorem 1. Also here we can assume without loss of generality that $\xi_\alpha^+ = 0$ and $\xi_\alpha^- = \xi_\alpha = (m_\alpha, 0, \dots, 0)$.

By Proposition 5.1, we have:

$$(5.16) \quad \lim_{\alpha \rightarrow N} m_\alpha = \lim_{\alpha \rightarrow N} |\xi_\alpha| = \infty, \quad \text{and} \quad \lim_{\alpha \rightarrow N} m_\alpha^{N-\alpha} = \lim_{\alpha \rightarrow N} |\xi_\alpha|^{N-\alpha} = 1.$$

Again we define then R_α to be the orthogonal reflection of $\mathbb{R}^N$ that sends 0 to ξ_α , that is for each $x \in \mathbb{R}^N$

$$R_\alpha(x) = (m_\alpha - x_1, x_2, \dots, x_N).$$

We also define the functions $\check{u}_\alpha = \tilde{u}_\alpha \circ R_\alpha$, and $v_\alpha = \tilde{u}_\alpha + \check{u}_\alpha$, and the half-space

$$\Omega_\alpha = \{x \in \mathbb{R}^N \mid \xi_\alpha \cdot x < |\xi_\alpha|^2/2\}.$$

By construction, $v_\alpha \circ R_\alpha = v_\alpha$, and thus the function v_α is even with respect to $\partial\Omega_\alpha$.

By Proposition 5.1, we have $v_\alpha \rightarrow 0$ in $H^1(\mathbb{R}^N)$ as $\alpha \rightarrow N$. We will show that for α sufficiently close to N , we have $v_\alpha = 0$, that is, the solution $\tilde{u}_\alpha$ has an odd reflection symmetry with respect to the hyperplane $\partial\Omega_\alpha$.

Step 1. For every direction $\zeta \in \mathbb{R}^N$,

$$(v_\alpha | \partial_\zeta V)_{H^1(\mathbb{R}^N)} = 0, \quad \text{and} \quad (v_\alpha | \partial_\zeta V)_{H^1(\Omega_\alpha)} = o(\|v_\alpha\|_{H^1(\Omega_\alpha)}).$$

The proof is the same as Step 1 in Theorem 1.

Step 2. The function v_α satisfies the linear equation

$$(5.17) \quad \mathcal{L}_\alpha v_\alpha = 0 \quad \text{in } \mathbb{R}^N,$$

where the linear differential operator $\mathcal{L}_\alpha$ is defined by

$$\mathcal{L}_\alpha v = -\Delta v + v - (\tilde{I}_\alpha * (G_\alpha v))H_\alpha - (\tilde{I}_\alpha * K_\alpha)L_\alpha v,$$

with

$$\begin{aligned} G_\alpha &= \begin{cases} \frac{|\tilde{u}_\alpha|^{p-2} - |\check{u}_\alpha|^{p-2}}{\tilde{u}_\alpha + \check{u}_\alpha} & \text{where } \tilde{u}_\alpha \neq -\check{u}_\alpha, \\ p|\tilde{u}_\alpha|^{p-2}\tilde{u}_\alpha & \text{elsewhere,} \end{cases} \\ H_\alpha &= \frac{1}{2}(|\tilde{u}_\alpha|^{p-2}\tilde{u}_\alpha - |\check{u}_\alpha|^{p-2}\check{u}_\alpha), \\ K_\alpha &= \frac{1}{2}(|\tilde{u}_\alpha|^p + |\check{u}_\alpha|^p), \\ L_\alpha &= \begin{cases} \frac{|\tilde{u}_\alpha|^{p-2}\tilde{u}_\alpha + |\check{u}_\alpha|^{p-2}\check{u}_\alpha}{\tilde{u}_\alpha + \check{u}_\alpha} & \text{where } \tilde{u}_\alpha \neq -\check{u}_\alpha, \\ (p-1)|\tilde{u}_\alpha|^{p-2} & \text{elsewhere.} \end{cases} \end{aligned}$$

Again, the proof is identical to that of Step 2 of Theorem 1.

Step 3. Conclusion of the proof of Theorem 2.

The idea here is also very closely related to that of Theorem 1; the main difference is in the way one passes to the limit. As commented above, Theorem 2 follows if we show that $v_\alpha = 0$ for α sufficiently close N . Let us assume by contradiction that there is a sequence $(\alpha_n)_{n \in \mathbb{N}}$ in $(\underline{\alpha}, N)$, $\alpha_n \rightarrow N$ such that $v_{\alpha_n} \neq 0$. We define for each $n \in \mathbb{N}$ the normalized functions

$$w_n = \frac{v_{\alpha_n}}{\|v_{\alpha_n}\|}.$$

Without loss of generality, we can assume that the sequence $(w_n)_{n \in \mathbb{N}}$ converges weakly in $H^1(\mathbb{R}^N)$ to some function $w \in H^1(\mathbb{R}^N)$.

By Proposition 5.1, we have that

$$(5.18) \quad \begin{aligned} G_\alpha - p(V^{p-1} - V^{p-1}(\cdot - \xi_\alpha)) &\rightarrow 0 \quad \text{in } L^q(\mathbb{R}^N), \quad \text{if } \frac{1}{2} - \frac{1}{N} \leq \frac{p-1}{q} \leq \frac{1}{2} \\ H_\alpha - (V^{p-1} - V^{p-1}(\cdot - \xi_\alpha)) &\rightarrow 0 \quad \text{in } L^q(\mathbb{R}^N), \quad \text{if } \frac{1}{2} - \frac{1}{N} \leq \frac{p-1}{q} \leq \frac{1}{2}, \\ K_\alpha - (V^p + V^p(\cdot - \xi_\alpha)) &\rightarrow 0 \quad \text{in } L^q(\mathbb{R}^N), \quad \text{if } \frac{1}{2} - \frac{1}{N} \leq \frac{p}{q} \leq \frac{1}{2}, \\ L_\alpha - (p-1)(V^{p-2} + V^{p-2}(\cdot - \xi_\alpha)) &\rightarrow 0 \quad \text{in } L^q(\mathbb{R}^N), \quad \text{if } \frac{1}{2} - \frac{1}{N} \leq \frac{p}{q} \leq \frac{1}{2}. \end{aligned}$$

We test the equation (5.17) against $\varphi \in C_0^\infty(\mathbb{R}^N)$ and we obtain, in view of (5.18),

$$\begin{aligned}
 (5.19) \quad & \int_{\mathbb{R}^N} (\nabla w_n \cdot \nabla \varphi + w_n \varphi) \\
 &= p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(V^{p-1}(x) - V^{p-1}(x - \xi_{\alpha_n})) w_n(x) (V^{p-1}(y) - V^{p-1}(y - \xi_{\alpha_n})) \varphi(y)}{|x - y|^{N-\alpha_n}} dx dy \\
 &+ (p-1) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(V^p(x) + V^p(x - \xi_{\alpha_n})) (V^{p-2}(y) + V^{p-2}(y - \xi_{\alpha_n})) w_n(y) \varphi(y)}{|x - y|^{N-\alpha_n}} dx dy \\
 &\quad + o(1).
 \end{aligned}$$

We claim that

$$(5.20) \quad \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(V^{p-1}(x) - V^{p-1}(x - \xi_{\alpha_n})) w_n(x) (V^{p-1}(y) - V^{p-1}(y - \xi_{\alpha_n})) \varphi(y)}{|x - y|^{N-\alpha_n}} dx dy \rightarrow 0.$$

Indeed, we observe that $V^{p-1}w_n \rightarrow V^{p-1}w$ and $(V^{p-1}(y) - V^{p-1}(y - \xi_{\alpha_n}))\varphi(y) \rightarrow V^{p-1}(y)\varphi(y)$ in $L^q(\mathbb{R}^N)$, for $q \in [1, \frac{2N}{N-2})$. By Lemma 3.4,

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{V^{p-1}(x)w_n(x)(V^{p-1}(y) - V^{p-1}(y - \xi_{\alpha_n}))\varphi(y)}{|x - y|^{N-\alpha_n}} dx dy \\
 = \left(\int_{\mathbb{R}^N} V^{p-1}w_n \right) \left(\int_{\mathbb{R}^N} V^{p-1}\varphi \right).
 \end{aligned}$$

Moreover, by the evenness of w_n with respect to $\partial\Omega_\alpha$, and since $R_\alpha(\xi_\alpha) = 0$, we have $|R_\alpha(z) - \xi_\alpha| = |z|$. Recalling that V is radially symmetric, we have by changes of variable $\check{x} = R_\alpha(x)$ and $\check{y} = R_\alpha(y)$,

$$\begin{aligned}
 \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{V^{p-1}(x - \xi_{\alpha_n}) w_n(x) (V^{p-1}(y) - V^{p-1}(y - \xi_{\alpha_n})) \varphi(y)}{|x - y|^{N-\alpha_n}} dx dy \\
 = \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{V^{p-1}(\check{x}) w_n(\check{x}) (V^{p-1}(\check{y} - \xi_{\alpha_n}) - V^{p-1}(\check{y})) \check{\varphi}(\check{y} - \xi_{\alpha_n})}{|\check{x} - \check{y}|^{N-\alpha_n}} d\check{x} d\check{y},
 \end{aligned}$$

where $\check{\varphi}(y_1, y_2, \dots, y_N) = \varphi(-y_1, y_2, \dots, y_N)$. Again by Lemma 3.4 and by the radial symmetry of V ,

$$\begin{aligned}
 \lim_{n \rightarrow +\infty} \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{V^{p-1}(x - \xi_{\alpha_n}) w_n(x) (V^{p-1}(y) - V^{p-1}(y - \xi_{\alpha_n})) \varphi(y)}{|x - y|^{N-\alpha_n}} dx dy \\
 = \left(\int_{\mathbb{R}^N} V^{p-1}w_n \right) \left(\int_{\mathbb{R}^N} V^{p-1}\check{\varphi} \right) = \left(\int_{\mathbb{R}^N} V^{p-1}w_n \right) \left(\int_{\mathbb{R}^N} V^{p-1}\varphi \right),
 \end{aligned}$$

Hence (5.20) follows.

Reasoning analogously and recalling that φ has compact support, the second term in the right-hand side of (5.19) converges to

$$2(p-1) \left(\int_{\mathbb{R}^N} V^p \right) \left(\int_{\mathbb{R}^N} V^{p-2}w\varphi \right).$$

We conclude that w is a (weak) solution of

$$-\Delta w + w = 2(p-1) \|V\|_{L^p(\mathbb{R}^N)}^p V^{p-2}w.$$

By Step 1 and the nondegeneracy of (2.3) (recall that $V = V_2$ is a groundstate solution of (2.7) for $\mu = 2$), we have $w = 0$.

We now multiply the equation (5.17) by the function v_{α_n} , integrate and divide by $\|v_{\alpha_n}\|_{H^1(\mathbb{R}^N)}^2$, to obtain:

$$\begin{aligned} 1 &= \int_{\mathbb{R}^N} |\nabla w_n|^2 + |w_n|^2 = \int_{\mathbb{R}^N} (I_{\alpha_n} * (G_{\alpha_n} w_n)) H_{\alpha_n} w_n + (I_{\alpha_n} * K_{\alpha_n}) L_{\alpha_n} w_n^2 + o(1) \\ &= p \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(V^{p-1}(x) - V^{p-1}(x - \xi_{\alpha_n})) w_n(x) (V^{p-1}(y) - V^{p-1}(y - \xi_{\alpha_n})) w_n(y)}{|x - y|^{N-\alpha_n}} dx dy \\ &\quad + (p-1) \int_{\mathbb{R}^N} \int_{\mathbb{R}^N} \frac{(V^p(x) + V^p(x - \xi_{\alpha_n})) (V^{p-2}(y) + V^{p-2}(y - \xi_{\alpha_n})) w_n^2(y)}{|x - y|^{N-\alpha_n}} dx dy \\ &\quad + o(1). \end{aligned}$$

We argue again as in the proof of (5.20) to conclude that the first term in the right-hand side converges to 0. Again, Lemma 3.4 and (5.16) imply that the second term in the right-hand side converges to

$$4(p-1) \left(\int_{\mathbb{R}^N} V^p \right) \left(\int_{\mathbb{R}^N} V^{p-2} w^2 \right) = 0,$$

since $V^{p-2} w_n^2 \rightarrow V^{p-2} w^2 = 0$ strongly in $L^q(\mathbb{R}^N)$, for $q \in [1, \frac{N}{N-2})$. This yields the desired contradiction and concludes the proof of Theorem 2. $\square$

REFERENCES

- [1] A. Bahri and P.-L. Lions, *On the existence of a positive solution of semilinear elliptic equations in unbounded domains*, Ann. Inst. H. Poincaré Anal. Non Linéaire **14** (1997), no. 3, 365–413.
- [2] V. Benci and G. Cerami, *Positive solutions of some nonlinear elliptic problems in exterior domains*, Arch. Rational Mech. Anal. **99** (1987), no. 4, 283–300.
- [3] A. Castro, J. Cossio, and J. M. Neuberger, *A sign-changing solution for a superlinear Dirichlet problem*, Rocky Mountain J. Math. **27** (1997), no. 4, 1041–1053.
- [4] ———, *A minmax principle, index of the critical point, and existence of sign-changing solutions to elliptic boundary value problems*, Electron. J. Differential Equations **1998** (1998), no. 2, 18.
- [5] G. Cerami, S. Solimini, and M. Struwe, *Some existence results for superlinear elliptic boundary value problems involving critical exponents*, J. Funct. Anal. **69** (1986), no. 3, 289–306.
- [6] S. Cingolani, M. Clapp, and S. Secchi, *Multiple solutions to a magnetic nonlinear Choquard equation*, Z. Angew. Math. Phys. **63** (2012), no. 2, 233–248.
- [7] S. Cingolani and S. Secchi, *Multiple $\mathbb{S}^1$ -orbits for the Schrödinger-Newton system*, Differential and Integral Equations **26** (2013), no. 9/10, 867–884.
- [8] M. Clapp and D. Salazar, *Positive and sign changing solutions to a nonlinear Choquard equation*, J. Math. Anal. Appl. **407** (2013), no. 1, 1–15.
- [9] L. Diósi, *Gravitation and quantum-mechanical localization of macro-objects*, Phys. Lett. A **105** (1984), no. 4–5, 199–202.
- [10] M. Ghimenti, V. Moroz, and J. Van Schaftingen, *Least action nodal solutions for the quadratic Choquard equation*, to appear in Proc. Amer. Math. Soc., available at [arXiv:1511.04779](https://arxiv.org/abs/1511.04779).
- [11] M. Ghimenti and J. Van Schaftingen, *Nodal solutions for the Choquard equation*, J. Funct. Anal. **271** (2016), no. 1, 107–135.
- [12] K. R. W. Jones, *Newtonian quantum gravity*, Austral. J. Phys. **48** (1995), no. 6, 1055–1081.
- [13] M. K. Kwong, *Uniqueness of positive solutions of $\Delta u - u + u^p = 0$ in $\mathbb{R}^n$* , Arch. Rational Mech. Anal. **105** (1989), no. 3, 243–266.
- [14] E. H. Lieb, *Existence and uniqueness of the minimizing solution of Choquard’s nonlinear equation*, Studies in Appl. Math. **57** (1976/77), no. 2, 93–105.

- [15] ———, *Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities*, Ann. of Math. (2) **118** (1983), no. 2, 349–374.
- [16] E. H. Lieb and M. Loss, *Analysis*, 2nd ed., Graduate Studies in Mathematics, vol. 14, American Mathematical Society, Providence, RI, 2001.
- [17] P.-L. Lions, *The Choquard equation and related questions*, Nonlinear Anal. **4** (1980), no. 6, 1063–1072.
- [18] G. P. Menzala, *On regular solutions of a nonlinear equation of Choquard’s type*, Proc. Roy. Soc. Edinburgh Sect. A **86** (1980), no. 3-4, 291–301.
- [19] I. M. Moroz, R. Penrose, and P. Tod, *Spherically-symmetric solutions of the Schrödinger-Newton equations*, Classical Quantum Gravity **15** (1998), no. 9, 2733–2742.
- [20] V. Moroz and J. Van Schaftingen, *Groundstates of nonlinear Choquard equations: Existence, qualitative properties and decay asymptotics*, J. Funct. Anal. **265** (2013), no. 2, 153–184.
- [21] ———, *A guide to the Choquard equation*, available at [arXiv:1606.02158](https://arxiv.org/abs/1606.02158).
- [22] Y.-G. Oh, *On positive multi-lump bound states of nonlinear Schrödinger equations under multiple well potential*, Comm. Math. Phys. **131** (1990), no. 2, 223–253.
- [23] S. I. Pekar, *Untersuchungen über die Elektronentheorie der Kristalle*, Akademie-Verlag, Berlin, 1954.
- [24] R. Penrose, *On gravity’s role in quantum state reduction*, Gen. Rel. Grav. **28** (1996), no. 5, 581–600.
- [25] P. Tod and I. M. Moroz, *An analytical approach to the Schrödinger-Newton equations*, Nonlinearity **12** (1999), no. 2, 201–216.
- [26] M. I. Weinstein, *Modulational stability of groundstates of nonlinear Schrödinger equations*, SIAM J. Math. Anal. **16** (1985), no. 3, 472–491.
- [27] T. Weth, *Spectral and variational characterizations of solutions to semilinear eigenvalue problems*, Johannes Gutenberg-Universität, Mainz, 2001.
- [28] M. Willem, *Minimax theorems*, Progress in Nonlinear Differential Equations and their Applications, 24, Birkhäuser, Boston, Mass., 1996.

UNIVERSIDAD DE GRANADA, DEPARTAMENTO DE ANÁLISIS MATEMÁTICO, CAMPUS FUENTENUEVA,
18071 GRANADA, SPAIN

E-mail address: daruiz@ugr.es

UNIVERSITÉ CATHOLIQUE DE LOUVAIN, INSTITUT DE RECHERCHE EN MATHÉMATIQUE ET PHYSIQUE,
CHEMIN DU CYCLOTRON 2 BTE L7.01.01, 1348 LOUVAIN-LA-NEUVE, BELGIUM

E-mail address: Jean.VanSchaftingen@UCLouvain.be