On the Transmittance Between OAM Antennas

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***Abstract*—A quasi-analytical solution is derived for the transmittance between two OAM antennas. It makes use of the properties of the modified circle polynomials, related to Zernike functions, and of integration in the complex plane of the radial spectral coordinate. The resulting formulation can be estimated extremely fast and is shown to exhibit several expected properties. An asymptotic expression is proposed and validated.**

***Index Terms*—Aperture antennas, multiplexing, spectral domain analysis.**

I. Introduction

In recent years, several demonstrations have been carried out [1]–[3] regarding a multiplexing approach that exploits the orbital angular momentum (OAM). A macroscopic way of distinguishing different OAM modes consists of considering apertures that exhibit an $\exp(j\ n\ \alpha)$ field distribution, where α is the azimuthal angle.

The actual gain of capacity offered by this type of multiplexing has been a subject of debate [4]–[7]. One may regard this multiplexing technique as another way of carrying out spatial diversity and, as such, this technical solution has sometimes been compared with multiple input multiple output (MIMO) technology [4], [5]. A loss of multiplexing gain has also been pointed out in case the antennas are located in each other's far field [7]. Indeed, OAM operation is essentially limited to near-field links.

The present note combines several classical results of antenna theory to provide a simple formulation for the transmitance for each OAM mode versus antenna dimensions and distance between antennas.

II. Mathematical Formulation

A result from electromagnetic reciprocity [8] is that the open-circuit voltage of a receiving antenna can be obtained as the projection over the receiving aperture S_r between incident fields and the current distribution of the same antenna when it is transmitting

$$V_{o.c.} = \frac{1}{I_r} \int\int \left(\vec{K}_e^a \cdot \vec{E}^{inc} - \vec{K}_m^a \cdot \vec{H}^{inc} \right) dS_r \tag{1}$$

where K_e^a and K_m^a are the equivalent electric and magnetic currents on the receiving antenna in transmit mode with port current I_r, while E^{inc} and H^{inc} are the incident electric and magnetic fields. In the following, we will consider doubled electric currents only [8], as can be done if the aperture is made infinite, or at least supports all significant fields. A reference configuration, along with the used geometrical quantities is provided in Fig. 1. We will also assume that no multiple scattering takes place between antennas, such that the incident field can be computed through (near-field or far-field) radiation integrals involving the current distribution on the transmitting antenna. From there, a transmittance is given in the form of a mutual impedance between two OAM antennas with apertures S_t and S_r

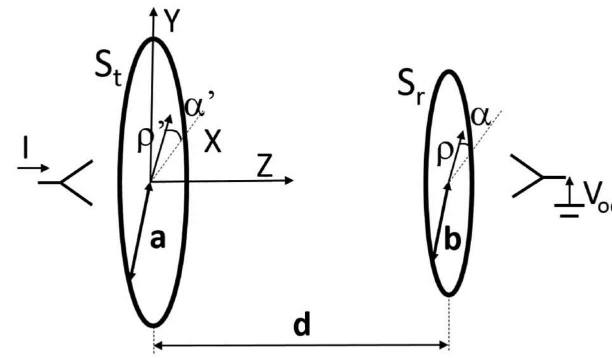

Fig. 1. Two apertures between which the transmittance is determined, along with related geometrical quantities.

$$H = \frac{-j\ 4\ k\ \eta}{I_r\ I_t} \int\int_{S_r} K_r(\vec{r}) \int\int_{S_t} K_t(\vec{r}\,')\ G(\vec{r},\vec{r}\,')\ dS_t\ dS_r \tag{2}$$

where K_t and K_r are the (equivalent) current distributions on transmitting and receiving antennas when they are active. I_t is the port current of the transmitting antenna, η is the free-space impedance, and k is the wavenumber. G is the scalar Green's function. To alleviate notations, the constant $C = -j\ 4\ k\ \eta/(I_r\ I_t)$ will be used. In (2), the outer integral stems from (1), whereas the inner integral results from radiation from S_t. Equation (2) has been obtained assuming that both antennas have the same polarization and that the distance d between the antennas is large enough for the polarization from the transmitting antenna to be homogeneous over the receiving aperture. This allows one to treat the problem in scalar form with G being the scalar Green's function. Section III will be devoted to the extension of results to vector fields.

In the following, the current distributions on the two apertures are decomposed into modified circle polynomials, F_m^n [9], which are directly related to Zernike functions [11], [12, Sec. 9.2] and to Jacobi polynomials [10], and dispose of orthogonality over the unit disk. The Appendix recalls a few important relations regarding the modified circle polynomials, which will be named as "F-functions" below. In the following, we will consider two OAM antennas with a fixed harmonic index n and two different F-function indices m' and m, for transmitting and receiving antennas, respectively. F-functions are now classically used [13] to represent source distributions over circular apertures

$$K_t(\vec{r}\,') = \sum_{n=-N}^{N} \sum_{m'=0}^{M} \gamma_{nm'}\ e^{j\ n\ \alpha'}\ F_{m'}^{|n|}(\rho'/a) \tag{3}$$

$$K_r(\vec{r}) = \sum_{n=-N}^{N} \sum_{m=0}^{M} \gamma_{nm}\ e^{-j\ n\ \alpha}\ F_{m}^{|n|}(\rho/b) \tag{4}$$

where (ρ', α') and (ρ, α) are local polar coordinates on the transmitting and receiving apertures, which have radii a and b, respectively. Hereafter, primed coordinates and indices will refer to sources (and to transmitting antenna) and unprimed values will refer to observation points (and to receiving antenna). Coefficients $\gamma_{nm'}$ and γ_{nm} are simply calculated through projection, by virtue of the orthogonality (cf. Appendix) of the F-functions over the unit disk. For a pure OAM

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mode, characterized by a single index n and by a radial distribution $A_n(\rho)$, we have

$$\gamma_{nm} = \frac{1}{2|n| + 4m + 2} \int_0^1 A_n(r\ a)\ F_m^{|n|}(r)\ r\ dr. \tag{5}$$

The transmittance between two OAM antennas with common azimuthal index n can be decomposed by projection into all pairs of transmittances denoted by the indices m' and m of the F-functions

$$H_{nm'm} = \sum_m \sum_{m'} \gamma_{nm}\ \gamma_{nm'}\ H_{nm'm}. \tag{6}$$

In the following, to alleviate notations, symbol $\mathcal{H}$ will actually represent the quantity $H_{nm'm}$. It is then trivial to determine the transmittance between two antennas with known distributions. Actually, the theory below also allows summation on indices n, such that it allows the determination of the transmittance between two apertures, not limited to OAM distributions, that are not necessarily located in each other's far-field regions.

The procedure followed below resembles the spectral-domain evaluation of reaction integrals between basis and testing functions, which is often used in the implementation of the method of moments [14]. The first step will hence consist of introducing the spectral-domain representation of the Green's function in the expression of the transmittance. Prior to this, the transmittance is rewritten making use of the F-functions

$$\mathcal{H} = H_{n,m,m'} = C \int\int_{S_r} \int\int_{S_t} e^{j\ n\ \alpha'}\ F_{m'}^{|n|}(\rho'/a)\ G(\vec{r}, \vec{r}\,')$$
$$F_m^{|n|}(\rho/b)\ e^{-j\ n\ \alpha}\ dS_t\ dS_r \tag{7}$$

where S_t and S_r correspond to the surfaces of the transmitting and receiving antennas. The Green's function can be written as a spectrum of plane waves, in either rectangular [15] or polar coordinates [12, Sec. 11.7], [16]

$$G(\vec{r}, \vec{r}\,') = \frac{e^{-j\ k\ ||\vec{r} - \vec{r}\,'||}}{4\ \pi\ ||\vec{r} - \vec{r}\,'||}$$
$$= \frac{1}{(2\pi)^2} \int\int \frac{e^{-j\ k_x (x - x')} e^{-j\ k_y (y - y')} e^{-j\ k_z\ d}}{2\ j\ k_z}\ dk_x\ dk_y \tag{8}$$
$$= \frac{1}{(2\pi)^2} \int\int \frac{e^{j\ \beta\ \rho'\ \cos(\alpha' - \phi)} e^{-j\ \beta\ \rho\ \cos(\alpha - \phi)} e^{-j\ k_z\ d}}{2\ j\ k_z}\ d\phi\ \beta\ d\beta \tag{9}$$

where k is the free-space wavenumber, $\beta = \sqrt{k_x^2 + k_y^2}$ and $k_z = \sqrt{k^2 - \beta^2}$. Result (9) is inserted into (7) and the following relation is applied twice [10]:

$$\int_0^{2\pi} e^{j\ n\ \alpha'}\ e^{j\ \beta\ \rho'\ \cos(\alpha' - \phi)}\ d\alpha = 2\pi\ J_n(\beta\ \rho')\ e^{j\ n\ \phi}\ j^n. \tag{10}$$

Then, the integral of ϕ-dependent factors trivially reads

$$I_\phi = \int_0^{2\pi} e^{j\ n\ \phi}\ e^{-j\ n\ \phi}\ d\phi = 2\ \pi. \tag{11}$$

In turn, the transmittance is rewritten as

$$\mathcal{H} = C\ \frac{\pi}{j} \int_\beta \int_{\rho'} J_n(\beta\ \rho')\ F_{m'}^{|n|}(\rho'/a)\ \rho'\ d\rho'$$
$$\frac{e^{-j\ k_z\ d}}{k_z} \int_\rho J_n(\beta\ \rho)\ F_m^{|n|}(\rho/b)\ \rho\ d\rho\ \beta\ d\beta. \tag{12}$$

Finally, knowing that F-functions have closed-form Hankel transforms over the unit circle [9], [16]

$$\int_0^1 F_m^n(r)\ J_n(u\ r)\ r\ dr = J_{n+2m+1}(u)/u \tag{13}$$

a single integral remains and we obtain

$$\mathcal{H} = \frac{C\ \pi}{j}\ a^2\ b^2 \int_{\beta=0}^{\infty} L_{m'}^{|n|}(\beta\ a)\ L_m^{|n|}(\beta\ b)\ \frac{e^{-j\ k_z\ d}}{k_z}\ \beta\ d\beta \tag{14}$$

with $L_m^n(x) = J_{n+2m+1}(x)/x$. Identifying the functions $L_{m'}^{|n|}$ and $L_m^{|n|}$ with the radiation patterns of the transmitting and receiving antennas (limited to the modes under consideration), and keeping in mind the trivial integral along ϕ, (14) extends to the evanescent part of the spectrum the result given in [19] for the coupling quotient between antennas (matching factors are not included here). To avoid the pole at $\beta = k$, one may deviate from the strict distinction between propagating and evanescent waves through analytic extension of wavenumber β. To implement this, a very simple integration path is chosen in the complex β plane, as a straight line forming a 45° angle with the real axis: $\beta = \beta_r(1 + j)$ with β_r real. One then arrives at

$$\mathcal{H} = \frac{C\ 2}{\pi}\ S_t\ S_r \int_{\beta_r=0}^{\infty} L_{m'}^{|n|}(\beta\ a)\ L_m^{|n|}(\beta\ b)\ \frac{e^{-j\ k_z\ d}}{k_z}\ \beta_r\ d\beta_r. \tag{15}$$

Results are shown in Fig. 2 for apertures of diameter $D = 2a = 2b = 40$ cm, while the wavelength is $\lambda = 1$ cm. The distance d is normalized ***with respect to d_{far}, which corresponds to the far-field distance $2D^2/\lambda$. Index n varies from 0 to 4 (top to bottom curves), while $m = m' = 0$. The transmittance is normalized by omitting the factor $C\ S_r\ S_t$.

As predicted in [7] and [17] for the special case of Gauss–Laguerre beams, the decay versus distance is asymptotically proportional to $1/d^{|n|+1}$. This has also been obtained numerically based on circular arrays in [18]. The formulation above leads to the same conclusion. Indeed, given the straight path chosen in the complex β plane, for large distances, the following approximation holds: $\exp(-j\ k_z\ d) \simeq \exp(-j\ k\ d) \exp(-\beta_r^2\ d/k)$ [20]. Still for large distances d, given the Gaussian envelope versus β_r, only small values of β_r need to be considered, such that the approximation $k_z \simeq k$ holds for the denominator. Next, for small values of x and $l \geq 0$, $J_l(x) \simeq (x/2)^l/l!$. Consequently, the interaction above can be computed using the following mathematical result:

$$\int_0^\infty e^{-\beta_r^2\ d/k} \beta_r^{1+2\ p} d\beta_r = \frac{p!}{2} \left(\frac{k}{d}\right)^{p+1}. \tag{16}$$

In the following, this result will be used for $p = |n| + m + m'$.

Exploiting the above results in (15) provides the following asymptotic expression for large distances:

$$\mathcal{H} \simeq C\ S_t\ S_r\ \frac{e^{-j\ k\ d}}{4\ \pi\ d} \left(\frac{k\ a\ b}{2\ d}\right)^p \left(\frac{b}{a}\right)^{m-m'} \frac{p!}{s!\ s'!}\ j^p \tag{17}$$

with $s = |n| + 2\ m + 1$ and $s' = |n| + 2\ m' + 1$. For the special case $m = m' = 0$, (17) confirms the $d^{-|n|-1}$ behavior recalled above. If, besides, $n = 0$, such that $p = 0$, then the expected $\exp(-j\ k\ d)/d$ dependence on distance is obtained. Fig. 3 shows transmittances for $m' = 2$, $m = 1$, and $n = 0, \ldots, 4$ (top to bottom). Given the integration path chosen, the integration along the β_r coordinate is easy, the integrand being very smooth. Integration using 200 points with a rectangles rule has been found sufficient. Much fewer points (about 50)

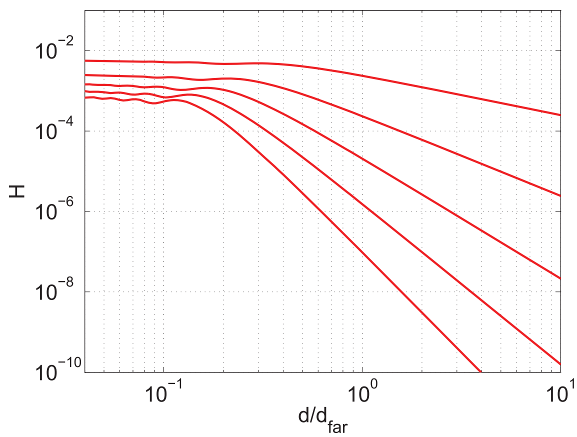

Fig. 2. Normalized transmittance versus distance normalized w.r.t. far-field distance. $m' = m = 0$ and $n = 0, \ldots, 4$ (top to bottom curves).

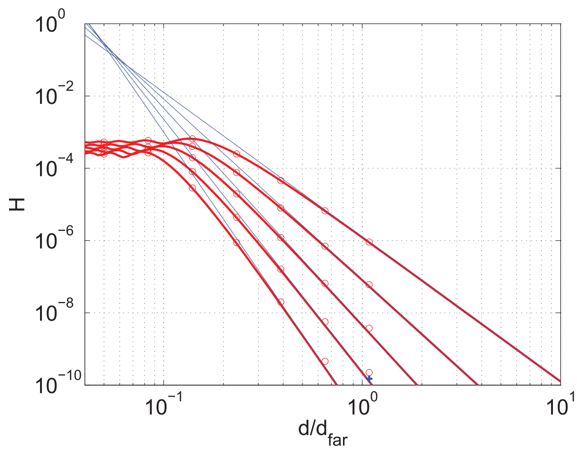

Fig. 3. Thick lines: normalized transmittance versus distance normalized w.r.t. far-field limit. $m' = 2$, $m = 1$ and $n = 0, \ldots, 4$ (top to bottom). Thin lines: asymptotic result for far-field conditions. Circles: brute-force integration with 75 points along all four coordinates. Cross: with 150 points.

could be used in the intermediate-to-far field region ($d/d_{\text{far}} > 0.1$). The thin lines correspond to the asymptotic result (17), which very accurately fits the full solution (15) in the far field. It should be noted that, in the near-to-intermediate fields (d smaller than about $d_{\text{far}}/10$), the decay is much slower versus distance and versus index n.

For validation purposes, the circles have been obtained through brute-force integration of (7) using a rectangle's rule with 75 samples along each of the four coordinates of integration. This very intensive integration provides an excellent match with the results obtained with the transmittance expressed by (15), at least for small distances. The slight discrepancy for large distances results from the more effective field cancellation taking place for high-order OAM at large distances. In such conditions, more points are needed in the brute-force approach. For example, if 150 points are used along every coordinate of integration, for $d/d_{\text{far}} = 1.08$ and $n = 3$, a closer match is obtained; the result is represented by a bold cross in the bottom of the figure. In other words, the brute-force solution serving as a reference has a worse convergence for large distances and high orders. Bearing in mind this difficulty, one can say that the proposed model for the transmittance has been successfully validated through a brute-force integration of the direct transmittance model.

III. Extension to Vector Fields

The above analysis stands for scalar fields. The extension to vector fields can be achieved via two changes. The first one consists of projecting the current distributions on transmitting and receiving antennas along two orthogonal polarizations and to estimate four transmittances. This entails another change, related to the plane-wave spectral decomposition of fields (9) now considered in the vector case. In this case, for each plane wave in the spectrum, characterized by wavenumbers k_x and k_y, one should project the source current on the plane perpendicular to the direction of propagation [21, Sec. 6.3] given by the complex vector wavenumber $\vec{k}$. The two changes above can be expressed by a multiplication for any wavenumber pair (k_x, k_y) by the following factor:

$$P = \left(\hat{e}_t - \frac{(\hat{e}_t \ \cdot \ \vec{k})\, \vec{k}}{k^2} \right) \ \cdot \ \vec{e}_r \tag{18}$$

where $\hat{e}_t$ and $\vec{e}_r$ are the unit vectors chosen to describe polarizations on transmit and receive sides, respectively. In the following those vectors will simply correspond to the unit vectors along X- or Y-axes. The operation between parentheses corresponds to the elimination of the component of the source current that is parallel to $\vec{k}$. The dot product with $\vec{e}_r$ simply corresponds to the scalar product with the current on the receiving antenna, according to the reciprocity relation reminded in (1). The introduction of factor P will impact the integration along the (k_x, k_y) coordinates. Here, we remind their expressions using polar coordinates: $k_x = \beta \ \cos\phi$ and $k_y = \beta \ \sin\phi$, where β can be extented to the complex plane. Below, we first consider the cross-polar and then the copolar cases. The former is obtained by considering for instance $\hat{e}_r = \hat{a}_x$ and $\hat{e}_t = \hat{a}_y$. In that case, we can write

$$P(k_x, k_y) = 0 - \beta \sin\phi \ \beta \ \cos\phi / k^2. \tag{19}$$

In Section II, the integration (11) along the azimuthal spectral coordinate ϕ has been found trivial; now, it requires some attention

$$I_\phi = \int_0^{2\pi} e^{jn\phi} \ e^{-jn\phi} \ \cos\phi \ \sin\phi \ d\phi = 0 \tag{20}$$

which means that, when polarization is included, the cross-polar transmittance always equals zero. Regarding the copolar transmittance, say when $\hat{e}_t = \hat{e}_r = \hat{a}_x$, the integration of ϕ-dependent factors reads

$$I_\phi = \int_0^{2\pi} e^{jn\phi} \ e^{-jn\phi} \ \left(1 - \frac{\beta^2}{k^2} \cos^2\phi\right) \ d\phi = 2\,\pi \ \left(1 - \frac{\beta^2}{2\,k^2}\right). \tag{21}$$

Hence, to fully include the effect of polarization, the factor between parentheses should be added to the integrand of (14). Equivalently, noting that $\beta = \beta_r \ (1 + j)$, the factor to be added in the integrand of (15) reads

$$P_c = 1 - j \ \frac{\beta_r^2}{k^2}. \tag{22}$$

This new factor will lead to a new term with an exponent of β_r incremented by two units. Based on (16), one can see that this term will be characterized by a faster decay in the intermediate-to-far field, with the exponent applied to distance d decremented by one unit, such that the vector-field correction has a negligible impact. Also in the near-field, the impact of the correction is very small. Considering the example of Fig. 2, we verified that the new term is at least two orders of magnitude smaller than the full solution for $d/d_{\text{far}} = 0.04$ and at least four orders of magnitude smaller for $d/d_{\text{far}} = 10$. Hence, down to very small distances, the scalar-field formulation is sufficient.

IV. Conclusion

An analytical derivation for intermediate-to-far field transmittance between OAM antennas has been provided in the form of a mutual impedance. The resulting model involves a single integral, which can be evaluated efficiently using a straight path in the complex plane of the radial spectral coordinate. The study clearly confirms the faster decay versus distance of OAM beams of higher orders, as already pointed out in several comments recently made about OAM communications in free space. The results are in principle also applicable to more general apertures, such that information could also be encoded on the radial dependence of the aperture distribution (m and m' indices). However, one should keep in mind the faster field decay for higher indices and the nonorthogonality of fields observed over the receiving antenna for same n and $m \neq m'$. The latter point means that the transmittance (7) is nonzero when the current distributions correspond to Zernike polynomials with different indices m and identical values of n. Hence, in practice, a field transmitted in a mode for a given value of m' will "leak" through all the receiving channels associated to all values of m, thereby preventing easy multiplexing of modes with different radial indices. The extension to vector fields has been provided through the introduction of an extra factor in the spectral expression for the transmittance. That correction does not lead to a significant change in the value of the transmittance.

Appendix: Modified Circle Polynomials

Series development of modified circle polynomials, named here "F-functions"

$$F_m^n(r) = \sum_{p=0}^{m} (-1)^p \, \frac{(m+n+p)!}{p!\,(m-p)!\,(n+p)!} \, r^{n+2p} \tag{23}$$

for $n \geq 0$.

Orthogonality [9]

$$\int_0^1 F_m^n(r)\, F_l^n(r)\, r\, dr = \frac{\delta_{ml}}{2(n+2m+1)}. \tag{24}$$

Relation with Zernike[1,2] functions R_m^n [12, Sec. 9.2], [16]

$$F_m^n(r) = (-1)^m \, R_{m_z}^n(r) \tag{25}$$

with $m_z = n + 2m$.

Relation with Jacobi polynomials $P_m^{\alpha,\beta}$ [9]:

$$F_m^n(r) = r^n \, P_m^{n,0}(1 - 2\,r^2) \tag{26}$$

with $0 \leq r \leq 1$.

Acknowledgment

***In memoriam of Julien Perruisseau-Carrier, 1979–2014, Professor at EPFL, Switzerland.

[1]One should warn the reader that Zernike functions are usually denoted as R_n^m instead of R_m^n, with m the azimuthal phase index.

[2]We prefer the "F-function" representation to the Zernike one for its easier indexing (no even vs. odd index considerations) and for its more compact Hankel transform (equation (13)).

References

[1] F. Tamburini *et al.*, "Encoding many channels on the same frequency through radio vorticity: First experimental test," *New J. Phys.*, vol. 14, p. 033001, 2012.

[2] A. Beniss *et al.*, "Flat plate for OAM generation in the millimeter band," in *Proc. 7th Eur. Conf. Antennas Propag.*, Gothenburg, Sweden, Apr. 2013, pp. 3203–3207.

[3] Y. Yan *et al.*, "High-capacity millimetre-wave communications with orbital angular momentum multiplexing," *Nat. Commun.*, vol. 5, p. 4876, Aug. 2014.

[4] O. Edfors and A. J. Johansson, "Is orbital angular momentum (OAM) based radio communication an unexploited area?," *IEEE Trans. Antennas Propag.*, vol. 60, no. 2, pp. 1126–1131, Feb. 2012.

[5] M. Tamagone, C. Craeye, and J. Perruisseau-Carrier, "Comment on 'Encoding many channels on the same frequency through radio vorticity: First experimental test'," *New J. Phys.*, vol. 14, p. 118001, Nov. 2012.

[6] F. Tamburini *et al.*, "Reply to Comment on 'Encoding many channels on the same frequency through radio vorticity: First experimental test'," *New J. Phys.*, vol. 14, p. 118002, Nov. 2012

[7] M. Tamagone, C. Craeye, and J. Perruisseau-Carrier, "Comment on Reply to Comments on 'Encoding many channels on the same frequency through radio vorticity: First experimental test'," *New J. Phys.*, vol. 15, p. 118001, Jul. 2013

[8] R. E. Colin and F. J. Zucker, *Antenna Theory*. New York, NY, USA: McGraw-Hill, 1969

[9] S. V. Savov, "A taylor series expansion method for calculation of the radiation integral for a circular aperture," *IEEE Trans. Antennas Propag.*, vol. 51, no. 12, pp. 3298–3302, Dec. 2003.

[10] Y. Rahmat-Samii and V. Galindo-Israel, "Shaped reflector antenna analysis using the jacobi-bessel series," *IEEE Trans. Antennas Propagat.*, vol. 28, no. 4, pp. 425–435, Jul. 1980.

[11] F. Zernike, "Beugungstheorie des schneidenverfahrens," *Physica*, vol. 1, p. 689, 1934.

[12] M. Born and E. Wolf, *Principles of Optics*. New York, NY, USA: Pergamon, p. 808, 1975.

[13] E. de Lera Acedo *et al.*, "Low-order beam models for the calibration of large aperture arrays for radio astronomy; the case of the SKA-low instrument," in *Proc. IEEE Int. Conf. Electromagn. Adv. Appl. (ICEAA)*, Torino, Italy, Sep. 2012 pp. 1182–1185.

[14] D. M. Pozar, "Input impedance and mutual coupling of rectangular microstrip antennas," *IEEE Trans. Antennas Propag.*, vol. 30, no. 6, pp. 1191–1196, Jun. 1982.

[15] H. Weyl, "Ausbreitung elektromagnetische wellen uber einem ebenen leiter," *Ann. Phys.*, vol. 365, pp. 481–500, 1919.

[16] J. Braat *et al.*, "Assessment of optical systems by means of point-spread functions," *Prog. Opt.*, vol. 51, pp. 349–468, Jan. 2008.

[17] Y. Luo and B. Lu, "Far-field properties in two off-axis superimposed Laguerre-Gaussian beams beyond the paraxial approximation," *J. Opt. Soc. Amer. A*, vol. 27, pp. 238–244, Feb. 2010.

[18] D. K. Nguyen *et al.*, "Discussion about the link budget for electromagnetic wave with orbital angular momentum," in *Proc. 8th Eur. Conf. Antennas Propag.*, The Hague, The Netherlands, Apr. 2014 pp. 1117–1121.

[19] A. D. Yaghjian, "Efficient computation of antenna coupling and fields within the near-field region," *IEEE Trans. Antennas Propag.*, vol. 30, no. 1, pp. 113–128, Jan. 1982.

[20] E. Martini *et al.*, "Harmonics-based inhomogeneous plane-wave method (HIPW)," *IEEE Trans. Antennas Propag.*, vol. 65, no. 5, pp. 2331–2336, May 2015.

[21] P.-S. Kildal, *Foundations of Antenna Engineering*. Gothenburg, Sweden: Kildal Antenna AB, p. 419, Apr. 2015.