

COAREA FORMULAE AND CHAIN RULES FOR THE JACOBIAN DETERMINANT IN FRACTIONAL SOBOLEV SPACES

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ABSTRACT. We prove weak and strong versions of the coarea formula and the chain rule for distributional Jacobian determinants Ju for functions u in fractional Sobolev spaces $W^{s,p}(\Omega)$, where Ω is a bounded domain in $\mathbb{R}^n$ with smooth boundary. The weak forms of the formulae are proved for the range $sp > n - 1$, $s > \frac{n-1}{n}$, while the strong versions are proved for the range $sp \geq n$, $s \geq \frac{n}{n+1}$. We also provide a chain rule for distributional Jacobian determinants of Hölder functions and point out its relation to two open problems in geometric analysis.

1. SETTING AND STATEMENT OF MAIN RESULT

Let $n \geq 2$, and $\Omega \subset \mathbb{R}^n$ bounded and open with smooth boundary. The fractional Sobolev space $W^{s,p}(\Omega)$ with $s \in (0, 1)$, $p \in [1, \infty)$ is defined as the set of functions $u \in L^p(\Omega)$ such that

$$[u]_{W^{s,p}}^p := \int_{\Omega \times \Omega} \frac{|u(x) - u(y)|^p}{|x - y|^{n+sp}} dx dy < \infty.$$

The norm on $W^{s,p}(\Omega)$ is defined by $\|u\|_{W^{s,p}} = \|u\|_{L^p} + [u]_{W^{s,p}}$. For vector-valued functions $W^{s,p}(\Omega; \mathbb{R}^m)$ is defined componentwise.

Here we are concerned with the *distributional Jacobian* Ju for $u \in W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$. For smooth u , this may be defined as follows: We set

$$\mathbf{j}u(x) := \frac{1}{n} (\text{cof } \nabla u(x))^T u(x),$$

where $\text{cof } \nabla u(x) \in \mathbb{R}^{n \times n}$ denotes the cofactor matrix of $\nabla u(x)$. Then we see that $\det \nabla u = \text{div } \mathbf{j}u$, so that we have

$$\begin{aligned} \langle Ju, \psi \rangle &:= \int_{\Omega} \det \nabla u(x) \psi(x) dx \\ &= \int_{\Omega} \text{div } \mathbf{j}u(x) \psi(x) dx \\ &= - \int_{\Omega} \mathbf{j}u(x) \cdot \nabla \psi(x) dx \end{aligned}$$

for $\psi \in C_c^1(\Omega)$. For $u \in L^\infty \cap W^{1,n-1}(\Omega; \mathbb{R}^n)$, we may obviously define $\mathbf{j}u$ as an object in $L^1(\Omega; \mathbb{R}^n)$, and Ju is defined distributionally via $Ju = \text{div } \mathbf{j}u$. In this paper, we follow [BN11] to obtain a well-defined notion of $\mathbf{j}u \in L^1(\Omega; \mathbb{R}^n)$ for u in $W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$ through multilinear interpolation, see Section 2. The thus defined distribution $Ju = \text{div } \mathbf{j}u$ is our object of investigation; we are going to derive coarea type formulae and chain rules for it, supposing u belongs to a certain family of subspaces of $W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$.

To be more precise, let $u^a(x) := \frac{u(x)-a}{|u(x)-a|}$ for $a \in \mathbb{R}^n$. We will prove the following theorems:

Theorem 1 (Weak coarea formula and chain rule). *Let $s \in (\frac{n-1}{n}, 1)$, $sp \in (n-1, \infty]$, and $u \in W^{s,p}(\Omega; \mathbb{R}^n)$. Then for almost every $a \in \mathbb{R}^n$, we have $u^a \in W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$ and the following two statements hold:*

(i) (Weak coarea formula) *For every $\psi \in C_c^1(\Omega)$, we have*

$$\langle Ju, \psi \rangle = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \langle Ju^a, \psi \rangle da.$$

(ii) (Weak chain rule) *For every $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$, and every $\psi \in C_c^1(\Omega)$, we have*

$$\langle J(F \circ u), \psi \rangle = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \det \nabla F(a) \langle Ju^a, \psi \rangle da.$$

Here $\omega_n = \mathcal{L}^n(B(0,1))$ is the volume of the n -dimensional unit ball.

Remark 1. The validity of the weak coarea formula and chain rule for the critical case $u \in W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$ is not treated here and remains an open question.

In our second theorem we are going to use the following notation: For a distribution $T : C_c^1(\Omega) \rightarrow \mathbb{R}$, we denote its total variation by

$$|T|_{TV} := \sup\{\langle T, \psi \rangle : \psi \in C_c^1(\Omega), |\psi| \leq 1\}.$$

By the Riesz-Radon representation theorem, T can be extended to a Radon measure on Ω if $|T|_{TV} < \infty$.

Theorem 2 (Strong coarea formula and chain rule). *Let $u \in C^0 \cap W^{\frac{n}{n+1},n+1}(\Omega; \mathbb{R}^n)$. Assume that $|Ju|_{TV} < \infty$. Then the following two statements hold:*

(i) (Strong coarea formula.) *For almost every $a \in \mathbb{R}^n$, Ju^a can be extended to a Radon measure and*

$$|Ju|_{TV} = \frac{1}{\omega_n} \int_{\mathbb{R}^n} |Ju^a|_{TV} da.$$

(ii) (Strong chain rule.) *Let $F \in C^\infty(\mathbb{R}^n; \mathbb{R}^n)$ be globally Lipschitz. Then $|J(F \circ u)|_{TV} < \infty$, and for every $\psi \in C_c^1(\Omega)$, we have that*

$$\langle J(F \circ u), \psi \rangle = \int_{\Omega} \det \nabla F(u(x)) \psi(x) dJu(x).$$

Weak and strong versions of the coarea formulae and chain rule for the Jacobians of (non-fractional) Sobolev functions have been treated before in [JS02, DL03].

Finally, in Section 5 below, we want to point out the relevance of chain rules for two well-known open problems in geometric analysis: the $C^{1,\alpha}$ isometric immersion problem and the Hölder embedding problem for the Heisenberg group. In order to avoid any misunderstanding, let us state clearly that we do not have any new results on these problems, but only offer a new vantage point. In order to do so, we formulate a chain rule for Hölder functions u , which allows for a slightly larger domain of Ju . In particular, we may define it on characteristic functions χ_E of sets of finite perimeter E such that $\overline{E} \subset \Omega$.

Theorem 3. *Let $\alpha \in (\frac{n-1}{n}, 1)$, $u \in C^{0,\alpha}(\Omega; \mathbb{R}^n)$, $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ globally Lipschitz and E as above. Then*

$$\langle J(F \circ u), \chi_E \rangle = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \det \nabla F(a) \langle Ju^a, \chi_E \rangle da.$$

Plan of the paper. In Section 2, we prove the estimate that defines the distributional Jacobian Ju for $u \in W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$. Theorem 2 and 3 will be fairly straightforward consequences of these estimates, and they will be proved in Sections 4 and 5.1 respectively. The proof of the weak chain rule (Theorem 1) is the most interesting, and will be carried out first, in Section 3. In Section 5.2, we will explain the relation between chain rules and the two open problems in geometric analysis mentioned above.

Some notation. The symbol D denotes the distributional derivative, while $\nabla = (\partial_1, \dots, \partial_n)$ is the approximate gradient. The function spaces $W^{s,p}(\Omega; \mathbb{R}^n)$ will be abbreviated by $W^{s,p}$ when it is clear from the context what is meant. The Hölder spaces $C^{0,\alpha}(\Omega; \mathbb{R}^m)$ are defined through the norm

$$\|u\|_{C^{0,\alpha}} := \|u\|_{C^0} + \sup_{x \neq y} \frac{|u(x) - u(y)|}{|x - y|^\alpha},$$

and $C^{0,\alpha}(\Omega; \mathbb{R}^m) := \{u \in C^0(\Omega; \mathbb{R}^m) : \|u\|_{C^{0,\alpha}} < \infty\}$. The space $BV(\Omega)$ is the set of all functions $u \in L^1(\Omega)$ such that

$$\sup \left\{ \int_{\Omega} u(x) \operatorname{div} \psi(x) dx : \psi \in C_c^1(\Omega), |\psi| \leq 1 \right\} < \infty.$$

For $u \in BV(\Omega)$, the distributional derivative Du is understood as a matrix-valued Radon measure. The symbol “ C ” is used as follows: A statement such as $f \leq C(\alpha)g$ is to be read as “there exists a constant $C > 0$ that depends only on α such that $f \leq Cg$ ”. For $f \leq Cg$, we also write $f \lesssim g$. The k -dimensional Hausdorff measure is denoted by $\mathcal{H}^k$. The Brouwer degree of $u \in C^0(\Omega; \mathbb{R}^n)$ in $a \in \mathbb{R}^n \setminus u(\partial\Omega)$ is denoted by $\deg(u, \Omega, a)$.

2. ESTIMATE DEFINING THE DISTRIBUTIONAL JACOBIAN

We define the distributional Jacobian Ju as in [BN11]. This is equivalent to a definition through multilinear interpolation by the method of trace spaces (see e.g. Chapter 28 of [Tar07]).

For smooth functions $\mathbf{j}u = \frac{1}{n}(\operatorname{cof} \nabla u)^T u$ has been defined above, and $Ju = \operatorname{div} \mathbf{j}u$. The following identity will be useful in the sequel: For $u \in C^1(\Omega; \mathbb{R}^n)$, $\psi \in C^1(\Omega)$, we have

$$\sum_{j=1}^n (\operatorname{cof} \nabla u)_{ij} \partial_{x_j} \psi = \det(\nabla u_1, \dots, \nabla u_{i-1}, \nabla \psi, \nabla u_{i+1}, \dots, \nabla u_n). \quad (1)$$

The following proposition is a statement of three estimates for $\mathbf{j}u$, the first of which can be found as Lemma 4 in [BN11]; we use the same method of proof.

Proposition 1. *Let $\Omega \subset \mathbb{R}^n$ be open bounded with Lipschitz boundary, and $\alpha \in (\frac{n-1}{n}, 1)$. Then for all $u, v \in C^1(\Omega; \mathbb{R}^n)$ and every $\psi \in C_c^1(\Omega)$, we have that*

$$\begin{aligned} & \left| \langle Ju - Jv, \psi \rangle \right| \\ & \lesssim \min \left(\|u - v\|_{W^{\frac{n-1}{n}, n}} \left(\|u\|_{W^{\frac{n-1}{n}, n}} + \|v\|_{W^{\frac{n-1}{n}, n}} \right)^{n-1} \|\nabla \psi\|_{L^\infty}, \right. \\ & \quad \|u - v\|_{W^{\frac{n}{n+1}, n+1}} \left(\|u\|_{W^{\frac{n}{n+1}, n+1}} + \|v\|_{W^{\frac{n}{n+1}, n+1}} \right)^{n-1} \|\psi\|_{W^{\frac{n}{n+1}, n+1}}, \\ & \quad \left. \|u - v\|_{C^{0, \alpha}} (\|u\|_{C^{0, \alpha}} + \|v\|_{C^{0, \alpha}})^{n-1} \|\nabla \psi\|_{L^1} \right). \end{aligned}$$

Proof. Let $\tilde{u}, \tilde{v}$ be extensions of u, v to $\mathbb{R}^n$ such that

$$\begin{aligned} \|\tilde{u}\|_{W^{\frac{n-1}{n}, n}(\mathbb{R}^n; \mathbb{R}^n)} & \lesssim \|u\|_{W^{\frac{n-1}{n}, n}(\Omega; \mathbb{R}^n)} \\ \|\tilde{u}\|_{W^{\frac{n}{n+1}, n+1}(\mathbb{R}^n; \mathbb{R}^n)} & \lesssim \|u\|_{W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)} \\ \|\tilde{u}\|_{C^{0, \alpha}(\mathbb{R}^n; \mathbb{R}^n)} & \lesssim \|u\|_{C^{0, \alpha}(\Omega; \mathbb{R}^n)}, \end{aligned}$$

with analogous estimates for $\tilde{v}$. Let $\eta \in C_c^\infty(\mathbb{R}^n)$ be a standard mollifier, i.e., $\int_{\mathbb{R}^n} \eta(x) dx = 1$, and set $\eta_t := t^{-n} \eta(\cdot/t)$. We may define extensions $U, V : \Omega \times [0, 1) \rightarrow \mathbb{R}^n$ of u, v by setting

$$U(x, t) := (\eta_t * \tilde{u})(x),$$

again with an analogous definition for V . We write $\tilde{\nabla} = (\nabla, \partial_t)$. By well known trace estimates (see e.g. [Lun18, Lio63]), the definition above implies

$$\begin{aligned} \|U\|_{W^{1, n}(\Omega \times [0, 1); \mathbb{R}^n)} & \lesssim \|u\|_{W^{\frac{n-1}{n}, n}(\Omega; \mathbb{R}^n)} \\ \|U\|_{W^{1, n+1}(\Omega \times [0, 1); \mathbb{R}^n)} & \lesssim \|u\|_{W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)} \\ \|\tilde{\nabla} U(\cdot, t)\|_{L^\infty(\Omega; \mathbb{R}^n)} & \lesssim t^{\alpha-1} \|u\|_{C^{0, \alpha}(\Omega; \mathbb{R}^n)}, \end{aligned} \tag{2}$$

with analogous estimates for V . Furthermore we may extend ψ to a function $\Psi \in C_c^1(\Omega \times [0, 1))$ such that

$$\begin{aligned} \|\Psi\|_{C^1(\Omega \times [0, 1))} & \lesssim \|\psi\|_{C^1(\Omega)} \\ \|\Psi\|_{W^{1, n+1}(\Omega \times [0, 1))} & \lesssim \|\psi\|_{W^{\frac{n}{n+1}, n+1}(\Omega)} \\ \sup_{t \in (0, 1)} \|\tilde{\nabla} \Psi(\cdot, t)\|_{L^1(\Omega)} & \lesssim \|\nabla \psi\|_{L^1(\Omega)}. \end{aligned} \tag{3}$$

Now we have, writing $JU = J(U(\cdot, t))$, that

$$\begin{aligned} \langle Ju - Jv, \psi \rangle & = - \int_0^1 \partial_t \langle JU - JV, \Psi \rangle dt \\ & = \int_0^1 \int_\Omega \partial_t ((\mathbf{j}U - \mathbf{j}V) \cdot \nabla \Psi) dx dt \end{aligned} \tag{4}$$

We will now rewrite the expression $\partial_t ((\mathbf{j}U - \mathbf{j}V) \cdot \nabla \Psi) dx$ using differential forms, and claim that it can be written as a sum of exact forms plus terms that can each be written as a product of a one-homogeneous function in $\tilde{\nabla} U - \tilde{\nabla} V$ with an n -homogeneous function

in $(\tilde{\nabla}U, \tilde{\nabla}V, \tilde{\nabla}\Psi)$. Indeed, with $U = (U_1, \dots, U_n)$ and $dU_i = \sum_{j=1}^n \partial_j U_i dx_j$ we have by (1) that

$$\begin{aligned} \mathbf{j}U \cdot \nabla \Psi dx_1 \wedge \dots \wedge dx_n &= \frac{1}{n} \sum_{i=1}^n U_i dU_1 \wedge \dots \wedge dU_{i-1} \wedge d\Psi \wedge dU_{i+1} \wedge \dots \wedge dU_n \\ &=: \frac{1}{n} \sum_{i=1}^n F_i(U, \Psi). \end{aligned}$$

By the product rule, and denoting derivatives with respect to t by a prime, we get

$$\begin{aligned} \partial_t F_i(U, \Psi) &= U'_i dU_1 \wedge \dots \wedge dU_{i-1} \wedge d\Psi \wedge dU_{i+1} \wedge \dots \wedge dU_n \\ &\quad + \sum_{j \neq i} G_{ij}(U, \Psi) \\ &\quad + U_i dU_1 \wedge \dots \wedge dU_{i-1} \wedge d\Psi' \wedge dU_{i+1} \wedge \dots \wedge dU_n, \end{aligned} \tag{5}$$

where

$$G_{ij}(U, \Psi) = U_i dU_1 \wedge \dots \wedge dU'_j \wedge \dots \wedge dU_{i-1} \wedge d\Psi \wedge dU_{i+1} \wedge \dots \wedge dU_n.$$

We treat the terms G_{ij} with an integration by parts,

$$\begin{aligned} G_{ij}(U, \Psi) &= (-1)^{j-1} d(U_i U'_j dU_1 \wedge \dots \wedge dU_{j-1} \wedge dU_{j+1} \wedge \dots \wedge dU_{i-1} \wedge d\Psi \wedge dU_{i+1} \wedge \dots \wedge dU_n) \\ &\quad + U'_j dU_1 \wedge \dots \wedge dU_{j-1} \wedge d\Psi \wedge dU_{j+1} \wedge \dots \wedge dU_n. \end{aligned}$$

In the same way,

$$\begin{aligned} U_i dU_1 \wedge \dots \wedge dU_{i-1} \wedge dU_{i+1} \wedge \dots \wedge dU_n \wedge d\Psi' &= (-1)^{i-1} d(U_i \Psi' dU_1 \wedge \dots \wedge dU_{i-1} \wedge dU_{i+1} \wedge \dots \wedge dU_n) \\ &\quad + \Psi' dU_1 \wedge \dots \wedge dU_n. \end{aligned}$$

Inserting these last two equations in (5), we obtain

$$\begin{aligned} \partial_t \mathbf{j}U \cdot \nabla \Psi dx &= \sum_{i=1}^n U'_i dU_1 \wedge \dots \wedge dU_{i-1} \wedge d\Psi \wedge dU_{i+1} \wedge \dots \wedge dU_n \\ &\quad + \Psi' dU_1 \wedge \dots \wedge dU_n \\ &\quad + dH(U, \Psi), \end{aligned}$$

where $dH(U, \Psi)$ is the sum of all the exact forms that appeared in the preceding calculations.

We turn to the computation of $\partial_t(\mathbf{j}U - \mathbf{j}V) \cdot \nabla \Psi$. We have that

$$\begin{aligned} &U'_i dU_1 \wedge \dots \wedge dU_{i-1} \wedge d\Psi \wedge dU_{i+1} \wedge \dots \wedge dU_n \\ &\quad - V'_i dV_1 \wedge \dots \wedge dV_{i-1} \wedge d\Psi \wedge dV_{i+1} \wedge \dots \wedge dV_n \\ &= (U'_i - V'_i) dU_1 \wedge \dots \wedge dU_{i-1} \wedge d\Psi \wedge dU_{i+1} \wedge \dots \wedge dU_n \\ &\quad + \sum_{j \neq i} V'_i dV_1 \wedge \dots \wedge dV_{j-1} \wedge (dU_j - dV_j) \wedge \\ &\quad \quad \quad \wedge dU_{j+1} \wedge \dots \wedge dU_{i-1} \wedge d\Psi \wedge dU_{i+1} \wedge \dots \wedge dU_n \end{aligned}$$

and similarly

$$\begin{aligned} \Psi' dU_1 \wedge \cdots \wedge dU_n - \Psi' dV_1 \wedge \cdots \wedge dV_n \\ = \sum_{i=1}^n \Psi' dV_1 \wedge \cdots \wedge dV_{i-1} \wedge (dU_i - dV_i) \wedge dU_{i+1} \wedge \cdots \wedge dU_n. \end{aligned}$$

This proves the claim about the structure of $\partial_t(\mathbf{j}U - \mathbf{j}V) \cdot \nabla \Psi dx$ that we have made above, and we may summarize by saying that it can be written as a sum of exact forms plus terms that can be estimated by

$$C|\tilde{\nabla}U - \tilde{\nabla}V|(|\tilde{\nabla}U| + |\tilde{\nabla}V|)^{n-1}|\tilde{\nabla}\Psi|.$$

Inserting this in (4), we obtain the following three estimates by Hölder's inequality:

$$\begin{aligned} |\langle Ju - Jv, \psi \rangle| \\ \lesssim \min \left(\|\tilde{\nabla}\Psi\|_{L^\infty} \|\tilde{\nabla}U - \tilde{\nabla}V\|_{L^n} \left(\|\tilde{\nabla}U\|_{L^n} + \|\tilde{\nabla}V\|_{L^n} \right)^{n-1}, \right. \\ \left. \|\tilde{\nabla}\Psi\|_{L^{n+1}} \|\tilde{\nabla}U - \tilde{\nabla}V\|_{L^{n+1}} \left(\|\tilde{\nabla}U\|_{L^{n+1}} + \|\tilde{\nabla}V\|_{L^{n+1}} \right)^{n-1}, \right. \\ \left. \int_0^1 \|\tilde{\nabla}\Psi(\cdot, t)\|_{L^1} \|\tilde{\nabla}U(\cdot, t) - \tilde{\nabla}V(\cdot, t)\|_{L^\infty} \left(\|\tilde{\nabla}U(\cdot, t)\|_{L^\infty} + \|\tilde{\nabla}V(\cdot, t)\|_{L^\infty} \right)^{n-1} dt \right) \end{aligned}$$

Using (2), (3) this becomes

$$\begin{aligned} |\langle Ju - Jv, \psi \rangle| \lesssim \min \left(\|\psi\|_{C^1} \|u - v\|_{W^{\frac{n-1}{n}, n}} \left(\|u\|_{W^{\frac{n-1}{n}, n}} + \|v\|_{W^{\frac{n-1}{n}, n}} \right)^{n-1}, \right. \\ \left. \|\psi\|_{W^{\frac{n}{n+1}, n+1}} \|u - v\|_{W^{\frac{n}{n+1}, n+1}} \left(\|u\|_{W^{\frac{n}{n+1}, n+1}} + \|v\|_{W^{\frac{n}{n+1}, n+1}} \right)^{n-1}, \right. \\ \left. \int_0^1 t^{n(\alpha-1)} dt \|\nabla\psi\|_{L^1} \|u - v\|_{C^{0, \alpha}} (\|u\|_{C^{0, \alpha}} + \|v\|_{C^{0, \alpha}})^{n-1} \right). \end{aligned}$$

By the assumption that $n(\alpha - 1) > -1$, the integral in the last line is finite, proving the lemma. $\square$

By the proposition, the following definition of Ju , which gives rigorous meaning to the statement of the theorems of the preceding section, is well-defined.

Definition 1. Let $u \in W^{\frac{n-1}{n}, n}(\Omega; \mathbb{R}^n)$. The distributional Jacobian $Ju : C_c^1(\Omega) \rightarrow \mathbb{R}$ is defined by

$$\langle Ju, \psi \rangle := \lim_{k \rightarrow \infty} \langle Ju_k, \psi \rangle$$

where u_k is any sequence in $C^1(\Omega; \mathbb{R}^n)$ that approximates u in $W^{\frac{n-1}{n}, n}(\Omega; \mathbb{R}^n)$. For $u \in C^{0, \alpha}(\Omega; \mathbb{R}^n)$, Ju may be extended to an element of the dual of $BV(\Omega)$ by

$$\langle Ju, \psi \rangle := \lim_{k \rightarrow \infty} \langle Ju_k, \psi_k \rangle,$$

where u_k is a sequence in $C^1(\Omega; \mathbb{R}^n)$ such that $u_k \rightarrow u$ in $C^{0, \alpha-\delta}$ for some $\delta > 0$ with $\alpha - \delta > \frac{n-1}{n}$, and ψ_k is a sequence in $C_c^1(\Omega)$ with $\psi_k \rightarrow \psi$ weakly-* in $BV(\Omega)$, i.e.,

$$\psi_k \rightarrow \psi \text{ in } L^1(\Omega)$$

$$\int_{\Omega} \varphi(x) |\nabla \psi_k|(x) dx \rightarrow \int_{\Omega} \varphi(x) d|D\psi|(x) \text{ for any } \varphi \in C_0(\Omega),$$

where $C_0(\Omega)$ denotes the closure of $C_c^0(\Omega)$ with respect to the C^0 -norm.

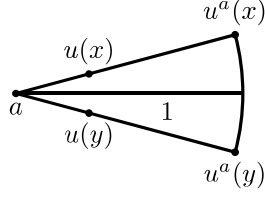


FIGURE 1. For $a \in \mathbb{R}^n$, $u^a : \Omega \rightarrow \mathbb{R}^n$ is the stereographic projection of u onto the unit sphere. If u is smooth and a is a regular value of u , then $Ju^a = \omega_n \sum_{x \in u^{-1}(a)} \text{sgn det } \nabla u(x) \delta_x$, see the appendix.

3. PROOF OF THE WEAK CHAIN RULE

Lemma 1. *Let $s \in (\frac{n-1}{n}, 1]$, $sp \in (n-1, \infty]$, and $u_k \rightarrow u$ in $W^{s,p}(\Omega; \mathbb{R}^n)$. Then the functions*

$$a \mapsto \|u_k^a - u^a\|_{W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)}^n$$

converge to 0 in $L^1_{\text{loc}}(\mathbb{R}^n)$.

Remark 2. Note that we require convergence in the strictly smaller space $W^{s,p}(\Omega; \mathbb{R}^n)$ to get convergence of almost every u_k^a to u^a in $W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$, at least for a subsequence.

Proof. Pick any subsequence of u_k . Without relabeling, choose a subsequence such that $u_k \rightarrow u$ almost everywhere. Then, noting that $|u^a(x)| = 1$, by the dominated convergence theorem,

$$\int_{B(0,R)} \|u_k^a - u^a\|_{L^n(\Omega; \mathbb{R}^n)}^n dx \rightarrow 0.$$

For the nonlocal part of the $W^{\frac{n-1}{n},n}$ -norm, we use the estimate

$$|u^a(x) - u^a(y)| \leq \min \left(\frac{|u(x) - u(y)|}{\min(|u(x) - a|, |u(y) - a|)}, 2 \right)$$

in the triple integral

$$\begin{aligned} & \int_{B(0,R)} [u_k^a - u^a]_{W^{\frac{n-1}{n},n}}^n da \\ &= \int_{B(0,R)} \int_{\Omega} \int_{\Omega} \frac{|u_k^a(x) - u_k^a(y) - u^a(x) + u^a(y)|^n}{|x - y|^{2n-1}} dx dy da \\ &= \underbrace{\int_{B(0,R)} \int_{|x-y|>\delta} \int_{\Omega} \frac{|u_k^a(x) - u_k^a(y) - u^a(x) + u^a(y)|^n}{|x - y|^{2n-1}} dx dy da}_{I_k} \\ & \quad + \underbrace{\int_{B(0,R)} \int_{|x-y|\leq\delta} \int_{\Omega} \frac{|u_k^a(x) - u_k^a(y) - u^a(x) + u^a(y)|^n}{|x - y|^{2n-1}} dx dy da}_{II_k}, \end{aligned}$$

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where $\delta > 0$ is independent of k . Again by dominated convergence, we have $I_k \rightarrow 0$. For II_k , we write

$$\begin{aligned} II_k &\lesssim \int_{B(0,R)} \int \int_{|x-y| \leq \delta} \frac{|u_k^a(x) - u_k^a(y)|^n + |u^a(x) - u^a(y)|^n}{|x-y|^{2n-1}} dx dy da \\ &\lesssim \underbrace{\int \int_{|x-y| \leq \delta} \frac{1}{|x-y|^{2n-1}} \int_{B(0,R)} \min \left(\frac{|u(x) - u(y)|^n}{\min(|u(x) - a|^n, |u(y) - a|^n)}, 2 \right) da dx dy}_{III} \\ &\quad + \underbrace{\int \int_{|x-y| \leq \delta} \frac{1}{|x-y|^{2n-1}} \int_{B(0,R)} \min \left(\frac{|u_k(x) - u_k(y)|^n}{\min(|u_k(x) - a|^n, |u_k(y) - a|^n)}, 2 \right) da dx dy}_{IV_k} \end{aligned}$$

We note that III is independent of k , and the interior integral can be estimated by

$$\begin{aligned} &\int_{B(0,R)} \min \left(\frac{|u(x) - u(y)|^n}{\min(|u(x) - a|^n, |u(y) - a|^n)}, 2 \right) da \\ &\lesssim |u(x) - u(y)|^n (|\log |u(x) - u(y)|| + \log R) \\ &\leq C(\varepsilon)(1 + \log R) \max(|u(x) - u(y)|^{n-\varepsilon}, |u(x) - u(y)|^{n+\varepsilon}) \end{aligned}$$

for $\varepsilon > 0$ chosen small enough such that

$$\begin{aligned} \frac{n-1}{n-\varepsilon} &< s \\ s - \frac{n}{p} &> -\frac{1}{n+\varepsilon}. \end{aligned}$$

This choice of ε implies

$$W^{s,p}(\Omega; \mathbb{R}^n) \subset W^{\frac{n-1}{n+\varepsilon}, n+\varepsilon} \cap W^{\frac{n-1}{n-\varepsilon}, n-\varepsilon}(\Omega; \mathbb{R}^n)$$

by standard embeddings for fractional Sobolev spaces (see e.g. [Tri06], Theorem 3.3.1). Thus we get

$$III \leq C(s, p, \Omega, R) \int \int_{|x-y| \leq \delta} \frac{\max(|u(x) - u(y)|^{n+\varepsilon}, |u(x) - u(y)|^{n-\varepsilon})}{|x-y|^{2n-1}} dx dy.$$

As $u \in W^{\frac{n-1}{n+\varepsilon}, n+\varepsilon} \cap W^{\frac{n-1}{n-\varepsilon}, n-\varepsilon}$, we have $III \rightarrow 0$ as $\delta \rightarrow 0$.

For IV_k , we arrive at the same estimate

$$IV_k \leq C(s, p, \Omega, R) \int \int_{|x-y| \leq \delta} \frac{\max(|u_k(x) - u_k(y)|^{n+\varepsilon}, |u_k(x) - u_k(y)|^{n-\varepsilon})}{|x-y|^{2n-1}} dx dy.$$

Since $u_k \rightarrow u$ strongly in $W^{\frac{n-1}{n+\varepsilon}, n+\varepsilon} \cap W^{\frac{n-1}{n-\varepsilon}, n-\varepsilon}$, the integrand is compact in L^1 and thus uniformly integrable, and we have $\sup_k IV_k \rightarrow 0$ as $\delta \rightarrow 0$. $\square$

Definition 2. For a distribution $T \in \mathcal{D}'(\Omega)$, we define

$$\|T\|_{W^{-1,1}(\Omega)} = \sup\{\langle T, \psi \rangle : \psi \in C_c^\infty(\Omega), \text{Lip } \psi \leq 1\},$$

where $\text{Lip } \psi$ denotes the Lipschitz constant of ψ , and we set $W^{-1,1}(\Omega) = \{T \in \mathcal{D}'(\Omega) : \|T\|_{W^{-1,1}(\Omega)} < \infty\}$.

In the following lemma, we will consider the extension of a function $u \in W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$ to $\Omega \times [0, 1)$ that we already used in the proof of Proposition 1: Let $\tilde{u} \in W^{\frac{n-1}{n},n}(\mathbb{R}^n; \mathbb{R}^n)$ be an extension of u to $\mathbb{R}^n$ such that

$$\|\tilde{u}\|_{W^{\frac{n-1}{n},n}(\mathbb{R}^n; \mathbb{R}^n)} \lesssim \|u\|_{W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)}$$

and define $U \in W^{1,n}(\Omega \times [0, 1); \mathbb{R}^n)$ by

$$U(x, t) = \eta_t * \tilde{u}(x).$$

The following Lemma then shows a certain continuity of the Ju^a under mollification. A similar statement can be found in [HL00, Theorem 1.1].

Lemma 2. *Let $\Omega \subseteq \mathbb{R}^n$ be open, with Lipschitz boundary. Let $u \in W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$, and U the extension to $\Omega \times [0, 1)$ defined above. Let $t_k \downarrow 0$ and take $u_k = U(\cdot, t_k) \in W^{\frac{n-1}{n},n} \cap C^\infty(\Omega; \mathbb{R}^n)$. Then there is a family of distributions $(T^a)_{a \in \mathbb{R}^n} \subset W^{-1,1}(\Omega)$ such that*

$$\int_{\mathbb{R}^n} \|Ju_k^a - T^a\|_{W^{-1,1}(\Omega)} da \rightarrow 0.$$

Remark 3. Combining Lemma 1 and 2, we obtain that $T^a = Ju^a$ for almost every a if $u \in W^{s,p}(\Omega; \mathbb{R}^n)$, $sp > n - 1$, $s > \frac{n-1}{n}$.

Proof of Lemma 2. By the coarea formula for U (see [AFP00, Theorem 2.93]) we have

$$\int_{\mathbb{R}^n} \mathcal{H}^1(U^{-1}(a)) da = \int_{\Omega \times [0,1)} |JU| dx \leq \|U\|_{W^{1,n}(\Omega \times [0,1); \mathbb{R}^n)}^n. \quad (6)$$

In addition, because U is smooth, almost all $a \in \mathbb{R}^n$ are regular values of U , so that $U^{-1}(a) \cap (\Omega \times (0, 1))$ is a 1-manifold, i.e. a countable union of curves that are either closed or terminate on either $\Omega \times \{0\}$, on $\Omega \times \{1\}$, or on $\partial\Omega \times [0, 1]$.

Now $u_k = U(\cdot, t_k) \in W^{\frac{n-1}{n},n} \cap C^\infty(\Omega; \mathbb{R}^n)$. We note that almost every $a \in \mathbb{R}^n$ is a regular value of both U and of all u_k .

For such an a , $Ju_k^a = \omega_n \sum_{x \in u_k^{-1}(a)} \sigma(x) \delta_x$, with $\sigma(x) = \pm 1$, see the appendix. We can then estimate for $t_k \leq t_l$

$$\|Ju_k^a - Ju_l^a\|_{W^{-1,1}(\Omega)} \leq \omega_n \mathcal{H}^1(U^{-1}(a) \cap (\Omega \times (t_k, t_l))). \quad (7)$$

To see this, consider a point $x \in \text{supp } Ju_k^a$. Then there is a unique curve $c \subseteq U^{-1}(a)$ passing through (x, t_k) transversally. We follow this curve into the set $\Omega \times (t_k, t_l)$ until we intersect either

- (i) $\Omega \times \{t_k\}$
- (ii) or $\Omega \times \{t_l\}$
- (iii) or $\partial\Omega \times (0, 1)$,

whichever comes first. See Figure 2 for a graphical analogy. In case (i), call the intersection point (y, t_k) . Then $y \in \text{supp } Ju_k^a$ and $\sigma(y) = -\sigma(x)$. In case (ii), call the point (y, t_l) . Then $y \in \text{supp } Ju_l^a$ and $\sigma(y) = \sigma(x)$. In case (iii), the intersection point is (y, t) , with $y \in \partial\Omega$ and $t \in [t_k, t_l]$ (a unique intersection point exists because c has finite length). For a point $x \in \text{supp } Ju_l^a$ we can also find a corresponding second point (y, t_k) or (y, t_l) or (y, t) as above.

All the points that are covered by the first two cases occur in pairs (x_i, y_i) such that their contributions appear with opposite signs when computing $Ju_k^a - Ju_l^a$. We collect the

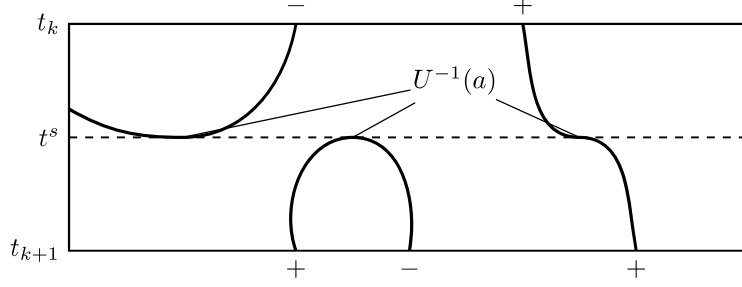


FIGURE 2. The level set $U^{-1}(a)$ for a regular value $a \in \mathbb{R}^n$. Note that a is a singular value of u_{t^s} only for a null set of $t^s \in (0, 1)$.

corresponding indices i in a set I' , and the set I'' serves as index set of the points occuring in case (iii) above. Thus we have for some $\sigma_i, \sigma_j \in \{\pm 1\}$

$$Ju_k^a - Ju_l^a = \omega_n \left(\sum_{i \in I'} \sigma_i (\delta(x_i) - \delta(y_i)) + \sum_{j \in I''} \sigma_j \delta(x_j) \right),$$

with $\sum_{i \in I'} |x_i - y_i| + \sum_{j \in I''} \text{dist}(x_j, \partial\Omega) \leq \mathcal{H}^1(U^{-1}(a) \cap (\Omega \times (t_k, t_l)))$. If $\psi \in C_c^1(\mathbb{R}^n)$ is a test function with $\text{Lip } \psi \leq 1$, we have

$$\begin{aligned} |\langle Ju_k^a - Ju_l^a, \psi \rangle| &\leq \omega_n \left(\sum_{i \in I'} |\psi(x_i) - \psi(y_i)| + \sum_{j \in I''} |\psi(x_j)| \right) \\ &\leq \omega_n \mathcal{H}^1(U^{-1}(a) \cap (\Omega \times (t_k, t_l))), \end{aligned}$$

which implies (7).

By (6) and (7), the maps $a \mapsto Ju_k^a$ form a Cauchy sequence in $L^1(\mathbb{R}^n; W^{-1,1}(\Omega))$. The statement follows from the completeness of that space. $\square$

Proof of Theorem 1. The weak coarea formula holds in $C^\infty(\Omega; \mathbb{R}^n)$ (see the appendix). For $u \in W^{s,p}(\Omega; \mathbb{R}^n)$ consider the extension $U \in W^{1,n} \cap C^\infty(\Omega \times [0, 1]; \mathbb{R}^n)$ defined above Lemma 2 and set $u_k = U(\cdot, t_k)$ for some sequence $t_k \downarrow 0$.

Now $u_k \rightarrow u$ in $W^{s,p}(\Omega; \mathbb{R}^n)$, and by Lemma 1 the maps $a \mapsto u_k^a$ converge to $a \mapsto u^a$ in $L_{\text{loc}}^1(\mathbb{R}^n; W^{\frac{n-1}{n}, n}(\Omega; \mathbb{R}^n))$. Thus $u_k^a \rightarrow u^a$ in $W^{\frac{n-1}{n}, n}(\Omega; \mathbb{R}^n)$ for almost every $a \in \mathbb{R}^n$ after extracting a subsequence. Fix a test function $\psi \in C_c^\infty(\mathbb{R}^n)$. Then

$$\langle Ju, \psi \rangle = \lim_{k \rightarrow \infty} \langle Ju_k, \psi \rangle = \lim_{k \rightarrow \infty} \frac{1}{\omega_n} \int_{\mathbb{R}^n} \langle Ju_k^a, \psi \rangle da.$$

By Proposition 1 and Lemma 1, we have $\langle Ju_k^a, \psi \rangle \rightarrow \langle Ju^a, \psi \rangle$ for almost every $a \in \mathbb{R}^n$. On the other hand, by Lemma 2, $\int_{\mathbb{R}^n} \|Ju_k^a - T^a\|_{W^{-1,1}} da \rightarrow 0$ for some family of distributions $(T^a)_{a \in \mathbb{R}^n}$. As we have already stated in Remark 3, it follows that $T^a = Ju^a$ for almost every a , and in particular

$$\lim_{k \rightarrow \infty} \frac{1}{\omega_n} \int_{\mathbb{R}^n} \langle Ju_k^a, \psi \rangle da = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \langle Ju^a, \psi \rangle da.$$

We show the weak chain rule in the same way. It clearly holds for all u_k . Since $F \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ is globally Lipschitz, $F \circ u_k \rightarrow F \circ u$ in $W^{s,p}(\Omega; \mathbb{R}^n)$, so that

$$\langle J(F \circ u), \psi \rangle = \lim_{k \rightarrow \infty} \langle J(F \circ u_k), \psi \rangle = \lim_{k \rightarrow \infty} \frac{1}{\omega_n} \int_{\mathbb{R}^n} \det \nabla F(a) \langle Ju_k^a, \psi \rangle da.$$

Once again,

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^n} \|Ju_k^a - Ju^a\|_{W^{-1,1}(\Omega)} da \rightarrow 0,$$

and since $\det \nabla F \in L^\infty(\mathbb{R}^n)$, we have

$$\lim_{k \rightarrow \infty} \frac{1}{\omega_n} \int_{\mathbb{R}^n} \det \nabla F(a) \langle Ju_k^a, \psi \rangle da = \frac{1}{\omega_n} \int_{\mathbb{R}^n} \det \nabla F(a) \langle Ju^a, \psi \rangle da.$$

□

4. PROOF OF THE STRONG CHAIN RULE

As a consequence of Proposition 1, if $u \in W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)$, we can extend Ju to a bounded linear functional on $W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)$:

Corollary 1. *For $u \in W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)$, we have that $Ju \in \left(W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)\right)^*$.*

From the corollary, we immediately obtain the proof of Theorem 2:

Proof of Theorem 2. For a smooth approximation $u_k \rightarrow u$ in $W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)$, we have

$$\det DF \circ u_k \rightarrow \det DF \circ u \quad \text{in } W^{\frac{n}{n+1}, n+1}(\Omega)$$

as well as $F \circ u_k \rightarrow F \circ u$ in $W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)$. (For our present purpose, it would be enough to have the latter convergence in $W^{\frac{n-1}{n}, n}(\Omega; \mathbb{R}^n)$.) Thus, for every $\psi \in C_c^1(\Omega)$, we have that

$$\begin{aligned} \langle J(F \circ u), \psi \rangle &= \lim_{k \rightarrow \infty} \langle J(F \circ u_k), \psi \rangle \\ &= \lim_{k \rightarrow \infty} \langle Ju_k, \psi(\det DF) \circ u_k \rangle \\ &= \langle Ju, \psi(\det DF) \circ u \rangle. \end{aligned} \tag{8}$$

In other words, we have $J(F \circ u) = Ju(\det DF) \circ u$ as Radon measures. This is just the claim (ii) of Theorem 2.

By Theorem 13 in [DL03], it follows that the strong coarea formula holds too. One needs to note that the assumptions of that Theorem include

$$u \in L^\infty \cap W^{1, n-1} \quad \text{and} \quad Ju \text{ is a Radon measure,}$$

but in the proof only the latter condition is used; it goes through for $u \in C^0 \cap W^{\frac{n}{n+1}, n+1}(\Omega; \mathbb{R}^n)$. Also, the requirement that (8) holds for every $F \in C_c^1(\mathbb{R}^n; \mathbb{R}^n)$ (as required in the statement of the quoted theorem) can be replaced by its validity for every $F \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, since the proof in [DL03] works by approximation, and that is possible even with the weaker requirement. □

5. CHAIN RULES FOR HÖLDER FUNCTIONS

5.1. Proof of the chain rule for Hölder functions. Let $\Omega \subset \mathbb{R}^n$ be as before, and $\alpha \in (\frac{n-1}{n}, 1)$. The Hölder space $C^{0,\alpha}(\Omega; \mathbb{R}^n)$ is a subset of $W^{\frac{n-1}{n},n}(\Omega; \mathbb{R}^n)$ (see [Tri06]), and thus the weak chain rule holds. Slightly more than that can be said in this case: We have already noted in Definition 1 that for $u \in C^{0,\alpha}$, Ju is an element of the dual of BV . Our aim is now to extend the weak chain rule to the case $\psi \in BV(\Omega)$. Since u^a is not a Hölder function, we cannot immediately make sense of the right hand side in that equation. However if we fix the test function to be the characteristic function of a Lipschitz set E , we have the following:

Lemma 3. *Let $u, v \in C^{0,\alpha}(\Omega; \mathbb{R}^n)$, and $a \in \mathbb{R}^n$. Then $\mathbf{j}u^a \in C^0(\Omega \setminus u^{-1}(a); \mathbb{R}^n)$ and $\mathbf{j}v^a \in C^0(\Omega \setminus v^{-1}(a); \mathbb{R}^n)$, and for every Lipschitz set E with $\bar{E} \subset \Omega$ and $a \notin u(\partial E) \cup v(\partial E)$, we have that*

$$\langle Ju^a - Jv^a, \chi_E \rangle \leq C (\|u\|_{C^{0,\alpha}} + \|v\|_{C^{0,\alpha}})^{n-1} \|u - v\|_{C^{0,\alpha}} \text{Per}(E),$$

where $\text{Per}(E) = |D\chi_E|(\Omega)$ denotes the perimeter of E and the constant C depends only on $\text{dist}(a, u(\partial E))$, $\text{dist}(a, v(\partial E))$.

Proof. Choose $r > 0$ such that $\min(\text{dist}(a, u(\partial E)), \text{dist}(a, v(\partial E))) > 2r$, set

$$\Omega_r := \Omega \setminus (u^{-1}(B(a, r)) \cup v^{-1}(B(a, r))),$$

and apply the previous corollary to the function $u^a \in C^{0,\alpha}(\Omega_r; \mathbb{R}^n)$. The claim follows from the elementary inequality

$$\|u^a\|_{C^{0,\alpha}(\Omega_r; \mathbb{R}^n)} \lesssim r^{-1} \|u\|_{C^{0,\alpha}(\Omega; \mathbb{R}^n)}.$$

□

For a given Lipschitz set E , we have that $\mathcal{L}^n(u(\partial E)) = 0$ (see e.g. [Olb17] Lemma 2.7), and hence $\langle Ju^a, \chi_E \rangle = -\langle \mathbf{j}u^a, D\chi_E \rangle$ is defined for almost every $a \in \mathbb{R}^n$. In fact, the weak coarea formula and chain rule hold:

Proof of Theorem 3. Choose $\tilde{\Omega}$ such that $E \Subset \tilde{\Omega} \Subset \Omega$. For $\varepsilon > 0$ small enough, we have that $\text{dist}(\tilde{\Omega}, \partial\Omega) > \varepsilon$ and we may assume that $u_\varepsilon := u * \eta_\varepsilon$ is defined in $\tilde{\Omega}$. Choose $\tilde{\alpha} \in (\frac{n-1}{n}, \alpha)$. With this choice we have that $u_\varepsilon \rightarrow u$, $F \circ u_\varepsilon \rightarrow F \circ u$ in $C^{0,\tilde{\alpha}}(\tilde{\Omega}; \mathbb{R}^n)$, and

$$\langle J(F \circ u_\varepsilon), \chi_E \rangle \rightarrow \langle J(F \circ u), \chi_E \rangle.$$

Now for $\varepsilon > 0$, u_ε is smooth, which means that by classical change of variables formula we obtain

$$\begin{aligned} \langle J(F \circ u_\varepsilon), \chi_E \rangle &= \int_E \det \nabla F(u_\varepsilon(x)) \det \nabla u_\varepsilon(x) dx \\ &= \int_{\mathbb{R}^n} \det \nabla F(a) \deg(u_\varepsilon, E, a) da. \end{aligned}$$

Supposing $a \notin u_\varepsilon(\partial E)$, we have that

$$\begin{aligned} \deg(u_\varepsilon, E, a) &= \frac{1}{\omega_n} \int_{\partial E} (u_\varepsilon)^a \cdot \text{cof } \nabla(u_\varepsilon)^a \nu d\mathcal{H}^{n-1} \\ &= \frac{1}{\omega_n} \langle J(u_\varepsilon)^a, \chi_E \rangle, \end{aligned}$$

where ν denotes the unit outer normal of ∂E , see the appendix. By Theorem 1.2 in [Olb17], we have that $\deg(u_\varepsilon, E, \cdot)$ is a convergent sequence in $L^1(\mathbb{R}^n)$, and by Lemma 3 we have that

$$\langle J(u_\varepsilon)^a, \chi_E \rangle \rightarrow \langle Ju^a, \chi_E \rangle$$

for almost every $a \in \mathbb{R}^n$. This implies that

$$\int_{\mathbb{R}^n} \det \nabla F(a) \langle J(u_\varepsilon)^a, \chi_E \rangle da \rightarrow \int_{\mathbb{R}^n} \det \nabla F(a) \langle Ju^a, \chi_E \rangle da,$$

proving our claim. $\square$

Remark 4. For $\alpha \in (\frac{n}{n+1}, 1)$, the strong chain rule holds for $u \in C^{0,\alpha}(\Omega; \mathbb{R}^n)$ by the inclusion $C^{0,\alpha} \subset W^{\frac{n}{n+1}, n+1}$.

5.2. Two open questions linked to the validity of the strong chain rule. A very important and completely open question is whether or not the strong chain rule holds for functions $u \in C^0 \cap W^{s,p}(\Omega; \mathbb{R}^n)$ with $s \in [\frac{n-1}{n}, \frac{n}{n+1})$ and $sp \in [n-1, n)$ such that Ju can be represented by a non-negative Radon measure. A positive answer to this question would prove in particular two open problems of high profile, concerning the rigidity of $C^{1,\alpha}$ isometric immersions and the existence of $C^{0,\alpha}$ embeddings of the two-dimensional disk into the Heisenberg group. In the present section, we want to briefly explain the open problems and their relation to the strong chain rule.

5.2.1. The $C^{1,\alpha}$ Weyl problem. Let g be a (smooth) Riemannian metric with positive curvature on S^2 . By the Nash-Kuiper theorem, any short immersion $f : S^2 \rightarrow \mathbb{R}^3$ can be approximated arbitrarily well in C^0 by an isometric immersion $\tilde{f} \in C^1(S^2; \mathbb{R}^3)$. In particular, there exists a very large set of solutions to the isometric immersion problem in the class of C^1 -immersions. This is in stark contrast to the situation when one requires higher regularity: By a result by Pogorelov [Pog73], there exists a unique solution in the class of C^2 -immersions (up to Euclidean motions). In other words, C^2 isometric immersions are *rigid*.

Concerning spaces between C^1 and C^2 , Borisov has shown in a series of works [Bor58a, Bor58b, Bor59, Bor60, Bor65, Bor04] that $C^{1,\alpha}$ isometric immersions are rigid if $\alpha > \frac{2}{3}$, see also [CDLSJ12] for a much shorter proof. On the other side, it has been shown that for $\alpha < \frac{1}{5}$, there exists again a very large set of $C^{1,\alpha}$ (local) isometric immersions [DLISJ18]. The question whether or not $C^{1,\alpha}$ isometric immerisions are rigid in the parameter range $\alpha \in [\frac{1}{5}, \frac{2}{3})$ is open.

The rigidity proof in [CDLSJ12] is based on proving that the immersed surfaces are of *bounded extrinsic curvature*. The core of the proof consists in showing the following: Let $U \subset \mathbb{R}^2$ be a coordinate chart of S^2 with Lipschitz boundary, and $\psi \in C_c^1(U)$. Let $\nu : U \rightarrow \mathbb{R}^2$ be a representation of the surface normal in coordinate charts. The boundedness of extrinsic curvature follows from the identity

$$\int_U \psi(\nu(x)) \kappa_g(x) dA_g(x) = \int_{\mathbb{R}^2} \psi(y) \deg(\nu, U, y) dy, \quad (9)$$

where κ_g is the Gauss curvature, and dA_g the surface element of the manifold (S^2, g) in the coordinate chart U . By the positivity of Gauss curvature, the above identity allows for an estimate of the extrinsic curvature.

Let $f^\psi = (f_1^\psi, f_2^\psi)$ denote a solution of $\operatorname{div} f^\psi = \psi$, and let us assume that $f^\psi \in C^1(U; \mathbb{R}^2)$. Let $F_1^\psi(x_1, x_2) = (f_1^\psi(x_1, x_2), x_2)$ and $F_2^\psi(x_1, x_2) = (x_1, f_2^\psi(x_1, x_2))$. Now suppose that $\nu \in C^{0,\alpha}(\Omega; \mathbb{R}^2)$ with $\alpha > \frac{1}{2}$. Note that $C^{0,\alpha} \subset W^{1/2,2}$. By the weak chain rule,

$$\sum_{i=1}^2 \left\langle J(F_i^\psi \circ \nu), \chi_U \right\rangle = \int_{\mathbb{R}^2} \psi(y) \deg(\nu, U, y) dy.$$

For $\alpha > \frac{2}{3}$, we have by the strong chain rule that

$$\sum_i J(F_i^\psi \circ \nu) = \sum_i \det DF_i^\psi(\nu) J\nu.$$

Testing this equation with χ_U gives precisely (9), and the boundedness of extrinsic curvature follows. If one were able to show the strong chain rule also for $\alpha > \frac{1}{2} = \frac{n-1}{n}$, the same would be true for this larger parameter range.

5.2.2. The Hölder mapping problem. The Heisenberg group $\mathbb{H}$ can be thought of as one of the simplest examples of a sub-Riemannian manifold. As a metric space, it can be defined as the pair $(\mathbb{R}^3, d_{CC})$, where the so-called Carnot-Carathéodory distance d_{CC} is defined as follows: For $p \in \mathbb{R}^3$, define the one-form $\Theta_p = dx_3 + \frac{1}{2}(x_1 dx_2 - x_2 dx_1)$. A Lipschitz curve γ in $\mathbb{R}^3$ is said to be admissible if $\Theta(\dot{\gamma}) = 0$ almost everywhere. Then d_{CC} is defined by taking the infimum of lengths of admissible curves,

$$d_{CC}(p, q) = \inf\{\operatorname{length}(\gamma) \mid \gamma : [0, T] \rightarrow \mathbb{R}^3 \text{ admissible}, \gamma(0) = p, \gamma(T) = q\}.$$

It can be shown that the so-called Korányi metric d_K is Bi-Lipschitz equivalent to d_{CC} ,

$$d_K(p, q)^4 = ((p_1 - q_1)^2 + (p_2 - q_2)^2)^2 + (p_3 - q_3 + p_1 q_2 - p_2 q_1)^2.$$

The Hölder mapping problem is to find a $C^{0,\alpha}$ embedding of the two-dimensional disk into $\mathbb{H} = (\mathbb{R}^3, d_K)$. It has been shown by Gromov that this is impossible for $\alpha > \frac{2}{3}$. On the other hand, any smooth embedding with respect to the Euclidean distance in $\mathbb{R}^3$ is a $C^{0,1/2}$ embedding with respect to d_{CC} . Existence for the range $\alpha \in (\frac{1}{2}, \frac{2}{3}]$ is an open question.

The argument for non-existence if $\alpha > \frac{2}{3}$ is as follows: If $\gamma : [0, T] \rightarrow \mathbb{H}$ is a $C^{0,\alpha}$ curve with $\alpha > \frac{1}{2}$, then the x_3 -component of γ is completely determined by the x_1 and x_2 components,

$$\gamma_3(t) - \gamma_3(0) = \frac{1}{2} \int_0^t (\gamma_1 d\gamma_2 - \gamma_2 d\gamma_1),$$

where the right hand side has to be understood as a Lebesgue-Stieltjes integral (see Lemma 3.1 in [LDZ13] and references therein). It follows that for $U \subset \mathbb{R}^2$ and a set $\Gamma \subset U$ that is Bi-Lipschitz equivalent to S^1 , and $v \in C^{0,\alpha}(U; \mathbb{H})$ we have that

$$\frac{1}{2} \int_{\Gamma} (v_1 dv_2 - v_2 dv_1) = 0. \tag{10}$$

On the other hand, one can show the following: Let $\pi : \mathbb{H} \rightarrow \mathbb{R}^2$ denote the projection onto the first two components. If $U \subset \mathbb{R}^2$ is open, and if $v : U \rightarrow \mathbb{H}$ is a $C^{0,\alpha}$ embedding with $\alpha > \frac{1}{2}$, then for every open subset $V \subset U$, there exists a closed Lipschitz curve Γ in V such that the curve $\pi \circ v \circ \Gamma$ defines a non-vanishing current, see Lemma 3.3 in [LDZ13]. We reformulate the latter: Let W be the union of the bounded components of $\mathbb{R}^2 \setminus \Gamma$. Let

$u := \pi \circ v$. Then the conclusion of Lemma 3.3 in [LDZ13] can be rephrased by saying that there exists a function $\psi \in C_c^1(\mathbb{R}^2 \setminus u(\partial W))$ such that

$$\int_{\mathbb{R}^2} \psi(y) \deg(u, W, y) dy \neq 0. \quad (11)$$

As above, we may rewrite $\psi = \sum_{i=1}^2 \det \nabla F_i^\psi$, and we obtain by the weak chain rule

$$\int_{\mathbb{R}^2} \psi(y) \deg(u, W, y) dy = \sum_{i=1}^2 \left\langle J(F_i^\psi \circ u), \chi_W \right\rangle. \quad (12)$$

Note that (10) implies that $Ju = 0$. For $\alpha > \frac{2}{3}$ the strong chain rule holds; hence (11) and (12) form a contradiction. Again, a proof of the strong chain rule in the range $\alpha \in (\frac{1}{2}, \frac{2}{3}]$ would immediately imply the result for the larger range.

APPENDIX A. THE WEAK COAREA FORMULA AND CHAIN RULE FOR SMOOTH FUNCTIONS

For $u \in C^1(\Omega; \mathbb{R}^n)$ let $\deg(u, \Omega, \cdot)$ denote the Brouwer degree, defined at regular points $y \in \mathbb{R}^n \setminus u(\partial\Omega)$ by

$$\deg(u, \Omega, y) := \sum_{x \in u^{-1}(y)} \operatorname{sgn} \det \nabla u(x).$$

It is well known that for $\psi \in C_c^1(\mathbb{R}^n)$, we have the change of variables formula

$$\int_{\mathbb{R}^n} \deg(u, \Omega, y) \psi(y) dy = \int_{\Omega} \psi(u(x)) \det \nabla u(x) dx.$$

The Brouwer degree is constant on the connected components of $\mathbb{R}^n \setminus u(\partial\Omega)$. For $a \in \mathbb{R}^n \setminus u(\partial\Omega)$, choose ε_a such that $\operatorname{dist}(a, u(\partial\Omega)) > \varepsilon_a$. We may choose the test function $\psi_a \in C_c^0(\mathbb{R}^n)$ such that $\int_{\mathbb{R}^n} \psi_a(y) dy = 1$, $\psi_a = \operatorname{div} V_a$ for some $V_a \in C^1(\mathbb{R}^n; \mathbb{R}^n)$ with $V_a(y) = \frac{1}{n\omega_n} \frac{y-a}{|y-a|^n}$ for $y \in \mathbb{R}^n \setminus B(a, \varepsilon_a)$. The latter implies $\psi_a = 0$ on $\mathbb{R}^n \setminus B(a, \varepsilon_a)$, and

$$\begin{aligned} \deg(u, \Omega, a) &= \int_{\mathbb{R}^n} \deg(u, \Omega, y) \psi_a(y) dy \\ &= \int_{\Omega} (\operatorname{div} V_a)(u(x)) \det \nabla u(x) dx \\ &= \int_{\Omega} \operatorname{div} ((\operatorname{cof} \nabla u(x))^T V_a(u(x))) dx \\ &= \frac{1}{n\omega_n} \int_{\partial\Omega} (\operatorname{cof} \nabla u(x))^T \frac{u(x) - a}{|u(x) - a|^n} \cdot \nu(x) d\mathcal{H}^{n-1}(x) \\ &= \frac{1}{\omega_n} \int_{\partial\Omega} \mathbf{j}u^a(x) \cdot \nu(x) d\mathcal{H}^{n-1}(x), \end{aligned} \quad (13)$$

where ν denotes the unit outer normal of Ω and $u^a(x) = \frac{u(x) - a}{|u(x) - a|}$.

Now let $\psi \in C_c^1(\Omega)$, and assume first that $\psi \geq 0$. We write

$$E_t := \{x \in \Omega : \psi(x) > t\}$$

and have that $\psi(x) = \int_0^\infty \chi_{E_t}(x) dt$ for every $x \in \Omega$. Using the above, Fubini's Theorem and the BV coarea formula, we get

$$\begin{aligned}
\langle Ju, \psi \rangle &= \int_{\Omega} \psi(x) \det \nabla u(x) dx \\
&= \int_{\Omega} \int_0^\infty \chi_{E_t}(x) \det \nabla u(x) dt dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty \deg(u, E_t, a) dt da \\
&= \frac{1}{\omega_n} \int_{\mathbb{R}^n} \int_0^\infty \int_{\partial E_t} \mathbf{j}u^a(x) \cdot \nu(x) d\mathcal{H}^{n-1}(x) dt da \\
&= -\frac{1}{\omega_n} \int_{\mathbb{R}^n} \int_0^\infty \int_{\Omega} \mathbf{j}u^a(x) \cdot d(D\chi_{E_t})(x) dt da \\
&= -\frac{1}{\omega_n} \int_{\mathbb{R}^n} \int_{\Omega} \mathbf{j}u^a(x) \cdot \nabla \psi(x) dx da \\
&= \frac{1}{\omega_n} \int_{\mathbb{R}^n} \langle Ju^a, \psi \rangle da.
\end{aligned}$$

This is the weak coarea formula. The general case (without the restriction $\psi \geq 0$) is obtained by decomposing into a non-negative and a non-positive part, $\psi = \max(\psi, 0) + \min(\psi, 0)$, and noting that the manipulations above still go through for both parts of ψ (even though they may not be C^1). The weak chain rule is proved in the same way:

$$\begin{aligned}
\langle J(F \circ u), \psi \rangle &= \int_{\Omega} \psi(x) \det \nabla F(u(x)) \det \nabla u(x) dx \\
&= \int_{\Omega} \int_0^\infty \chi_{E_t}(x) \det \nabla F(u(x)) \det \nabla u(x) dt dx \\
&= \int_{\mathbb{R}^n} \int_0^\infty \det \nabla F(a) \deg(u, E_t, a) dt da \\
&= -\frac{1}{\omega_n} \int_{\mathbb{R}^n} \int_{\Omega} \det \nabla F(a) \mathbf{j}u^a(x) \cdot \nabla \psi(x) dx da \\
&= \frac{1}{\omega_n} \int_{\mathbb{R}^n} \det \nabla F(a) \langle Ju^a, \psi \rangle da.
\end{aligned}$$

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