Asymptotic analysis of relaxed manifold constrained harmonic extension problems

Benoît Van Vaerenbergh

Thesis submitted in partial fulfillment of the requirements for the Degree of Doctor in Sciences

June 2025

Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
Louvain-la-Neuve
Belgium

Thesis committee:

Pr. Marino Gran (jury president)	UCLouvain
Pr. Augusto Ponce (secretary)	UCLouvain
Pr. Heiner Olbermann	UCLouvain
Pr. Petru Mironescu	Université Claude Bernard Lyon 1
Pr. Jan Kristensen	University of Oxford
Pr. Jean Van Schaftingen (supervisor)	UCLouvain

Asymptotic analysis of
relaxed manifold constrained harmonic extension problems
by Benoît Van Vaerenbergh

Benoît Van Vaerenbergh
Institut de Recherche en Mathématique et Physique
Université catholique de Louvain
Chemin du Cyclotron
1348 Louvain-la-Neuve
Belgium

This work was supported by the F.R.S-FNRS through the F.R.I.A. grant.

Introduction | 1

AT THE START of the ideas of this thesis, there is this optimization result in geometric analysis: in a connected smooth compact Riemannian manifold N, every homotopy class can be represented by a length minimizing curve $\mathbb{S}^1 \to N$ having the circle as domain. This mathematical fact idealizes the following fact in our physical world:

> by knoting a string around an object, if one cannot remove the string from the object without untying the knot, one can always find a configuration of the string without untying the knot that minimizes the length needed to encircle the object.

Riemannian manifolds are one of the tools that mathematicians use to think about objects where length can be measured both globally and infinitesimally; homotopy classes encode mathematically the set of configuration reachable without untying the knot; they are thought as deformations of the closed string. In very mathematical terms, writing the minimization problem for a fixed homotopy class $[\gamma] \in [\mathbb{S}^1, N]$, the infimum

$$\lambda([\gamma]) \doteq \inf \left\{ \int_{\mathbb{S}^1} |\dot{\tilde{\gamma}}| : \tilde{\gamma} \in \dot{\mathrm{W}}^{1,1}(\mathbb{S}^1, N) \quad \tilde{\gamma} \in [\gamma] \right\}, \tag{1.1.1}$$

makes sense as $\dot{\mathrm{W}}^{1,1}(\mathbb{S}^1, N)$ maps, *Sobolev mappings* with integrable weak derivative are continuous and the infimum is achieved. Sobolev maps are the tools used here to tackle optimization problems with geometric flavour; we refer to Fig. 1.1 for a brief tour of the concept.

A proof of the existence of at least one map reaching the infimum (1.1.1) when looking at a nontrivial homotopy class $[\gamma]$ goes by considering the squared energy

$$\inf \left\{ \int_{\mathbb{S}^1} \frac{|\dot{\tilde{\gamma}}|^2}{2} : \tilde{\gamma} \in \dot{\mathrm{W}}^{1,2}(\mathbb{S}^1, N) \quad \tilde{\gamma} \in [\gamma] \right\}. \tag{1.1.2}$$

In higher dimensions, problem (1.1.2) generalizes in at least three ways. This introduction will focus on the first two.

- The *Dirichlet problem* which is the one mostly chosen in this thesis. Given a compact manifold M with boundary ∂M, for some $g : \partial M \to N$, minimize

$$\inf \left\{ \int_{M} \frac{|\mathrm{D}u|^2}{2} : \begin{matrix} u \in \dot{\mathrm{W}}^{1,2}(M, N) \\ \operatorname{tr}_{\partial M} u = g \end{matrix} \right\}. \tag{1.1.3}$$

$s > 0$ is the *amount* of differentiability

$p \geq 1$ parametrizes *integrability* of the differential

$$\mathrm{W}^{s,p}(M, N)$$

W stands for a *weak* notion of differentiability

M is the domain of the maps

N is the codomain of the maps

Figure 1.1: Sobolev maps $M \to N$ are elements of the space $\mathrm{W}^{s,p}(M, N)$ where M and N are manifolds. When $s \geq 1$, they carry a weak differential or derivative that we will denote $\mathrm{D}u$. The presence of a dot above the W in the writing $\dot{\mathrm{W}}^{s,p}(M, N)$ means the we only measure the derivative with the integral.

The precise meaning of $|\mathrm{D}u|$ is the following. One could use intrinsic geometry [29] but we will instead assume that the manifold is isometrically embedded [58, 59], *i.e.* $N \subset \mathbb{R}^\nu$ for some $\nu \in \mathbb{N}$. The meaning of $|\mathrm{D}u|$ is square root of the sum of the square of the components of the matrix of the weak derivative $\mathrm{D}u$ of the map $u : M \to \mathbb{R}^\nu$. If $M = \mathbb{B}^d$, $\mathrm{D}u = (\partial_i u_j)_{\substack{i=1,\dots,d \\ j=1,\dots,\nu}}$.

One refers to this problem as *manifold constrained harmonic extension* by analogy with the $\dot{\mathrm{W}}^{1,2}$-energy minimized by solutions $f : M \to \mathbb{R}$ with vanishing Laplacian, called harmonic functions.

Typically, if $\partial M = \mathbb{S}^1$, one would chose g to be a solution of (1.1.1) to get a higher dimensional representative of the homotopy class $[\gamma]$.

- One can also try to produce distinguished sections of a (principal) fiber bundle: if N is a principal or associated bundle over a compact Riemannian manifold M without boundary,

$$\inf \left\{ \int_M \frac{|\mathrm{D}u|^2}{2} : u \in \Gamma_{\dot{\mathrm{W}}^{1,2}}(M, N) \right\} \tag{1.1.4}$$

where $\Gamma_{\dot{\mathrm{W}}^{1,2}}(M, N)$ denotes bundle sections of $\dot{\mathrm{W}}^{1,2}$-regularity. If the bundle is trivial, one can produce constant sections, leaving the infimum zero. One is then led to add more constraints. Here, many meanings of Du coexist. On can take a connection, or the differential, or a combination of the two, or extrinsic Dirichlet energies, *cfr* [41].

- Lastly, one can constrain the homotopy of maps. Given a manifold M and a continuous map $v : M \to N$, consider

$$\inf \left\{ \int_M \frac{|\mathrm{D}u|^2}{2} : \begin{array}{c} u \in \dot{\mathrm{W}}^{1,2}(M,N) \text{ is continuous} \\ \text{and homotopic to } v \end{array} \right\}. \tag{1.1.5}$$

We refer to [79, 68, 57] for this problem that we will not investigate futher here.

Of course, one could devise similar problems for the area of the graph of mappings $u : M \to N$ which generalizes the length of curves (1.1.1); we will discuss those in a dedicated section, Section 1.6. Note that at least in the unconstrained case, *i.e.* $N = \mathbb{R}^\nu$, the area of maps $\mathbb{B}^d \to \mathbb{R}^\nu$ is minimized through a Dirichlet energy (1.1.3) [74, Ch. 1] which also motivates this first investigation.

1.2 Emptiness of admissible class

Led with problems (1.1.3) and (1.1.4) we have the following negative results.

Lemma 1.2.1. *There exist manifolds M, N and $g \in C^\infty(\partial M, N)$ such that*

$$\dot{\mathrm{W}}^{1,2}_g(M,N) \doteq \{u \in \dot{\mathrm{W}}^{1,2}(M,N) : \mathrm{tr}_{\partial M}\, u = g\}$$

is empty.

Lemma 1.2.2. *There exists a principal bundle N over M such that $\Gamma_{\dot{\mathrm{W}}^{1,2}}(M,N)$ is empty.*

The obstructions to existence are of topological nature. Indeed for Lemma 1.2.1 one takes $M = \mathbb{B}^2$ the unit ball in $\mathbb{R}^2$ centered at the origin. If $u \in \dot{\mathrm{W}}^{1,2}_g(\mathbb{B}^2, N)$ for some N and g to be chosen below, we note that

$$r \in (0,1] \mapsto \lambda([\mathrm{tr}_{\mathbb{S}^1}\, u(r\cdot)])$$

is constant and equal to $\lambda([g])$ as $\mathrm{tr}_{\mathbb{S}^1}\, u(r\cdot)$ and g are homotopic in some suitable sense. Then, by Jensen's inequality,

$$\int_{\mathbb{B}^2} |\mathrm{D}u|^2 = \int_0^1 \int_{\mathbb{S}^1_r} |\mathrm{D}u|^2 \geq \int_0^1 \frac{1}{2\pi r} \left(\int_{\mathbb{S}^1} |\mathrm{D}u(r\cdot)| \right)^2 \geq \frac{\lambda([g])^2}{2\pi} \int_0^1 \frac{\mathrm{d}r}{r}.$$

The latter integral does not converge. The contradiction arises if we choose a $g : \mathbb{S}^1 \to N$ such that $\lambda([g])$ is not zero. Such a g exists if the manifold N is compact and not simply connected, *i.e.* the fundamental group $\pi_1(N)$, which is equivalent to $[\mathbb{S}^1, N]$ contains only one trivial element. Taking $N = \mathbb{S}^1$, the identity map $\gamma : z \in \mathbb{S}^1 \to z \in N$ is not homotopic to a constant; in fact, we have $\lambda([\gamma]) = 2\pi$.

We shall not enter here into the details of the homotopy of Sobolev functions, which provides a rigorous meaning to the term "suitable sense" mentioned in the previous paragraph. Nevertheless, let us mention the underlying technology used to give a precise meaning: VMO functions for "vanishing mean oscillation" [19, 1, 27].

Concerning Lemma 1.2.2, one knows from the Hairy ball theorem [45, Thm 13.32]—a result also known as the hedgehog theorem— that the unit tangent bundle $N = \mathbb{S}^1 \cap \mathrm{T}\mathbb{S}^2$, which can be given the structure of a principal $\mathbb{S}^1$ bundle over the sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, do not carry a global continous section; in short, $\Gamma_{C^0}(S^2, N) = \emptyset$. One can then show that [27]

$$\Gamma_{C^0}(\mathbb{S}^2, N) = \emptyset \text{ if and only if } \Gamma_{\dot{\mathrm{W}}^{1,2}}(\mathbb{S}^2, N) = \emptyset.$$

In fact the theorem generalizes to higher even dimensions.

The starting point of this thesis is the following result due to Hardt and Lin [38].

Theorem 1.2.3. *If $p \in (1, 2)$, for every $g \in \dot{\mathrm{W}}^{1-1/p,p}(\partial M, N)$, there exists $u \in \dot{\mathrm{W}}^{1,p}(M, N)$ such that $\mathrm{tr}_{\partial M}\, u = g$. Moreover,*

$$\int_M |\mathrm{D}u|^p \leq \frac{C_{M,N}}{2-p} \mathcal{W}^{1-1/p,p}(g; \partial M) \tag{1.2.1}$$

where $C_{M,N} > 0$ depends only on M and N.

The fractional Sobolev space is defined using the fractional Sobolev energy: $g \in \dot{\mathrm{W}}^{1-1/p,p}(\partial M, N)$ if

$$\mathcal{W}^{1-1/p,p}(g; \partial M) \doteq \iint_{\partial M \times \partial M} \frac{\mathrm{dist}_N(g(y), g(x))^p \,\mathrm{d}x \,\mathrm{d}y}{\mathrm{dist}_{\partial M}(x, y)^{p-1+\dim(\partial M)}} \tag{1.2.2}$$

is finite. Smooth bounded maps have finite fractional energy. To be precise, (1.2.1) is the converse of Gagliardo's trace theorem [32], which identifies the image of the trace operator for Sobolev functions as the fractional Sobolev space, often referred to as the *trace space* in the literature. When $p \in (1, 2)$, Theorem 1.2.3 asserts that there is no obstruction to the extension in the Sobolev space $\dot{\mathrm{W}}^{1,p}(M, N)$; in particular we deduce that this space contains noncontinuous mappings. For $p \geq 2$ one can decide if a statement such as Theorem 1.2.3 holds or not with or without an estimate; necessary and sufficient conditions can in fact be given [76], they involve higher order homotopy groups, $\pi_k(N) \simeq [\mathbb{S}^k, N]$. In the litterature this question is known as the extension problem [12, 43, 13, 53, 51, 76, 20]

In this thesis we precise the case $p = 1$ which was already known [38, 53]:

Theorem 1.2.4. *For any measurable $g : \partial M \to N$, there exists $u \in \dot{\mathrm{W}}^{1,1}(M, N)$ such that $\operatorname{tr}_{\partial M} u = g$. Moreover,*

$$\int_M |\mathrm{D}u| \le C_M \iint_{\partial M \times \partial M} \operatorname{dist}_N(g(x), g(y)) \,\mathrm{d}x \,\mathrm{d}y. \tag{1.2.3}$$

where $C_M > 0$ depends only on M, provided the right-hand side is finite.

In fact we also treat noncompact manifold N without boundaries, give extension results in the model cases of the strip $\mathbb{R}^d \times (-1, 1)$ and the halfspace.

In simple situations, neglecting the estimate (1.2.1), one can easily construct $\dot{\mathrm{W}}^{1,p}$ maps having specific traces. If $M = \mathbb{B}^2$ and $\gamma : \partial\mathbb{B}^2 \to N$ is a minimizing geodesic parametrized at constant speed, one can show that the *vortex map*, also known as the “hedgehog” map,

$$u(x) \doteq \gamma\Big(\frac{x}{|x|}\Big) \in \dot{\mathrm{W}}^{1,p}(\mathbb{B}^2, N) \tag{1.2.4}$$

satisfies

$$\int_{\mathbb{B}^2} |\mathrm{D}u|^p = \frac{1}{2-p} \int_{\mathbb{S}^1} |\dot{\gamma}|^p.$$

At the scale of Lorentz spaces, in dimension 2, one can in fact show the following result [62, 63, 20].

Theorem 1.2.5. *Let $M^2 = \mathbb{B}^2, \mathbb{R} \times [0, \infty)$ or the hyperbolic space $\mathbb{H}^2$. For any $g \in \dot{\mathrm{W}}^{1/2,2}(\partial M^2, N)$ there exists $u \in \dot{\mathrm{W}}^{1,1}(M^2, N)$ satisfying for all $t > 0$*

$$t^2 |\{x \in M^2 : |\mathrm{D}u(x)| > t\}| \le \Theta\Big(\mathcal{W}^{1/2,2}(g; \partial M^2)\Big) \tag{1.2.5}$$

where $\Theta : [0, \infty) \to [0, \infty)$ is a convex function.

The functions for which the Marcinkiewicz energy

$$\mathcal{W}^{1,(2,\infty)}(u) \doteq \sup_{t>0} t^2 |\{x \in M^2 : |\mathrm{D}u(x)| > t\}| \tag{1.2.6}$$

is finite is called the weak-L^2 Sobolev space. Classically one writes $|\mathrm{D}u| \in \mathrm{L}^{2,\infty}(M)$ if the Marcinkiewicz energy is finite; $\mathrm{L}^{2,\infty}(M)$ refers to the Lorentz or weak L^2-space. We record here that Theorem 1.2.5 admits higher dimension variants for $M^d = \mathbb{B}^d, \mathbb{R}^{d-1} \times [0, \infty)$ and $\mathbb{H}^d$ for the Lorentz space $\mathrm{L}^{d,\infty}$. However to our knowledge an extension result for the Marcinkiewicz energy $\mathcal{W}^{1,(2,\infty)}(u)$ is not known in any dimension.

In dimension 2, we show the following, which admits higher dimension versions [20].

Lemma 1.2.6. *Let $\Omega \subset \mathbb{R}^2$ be an open subset of the plane. Let $a_1, \ldots, a_k \in \Omega$ be points. If $u \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_i\}_{i=1,\ldots,k}, N)$, then*

$$\sum_{i=1}^{k} \frac{\lambda([u, a_i])^2}{4\pi} \leq \varlimsup_{t \to \infty} t^2 \mathcal{H}^2(\{x \in \Omega : |\mathrm{D}u|(x) \geq t\}). \tag{1.2.7}$$

This lemma should motivate the study of the quantity $\mathcal{W}^{1,(2,\infty)}(u)$. Defining the systole of the manifold,

$$\operatorname{sys}(N) \doteq \inf\{\lambda(\gamma) : \gamma \in C^1(\mathbb{S}^1, N) \text{ is not contractible}\}, \tag{1.2.8}$$

we see that the $\mathcal{W}^{1,(2,\infty)}(u)$ energy also controls the number of singularities: in the context of Lemma 1.2.6, $k \operatorname{sys}(N)^2/4\pi \leq \mathcal{W}^{1,(2,\infty)}(u; \Omega)$.

For the global section problem (1.1.4), let us mention that, to the best of our knowledge, a positive result corresponding to Lemma 1.2.2 in the form of Theorem 1.2.3 asserting the nonemptiness of $\Gamma_{\dot{\mathrm{W}}^{1,p}}(M, N)$ –that is, sections of $\dot{\mathrm{W}}^{1,p}$ regularity– when $p \in [1, 2)$ is to this day a folklore type result. We wish to raise here the question of the estimate relating the $\dot{\mathrm{W}}^{1,p}-$energy with the data of the bundle that would accompany such an existence result for a $\dot{\mathrm{W}}^{1,p}-$valued section in a fiber bundle; an interesting answer would also provide estimates on the Hausdorff dimension of the singular set of the section.

1.3 $p-$harmonic maps at fixed p

The extension result, Theorem 1.2.3, suggests to study (1.1.3) when 2 is replaced by a parameter $p \in [1, 2)$: given $g : \partial M \to N$ minimize

$$\mathcal{W}^{1,p}(u; M) \doteq \int_M \frac{|\mathrm{D}u|^p}{p} \tag{1.3.1}$$

among all $u \in \dot{\mathrm{W}}^{1,p}(M, N)$ satisfying $\operatorname{tr}_{\partial M} u = g$. If $p \in (1, 2)$, the direct method of calculus of variations yields the existence of a minimizer. Minimizers are called minimizing $p-$harmonic maps or $p-$harmonic maps for short. If $p = 1$, minimizing sequences are not compact in $\dot{\mathrm{W}}^{1,1}(M, N)$ and almost nothing is known regarding minimizers, see Section 1.6.

Although the functional (1.3.1) is convex, the manifold constraint removes any convexity from the problem, shelving any question of uniqueness of $p-$harmonic maps. The obstruction is one-dimensional: for the identity map $g : \{-1, 1\} \to \{-1, 1\} \subset \mathbb{S}^1 \subset \mathbb{C}$, there are two geodesic $[-1, 1] \to \mathbb{S}^1_+$ and $[-1, 1] \to \mathbb{S}^1_-$ having g as boundary data. However, in very symetric situations one has the following, originally due to Robert HARDT and Fanghua LIN and Changyou WANG [40], *cfr* [18, Thm 13.6].

Theorem 1.3.1. *Let $p \in (1, 2)$. If $g : \mathbb{S}^1 \to \mathbb{S}^1$ is the identity, then the unique minimizing $p-$harmonic map with trace g is the vortex map $\frac{x}{|x|}$.*

To our knowledge this is the only explicit example up to isometries of the plane of nonconstant minimizing $p-$harmonic map. One could be interested in other examples such as $\gamma(\frac{x}{|x|})$ where $\gamma \in C^1(\mathbb{S}^1, N)$ is a length minimizing geodesic. Though it is not minimizing in general as degree two geodesics do not generate $p-$harmonic mappings [37, Lem 7.1]. However, in general, it is easy to conjecture that for a given manifold one could associate a geodesic γ such that the vortex map $\gamma(\frac{x}{|x|})$ minimizes the Dirichlet $p-$energy $\mathcal{W}^{1,p}(\cdot; \mathbb{B}^2)$.

We note that minimizers form a compact family:

Lemma 1.3.2. *Let $p \in (1, \infty)$. Let $g \in \dot{\mathrm{W}}^{1-1/p,p}(\partial M, N)$, the set*

$$\left\{u \in \dot{\mathrm{W}}^{1,p}(M, N) \text{ is a minimizing } p-\text{harmonic map of trace } g\right\}$$

is compact in $\dot{\mathrm{W}}^{1,p}(M, N)$.

Lemma 1.3.2 is a consequence of the inequalities of HANNER [36], *cfr* [80, Thm 4.1.9(c)]. This result implies, by the DE LA VALLÉE POUSSIN criterion that there exists $\theta \colon [0, \infty) \to [0, \infty)$, convex such that $\lim_{t\to\infty} \theta(t)/t = +\infty$ so that

$$\int_M \theta(|\mathrm{D}u|^p)$$

is uniformly bounded when u varies in the set of $p-$minimizers.

Locally one can improve the result, *cfr* [56, Corollary 2.21].

Theorem 1.3.3. *Let $p \in (1, \dim M)$. If $u \in \dot{\mathrm{W}}^{1,p}(M, N)$ is a minimizing $p-$harmonic map, then for each $q \in (p, \lfloor p \rfloor + 1)$, $|\mathrm{D}u|^q$ is locally integrable.*

The theorem is in fact more quantitative as it relates the L^q norm with the p-energy.

One can wonder if in $2d$, the Marcinkiewicz energy $\mathcal{W}^{1,(2,\infty)}(u)$ is finite. This would imply Theorem 1.3.3.

Concerning continuous scales, $p-$harmonic mappings are continuous away from a singular set [38].

Theorem 1.3.4. *Let $p \in (1,\infty)$. Let $g : \partial M \to N$ be twice continuously differentiable. If $u \in \dot{\mathrm{W}}^{1,p}(M,N)$ is a minimizing $p-$harmonic map, there exist a compact set* $\mathrm{Sing}(u) \subset M$ *whose Hausdorff dimension does not exceed* $\dim M - \lfloor p \rfloor - 1$ *and $\alpha \in (0,1)$ such that $u \in C^{0,\alpha}(M \cup \partial M \setminus \mathrm{Sing}(u))$ and* $\mathrm{D}u \in C^{0,\alpha}_{\mathrm{loc}}(M \setminus \mathrm{Sing}(u))$.

If $\dim M = 2$, $\mathrm{Sing}(u)$ therefore consists in a finite number of points in M. This is in accordance with our only example Theorem 1.3.1. If $p \neq 2$, the Hölder regularity is actually the best one can hope for as there exists a $p-$harmonic function $M \to \mathbb{R}$ which is $C^{1,\alpha}$ but not twice differentiable [75]; we were unable to find $\dot{\mathrm{W}}^{2,r}$-estimates in $2d$ that would imply the $C^{1,\alpha}$-ones. When $p = 2$, one can in fact show that harmonic maps are as smooth as the manifold N: if the manifold is smooth in the $C^\infty-$sense, then the minimizing 2-harmonic mappings are smooth [67, 71]. Concerning rectifiability of the singular set or its stratification —the fact that it writes as a union of rectifiable sets of integer dimension— while much is known from 2-harmonic mappings, the case when $p \neq 2$ has not received much attention; we record the result for $p-$harmonic mappings $p \in (1,2)$, $\mathbb{B}^3 \to \mathbb{R}P^2$ with values in a model of the projective plane [24] which asserts that the singular set consists in the union of a finite set and a one-dimensional set.

Calculus of Variations, the field of mathematics to which this thesis belongs, started with the computation of *variations* to deduce Euler-Lagrange equations. We work with domains, open bounded Lipschitz sets of $\mathbb{R}^d$ mostly denoted Ω instead of manifold to avoid the heavy vocabulary of Riemannian geometry. The precise meaning of Lipschitz domains is that one can cover the boundary of the open set by Lipschitz charts. While the setting is *flat* in the sense that is it a submanifold of $\mathbb{R}^d$ where any notion of curvature vanishes, it has the advantage to cover the square, the cube etc. If one uses variations in the domain, so called *inner variations* and the invariance of the integrand (1.3.1) of $\mathcal{W}^{1,p}(u;\Omega)$ in the variable of the domain, then one obtains the following [64] Noether theorem.

Theorem 1.3.5. *Let $p \in (1,\infty)$. Let $\Omega \subset \mathbb{R}^d$. If $u \in \dot{\mathrm{W}}^{1,p}(\Omega,N)$ is a $p-$minimizing harmonic map, then the* stress-energy *tensor*

$$T_p(\mathrm{D}u) \doteq \frac{|\mathrm{D}u|^p}{p}\mathrm{I}_{d\times d} - \frac{\mathrm{D}u \otimes \mathrm{D}u}{|\mathrm{D}u|^{2-p}} \tag{1.3.2}$$

has divergence-free collumns.

The divergence is here understood in the distributional sense. Here, using that $N \subset \mathbb{R}^\nu$,

$$\mathrm{D}u \otimes \mathrm{D}u = (\partial_i u \cdot \partial_j u)_{i,j\doteq 1,\ldots,d}$$

is a $d \times d$-matrix. The tensor is symmetric as a matrix and therefore it is also divergence free in its rows.

Using a projection $\Pi : N_\delta \to N$ from a tubular neighborhood N_δ to N one can compute *outer variations*,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{W}^{1,p}(\Pi(u+t\phi))\Big|_{t=0} = 0,$$

one obtains the $p-$harmonic map equation [38].

Theorem 1.3.6. *Let $p \in (1,\infty)$. If $u \in \dot{\mathrm{W}}^{1,p}(\Omega, N)$ is a $p-$minimizing harmonic map,*

$$-\mathrm{div}\left(\frac{\mathrm{D}u}{|\mathrm{D}u|^{2-p}}\right) = \frac{\mathrm{B}_u(\mathrm{D}u,\mathrm{D}u)}{|\mathrm{D}u|^{2-p}} \tag{1.3.3}$$

where $B_u : \mathrm{T}_u N \times \mathrm{T}_u N \to \mathbb{R}^\nu$ is the second fundamental form of the imbedding $N \subset \mathbb{R}^\nu$.

The divergence is taken row-wise. When $p = 2$ and Ω is an interval the equation reduces to the geodesic equation.

1.4 $p-$harmonic energies and their $p \nearrow 2$ limits

Since, for each $p \in [1,2)$, the Sobolev space $\dot{\mathrm{W}}^{1,p}_g(M,N)$ of mappings of trace $g \in \dot{\mathrm{W}}^{1-1/p,p}(\partial M, N)$ contains at least one element (*cfr* Theorem 1.2.3), one would be tempted to take a $p \nearrow 2$ limit in some sense. Here is the one we chose.

First, we will take $g \in \dot{\mathrm{W}}^{1-1/2,2}(\partial M, N)$. This is somehow necessary if one wants to compare different levels of $p \in (1,2)$. One has $\dot{\mathrm{W}}^{1-1/2,2}(\partial M,N) \subset \dot{\mathrm{W}}^{1-1/p,p}(\partial M,N)$ for each $p \in (1,2]$ and if $g \in \dot{\mathrm{W}}^{1-1/2,2}(\partial M,N)$

$$\mathcal{W}^{1-1/p,p}(u;\partial M) \xrightarrow{p\nearrow 2} \mathcal{W}^{1-1/2,2}(u;\partial M).$$

In the spirit of (1.1.3) we seek for a distinguished element in the limit. Let $u_p \in \dot{\mathrm{W}}^{1,p}_g(M,N)$ be a minimizing $p-$harmonic map of trace g. From Theorem 1.2.3,

$$\limsup_{p\nearrow 2}(2-p)\mathcal{W}^{1,p}(u_p;M) \le C_M \mathcal{W}^{1-1/2,2}(g;\partial M). \tag{1.4.1}$$

This last equation invites us to give a description of description of a limiting measure

$$\mu_* = \lim_{n\to\infty} (2-p_n)\frac{|\mathrm{D}u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^{\dim(M)} \tag{1.4.2}$$

which always exists by weak$-*$ compactness and the consideration of an appropriate sequence $(p_n)_n$ converging by lower values to the critical parameter 2. In the sequel, we will refer to those results as *first order results* while assertions concerning the convergence of $p-$harmonic mappings, *i.e.* the computation of $\lim_{n\to\infty} u_{p_n}$ along subsequences, will be described as *second order convergence results.* This vocabulary is inspired by the Taylor expansion theorem.

We finally conclude this section noting that, in dimension 2, when $g : \mathbb{S}^1 \to \mathbb{S}^1$ is the identity map, Theorem 1.3.1 suggests to take the vortex map $\frac{x}{|x|}$ in the limit $p \nearrow 2$ as constant sequences converge.

1.4.1 Dimension 2

In the first chapter of the thesis we chose to work with $M = \Omega \subset \mathbb{R}^2$ a 2-dimensional open Lipschitz set. We show

$$\lim_{p\nearrow 2}(2-p)\mathcal{W}^{1,p}(u_p;\Omega) = \mathcal{E}^{1,2}_{\mathrm{sg}}(g). \tag{1.4.3}$$

The singular energy of $g \in \dot{\mathrm{W}}^{1/2,2}(\partial\Omega, N)$ is defined by

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(g) = \inf \sum_{i=1}^{k} \frac{\lambda([\gamma_i])^2}{4\pi} \tag{1.4.4}$$

where the infimum is taken over of k-tuple ($k \in \mathbb{N}\setminus\{0\}$) of curves $\gamma_i \in C^1(\mathbb{S}^1, N)$, $i = 1,\dots,k$ and distinct points $a_i \in \Omega$ such that there exists for some $\rho > 0$,

$$u \in \dot{\mathrm{W}}^{1,2}(\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i,\rho), N),$$

so that $\operatorname{tr}_{\partial\Omega} u = g$ and $\operatorname{tr}_{\mathbb{S}^1} u(a_i + \rho\cdot) = \gamma_i$ for all $i = 1,\dots,k$. We say that $(\gamma_i)_i$ is a topological resolution of g.

If we further specialize to $\Omega = \mathbb{B}^2$ then degree theory [49, 19] allows us to write $\mathcal{E}^{1,2}_{\mathrm{sg}}(g) = \pi|\deg(g)|$.

Here is a second order convergence result.

Theorem 1.4.1. *Let $p_n \nearrow 2$. Suppose $(u_n)_n \subset \dot{\mathrm{W}}^{1,p_n}(\Omega; N)$ is an asymptotically minimizing sequence in the sense that $\mathrm{tr}_{\partial\Omega}\, u_n = g$ for all n and*

$$\mathcal{W}^{1,p_n}(u_n, \Omega) - \inf\left\{\mathcal{W}^{1,p_n}(v, \Omega) : \begin{array}{c} v \in \dot{\mathrm{W}}^{1,p_n}(\Omega; N) \\ \mathrm{tr}_{\partial\Omega}\, v = g \end{array}\right\} \xrightarrow{n\to\infty} 0. \tag{1.4.5}$$

Then, passing to a subsequence, u_n converges strongly in $\mathrm{W}^{1,1}$ to a limit map $u_* \in \mathrm{W}^{1,1}(\Omega; N)$ with $\mathrm{tr}_{\partial\Omega}\, u_* = g$ satisfying

$$u_* \in \dot{\mathrm{W}}^{1,2}\Big(\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \rho); N\Big) \quad \textit{for all } \rho > 0, \tag{1.4.6}$$

where $\kappa \in \mathbb{N}$ and $a_1, \dots, a_\kappa \in \Omega$. Moreover, for all $\rho > 0$ sufficiently small,

(i) *u_* is a minimizing 2-harmonic map on $\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}^2(a_i, \rho)$ with respect to its own boundary conditions,*

(ii) $\displaystyle\int_{\Omega\setminus\bigcup_{i=1}^{\kappa} \mathbb{B}^2(a_i,\rho)} |\mathrm{D}u_* - \mathrm{D}u_n|^{p_n}\, \mathrm{d}x \xrightarrow{n\to\infty} 0,$

(iii) *u_* is renormalisable in that the renormalized energy*

$$\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) \doteq \lim_{\rho\to 0} \int_{\Omega \bigcup_{i=1}^{k} \mathbb{B}^2(a_i,\rho)} \frac{|\mathrm{D}u_*|^2}{2}\, \mathrm{d}x - \mathcal{E}^{1,2}_{\mathrm{sg}}(g) \log\frac{1}{\rho} \tag{1.4.7}$$

exists and is finite, and we have

$$\lim_{n\to\infty}\left[\mathcal{W}^{1,p_n}(u_{p_n}, \Omega) - \frac{2}{p_n(2-p_n)}\mathcal{E}^{1,2}_{\mathrm{sg}}(g)\right] = \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa},$$

where

$$\mathrm{H}([u_*, a_i])_{i=1}^{\kappa} \doteq \lim_{\epsilon\searrow 0} \sum_{i=1}^{\kappa} \frac{\lambda([\mathrm{tr}_{\mathbb{S}^1}\, u_*(a_i + \epsilon\cdot)])^2}{4\pi} \log\frac{2\pi}{\lambda([\mathrm{tr}_{\mathbb{S}^1}\, u_*(a_i + \epsilon\cdot)])}. \tag{1.4.8}$$

This theorem suggests what would replace the emptiness of $\dot{\mathrm{W}}^{1,2}_g(\Omega, N)$: mappings whose renormalized energy (4.1.8) is finite. Renormalised mappings where introduced for maps $\Omega \to N$ by Antonin Monteil, Rémy Rodiac and Jean Van Schaftingen in [55, 54]. Before them in the case of the circle, $N = \mathbb{S}^1$, such an energy already appeared, *cfr* [14, Thm I.8]. We write $\mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega, N)$ for the set of mappings in $\dot{\mathrm{W}}^{1,1}(\Omega, N)$ for which the renormalized energy (4.1.8) is finite. By (i) the limit u_* has some local minimality properties. On the global side we show that the limit map u_* satisfies

$$\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa} = \inf\left\{\mathrm{E}^{1,2}_{\mathrm{ren}}(\bar{u}) + \mathrm{H}([\bar{u}, \bar{a}_i])_{i=1}^{\bar{\kappa}}\right\},$$

where the infimum is taken over all $\bar{u} \in \dot{\mathrm{W}}^{1,2}_{\mathrm{ren}}(\Omega; \mathcal{N})$ with $\operatorname{tr}_{\partial\Omega} \bar{u} = g$, where $\bar{u}$ has $\bar{\kappa} \in \mathbb{N}$ singularities $\bar{a}_1, \dots, \bar{a}_{\bar{\kappa}}$, forming a minimal topological resolution in that

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(g) = \lim_{\epsilon \searrow 0} \sum_{i=1}^{\bar{\kappa}} \frac{\lambda([\operatorname{tr}_{\mathbb{S}^1} \bar{u}(a_i + \epsilon \cdot)])^2}{4\pi}.$$

The limit is artificial as the sequence becomes constant for small choices of $\epsilon > 0$.

We will compare other results of this type below. This is the aim of this introduction, at least; we will attempt to name the main functionals and chosen frameworks. Here, *comparing* means explaining the concepts and discussing them using the lenses of first order/second order convergence, as well as providing insights in higher dimensions when possible. We will also point out potential gaps in the scientific litterature. In a nonlinear context, it is generally difficult to compare notions of convergence in different spaces, so we will primarily rely on analogy to discuss results.

Let us first mention here that Theorem 1.4.1 is a generalization of the work of [39] which only treats the case $N = \mathbb{S}^1$. Let us however note that the authors of [39] obtain convergence in the optimal space $C^{1,\alpha}$, see Theorem 1.3.4. Our approach is however more variational as we cover almost minimizers, *cfr* (1.4.5).

In the spirit of Theorem 1.2.5 and as a complement to Theorem 1.4.1 we prove that the sequence satisfies

$$\limsup_{n\to\infty} \sup_{t>0} t^{p_n} |\{|\mathrm{D}u_n| > t\}| \text{ is finite.} \tag{1.4.9}$$

This implies that the Marcinkiewicz energy $\mathcal{W}^{1,(2,\infty)}(u_*)$ is finite, a result already derived in [55] and whose foundation was laid by Sylvia Serfaty and Ian Tice [69] in the case of the circle.

Concerning (ii) we do not know if the stronger conclusion

$$\int_\Omega |\mathrm{D}u_* - \mathrm{D}u_n|^{p_n} \xrightarrow{n\to\infty} 0 \tag{1.4.10}$$

holds true. First, it is compatible with the fact that $\dot{\mathrm{W}}^{1,2}_g(\Omega, N)$ can be empty. One can show that for $a, b \in \mathbb{B}^2$ there exists $c, C > 0$ such that for all $p \in (1,2)$

$$c\frac{|b-a|^{2-p}}{2-p} \leq \int_{\mathbb{B}^2} \left| \frac{1}{|x-a|} - \frac{1}{|x-b|} \right|^p \mathrm{d}x \leq C\frac{|b-a|^{2-p}}{2-p}$$

which suggests that (1.4.10) is related to the rate of convergence of singularities of $p-$harmonic map, *cfr* Theorem 1.3.4. Few quantitative results are known, *cfr* [50, Thm 8.1].

Finally one can describe the concentration of the measure, first order convergence, in the context of Theorem 1.4.1: we have weakly* as measures on Ω,

$$(1.4.11)\quad (2-p_n)\frac{|\mathrm{D}u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^2 \xrightarrow{n\to+\infty} \lim_{\epsilon \searrow 0} \sum_{i=1}^{\kappa} \frac{\lambda([\mathrm{tr}_{\mathbb{S}^1} u_*(a_i+\epsilon\cdot)])^2}{4\pi}\delta_{a_i}.$$

1.4.2 Dimension 3

In dimension 3, we work with a 3-dimensional Lipschitz domain $\Omega \subset \mathbb{R}^3$ and and a boundary data $g \in \dot{\mathrm{W}}^{1/2,2}(\partial\Omega, N)$. We work with the assumption that $\pi_1(N)$ is a finite set; the latter condition is equivalent —and mostly used as such in our proofs— to the fact that the universal covering of N is a compact manifold.

Theorem 1.4.2. *Let N have a finite fundamental group and $(u_n)_n$ be a sequence of minimizing p_n-harmonic mappings all sharing the same trace g, then there exists a measure μ_* on Ω such that along a subsequence,*

$$(2-p_n)\frac{|\mathrm{D}u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^3 \xrightarrow{n\to+\infty} \mu_*$$

weakly* as measures on Ω.

We have that μ_* is the supporting measure of a stationary $1-$dimensional varifold V on Ω.

Let us first explain the vocabulary. If $\Omega \subset \mathbb{R}^d$, a varifold on Ω is a generalization of embedded surface in Ω in a measure theoretic sense. Their introduction was justified by Plateau's problem: minimizing the area of a surface under some constrain. Precisely, for $\ell \in \{1,\dots,d\}$, a ℓ-dimensional varifold on Ω, say V, is a measure on $\Omega \times G(\ell,d)$ where $G(\ell,d)$ is the Grassmanian of ℓ-dimensional vector space in $\mathbb{R}^d$. Classically they are thought as the orthogonal projections:

$$(1.4.12)\qquad G(\ell,d) \doteq \left\{A \in \mathbb{R}^{d\times d} : A = A^t, \quad A^2 = A, \quad \mathrm{tr}(A) = \ell\right\}$$

Following [48, Section 6.2], if M is a k-dimensional continuously embedded submanifold of Ω with locally finite k-Hausdorff measure, the measure defined for all $B \subset \Omega \times G(k,d)$

$$\imath(M)(B) = \mathcal{H}^k(\{x \in M : (x, \mathrm{T}_x M) \in B\})$$

defines a k-dimensional varifold. Given a k-dimensional varifold V on Ω, its *supporting measure* is defined for all $A \subset \Omega$ by $\mu(A) = V(A \times G(k,d))$; in the case of $\imath(M)$, the supporting measure is $\mathcal{H}^k \llcorner M$. The number $\mu(\Omega)$ is the volume of the varifold; one can show that if the varifold is a critical point of the volume the varifold is *stationary*, which writes

$$\int_\Omega \langle \mathrm{D}X(x) | A(x) \rangle_{\mathbb{R}^{d\times d}} \, \mathrm{d}V(x, A) = 0$$

for all $X \in C_c(\Omega, \mathbb{R}^d)$. The left hand-side is often denoted $\delta V(X)$, it is the first variation of the varifold.

In the context of Theorem 1.4.2, we have that there exists a measure V_*, which is a stationary varifold, on $\Omega \times G(1,3)$ such that $\mu_*(A) = V_*(A \times G(1,3))$. Using that the dimension is equal to one and stationarity of the varifold allows us to conclude that there exists at most a countable union of segments $(L_j)_{j\in J}$ lying in $\overline{\Omega}$, mappings $(\gamma_j)_{j\in J}$ in $C(\mathbb{S}^1, \mathcal{N})$, such that the support S_* of the measure satisfies $S_* \cap \Omega = \bigcup_{j\in J} L_j \cap \Omega$. Moreover, for all $E \subset \Omega$,

$$\mu_*(E) = \sum_{j\in J} \mathcal{E}_2^{\mathrm{sg}}(\gamma_j) \mathcal{H}^1(L_j \cap E) \tag{1.4.13}$$

In fact, the limiting measure in not only “stationary” but has minimizing properties locally in Ω.

Concerning second order convergence we have:

Theorem 1.4.3. *In the context of Theorem 1.4.2, there exists $u_* \in \dot{\mathrm{W}}^{1,q}(\Omega, N)$ for all $q \in [1,2)$ such that along a subsequence, $u_n \to u_*$ almost everywhere in Ω*

- *For each $q \in [1,2)$, $u_n \to u_*$ in $\dot{\mathrm{W}}^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^\nu)$ and*

$$\limsup_{n\to+\infty} \int_\Omega (\mathrm{dist}(x, \partial\Omega) |\mathrm{D}u_n(x)|)^q \, \mathrm{d}x \text{ is finite.}$$

- *$u_* \in \dot{\mathrm{W}}^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$ is a locally minimizing harmonic map in $\Omega \setminus S_*$. Moreover there exists $S_0 \subset \Omega$, a finite set such that $u_* \in C^\infty(\Omega \setminus (S_0 \cup S_*), N)$.*

Given u_* we may explain which are the $(\gamma_j)_{j\in J}$ in (1.4.13). Given a segment L_j we consider a small disk D^2 embedded in $\mathbb{R}^3$ perpendicular to L_j, intersecting only the segment L_j and such that $L_j \cap D^2$ is the center of D^2, then one may take $\gamma_j = \mathrm{tr}_{\partial D^2} u_*$, provided $D^2 \cap S_0 = \emptyset$.

If the domain is a ball and if we take a boundary data $g : \partial\Omega \to N$ which is smooth outside a finite set of points to mimic the defects, *i.e.* the singularities $\{a_i\}_{i=1}^k$ observed in dimension 2, one obtains the following strenghtening.

Theorem 1.4.4. *If $\Omega = \mathbb{B}^3$ and g is a C^1 map outside a finite set of points $\{a_1, \dots, a_m\} \subset \partial\Omega = \mathbb{S}^2$ around each of which g is homotopically nontrivial and for $x \in \partial\Omega$ lying in a neighborhood of a_i $|\mathrm{D}_{\mathbb{S}^2} g(x)| \leq C(g)\,\mathrm{dist}_{\mathbb{S}^2}(x, a_i)^{-1}$ near a_i and if for each $n \in \mathbb{N}$, Du_n is continuous on $\partial\Omega \setminus \{a_1, \dots, a_m\}$, then the following assertions hold.*

- *For each $q \in [1, 2)$, $u_* \in \dot{\mathrm{W}}^{1,q}(\Omega, \mathcal{N})$ is of trace g, $u_n \to u_*$ in $W^{1,q}(\Omega, \mathbb{R}^\nu)$ and*

$$\limsup_{n\to+\infty} \int_\Omega |\mathrm{D}u_n|^q \ \textit{is finite,}$$

where $C = C(g, \Omega, \mathcal{N}) > 0$.

- *$S_* \cap \partial\Omega = \{a_1, \dots, a_m\}$ and S_* lies in the convex hull of $\{a_1, \dots, a_m\}$.*
- *The set J is finite and $S_* = \bigcup_{j\in J} L_j$ and the quantity*

$$\sum_{j\in J} \mathcal{E}_2^{\mathrm{sg}}(\gamma_j)\mathcal{H}^1(L_j)$$

is globally minimizing.

In the context of the last item, the precise meaning is that

$$\sum_{j\in J} \mathcal{E}_2^{\mathrm{sg}}(\gamma_j)\mathcal{H}^1(L_j) \quad \text{minimizes} \quad \sum_{i\in I} \mathcal{E}_2^{\mathrm{sg}}(f_i)\mathcal{H}^1(K_i)$$

among all configurations of closed straight line segments $(K_i)_{i\in I}$ and topological charges $(f_i)_{i\in I} \subset C(\mathbb{S}^1, \mathcal{N})$ satisfying the following conditions: I is finite; each K_i has a positive length; the interiors of the segments are pairwise disjoint; $K = \bigcup_{i\in I} K_i$ is a compact subset of $\overline{\Omega}$ and $K \cap \partial\Omega$ is a finite set of points; $\partial K \subset \partial\Omega$, in particular, the endpoints only exist outside Ω; there exists a continous map $v : \Omega \setminus (K \cup F) \to N$, where F is a locally finite subset of $\Omega \setminus K$, such that $\mathrm{tr}_{\partial\Omega}(v) = g$ and v restricted to small circles transversal to K_i is, up to orientation, homotopic to f_i and we say that S_* solves a homotopical Plateau problem in codimension 2.

The fact that $\mathrm{D}u_n$ is continuous in Theorem 1.4.4 is a folklore result that deserves a dedicated proof which is an ungoing project.

We note that this second $3d$ results does not identify a functional, such as a renormalized energy, that the map u_* is *globally* minimizing; this contrasts with the $2d$ cases. We record that in the case of the circle (which is not covered by our result) one has been proposed [7].

1.4.3 Comparison with existing results in the litterature

In our setting of trace constrained $p-$harmonic mappings with $p \nearrow 2$, we record the results of HARDT and LIN [39] [46]. In dimension 2, our results are the same, as already mentioned. In the case $N = \mathbb{S}^1$, they obtain a version of Theorem 1.4.1 with optimal $C^{1,\alpha}$ convergence away from the singularities and they show that no collision of singularities occur with the boundary in the limit. This is strongly related to the choice of $N = \mathbb{S}^1$ and is not possible to exclude in general [55, Sec. 6].

In dimension 3, we write their result, that they are able to extend to mappings $u_n : \mathbb{B}^{m+k} \to \mathbb{S}^{k-1}$ in higher dimensions, and which is very close to Theorem 1.4.2.

Theorem 1.4.5. *Let $\Omega \subset \mathbb{R}^3$ be a smooth open domain. Let $N = \mathbb{S}^1$. Let $\Gamma \subset \partial\Omega$ be a finite set. Let $g : \partial\Omega \setminus \Gamma \to \mathbb{S}^1$ be a smooth map outside Γ. Along a subsequence,*

$$(2 - p_n)\frac{|\mathrm{D}u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^3 \xrightarrow{n\to+\infty} \mu_*$$

weakly as measures on Ω.*

We have that μ_* is the supporting measure of a $1-$dimensional current C on Ω which is area minimizing and whose boundary is Γ.

While varifolds are the measure theoretic generalization of surfaces, currents are the distributional extension of the concept of orientable embedded manifold. A k-current is a linear map on k-forms on Ω. Stokes' theorem allows one to define the boundary.

The assumptions of Theorem 1.4.5 are disjoint with our hypothesis of finite fundamental group; indeed $\pi_1(\mathbb{S}^1) = \mathbb{Z}$ is a infinite Abelian group. However, to our understanding, no second order convergence in that case is described there.

Though, Daniel STERN [73] proves the following:

Theorem 1.4.6. *Let M be a closed Riemannian m-dimensional manifold. Let $N = \mathbb{S}^1$. Let u_n be a sequence of stationary p_n-harmonic maps such* *that* $p_n \nearrow 2$. *Along a subsequence,*

$$(2-p_n)\frac{|\mathrm{D}u_n|^{p_n}}{p_n}\,\mathrm{d}\mathcal{H}^m \xrightarrow{n\to+\infty} \mu_*$$

*weakly** *as measures on* M *and* μ_* *decompose in the supporting measure of a stationary* $(m-2)$*-dimensional varifold* V *and* $|h|^2\,\mathrm{d}\mathcal{H}^m$ *where* h *is a harmonic* 1*-form.*

Moreover, if the first homology space is trivial, i.e. $\mathrm{H}^1(M,\mathbb{R})=\{0\}$, *or if* $\sup_n \|\mathrm{d}u_n\|_{\mathrm{L}^1(M)}<\infty$, *then, along a subsequence, the maps* u_n *converge weakly in* $\dot{\mathrm{W}}^{1,q}(M)$ *for all* $q\in(1,2)$, *and strongly in* $\dot{\mathrm{W}}^{1,2}_{loc}(M\setminus \mathrm{spt}(V))$, *to a limiting map* $u_*\in\bigcap_{q\in(1,2)}\dot{\mathrm{W}}^{1,q}(M,S^1)$ *that is harmonic away from* $\mathrm{spt}(V)\subset M$.

Stationary $p-$harmonic mappings $u\in\dot{\mathrm{W}}^{1,p}(M,N)$ are mappings with divergence-free stress-energy tensor, *cfr* Theorem 1.3.5. The framework of Stern does not face any topological obstruction in the sense that $\dot{\mathrm{W}}^{1,2}(M,\mathbb{S}^1)$ is not an empty space. Minimizing $p-$harmonic mappings turn out to be constant in his setting. Therefore, Stern shows that there exist nontrivial sequences of stationary p_n-harmonic mappings so that one deduces that the varifold is not trivial.

In fact, we are allowed to hope that it should be possible to produce first order convergence results for manifold N with Abelian $\pi_1(N)$ [25] which would cover all the Lie groups as we know that for $k\in\mathbb{N}$, $\pi_{2k+1}(G)$ is a free Abelian group and $\pi_{2k}(G)=\{0\}$, *cfr* [52, Thm 2.7]. There exist manifolds N carrying noncommutative and nonfinite fundamental group $\pi_1(N)$: the Klein bottle K is an example which is also a closed compact $2d-$surface; $\pi_1(K)=\langle a,b\,|\,aba^{-1}b\rangle$. This means that a general result covering all compact manifolds, whatever the topology of it, is still an open topic.

1.5 Ginzburg-Landau approach

Concerning the two problems which motivated our study, *cfr* (1.1.3) and (1.1.4), the former approach to tackle the nonemptiness of the $\dot{\mathrm{W}}^{1,2}-$class was to remove the constraint and penalize that the maps leave the constraint N. In the context of the extension problem (1.1.3), for $g\in\dot{\mathrm{W}}^{1/2,2}(\partial M,N)$. Let $\epsilon>0$ and minimize for $u\in\mathrm{W}^{1,2}(M,\mathbb{R}^\nu)$ the Ginzburg-Landau functional

$$\mathcal{G}_\epsilon(u)\doteq\int_M\frac{|\mathrm{D}u|^2}{2}+\frac{F(u)}{\epsilon^2} \tag{1.5.1}$$

under the constraint $\mathrm{tr}_{\partial M} u = g$ where $F : \mathbb{R}^\nu \to [0, +\infty)$ is such that $F^{-1}(N) = \{0\}$. Recall that we have assumed an isoperimetric embedding: $N \subset \mathbb{R}^\nu$ and minimizers of (1.5.1) will generally depend on the embedding and the chosen function F; see the $\mathrm{Q}_F(u)$ below in Theorem 1.5.1(iii). Examples of typical functions F are $F(u) = \mathrm{dist}(u, N)^2$ and for $N = \mathbb{S}^1$, $F(u) = (1 - |u|^2)^2/4$ which is the historical example [15]. Typically F is not convex so that the problem (1.5.1) is not convex; for the $p-$harmonic setting the energy $\mathcal{W}^{1,p}(u; \Omega)$ is convex and the nonconvexity of the problem comes from the constraint. Let us mention that for the construction of global section $\Gamma_{\dot{\mathrm{W}}^{1,2}}$, if the considered bundle N is a principal G-bundle, one uses the construction of the associated vector bundle and produces an example similar to (1.5.1), *cfr* [41].

In dimension 2, let us quote the result covering the most general manifold. It is a second order convergence result.

Theorem 1.5.1. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain. Let N be a compact Riemannian manifold isometrically embedded in $\mathbb{R}^\nu$. Let $F : \mathbb{R}^\nu \to [0, \infty)$ be a continuous map comparable to $\mathrm{dist}(\cdot, N)^2$ near $N \subset \mathbb{R}^\nu$. Let $g \in \dot{\mathrm{W}}^{1/2,2}(\Omega, N)$. If $u_n \in \dot{\mathrm{W}}^{1,2}(\Omega, \mathbb{R}^\nu)$ satisfies $\mathrm{tr}_{\partial\Omega} u_n = g$ for all n and*

$$(1.5.2) \qquad \mathcal{G}_{\epsilon_n}(u_n; \Omega) - \inf \left\{ \mathcal{G}_{\epsilon_n}(u_*; \Omega) : \begin{array}{c} v \in \dot{\mathrm{W}}^{1,p_n}(\Omega; \mathcal{N}) \\ \mathrm{tr}_{\partial\Omega} v = g \end{array} \right\} \xrightarrow{n \to \infty} 0,$$

then, passing to a subsequence, u_n converges strongly in $\mathrm{W}^{1,1}$ to a limit map $u_* \in \mathrm{W}^{1,1}(\Omega; \mathcal{N})$ with $\mathrm{tr}_{\partial\Omega} u_* = g$ satisfying

$$(1.5.3) \qquad u_* \in \dot{\mathrm{W}}^{1,2}\Big(\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \rho); \mathcal{N}\Big) \quad \text{for all } \rho > 0,$$

where $\kappa \in \mathbb{N}$ and $a_1, \dots, a_\kappa \in \Omega$. Moreover, for all $\rho > 0$ sufficiently small,

(i) *u_* is a minimizing 2-harmonic map on $\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}^2(a_i, \rho)$ with respect to its own boundary conditions,*

(ii) $$\int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}^2(a_i,\rho)} |\mathrm{D}u_* - \mathrm{D}u_n|^2 \, \mathrm{d}x \xrightarrow{n \to \infty} 0,$$

(iii) *u_* is renormalizable, i.e. $\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*)$ is finite, cfr (4.1.8). We moreover have*

$$\lim_{n \to \infty} \left[\mathcal{G}_{\epsilon_n}(u_n; \Omega) - \mathcal{E}^{1,2}_{\mathrm{sg}}(g) \log \frac{1}{\epsilon_n} \right] = \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \lim_{\delta \to 0} \mathrm{Q}_F(\mathrm{tr}_{\mathbb{S}^1} u(a_i + \delta \cdot))_{i=1}^{\kappa},$$

where $\mathrm{Q}_F(\mathrm{tr}_{\mathbb{S}^1} u(a_i + \delta\cdot))_{i=1}^{\kappa}$ *depends only on* F *and* $(\mathrm{tr}_{\mathbb{S}^1} u(a_i + \delta\cdot))_{i=1}^{\kappa}$*,*

(iv) $\displaystyle\lim_{n\to 0} \int_\Omega \frac{F(u)}{\epsilon_n^2} = 0,$

(v) $\displaystyle\limsup_{n\to+\infty} \mathcal{W}^{1,(2,\infty)}(u_n;\Omega)$ *is finite,*

(vi) $\displaystyle\frac{|\mathrm{D}u_n|^2}{2\log\frac{1}{\epsilon_n}}\,\mathrm{d}\mathcal{L}^2 \xrightarrow{n\to+\infty} \sum_{i=1}^{\kappa} \frac{\lambda(\gamma_i)^2}{4\pi}\delta_{a_i}$ *weakly* as measures.*

Theorem 1.5.1 in this form is due to Monteil, Rodiac and Van Schaftingen [54]. It is a generalization of several previously known results which would be difficult to exhaustively quote. So let us mention the historical book [14] and the more recent work [6] summarizing all the ideas. The term $\mathrm{Q}_F(\mathrm{tr}_{\mathbb{S}^1} u(a_i+\delta\cdot))_{i=1}^{\kappa}$ in (iii) is a sum over $i = 1,\ldots,\kappa$ of $\mathrm{Q}_F([\mathrm{tr}_{\mathbb{S}^1}(a_i+\delta\cdot)])$ where for a closed geodesic $\gamma \in C^1(\mathbb{S}^1,\mathcal{N})$

$$\mathrm{Q}_F([\gamma]) \doteq \lim_{R\to\infty} \inf\left\{ \mathcal{G}_\epsilon(v;\mathbb{B}_R^2)\big|_{\epsilon=1} : \begin{array}{c} v\in \mathrm{W}^{1,2}(\mathbb{B}_R^2,\mathbb{R}^\nu) \\ \mathrm{tr}_{\mathbb{S}^1} u(R\cdot) = \gamma \end{array}\right\} - \frac{\lambda(\gamma)^2}{4\pi}\log R.$$

When $F(u) = (1-|u|^2)^2/4$ and $N = \mathbb{S}^1$, this last quantity is refered to as "γ/π" in the litterature where γ is a number, *cfr* [14, Lemma IX.1].

The boundedness described by (v) is an integrability result comparable to (1.4.9), the weak L^p-bound in $2d$. One indeed has for each $q \in [1,2)$

$$\int_\Omega |\mathrm{D}u|^q \leq \frac{C}{2-q}\left(\mathcal{W}^{1,(2,\infty)}(u;\Omega)\right)^{\frac{q}{2}}, \tag{1.5.4}$$

where $C > 0$ depends only on $\mathcal{H}^2(\Omega)$. Its first appearance in the literature is in the case $N = \mathbb{S}^1$ [70]. A result such as (v) is not possible in the framework of Theorem 1.4.1 as we look at almost minimizers (1.4.5). For minimizing $p-$harmonic map however we don't know if (v) holds; it would sharpen the bound (1.4.9) and shed light on the integrability result mentioned in Theorem 1.3.3.

The first order convergence (vi) is very comparable to (1.4.11).

1.5.1 The case of the circle

In the case of the circle, $N = \mathbb{S}^1$, one can be more precise. For instance one can write a formula for the limit u_* [14]. See [6] for a condensed $2d$ gamma convergence approach. Also, quantitative estimates are available for the convergence of the singular points [44] coded by Dirac masses which is a feature that we don't see in the $p-$harmonic setting for general manifold N. Another feature one would like to see for the $p-$harmonic setting is the equipartion of the energy. In [66], it is explained that if $u_\epsilon : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ is an $\epsilon-$minimizer of the Ginzburg Landau functional (1.5.1), then for all $h \in \mathbb{S}^1$,

$$\lim_{\epsilon\to 0} \frac{1}{\log\frac{1}{\epsilon}} \int_\Omega \frac{|\mathrm{D}u_\epsilon[h]|^2}{2} = \frac{1}{2} \times \lim_{\epsilon\to 0} \frac{1}{\log\frac{1}{\epsilon}} \int_\Omega \frac{|\mathrm{D}u_\epsilon|^2}{2}. \tag{1.5.5}$$

1.5.2 Higher dimensions

In higher dimensions, at the level of first order results we mention the results of [25]; they work with an Abelian $\pi_1(N)$. It is shown there that

Theorem 1.5.2. *Let $\Omega \subset \mathbb{R}^d$ be an open Lipschitz domain. Let $g \in \dot{\mathrm{W}}^{1/2,2}(\partial\Omega, N)$. If $u_\epsilon \in \dot{\mathrm{W}}^{1,2}(\Omega, \mathbb{R}^\nu)$ minimizes $\mathcal{G}_\epsilon(u)$ for $\epsilon > 0$, then the measures*

$$\mu_\epsilon = \Big(\frac{|\mathrm{D}u_\epsilon|^2}{2} + \frac{F(u_\epsilon)}{\epsilon^2}\Big)\frac{\mathrm{d}\mathcal{H}^d}{\log\frac{1}{\epsilon}}$$

converge weakly along a subsequence to the supporting measure μ_* of a flat chain S_* minimizing the area in its homology class.*

We refer to the precited paper for the vocabulary employed here: flat chains are neither varifold or currents but manifold carrying a multiplicity with values in a group; here the group is $\pi_1(N)$. We stress that again the object in the limit is area minimizing in some sense. Here the word "homology" refers to the fact that the area (or mass) is minimized among boundary perturbations: $\mathrm{M}(S) \leq \mathrm{M}(S + \partial R)$ for all R. Concerning the Abelianity of the $\pi_1(N)$, we quote the authors [25, Introduction]:

> "Unfortunately, the very statement of [our] Theorem does not make sense in the non-Abelian setting, because homology requires the coefficient group to be Abelian."

We record that, in the case of finite $\pi_1(N)$, our Theorem 1.4.2 in the $p \nearrow 2$–setting overcomes this difficulty in $3d$. In general –meaning $p \nearrow 2$ in any dimension– however, we seem to lack of vocabulary to even describe what would be the limiting object S in the non-Abelian setting.

Concerning second order results the circle is covered in any dimension [65][47][11].

Theorem 1.5.3. *Let $\Omega \subset \mathbb{R}^d$, $d \geq 3$ be an open Lipschitz domain. Take $F(u) = (1 - |u|^2)^2/4$. Let $g \in \dot{\mathrm{W}}^{1,2}(\partial\Omega, \mathbb{S}^1)$. If $u_\epsilon \in \dot{\mathrm{W}}^{1,2}(\Omega, \mathbb{R}^2)$ minimizes $\mathcal{G}_\epsilon(u)$ of trace g for $\epsilon > 0$, then along a subsequence $\epsilon_n \to 0$ there exists $u_* \in \dot{\mathrm{W}}^{1,1}(\Omega, \mathbb{S}^1)$ of trace g,* *(i)* $u_{\epsilon_n} \to u$ *strongly in* $\mathrm{W}^{1,1}(\Omega, \mathbb{R}^2)$*,*

(ii) There exists a closed singular set $S \subset \bar{\Omega}$ *of finite* $(d-2)$*-Hausdorff measure which is the support of* μ_* *such that, for every* $k \in \mathbb{N}$*,* $u_{\epsilon_n} \to u_*$ *in* $C^k_{\mathrm{loc}}(\Omega \setminus S_*, \mathbb{R}^2)$ *,*

(iii) μ_* *is the supporting measure of a stationary varifold,*

(iv) $\sup_n \int_\Omega |\mathrm{D}u_{\epsilon_n}|^{\frac{d}{d-1}}$ *is finite,*

(v) $\sup_n \int_K |\mathrm{D}u_{\epsilon_n}|^q$ *is finite for every* $q \in [1, 2)$ *and every compact* $K \subset \Omega$*.*

The estimate (iv) is optimal [16, Section 9]; it is one of the huge differences with the $p-$harmonic setting. The difference between (iv) and (v) is that the boundary is not covered.

Concerning other manifolds, not much is known in terms of second order analysis. In dimension 3, [22] writes it for specific models of the projective plane and a specific choice of f which inspired our work in $3d$ in the form of Theorem 1.4.1. We note that the fundamental group of the projective plane $\mathbb{R}\mathrm{P}^2$ is the two-element Abelian group $\mathbb{Z}_2$ at the intersection of the two covered settings. The work [22] does not contain anything about integrability, corresponding to the two last estimates of Theorem 1.5.3.

1.5.3 Bundle setting

In the Ginzburg-Landau case, answers in the case of principal $U(1)$-can be given. The litterature covers two cases : on the one side, one can perform a double minimization on sections and on connections on the bundle [60, 23, 61]. We describe the fixed connection setting as the function (1.5.1) is unchanged. If one minimizes also on the connection, a L^2-curvature term is often added, properly multiplied by a power of ϵ^2. The following theorem covers the problem of global sections [41][35].

Theorem 1.5.4. *Let* M^2 *be a closed Riemannian surface. Let L be a* $2-$*dimensional line bundle over* M^2 *with Hermitian complex structure and Riemannian structure* $g_L : \mathrm{T}L \times \mathrm{T}L \to \mathbb{R}$*. Fix a connection* A *on* L *acting by* $\mathrm{D}_A u = \mathrm{d}u + Au$ *on local sections* $u : M^2 \to L$*. If for each* $\epsilon > 0$*,* $u_\epsilon \in \Gamma_{\mathrm{W}^{1,2}_A}(M^2, L)$ *is a minimizer of*

$$\mathcal{G}_\epsilon(u) \doteq \int_M \frac{|\mathrm{D}_A u|^2}{2} + \frac{(1 - g_L(u,u))^2}{4\epsilon^2},$$

then, there exist a subsequence $u_{\epsilon_n} \to u_*$ *a.e. and a set of* $d = |\chi(L)|$ *points* $a_1, \dots, a_d \in M^2$ *where* $\chi(L)$ *is the Euler characteristic of* L *such that*

- $\mathcal{G}_{\epsilon_n}(u_{\epsilon_n}) - \pi|\chi(L)| \log \frac{1}{\epsilon_n} - |\chi(L)|\gamma \xrightarrow{n\to\infty} \mathcal{E}^{1,2}_{\mathrm{ren}}(u_*)$ *where* $\mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) = \lim_{\rho\to 0} \int_{M^2 \setminus \bigcup_{i=1}^d \mathbb{B}(a_i,\rho)} \frac{|\mathrm{D}_A u_*|^2}{2} - \pi|\chi(L)| \log \frac{1}{\rho}$,
- $\sup_n \int_{M^2} |\mathrm{D}_A u_{\epsilon_n}|^q$ *is finite for each* $q \in [1, 2)$,
- $u_{\epsilon_n} \to u_*$ *in* $\Gamma_{C^k}(M^2 \setminus \{a_1, \dots, a_d\}, L)$ *for all* $k \in \mathbb{N}$,
- *the points* $a_1, \dots, a_d$ *minimize* $\mathcal{W}(b_1, \dots, b_d) = \inf \mathcal{E}^{1,2}_{\mathrm{ren}}(v)$ *where the infimum is taken over the* v *which are sections smooth outside* $b_1, \dots, b_d \in M^2$.

Line bundles are $2d$ real vector spaces attached to the manifolds with a complex structure. Classical examples are the tangent and cotangent bundles. They are numbered by the second homology group $\mathrm{H}^2(M^2, \mathbb{Z})$ through the homology class of the curvature $\mathrm{K}_A = \mathrm{d}A$; the integral of the curvature is a multiple of the Euler characteristic. We point out that [41] is a Γ-convergence result which go beyond the connection framework and explore other derivations than the connection D_A. If $M^2 = \mathbb{S}^2$ and $L = \mathrm{T}S^2$, it can be shown that u_* has two singularities $a_1, a_2 \in \mathbb{S}^2$ which are diametrically opposed points on $\mathbb{S}^2$. For instance if $L = \mathrm{T}\mathbb{S}^2$ then $\chi(\mathrm{T}\mathbb{S}^2) = \chi(\mathbb{S}^2) = 2$ is the classical Euler characteristic we deduce that the canonical replacement to the absence of global section of unit norm (*cfr* (1.1.4)) is a two-defect section. However there is no canonical section as the problem of limiting renormalized canonical sections is in general not unique, *cfr* [41, Ex. 6.7]. We note that [35, Thm 2] describe placement of singularities when $|\chi(L)| \neq 2$. For a general fiber bundle we expect the Euler characteristic to be replaced by some kind of singular energy of the fibered bundle.

1.6 Area–type energies

The preceding sections are devoted to the Dirichlet energy appearing in (1.1.3) and (1.1.4). To produce a distinguished representative, one could be tempted to directly minimize the equivalent of the length (1.1.1) in higher dimension which is the area of the graph of maps $u : M \to N$. As this section will aim to demonstrate, the frontiers of mathematical knowledge have not yet progressed sufficiently or the faced problems described hereafter are simply ill-posed: in addition to topological conditions, one encounters obstructions of analytical nature, in the sense that it is even difficult to make sense of the problem itself using the tools of contemporary analysis.

We will directly discard the pullback by u of the volume for of M which, in dimension 1, gives the length. Indeed if $u : \mathbb{B}^2 \to \mathbb{S}^1$ is C^1 then pointwise $\det \mathrm{D}u = 0$. So, one has to look for suitable relaxation of the Jacobian determinant $\det \mathrm{D}u$: in dimension 2, for $p \in (1,2)$,

(1.6.1)
$$\mathrm{TV}(u;\mathbb{B}^2) \doteq \inf \left\{ \liminf_{n\to\infty} \int_{\mathbb{B}^2} |\det \mathrm{D}u_n| : \begin{array}{c} u_n \in \mathrm{W}^{1,2}(\mathbb{B}^2;\mathbb{R}^2) \\ u_n \to u \\ \mathrm{W}^{1,p}(\mathbb{B}^2,\mathbb{R}^2) - \text{weakly} \end{array} \right\}.$$

The sequence is taken in $\mathrm{W}^{1,2}(\mathbb{B}^2;\mathbb{R}^2)$ to make sense of the integral as, if $u \in \mathrm{W}^{1,2}(\mathbb{B}^2;\mathbb{R}^2)$,

$$|\det \mathrm{D}u| \leq \frac{|\mathrm{D}u|^2}{2}. \tag{1.6.2}$$

In fact, by a suitable integration by part one can give a meaning to $\det \mathrm{D}u$ in a distributional sense and write a representation formula for $\mathrm{TV}(u;\mathbb{B}^2)$ for many maps in $\mathrm{W}^{1,p}(\mathbb{B}^2,\mathbb{R}^2)$. We have written the relaxation with the weak–$\mathrm{W}^{1,p}$ convergence because in that case one has representation formulas in contrast with a strong$-\mathrm{L}^1$ relaxation. We write here a corollary of [31], relying on the regularity theory of minimizing $p-$harmonic mappings (see Theorem 1.3.4).

Theorem 1.6.1. *Let $p \in (1,2)$. If $u \in \dot{\mathrm{W}}^{1,p}(\mathbb{B}^2,\mathbb{S}^1)$ is a minimizing $p-$harmonic map of continous trace $\mathrm{tr}_{\mathbb{S}^1} u$, then*

$$\mathrm{TV}(u;\mathbb{B}^2) = \pi \sum_{i=1}^{k} |\deg(u,a_i)|,$$

where $a_1,\ldots,a_k \in \mathbb{B}^2$ are the singularities of u (cfr Theorem 1.3.4).

As a corollary, the vortex map $u_{\mathrm{V}}(x) = \frac{x}{|x|}$ satisfies $\mathrm{TV}(u_{\mathrm{V}};\mathbb{B}^2) = \pi$. While the total variation of the Jacobian determinant has received some attention, it lacks of coercivity: the fact that sequences of bounded total variations of the Jacobian variation admit a weakly converging subsequence in any reasonable topology.

Also, it is, as the Ginzburg-Landau energy, of *extrinsic character.* For a circle valued $u \in \dot{\mathrm{W}}^{1,p}(\mathbb{B}^2,\mathbb{S}^1)$ the sequence in the definition 1.6.1 is taken in the Euclidian space $\mathbb{R}^2$.

First we note that in dimension 1,

$$\inf\left\{\int_{\mathbb{S}^1} \frac{\sqrt{1+\delta^2|\dot{\tilde{\gamma}}|^2}-1}{\delta^2} : \tilde{\gamma} \in \dot{\mathrm{W}}^{1,1}(\mathbb{S}^1, N) \quad \tilde{\gamma} \in [\gamma]\right\} \tag{1.6.3}$$

is, when $\delta = 1$, the length of the graph of γ; for every $\delta > 0$, it is equal to the quantity (1.1.2). Therefore in dimension 1 the 3 problems (1.1.2), (1.1.1) and (1.6.3) are equal.

If $u : \Omega \subset \mathbb{R}^d \to N$, the area of its graph can be shown to be

$$\int_{\Omega} \sqrt{\det(I + \delta^2 \mathrm{D}u^t \mathrm{D}u)}, \tag{1.6.4}$$

it corresponds when $\delta = 1$ to the pullback of the volume form of $\Omega \times \mathbb{R}^d$ by $(u(x), x)$. The Cauchy–Binet formula [34, Ex. 3 p. 80] allows us to write the argument of the square root as a sum of the squares of all $k \times k$-subdeterminants of $\mathrm{D}u$, $k \in \{1, \dots, d\}$, meaning that

$$\det(I + \delta^2 \mathrm{D}u^t \mathrm{D}u) = 1 + \delta^2 |\mathrm{D}u|^2 + \text{nonnegative multilinear terms.}$$

In particular it is coercive: a sequence of bounded area will have finite uniformaly bounded $\dot{\mathrm{W}}^{1,1}$–energy; we will come back to this fact. If $d = 2$, the multilinear terms are all the square of the 2×2 minors of $\mathrm{D}u$. For mappings $u : \mathbb{B}^2 \to \mathbb{R}^2$, $\delta^4 (\det \mathrm{D}u)^2$ is the only multilinear term. When $d = 2$, using (1.6.2), we get a finite area (1.6.4) for $\dot{\mathrm{W}}^{1,2}(\mathbb{B}^2, \mathbb{R}^\nu)$ mappings.

Again, one is led to study a relaxed area functional: for $u : \Omega \to N$,

$$\mathcal{A}_{\delta,\mathrm{L}^1}(u;\Omega) \doteq \inf\left\{\lim_{n\to\infty} \int_{\Omega} \sqrt{\det(I + \delta^2 \mathrm{D}u_n^t \mathrm{D}u_n)} : \begin{array}{c} u_n \in C^1(\Omega, \mathbb{R}^\nu) \\ u_n \to u \in \mathrm{L}^1(\Omega, \mathbb{R}^\nu) \end{array}\right\}.$$

This definition of relaxed functional with L^1−convergence is motivated by the fact that it defines an L^1−lower semicontinous functional which is one on the crucial blocks for existence of minimizers by the direct method.

The domain *i.e.* the set of maps for which $\mathcal{A}_{\delta,\mathrm{L}^1}(u;\Omega)$ is finite is not known. As the following theorem shows, the domain of $\mathcal{A}_{\delta,\mathrm{L}^1}(u;\Omega)$ contains at least one map belonging to $\dot{\mathrm{W}}^{1,p}(\mathbb{B}^2, \mathbb{S}^1), p < 2$ but not $\dot{\mathrm{W}}^{1,2}(\mathbb{B}^2, \mathbb{S}^1)$.

Theorem 1.6.2. *If $u_{\mathrm{V}}(x) = \frac{x}{|x|}$ for $x \in \mathbb{B}^2$ is the vortex map, then $\mathcal{A}_{\delta,\mathrm{L}^1}(u_{\mathrm{V}}; \mathbb{B}^2)$ is finite and can be computed for all $\delta > 0$ [10]. If δ is sufficiently small [2],*

$$\mathcal{A}_{\delta,\mathrm{L}^1}(u_{\mathrm{V}}; \mathbb{B}^2) = \int_{\mathbb{B}^2} \sqrt{1 + \delta^2 |\mathrm{D}u_{\mathrm{V}}|^2} + \pi. \tag{1.6.5}$$

The contribution of [10] is the fact that for all $\delta > 0$,

$$\mathcal{A}_{\delta,\mathrm{L}^1}(u_{\mathrm{V}};\mathbb{B}^2) = \int_{\mathbb{B}^2} \sqrt{1+\delta^2|\mathrm{D}u_{\mathrm{V}}|^2} + \text{“some Plateau problem”}$$

and a precise description of the second term. The highest threshold $\delta_0 > 0$ for which $\delta \in (0,\delta_0)$ implies (1.6.5) is not known. Though the estimate $\delta_0 < 2$ is known. One notes that, since in $2d$, $|\mathrm{D}u_{\mathrm{V}}|(x) = |x|^{-1}$, the integral in (1.6.5) can actually be further simplified using the fundamental theorem of calculus.

It seems that the study of the area functional in its L^1-relaxation shape is quite difficult. If one was to provide yet another argument in addition to the result given above, we would say that the functional does not even satisfy the reasonable condition of subadditivity on its domain in dimension $d \geq 3$. If one takes a slightly stronger topology, namely strict convergence of mappings of bounded variations, the situation becomes a bit less cloudy; we however lose the natural lower semicontinuity property. We only quote here the $2d$ case [9] while higher dimensions are available [28, 2].

Theorem 1.6.3. *If* $\gamma \in C^1(\mathbb{S}^1,\mathbb{R}^2)$, *define* $u(x) = \gamma(\frac{x}{|x|})$, $x \in \mathbb{B}^2$ *and*

$$\mathcal{A}_{\delta,\mathrm{BV}}(u;\mathbb{B}^2) = \inf\left\{ \lim_{n\to\infty}\int_{\mathbb{B}^2} \sqrt{\det(I+\delta^2 \mathrm{D}u_n^t\mathrm{D}u_n)} : \begin{array}{c} u_n \in C^1(\Omega,\mathbb{R}^\nu) \\ u_n \to u \in \mathrm{L}^1(\mathbb{B}^2,\mathbb{R}^\nu) \\ \int_{\mathbb{B}^2}|\mathrm{D}u_n| \to \int_{\mathbb{B}^2}|\mathrm{D}u| \end{array}\right\}, \tag{1.6.6}$$

then for all $\delta > 0$

$$\mathcal{A}_{\delta,\mathrm{BV}}(u;\mathbb{B}^2) = \int_{\mathbb{B}^2}\sqrt{1+\delta^2|\mathrm{D}u|^2} + \pi|\deg\gamma|. \tag{1.6.7}$$

While $\mathcal{A}_{\delta,\mathrm{L}^1}(u,\mathbb{B}^d)$ is coercive and weakly lower semicontinuous, a sequence $u_n : \mathbb{B}^d \to \mathbb{R}^\nu$ of uniformly bounded $\mathcal{A}_{\delta,\mathrm{L}^1}(u_n,\mathbb{B}^d)$ is precompact in $\dot{\mathrm{BV}}(\mathbb{B}^d,\mathbb{R}^\nu)$, the space of mappings of bounded variations whose gradient is a measure and to our knowledge we are not able to give a meaning to $\mathcal{A}_{\delta,\mathrm{L}^1}$ unless we neglect the determinant terms as we will do in the next section.

1.6.1 The codimension 1 area functional

Faced with all these difficulties, let us introduce the following approximation of the area:

$$A^1_\delta(u;\Omega) \doteq \int_\Omega \sqrt{1+\delta^2|\mathrm{D}u|^2}. \tag{1.6.8}$$

It is defined for all $u \in \dot{\mathrm{W}}^{1,1}(\Omega, N)$ and corresponds to an area, that of the graph $(x, |u|(x))$, which loses its geometric flavor. If we introduce its natural L^1 relaxation:

(1.6.9)
$$\mathcal{A}^1_\delta(u;\Omega) \doteq \inf\left\{\lim_{n\to\infty}\int_\Omega \sqrt{1+\delta^2|\mathrm{D}u_n|^2} : \begin{array}{c} u_n \in C^1(\Omega,\mathbb{R}^\nu) \\ u_n \to u \text{ in } \mathrm{L}^1(\Omega,\mathbb{R}^\nu)\end{array}\right\}.$$

The computation of the relaxation of $\mathcal{A}^1_\delta(u;\Omega)$ is a classic result in the theory of functionals with linear growth. We are interested in a Dirichlet type problem; therefore we restrict our attention to a fixed trace problem. Take $g : \partial\Omega \to N$ an integrable map and define

(1.6.10)
$$\mathcal{A}^1_{\delta,g}(u;\Omega) \doteq \inf\left\{\lim_{n\to\infty}\int_\Omega \sqrt{1+\delta^2|\mathrm{D}u_n|^2} : \begin{array}{c} u_n \in \dot{\mathrm{W}}^{1,1}(\Omega,\mathbb{R}^\nu) \\ \operatorname{tr}_{\partial\Omega} u_n = g \\ u_n \to u \text{ in } \mathrm{L}^1(\Omega,\mathbb{R}^\nu)\end{array}\right\}.$$

We recall that by Theorem 1.2.3 there exists $u \in \dot{\mathrm{W}}^{1,1}(\Omega,\mathbb{R}^\nu)$ such that $\operatorname{tr}_{\partial\Omega} u = g$.

Theorem 1.6.4. *The domain of $\mathcal{A}^1_{\delta,g}(\cdot;\Omega)$ and $\mathcal{A}^1_\delta(\cdot;\Omega)$ are the mappings $u : \Omega \to \mathbb{R}^\nu$ of bounded variations denoted $\dot{\mathrm{B}}\mathrm{V}(\Omega,\mathbb{R}^\nu)$. Moreover, if $u \in \dot{\mathrm{B}}\mathrm{V}(\Omega,\mathbb{R}^\nu)$,*

$$\mathcal{A}_{\delta,g}(u,\Omega) = \int_\Omega \sqrt{1+\delta^2|\mathrm{D}^{\mathrm{a}}u|^2} + \delta|\mathrm{D}^{\mathrm{s}}u|(\Omega) + \delta\int_{\partial\Omega} |\operatorname{tr}_{\partial\Omega} u - g|\,\mathrm{d}\mathcal{H}^1 \tag{1.6.11}$$

where $\mathrm{D}u = \mathrm{D}^{\mathrm{a}}u\,\mathcal{L}^d + \mathrm{D}^{\mathrm{s}}u$ is the Lebesgue decomposition of the differential.

The functional is lower semicontinuous. However, aside from the fact that this functional is not intrinsic when restricted to mappings $\Omega \to N \subset \mathbb{R}^\nu$, we face a new problem that we would have also had for the other area functions: the non attainment of boundary values. If one takes a minimizing sequence of $\mathcal{A}^1_{\delta,g}(\cdot;\Omega)$, one obtains a generalized minimizer $u \in \dot{\mathrm{B}}\mathrm{V}(\Omega,\mathbb{R}^\nu)$ satisfying for all $v \in \dot{\mathrm{B}}\mathrm{V}(\Omega,\mathbb{R}^\nu)$

$$\mathcal{A}^1_{\delta,g}(u;\Omega) \le \mathcal{A}^1_{\delta,g}(v;\Omega). \tag{1.6.12}$$

This does not imply that $\operatorname{tr}_{\partial\Omega} u = g$ and in fact the two last terms are not strictly convex: this confers to the question of uniqueness of generalized minimizers a nontrivial taste. Here is a positive result [8].

Theorem 1.6.5. *Let $\Omega \subset \mathbb{R}^d$ be a Lipschitz domain. Let $g \in \mathrm{L}^1(\partial\Omega,\mathbb{R}^\nu)$. Any generalized minimizer $u \in \dot{\mathrm{B}}\mathrm{V}(\Omega,\mathbb{R}^\nu)$ satisfying (1.6.12) satisfies $u \in \dot{\mathrm{W}}^{1,1}(\Omega,\mathbb{R}^\nu)$ and $|\mathrm{D}u|\log(1+|\mathrm{D}u|^2)$ is locally integrable in Ω.*

Theorem 1.6.5 tells us that we may discard the singular part of the gradient in 4.2.42 for generalized minimizers. The method of proof is based on finite differences and makes it unsuitable for manifold constrained mappings. Concerning nonuniqueness, the authors show that there exist $u_0 \in \dot{\mathrm{BV}}(\Omega, \mathbb{R}^\nu)$ and $h \in \mathbb{R}^\nu$ such that all generalized minimizers write $u_0 + th$ for some $t \in [-1, 1]$.

1.6.2 Asymptotic analysis

If we were to summarize the previous sections, we would say that the codimension 1 area functional is arguably the easiest to understand despite everything we have described. We have also seen that when $\delta > 0$ is small, one can expect better behavior of the graph's area. Moreover, one observes that

$$f_\delta(z) \doteq \frac{\sqrt{1 + \delta^2 |z|^2} - 1}{\delta^2} \xrightarrow{\delta \to 0} \frac{|z|^2}{2}. \tag{1.6.13}$$

In collaboration with Christopher Irving, in the $2d$ setting, we have given this limit a formal meaning, *cfr* Theorem 1.7.1 below. This therefore supports the fact that the area behaves better when $\delta > 0$ is small.

1.7 Universality, nonlocal energies and other functionals

In fact we prove, the following *universality result.*

Theorem 1.7.1. *Let $(f_n : [0, +\infty) \to [0, +\infty))_n$ be a sequence of convex function converging pointwise to $|z|^2/2$. Let us assume that*

$$t \in [0, +\infty) \mapsto f_n \circ \sqrt{t} \text{ is concave} \tag{1.7.1}$$

and that the vortex energy

$$\mathcal{V}(f_n) \doteq \frac{1}{\pi} \int_{\mathbb{B}^2} f_n(|\mathrm{D}u_{\mathrm{V}}|) \tag{1.7.2}$$

is finite. Suppose $(u_n)_n \subset \dot{\mathrm{W}}^{1,1}(\Omega; \mathcal{N})$ is an asymptotically minimizing sequence in the sense that $\mathrm{tr}_{\partial\Omega}\, u_n = g$ for all n and

$$\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \inf\left\{ \int_\Omega f_n(|\mathrm{D}v|)\,\mathrm{d}x : \begin{array}{c} v \in \dot{\mathrm{W}}^{1,1}(\Omega; N) \\ \mathrm{tr}_{\partial\Omega}\, v = g \end{array} \right\} \xrightarrow{n\to\infty} 0. \tag{1.7.3}$$

Then, passing to a subsequence, u_n converges strongly in $\dot{\mathrm{W}}^{1,1}$ to a limit map $u_* \in \dot{\mathrm{W}}^{1,1}(\Omega; N)$ with $\mathrm{tr}_{\partial\Omega}\, u_* = g$ satisfying

$$u_* \in \dot{\mathrm{W}}^{1,2}\Big(\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \rho); N\Big) \quad \text{for all } \rho > 0, \tag{1.7.4}$$

where $\kappa \in \mathbb{N}$ *and* $a_1, \ldots, a_\kappa \in \Omega$*. Moreover, for all* $\rho > 0$ *sufficiently small,*

(i) u_* *is a minimizing harmonic map on* $\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}^2(a_i, \rho)$ *with respect to its own boundary conditions,*

(ii) $\displaystyle\int_{\Omega\setminus\bigcup_{i=1}^{\kappa} \mathbb{B}^2(a_i,\rho)} f_n(|\mathrm{D}u_* - \mathrm{D}u_n|)\,\mathrm{d}x \xrightarrow{n\to\infty} 0,$

(iii) u_* *is renormalizable and we have*

$$\lim_{n\to\infty} \left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathcal{E}^{1,2}_{\mathrm{sg}}(g)\right] = \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa},$$

where $\mathcal{V}(f_n) \doteq \dfrac{1}{\pi}\displaystyle\int_{\mathbb{B}^2} f(|\mathrm{D}u_{\mathrm{V}}|)$ *is the vortex energy.*

Theorem 1.7.1 is a generalization of Theorem 1.4.1 concerning $p-$harmonic maps. It also covers the area (1.6.13). Theorems 1.4.1 and 1.5.3 suggests that a harmonic extension of Dirichlet data in dimension $2d$ is achieved by minimizing the renormalized energy; here, we support this fact by arguing that this renormalized limit does not depend on the way it is approached, which gives it a universal character. While the convexity assumption of $(f_n)_n$ seems natural, as does the assumption of having the vortex within the domain of the functional *cfr* (1.7.2), we do not know whether the condition of being sub-quadratic (1.7.1) is optimal. Our result also supports nonstrictly convex functionals such as

$$f_n(z) = \begin{cases} \frac{1}{2}|z|^2 & \text{if } |z| < \kappa_n, \\ \kappa_n(|z| - \kappa_n/2) & \text{if } |z| \geq \kappa_n \end{cases} \text{where } \kappa_n \to +\infty.$$

Let us note here that the working assumption (1.7.3), which deals with asymptotic minimizers, saves the result from covering only the empty set in this latter case, since—as mentioned in the previous section—the minimizers do not always reach their boundary data. Of course, this result would deserve a BV case, but the essential ingredients to provide a proof with our method, namely a density theorem for smooth maps away from points, are not available in the case of mappings of bounded variations valued in a manifold.

Let us nuance the universal nature of the above theorem, as not all minimization problems take the canonical form

$$\inf\left\{\int_\Omega f_n(|\mathrm{D}v|)\,\mathrm{d}x : \begin{array}{c} v \in \dot{\mathrm{W}}^{1,1}(\Omega; N) \\ \mathrm{tr}_{\partial\Omega}\, v = g \end{array}\right\}.$$

For example, still in dimension 2, one could study the fractional Sobolev spaces through

$$\mathcal{W}^{s,p}(u;\Omega) \doteq \iint_{\Omega\times\Omega} \frac{\mathrm{dist}_N(u(y),u(x))^p \,\mathrm{d}x\,\mathrm{d}y}{\mathrm{dist}_{\partial M}(x,y)^{sp+\dim(\Omega)}}$$

as $(s,p) \to (1,2)$ with $sp < 2$ [7]. These energies also measure the oscillation of the function, as the integral of the gradient does, but in a way that is not local: changing the value in a small neighborhood of a point modifies the integrand over a large portion of the interaction domain. Of course, one could also study the case where the domain is the "complement of the complements"

$$(\mathbb{R}^2 \times \mathbb{R}^2) \setminus \big((\mathbb{R}^2 \setminus \Omega) \times (\mathbb{R}^2 \setminus \Omega)\big),$$

which contains the Cartesian product $\Omega \times \Omega$ and, from our perspective, would probably make the definition of boundary values simpler. In general, one can ask the question of a result similar to Theorem 1.7.1 for Bourgain-Brezis-Mironescu-type functionals [17], such as

$$\mathcal{B}_\epsilon(u;\Omega) \doteq \frac{1}{\epsilon^{d+2}\log\frac{1}{\epsilon}} \iint_{\Omega\times\Omega} \rho\Big(\frac{|x-y|}{\epsilon}\Big)|u(x)-u(y)|^2 \,\mathrm{d}x\,\mathrm{d}y \quad \epsilon > 0$$

where $\rho : [0,+\infty) \to [0,+\infty)$ is integrable. When $N = \mathbb{S}^1$ this has been implemented at first order but in any dimensions by Margherita Solci [72]. Here we state a corollary in our terms.

Theorem 1.7.2. *Let $\Omega \subset \mathbb{R}^d$ be an open Lipschitz domain. Let $g : \mathbb{R}^d \setminus \Omega \to \mathbb{S}^1$. For $\epsilon > 0$, let u_ϵ be such that in $\mathbb{R}^d \setminus \Omega$, $u_\epsilon = g$. Assume $\rho \geq \rho_0 > 0$ in a neighborhood of the origin. If u_ϵ is an almost minimizer of $\mathcal{B}_\epsilon(u;\Omega)$ then there exists a sequence $(u_{\epsilon_n})_n$ such that the Jacobian determinant of the piecewise affine interpolation of means on cubes of size ϵ_n of u_{ϵ_n} converges in flat norm to an integral $(d-2)-$current which is mass (or area) minimizing.*

We refer to the aforementioned paper for the vocabulary used here, as well as for the notion of convergence. The point is that the convergence is not easy to describe. The author relies, in fact, on a comparable result in a discrete context [3]. There, among the studied functionals, we record

$$\mathcal{X}_\epsilon(u) \doteq \frac{1}{\log\frac{1}{\epsilon}} \sum_{i\sim j} \frac{|u(\epsilon i)-u(\epsilon j)|^2}{2} \tag{1.7.5}$$

where $u : \epsilon\mathbb{Z}^d \cap \Omega \to \mathbb{S}^1$ and $i \sim j$ means that $i,j \in \mathbb{Z}^d$ are neighbors on the grid: the distance between them is of one unit. In fact, in $2d$, in [26], the authors are able to write a second convergence result for a cousin functional of (1.7.5), having as domain sections of a principal $U(1)$-bundle defined on the vortices of a suitable triangulation converging to a minimizer of a renormalized energy. Roughly speaking, one could say that it is a discrete cousin of Theorem 1.5.4.

Let us also mention an approach inspired by homogenization and the Ginzburg-Landau functional [4], where the authors study

$$\mathcal{H}_\epsilon(u;\Omega) \doteq \frac{1}{\log\frac{1}{\epsilon}} \int_\Omega a\left(\frac{x}{\delta_\epsilon}\right) \frac{|\mathrm{D}u(x)|^2}{2} + \frac{F(u(x))}{\epsilon^2}\,\mathrm{d}x$$

in the case $N = \mathbb{S}^1$ and in dimension 2, and demonstrate first-order convergence. One can also synthetize the Ginzburg-Landau functional and fractional energies [5].

Similarly, in [30], the authors define, also in the case of the circle and on two-dimensional domains, for all $u \in \dot{\mathrm{SBV}}(\Omega,\mathbb{R}^2)$ (the set of special mappings of bounded variations carrying only a jump part in their singular part whose support is denoted J_u),

$$\mathcal{F}(u) = \frac{1}{\log\frac{1}{\epsilon}} \int_\Omega \frac{|\mathrm{D}^{\mathrm{a}}u|^2}{2} + \frac{\mathcal{H}^1(\mathrm{J_u})}{\epsilon}$$

where $\mathrm{D}^{\mathrm{a}}u$ is defined as in Theorem 1.6.4. They provide first order Γ-convergence results.

We conclude this section on functionals converging to the Dirichlet energy by mentionning the result of [33] where a direct variational approach is given in the context of circle valued mappings.

1.8 Organization of the present thesis

Aside from the introduction, this thesis comprises four chapters, which are scientific articles written during the course of the doctoral program. Their format was designed for publication in specialized mathematics journals.

The first chapter consists in

> J. Van Schaftingen and B. Van Vaerenbergh. "Asymptotic behavior of minimizing p-harmonic maps when $p \nearrow 2$ in dimension 2". English. In: *Calc. Var. Partial Differ. Equ.* 62.8 (2023). Id/No 229, p. 45

A detailled form of Theorem 1.4.1 is proven. In the second chapter we reproduce

> B. Bulanyi, J. Van Schaftingen, and B. Van Vaerenbergh. "Limiting behavior of minimizing p-harmonic maps in 3d as p goes to 2 with finite fundamental group". English. In: *Arch. Ration. Mech. Anal.* 249.3 (2025). Id/No 29, p. 103

where the results of Section 1.4.2 are gathered. In

> C. Irving and B. Van Vaerenbergh. *Universality of renormalisable mappings in two dimensions: the case of polar convex integrands.* Preprint, arXiv:2411.17520 [math.AP] (2024). 2024

the universality result Theorem 1.7.1 is derived. Finally in the fourth chapter, the extension result

> J. Van Schaftingen and B. Van Vaerenbergh. *Extensions of traces for Sobolev mappings into manifolds at the endpoint* $p = 1$. 2025

mentioned in Theorem 1.2.3 is explained as well as many variants.

Bibliography of Chapter 1

[1] A. Abbondandolo. "On the homotopy type of VMO". In: *Topol. Methods Nonlinear Anal.* 7.2 (1996), pp. 431–436.

[2] E. Acerbi and G. Dal Maso. "New lower semicontinuity results for polyconvex integrals". In: *Calc. Var. Partial Differ. Equ.* 2.3 (1994), pp. 329–371.

[3] R. Alicandro and M. Cicalese. "Variational analysis of the asymptotics of the XY model". In: *Arch. Ration. Mech. Anal.* 192.3 (2009), pp. 501–536.

[4] R. Alicandro, A. Braides, M. Cicalese, L. De Luca, and A. Piatnitski. "Topological singularities in periodic media: Ginzburg-Landau and core-radius approaches". In: *Arch. Ration. Mech. Anal.* 243.2 (2022), pp. 559–609.

[5] R. Alicandro, A. Braides, M. Solci, and G. Stefani. *Topological singularities arising from fractional-gradient energies.* Preprint, arXiv:2309.10112 [math.AP] (2023). 2023.

[6] R. Alicandro and M. Ponsiglione. "Ginzburg-Landau functionals and renormalized energy: a revised γ-convergence approach". In: *J. Funct. Anal.* 266.8 (2014), pp. 4890–4907.

[7] M. Badran. *Harmonic maps to the circle with higher dimensional singular set.* Preprint, arXiv:2411.14186 [math.DG] (2024). 2024.

[8] L. Beck and T. Schmidt. "On the Dirichlet problem for variational integrals in BV". In: *J. Reine Angew. Math.* 674 (2013), pp. 113–194.

[9] G. Bellettini, S. Carano, and R. Scala. "The relaxed area of $\mathbb{S}^1$-valued singular maps in the strict BV-convergence". In: *ESAIM, Control Optim. Calc. Var.* 28 (2022). Id/No 56, p. 38.

[10] G. Bellettini, A. Elshorbagy, and R. Scala. *The L^1-relaxed area of the graph of the vortex map.* 2021.

[11] F. Bethuel, H. Brezis, and G. Orlandi. "Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions". In: *Journal of Functional Analysis* 186.2 (2001), pp. 432–520.

[12] F. Bethuel and F. Demengel. "Extensions for Sobolev mappings between manifolds". In: *Calc. Var. Partial Differ. Equ.* 3.4 (1995), pp. 475–491.

[13] F. Bethuel. "A new obstruction to the extension problem for Sobolev maps between manifolds". In: *J. Fixed Point Theory Appl.* 15.1 (2014), pp. 155–183.

[14] F. Bethuel, H. Brezis, and F. Hélein. *Ginzburg-Landau vortices.* Reprint of the 1994 hardback edition. Mod. Birkhäuser Class. Cham: Birkhäuser Springer, 2017.

[15] F. Bethuel, H. Brézis, and F. Helein. "Limite singulière pour la minimisation de fonctionnelles du type Ginzburg-Landau. (Singular limit for the minimization of Ginzburg-Landau functionals)". In: *C. R. Acad. Sci., Paris, Sér. I* 314.12 (1992), pp. 891–895.

[16] J. Bourgain, H. Brezis, and P. Mironescu. "$H^{1/2}$ maps with values into the circle: minimal connections, lifting, and the Ginzburg-Landau equation". In: *Publ. Math., Inst. Hautes Étud. Sci.* 99 (2004), pp. 1–115.

[17] J. Bourgain, H. Brezis, and P. Mironescu. "Another look at Sobolev spaces". In: *Optimal control and partial differential equations. In honour of Professor Alain Bensoussan's 60th birthday. Proceedings of the conference, Paris, France, December 4, 2000.* Amsterdam: IOS Press; Tokyo: Ohmsha, 2001, pp. 439–455.

[18] H. Brezis and P. Mironescu. *Sobolev maps to the circle. From the perspective of analysis, geometry, and topology.* Vol. 96. Prog. Nonlinear Differ. Equ. Appl. New York, NY: Birkhäuser, 2021.

[19] H. Brézis and L. Nirenberg. "Degree theory of BMO. I: Compact manifolds without boundaries". In: *Sel. Math., New Ser.* 1.2 (1995), pp. 197–263.

[20] B. Bulanyi and J. Van Schaftingen. "Singular extension of critical Sobolev mappings under an exponential weak-type estimate". In: *J. Funct. Anal.* 288.1 (2025). Id/No 110681, p. 27.

[21] B. Bulanyi, J. Van Schaftingen, and B. Van Vaerenbergh. "Limiting behavior of minimizing p-harmonic maps in 3d as p goes to 2 with finite fundamental group". In: *Arch. Ration. Mech. Anal.* 249.3 (2025). Id/No 29, p. 103.

[22] G. Canevari. "Line defects in the small elastic constant limit of a three-dimensional Landau-de Gennes model". In: *Arch. Ration. Mech. Anal.* 223.2 (2017), pp. 591–676.

[23] G. Canevari, F. L. Dipasquale, and G. Orlandi. "The Yang-Mills-Higgs functional on complex line bundles: Γ-convergence and the London equation". In: *Arch. Ration. Mech. Anal.* 247.6 (2023). Id/No 104, p. 67.

[24] G. Canevari and G. Orlandi. "Improved partial regularity for manifold-constrained minimisers of subquadratic energies". In: *Commun. Math. Phys.* 374.3 (2020), pp. 1483–1495.

[25] G. Canevari and G. Orlandi. "Topological singular set of vector-valued maps. II: *Γ*-convergence for Ginzburg-Landau type functionals". In: *Arch. Ration. Mech. Anal.* 241.2 (2021), pp. 1065–1135.

[26] G. Canevari and A. Segatti. "Defects in nematic shells: a Γ-convergence discrete-to-continuum approach". In: *Arch. Ration. Mech. Anal.* 229.1 (2018), pp. 125–186.

[27] G. Canevari, A. Segatti, and M. Veneroni. "Morse's index formula in VMO for compact manifolds with boundary". In: *J. Funct. Anal.* 269.10 (2015), pp. 3043–3082.

[28] S. Carano and D. Mucci. “Strict BV relaxed area of Sobolev maps into the circle: the high dimension case”. In: *NoDEA, Nonlinear Differ. Equ. Appl.* 31.4 (2024). Id/No 54, p. 36.

[29] A. Convent and J. Van Schaftingen. “Intrinsic co-local weak derivatives and Sobolev spaces between manifolds”. In: *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)* 16.1 (2016), pp. 97–128.

[30] L. De Luca, R. Scala, and N. van Goethem. “A new approach to topological singularities via a weak notion of Jacobian for functions of bounded variation”. In: *Indiana Univ. Math. J.* 73.2 (2024), pp. 723–779.

[31] I. Fonseca, N. Fusco, and P. Marcellini. “On the total variation of the Jacobian.” In: *J. Funct. Anal.* 207.1 (2004), pp. 1–32.

[32] E. Gagliardo. “Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili”. In: *Rend. Semin. Mat. Univ. Padova* 27 (1957), pp. 284–305.

[33] F. Gaia and T. Rivière. “A variational approach to S^1-harmonic maps and applications”. In: *J. Funct. Anal.* 285.11 (2023). Id/No 110147, p. 58.

[34] M. Giaquinta, G. Modica, and J. Souček. *Cartesian currents in the calculus of variations I. Cartesian currents.* Vol. 37. Ergeb. Math. Grenzgeb., 3. Folge. Berlin: Springer, 1998.

[35] D. Golovaty, A. Montero, E. Sandier, and P. Sternberg. “On Sections of Complex Line Bundles Over Surfaces Minimizing a Ginzburg–Landau Energy”. In: *J. Nonlinear Sci.* 35.1 (2025), Paper No. 22.

[36] O. Hanner. “On the uniform convexity of L^p and l^p”. In: *Ark. Mat.* 3 (1956), pp. 239–244.

[37] R. Hardt and B. Chen. “Prescribing singularities for p-harmonic maps”. In: *Indiana Univ. Math. J.* 44.2 (1995), pp. 575–602.

[38] R. Hardt and F. Lin. “Mappings minimizing the L^p norm of the gradient”. In: *Commun. Pure Appl. Math.* 40.5 (1987), pp. 555–588.

[39] R. Hardt, F. Lin, and C. Wang. “Singularities of p-energy minimizing maps”. In: *Commun. Pure Appl. Math.* 50.5 (1997), pp. 399–447.

[40] R. Hardt, F. Lin, and C. Wang. “The p-energy minimality of $x/|x|$”. In: *Commun. Anal. Geom.* 6.1 (1998), pp. 141–152.

[41] R. Ignat and R. L. Jerrard. “Renormalized energy between vortices in some Ginzburg-Landau models on 2-dimensional Riemannian manifolds”. In: *Arch. Ration. Mech. Anal.* 239.3 (2021), pp. 1577–1666.

[42] C. Irving and B. Van Vaerenbergh. *Universality of renormalisable mappings in two dimensions: the case of polar convex integrands.* Preprint, arXiv:2411.17520 [math.AP] (2024). 2024.

[43] T. Isobe. “Obstructions to the extension problem of Sobolev mappings”. In: *Topol. Methods Nonlinear Anal.* 21.2 (2003), pp. 345–368.

[44] R. Jerrard and D. Spirn. “Refined Jacobian estimates for Ginzburg-Landau functionals”. In: *Indiana Univ. Math. J.* 56.1 (2007), pp. 135–186.

[45] J. M. Lee. *Introduction to topological manifolds.* 2nd ed. Vol. 202. Grad. Texts Math. New York, NY: Springer, 2011.

[46] F. H. Lin. “Topology of Sobolev mappings and associated variational problems”. In: *Surveys in geometric analysis and relativity. Dedicated to Richard Schoen in honor of his 60th birthday.* Somerville, MA: International Press; Beijing: Higher Education Press, 2011, pp. 363–393.

[47] F. Lin and T. Rivière. "Complex Ginzburg-Landau equations in high dimensions and codimension two area minimizing currents". In: *J. Eur. Math. Soc. (JEMS)* 1.3 (1999), pp. 237–311.

[48] F. Lin and X. Yang. *Geometric measure theory. An introduction.* Vol. 1. Adv. Math., Beijing/Boston. Beijing: Science Press; Boston, MA: International Press, 2002.

[49] J. Mawhin. "A simple approach to Brouwer degree based on differential forms". In: *Adv. Nonlinear Stud.* 4.4 (2004), pp. 535–548.

[50] K. Mazowiecka, M. Miśkiewicz, and A. Schikorra. *On the Size of the Singular Set of Minimizing Harmonic Maps.* Oct. 2024.

[51] K. Mazowiecka and J. Van Schaftingen. "Quantitative characterization of traces of Sobolev maps". In: *Commun. Contemp. Math.* 25.2 (2023). Id/No 2250003, p. 31.

[52] J. W. Milnor. *Morse theory. Based on lecture notes by M. Spivak and R. Wells.* Reprint of the 1963 original published by Princeton University Press. Vol. 66. Texts Read. Math. New Delhi: Hindustan Book Agency, 2013.

[53] P. Mironescu and J. Van Schaftingen. "Trace theory for Sobolev mappings into a manifold". In: *Ann. Fac. Sci. Toulouse, Math. (6)* 30.2 (2021), pp. 281–299.

[54] A. Monteil, R. Rodiac, and J. Van Schaftingen. "Ginzburg-Landau relaxation for harmonic maps on planar domains into a general compact vacuum manifold". In: *Arch. Ration. Mech. Anal.* 242.2 (2021), pp. 875–935.

[55] A. Monteil, R. Rodiac, and J. Van Schaftingen. "Renormalised energies and renormalisable singular harmonic maps into a compact manifold on planar domains". In: *Math. Ann.* 383.3-4 (2022), pp. 1061–1125.

[56] A. Naber, D. Valtorta, and G. Veronelli. "Quantitative regularity for p-harmonic maps". In: *Commun. Anal. Geom.* 27.1 (2019), pp. 111–159.

[57] N. Nakauchi. "Homomorphism between homotopy groups induced by elements of the Sobolev space $L^{1,p}(M,N)$". In: *Manuscr. Math.* 78.1 (1993), pp. 1–7.

[58] J. Nash. "C^1 isometric imbeddings". In: *Ann. Math. (2)* 60 (1954), pp. 383–396.

[59] J. Nash. "The imbedding problem for Riemannian manifolds". In: *Ann. Math. (2)* 63 (1956), pp. 20–63.

[60] G. Orlandi. "Asymptotic behavior of the Ginzburg-Landau functional on complex line bundles over compact Riemann surfaces". In: *Rev. Math. Phys.* 8.3 (1996), pp. 457–486.

[61] D. Parise, A. Pigati, and D. Stern. "Convergence of the self-dual $U(1)$-Yang-Mills-Higgs energies to the $(n-2)$-area functional". In: *Commun. Pure Appl. Math.* 77.1 (2024), pp. 670–730.

[62] M. Petrache and T. Rivière. "Global gauges and global extensions in optimal spaces". In: *Anal. PDE* 7.8 (2014), pp. 1851–1899.

[63] M. Petrache and J. Van Schaftingen. "Controlled singular extension of critical trace Sobolev maps from spheres to compact manifolds". In: *Int. Math. Res. Not.* 2017.12 (2017), pp. 3647–3683.

[64] F. Rindler. *Calculus of variations.* Universitext. Cham: Springer, 2018.

[65] T. Riviere. "Lines vortices in the U (1)-Higgs model". In: *ESAIM: Control, Optimisation and Calculus of Variations* 1 (1996), pp. 77–167.

[66] E. Sandier and S. Serfaty. "A product-estimate for Ginzburg-Landau and corollaries". In: *J. Funct. Anal.* 211.1 (2004), pp. 219–244.

[67] R. Schoen and K. Uhlenbeck. "A regularity theory for harmonic maps". In: *J. Differ. Geom.* 17 (1982), pp. 307–335.

[68] R. Schoen and S.-T. Yau. “Existence of incompressible minimal surfaces and the topology of three dimensional manifolds with non-negative scalar curvature”. In: *Ann. Math. (2)* 110 (1979), pp. 127–142.

[69] S. Serfaty and I. Tice. “Lorentz space estimates for the Ginzburg-Landau energy”. In: *J. Funct. Anal.* 254.3 (2008), pp. 773–825.

[70] S. Serfaty and I. Tice. “Lorentz space estimates for the Ginzburg-Landau energy”. In: *J. Funct. Anal.* 254.3 (2008), pp. 773–825.

[71] L. Simon. *Theorems on regularity and singularity of energy minimizing maps. Based on lecture notes by Norbert Hungerbühler.* Basel: Birkhäuser, 1996.

[72] M. Solci. “Nonlocal-interaction vortices”. In: *SIAM J. Math. Anal.* 56.3 (2024), pp. 3430–3451.

[73] D. Stern. “p-harmonic maps to S^1 and stationary varifolds of codimension two”. In: *Calc. Var. Partial Differ. Equ.* 59.6 (2020). Id/No 187, p. 46.

[74] M. Struwe. *Plateau’s problem and the calculus of variations.* Vol. 35. Math. Notes (Princeton). Princeton, NJ: Princeton University Press, 1988.

[75] P. Tolksdorf. “Regularity for a more general class of quasilinear equations”. In: *J. Differ. Equations* 51 (1984), pp. 126–150.

[76] J. Van Schaftingen. *The extension of traces for Sobolev mappings between manifolds.* Preprint, arXiv:2403.18738 [math.AP] (2024). 2024.

[77] J. Van Schaftingen and B. Van Vaerenbergh. *Extensions of traces for Sobolev mappings into manifolds at the endpoint* $p = 1$. 2025.

[78] J. Van Schaftingen and B. Van Vaerenbergh. “Asymptotic behavior of minimizing p-harmonic maps when $p \nearrow 2$ in dimension 2”. In: *Calc. Var. Partial Differ. Equ.* 62.8 (2023). Id/No 229, p. 45.

[79] B. White. “Homotopy classes in Sobolev spaces and the existence of energy minimizing maps”. In: *Acta Math.* 160.1-2 (1988), pp. 1–17.

[80] M. Willem. *Functional analysis. Fundamentals and applications.* 2nd edition. Cornerstones. Cham: Birkhäuser, 2022.

Asymptotic behavior of minimizing p-harmonic maps when $p \nearrow 2$ in dimension 2

2

Abstract. We study p–harmonic maps with Dirichlet boundary conditions from a planar domain into a general compact Riemannian manifold. We show that as p approaches 2 from below, they converge up to a subsequence to a minimizing singular renormalizable harmonic map. The singularities are imposed by topological obstructions to the existence of harmonic mappings; the location of the singularities being governed by a renormalized energy. Our analysis is based on lower bounds on growing balls and also yields some uniform weak-L^p bounds (also known as Marcinkiewicz or Lorentz $L^{p,\infty}$).

2.1 Introduction and main results

Given a bounded Lipschitz planar domain $\Omega \subset \mathbb{R}^2$, a Riemmanian manifold $\mathcal{N} \subset \mathbb{R}^\nu$, and $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, a minimizing 2-harmonic map $u \in W^{1,2}(\Omega, \mathcal{N}) \doteq W^{1,2}(\Omega, \mathbb{R}^\nu) \cap \{\text{for a.e. } x \in \Omega, u(x) \in \mathcal{N}\}$ is a minimizer of

$$\inf\left\{\int_\Omega \frac{|Du|^2}{2} : \begin{array}{c} u \in W^{1,2}(\Omega, \mathcal{N}) \\ \operatorname{tr}_{\partial\Omega} u = g \end{array}\right\}. \tag{2.1.1}$$

If $W^{1,2}_g(\Omega, \mathcal{N}) \doteq W^{1,2}(\Omega, \mathcal{N}) \cap \{\operatorname{tr}_{\partial\Omega} u = g\}$ contains at least one map, a minimizer does exist by the direct method of the calculus of variations. However in general due to topological obstructions [4][38, Section 6.3], it can be that $W^{1,2}_g(\Omega, \mathcal{N}) = \varnothing$ when $\mathcal{N}$ is not simply connected.

Meanwhile, when $p \in (1, 2)$, minimizing p-harmonic map $u_p \in W^{1,p}(\Omega, \mathcal{N})$, *i.e.* minimizers of

$$\inf\left\{\int_\Omega \frac{|Du|^p}{p} : \begin{array}{c} u \in W^{1,p}(\Omega, \mathcal{N}) \\ \operatorname{tr}_{\partial\Omega} u = g \end{array}\right\} \tag{2.1.2}$$

This chapter originally appeared as J. Van Schaftingen and B. Van Vaerenbergh. "Asymptotic behavior of minimizing p-harmonic maps when $p \nearrow 2$ in dimension 2". English. In: *Calc. Var. Partial Differ. Equ.* 62.8 (2023). Id/No 229, p. 45

always does exist. This follows by an extension theorem of Robert HARDT and Fang-Hua LIN [20] (see proposition 2.1.1 below) that asserts the existence of a least one map realizing the constraints. Following HARDT and LIN [21] (see also Daniel STERN [37]) in the case of the circle $\mathcal{N} = \mathbb{S}^1$, we want to construct 2-harmonic mappings as the limit of p–harmonic maps as $p \nearrow 2$ for a general manifold. Other non-simply connected targets for harmonic mappings appear in several contexts: projective plane in liquid crystal models [10, Section 1.A], the group of rotations in elasticity (Cosserat materials) [32] and quotient of the group of rotations by discrete subgroups in computer graphics (frame fields) [2, 25].

We point out that by the embedding theorem of John NASH [31] any Riemannian manifold $\mathcal{N}$ —compact or not— can be embedded as a closed set (see Olaf MÜLLER [30]) of some Euclidian space $\mathbb{R}^\nu$ and that one can define Sobolev spaces $W^{1,p}(\Omega, \mathcal{N})$ intrincly *i.e.* independently of the choice of a closed embedding (see Alexandra CONVENT and Jean VAN SCHAFTINGEN [15]).

Since we are mainly interested in the case where the infimum in (2.1.1) is infinite, the infimum in (2.1.2) should blow up as $p \nearrow 2$. Our first result describes the asymptotic behavior of the infimum of (2.1.2).

Theorem A. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\mathcal{N}$ a compact Riemannian manifold. For each $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, if, for each $p \in (1, 2)$, $u_p \in W^{1,p}(\Omega, \mathcal{N})$ is a minimizing p-harmonic map with trace $\operatorname{tr}_{\partial\Omega} u_p = g$, then*

$$\lim_{p \nearrow 2}(2 - p) \int_\Omega \frac{|Du_p|^p}{p} = \mathcal{E}^{1,2}_{\mathrm{sg}}(g) \in \{0\} \cup \Big[\frac{\operatorname{Sys}(\mathcal{N})^2}{4\pi}, +\infty\Big). \tag{2.1.3}$$

The *systole* $\operatorname{Sys}(\mathcal{N})$ of the manifold $\mathcal{N}$ is the least length of a non-contractible map $\mathbb{S}^1 \to \mathcal{N}$ (see definition 2.2.3; see [34, 18, 3] for early apparitions in the literature). If the manifold $\mathcal{N}$ is simply connected (1-connected or $\pi_1(\mathcal{N}) \simeq \{0\}$), the right-hand side of (2.1.3) is understood to be zero. Every compact manifold has a positive sytole. By *Riemannian manifold*, we mean a complete connected smooth Riemannian manifold without boundary and of finite dimension.

The *singular energy* $\mathcal{E}^{1,2}_{\mathrm{sg}}(g)$ introduced by Antonin MONTEIL, Rémy RODIAC and Jean VAN SCHAFTINGEN [29] quantifies the nontriviality of the free homotopy class of the map g:

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(g) = \inf\left\{ \frac{1}{4\pi}\sum_{i=1}^{k}\int_{\mathbb{S}^1} |\gamma_i'|^2 : \begin{array}{c} u \in C^1(\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho), \mathcal{N}), \rho > 0 \text{ small} \\ \gamma_i \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ geodesic homotopic to } u|_{\partial\mathbb{B}(a_i;\rho)} \\ u|_{\partial\Omega} \text{ homotopic to } g \end{array} \right\} \tag{2.1.4}$$

If $\mathrm{Sys}(\mathcal{N}) > 0$, the singular energy vanishes if and only if there exists $u \in W^{1,2}(\Omega, \mathcal{N})$ such that $\mathrm{Tr}_{\partial\Omega} u = g$ as explained in section 2.3. The singular energy is defined using minimal length of geodesics in $\mathcal{N}$ in chosen homotopy classes (see definition 2.2.8), where the homotopy is understood in the sense of VMO (vanishing mean oscillation, see [9, 11]). The singular energy only depends on the homotopy class of g: if $g_1, g_2 \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ are freely homotopic, one has $\mathcal{E}^{1,2}_{\mathrm{sg}}(g_1) = \mathcal{E}^{1,2}_{\mathrm{sg}}(g_2)$.

Theorem A is closely related to the extension of trace for Sobolev mappings of Hardt and Lin (see [20, Section 6] for compact $\mathcal{N}$; see [38] for the general case):

Proposition 2.1.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\mathcal{N}$ a Riemannian manifold. There exists a constant $C > 0$ depending on $\mathcal{N}$ and Ω such that for all $p \in [1, 2)$ and $g \in W^{1/p,p}(\partial\Omega, \mathcal{N})$, there exists $u \in W^{1,p}(\Omega, \mathcal{N})$ such that*

$$(2-p)\int_{\Omega} \frac{|Du|^p}{p} \leq C \iint_{\partial\Omega\times\partial\Omega} \frac{\mathrm{d}_{\mathcal{N}}(g(x), g(y))^p}{|x-y|^p}\,\mathrm{d}x\,\mathrm{d}y.$$

Briefly, we will write that $W^{1,p}_g(\Omega, \mathcal{N}) \neq \varnothing$ for $p \in [1, 2)$. We repeat that it can be that for $p = 2$, $W^{1,2}_g(\Omega, \mathcal{N}) = \varnothing$.

We next describe the asymptotics of families of p–harmonic maps.

Theorem B. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $\mathcal{N}$ a compact Riemannian manifold, $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ and $p_n \in (1, 2)$ an increasing sequence converging to 2.*
If $u_{p_n} \in W^{1,p_n}(\Omega, \mathcal{N})$ is a sequence of minimizing p_n-harmonic maps of trace $\mathrm{Tr}_{\partial\Omega} u_n = g$, then there exists a renormalizable harmonic map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ with trace $\mathrm{Tr}_{\partial\Omega} u_* = g$ such that up to some subsequence

$$u_{p_n} \xrightarrow{n\to+\infty} u_*$$

almost everywhere, and there exists $\kappa \leq 4\pi\mathcal{E}^{1,2}_{\mathrm{sg}}(g)/\mathrm{Sys}(\mathcal{N})^2$ distinct points $\{a_i\}_{i=1,\ldots,\kappa} \subset \Omega$ such that *(i) for $\rho > 0$ small enough $u_*|_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \in W^{1,2}(\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho), \mathcal{N})$ is a minimizing harmonic map with respect to its own boundary condition,*

(ii) the map u_* minimizes the renormalized energy of mappings

$$\lim_{\rho \searrow 0} \int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \frac{|Du_*|^2}{2} - \mathcal{E}^{1,2}_{\mathrm{sg}}(g) \log \frac{1}{\rho} + \mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa}, \tag{2.1.5}$$

(iii) the charges $([u_*, a_i])_{i=1,\dots,\kappa}$ and the points $(a_i)_{i=1,\dots,\kappa}$ minimize the renormalized energy of the configuration of points

$$\begin{aligned} &\mathcal{E}^{1,2}_{\mathrm{geom},\gamma_1,\dots,\gamma_k}([u_*, a_1], \dots, [u_*, a_k]) + \mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa} \\ &\quad = \inf \Big\{ \mathcal{E}^{1,2}_{\mathrm{geom},\gamma_1,\dots,\gamma_k}(x_1, \dots, x_k) + \mathrm{H}(\gamma_i)_{i=1,\dots,\kappa} : \\ &\qquad\qquad x_1, \dots, x_k \in \Omega \\ &\qquad\qquad \gamma_1, \dots, \gamma_k \textit{ achieve the infimum in } (2.1.4) \Big\}. \end{aligned} \tag{2.1.6}$$

Renormalizable maps $v \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ were introduced in [29] (see definition 2.2.15) and are measurable maps $v : \Omega \to \mathcal{N}$ for which

$$\mathcal{E}^{1,2}_{\mathrm{ren}}(v) = \lim_{\rho \searrow 0} \int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \frac{|Dv|^2}{2} - \mathcal{E}^{1,2}_{\mathrm{sg}}(g) \log \frac{1}{\rho}$$

is finite. Here and after $|Dv|$ refers to the Frobenius norm of the weak derivative Dv.

The term $\mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa}$ in theorem B is defined in (2.3.2) in proposition 4.4.1 only depends on the homotopy class of u_* *near* each a_i, $i = 1, \dots, \kappa$. That is, for $i = 1, \dots, \kappa$ and $\rho > 0$ small, $\mathrm{tr}_{\mathbb{S}^1(a_i;\rho)} u_*$ defines a map in $W^{1/2,2}(\mathbb{S}^1(a_i;\rho), \mathcal{N}) \subset \mathrm{VMO}(\mathbb{S}^1(a_i;\rho), \mathcal{N})$ which has a well-defined homotopy class $[u_*, a_i]$ on which depends $\mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa}$ (see (2.2.2) for details). The geometrical renormalized energy, $\mathcal{E}^{1,2}_{\mathrm{geom}}$, energy is defined in (2.2.30).

Theorem B generalizes Hardt and Lin's result was obtained for the circle, $\mathcal{N} = \mathbb{S}^1$ [21]; we relax both assumptions that the domain is simply connected and that the target is the unit circle. It will also appear that our method is quite different and robust enough to describe almost minimizers (also knows as quaziminimizers), that is, sequences of mappings $u_p \in W^{1,p}_g(\Omega, \mathcal{N})$ that are $o(1)$ close to the infimum (2.1.2) when $p \nearrow 2$.

Other approaches to deal with the emptiness of the $W^{1,2}_g(\Omega, \mathcal{N})$-class in dimension 2 exist: for instance, the well-studied Ginzburg–Landau family of functionals appeared before the p-harmonic map relaxation in the literature (see Fabrice Bethuel, Haïm Brezis and Frédéric Hélein [6] and references therein for the unit circle; more recently for general manifolds [12, 28]): for every $\varepsilon > 0$, one considers a minimizer $u_\varepsilon \in W^{1,2}_g(\Omega, \mathbb{R}^\nu)$ of

$$\inf \left\{ \int_\Omega \frac{|Du|^2}{2} + \frac{F(u)}{\varepsilon^2} : u \in W^{1,2}_g(\Omega, \mathbb{R}^\nu) \right\}$$

where $F(u)$ behaves as $\operatorname{dist}(u, \mathcal{N})^2$ near $\mathcal{N}$ (see [28] for detailed assumptions on F), $\mathcal{N}$ is a connected compact submanifold of $\mathbb{R}^\nu$ and studies the limit of $(u_\varepsilon)_{\varepsilon>0}$ when $\varepsilon \searrow 0$. More precisely, one gets *minimizing renormalizable singular harmonic maps* introduced in [29] and the limit happens to be in various notions of topologies (see [28]). If we write $u_* = \lim_{\varepsilon \searrow 0} u_\varepsilon \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$, Monteil, Rodiac and Van Schaftingen show that u_* minimizes a related renormalized energy which is the sum of a contribution depending on the charges and the location of the singularities, and another contribution depending on the charges and the potential F. Interestingly, the *renormalized energy*, $\mathcal{E}^{1,2}_{\mathrm{ren}}(u_*)$, is common to both relaxation, see [28] for details. The Ginzburg-Landau functional has an extrinsic character as one needs to embed isometrically the manifold $\mathcal{N}$ into a Euclidean space $\mathbb{R}^\nu$ and to choose a potential F before beginning the asymptotic analysis. Due to the nature of the Ginzburg-Landau functional, the convergence is different: in [28], it is shown that Du_ε strongly converges in L^2 away from the singularities of u_* (compare with the convergence in theorem B).

In order to perform the analysis described in theorems A and B, we adapt the approach that was used for studying the Ginzburg-Landau family of functionals in [29] and thereby we do not rely on the regularity of p-harmonic maps. Our method is therefore different from Hardt and Lin's [21] that heavily uses the structure of the circle $\mathcal{N} = \mathbb{S}^1$, estimates on liftings to its universal covering as well as monotonicity formulas. Building an *upper* and a *lower bound* on the *renormalized energy* $\mathcal{E}^{1,2}_{\mathrm{ren}}$ (see respectively propositions 4.4.1 and 2.5.4), we can treat the case of quaziminimizer of the p-Dirichlet functional *in the spirit of* Γ-*convergence* (also known as G-convergence or variational convergence) through a variational approach (see 2.6.1). The scheme of the proof of the lower bound follows Jerrard [22] and Sandier [35], as it was adapted to general target manifolds [28], by an *amalgamation/merging ball lemma* (see lemma 2.4.5) that we made more explicit for our purposes (see proposition 2.4.3). We finally strengthen the result for minimizers in proposition 2.6.1.

Our approach also has the advantage of yielding some improved estimates in the Marcinkiewicz $L^{p,\infty}(\Omega)$ space (or weak Lebesgue space, weak-L^p space or Lorentz space [26][13, Chapter 5]) which consists in $v : \Omega \to \mathbb{R}$ measurable verifying $\sup_{t>0} t^p\, \mathcal{H}^2(\{|v| > t\} \cap \Omega) < +\infty$. Here and after, $\mathcal{H}^2(A)$ refers to the 2-dimensional Hausdorff measure of measurable sets $A \subset \mathbb{R}^2$; equivalently [1, Theorem 4.1.1], it corresponds to the Lebesgue measure on $\mathbb{R}^2$. We obtain (see proposition 2.5.8)

$$\varlimsup_{p\to 2} \sup_{t>0} t^p\, \mathcal{H}^2(\{|Du_p| > t\} \cap \Omega) < +\infty, \tag{2.1.7}$$

which is some kind of *asymptotic upper bound in* $L^{p,\infty}(\Omega)$ for the Frobenius norm of the derivative of p-harmonic maps when $p \nearrow 2$; in view of theorem A, the corresponding strong L^p bound fails when $W^{1,2}_g(\Omega, \mathcal{N}) = \emptyset$. Bounds similar to (2.1.7) have been obtained for the Ginzburg–Landau relaxation for harmonic maps into the circle [36] and into a compact submanifold [28].

The compactness condition on the manifold $\mathcal{N}$ can be in fact weakened in our analysis. This covers some non-compact manifolds such as the cylinder $\mathbb{S}^1 \times \mathbb{R}$ but not all non-compact manifolds; we refer to (2.2.29) for a non-compact manifold which is not covered by our analysis. We devote section 2.7 to the coverable non-compact cases.

2.2 Setting: Quantifying energies and other relevant quantities

In this section we describe the length λ, the length spectrum, the singular energy, the renormalized energy and relation between them. All the homotopies are assumed to be *free* and are thought in VMO (see [9, 11]).

2.2.1 Length spectrum and systole

Let $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ be a map whose (essential) image lives in a Riemannian manifold $\mathcal{N}$ and of vanishing mean oscillation [9]. Following [29], we define the *length of its (free) homotopy class* by

$$\lambda(\gamma) = \inf\{\ell(\tilde\gamma) : \tilde\gamma \in W^{1,1}(\mathbb{S}^1, \mathcal{N}) \text{ is homotopic to } \gamma\} \tag{2.2.1}$$

where $\ell(\tilde\gamma)$ is the Riemannian length of the closed curve $\tilde\gamma$. In the definition, the homotopy refers to a (free) homotopy in the class of vanishing mean oscillation $\mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$. Moreover, every map in $\mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ is (freely) homotopic in $\mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ to a smooth and hence $W^{1,1}$-map. Therefore, the infimum is finite.

The length λ depends only on the VMO–homotopy class of γ: if γ_1 is homotopic to γ_2 then $\lambda(\gamma_1) = \lambda(\gamma_2)$.

If $\mathcal{N} = \mathbb{S}^1$, $\lambda(\gamma) = 2\pi|\deg\gamma|$, where $\deg\gamma$ is the topological degree of the map $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathbb{S}^1)$. If the manifold has the form $\mathcal{N} = \mathcal{N}_1 \times \mathcal{N}_2$, then one writes $\gamma = (\gamma_1, \gamma_2)$ and observes that $\lambda_{\mathcal{N}}(\gamma)^2 = \lambda_{\mathcal{N}_1}(\gamma_1)^2 + \lambda_{\mathcal{N}_2}(\gamma_2)^2$.

We now repeat a well-known construction for the topological degree that we adapt to the length λ. If $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{B}(a;\rho) \setminus \{a\}, \mathcal{N})$, it is possible to define the "homotopy of u near a", $[u, a]$. If $\gamma_a : \mathbb{S}^1 \to \mathbb{B}(a;\rho) \setminus \{a\}$ is a smooth curve isotopic to $z \in \mathbb{S}^1 \mapsto a + \rho z$ then $\mathrm{Tr}_{\gamma_a} u = \mathrm{Tr}_{\mathbb{S}^1}(u \circ \gamma)$ has a well-defined homotopy class in VMO that does not depend on the choice of the curve γ_a. This class of equivalence is denoted $[u, a]$. As $\gamma_a^t : z \in \mathbb{S}^1 \mapsto a + t\rho z \in \mathbb{R}^2$ is a family indexed by $t \in (0, 1)$ of admissible curves, we may define

$$\lambda([u, a]) \doteq \lambda(\mathrm{Tr}_{\mathbb{S}^1} u(a + r\cdot)) \tag{2.2.2}$$

for small $r > 0$. Any curve γ_a with the precited properties will then satisfy $\lambda([u, a]) = \lambda(\mathrm{Tr}_{\mathbb{S}^1}(u \circ \gamma_a))$ and this definition does not depend on $\rho > 0$.

Definition 2.2.1. *The* length spectrum *is the set of nonnegative real numbers* $\{\lambda(\gamma) : \gamma \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})\}$.

As our definition of smooth manifold implies that $\mathcal{N}$ is second countable, we obtain that this set is countable [24, Proposition 1.16] but can have accumulation points. In the compact case, it can be shown [29, Proposition 3.2] that

Lemma 2.2.2. *If the Riemannian manifold* $\mathcal{N}$ *is compact, the length spectrum is a discrete set of the real line.*

The compactness assumption is not necessary: the spectrum of the Euclidean space $\mathbb{R}^\mu$ and the infinite cylinder $\mathcal{N} \times \mathbb{R}$ (where $\mathcal{N}$ is compact manifold) have a discrete length spectrum but are not compact.

The *systole* of the manifold $\mathcal{N}$ is defined by

$$\mathrm{Sys}(\mathcal{N}) \doteq \inf\{\lambda(\gamma) : \gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N}) \text{ is not contractible}\}. \tag{2.2.3}$$

We set $\mathrm{Sys}(\mathcal{N}) = +\infty$ if the manifold $\mathcal{N}$ is simply connected. If $\mathcal{N}$ is compact, the systole is the first nonzero value of the length spectrum; this fact is sometimes used as a definition of the systole.

Our results heavily rely on the fact that the manifold $\mathcal{N}$ satisfies $\mathrm{Sys}(\mathcal{N}) > 0$. They cover Euclidean spaces $\mathbb{R}^\mu$, compact manifolds and infinite cylinder $\mathcal{N} \times \mathbb{R}^\mu$ ($\mathcal{N}$ compact). It does not cover infinite horns such as

$$\mathcal{N} = \{(ve^t, t) : t \in \mathbb{R}, v \in \mathbb{S}^1\} \subset \mathbb{R}^3. \tag{2.2.4}$$

In the homotopy class of $\mathbb{S}^1 \to \mathbb{S}^1 \times \{0\} \subset \mathcal{N}$, the infimum (2.2.1) is equal to zero and is not achieved; hence, $\operatorname{Sys}(\mathcal{N}) = 0$.

Here is a first link between the p-Dirichlet energy and the length λ.

Lemma 2.2.3 (Local lower bound on circles)**.** *If $p \in [1, \infty)$ and if $u \in W^{1,p}(\partial\mathbb{B}(a;\rho), \mathcal{N})$, with $\mathbb{B}(a;\rho) \subset \mathbb{R}^2$, then*

$$\frac{\lambda(u)^p}{p(2\pi\rho)^{p-1}} \le \int_{\partial\mathbb{B}(a;\rho)} \frac{|u'|^p}{p} \tag{2.2.5}$$

In the statement of lemma 2.2.3, u' refers to the tangential derivative along the circle $\partial\mathbb{B}(a;\rho)$.

Proof of lemma 2.2.3. By Hölder's inequality,

$$\frac{1}{\mathcal{H}^1(\partial\mathbb{B}(a;\rho))^{p-1}} \left(\int_{\partial\mathbb{B}(a;\rho)} |u'| \right)^p \le \int_{\partial\mathbb{B}(a;\rho)} |u'|^p.$$

Since the map u is admissible in the greatest lower bound (2.2.1) that defines λ, we get the conclusion (2.2.5) as $\mathcal{H}^1(\partial\mathbb{B}(a;\rho)) = 2\pi\rho$. □

The following proposition will play the role of an infinitesimal lower bound. It generalizes [21, Lemma 2.3].

Proposition 2.2.4. *Let $0 < \sigma < \rho$ and $u \in W^{1,2}(\mathbb{B}(a;\rho) \setminus \mathbb{B}(a;\sigma), \mathcal{N})$. Then, if $p \in [1, \infty) \setminus \{2\}$,*

$$\frac{(\rho^{2-p} - \sigma^{2-p})\lambda(\operatorname{Tr}_{\partial\mathbb{B}(a;\rho)} u)^p}{(2\pi)^{p-1} p(2-p)} \le \int_{\mathbb{B}(a;\rho)\setminus\mathbb{B}(a;\sigma)} \frac{|Du|^p}{p}. \tag{2.2.6}$$

Proof of proposition 2.2.4. Integrating the estimate of lemma 2.2.3 over (σ, ρ), we get

$$\begin{aligned}\int_{\mathbb{B}(a;\rho)\setminus\mathbb{B}(a;\sigma)} \frac{|Du|^p}{p} &\ge \int_\sigma^\rho \frac{1}{p(2\pi r)^{p-1}} \lambda(\operatorname{Tr}_{\mathbb{S}^1} u(a + r\cdot))^p \,\mathrm{d}r \\ &= \frac{\rho^{2-p} - \sigma^{2-p}}{2-p} \frac{\lambda(\operatorname{Tr}_{\partial\mathbb{B}(a;\rho)} u)^p}{(2\pi)^{p-1} p}\end{aligned}$$

by homotopy invariance. □

Letting $\sigma \searrow 0$ in (2.2.6), we get that if $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{B}(a;\rho) \setminus \{a\}, \mathcal{N})$,

$$\frac{\rho^{2-p}\lambda([u,a])^p}{(2\pi)^{p-1}p(2-p)} \le \int_{\mathbb{B}(a;\rho)} \frac{|Du|^p}{p}. \tag{2.2.7}$$

Letting $p \nearrow 2$ in (2.2.6) yields

$$\frac{\lambda(\operatorname{Tr}_{\partial\mathbb{B}(a;\rho)} u)^2}{4\pi} \log\frac{\rho}{\sigma} \le \int_{\mathbb{B}(a;\rho)\setminus\mathbb{B}(a;\sigma)} \frac{|Du|^2}{2}. \tag{2.2.8}$$

In the case of the circle $\mathcal{N} = \mathbb{S}^1$, this estimate is known as *lower bounds on annuli* (see [35, Lemma 1.1]).

Corollary 2.2.5. *Let* $\mathcal{N}$ *be a Riemannian manifold of positive systole. Let* $p \in [1,\infty) \setminus \{2\}$ *and* $0 \le \sigma < \rho$ *and* $u \in W^{1,p} \cap W^{1,2}_{\mathrm{loc}}(\mathbb{B}(a;\rho) \setminus \mathbb{B}(a;\sigma), \mathcal{N})$*. If*

$$\int_{\mathbb{B}(a;\rho)\setminus\mathbb{B}(a;\sigma)} \frac{|Du|^p}{p} < \frac{\big(\rho^{2-p} - \sigma^{2-p}\big)\operatorname{Sys}(\mathcal{N})^p}{(2\pi)^{p-1}p(2-p)},$$

then $\operatorname{Tr}_{\partial\mathbb{B}(a;r)} u$ *is homotopic to a constant function for each* $r \in (\rho, \sigma)$*.*

Proof of corollary 2.2.5. The assumption implies that $\lambda(\operatorname{Tr}_{\partial B(a;r)} u) < \operatorname{Sys}(\mathcal{N})$ which yields $\lambda(\operatorname{Tr}_{\partial\mathbb{B}(a;r)} u) = 0$ and hence the conclusion. $\square$

Observe that (2.2.7) fails at $p = 2$. However at the Marcinkiewicz scale we observe the following.

Corollary 2.2.6. *Let* $\Omega \subset \mathbb{R}^2$ *be an open subset of the plane. Let* $a_1, \dots, a_k \in \Omega$ *be points. If* $u \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_i\}_{i=1,\dots,k}, \mathcal{N})$*, then*

$$\sum_{i=1}^{k} \frac{\lambda([u,a_i])^2}{4\pi} \le \varlimsup_{t\to\infty} t^2\, \mathcal{H}^2(\{|Du| \ge t\} \cap \Omega). \tag{2.2.9}$$

We observe that at this scale no constant depending on the domain Ω appears.

Proof of corollary 2.2.6. Given $t_0 > 0$, we define

$$M(t_0) \doteq \sup_{t > t_0} t^2\, \mathcal{H}^2(\{|Du| \ge t\}) \cap \Omega).$$

There exists a $\rho > 0$ such that $G_\rho \doteq \bigcup_{i=1}^k \mathbb{B}(a_i; \rho\lambda([u,a_i])) \subset \Omega$, the disks $\mathbb{B}(a_i; \rho\lambda([u,a_i]))$ are disjoint and

$$\mathcal{H}^2(G_\rho) = \pi\rho^2 \sum_{i=1}^{k} \lambda([u,a_i])^2 \le \frac{M(t_0)}{t_0^2}. \tag{2.2.10}$$

Summing (2.2.7) k times with $p = 1$, we obtain

$$\begin{aligned}\sum_{i=1}^{k} \rho\lambda([u, a_i])^2 &\leq \int_{G_\rho} |Du| \\ &= \int_0^\infty \mathcal{H}^2(\{|Du| \geq t\} \cap G_\rho)\,\mathrm{d}t \\ &= \int_0^\infty \min(\mathcal{H}^2(G_\rho), \frac{M(t_0)}{t^2})\,\mathrm{d}t,\end{aligned} \tag{2.2.11}$$

by the choice (2.2.10). Therefore

$$\rho \sum_{i=1}^{k} \lambda([u, a_i])^2 \leq 2\Big[M(t_0)\pi\rho^2 \sum_{i=1}^{k} \lambda([u, a_i])^2\Big]^{\frac{1}{2}}$$

and the conclusion (2.2.9) follows by the arbitrariness of t_0. □

2.2.2 Singular energy of the boundary data, $\mathcal{E}_{\mathrm{sg}}^{1,p}$

The following concept was introduced in [29] and extend λ to general boundary of domains.

Definition 2.2.7 (Topological resolution)**.** *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, a Riemannian manifold $\mathcal{N}$ and $g \in \mathrm{VMO}(\partial\Omega, \mathcal{N})$. A finite family of closed curves $\gamma_i \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$, $i = 1, \ldots, k$, with $k \in \mathbb{N} \doteq \{0, 1, 2, \ldots\}$, called* charges *constitutes a* topological resolution *of the map g whenever there exist k non-intersecting closed balls $\mathbb{B}(a_i; \rho) \subset \Omega$ with $\rho > 0$ and a map*

$$u \in W^{1,2}(\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i; \rho), \mathcal{N})$$

such that $\mathrm{Tr}_{\partial\Omega} u$ and g are homotopic in $\mathrm{VMO}(\partial\Omega, \mathcal{N})$ and $\mathrm{Tr}_{\mathbb{S}^1} u(a_i + \rho\cdot)$ and γ_i are homotopic in $\mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ for $i = 1, \ldots, k$.

We allow $k = 0$ in the definition. In that case the topological resolution is said to be *trivial* and the associated map $u \in W^{1,2}(\Omega, \mathcal{N})$ is an extension of g.

We refer to the points $a_i \in \Omega$ $i = 1, \ldots, k$ as *singularities.*

If for some $j = 1, \ldots, k$, it happens that $\lambda(\gamma_j) = 0$ then, provided that $\mathrm{Sys}(\mathcal{N}) > 0$ holds, there exists $w \in W^{1,2}(\mathbb{B}(a_j; \rho), \mathcal{N})$ such that $\mathrm{Tr}_{\mathbb{S}^1} u(a_j + \rho\cdot) = \gamma_j$ and one can replace in the definition u by a new map

$$\bar{u} \in W^{1,2}(\Omega \setminus \bigcup_{\substack{i=1 \\ i\neq j}}^{k} \mathbb{B}(a_i; \rho), \mathcal{N}) \tag{2.2.12}$$

where $\bar{u} = u$ on $\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i;\rho)$ and $\bar{u} = w$ on $\mathbb{B}(a_j;\rho)$.

Definition 2.2.8 (Singular energy, $\mathcal{E}^{1,p}_{\mathrm{sg}}$)**.** *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a Riemannian manifold $\mathcal{N}$, $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ and $p \geq 1$. We set*

$$\mathcal{E}^{1,p}_{\mathrm{sg}}(g) \doteq \inf \Big\{ \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^p}{p(2\pi)^{p-1}} : \\ k \in \mathbb{N}, (\gamma_i)_{i=1,\dots,k} \textit{ is a topological resolution of } g \Big\}.$$

On the one hand, if the systole $\mathrm{Sys}(\mathcal{N})$ is positive, then $\mathcal{E}^{1,p}_{\mathrm{sg}}(g) = 0$ if and only if there exists a map $u \in W^{1,2}_g(\Omega, \mathcal{N})$ that extends the map g. This follows by iterating (2.2.12) On the other hand, if $\mathrm{Sys}(\mathcal{N}) > 0$ and $\mathcal{E}^{1,p}_{\mathrm{sg}}(g) > 0$ then

$$\mathcal{E}^{1,p}_{\mathrm{sg}}(g) \geq \frac{\mathrm{Sys}(\mathcal{N})^p}{p(2\pi)^{p-1}}. \tag{2.2.13}$$

We also have the following decreasing property of the singular energy $\mathcal{E}^{1,p}_{\mathrm{sg}}$ of importance in the circle construction (see proposition 2.4.3).

Proposition 2.2.9 (Decreasing property of the singular energy)**.** *If two Lipschitz bounded domains satisfy $\Omega_0 \subset \Omega_1$ and $u \in W^{1,2}(\Omega_1 \setminus \Omega_0, \mathcal{N})$, then $\mathcal{E}^{1,p}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega_0} u) \geq \mathcal{E}^{1,p}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega_1} u)$.*

Proof. See [29, Proposition 2.6]. □

The positivity of the systole yields a continuity statement in p of the p-singular energy :

Lemma 2.2.10 (Continuity in p of $\mathcal{E}^{1,p}_{\mathrm{sg}}$)**.** *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, a Riemannian manifold $\mathcal{N}$ and $g \in \mathrm{VMO}(\partial\Omega, \mathcal{N})$. If $\mathrm{Sys}(\mathcal{N}) > 0$, then*

$$p \in [1, \infty) \mapsto \mathcal{E}^{1,p}_{\mathrm{sg}}(g) \tag{2.2.14}$$

is locally bounded and locally Lipschitz continuous.

The positive systole assumption is crucial as it allows us to state an equivalence between $\ell^p(\mathbb{N})$ and $\ell^q(\mathbb{N})$ norms of the vector $(\lambda_1, \dots, \lambda_k)$, see (2.2.15).

Proof of lemma 2.2.10. Set $1 \leq p < q < \infty$, $c_p \doteq 1/(2\pi)^{p-1}p$ and $f(p) \doteq \mathcal{E}^{1,p}_{\mathrm{sg}}(g)/c_p$. For every $\lambda_i \in \{\lambda(\gamma) : \gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})\}$, $i = 1, \dots, k$, we have either $\lambda_i \geq \mathrm{Sys}(\mathcal{N})$ or $\lambda_i = 0$, and thus $\mathrm{Sys}(\mathcal{N})^{q-p}\lambda_i^p \leq \lambda_i^q$, so that

$$\mathrm{Sys}(\mathcal{N})^{q-p}\sum_{i=1}^k \lambda_i^p \leq \sum_{i=1}^k \lambda_i^q \leq \Big[\sum_{i=1}^k \lambda_i^p\Big]^{\frac{q}{p}} \tag{2.2.15}$$

which implies

$$\mathrm{Sys}(\mathcal{N})^{q-p}f(p) \leq f(q) \leq (f(p))^{q/p}$$

so that f is locally bounded. Moreover,

$$(\mathrm{Sys}(\mathcal{N})^{q-p} - 1)f(p) \leq f(q) - f(p) \leq f(p)\big(f(p)^{\frac{q-p}{p}} - 1\big) \tag{2.2.16}$$

so that f is locally Lipschitz-continuous as $f(p) \geq \mathrm{Sys}(\mathcal{N})^{\frac{p}{p-1}}$. $\square$

Using (2.2.16), it is possible to estimate the local Lipschitz constant of (2.2.14). If $I \subset [1, \infty)$ is compact and $M = \sup_{p\in I}(2\pi)^{p-1}p\mathcal{E}^{1,p}_{\mathrm{sg}}(g)$

$$[(2\pi)^{p-1}p\mathcal{E}^{1,p}_{\mathrm{sg}}(g)]_{\mathrm{Lip}(I)} \leq \max(\big|\mathrm{Sys}(\mathcal{N})\log\mathrm{Sys}(\mathcal{N})\big|, \big|M\log M\big|). \tag{2.2.17}$$

Aside from the continuity statement in p of the map (2.2.14), the singular energy $\mathcal{E}^{1,p}_{\mathrm{sg}}(g)$ is locally constant in $W^{1/2,2}$:

Lemma 2.2.11. *Given $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, there exists $\delta = \delta(g, \mathcal{N}) > 0$ such that if $h \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ satisfies $\|g - h\|_{W^{1/2,2}(\partial\Omega)} \leq \delta$ then $\mathcal{E}^{1,p}_{\mathrm{sg}}(g) = \mathcal{E}^{1,p}_{\mathrm{sg}}(h)$.*

This follows from the fact that given $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ there exists $\delta = \delta(g) > 0$ such that if $h \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ satisfies $\|g - h\|_{W^{1/2,2}} \leq \delta$, then they are homotopic (for a proof of this fact : [9, Lemma A.19] combined with $W^{1/2,2} \subset \mathrm{VMO}$) and $\mathcal{E}^{1,p}_{\mathrm{sg}}(g)$ is homotopy invariant.

Definition 2.2.12 (Minimal topological resolution)**.** *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, a Riemannian manifold $\mathcal{N}$ and $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$. A topological resolution of the map g is* minimal topological resolution *of g whenever the singularities $(\gamma_i)_{i=1,\dots,k}$ satisfy $\lambda(\gamma_i) > 0$ and*

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(g) = \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi} > 0.$$

By (2.2.13), if $\mathrm{Sys}(\mathcal{N}) > 0$, the number k of singularities of a minimal topological resolution is at most

$$k \leq 4\pi\mathcal{E}^{1,2}_{\mathrm{sg}}(g)/\mathrm{Sys}(\mathcal{N})^2.$$

Definition 2.2.13 (Atomicity)**.** *A map $\gamma \in W^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ is said to be atomic if*
$$\mathcal{E}^{1,2}_{\mathrm{sg}}(\gamma) = \frac{\lambda(\gamma)^2}{4\pi}.$$

For instance, when $\mathcal{N} = \mathbb{S}^1$ the identity map $\mathrm{Id}_{\mathbb{S}^1} : z \in \mathbb{S}^1 \to z \in \mathbb{S}^1$ and its conjugate are atomic maps.

We conclude this section with the following observation.

Lemma 2.2.14 (Atomicity in minimal topological resolutions)**.** *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, a Riemannian manifold $\mathcal{N}$ and $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ as well as a minimal topological resolution $(\gamma_i)_{i=1,\dots,k}$ of the map g. For each $i = 1, \dots, k$ the map γ_i is atomic.*

Here and in the sequel we use the notation

(2.2.18)
$$\rho(\{a_i\}_{i=1,\dots,\kappa}) \doteq \min\{\mathrm{dist}(\partial\Omega, a_i), |a_i - a_j| : i \neq j,\ i, j = 1, \dots, \kappa\}$$

for a quantity that will play the role of a non-intersecting radius of balls.

Proof of lemma 2.2.14. Fix $j \in \{1, \dots, k\}$ and $\rho \in (0, \rho(a_i)_{i=1,\dots,k})$. Let u be the map that carries the topological resolution. By definition of the singular energy,

$$\begin{aligned}\mathcal{E}^{1,2}_{\mathrm{sg}}(g) = \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^2}{4\pi} &\geq \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial(\bigcup_{i\neq j} \mathbb{B}(a_i;\rho))} u) + \frac{\lambda(\gamma_j)^2}{4\pi} \\ &\geq \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial(\bigcup_{i\neq j} \mathbb{B}(a_i;\rho)} u) + \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\mathbb{B}(a_j;\rho)} u) \geq \mathcal{E}^{1,2}_{\mathrm{sg}}(g).\end{aligned}$$

This forces $\frac{\lambda(\gamma_j)^2}{4\pi} = \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\mathbb{B}(a_j;\rho)} u) = \mathcal{E}^{1,2}_{\mathrm{sg}}(\gamma_j)$. □

2.2.3 Renormalized energy of a mapping, $\mathcal{E}^{1,2}_{\mathrm{ren}}$

Minimizing p-harmonic maps when $p \nearrow 2$ will converge to minimizers of the following renormalized energy introduced in [29]:

Definition 2.2.15 (Renormalized energy and renormalizable Sobolev maps)**.** *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a Riemannian manifold $\mathcal{N}$. A measurable map $u : \Omega \to \mathcal{N}$ is said to be* renormalizable *whenever there exist $k \in \mathbb{N}$ distinct points $a_i \in \Omega$ such that $u \in W^{1,2}(\Omega \setminus \{a_1, \dots, a_k\}, \mathcal{N})$ and $\lambda([u, a_i]) > 0$ such that its* renormalized energy

(2.2.19)
$$\mathcal{E}^{1,2}_{\mathrm{ren}}(u) \doteq \lim_{\rho \searrow 0} \int_{\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i;\rho)} \frac{|Du|^2}{2} - \sum_{i=1}^{k} \frac{\lambda([u, a_i])^2}{4\pi} \log\frac{1}{\rho}$$

is finite.

For example, if $\Omega = \mathbb{B}$ is the unit ball of $\mathbb{R}^2$, $\mathcal{N} = \mathbb{S}^1$ and $g = \mathrm{Id}_{\mathbb{S}^1} : \mathbb{S}^1 \doteq \partial\mathbb{B} \to \mathbb{S}^1$ is the identity map, then the *vortex map* or *hedgehog map* $u_h(x) = {}^{x}/_{|x|}$ belongs to $W^{1,2}_{\mathrm{ren}}(\mathbb{B}, \mathbb{S}^1)$. Moreover $\mathcal{E}^{1,2}_{\mathrm{ren}}(u_h) = 0$.

We point out that the existence of such maps is not granted when the manifold $\mathcal{N}$ is not compact, see proposition 2.2.20.

The inferior limit $\underline{\lim}$ in (2.2.19) is actually a limit as

$$\rho \in (0, \rho(a_i)_{i=1,\dots,k}) \mapsto \int_{\Omega\setminus\bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} \frac{|Du|^2}{2} - \sum_{i=1}^k \frac{\lambda([u,a_i])^2}{4\pi} \log\frac{1}{\rho} \tag{2.2.20}$$

is non-increasing, see [29, Proposition 2.10] for a proof that only assumes $\mathrm{Sys}(\mathcal{N}) > 0$.

We finally give an expression of the renormalized energy that does not make use of the limit operator. It splits the expression of $\mathcal{E}^{1,2}_{\mathrm{ren}}$ in two parts (an L^2-part and a renormalized part).

Proposition 2.2.16 (Integral representation of the renormalized energy)**.** *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a Riemannian manifold $\mathcal{N}$. If $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ is a renormalizable map associated to the distinct points $a_i \in \Omega$, $i = 1, \dots, k$, then, for every $\sigma \in (0, \rho(\{a_i\}_{i=1,\dots,k}))$,*

$$\begin{aligned} &\mathcal{E}^{1,2}_{\mathrm{ren}}(u) + \sum_{i=1}^k \frac{\lambda([u,a_i])^2}{4\pi}\log\frac{1}{\sigma} \\ &= \int_{\Omega\setminus\bigcup_{i=1}^k \mathbb{B}(a_i;\sigma)} \frac{|Du|^2}{2} + \sum_{i=1}^k \int_0^\sigma \left[\int_{\partial\mathbb{B}(a_i;r)} \frac{|Du|^2}{2}\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])^2}{4\pi r}\right]\mathrm{d}r. \end{aligned} \tag{2.2.21}$$

Conversely, if $u \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus \{a_i\}_{i=1,\dots,k}, \mathcal{N})$ and there exists a $\sigma \in (0, \rho(a_i)_{i=1,\dots,k})$ such that the right-hand side of (2.2.21) *is finite, then $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$.*

We recall that the non-intersection radius $\rho(\{a_i\}_{i=1,\dots,k})$ was defined in (2.2.18).

Proof of propostion 2.2.16. We have by the existence of the limit in the expression of the renormalized energy (see (2.2.20))

$$\begin{aligned}
\mathcal{E}^{1,2}_{\mathrm{ren}}(u) &+ \sum_{i=1}^{k} \frac{\lambda([u,a_i])^2}{4\pi} \log \frac{1}{\sigma} \\
&= \lim_{\rho \searrow 0} \int_{\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i;\sigma)} \frac{|Du|^2}{2} \\
&\quad + \sum_{i=1}^{k} \left[\int_{\mathbb{B}(a_i;\sigma)(a_i) \setminus \mathbb{B}(a_i;\rho)} \frac{|Du|^2}{2} - \frac{\lambda([u,a_i])^2}{4\pi} \log \frac{\sigma}{\rho} \right] \\
&= \int_{\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i;\sigma)} \frac{|Du|^2}{2} \\
&\quad + \sum_{i=1}^{k} \lim_{\rho \searrow 0} \int_{\rho}^{\sigma} \left[\int_{\partial \mathbb{B}(a_i;r)} \frac{|Du|^2}{2} \,\mathrm{d}\mathscr{H}^1 - \frac{\lambda([u,a_i])^2}{4\pi r} \right] \mathrm{d}r.
\end{aligned}$$

The terms in bracket are nonnegative by lemma 2.2.3. As a consequence Levi's monotone convergence theorem applies. The reciprocal follows from the same computation. □

When $p \in [1,2)$, we introduce the following p-renormalized energy

$$\begin{aligned}
\mathcal{E}^{1,p}_{\mathrm{ren}}(u) &\doteq \int_{\Omega} \frac{|Du|^p}{p} - \sum_{i=1}^{k} \frac{\lambda([u,a_i])^p}{(2\pi)^{p-1}p} \frac{1}{2-p} \\
&= \lim_{\rho \searrow 0} \int_{\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i;\rho)} \frac{|Du|^p}{p} - \sum_{i=1}^{k} \frac{\lambda([u,a_i])^p}{(2\pi)^{p-1}p} \frac{1-\rho^{2-p}}{2-p}.
\end{aligned} \tag{2.2.22}$$

defined for maps $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$. In [29, Theorem 5.1], it is shown that if $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ then $Du \in L^{2,\infty}(\Omega)$, see also corollary 2.5.13. In particular $W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N}) \subset W^{1,p}(\Omega, \mathcal{N})$ for $p \in [1,2)$ since the weak-L^2 (Marcinkiewicz) space satisfies $L^{2,\infty}(\Omega) \subset L^p(\Omega)$, see *e.g.* [13, Theorem 5.9].

Proposition 2.2.17. *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a Riemannian manifold $\mathcal{N}$ with* $\mathrm{Sys}(\mathcal{N}) > 0$. *If $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ then*

$$\mathcal{E}^{1,p}_{\mathrm{ren}}(u) \xrightarrow{p \nearrow 2} \mathcal{E}^{1,2}_{\mathrm{ren}}(u).$$

The proof of proposition 2.2.17 relies on the following Hölder-type estimate :

Lemma 2.2.18. *Let $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{B}(a;\rho) \setminus \{a\})$. If $\lambda([u,a]) > 0$, then, for every $r \in (0,\rho)$,*

$$\int_{\partial\mathbb{B}(a;r)} \frac{|Du|^p}{p} \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a])^p}{p(2\pi r)^{p-1}} \le \Big(\frac{2\pi r}{\lambda([u,a])}\Big)^{2-p} \Big[\int_{\partial\mathbb{B}(a;r)} \frac{|Du|^2}{2} \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a])^2}{4\pi r}\Big] \tag{2.2.23}$$

and

$$\int_{\mathbb{B}(a;\rho)} \frac{|Du|^p}{p} - \frac{\lambda([u,a])^p}{p(2\pi|x-a|)^p} \,\mathrm{d}x \le \Big(\frac{2\pi\rho}{\lambda([u,a_i])}\Big)^{2-p} \int_{\mathbb{B}(a;\rho)} \frac{|Du|^2}{2} - \frac{\lambda([u,a])^2}{8\pi^2|x-a|^2} \,\mathrm{d}x \tag{2.2.24}$$

provided the right-hand sides are finite.

Proof of lemma 2.2.18. By Young's inequality with exponents $1/(2/p) + 1/(2/(2-p)) = 1$,

$$|Du|^p \Big(\frac{\lambda([u,a_i])}{2\pi r}\Big)^{2-p} \le \frac{p}{2}|Du|^2 + \Big(1 - \frac{p}{2}\Big)\Big(\frac{\lambda([u,a_i])}{2\pi r}\Big)^2.$$

Hence,

$$\frac{|Du|^p}{p} - \frac{1}{p}\Big(\frac{\lambda([u,a])}{2\pi r}\Big)^p \le \Big(\frac{2\pi\rho}{\lambda([u,a])}\Big)^{2-p} \Big[\frac{|Du|^2}{2} - \frac{1}{2}\Big(\frac{\lambda([u,a])}{2\pi r}\Big)^2\Big].$$

Integrating on $\partial\mathbb{B}(a_i;r)$ we obtain (2.2.23).

The second conclusion follows by integration. □

Proof of proposition 2.2.17. Let $u \in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$ and let $(a_1,\dots,a_k) \in \Omega^k$ be the assumed distinct points associated with it (see definition 2.2.15). Fix $\rho \in (0, \rho(a_i)_{i=1,\dots,k})$.

By Young's inequality,

$$\frac{|Du|^p}{p} \le \frac{|Du|^2}{2} + \frac{2-p}{2p} \le \frac{|Du|^2}{2} + \frac{1}{2} \tag{2.2.25}$$

By Lebesgue's dominated convergence theorem with (2.2.25) as a Lebesgue dominant, since $\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho)$ has finite Lebesgue measure, we get

$$\int_{\Omega\setminus\bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} \frac{|Du|^p}{p} \xrightarrow{p\nearrow 2} \int_{\Omega\setminus\bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} \frac{|Du|^2}{2}. \tag{2.2.26}$$

We now study the other part of the renormalized energy. The Lebesgue's convergence theorem will be used, with a Lebesgue dominant provided by (2.2.23) in lemma 2.2.18.

By the converse of Lebesgue's dominated convergence theorem, for almost every $r \in (0, \rho)$,

$$\int_{\partial \mathbb{B}(a_i;r)} \frac{|Du|^p}{p} \xrightarrow{p \nearrow 2} \int_{\partial \mathbb{B}(a_i;r)} \frac{|Du|^2}{2}. \tag{2.2.27}$$

In view of the bound (2.2.23) and of the convergence (2.2.27), we get by Lebesgue's dominated convergence theorem that

$$\begin{aligned} \int_0^\rho \int_{\partial \mathbb{B}(a_i;r)} \frac{|Du|^p}{p} \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])^p}{p(2\pi r)^{p-1}} \,\mathrm{d}r \\ \xrightarrow{p \nearrow 2} \int_0^\rho \int_{\partial \mathbb{B}(a_i;r)} \frac{|Du|^2}{2} \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])^2}{4\pi r} \,\mathrm{d}r. \end{aligned} \tag{2.2.28}$$

Combining (2.2.26) and (2.2.28) and proposition 2.2.16, we get that $\mathcal{E}^{1,p}_{\mathrm{ren}}(u) \to \mathcal{E}^{1,2}_{\mathrm{ren}}(u)$ when $p \nearrow 2$. □

2.2.4 Existence of renormalizable mappings

If the manifold $\mathcal{N}$ is compact there always exists a renormalizable mapping (see proposition 2.2.19) while in the non-compact case we descibe a manifold carrying no nonconstant renormalizable mappings (see proposition 2.2.20)

Proposition 2.2.19. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain, $\mathcal{N}$ be a Riemannian manifold and $g \in W^{1/2,2}(\Omega, \mathcal{N})$. If the manifold $\mathcal{N}$ is compact, there exists $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ such that $\operatorname{tr}_{\partial\Omega} u = g$.*

We recall that in the present paper every Riemannian manifold is assumed to be connected as explained in the introduction (section 2.1).

Proof of proposition 2.2.19. Taking $\delta > 0$ sufficiently small the set $\Omega_\delta \doteq \{x \in \overline{\Omega} : \operatorname{dist}(x, \partial\Omega) \le \delta\}$ is equivalent by biLipschitz homeomorphism to $\partial\Omega \times [0, 1]$. As every $W^{1/2,2}(\Omega, \mathcal{N})$-map is smoothly homotopic to a smooth map, we realize this homotopy on $\Omega_\delta \simeq \partial\Omega \times [0, 1]$. We therefore assume that g is smooth.

We may also assume that Ω is simply connected. By the connectedness of $\mathcal{N}$ one can connect the image $g(\partial\Omega)$ by paths realized by segments in Ω. Taking a finite number of such paths one partitions Ω in simply connected rooms delimited by the union of $\partial\Omega$ and those paths.

Assuming that Ω is simply connected, $\partial\Omega$ is equivalent to the circle $\mathbb{S}^1$. By compactness of the manifold $\mathcal{N}$, there exists a smooth map $\gamma : \mathbb{S}^1 \to \mathcal{N}$ such that γ is homotopic to g and $\lambda(g) = \ell(\gamma) = 2\pi\|\gamma'\|_\infty$. This is an instance of the existence of constant speed geodesics in each homotopy class (see [23, Proposition 6.28]). Eventually γ is token constant. There exists a small $\varepsilon > 0$ and $a \in \Omega$ such that $\Omega \setminus \mathbb{B}(a;\varepsilon)$ is equivalent to $\mathbb{S}^1 \times [0,1]$ and we realize the homotopy between g and γ there.

Combining the above described homotopies, we obtain a map $u \in C^\infty(\Omega \setminus \mathbb{B}(a;\varepsilon), \mathcal{N}) \cap W^{1,2}(\Omega \setminus \mathbb{B}(a;\varepsilon), \mathcal{N})$. We futher define for all $x \in \mathbb{B}(a;\varepsilon)$

$$u(x) \doteq \gamma\Big(\frac{x-a}{|x-a|}\Big)$$

and observe that $|Du(x)| = \|\gamma'\|_\infty/|x|$ so that for $\rho \in (0,\varepsilon)$

$$\int_{\mathbb{B}(a;\varepsilon)\setminus\mathbb{B}(a;\rho)} |Du(x)|^2 = \frac{\lambda([g])^2}{2\pi}\log\frac{\varepsilon}{\rho},$$

showing the renormalized character of u near a. □

Proposition 2.2.20. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz simply connected domain and $(\mathcal{N}, g)$ be the Riemannian manifold defined by $\mathcal{N} \doteq \mathbb{S}^1 \times \mathbb{R} \subset \mathbb{R}^3$ with metric defined for every $(v,t) \in \mathbb{S}^1 \times \mathbb{R}$ and $h \in T_{v,t}\mathbb{S}^1 \times \mathbb{R} \subseteq \mathbb{R}^3$ by*

$$g_{(v,t)}(h) \doteq (1 + f(t))|h|^2 \text{ where } f(t) \doteq \frac{1}{\sqrt{1+t^2}}. \tag{2.2.29}$$

If $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$, then $\mathrm{tr}_{\partial\Omega}\, u$ is homotopic to a constant.

In proposition 2.2.20, the manifold $\mathcal{N}$ is not presented as an isometrically embedded submanifold of $\mathbb{R}^3$. The proof relies on the fact that f is not integrable in the complement of any compact set.

Proof of proposition 2.2.20. The proof first reduces to the case of $\Omega = \mathbb{B}_1$ by a change of variable in the Dirichlet energy. Given a map $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ such that $\mathrm{tr}_{\partial\Omega}\, u$ is not homotopic to a constant there exist $a \in \Omega$ and $\sigma > 0$ such that $u \in W^{1,2}_{\mathrm{loc}}(\mathbb{B}(a;\sigma) \setminus \{a\}, \mathcal{N})$. We may further assume that $a = 0, \sigma = 0$ and $\mathrm{tr}_{\mathbb{S}^1}\, u$ is not homotopic to a constant. We finally write $\lambda \doteq \lambda([u,0]) = 2\pi\,|\deg([u,0])|$. We denote $\pi_{\mathbb{S}^1} : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{S}^1$ the projection on the circle, $\pi_{\mathbb{R}} : \mathbb{S}^1 \times \mathbb{R} \to \mathbb{R}$ the projection on the real line and claim that $\pi_{\mathbb{R}} \circ u$ is unbounded. If $\pi_{\mathbb{S}^1} \circ \gamma$ is not homotopic to a constant, by lemma 2.2.3,

$$\begin{aligned}\int_{\mathbb{S}^1} |\gamma'|_g^2 &\geq \int_{\mathbb{S}^1} (1 + f \circ \pi_{\mathbb{R}})|(\pi_{\mathbb{S}^1} \circ \gamma)'|^2 + \int_{\mathbb{S}^1} |(\pi_{\mathbb{R}} \circ \gamma)'|^2 \\ &\geq \frac{\lambda^2}{2\pi}(1 + \inf_{\mathbb{S}^1} f \circ \pi_{\mathbb{R}} \circ \gamma) + \frac{1}{2\pi}(\sup_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ \gamma| - \inf_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ \gamma|)^2.\end{aligned}$$

Let us fix $t > 0$. If

$$\int_{\mathbb{S}^1} |\gamma'|_g^2 \le \frac{\lambda^2}{2\pi} + t$$

then $t \ge \frac{\lambda^2}{2\pi} \inf_{\mathbb{S}^1} f \circ \pi_{\mathbb{R}} \circ \gamma + \frac{1}{2\pi}(\sup_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ \gamma| - \inf_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ \gamma|)^2$, meaning that

$$\sup_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ \gamma|^2 \ge \left(\frac{\lambda^2}{2\pi t}\right)^2 - 1 \text{ and } \inf_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ \gamma| \ge \sup_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ \gamma| - \sqrt{2\pi t} \ge D(t)$$

where $D(t) \doteq ((\frac{\lambda^2}{2\pi t})^2 - 1)_+^{\frac{1}{2}} - \sqrt{2\pi t}$. Since $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ there exists $M > 0$ such that for all $\rho > 0$ there exists $r \in (\rho, 1)$ satisfying

$$\log \frac{1}{\rho} \int_{\mathbb{S}^1} |u|_{\mathbb{S}^1}(r\cdot)'|_g^2 \le \int_{\mathbb{B}_1 \setminus \mathbb{B}_\rho} \frac{|Du|_g^2}{2} \le M + \frac{\lambda^2}{2\pi} \log \frac{1}{\rho}.$$

We therefore deduce that for each $\rho \in (0, 1)$ there exists $r \in (\rho, 1)$ such

$$\inf_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ u|_{\mathbb{S}^1_r}| \ge D(\frac{M}{\log \frac{1}{\rho}})$$

which implies the uniform divergence of the map u as the right-hand side is unbounded as ρ approaches zero.

We now claim that the set $\pi_{\mathbb{R}} \circ u(\mathbb{B}_1 \setminus \mathbb{B}_\rho)$ contains the whole interval $[0, \inf_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ u|_{\mathbb{S}^1}(r\cdot)|]$. We assume that the map is continuous on $\mathbb{B}_1 \setminus \mathbb{B}_\rho$ by approximation : this follows by [7] and the fact that $\mathcal{N}$ satisfies the *trimming property* [7, Proposition 6.3] ; we leave the details to the reader. Assume there exist a point $x \in \mathbb{S}^1 \times \mathbb{R}$ such that $x \notin u(\overline{\mathbb{B}_1 \setminus \mathbb{B}_\rho})$. We then observe that $\mathbb{S}^1 \times [0, 1] \setminus \{a\}$ (where $a \in \mathbb{S}^1 \times (0, 1)$) is homotopic to two copies of $\mathbb{S}^1$ joint by a segment. This implies that using the above described homotopy one can construct a retraction of the image $u(\overline{\mathbb{B}_1 \setminus \mathbb{B}_\rho})$ to a continous mapping $\mathbb{S}^1 \times [0, 1] \simeq \mathbb{B}_1 \setminus \mathbb{B}_\rho \to \mathcal{N} \xrightarrow{\pi_{\mathbb{S}^1}} \mathbb{S}^1$ that is an homotopy between two nonzero degree maps in $t = 0, 1$ but is constant near $t = 1/2$.

Since $|Du|_g^2 = |Du|^2 + (f \circ \pi_{\mathbb{R}} \circ u)|Du|^2$ and $|Du|^2 \ge 2|\partial_1 u \times_{\mathbb{R}^3} \partial_2 u|$, we obtain by the area formula [17, Theorem 1.6] that

$$\begin{aligned}
\int_{\mathbb{B}_1 \setminus \mathbb{B}_\rho} \frac{|Du|_g^2}{2} - \frac{\lambda^2}{2\pi} \log \frac{1}{\rho} &\ge \int_{\mathbb{B}_1 \setminus \mathbb{B}_\rho} f \circ \pi_2 \circ u |\partial_1 u \wedge \partial_2 u| \\
&\ge \int_{\mathcal{N}} f \circ \pi_2(y) \mathcal{H}^0(\mathbb{B}_1 \setminus \mathbb{B}_\rho \cap u^{-1}(\{y\}))\, \mathrm{d}y \\
&\ge \int_0^{\inf_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ u|_{\mathbb{S}^1}(r\cdot)|} f \,\mathrm{d}y
\end{aligned}$$

by the surjectivity of $\pi_{\mathbb{R}} \circ u(\mathbb{B}_1 \setminus \mathbb{B}_\rho)$ on the interval $[0, \inf_{\mathbb{S}^1} |\pi_{\mathbb{R}} \circ u|_{\mathbb{S}^1}(r\cdot)|]$. When $\rho \searrow 0$, the left-hand side is bounded as the map u is assumed to be renormalizable while by the choice of $f \notin L^1(\mathbb{R})$ the right-hand side is unbounded, contradicting the fact that $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$. □

2.2.5 Minimizing renormalizable singular harmonic maps

In this section we recall the notion of minimizing renormalizable harmonic map.

Definition 2.2.21. *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a Riemannian manifold $\mathcal{N}$ with* $\mathrm{Sys}(\mathcal{N}) > 0$. *A map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ associated with k distinct points $\{a_i\}_{i=1,\dots,k} \subset \Omega$ is a* minimal renormalizable singular harmonic map *(see [29, Definition 7.8]) whenever for every map $v \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ associated with k distinct points $\{b_i\}_{i=1,\dots,k} \subset \Omega$ such that $\mathrm{tr}_{\partial\Omega}\, u_* = \mathrm{tr}_{\partial\Omega}\, v$ and as sets*

$$\{([u_*, a_1], a_1), \dots, ([u_*, a_k], a_k)\} = \{([v, b_1], b_1), \dots, ([v, b_k], b_k)\},$$

we have $\mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) \leq \mathcal{E}^{1,2}_{\mathrm{ren}}(v)$.

When the manifold $\mathcal{N}$ is compact, minimal renormalizable maps enjoy of the following properties. Our analysis does not rely on those properties.

Proposition 2.2.22. *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a compact Riemannian manifold $\mathcal{N}$. Let $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ associated with k points $\{a_i\}_{i=1,\dots,k} \subset \Omega$ be a* minimizing renormalizable singular harmonic map. *If $\mathcal{N}$ is compact, then*

(i) $u_* \in C^\infty(\Omega \setminus \{a_i\}_{i=1,\dots,k}, \mathcal{N})$,

(ii) for each $i = 1, \dots, k$, $\sup_{x \in \mathbb{B}(a_i; \rho(\{a_i\}_{i=1,\dots,k}))} |x - a_i||Du_*(x)| < +\infty$,

(iii) for each $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,k}))$, u_* is a minimizing harmonic map on $\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i; \rho)$ with respect to its own boundary conditions provided $\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i; \rho)$ is a Lipschitz domain.

2.2.6 Renormalized energy of configuration of points

Following [29, (2.8) and (2.11)], one can define the following renormalized energy of a configuration of points for a topological resolution $(\gamma_1, \dots, \gamma_k)$

of $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$

$$\begin{aligned}(2.2.30)\quad & \mathcal{E}^{1,2}_{\mathrm{top},g,\gamma_1,\dots,\gamma_k}(a_1,\dots,a_k)\\ &\doteq \inf\left\{ \int_{\Omega\setminus\bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} \frac{|Du|^2}{2} - \sum_{i=1}^k \frac{\lambda([u,a_i])^2}{4\pi}\log\frac{1}{\rho} : \right.\\ &\qquad \left. \begin{array}{c} \rho \in (0,\rho(\{a_i\}_{i=1,\dots,k})) \\ u \in W^{1,2}(\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho), \mathcal{N}), \\ \mathrm{Tr}_{\partial\Omega} u = g,\ \mathrm{Tr}_{\mathbb{S}^1} u(a_i + \rho\cdot) \text{ is homotopic to } \gamma_i \end{array}\right\}.\end{aligned}$$

The renormalized energy is locally Lipschitz function of configurations of distinct points, which is bounded from below when the singularities $[u,a_1],\dots,[u,a_i]$ form a minimal topological resolution of the boundary condition g.

The renormalized energy of the configuration formed by the singularities provides a lower bound on the renormalized energy of a mapping [29, Proposition 7.2].

Proposition 2.2.23. *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a compact Riemannian manifold $\mathcal{N}$. If $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$, if $\{a_i\}_{i=1,\dots,k} \subset \Omega$ are the associated distinct singular points and if $g = \mathrm{Tr}_{\partial\Omega} u_*$, then*

$$\mathcal{E}^{1,2}_{\mathrm{top},g,\gamma_1,\dots,\gamma_k}(a_1,\dots,a_k) \le \mathcal{E}^{1,2}_{\mathrm{ren}}(u_*).$$

where $\gamma_i \in [u_*,a_i]$ $i=1,\dots,k$ are constant speed geodesic.

Conversely, given any configuration of points and choice of geodesics there exists a singular Sobolev map having homotopic singularities [29, Proposition 8.2].

Proposition 2.2.24. *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a compact Riemannian manifold $\mathcal{N}$. Given distinct points $a_1,\dots,a_k \in \Omega$ and geodesics $\gamma_1,\dots,\gamma_k : \mathbb{S}^1 \to \mathcal{N}$, there exists $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$ with trace $\mathrm{Tr}_{\partial\Omega} u_* = g$ with associated singular points $a_1,\dots,a_k$ such that γ_i and $[u_*,a_i]$ are homotopic and*

$$\mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) \le \mathcal{E}^{1,2}_{\mathrm{top},g,\gamma_1,\dots,\gamma_k}(a_1,\dots,a_k).$$

2.3 Upper bound

The renormalized energy introduced, we give the upper bound on sequences of minimizing p-harmonic maps.

Proposition 2.3.1 (Upper bound). *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, a Riemannian manifold $\mathcal{N}$ and $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$. For all renormalizable $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ associated to distinct points $a_i \in \Omega$, $i = 1, \dots, k$, if $([u, a_i], a_i)_{i=1,\dots,k}$ is minimal topological resolution of $\mathrm{Tr}_{\partial\Omega} u$, then*

$$\varlimsup_{p \nearrow 2} \left[\int_\Omega \frac{|Du|^p}{p} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u)}{2-p} \right] \le \mathcal{E}^{1,2}_{\mathrm{ren}}(u) + \mathrm{H}([u, a_i])_{i=1,\dots,k} \tag{2.3.1}$$

where

$$\mathrm{H}([u, a_i])_{i=1,\dots,k} \doteq \sum_{i=1}^{k} \frac{\lambda([u, a_i])^2}{8\pi} \left[1 + \log \Bigl(\frac{2\pi}{\lambda([u, a_i])} \Bigr)^2 \right]. \tag{2.3.2}$$

Hardt and Lin established this result in the case of the circle $\mathcal{N} = \mathbb{S}^1$ [21, Theorem 2.10]. We observe that minimal topological resolutions when $\mathcal{N} = \mathbb{S}^1$ either all satisfy $\deg([u, a_i]) = 1$ or all satisfy $\deg([u, a_i]) = -1$, so that for every $k \in \mathbb{N}$, $\mathrm{H}([u, a_i])_{i=1,\dots,k} = k\pi/2$ as $\lambda([u, a_i]) = 2\pi$. When Ω is simply connected, one further knows that $k = \deg(g, \partial\Omega)$.

The assumptions of proposition 4.4.1 are consistent with the existence of minimizing p-harmonic maps $u_p \in W^{1,p}_g(\Omega, \mathcal{N})$ (see proposition 2.1.1) and the proposition implies the first order bound

$$\varlimsup_{p \nearrow 2} (2-p) \int_\Omega \frac{|Du_p|^p}{p} \le \sum_{i=1}^{k} \frac{\lambda([u, a_i])^2}{4\pi}. \tag{2.3.3}$$

In fact, by proposition 2.5.1,

$$\lim_{p \nearrow 2} (2-p) \int_\Omega \frac{|Du_p|^p}{p} = \mathcal{E}^{1,2}_{\mathrm{sg}}(g). \tag{2.3.4}$$

In particular the limit in (2.3.4) only depends on the homotopy class of g.

Proof of proposition 4.4.1. We obtain

$$\varlimsup_{p \nearrow 2} \left[\int_\Omega \frac{|Du|^p}{p} - \frac{1}{2-p} \sum_{i=1}^{k} \frac{\lambda([u, a_i])^p}{(2\pi)^{p-1} p} \right] \le \mathcal{E}^{1,2}_{\mathrm{ren}}(u) \tag{2.3.5}$$

by proposition 2.2.17 and the fact that $W^{1,2}_{\mathrm{ren},g}(\Omega, \mathcal{N}) \subset W^{1,p}_g(\Omega, \mathcal{N})$. To obtain 2.3.1 we subtract $\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u)/(2-p)$ and use the fact that

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_\Omega u) = \sum_{i=1}^{k} \frac{\lambda([u, a_i])^2}{4\pi}$$

and

$$\lim_{p\nearrow 2}\frac{1}{2-p}\left[\sum_{i=1}^{k}\frac{\lambda([u,a_i])^p}{(2\pi)^{p-1}p}-\sum_{i=1}^{k}\frac{\lambda([u,a_i])^2}{4\pi}\right] = \frac{1}{2}\left[\sum_{i=1}^{k}\frac{\lambda([u,a_i])^2}{4\pi}-\sum_{i=1}^{k}\frac{\lambda([u,a_i])^2}{2\pi}\log\frac{\lambda([u,a_i])}{2\pi}\right]. \quad \square \tag{2.3.6}$$

The compactness condition on the manifold $\mathcal{N}$ can be in fact weakened in proposition 4.4.1 by the weaker condition that there exists at least a renormalizable map u of trace g *i.e.* $W^{1,2}_{\mathrm{ren},g}(\Omega,\mathcal{N}) \neq \varnothing$. Every compact manifold $\mathcal{N}$ satisfies $W^{1,2}_{\mathrm{ren},g}(\Omega,\mathcal{N}) \neq \varnothing$ (proposition 2.2.19) while their exist non-compact manifolds carrying no renormalizable mappings (proposition 2.2.20)

2.4 Lower bound

In order to state the proposition that will give us the lower bound we need to extend maps $u \in W^{1,p}(\Omega,\mathcal{N})$ of trace $g \in W^{1/2,2}(\Omega,\mathcal{N})$ to a larger domain Ω_δ.

2.4.1 Nonlinear extension of Sobolev maps

Unlike the linear case $W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)$, the trace operator is only locally surjective in the following sense (see [4]):

Proposition 2.4.1. *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$ and a Riemannian manifold $\mathcal{N}$. There exists a $\delta = \delta(\partial\Omega) > 0$ which depends only on the boundary $\partial\Omega$ such that for every boundary data $g \in W^{1/2,2}(\partial\Omega,\mathcal{N})$ one can construct a map $U \in W^{1,2}(\partial\Omega_\delta,\mathcal{N})$ satisfying* $\mathrm{Tr}_{\partial\Omega}U = g$.

Here $\partial\Omega_\delta$ means $\{x \in \mathbb{R}^2 : \mathrm{dist}(x,\partial\Omega) < \delta\}$. The proof of this proposition is a variant of [27, Theorem 1]. We point out that there is no quantitative estimate on U [38].

Given the previous lemma, one can extend a map $u \in W^{1,p}_g(\Omega,\mathcal{N})$, $p \in [1,2]$, to a map $\bar{u} \in W^{1,p}_g(\Omega_\delta,\mathcal{N})$ where $\Omega_\delta \doteq \{x \in \mathbb{R}^2 : \mathrm{dist}(x,\partial\Omega) < \delta\}$. Indeed, given the map U of lemma 2.4.1 we set

$$\bar{u} \doteq \begin{cases} u & \text{on } \Omega \\ U\big|_{\Omega_\delta\setminus\Omega} & \text{on } \Omega_\delta\setminus\Omega. \end{cases} \tag{2.4.1}$$

Along the boundary $\partial\Omega$ the traces of U and u coincide. Hence by the integration by parts formula it follows that the extended map $\bar{u}$ possesses a weak derivative on the extended domain Ω_δ [38]. For the sake of simplicity we will denote u the extended map instead of $\bar{u}$. We note that this extension is not canonical as the map U is in general not unique.

2.4.2 Approximation by smooth maps except at finitely many points

The dense class we will use takes its roots in the work of Fabrice Bethuel [5, Theorem 2] (see also [38]). The following density result is due to Pierre Bousquet, Augusto Ponce and Jean Van Schaftingen [8, 7] (see also [33]).

Proposition 2.4.2. *Fix $p \in (1,2)$, an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, a Riemannian manifold $\mathcal{N}$ and a map $u \in W^{1,p}(\Omega, \mathcal{N})$. Assume that there exists an open set $\omega \subset\subset \Omega$ with $u \in W^{1,2}(\Omega \setminus \bar{\omega}, \mathcal{N})$.*
Then, for every $\varepsilon > 0$ and every $\theta \in (0,1)$, setting

$$\omega_\theta \doteq \{x \in \Omega : \operatorname{dist}(\omega, x) < \theta \operatorname{dist}(\omega, \Omega)\},$$

there exists a map $v \in W^{1,p}(\omega_\theta, \mathcal{N})$ such that

(i) for some points $\{a_i\}_{i=1,\dots,k} \subset \omega_\theta$ ($k \in \mathbb{N}$) such that $v \in C^\infty(\omega_\theta \setminus \{a_i\}_{i=1,\dots,k}, \mathcal{N})$,

(ii) $\|u - v\|_{W^{1,p}(\omega_\theta)} \leq \varepsilon$,

(iii) $\operatorname{Tr}_{\partial\omega_\theta} u$ and $\operatorname{Tr}_{\partial\omega_\theta} v$ are homotopic.

We point out that no compactness assumption on the manifold $\mathcal{N}$ is done.

Sketch of the proof of proposition 2.4.2. Assuming first that $u \in L^\infty \cap W^{1,p}(\omega, \mathcal{N})$, we observe that the proof [5, 8, 33] of the density of smooth functions except at a finite number of points (elements of $C^\infty(\omega \setminus \{a_i\}_{i=1,\dots,k}, \mathcal{N})$) in $L^\infty \cap W^{1,p}(\omega, \mathcal{N})$ works by considering small disks $\mathbb{B}(a_n; \rho_n)$ and adaptively mollifying whether the mean oscillation of u on the disk depending is small or not. The condition is always satisfied if $u \in W^{1,2}(\mathbb{B}(a_n; \rho_n), \mathcal{N})$. In our case this implies that the modification of u in $\Omega \setminus \omega_\theta$ for some small $\theta \in (0,1)$ will not change the homotopy.

By [7, Theorem 1], as $p \notin \mathbb{N}$, $L^\infty \cap W^{1,p}(\Omega, \mathcal{N})$ is dense in $W^{1,p}(\Omega, \mathcal{N})$ and the proof of it shows that the modification done on u preserves the $W^{1,2}$-character on open sets. By this fact, we may assume that $u \in L^\infty \cap W^{1,p}(\Omega, \mathcal{N})$ and $u \in W^{1,2}(\Omega \setminus \bar{\omega}, \mathcal{N})$ by first approximating it by bounded maps. $\square$

2.4.3 Expansion of circles method

We now describe the expansion of disks method of Robert JERRARD [22] (see also Etienne SANDIER [35]) that we adapt to the p-energy. We recall that $\mathbb{B}(a;\rho)$ denotes the open disk of radius $r \geq 0$ centered in $a \in \mathbb{R}^2$. In particular we warn the reader that the empty set can be written $\mathbb{B}(a;0)$.

Proposition 2.4.3. *Let* $a_1,\dots,a_k \in \bar{\Omega}$ *where* $k \in \mathbb{N}\setminus\{0\}$ *and* $u \in W^{1,p}(\Omega,\mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\bar{\Omega}\setminus\{a_i\}_{i=1,\dots,k}),\mathcal{N})$. *If* $\operatorname{dist}(\{a_i\}_{i=1,\dots,k},\partial\Omega) > 0$, *for every* $\delta \in (0,\operatorname{dist}(\{a_i\}_{i=1,\dots,k},\partial\Omega)]$ *there exists a collection* $\mathcal{S}$ *of circles* $\mathbb{S}^1(a;\rho) \subset \Omega$ *such that* $\cup\mathcal{S}$ *is a finite union of disjoint annuli i.e. sets of the form* $\mathbb{A}^2(c;\rho,\sigma) \doteq \mathbb{B}(c;\sigma)\setminus\mathbb{B}(c;\rho) \subset \Omega$. *We have for some centers* $c_i \in \Omega$, $0 \leq \underline{r}_i < \overline{r}_i$ *for* $i = 1,\dots,N$

$$\cup\mathcal{S} = \bigcup_{i=1}^{N}\{\mathbb{S}^1(c_i;r) : \underline{r}_i \leq r < \overline{r}_i\} \tag{2.4.2}$$

and the union is disjoint. Moreover,

(i) $\cup\mathcal{S}$ *is contained in a finite number of disks* $\mathbb{B}_1,\dots,\mathbb{B}_l$ *such that* $\displaystyle\sum_{i=1}^{l}\operatorname{diam}(\mathbb{B}_i) = \delta$ *and* $\partial\mathbb{B}_i \in \mathcal{S}$ *for each* $i = 1,\dots,l$,

(ii) for every $\varepsilon \in (0,\delta]$ *there exists a finite number of disks* $\mathbb{S}^1(c_i,\rho_i) \in \mathcal{S}$ $i = 1,\dots,n$ *such that* $\displaystyle\sum_{i=1}^{n}\rho_i = \varepsilon$, $\{a_i\}_{i=1,\dots,k} \subset \displaystyle\bigcup_{i=1}^{n}\mathbb{B}(c_i,\rho_i)$

(iii) for any subcollection $\{\mathbb{S}^1(c_i^*;\rho_i^*)\}_{i=1,\dots,k_*} \subset \{\mathbb{S}^1(c_i,\rho_i)\}_{i=1,\dots,n}$, *there exists a disjoint collection of annuli*

$$\{\mathbb{A}^2(c_i;\underline{r}_i,\overline{r}_i)\}_{i=1,\dots,n} \subset \bigcup_{i=1}^{k_*}\mathbb{B}(c_i^*,\rho_i^*),$$

such that

$$\Big[\sum_{i=1}^{k_*}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\operatorname{Tr}_{\mathbb{S}^1(c_i^*;\rho_i^*)}u)\Big]^{p-1}\Big[\sum_{i=1}^{k_*}\rho_i^*\Big]^{2-p} \leq (2-p)\sum_{i=1}^{n}\int_{\underline{r}_i}^{\overline{r}_i}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\operatorname{Tr}_{\mathbb{S}^1(c_i;r)}u)^{p-1}\frac{\mathrm{d}r}{r^{p-1}}. \tag{2.4.3}$$

Lattice structure of $\mathcal{S}$. In fact, the proof of proposition 2.4.3 shows that the collection of circles $\mathcal{S}$ has a finite lattice structure which is a finite union of oriented trees whose roots are the elements $\{a_i\}_{i=1,\dots,k}$ (see item (ii)), the top of each tree is the collection $\partial\mathbb{B}_i$ of item (i) of the proposition. An element of the lattice corresponds to a circle of the collection $\mathcal{S}$. The orientation encodes the following partial order relation: $\mathbb{S}^1(a;\rho) \preccurlyeq \mathbb{S}^1(b;\sigma)$ if and only if $\mathbb{B}(a;\rho) \subset \mathbb{B}(b;\sigma)$. Edges on the lattice correspond to a linear parametrized subfamily of $\mathcal{S}$: each edge is described by $\{\mathbb{S}^1(a, s\rho) : s \in (t_*, t^*)\}$ for some $a \in \Omega$, $0 < t_* < t^* < \infty$; these are the annuli of (2.4.2). The tree structure carries some topological information: given a element $\mathbb{S}^1_*(a_*, \rho_*) \in \mathcal{S}$ of the oriented lattice and any family F of elements of the lattice such that each $\mathbb{S}^1(a) \in F$ satisfies $\mathbb{S}^1(a;\rho) \preccurlyeq \mathbb{S}^1_*(a_*, \rho_*)$ for each $a_i \in \mathbb{B}(a_*, \rho_*)$ and there exists a unique $\mathbb{S}^1(a_i;\rho_i) \in F$ such that $a_i \in \mathbb{B}(a_*;\rho_*)$, then $(\mathrm{Tr}_{\mathbb{S}^1} u(a+\rho\cdot))_{\mathbb{S}^1(a;\rho)\in F}$ is a topological resolution of $(\mathrm{Tr}_{\mathbb{S}^1} u(a_* + \rho_*\cdot))$.

Particular cases. We discuss two particular cases of proposition 2.4.3.

- *Single singularity.* If $a \in \Omega$, $u \in W^{1,p}(\Omega, \mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\bar\Omega \setminus \{a\}, \mathcal{N})$. Then for $\delta \in (0, \mathrm{dist}(\{a\}, \partial\Omega))$, proposition 2.4.3 yields $\mathcal{S} = \{\mathbb{S}^1(a;r) : r \in [0,\delta]\}$ and $\cup\mathcal{S} = \mathbb{B}(a;\delta) \subset \Omega$. In this precise case (2.4.3) is in fact an equality.

- *Pair of singularities.* If $a, b \in \Omega$, $u \in W^{1,p}(\Omega, \mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\bar\Omega \setminus \{a, b\}, \mathcal{N})$. Then for $\delta \in (0, \mathrm{dist}(\{a, b\}, \partial\Omega))$, proposition 2.4.3 yields a collection of circles whose union consists in two disks centered in a, b respectively and an annulus surrounding the two disks whose radii are chosen in order to (2.4.3) to hold.

We record the following corollary that extends proposition 2.2.4 to general domains.

Corollary 2.4.4. *Let $a_1, \dots, a_k \in \Omega$ and $u \in W^{1,p}(\Omega, \mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\bar\Omega \setminus \{a_i\}_{i=1,\dots,k}, \mathcal{N})$. There exists $l \in \mathbb{N}$ disjoint disks $\mathbb{B}_i$ $i = 1, \dots, l$ such that its sum of the radii is $\delta \in (0, \mathrm{dist}(\{a_i\}_{i=1,\dots,k}, \partial\Omega)]$ and*

$$\delta^{2-p}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u)^{p-1} \le (2-p)\frac{(p-1)^{p-1}}{(2\pi/p)^{2-p}} \int_{\bigcup_{i=1}^{l} \mathbb{B}_l} \frac{|Du|^p}{p}$$

Proof of the corollary 2.4.4. Using proposition 2.4.3, we define $\mathcal{B}$ to be the collection of proposition 2.4.3 (i). By construction the disks are disjoint. Using proposition 2.4.3(ii) with the collection of assertion (i), proposition 2.2.4 and proposition 2.2.9, we get the desired result. □

Lemma 2.4.5 (*Merging disks lemma*)**.** *Let $\mathcal{B}_*$ be a finite collection of closed disks in $\mathbb{R}^2$. There exists a finite collection of disks $\mathcal{B}^*$ —called the merged collection— of closed disks such that* *(i) two distinct disks* $\mathbb{B}_1, \mathbb{B}_2 \in \mathcal{B}_*$ *satisfy* $\mathbb{B}_1 \cap \mathbb{B}_2 = \varnothing$,

(ii) B^* *covers* $\mathcal{B}_*$*: for every* $\mathbb{B} \in \mathcal{B}_*$*, there exists* $\mathbb{B}' \in \mathcal{B}^*$ *such that* $\mathbb{B} \subseteq \mathbb{B}'$,

(iii) the sum of the radii is conserved i.e. $\displaystyle\sum_{\mathbb{B}(a;\rho)\in\mathcal{B}_*} \rho = \sum_{\mathbb{B}(a;\rho)\in\mathcal{B}^*} \rho.$

The proof of the lemma is well–known, see *e.g.* [35, p. 386][22, lemma 3.1].

Proof of lemma 2.4.5. For the first part of the proof, we assume that there are $k = 2$ intersecting disks $\mathbb{B}(a_1; \rho_1)$ and $\mathbb{B}(a_2; \rho_2)$. We then set

$$\mathcal{B}^* = \{\mathbb{B}(c; \rho_1 + \rho_2)\} \text{ where } c \doteq \frac{\rho_1 a_1 + \rho_2 a_2}{\rho_1 + \rho_2}.$$

If $k \geq 3$ one proceeds by induction. □

Proof of proposition 2.4.3. We fix

$$u \in W^{1,p}(\Omega, \mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\bar{\Omega} \setminus \{a_i\}_{i=1,\dots,k}, \mathcal{N}).$$

For simplicity we write $\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;\rho))$ for $\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathrm{tr}_{\mathbb{S}^1(a;\rho)}\, u)$ and $\mathcal{E}^{1,p'}_{\mathrm{sg}}(a)$ for $\mathcal{E}^{1,p'}_{\mathrm{sg}}([u,a])$. Fix $t_0 > 0$ to be chosen later. We set

$$\mathrm{S}(0) \doteq \bigcup_{0<s<t_0} \mathcal{S}_s \quad \text{with} \quad \mathcal{S}_s \doteq \bigcup_{i=1}^{k} \{\mathbb{S}(a_i, s\mathcal{E}^{1,p'}_{\mathrm{sg}}(a_i))\}. \tag{2.4.4}$$

We choose the largest $t_0 > 0$ such that the collection of circles satisfies for each $s \in (0, t_0)$,

(I) the sum of the radii of the circles in the collection $\mathcal{S}_s$ does not exceed δ,

(II) the collection $\mathcal{S}_s$ is a pairwise disjoint collection of circles agglomerated in disjoint annuli.

If the process does not satisfy anymore (I) at t_0 then we stop the process. Note that for all $s \in (0, t_0)$ every circle $\mathbb{S}^1(a_s, \rho_s) \in \mathcal{S}_s$ satisifies

$$\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a_s; \rho_s))s = \rho_s. \tag{2.4.5}$$

If at time t_0 some circles intersect, by lemma 2.4.5, we merge them and we obtain new disks $\{\mathbb{S}^1(a_{t_0}; \rho_{t_0})\}$ such that

$$\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a_s; \rho_s))s \leq \rho_s \tag{2.4.6}$$

by the monotonicity of the singular energy (proposition 2.2.9). The collection $\{\mathbb{S}^1(a_{t_0};\rho_{t_0})\}$ is the disjoint union of the collection $\mathcal{S}^{=}_{t_0}$ circles such that (2.4.5) holds at t and the collection $\mathcal{S}^{<}_{t_0}$ of those disks that satisfies (2.4.6) with a strict inequality sign. This latter collection has the property to contain circles that can be decomposed in a finite number of circles such that (2.4.5) holds. We gather those in a collection that we denote $\mathcal{S}_{t_0,\mathrm{sub}}$. Indeed, before merging, (2.4.5) held for each disks. We define $\mathcal{S}_{t_0} \doteq \{\mathbb{S}^1(a_{t_0};\rho_{t_0})\} = \mathcal{S}^{<}_{t_0} \cup \mathcal{S}^{=}_{t_0}$ and $\mathcal{S}_{t_0^-} = \mathcal{S}^{<}_{t_0} \cup \mathcal{S}_{t_0,\mathrm{sub}}$.

Next we define

$$
\text{(2.4.7)} \quad \mathrm{S}(1) \doteq \bigcup_{t_0<s<t_1} \mathcal{S}_s \quad \text{with} \quad \mathcal{S}_s \doteq \mathcal{S}^{<}_{t_0} \cup \mathcal{S}^{=}_s
$$

$$
\text{where} \quad \mathcal{S}^{=}_s \doteq \bigcup \{\mathbb{S}(a_{t_0}, s\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a_{t_0};\rho_{t_0})) : \mathbb{S}^1(a_{t_0};\rho_{t_0}) \in \mathcal{S}^{=}_{t_0}\}.
$$

where $t_1 > t_0$ is the largest number such that (II) and (I) hold for $s \in (t_0, t_1)$ and

(III) $\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a_{t_0};\rho_{t_0}))s < \rho_{t_0}$ for all $\mathbb{S}^1(a_{t_0};\rho_{t_0}) \in \mathcal{S}^{<}_{t_0}$.

If at t_1, (I) happens we stop the construction. If at t_1, (II) occurs, we may repeat the merging process by distinguishing $\mathcal{S}_{t_1} = \mathcal{S}^{=}_{t_0} \cup \mathcal{S}^{<}_{t_0}$ circles that satifies (2.4.6) with an equality or strict inequality. If condition (III) (see (2.4.5)) is unsatisfied at t_1 for some circles $\{\mathbb{S}^1(a_j;\rho_j)\}_{j=1,\dots,l}$. We set $\mathcal{S}^{<}_{t_1} = \mathcal{S}^{<}_{t_0} \backslash \{\mathbb{S}^1(a_j;\rho_j)\}_{j=1,\dots,l}$ and redefine $\mathcal{S}^{=}_{t_1}$ to be $\mathcal{S}^{=}_{t_1} \cup \{\mathbb{S}^1(a_j;\rho_j)\}_{j=1,\dots,l}$. It is then possible to reiterate the construction (2.4.7).

After a finite number N of iterations the construction of S(0), S(1), ..., S(N) stops by the finite number of balls that decrease at each step and we obtain a $T > 0$ such that

(a) the sum of the radii at of $\mathcal{S}_T$ is equal to δ.

(b) for each $t \in (0, T]$, $(\mathrm{Tr}_{\mathbb{S}^1(a;\rho)} u)_{\mathbb{S}^1(a;\rho) \in \mathcal{S}_t}$ is a topological resolution of $\mathrm{Tr}_{\partial\Omega} u$

(c) for each $t \in (0, T]$, every disk in $\mathbb{S}^1(a_t;\rho_t) \in \mathcal{S}_t$ satisfies either (1) $\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a_t;\rho_t))t = \rho_t$, or (2) there exists $k_t \in \mathbb{N}$ disks $\{\mathbb{S}^1(a^i_t;\rho^i_t)\}_{i=1,\dots,k_t}$ such that
$$
\sum_{i=1}^{k_t} \rho^i_t = \rho_t, \quad \bigcup_{i=1}^{k_t} \mathbb{S}^1(a^i_t;\rho^i_t) \subset \mathbb{B}(a_t;\rho_t),
$$
$(\mathrm{Tr}_{\mathbb{S}^1(a^i_t;\rho^i_t)} u)_{i=1,\dots,k_t}$ is a topological resolution of $\mathrm{Tr}_{\mathbb{S}^1(a_t,\rho_t)} u$ and $\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a^i_t;\rho^i_t))t = \rho^i_t$ for each $i = 1, \dots, k_t$.

(d) for each $0 < s < t < T$, $\mathcal{S}_s \preccurlyeq \mathcal{S}_t$ in the sense that for each circle $\mathbb{S}^1(a,\rho) \in \mathcal{S}_s$ there exists $\mathbb{S}^1(a',\rho') \in \mathcal{S}_t$ such that $\mathbb{S}^1(a,\rho) \subset \mathbb{B}(a';\rho')$.

The local estimate. We prove an intermediate and localized version of (2.4.3). Fix some $\mathbb{S}^1(a^*;r^*) \in \mathcal{S}$. We set $\bar{T}$ the largest $t > 0$ such that $\mathcal{S}_t \cap \mathbb{B}(a^*, r^*) \neq \varnothing$. By construction $\mathcal{S}_{\bar{T}} \cap \mathbb{B}(a^*, r^*)$ consists in a disjoint and finite collection of annuli $\mathcal{A} = \{\mathbb{A}^2(a^i; r_i, r^i))\}_{i=1,\ldots,n_*}$. For simplicity we denote

$$\int_{\bigcup \mathcal{A}} \mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;r))^{p-1} \frac{\mathrm{d}r}{r^{p-1}} \doteq \sum_{i=1}^{n} \int_{r_i}^{r^i} \mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a^i;r))^{p-1} \frac{\mathrm{d}r}{r^{p-1}}.$$

We have by change of variables

$$\begin{aligned}\int_{\bigcup \mathcal{A}} &\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;r))^{p-1} \frac{\mathrm{d}r}{r^{p-1}} \\ &= \sum_{j=1}^{N} \sum_{\mathbb{S}^1(a;\rho) \in \mathbb{B}(a^*;r^*) \cap \mathcal{S}_{t_j}} \int_{t_j}^{t_{j+1}} \mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;\rho_j)) \frac{\mathrm{d}s}{s^{p-1}} \\ &\geq \sum_{j=1}^{N} \int_{t_j}^{t_{j+1}} \mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a^*;r^*)) \chi_{[0,\bar{T}]} \frac{\mathrm{d}s}{s^{p-1}}\end{aligned}$$

by definition of the singular energy. We thus have

$$\begin{aligned}\int_{\bigcup \mathcal{A}} &\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;r))^{p-1} \frac{\mathrm{d}r}{r^{p-1}} \\ &\geq \mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a^*;r^*)) \int_0^{\bar{T}} \frac{\mathrm{d}s}{s^{p-1}} = \mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a^*;r^*)) \frac{\bar{T}^{2-p}}{2-p}.\end{aligned}$$

The estimate (2.4.3) *in* (ii). For every $\varepsilon \in (0, \delta]$ there exists a finite number of disks $\mathbb{S}^1(c_i, \rho_i) \in \mathcal{S}$ $i = 1, \ldots, n$ such that $\sum_{i=1}^n \rho_i = \varepsilon$ and $\{a_i\}_{i=1,\ldots,k} \subset \bigcup_{i=1}^n \mathbb{B}(c_i, \rho_i)$. This is because $t \mapsto \sum_{\mathbb{S}^1(a;\rho) \in \mathcal{S}_t} \rho$ is increasing and continuous equal to zero at $t = 0$, equal to δ at $t = T$ and so the intermediate value theorem applies and yields the existence of a $\bar{T} \in (0, T]$ such that $\mathcal{S}_{\bar{T}} = \{\mathbb{S}^1(c_i, \rho_i)\}_{i=1,\ldots,n}$ and $\sum_{i=1}^n \rho_i = \rho$. By (c)(2) we can assume that for each $i = 1, \ldots, n$,

$$\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(c_i;\rho_i))\bar{T} = \rho_i.$$

Let us consider an arbitrary subcollection $\{\mathbb{S}^1(c_i^*;\rho_i^*)\}_{i=1,\ldots,k_*} \subset \{\mathbb{S}^1(c_i, \rho_i)\}_{i=1,\ldots,n}$, by the preceding paragraph there exist N disjoint collections $\mathcal{A}_j$ of cardinality n_j such that

$$\mathcal{A}_j = \{\mathbb{A}^2(c_i; \underline{r}_i, \overline{r}^i)\}_{i=1,\ldots,n_j}$$

and therefore

$$\begin{aligned}\sum_{i=1}^{N}\int_{\underline{r}_i}^{\overline{r}^i}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(c_i;r))^{p-1}\frac{\mathrm{d}r}{r^{p-1}} &= \int_{\bigcup_{i=1}^N\bigcup\mathcal{A}_i}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;r))^{p-1}\frac{\mathrm{d}r}{r^{p-1}}\\ &\geq \sum_{i=1}^{k_*}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(c_i^*;\rho_i^*))\frac{\bar{T}^{2-p}}{2-p}\\ &= \Big(\sum_{i=1}^{k_*}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(c_i^*;\rho_i^*)\Big)^{p-1}\frac{\Big(\sum_{i=1}^{k_*}\rho_i^*\Big)^{2-p}}{2-p}\end{aligned}$$

where $N=\sum_{j=1}^n n_j$ and the last equality follows from the choice of the disks. □

2.4.4 Going to the Marcinkiewicz scale

In this section we prove the following proposition. We recall that $A_+\doteq\max(0,A)$.

Proposition 2.4.6 (Mixed Marcinkiewicz estimate)**.** *Fix $p\in(1,2)$. Let $u\in W^{1,2}_{\mathrm{loc}}(\Omega\setminus\{a_i\}_{i=1,\dots,k},\mathcal{N})\cap W^{1,p}(\Omega,\mathcal{N})$ where $\{a_i\}_{i=1,\dots,k}\subset\Omega$. Let also $\delta\in(0,\mathrm{dist}(\{a_i\}_{i=1,\dots,k},\partial\Omega)]$. There exists a finite collection $\mathcal{B}$ of disjoint disks $\mathbb{B}\subset\Omega$ such that the sum of their radii does not exceed δ, and a nonnegative measurable function $U:\Omega\to\mathbb{R}$ supported in $\bigcup_{\mathbb{B}\in\mathcal{B}}\mathbb{B}$ such that*

(2.4.8)
$$\frac{p-1}{p}\int_{\bigcup_{\mathbb{B}\in\mathcal{B}}\mathbb{B}}(|Du|-U)_+^p+\frac{(2\pi\delta)^{2-p}}{p^{2-p}(p-1)^{p-1}}\Big(\sum_{\mathbb{B}\in\mathcal{B}}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\mathbb{B}}u)\Big)^{p-1}\leq\frac{(3-p)p}{2}\int_{\bigcup_{\mathbb{B}\in\mathcal{B}}\mathbb{B}}\frac{|Du|^p}{p}.$$

Moreover, $U\in\{0\}\cup[\mathrm{Sys}(\mathcal{N})/(2\pi\delta),\infty)$ almost everywhere,

$$\sup_{t>0}t^p|\{U>t\}|\leq\frac{(2-p)p^{p-1}}{2(p-1)^{p-1}(2\pi)^{2-p}}\int_\Omega\frac{|Du|^p}{p}, \tag{2.4.9}$$

and

$$\sup_{t>0}t^{p-1}\mathrm{Per}(\{U>t\})\leq\frac{(2-p)(2\pi)^{3-p}(p')^{p-1}}{\mathrm{Sys}(\mathcal{N})}\int_\Omega\frac{|Du|^p}{p}. \tag{2.4.10}$$

One consequence of (2.4.9)–(2.4.10) is that, combined with the first-order upper bound (2.3.3), it asserts that any family of minimizing p-harmonic maps $(u_p)_{p\in(1,2)}$ which is smooth outside a finite number of points and subject to $\mathrm{Tr}_{\partial\Omega} u_p = g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ satisfies

$$\varlimsup_{p\nearrow 2} \sup_{t>0} t^p |\{U_p > t\}| \le 4\pi \varlimsup_{p\nearrow 2} (2-p) \int_\Omega \frac{|Du_p|^p}{p} \le 4\pi \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u),$$

where U_p is the map given by proposition 2.4.6. Some of consequences of this fact are described in proposition 2.5.8. The perimeter Per of the level set $\{U > t\}$ is defined through the gradient measure of the characteristic function of the level set [40, Section 7.4] which is of bounded variation (BV):

$$\mathrm{Per}(\{U > t\}) \doteq \int_\Omega |D\chi_{\{U>t\}}|.$$

To prove the announced bound we will first write an estimate concerning real numbers (lemma 2.4.7). We will then integrate it in lemma 2.4.8 on $\partial\mathbb{B}(a;\rho)$ and then, with the help of the circle proposition 2.4.3, we will choose appropriately the shells on which we will integrate. The function U is explicitly given in the proof, see (2.4.15).

Lemma 2.4.7. *If $a, b \ge 0$ and $p \in [1,2]$, then*

$$\frac{a^p}{p} + a^{p-1}(b-a) + \bigl(1 - \frac{1}{p}\bigr)(b-a)_+^p \le \frac{3-p}{2} b^p.$$

Proof of lemma 2.4.7. Considering the functions $t \in [0,1] \mapsto (a(1-t) + bt)^p$ we have by the integral form of the Taylor expansion

$$b^p = a^p + pa^{p-1}(b-a) + p(p-1)\int_0^1 \frac{(b-a)^2(1-t)}{|(1-t)a + tb|^{2-p}}\,\mathrm{d}t. \tag{2.4.11}$$

We check that

$$\frac{(b-a)^2}{|(1-t)a + tb|^{2-p}} \ge \frac{(b-a)_+^2}{b^{2-p}} \quad\text{and}\quad \int_0^1 (1-t)\,\mathrm{d}t = \frac{1}{2}. \tag{2.4.12}$$

By Young's inequality, we have

$$(b-a)_+^p = \frac{(b-a)_+^p}{b^{\frac{p(2-p)}{2}}} b^{\frac{p(2-p)}{2}} \le \frac{p}{2}\frac{(b-a)_+^2}{b^{2-p}} + \frac{(2-p)}{2} b^p \tag{2.4.13}$$

as $1 = 1/(2/p) + 1/(2/(2-p))$. Combining (2.4.11)–(2.4.13) multiplied by $(p-1)/p$, we get the conclusion. ☐

Proposition 2.4.8. *Let $a \in \mathbb{R}^2$ and $0 < \rho < \sigma$. If $u \in W^{1,2}(\mathbb{B}(a;\sigma) \setminus \mathbb{B}(a;\rho), \mathcal{N})$, then we have for every $r \in (\rho,\sigma)$ and for any $0 \le \eta \le \lambda(\operatorname{Tr}_{\partial\mathbb{B}(a;\rho)} u)$*

(2.4.14)
$$\frac{\eta^p}{p(2\pi r)^{p-1}} + \Bigl(1 - \frac{1}{p}\Bigr) \int_{\partial\mathbb{B}(a;r)} \Bigl(|Du| - \frac{\eta}{2\pi r}\Bigr)_+^p \le \frac{(3-p)p}{2} \int_{\partial\mathbb{B}(a;r)} \frac{|Du|^p}{p}.$$

Proof of proposition 2.4.8. From lemma 2.4.7, we get, setting $b = |Du|$,

$$\begin{aligned}\frac{a^p}{p}(2\pi r) + a^{p-1} \int_{\partial\mathbb{B}(a;r)} (|Du| - a) + \Bigl(1 - \frac{1}{p}\Bigr) \int_{\partial\mathbb{B}(a;r)} (|Du| - a)_+^p \\ \le \frac{(3-p)p}{2} \int_{\partial\mathbb{B}(a;r)} \frac{|Du|^p}{p}.\end{aligned}$$

We on the Sobolev energy on circles (lemma 2.2.3) that

$$\int_{\partial\mathbb{B}(a;r)} |Du| \ge \lambda(\operatorname{Tr}_{\mathbb{S}^1} u(a + \rho\cdot)) \ge \eta = \int_{\partial\mathbb{B}(a;r)} \frac{\eta}{2\pi r}.$$

Taking $a = \eta/(2\pi r)$, we get the conclusion. □

Proof of proposition 2.4.6. From proposition 2.4.3 with

$$\delta = \operatorname{dist}(\{a_i\}_{i=1,\dots,N}, \partial\Omega),$$

we get the existence of a collection of circles $\mathcal{S}$ considered as a disjoint collection $\mathcal{A}$ of annuli. We write $\mathcal{A} = \{\mathbb{A}^2(c_i; \underline{r}_i, \overline{r}_i)\}_{1,\dots,N}$. Next, we define for all $x \in \Omega$

(2.4.15)
$$U(x) \doteq \sup\left\{ \frac{\Bigl((2\pi)^{p'-1} p' \,\mathcal{E}^{1,p'}_{\mathrm{sg}}(\operatorname{Tr}_{\partial\mathbb{B}(a;r)} u)\Bigr)^{\frac{1}{p'}}}{2\pi r} : \mathbb{S}^1(a;r) \in \mathcal{S}, x \in \mathbb{B}(a;r) \right\}$$

with the convention $\sup \varnothing = 0$. If $x \in \mathbb{A}^2(c_i; \underline{r}_i, \overline{r}^i)$ then by definition 2.2.8

$$U(x) = \frac{\Bigl((2\pi)^{p'-1} p' \,\mathcal{E}^{1,p'}_{\mathrm{sg}}(\operatorname{Tr}_{\partial\mathbb{B}(c_i;\underline{r}_i)} u)\Bigr)^{\frac{1}{p'}}}{2\pi|x - c_i|} \le \frac{\lambda(\operatorname{Tr}_{\partial\mathbb{B}(a_i;\underline{r}_i)} u)}{2\pi|x - c_i|}.$$

Integrating proposition 2.4.8 over all the annuli $\mathbb{S}(a;\rho) \in \mathcal{S}$ of proposition 2.4.3, we get

$$\begin{aligned}\frac{(3-p)p}{2}\int_{\cup\mathcal{S}}\frac{|Du|^p}{p}\\ &\geq \frac{(2\pi)^{2-p}}{p^{2-p}(p-1)^{p-1}}\sum_{i=1}^{N}\int_{\underline{r}_i}^{\overline{r}^i}\mathcal{E}^{1,p'}_{\mathrm{sg}}(u,\mathbb{S}^1(c_i;r))^{p-1}\frac{\mathrm{d}r}{r^{p-1}}\\ &\qquad + \frac{p-1}{p}\int_{\cup\mathcal{S}}(|Du|-U)_+^p\end{aligned}$$

We define $\mathcal{B} \doteq \{\mathbb{B}_i : i = 1,\dots,l\}$ where the disks are those of assertion (i) of proposition 2.4.3. By construction the disks are disjoint and by the choice of the circles of proposition 2.4.3 and by adding the integral of $|Du|^p$ over $\bigcup\mathcal{B}\setminus\cup\mathcal{S}$, we get the announced result (2.4.8).

Weak-L^p estimate (2.4.9). By construction of U,

$$\{U > t\} = \bigcup\mathcal{B}$$

where the union runs over a finite family $\mathcal{B}$ of pairwise disjoint disks $\mathbb{B}(a;\rho)$ such that

$$\rho t \leq \frac{\big((2\pi)^{p'-1}p'\,\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;\rho))\big)^{\frac{1}{p'}}}{2\pi}. \tag{2.4.16}$$

Moreover, one has

$$\rho^{2-p}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;\rho))^{p-1} \leq (2-p)\int_{\mathbb{B}(a;\rho)}\frac{|Du|^p}{p}. \tag{2.4.17}$$

Hence,

$$\begin{aligned}t^p|\{U>t\}| &\leq \pi\sum_{\mathbb{B}(a;\rho)\in\mathcal{B}} t^p\rho^2\\ &\leq \frac{((2\pi)^{p'-1}p')^{p-1}\pi}{(2\pi)^p}\sum_{\mathbb{B}(a;\rho)\in\mathcal{B}}\rho^{2-p}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;\rho))^{p-1}\\ &\leq \frac{(2-p)p^{p-1}}{2(p-1)^{p-1}(2\pi)^{2-p}}\int_\Omega\frac{|Du|^p}{p}.\end{aligned}$$

Perimeter estimate. Letting $\mathcal{B}$ as above, we have

$$\begin{aligned}\mathrm{Per}(\{U>t\}) &= \sum_{\mathbb{B}(a;\rho)\in\mathcal{B}} 2\pi\rho\\ &= 2\pi\sum_{\mathbb{B}(a;\rho)\in\mathcal{B}}\rho^{2-p}\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;\rho))^{p-1}\Big(\frac{\rho}{\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{S}^1(a;\rho))}\Big)^{p-1}.\end{aligned} \tag{2.4.18}$$

On the other hand, by (2.4.16) and by (2.2.13),

$$\frac{\rho}{\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{B}(a;\rho))} \le \frac{((2\pi)^{p'-1}p')^{\frac{1}{p'}}}{2\pi t \mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathbb{B}(a;\rho))^{\frac{1}{p}}} \le \frac{p'(2\pi)^{p'-2}}{t\mathrm{Sys}(\mathcal{N})^{\frac{1}{p-1}}}$$

and thus (2.4.18) becomes, in view of (2.4.17),

$$\mathrm{Per}(\{U > t\}) \le \frac{(2-p)(2\pi)^{3-p}(p')^{p-1}}{t^{p-1}\mathrm{Sys}(\mathcal{N})} \int_\Omega \frac{|Du|^p}{p}. \qquad \square$$

2.5 Convergence of bounded sequences

We establish a compactness result which will be applied to sequences of p-harmonic mappings as $p \nearrow 2$.

Proposition 2.5.1 (Compactness theorem). *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $\mathcal{N}$ a Riemannian manifold with $\mathrm{Sys}(\mathcal{N}) > 0$ and $(p_n)_n$ be a sequence in the interval $(1,2)$ converging to 2. Let $(u_n)_n$ be a sequence of maps such that $u_n \in W^{1,p_n}(\Omega,\mathcal{N})$, $\mathrm{Tr}_{\partial\Omega} u_n = \mathrm{Tr}_{\partial\Omega} u_0 \in W^{1/2,2}(\partial\Omega,\mathcal{N})$ for all n. If*

$$\sup_n \left[\int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)}{2-p_n}\right] < +\infty, \tag{2.5.1}$$

then, up to some unrelabeled subsequence, $(u_n)_n$ converges almost everywhere to some measurable $u_* : \Omega \to \mathcal{N}$.
Moreover,

(i) the map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega,\mathcal{N})$ satisfies $\mathrm{Tr}_{\partial\Omega} u_* = \mathrm{Tr}_{\partial\Omega} u_0$. Calling $\{a_i\}_{i=1,\dots,\kappa} \subset \Omega$ the associated singular points of the renormalizable map u_*, we have that for all $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,k}))$, $(\mathrm{Tr}_{\partial\mathbb{B}(a_i;\rho)} u_*(a_i + \rho\cdot))_{i=1,\dots,\kappa}$ is a minimal topological resolution of $\mathrm{Tr}_{\partial\Omega} u_*$ i.e.

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_*) = \sum_{i=1}^{\kappa} \frac{\lambda([u_*, a_i])^2}{4\pi},$$

and, moreover, $\lambda([u_*, a_i]) > 0$ for all $i = 1, \dots, \kappa$,

(ii) for all $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,k}))$, $(Du_n)_n$ is uniformly integrable on $\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)$ and for all $q \in [1,\infty)$,

$$\varlimsup_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i;\rho)} |u_n|^q + \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i;\rho)} |Du_n|^{p_n} < +\infty.$$

(iii) narrowly as measures on Ω,

$$(2-p_n)\frac{|Du_n|^{p_n}}{p_n} \xrightarrow{n\to\infty} \sum_{i=1}^{\kappa} \frac{\lambda([u_*,a_i])^2}{4\pi}\delta_{a_i}$$

(iv) one has

$$\lim_{n\to\infty} \int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_n)}{2-p_n} \tag{2.5.2}$$
$$\geq \mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*,a_i])_{i=1,\dots,\kappa}$$
$$\geq \mathcal{E}^{1,2}_{\mathrm{top},g,[u_*,a_1],\dots,[u_*,a_k]}(a_1,\dots,a_k) + \mathrm{H}([u_*,a_i])_{i=1,\dots,\kappa}.$$

(v) for each $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,k}))$,

$$\lim_{n\to\infty} \int_{\mathbb{B}(a_i;\rho)} \frac{|Du_n|^{p_n}}{p_n} - \frac{\lambda([u_*,a_i])^{p_n}}{(2\pi)^{p_n} p_n |\cdot - a_i|^{p_n-1}}$$
$$\geq \int_{\mathbb{B}(a_i;\rho)} \frac{|Du_*|^2}{2} - \frac{\lambda([u_*,a_i])^2}{8\pi^2|\cdot - a_i|}.$$

Recall the quantity H was defined in proposition 4.4.1 and the non-intersection radius $\rho(\{a_i\}_{i=1,\dots,k})$ was defined in (2.2.18).

By item (ii), we obtain that for all $\rho > 0$ and $p \in (1,2)$ $Du_n \to Du_*$ weakly in $L^p(\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho), \mathbb{R}^{\nu\times 2})$. This implies that $Du_n \to Du_*$ as distributions on $\Omega \setminus \{a_i\}_{i=1,\dots,\kappa}$.

The narrow convergence in (iii) is equivalent (see [1, Proposition 4.2.4]) to the fact that for any bounded and continous function $\xi : \Omega \to \mathbb{R}$,

$$(2-p_n)\int_\Omega \xi \,\mathrm{d}|Du_n| \xrightarrow{n\to\infty} \sum_{i=1}^{\kappa} \frac{\lambda([u_*,a_i])^2}{4\pi}\xi(a_i).$$

Narrow convergence can also be understood as the combination of weak convergence of measures and the condition that the total mass converges:

$$(2-p_n)|Du_n|(\Omega) \xrightarrow{n\to\infty} \sum_{i=1}^{\kappa} \frac{\lambda([u_*,a_i])^2}{4\pi}\delta_{a_i}(\Omega) = \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega} u_*).$$

2.5.1 Convergence of L^p-bounded sequences when $p \nearrow 2$

In this section 2.5.1, we prove a Fatou type result (see (2.5.4)) and a compactess result (see proposition 2.5.4) for sequence bounded in the sense of (2.5.3).

Proposition 2.5.2. *Let $\Omega \subset \mathbb{R}^2$ be an open and bounded set, $(p_n)_n$ be an increasing sequence converging to 2 and $u_n \to u$ a.e. be a sequence of measurable maps in $W^{1,p_n}(\Omega)$. If*

$$\sup_n \int_\Omega |Du_n|^{p_n} < +\infty, \tag{2.5.3}$$

then

$$\int_\Omega |Du_*|^2 \leq \varliminf_{n\to\infty} \int_\Omega |Du_n|^{p_n}. \tag{2.5.4}$$

Moreover, for every $a \in \Omega$ and almost every $\rho \in (0, \operatorname{dist}(a, \partial\Omega))$,

$$\int_{\partial\mathbb{B}(a;r)} |Du_*|^2 \, \mathrm{d}\mathcal{H}^1 \leq \varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a;r)} |Du_n|^{p_n} \, \mathrm{d}\mathcal{H}^1.$$

The second part of proposition 2.5.2 is based on the next lemma 2.5.3 that disintegrates on circles the lower semi-continuity statement that weak convergence implies. Its proof is well known, based on Fubini-Tonneli theorem and Fatou's lemma and we shall omit it.

Lemma 2.5.3. *Let $\Omega \subset \mathbb{R}^m$ be an open and bounded set, $p \in (1, \infty)$. Let $u_n \to u$ weakly in $W^{1,p}(\Omega)$. Then, for every $a \in \Omega$ and almost every $\rho \in (0, \operatorname{dist}(a, \partial\Omega))$,*

$$\int_{\partial\mathbb{B}(a;\rho)} |Du|^p \, \mathrm{d}\mathcal{H}^1 \leq \varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a;\rho)} |Du_n|^p \, \mathrm{d}\mathcal{H}^1$$

and there exists a subsequence n' depending on a and ρ such that $u_{n'} \to u$ $\mathcal{H}^1$-a.e converges and $u_{n'}$ is bounded in $W^{1,p}(\partial\mathbb{B}(a;\rho))$.

Proof of proposition 2.5.2. We only prove the second assertion. This is lemma 2.5.3 combined with the observation that

$$\begin{aligned}
\int_{\partial\mathbb{B}(a_i;r)} |Du_*|^2 \, \mathrm{d}\mathcal{H}^1 &= \lim_{p\nearrow 2} \int_{\partial\mathbb{B}(a_i;r)} |Du_*|^p \, \mathrm{d}\mathcal{H}^1 \\
&\leq \varliminf_{p\nearrow 2} \varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a_i;r)} |Du_n|^p \, \mathrm{d}\mathcal{H}^1 \\
&\leq \varliminf_{p\nearrow 2} \varliminf_{n\to\infty} \mathcal{H}^1(\partial\mathbb{B}(a_i;r))^{\frac{p(p_n-1)}{p_n}} \times \\
&\qquad \Big(\int_{\partial\mathbb{B}(a_i;r)} |Du_n|^{p_n} \, \mathrm{d}\mathcal{H}^1 \Big)^{\frac{p}{p_n}} \\
&\leq \varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a_i;r)} |Du_n|^{p_n} \, \mathrm{d}\mathcal{H}^1. \square
\end{aligned}$$

In the proof of proposition 2.5.1, we will use the following routine.

Proposition 2.5.4. *Let Ω be an open Lipschitz bounded domain of $\mathbb{R}^d$. Let $(u_n)_n$ be a sequence of maps $u_n \in W^{1,p_n}(\Omega,\mathbb{R}^\nu)$ sharing all the same trace on $\partial\Omega$. Let, for each m, $\mathcal{B}_m$ be a finite collection of disjoint disks contained in Ω whose union of forms a decreasing sequence of sets satisfying, as sets, $\cup\mathcal{B}_m \downarrow \{a_1,\dots,a_k\} \subset \Omega$. Let $p_n \in [1,2)$ be such that $p_n \nearrow 2$.*
If, for each m,

$$\sup_n \int_{\Omega\setminus B_m} |Du_n|^{p_n} < +\infty,$$

then up to some unrelabelled subsequence u_n converges almost everywhere to a map $u \in W^{1,2}_{\mathrm{loc}}(\Omega\setminus\{a_i\}_{i=1,\dots,k},\mathbb{R}^\nu)$. Moreover the subsequence satisfies

$$\overline{\lim_{n\to\infty}} \int_{\Omega\setminus\bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} |u_n|^{p_n} + |Du_n|^{p_n} < +\infty. \tag{2.5.5}$$

We will deduce proposition 2.5.4 from the following lemmata.

Lemma 2.5.5. *For each m, it is possible to change the values of each u_{p_n} on $\cup\mathcal{B}_m$ in order to get a sequence $\bar{u}_{p_n} \in W^{1,p_n}(\Omega,\mathbb{R}^\nu)$ that verifies*

$$\sup_n \int_\Omega |D\bar{u}_n|^{p_n} < \infty.$$

Proof of lemma 2.5.5. It follows from (linear) trace theory and its estimates [16]. □

Lemma 2.5.6. *Let $\Omega \subset \mathbb{R}^2$ be an open set. Let $p_n \in [1,2)$ be an increasing sequence such that $p_n \nearrow 2$. Assume a sequence $v_n \in W^{1,p_n}(\Omega,\mathbb{R}^\nu)$ verifies*

$$\sup_n \int_\Omega |Dv_n|^{p_n} < \infty \text{ and } v_n - v_0 \in W_0^{1,p_0}(\Omega,\mathbb{R}^\nu)$$

for each n. Then, it admits an unrelabelled subsequence v_n such that $v_n \to v$ almost everywhere Moreover, $v \in W^{1,2}(\Omega,\mathbb{R}^\nu)$ and, for each $p \in [1,2)$,

$$\sup_n \int_\Omega |v_n|^p + |Dv_n|^p < +\infty. \tag{2.5.6}$$

Proof of lemma 2.5.6. As $p_n \nearrow 2$, for every $q \in [1,2)$, the tail of the sequence $(v_n)_n$ is bounded in $W^{1,q}(\Omega,\mathbb{R}^\nu)$ by the Poincaré inequality. Hence, for some $q \in [1,2)$ we extract a subsequence of (v_n) still denote v_n such that $v_n \to v$ almost everywhere and in $L^q(\Omega,\mathbb{R}^\nu)$. We also get that

$Dv_n \to Dv$ weakly in $L^p(\Omega, \mathbb{R}^\nu)$. We next consider a sequence $q_m \nearrow 2$ and by a Cantor's diagonal argument such that the diagonal sequence v_n converges in $L^{q_m}(\Omega, \mathbb{R}^\nu)$ and $Dv_n \to Dv$ weakly in $L^{q_m}(\Omega, \mathbb{R}^\nu)$ for each m and thus for each $p \in [1, 2)$ we have

$$\sup_n \int_\Omega |Dv_n|^p < +\infty. \tag{2.5.7}$$

The fact that $Du \in L^2(\Omega, \mathbb{R}^{2\times\nu})$ then follows from a variant of [40, Theorem 6.1.7]. The estimate (2.5.6) is implied by the weak convergence. □

Proof of proposition 2.5.4. Lemmata 2.5.5 and 2.5.6 with a Cantor's diagonal argument yield the conclusion. □

2.5.2 Proof of proposition 2.5.1

We will make use of the following lemma from [29, Lemma 6.2].

Lemma 2.5.7. *Let* $\Omega \subset \mathbb{R}^2$*,* $a \in \mathbb{R}^2$ *and* $0 < \sigma < \tau$*. If* $u \in W^{1,2}(\mathbb{B}(a;\tau) \setminus \mathbb{B}(a;\sigma), \mathcal{N})$*, then,*

$$\int_{(\mathbb{B}(a;\tau)\setminus\mathbb{B}(a;\sigma))\cap\Omega} \frac{|Du|^2}{2} \geq \frac{\lambda^2(\operatorname{Tr}_{\partial\mathbb{S}^1} u(a+\tau\cdot))}{4\pi\nu_\tau^\sigma(a)} \log\frac{\tau}{\sigma} \times \left[1 - \frac{\sqrt{2\pi}}{\lambda(\operatorname{Tr}_{\partial\mathbb{S}^1} u(a+\tau\cdot))}\Big(\frac{1}{\log\frac{\tau}{\sigma}}\int_{\mathbb{B}(a;\tau)\setminus(\mathbb{B}(a;\sigma)\cup\Omega)} |Du|^2\Big)^{\frac{1}{2}}\right]^2$$

where

$$\nu_\tau^\sigma(a) = \frac{1}{2\pi\log\frac{\tau}{\sigma}} \int_{(\mathbb{B}(a;\tau)\setminus\mathbb{B}(a;\sigma))\cap\Omega} \frac{\mathrm{d}x}{|x-a|^2} \leq 1. \tag{2.5.8}$$

Proof of proposition 2.5.1. In the first part of the proof we assume that

$$u_n \in W^{1,p_n}(\Omega, \mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\bar{\Omega} \setminus \{a_i^n\}_{i=1,\dots,k^n}, \mathcal{N}),$$

where $k^n \in \mathbb{N}$ and $\{a_i^n\}_{i=1,\dots,k^n} \subset \Omega$ are such that

$$\delta = \inf_n \operatorname{dist}(\{a_i\}_{i=1,\dots,k^n}, \partial\Omega) > 0.$$

We will treat the general case by density (see section 2.5.2.5).

By our assumption (2.5.1),

$$\varlimsup_{n\to\infty} (2-p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \leq \mathcal{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega} u_0). \tag{2.5.9}$$

2.5.2.1 Convergence of the disks

Let $(\eta_m)_m$ be a decreasing sequence such that $\eta_m \in (0,\delta)$ for each m and $\eta_m \searrow 0$. For each m, n we apply proposition 2.4.3 and get the existence of a finite collection of disjoint disks $\mathcal{B}_{m,n}$ contained in Ω such that the sum of the diameter is η_m (more details are explained in corollary 2.4.4) and
(2.5.10)

$$(2-p_n)\int_{\bigcup_{\mathbb{B}\in\mathcal{B}_{m,n}}} \frac{|Du_n|^{p_n}}{p_n} \geq \Big(\sum_{\mathbb{B}\in\mathcal{B}_{m,n}} \mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\mathbb{B}} u_n)\Big)^{p_n-1} \eta_m^{2-p_n}.$$

We thus have by proposition 2.2.10, (2.5.9) and (2.5.10),

$$\begin{aligned}\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0) &= \lim_{n\to\infty} \mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)^{p_n-1}\\ &\leq \Big(\overline{\lim_{n\to\infty}} \sum_{\mathbb{B}\in\mathcal{B}_{m,n}} \mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\mathbb{B}} u_n)\Big)^{p_n-1}\\ &\leq \overline{\lim_{n\to\infty}}\,(2-p_n)\int_\Omega |Du_n|^{p_n} \leq \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0).\end{aligned}$$

Thus,

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0) = \lim_{n\to\infty}(2-p_n)\int_\Omega |Du_n|^{p_n}. \tag{2.5.11}$$

Letting $\mathcal{B}^{\mathrm{Top}}_{m,n}$ be the subcollection of disks $\mathbb{B}$ of $\mathcal{B}_{m,n}$ with the property that $\mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\mathbb{B}} u_n) > 0$; in view of (2.2.13) they satisfy the stronger bound

$$\mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\mathbb{B}} u_n) \geq \frac{\mathrm{Sys}(\mathcal{N})^{\frac{p_n}{p_n-1}}}{(2\pi)^{p_n'-1}p_n'} > 0, \tag{2.5.12}$$

we have that the number of elements of this collection satisfies by (2.5.10)

$$\overline{\lim_{n\to\infty}}\,\#\mathcal{B}^{\mathrm{Top}}_{m,n} \leq \frac{4\pi\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)}{\mathrm{Sys}(\mathcal{N})^2}. \tag{2.5.13}$$

Hence up to some subsequence in n the collection $\mathcal{B}^{\mathrm{Top}}_{m,n}$ converges to a limit collection $\mathcal{B}^{\mathrm{Top}}_m$ for each m in the sense that $\#\mathcal{B}^{\mathrm{Top}}_{m,n} \to \#\mathcal{B}_m$ for each m and the vector of the center of the disks and the radii converge to the associated vector of the limit collection. By a Cantor diagonal argument we may ensure that the subsequence does not depend on m. Then, (4.5.45) also holds for $\#\mathcal{B}_m$. Thus repeating the argument we get up to some subsequence a limit collection $\mathcal{B}$ such that $\mathcal{B}_m \to \mathcal{B}$. As $\eta_m \searrow 0$, this collection consists of $\kappa \in \mathbb{N}$ with $\kappa \leq \#\mathcal{B}$ points $\{a_i\}_{i=1,\dots,\kappa} \subset \bar{\Omega}$ such that

$$\mathrm{dist}(\{a_i\}_{i=1,\dots,\kappa}, \partial\Omega) \geq \delta.$$

2.5.2.2 Uniform bound away from singularities and convergence to u_*

For each m, n, we observe that

$$\begin{aligned}\int_{\Omega\setminus\bigcup\mathcal{B}_{m,n}} \frac{|Du_n|^{p_n}}{p_n} &= \int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)}{2-p_n}\\ &+ \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0) - \mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)^{p_n-1}}{2-p_n}\\ &+ \mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)^{p_n-1}\frac{1-\eta_m^{2-p_n}}{2-p_n}\\ &+ \left[\frac{\mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)^{p_n-1}\eta_m^{2-p_n}}{2-p_n} - \int_{\bigcup\mathcal{B}_{m,n}} \frac{|Du_n|^{p_n}}{p_n}\right].\end{aligned}$$

Hence, by our assumption (2.5.1), the local Lipschitz property of the singular energy (lemma 2.2.10) and the lower bound (2.5.10),

$$\varlimsup_{n\to\infty} \int_{\Omega\setminus\bigcup\mathcal{B}_{m,n}} \frac{|Du_n|^{p_n}}{p_n} \le \Lambda + \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)\log\frac{1}{\eta_m}, \tag{2.5.14}$$

where we write for future use

$$\begin{aligned}\Lambda \doteq \varlimsup_{n\to\infty}\left[\int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)}{2-p_n}\right]&\\ + \varlimsup_{p\nearrow 2} \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0) - \mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)^{p'-1}}{2-p}&.\end{aligned} \tag{2.5.15}$$

From (2.5.14), by proposition 2.5.4 we get a limit map $u_* \in W^{1,2}(\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho), \mathcal{N})$ for every $\rho \in (0,\bar\rho)$ such that up to some subsequence independent of m we have $u_n \to u_*$ as described by the proposition. This proves (ii).

In the sequel, we will use

$$\bar\rho \doteq \rho(\{a_i\}_{i=1,\dots,\kappa})$$

for the non-intersection radius defined in (2.2.18). By (2.5.15) for m large enough,

$$\begin{aligned}\int_{\Omega\setminus\bigcup_{i=1}^\kappa \mathbb{B}(a_i;2\eta_m)} \frac{|Du_*|^2}{2} &\le \varlimsup_{n\to\infty}\int_{\Omega\setminus\bigcup\mathcal{B}_{m,n}} \frac{|Du_n|^{p_n}}{p_n}\\ &\le \Lambda + \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)\log\frac{1}{\eta_m}.\end{aligned}$$

Hence, using definition 2.2.8 of the singular energy and lemma 4.5.11,

$$\begin{aligned}\Gamma \log \frac{\bar\rho}{2\eta_m}\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_*) &\le \Gamma \log \frac{\bar\rho}{2\eta_m}\sum_{i=1}^{\kappa}\frac{\lambda(\mathrm{Tr}_{\partial\mathbb{S}^1} u_*(a_i+\bar\rho\cdot))^2}{4\pi\nu_{\bar\rho}^{2\eta_m}(a_i)} \\ &\le \Lambda + \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)\log\frac{1}{\eta_m},\end{aligned} \tag{2.5.16}$$

where we have set

$$\Gamma \doteq \left[1 - \frac{\sqrt{2\pi}}{\mathrm{Sys}(\mathcal{N})}\Big(\log\frac{\bar\rho}{2\eta_m}\int_{\{x\in\Omega:\mathrm{dist}(\partial\Omega,x)<\delta\}}|Du_*|^2\Big)^{\frac12}\right]^2$$

and $\nu_{\bar\rho}^{2\eta_m}(a_i)$ is given by (2.5.8). From (2.5.16), we get in the limit $m\to\infty$,

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_*) \le \sum_{i=1}^{\kappa}\frac{\lambda(\mathrm{Tr}_{\partial\mathbb{S}^1} u_*(a_i+\bar\rho\cdot))^2}{4\pi} \le \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0). \tag{2.5.17}$$

But $\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0) = \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_*)$ as $\mathrm{Tr}_{\partial\Omega} u_* = \mathrm{Tr}_{\partial\Omega} u_0$ by assumption. Hence we deduce that $\nu_{\bar\rho}^{2\eta_m}(a_i) = 1$ for each $i = 1,\dots,\kappa$ and that equality holds in (2.5.17) which implies that $(\mathrm{Tr}_{\partial\mathbb{S}^1} u_*(a_i+\bar\rho\cdot))_{i=1,\dots,\kappa}$ is a minimimal topological resolution of $\mathrm{Tr}_{\partial\Omega} u_0$ *i.e.*

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_*) = \sum_{i=1}^{\kappa}\frac{\lambda(\mathrm{Tr}_{\partial\mathbb{S}^1} u_*(a_i+\bar\rho\cdot))^2}{4\pi} \tag{2.5.18}$$

and

$$\mathrm{dist}(\{a_i\}_{i=1\dots,\kappa},\partial\Omega) > \delta. \tag{2.5.19}$$

2.5.2.3 Narrow convergence (iii)

By (2.5.14) and the convergence results (see proposition 2.5.4(ii)), we have $u_n \to u_*$ almost everywhere and we may futher assume that

$$(2-p_n)|Du_n|^{p_n} \to \sum_{i=1}^{\kappa}\alpha_i\delta_{a_i}$$

weakly as measures where $\alpha_i \ge 0$. Moreover,

$$\sum_{i=1}^{\kappa}\alpha_i = \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u)$$

by (2.5.11). By section 2.5.2.1, for large m and n, we may assume that the number of disks contained in $\mathbb{B}(a_i;\bar\rho/2)$ is constant for each $i = 1,\dots,\kappa$.

Then, by the construction of the disks given by proposition 2.4.3(2.4.3), for each $i = 1, \dots, \kappa$, for large enough m,

$$\alpha_i = \varlimsup_{n\to\infty} (2-p_n) \int_{\mathbb{B}(a_i,\bar\rho)} \frac{|Du_n|^{p_n}}{p_n} \ge \varlimsup_{n\to\infty} \mathcal{E}^{1,p_n'}_{\mathrm{sg}} (\mathrm{Tr}_{\partial\mathbb{B}(a_i;\bar\rho)} u_n)^{p_n-1} \Big(\frac{\bar\rho}{2}\Big)^{2-p_n}$$

We consider a subsequence such that the superior limit is an actual limit for each $i = 1, \dots, \kappa$. By lemma 2.5.3, there exists $r \in (\bar\rho/2, \bar\rho)$ such that for a subsequence $u_{n_k^r}$ depending on r one has $u_{n_k^r} \rightharpoonup u_*$ weakly in $W^{1,p}(\partial\mathbb{B}(a_i;r))$. Using the Sobolev embedding and the Arzelà–Ascoli compactness criterion $u_n \to u_*$ uniformly on $\partial\mathbb{B}(a_i;r)$. Hence for k large enough $\mathrm{Tr}_{\partial\mathbb{B}(a_i;r)} u_{n_k^r}$ is homotopic to $\mathrm{Tr}_{\partial\mathbb{B}(a_i;r)} u_*$. We deduce that for k and m large enough $\mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}(a_i;\bar\rho)}\, u_n) = \mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}(a_i;r)}\, u_n) = \mathcal{E}^{1,p_n'}_{\mathrm{sg}}([u_*, a_i])$ and, using lemma 2.2.10,

$$\alpha_i \ge \varlimsup_{n\to\infty} \mathcal{E}^{1,p_n'}_{\mathrm{sg}} (\mathrm{Tr}_{\partial\mathbb{B}(a_i;\bar\rho)} u_*)^{p_n-1} \Big(\frac{\bar\rho}{2}\Big)^{2-p_n} = \mathcal{E}^{1,2}_{\mathrm{sg}}([u_*, a_i])$$

which, combined with the fact that $(\mathrm{Tr}_{\partial\mathbb{S}^1} u_*(a_i + \bar\rho\cdot))_{i=1,\dots,\kappa}$, is a minimimal topological resolution of $\mathrm{Tr}_{\partial\Omega} u_0$ results into that

$$\alpha_i = \mathcal{E}^{1,2}_{\mathrm{sg}}([u_*, a_i]) = \frac{\lambda([u_*, a_i])^2}{4\pi}$$

by lemma 2.2.14. This proves the weak convergence of measures. The narrow convergence (iii) is reached by (2.5.11).

2.5.2.4 Lower semicontinuity statements

We now observe that, by corollary 2.5.2, for each $i = 1, \dots, \kappa$ and almost every $r \in (0, \bar\rho)$,

$$\int_{\partial\mathbb{B}(a_i;r)} |Du_*|^2 \le \varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a_i;r)} |Du_n|^{p_n}. \tag{2.5.20}$$

In order to prove (v). For all $\rho \in (0, \bar\rho)$, we have by (2.5.20)

$$\begin{aligned}
&\int_0^\rho \int_{\partial\mathbb{B}(a_i;r)} \frac{|Du_*|^2}{2}\, \mathrm{d}\mathcal{H}^1 - \frac{\lambda(\mathrm{Tr}_{\partial\mathbb{B}(a_i;r)} u_*)^2}{4\pi r}\, \mathrm{d}r \\
&\quad \le \int_0^\rho \varliminf_{n\to\infty} \left[\int_{\partial\mathbb{B}(a_i;r)} \frac{|Du_n|^{p_n}}{p_n}\, \mathrm{d}\mathcal{H}^1 - \frac{\lambda(\mathrm{Tr}_{\partial\mathbb{B}(a_i;r)} u_*)^{p_n} (2\pi r)^{2-p_n}}{2p_n\pi r}\right] \mathrm{d}r \\
&\quad \le \varliminf_{n\to\infty} \int_0^\rho \left[\int_{\partial\mathbb{B}(a_i;r)} \frac{|Du_n|^{p_n}}{p_n}\, \mathrm{d}\mathcal{H}^1 - \frac{\lambda(\mathrm{Tr}_{\partial\mathbb{B}(a_i;r)} u_*)^{p_n} (2\pi r)^{2-p_n}}{2p_n\pi r}\right] \mathrm{d}r \\
&\quad = \varliminf_{n\to\infty} \int_{\mathbb{B}(a_i;\rho)} \frac{|Du_n|^{p_n}}{p_n} - \frac{\lambda(\mathrm{Tr}_{\partial\mathbb{B}(a_i;r)} u_*)^{p_n}}{2p_n\pi} \frac{(2\pi\rho)^{2-p_n}}{2-p_n}.
\end{aligned}$$

We prove (2.5.2). By the integral representation of the renormalized energy, proposition 2.2.16, we have

$$
\begin{aligned}
\mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) = &\int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \frac{|Du_*|^2}{2} - \sum_{i=1}^{\kappa} \frac{\lambda(\mathrm{Tr}_{\partial \mathbb{B}(a_i;\rho)} u_*)^2}{4\pi} \ln \frac{1}{\rho} \\
&\qquad + \sum_{i=1}^{\kappa} \int_0^{\rho} \int_{\partial \mathbb{B}(a_i;\rho)} \frac{|Du_*|^2}{2} \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda(\mathrm{Tr}_{\partial \mathbb{B}(a_i;\rho)} u_*)^2}{4\pi r} \,\mathrm{d}r \\
\le\ & \varliminf_{n\to\infty} \int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \frac{|Du_n|^{p_n}}{p_n} \\
&\qquad + \varliminf_{n\to\infty} \sum_{i=1}^{\kappa} \frac{\lambda(\mathrm{Tr}_{\partial \mathbb{B}(a_i;\rho)} u_*)^{p_n}}{2p_n\pi} \frac{(2\pi\rho)^{2-p_n} - (2\pi)^{2-p_n}}{2-p_n} \\
&\qquad + \sum_{i=1}^{\kappa} \varliminf_{n\to\infty} \int_{\mathbb{B}(a_i;\rho)} \frac{|Du_n|^{p_n}}{p_n} - \frac{\lambda(\mathrm{Tr}_{\partial \mathbb{B}(a_i;\rho)} u_*)^{p_n}}{2p_n\pi} \frac{(2\pi\rho)^{2-p_n}}{2-p_n} \\
\le\ & \varliminf_{n\to\infty} \int_{\Omega} \frac{|Du_n|^{p_n}}{p_n} - \sum_{i=1}^{\kappa} \frac{\lambda(\mathrm{Tr}_{\partial \mathbb{B}(a_i;\rho)} u_*)^{p_n}}{2p_n\pi} \frac{(2\pi)^{2-p_n}}{2-p_n}.
\end{aligned} \tag{2.5.21}
$$

Subtracting and adding $\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_n)/(2-p_n)$ in (2.5.21) and using the equality (2.5.18), we obtain the first inequality in 2.5.2 (see also (2.3.6)), whereas the second inequality in 2.5.2 follows from proposition 2.2.23.

2.5.2.5 Density argument

If $u_n \in W^{1,p_n}(\Omega, \mathcal{N})$, and $\mathrm{Tr}_{\partial\Omega} u_n = \mathrm{Tr}_{\partial\Omega} u_0 \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, we argue by density. By (2.4.1), we may assume, considering a larger domain, that

$$u_n \in W^{1,p_n}(\Omega', \mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\{x \in \Omega' : \mathrm{dist}(x, \partial\Omega') < \delta\}, \mathcal{N})$$

for some $\delta > 0$ and $u_n = u_0$ on $\{x \in \Omega' : \mathrm{dist}(x, \partial\Omega') < \delta\}$. Then, by proposition 2.4.2, we obtain for each n a finite set $\{a_i^n\}_{i=1,\dots,k^n} \subset \Omega'$ satisfying $\mathrm{dist}(\{a_i^n\}_{i=1,\dots,k^n}, \partial\Omega') > \delta$ such that

$$\bar{u}_n \in W^{1,p_n}(\Omega', \mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\Omega' \setminus \{a_i^n\}_{i=1,\dots,k^n}, \mathcal{N}),$$

$(\bar{u}_n - u_n) \to 0$ almost everywhere in Ω', $\|\bar{u}_n - u_n\|_{W^{1,p_n}(\Omega',\mathcal{N})} \to 0$ and in particular

$$\int_{\Omega'} |D\bar{u}_n|^{p_n} - \int_{\Omega'} |Du_n|^{p_n} \xrightarrow{n\to\infty} 0.$$

Under these conditions, the assumption (2.5.1) on $(u_n)_n$ implies

$$\sup_n \left[\int_{\Omega'} \frac{|D\bar{u}_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega'} \bar{u}_0)}{2-p_n} \right] < +\infty.$$

By the proof above, we obtain (i)–(v) by restriction from Ω' to Ω and by observing that the

$$\lim_{n\to\infty}\int_{\Omega'\setminus\Omega}|D\bar{u}_n|^{p_n}=\int_{\Omega'\setminus\Omega}|D\bar{u}_*|^2$$

is implied by $\|\bar{u}_n-u_n\|_{W^{1,p_n}(\Omega',\mathcal{N})}\to 0$. The condition

$$\operatorname{dist}(\{a_i\}_{i=1,\ldots,\kappa},\partial\Omega')>\delta$$

(see (2.5.19)) implies $\{a_i\}_{i=1,\ldots,\kappa}\subset\Omega$. □

2.5.3 Mixed Marcinkiewicz estimates of bounded sequences

We also obtain mixed Marcinkiewicz estimates for sequences of p-harmonic mappings with bounded renormalized p–energy.

Proposition 2.5.8. *Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain, $\mathcal{N}$ a Riemannian manifold with* $\operatorname{Sys}(\mathcal{N})>0$ *and let $p_n\in(1,2)$ be a sequence $p_n\nearrow 2$. Let $(u_n)_n$ be a sequence of maps such that $u_n\in W^{1,p_n}(\Omega,\mathcal{N})$,* $\operatorname{Tr}_{\partial\Omega}u_n=\operatorname{Tr}_{\partial\Omega}u_0\in W^{1/2,2}(\partial\Omega,\mathcal{N})$. *If $u_0\in W^{1,2}(\Omega_\delta\setminus\Omega,\mathcal{N})$ for some $\delta>0$, $u_n=u_0$ on $\Omega_\delta\setminus\Omega$ and*

$$\Lambda\doteq\varlimsup_{n\to\infty}\left[\int_\Omega\frac{|Du_n|^{p_n}}{p_n}-\frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\operatorname{Tr}_{\partial\Omega}u_0)}{2-p_n}\right]<+\infty, \tag{2.5.22}$$

then, for each n, there exists a measurable $U_n:\Omega\to\{0\}\cup[\operatorname{Sys}(\mathcal{N})/(2\pi\delta),\infty)$ such that

$$\varlimsup_{n\to\infty}\int_\Omega\frac{(|Du_n|-U_n)_+^{p_n}}{p_n}\le\Lambda+\mathcal{E}^{1,2}_{\mathrm{sg}}(\operatorname{Tr}_{\partial\Omega}u_0)\times \\ \left(\left[\mathcal{E}^{1,p}_{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega}u_0)\right]_{\mathrm{Lip}([3/2,2])}+\log\frac{1}{\delta}\right) \tag{2.5.23}$$

$$\varlimsup_{n\to\infty}\sup_{t>0}t^{p_n}\,\mathcal{H}^2(\{U_n>t\})\le\mathcal{E}^{1,2}_{\mathrm{sg}}(\operatorname{Tr}_{\partial\Omega}u_0), \tag{2.5.24}$$

$$\varlimsup_{n\to\infty}\sup_{t>0}t^{p_n-1}\operatorname{Per}(\{U_n>t\})\le\frac{4\pi}{\operatorname{Sys}(\mathcal{N})}\mathcal{E}^{1,2}_{\mathrm{sg}}(\operatorname{Tr}_{\partial\Omega}u_0). \tag{2.5.25}$$

In (2.5.23), $[\mathcal{E}^{1,p}_{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega}u_0)]_{\mathrm{Lip}([3/2,2])}$ refers to the Lipschitz constant of the map $p\mapsto\mathcal{E}^{1,p}_{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega}u_0)$. Lemma 2.2.10 guarantees that this quantity is finite and (2.2.17) gives an estimate on it. Together (2.5.23) and (2.5.24) will imply weak-L^p boundedness of the sequence Du_n, see corollary 2.5.9 below.

Under the conclusion of proposition 2.5.8, we have the following lower semi-continuity statements (lemma 2.5.10, lemma 2.5.11 and lemma 2.5.12). As a corollary (see corollary 2.5.13), we obtain that the map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ resulting of proposition 2.5.1 has its gradient that lies in weak-L^2. We refer to [29] for the original proof of this fact for a general manifold and [36] for the case of the circle.

We first record the following corollary that implies the result (2.1.7) mentioned in the introduction.

Corollary 2.5.9. *Any sequence $(u_n)_n$ that satisfies the assumptions of proposition 2.5.8 verifies*

$$\begin{aligned}&\varlimsup_{n\to\infty} \sup_{t>0} t^{p_n} \mathcal{H}^2(\Omega \cap \{|Du_n| > 2t\})\\ &\le \Lambda + \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)\Big(\big[\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)^{p-1}\big]_{\mathrm{Lip}([3/2,2])} + \log\frac{1}{\delta}\Big) + 4\pi\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0).\end{aligned}$$

Proof of corollary 2.5.9. Noting that $|Du_n| \le (|Du_n| - U_n)_+ + U_n$ where U_n is the map given by proposition 2.5.8, we have

$$\begin{aligned}&t^{p_n} \mathcal{H}^2(\Omega \cap \{|Du_n| > 2t\})\\ &\qquad \le t^{p_n} \mathcal{H}^2(\Omega \cap \{(|Du_n| - U_n)_+ > t\}) + t^{p_n} \mathcal{H}^2(\Omega \cap \{U_n > t\})\\ &\qquad \le \int_\Omega (|Du_n| - U_n)_+^{p_n} + t^{p_n} \mathcal{H}^2(\Omega \cap \{U_n > t\}). \quad \square\end{aligned}$$

Lemma 2.5.10. *Let us fix $\Omega \subset \mathbb{R}^m$ and $\{a_i\}_{i=1,\dots,\kappa} \subset \Omega$. Consider two sequences $(U_n)_n$ and $(v_n)_n$ such that, for each n, $U_n \in L^1(\Omega, \mathbb{R})$ and $v_n \in L^{p_n}(\Omega, \mathbb{R}^{m\times\nu})$ such that $U_n \to U$ a.e, $v_n \rightharpoonup v$ weakly in $L^p(\Omega \setminus \bigcup_{i=1}^\kappa \mathbb{B}(a_i;\rho), \mathbb{R}^{m\times\nu})$ for every $p \in (1,2)$ and $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,\kappa}))$ and*

$$\sup_n \int_\Omega \frac{(|v_n| - U_n)_+^{p_n}}{p_n} < +\infty.$$

Then,

$$\int_\Omega \frac{(|v| - U)_+^2}{2} \le \varliminf_{n\to\infty} \int_\Omega \frac{(|v_n| - U_n)_+^{p_n}}{p_n}.$$

Proof of lemma 2.5.10. Let us define for $s > 1, p \in (1,2), \xi \in \mathbb{R}$ the $C^1(\mathbb{R},\mathbb{R})$ non-decreasing convex function

$$\Phi_{s,p}(\xi) = \begin{cases} 0 & \text{if } \xi \le 0\\ \frac{\xi^p}{p} & \text{if } \xi \in (0,s)\\ \frac{s^p}{p} + s^{p-1}(\xi - s) & \text{if } \xi \ge s\end{cases}$$

for which one has

$$\Phi'_{s,p}(\xi) = \begin{cases} 0 & \text{if } \xi \leq 0 \\ \xi^{p-1} & \text{if } \xi \in (0,s) \\ s^{p-1} & \text{if } \xi \geq s. \end{cases}$$

Since v is measurable, there exists $\zeta \in L^\infty$ such that $|\xi| = 1$ and $\langle v, \zeta \rangle = |v|$. By convexity,

$$\begin{aligned} \Phi_{s,p}(|v| - U_n) &= \Phi_{s,p}(\langle v, \zeta \rangle - U_n) \\ &\leq \Phi_{s,p}(\langle v_n, \zeta \rangle - U_n) - \Phi'_{s,p}(\langle v, \zeta \rangle - U_n)[\langle v_n - v, \zeta \rangle]. \end{aligned}$$

Since $v_n \rightharpoonup v_n$ weakly in L^q for some $p < q < 2$, we have and since $\Phi'_{s,p}(\langle v, \zeta \rangle - U_n)$ is bounded and converges almost everywhere to

$$(\Phi'_{s,p}(\langle v, \zeta \rangle - U)$$

we have for all $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,\kappa}))$

$$\lim_{n\to\infty} \int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \Phi'_{s,p}(\langle v, \zeta \rangle - U_n)[\langle v_n - v, \zeta \rangle] = 0.$$

and thus

$$\begin{aligned} \int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \Phi_{s,p}(|v| - U) &\leq \varliminf_{n\to\infty} \int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \Phi_{s,p}(|v| - U_n) \\ &\leq \varliminf_{n\to\infty} \int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \Phi_{s,p}(\langle v_n, \xi \rangle - U_n) \\ &\leq \varliminf_{n\to\infty} \int_{\Omega} \frac{(|v_n| - U_n)_+^p}{p} \\ &\leq \varliminf_{n\to\infty} \int_{\Omega} \frac{(|v_n| - U_n)_+^{p_n}}{p_n} \end{aligned}$$

where we used Hölder's inequality to obtain the last inequality. Next, letting $p \nearrow 2$, $s \nearrow \infty$ and $\rho \searrow 0$, we conclude by Fatou's lemma. $\square$

Lemma 2.5.11. *Let $(p_n)_n$ be a sequence such that, for each n, $p_n \in (1,2)$ and converging to $p \in (1,2]$, let $U_n : \Omega \to [0,\infty)$ be a sequence of measurable maps such that $U_n \geq 1$. If*

$$\sup_n \sup_{t>1} t^{p_n - 1} \mathrm{Per}(\{U_n > t\}) < +\infty, \tag{2.5.26}$$

then a subsequence of U_n converges almost everywhere on Ω. Moreover,

$$\sup_{t>1} t^{p-1} \mathrm{Per}(\{U > t\}) \leq \varliminf_{n\to\infty} \sup_{t>1} t^{p_n - 1} \mathrm{Per}(\{U_n > t\}).$$

Proof of lemma 2.5.11. We call Λ the supremum in (2.5.26) and we define for $T \in (1,\infty)$, $U_n^T \doteq T^{-1} \vee (T \wedge U)$. By our assumption and the coarea formula [1, Theorem 10.3.3], for $T > 1$,

$$\begin{aligned}\varlimsup_{n\to\infty} \int_\Omega |D(U_n^T)| &= \varlimsup_{n\to\infty} \int_{1/T}^T \operatorname{Per}(\{U_n > t\})\,\mathrm{d}t \\ &\le \Lambda \lim_{n\to\infty} \frac{T^{2-p_n} - T^{p_n-2}}{2-p_n} \\ &= \Lambda \begin{cases} 2\log T & \text{if } p = 2 \\ \dfrac{T^{2-p} - T^{p-2}}{2-p} & \text{if } p \in (1,2).\end{cases}\end{aligned}$$

For fixed $T > 0$, as $U_n^T \le T$ we obtain by the compact embedding for mappings of bounded variation (BV) [1, Theorem 10.1.4] and the partial converse of Lebesgue's dominated convergence theorem [40, Proposition 4.2.10] that, up to some subsequence in n that depends on T, $U_n \wedge T$ converges in L^1 and almost everywhere. Considering a sequence $T_n \to \infty$, a Cantor diagonal argument yields, after the extraction of a subsequence, that for all $T > 0$ U_n^T converges almost everywhere to U^T where $U : \Omega \to \mathbb{R}$ is a measurable map and $U^T \doteq T^{-1} \vee (T \wedge U)$. Moreover we have

$$\int_\Omega |U_n^T - U^T| = \int_{1/T}^T \mathcal{H}^2(\{U_n > t\}\Delta\{U > t\})\,\mathrm{d}t \tag{2.5.27}$$

where Δ refers to the symmetric difference of two sets. Up to a subsequence, we can assume that for almost every $t \in (0,\infty)$,

$$\lim_{n\to\infty} \mathcal{H}^2(\{U_n > t\}\Delta\{U > t\}) = 0.$$

By BV-theory [40, Theorem 7.3.2], for almost every $t \in (0,\infty)$,

$$\begin{aligned}\operatorname{Per}(\{U > t\}) &= \int_\Omega |D\chi_{\{U>t\}}| \\ &\le \varliminf_{n\to\infty} \int_\Omega |D\chi_{\{U_n>t\}}| = \varliminf_{n\to\infty} \operatorname{Per}(\{U_n > t\}),\end{aligned} \tag{2.5.28}$$

and, for all $t > 0$, $\operatorname{Per}(\{U > t\}) \le \liminf_{s\searrow t} \operatorname{Per}(\{U > s\})$, which concludes the proof. □

Lemma 2.5.12. *Let $(X, |\cdot|)$ be a measure space. Assume $U_n \to U$ in measure, $p_n \in [1,\infty)$ and $p_n \to p \in [1,\infty)$. Then,*

$$\sup_{t>0} t^p\, \mathcal{H}^2(\{|U| > t\}) \le \varliminf_{n\to\infty} \sup_{t>0} t^{p_n}\, \mathcal{H}^2(\{|U_n| > t\}).$$

We omit the proof of lemma 2.5.12.

Proof of proposition 2.5.8. As in the proof of proposition 2.5.4, we assume without loss of generality that

$$u_n \in W^{1,p_n}(\Omega, \mathcal{N}) \cap W^{1,2}_{\mathrm{loc}}(\bar{\Omega} \setminus \{a_i^n\}_{i=1,\dots,k^n}, \mathcal{N})$$

where $\{a_i^n\}_{i=1,\dots,k^n} \subset \Omega$ are such that

$$\delta = \inf_n \operatorname{dist}(\{a_i\}_{i=1,\dots,k^n}, \partial\Omega) > 0.$$

The general case follows by density as in section 2.5.2.5 in the proof of proposition 2.5.4.

We first note that

$$\varlimsup_{n\to\infty} (2-p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \le \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0).$$

For each n let $U_n : \Omega \to \{0\} \cup [\mathrm{Sys}(\mathcal{N})/(2\pi\delta), \infty]$ be given by proposition 2.4.6. We then have

$$\begin{aligned}\varlimsup_{n\to\infty} \frac{p_n-1}{p_n} &\int_\Omega (|Du_n| - U_n)_+^{p_n} \\ &\le \varlimsup_{n\to\infty} \frac{(3-p_n)p_n}{2} \int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,p_n'}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)^{p_n-1}(2\pi\delta)^{2-p_n}}{(p_n-1)^{p_n-1} p_n^{2-p_n}}\end{aligned}$$

and thus

$$\begin{aligned}\varlimsup_{n\to\infty} \int_\Omega \frac{(|Du_n| - U_n)_+^{p_n}}{p_n} \le \varlimsup_{n\to\infty} &\left[\int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)}{2-p_n}\right] \\ &+ \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0)\Big(\big[\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)^{p-1}\big]_{\mathrm{Lip}([3/2,2])} + \log\frac{1}{\delta}\Big)\end{aligned}$$

Lemma 2.2.10 and the fact that $\mathrm{Sys}(\mathcal{N}) > 0$ imply that

$$\big[\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u_0)^{p-1}\big]_{\mathrm{Lip}([3/2,2])}$$

is finite. In addition, by (2.4.9), (2.4.10) and lemmata 2.5.11 and 2.5.12

$$\varlimsup_{n\to\infty} \sup_{t>0} t^{p_n} \{U_n > t\} \le \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0),$$

$$\varlimsup_{n\to\infty} \sup_{t>0} t^{p_n-1} \mathrm{Per}(\{U_n > t\}) \le \frac{4\pi}{\mathrm{Sys}(\mathcal{N})} \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_0). \square$$

As a corollary (see corollary 2.5.13 below) of the result of section 2.5.3, we obtain that the map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ resulting of proposition 2.5.1 has its gradient that lies in weak-L^2.

Corollary 2.5.13. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\mathcal{N}$ a Riemannian manifold with* $\operatorname{Sys}(\mathcal{N}) > 0$*. If $u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$, then $|Du| \in L^{2,\infty}(\Omega, \mathbb{R})$. Moreover, there exists a positive $U \in L^{2,\infty}(\Omega, \mathbb{R})$,*

$$\sup_{t>0} t^2\, \mathcal{H}^2(\{U > t\}) \le \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u)$$

and

$$\sup_{t>0} t \operatorname{Per}(\{U > t\}) \le \frac{4\pi}{\operatorname{Sys}(\mathcal{N})} \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u)$$

$$\begin{aligned}\int_\Omega \frac{(|Du| - U)_+^2}{2} \le \mathcal{E}^{1,2}_{\mathrm{ren}}(u) + \mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u) \times &\\ \Big(\big[\mathcal{E}^{1,p'}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega}\, u)\big]_{\mathrm{Lip}([3/2,2])} + |\log \operatorname{dist}(\{a_i\}_{i=1,\dots,k}, \partial\Omega)| \Big)&\end{aligned}$$

where $\{a_i\}_{i=1,\dots,k} \subset \Omega$ are the associated singularities of the renormalizable map u.

Proof of corollary 2.5.13. The sequence defined by $u_n = u$ for all n satisfies the assumption (2.5.22) of proposition 2.5.8 by proposition 2.2.17. Combining the estimates of proposition 2.5.8 with the ones of lemma 2.5.10, lemma 2.5.11 and lemma 2.5.12 we obtain the conclusion thanks to proposition 2.2.17. □

2.6 Convergence of minimizers

Proposition 2.6.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded Lipschitz domain and $\mathcal{N}$ a compact Riemannian manifold.*

Let $(p_n)_n$ be a sequence satisfying, for each n, $p_n \in (1, 2)$ and let $(u_n)_n$ be a sequence of minimizing p_n-harmonic maps such that, for all n, $u_n \in W^{1,p_n}(\Omega, \mathcal{N})$ and $\mathrm{Tr}_{\partial\Omega} u_n = \mathrm{Tr}_{\partial\Omega} u_0 \in W^{1/2,2}(\partial\Omega, \mathcal{N})$.

If $p_n \nearrow 2$, then up to some subsequence $(u_n)_n$ converges almost everywhere to a renormalizable map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ of trace $\mathrm{Tr}_{\partial\Omega} u_* = \mathrm{Tr}_{\partial\Omega} u_0$. We denote the associated singular points of the renormalizable map u_* by $\{a_i\}_{i=1,\dots,\kappa} \subset \Omega$. In addition to (i)–(v) *of proposition 2.5.1, we have*

(v) $Du_n \to Du_*$ *almost everywhere in Ω and*

$$\int_\Omega |Du_*|^{p_n} - \int_\Omega |Du_n|^{p_n} \xrightarrow{n\to\infty} 0,$$

(vi) $$\begin{aligned}\lim_{n\to\infty} \int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_n)}{2 - p_n} &= \mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u, a_i])_{i=1,\dots,\kappa} \\ &= \mathcal{E}^{1,2}_{\mathrm{top},\gamma_1,\dots,\gamma_k}([u_*, a_1], \dots, [u_*, a_k]) + \mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa},\end{aligned}$$

(vii) $\mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u, a_i])_{i=1,\dots,\kappa}$
$$= \inf \left\{ \mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u, a_i])_{i=1,\dots,\kappa} : \begin{array}{c} u \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N}) \\ \mathrm{Tr}_{\partial\Omega} u_* = g \end{array} \right\},$$

(viii) the charges $([u_*, a_i])_{i=1,\dots,\kappa}$ *and points* $(a_i)_{i=1,\dots,\kappa}$ *minimize the renormalized energy of the configuration of points*

$$\begin{aligned} \mathcal{E}^{1,2}_{\mathrm{top},\gamma_1,\dots,\gamma_\kappa}&([u_*, a_1], \dots, [u_*, a_k]) + \mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa} \\ &= \inf \Big\{ \mathcal{E}^{1,2}_{\mathrm{top},\gamma_1,\dots,\gamma_\kappa}(x_1, \dots, x_k) + \mathrm{H}(\gamma_i)_{i=1,\dots,\kappa} : \\ &\qquad \begin{array}{c} x_1, \dots, x_k \in \Omega \\ (\gamma_1, \dots, \gamma_k) \text{ is a minimal topological resolution} \end{array} \Big\} \end{aligned} \tag{2.6.1}$$

(ix) for each $i = 1, \dots, \kappa$*, and* $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,\kappa}))$

$$\begin{aligned} \int_{\mathbb{B}(a_i;\rho)} \frac{|Du_n|^{p_n}}{p_n} &- \frac{\lambda([u_*, a_i])^{p_n}}{(2\pi)^{p_n} p_n |\cdot - a_i|^{p_n}} \\ &\xrightarrow{n\to\infty} \int_{\mathbb{B}(a_i;\rho)} \frac{|Du_*|^2}{2} - \frac{\lambda([u_*, a_i])^2}{8\pi^2 |\cdot - a_i|^2}. \end{aligned} \tag{2.6.2}$$

The conclusion of proposition 2.6.1 implies that $u_n \to u_*$ in $W^{1,p}(\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho), \mathcal{N})$, for each $p \in [1, 2)$ and even more (see (2.6.11)):

$$\int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} |Du_n - Du_*|^{p_n} \xrightarrow{n\to\infty} 0. \tag{2.6.3}$$

We will refer to (2.6.3) as strong convergence of the gradients. Meanwhile we cannot hope for

$$\varliminf_{n\to\infty} \int_\Omega |Du_n|^{p_n} < +\infty \tag{2.6.4}$$

which would imply that $W^{1,2}_g(\Omega, \mathcal{N})$ is not empty by an application of Fatou lemma.

Note that even if

$$\int_\Omega |Du_n - Du_*|^{p_n} \xrightarrow{n\to\infty} 0 \tag{2.6.5}$$

holds, it would not imply (2.6.4). Therefore one could wonder if

$$\int_{\mathbb{B}(a_i;\rho)} |Du_n - Du_*|^{p_n} \xrightarrow{n\to\infty} 0$$

holds true near the singularities a_i of the map as (2.6.3) already holds true. As the following calculation shows, it could be related to the rate of convergence of singularities of p-harmonic maps (in [20] it is shown that p-harmonic mappings are smooth outside a finite number of points) to the points a_i of the renormalized maps. As (2.6.2) suggests, the norm of a derivative of a p-harmonic map behaves as $1/|x - a_p|$ near one of its singularities a_p and a renormalized map satisfy $\sup_{x\in\mathbb{B}(a;\rho)} |x - a||Du_*|(x) < \infty$ near on of its singularities a (see proposition 2.2.22(ii)). On the other hand, we have

$$2^{1-p}3^{p-2}\pi\frac{|a_p|^{2-p}}{2-p} \le \int_{\mathbb{B}(0;1)} \left|\frac{1}{|x|} - \frac{1}{|x-a_p|}\right|^p \,\mathrm{d}x \le 2^5\pi\frac{|a_p|^{2-p}}{2-p} \tag{2.6.6}$$

if $|a_p| < 1/2$.

Proof of (2.6.6). *Lower bound.* For all $x \in \mathbb{B}(a;|a|/3)$, $|x - a| < |x|/4$ and

$$\left|\frac{1}{|x|} - \frac{1}{|x-a|}\right|^2 \ge \frac{1}{|x-a|}\left(\frac{1}{|x-a|} - \frac{2}{|x|}\right) \ge \frac{1}{2|x-a|^2}$$

and

$$\int_{\mathbb{B}(a;|a|/3)} \left|\frac{1}{|x|} - \frac{1}{|x-a|}\right|^p \,\mathrm{d}x \ge \frac{2\pi|a|^{2-p}}{3^{2-p}2^p(2-p)}.$$

Upper bound. Setting $a_t = (1-t)0 + ta$ for $t \in (0,1)$, there exists a $t \in (0,1)$ such that for all $x \in \mathbb{B}(0;1) \setminus \mathbb{B}(a;2|a|)$

$$\left|\frac{1}{|x|} - \frac{1}{|x-a|}\right| \le \frac{|a|}{|x-a_t|^2}.$$

Also,

$$\begin{aligned}\int_{\mathbb{B}(0;1)} \left|\frac{1}{|x|} - \frac{1}{|x-a|}\right|^p \,\mathrm{d}x &\le |a|^p \int_{\mathbb{B}(0;1)\setminus\mathbb{B}(a_t;2|a|)} \frac{\mathrm{d}x}{|x-a_t|^{2p}} \\ &\quad + 2^p \int_{\mathbb{B}(0;4|a|)} \frac{\mathrm{d}x}{|x|^p} \\ &\le \frac{2^{4p}\pi|a|^{2-p}}{2p-2} + 2^{p+1+2(2-p)}\pi\frac{|a|^{2-p}}{2-p}. \square\end{aligned}$$

For each a_i with $i = 1,\ldots,\kappa$, the points in proposition 2.6.1, the following lemma (lemma 2.6.2) implies that on almost every disk $\partial\mathbb{B}(a_i;\rho)$ and all $i = 1,\ldots,\kappa$, for n large enough depending on ρ, $u_n|_{\partial\mathbb{B}(a;\rho)}$ is in the same homotopy class that $u_*|_{\partial\mathbb{B}(a;\rho)}$. Since $u_* \in W^{1,2}_{\mathrm{loc}}(\mathbb{B}(a_i;\rho(\{a_i\}_{i=1,\ldots,\kappa})) \setminus \{a_i\}_{i=1,\ldots,\kappa})$, we obtain that for n large enough $u_n|_{\partial\mathbb{B}(a_i;\rho)} \in [u_*, a_i]$. In particular, for n large enough, $\lambda(\operatorname{tr}_{\mathbb{S}^1(a;\rho)} u_n) = \lambda([u_*, a_i])$.

Lemma 2.6.2. *Let $p \in [1, \infty)$ and $\Omega \subset \mathbb{R}^2$. Let $(u_n)_n$ be a sequence converging to a map u in $W^{1,p}(\Omega, \mathcal{N})$. Then for all $a \in \Omega$ there exists an unrelabelled subsequence depending on a such that for almost every $\rho \in (0, \operatorname{dist}(a, \partial\Omega))$, $u_n \to u$ uniformly on $\partial\mathbb{B}(a; \rho)$.*

Proof of lemma 2.6.2. Fix $a \in \Omega$. Integration in polar coordinates combined with the partial converse to the Lebesgue dominated convergence theorem yields a subsequence of $(u_n)_n$ converges in $W^{1,p}(\partial\mathbb{B}(a;\rho), \mathcal{N})$. By the Sobolev embedding in dimension 1, along this subsequence $u_n \to u$ uniformly on $\partial\mathbb{B}(a; \rho)$. $\square$

2.6.1 Uniform convexity of the $p \nearrow 2$ L^p-convergence

We will obtain the strong convergence of the gradients using this proposition concerning the notion of convergence that induces

$$\int_X |u_p - u|^p \xrightarrow{p \nearrow 2} 0.$$

In the case of Lebesgue spaces, this proposition would have been a consequence of *uniform convexity* [14][40, Theorem 5.4.2].

Lemma 2.6.3. *Let X be a measure space. Let $v_p \in L^p(X, \mathbb{R}^\mu)$ for $p \in [1, \infty)$ and $v \in L^q(X, \mathbb{R}^\mu)$, for some $q \in (1, \infty)$. If*

$$\lim_{p \to q} \int_X |v_p|^p = \int_X |v|^q \text{ and } 2^q \int_X |v|^q \leq \varliminf_{p \to q} \int_X |v + v_p|^p, \tag{2.6.7}$$

then

$$\lim_{p \to q} \int_X |v_p - v|^p = 0.$$

The proof of lemma 2.6.3 rely on Hanner's inequality [19][40, Theorem 4.1.9(c)] (see (2.6.8) in the proof).

Proof of lemma 2.6.3. We assume $q \in (1, 2]$ and write $\|\cdot\|_r$ for $\|\cdot\|_{L^r(X, \mathbb{R}^\mu)}$. Since

$$\varlimsup_{p \to q} \|v_p - v\|_p \leq 2\|v\|_q < \infty,$$

we consider a sequence $p_n \to q$ such that the superior limit is actually a limit that we denote $\eta \geq 0$. We further assume that $p_n \leq q \leq 2$. For each n, Hanner's inequality for exponent below 2 implies

$$\begin{aligned} 2^{p_n}(\|v_n\|_{p_n}^{p_n} + \|v\|_{p_n}^{p_n}) \geq \big(\|v + v_n\|_{p_n} + \|v - v_n\|_{p_n}\big)^{p_n} \\ + \big|\|v + v_n\|_{p_n} + \|v - v_n\|_{p_n}\big|^{p_n}. \end{aligned} \tag{2.6.8}$$

Letting $n \to \infty$, we obtain

$$2\|v\|_p^p \geq \big(\|v\|_p + \eta\big)^p + \big|\|v\|_p - \eta\big|^p.$$

By lemma 2.6.4 below this forces $\eta = 0$ if $\|v\|_p \neq 0$ as η is nonnegative. If $q \geq 2$ or $p_n \geq q$ one proceeds the same way by if necessary using Hanner's inequality for exponent above 2. □

Lemma 2.6.4. *If $p \in (1, 2]$ and if $x \geq 0$ satisfy $2 \geq (1 + x)^p + |1 - x|^p$ then $x = 0$.*

2.6.2 Proof of convergence of minimizers (proposition 2.6.1)

Proof of proposition 2.6.1. By proposition 4.4.1, the sequence $(u_n)_n$ satisfies the assumption of the compactness proposition 2.5.1. This shows the existence of the limit map $u_* \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ of trace $\mathrm{Tr}_{\partial\Omega} u_* = \mathrm{Tr}_{\partial\Omega} u_0$. We call $\{a_i\}_{i=1,\dots,\kappa} \subset \Omega$, the associated singular points of the renormalizable map u_*. Let $v \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$ such that $\mathrm{Tr}_{\partial\Omega} v = \mathrm{Tr}_{\partial\Omega} u_0$ and its singularities $\{a_i^v\}_{i=1,\dots,k} \subset \Omega$ satisfy for all $\rho \in (0, \rho(a_i^k)_{i=1,\dots,k})$,

$$\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} v) = \sum_{i=1}^{k} \frac{\lambda(\mathrm{Tr}_{\partial\mathbb{B}(a_i;\rho)} v)^2}{4\pi}.$$

Hence, by 2.5.2 in proposition 2.5.1, the minimality assumption and the continuity proposition 2.2.17, we get

$$\begin{aligned}(2.6.9)\quad \mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) &- \sum_{i=1}^{\kappa} \frac{\lambda([u_*, a_i])^2}{4\pi} \log \frac{\lambda([u_*, a_i])}{2\pi} \\ &\leq \varliminf_{n\to\infty} \int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\partial\Omega} u_n)}{2 - p_n} - \frac{\mathcal{E}^{1,2}_{\mathrm{sg}}(\mathrm{Tr}_{\Omega} u_*)}{2} \\ &\leq \mathcal{E}^{1,2}_{\mathrm{ren}}(v) - \sum_{i=1}^{k} \frac{\lambda([v, a_i])^2}{4\pi} \log \frac{\lambda([v, a_i])}{2\pi}.\end{aligned}$$

This shows that u_* has the minimizing property vii. To get (vi), we combine the integral representation of the renormalized energy (proposition 2.2.16) and vii in proposition 2.5.1: for all $v \in W^{1,2}_{\mathrm{ren}}(\Omega, \mathcal{N})$

$$\begin{aligned}(2.6.10)\quad \mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) &+ \mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa} \\ &\leq \int_{\Omega\setminus\bigcup_{i=1}^{\kappa} \mathbb{B}(a_i;\rho)} \frac{|Dv|^2}{2} - \log\frac{1}{\rho} \sum_{i=1}^{\kappa} \frac{\lambda([v, a_i])^2}{4\pi} \\ &\quad + \sum_{i=1}^{\kappa} \int_0^\rho \left[\int_{\partial\mathbb{B}(a_i;r)} \frac{|Dv|^2}{2}\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([v, a_i])^2}{4\pi r}\right] \mathrm{d}r.\end{aligned}$$

Minimizing over u as in the definition (2.2.30) of the renormalized geometrical energy and letting $\rho \searrow 0$ we obtain $\mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa} \leq \mathcal{E}^{1,2}_{\mathrm{top},[u_*,a_1],\dots,[u_*,a_\kappa]}(v) + \mathrm{H}([u_*, a_i])_{i=1,\dots,\kappa}$. The reverse inequality is given by proposition 2.5.1(iv). The assertion viii follows from the arbitrariness of v.

Let us now show that for all $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,\kappa}))$,

$$\overline{\lim_{n\to\infty}} \int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} |Du_n|^{p_n} \leq \int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} |Du_*|^2 \tag{2.6.11}$$

because then we will get the strong convergence of the gradients in the sense of (2.6.3) in view of lemma 2.6.3 and the weak convergence of Du_n given by proposition 2.5.1. It implies by the partial converse of the Lebesgue dominated convergence theorem that $Du_n \to Du_*$ almost everywhere.

Let us observe by the integral expression of the renormalized energy, proposition 2.2.16 and (2.6.9) evaluated at $v = u_*$,

$$\begin{aligned}
\int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} \frac{|Du_*|^2}{2} &= \mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) + \sum_{i=1}^k \frac{\lambda([u_*, a_i])^2}{4\pi} \log \frac{1}{\rho} \qquad (2.6.12)\\
&\quad - \sum_{i=1}^\kappa \int_0^\rho \left[\int_{\partial\mathbb{B}(a_i;r)} \frac{|Du_*|^2}{2} \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r} \right] \mathrm{d}r\\
&= \lim_{n\to\infty} \int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \sum_{i=1}^\kappa \frac{\lambda([u_*, a_i])^{p_n}}{2\pi p_n} \frac{(2\pi)^{2-p_n}}{2-p_n}\\
&\quad + \lim_{n\to\infty} \sum_{i=1}^\kappa \frac{\lambda([u_*, a_i])^{p_n}}{2p_n\pi} \frac{(2\pi)^{2-p_n} - (2\pi\rho)^{2-p_n}}{2-p_n}\\
&\quad - \sum_{i=1}^\kappa \int_0^\rho \left[\int_{\partial\mathbb{B}_r(a_i)} \frac{|Du_*|^2}{2} \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r} \right] \mathrm{d}r.
\end{aligned}$$

Hence,

$$\begin{aligned}
\int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} \frac{|Du_*|^2}{2} &\geq \overline{\lim_{n\to\infty}} \int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} \frac{|Du_n|^{p_n}}{p_n} \qquad (2.6.13)\\
&\quad + \underline{\lim_{n\to\infty}} \int_{\bigcup_{i=1}^\kappa \mathbb{B}(a_i;\rho)} \frac{|Du_n|^{p_n}}{p_n} - \sum_{i=1}^\kappa \frac{\lambda([u_*, a_i])^{p_n}}{2p_n\pi} \frac{(2\pi\rho)^{2-p_n}}{2-p_n}\\
&\quad - \sum_{i=1}^\kappa \int_0^\rho \left[\int_{\partial\mathbb{B}(a_i;r)} \frac{|Du_*|^2}{2} \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r} \right] \mathrm{d}r.\\
&\geq \overline{\lim_{n\to\infty}} \int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i;\rho)} \frac{|Du_n|^{p_n}}{p_n},
\end{aligned}$$

where we used for the last inequality the limit (v). Moreover, we have equality in (2.6.13) by the weak convergence of the weak derivatives Du_n. Therefore, we deduce (2.6.2) as (v) allow us to localize on balls $\mathbb{B}(a_i;\rho)$ for $\rho \in (0, \rho(\{a_i\}_{i=1,\dots,\kappa})$.

By proposition 2.2.17, and by (2.6.13),

$$\mathcal{E}^{1,2}_{\mathrm{ren}}(u_*) = \lim_{n\to\infty} \int_\Omega \frac{|Du_*|^{p_n}}{p_n} - \sum_{i=1}^{k} \frac{\lambda([u_*,a_i])^{p_n}}{(2\pi)^{p_n-1}p_n}\frac{1-\rho^{2-p_n}}{2-p_n}, \tag{2.6.14}$$

$$= \lim_{n\to\infty} \int_\Omega \frac{|Du_n|^{p_n}}{p_n} - \sum_{i=1}^{\kappa} \frac{\lambda([u_*,a_i])^{p_n}}{(2\pi)^{p_n-1}p_n}\frac{1-\rho^{2-p_n}}{2-p_n}, \tag{2.6.15}$$

and thus we get v, concluding the proof. □

2.7 The case of non-compact manifolds

We point out assumptions about the complete Riemannian manifold $\mathcal{N}$ to which the present method of proof applies without changing a single thing.

(i) (*Positive systole*) The manifold $\mathcal{N}$ should have a positive systole $\mathrm{Sys}(\mathcal{N})$. This result is crucial to obtain the continuity of $\mathcal{E}^{1,p}_{\mathrm{sg}}(g)$ in p, see lemma 2.2.10 and to count the balls in the proof of proposition 2.5.1, see (2.5.12).

(ii) (*Nonemptiness of* $W^{1,2}_{\mathrm{ren},g}$) For $g \in W^{1/2,2}(\partial\Omega,\mathcal{N})$, $W^{1,2}_{\mathrm{ren},g}(\Omega,\mathcal{N})$ is not empty. Proposition 2.2.20 shows that it can be empty in the non-compact case. This guarantees the existence of competitors, see proposition 4.4.1. One can check that the manifold $\mathcal{N} \doteq \mathbb{S}^1 \times \mathbb{R}$ and the map $g = \mathrm{Id}_{\mathbb{S}^1} \times 0$ verify $W^{1,2}_{\mathrm{ren},g}(\mathbb{B}(0,1),\mathcal{N}) \neq \varnothing$.

(iii) ($W^{1,2}$*-extension of the boundary data*) There exists a $\delta > 0$ such that for the boundary data $g \in W^{1/2,2}(\partial\Omega,\mathcal{N})$ one can construct a map $U \in W^{1,2}(\partial\Omega_\delta,\mathcal{N})$ such that $\mathrm{Tr}_{\partial\Omega}U = g$ where $\partial\Omega_\delta$ means $\{x \in \mathbb{R}^2 : \mathrm{dist}(x,\partial\Omega) < \delta\}$. This replaces proposition 2.4.1 and holds if $g \in L^\infty \cap W^{1/2,2}(\partial\Omega,\mathcal{N})$ by proposition 2.4.1 for instance.

Bibliography of Chapter 2

[1] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces: Applications to PDEs and optimization.* Second. Vol. 17. MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2014, pp. xii+793.

[2] P.-A. Beaufort, J. Lambrechts, F. Henrotte, C. Geuzaine, and J.-F. Remacle. "Computing cross fields : A PDE approach based on the Ginzburg-Landau theory". In: *Procedia Engineering* 203 (2017). 26th International Meshing Roundtable, IMR26, 18-21 September 2017, Barcelona, Spain, pp. 219–231.

[3] M. Berger. "Systoles et applications selon Gromov". In: 216. Séminaire Bourbaki, Vol. 1992/93. 1993, Exp. No. 771, 5, 279–310.

[4] F. Bethuel and F. Demengel. "Extensions for Sobolev mappings between manifolds". In: *Calc. Var. Partial Differ. Equ.* 3.4 (1995), pp. 475–491.

[5] F. Bethuel. "The approximation problem for Sobolev maps between two manifolds". In: *Acta Math.* 167.3-4 (1991), pp. 153–206.

[6] F. Bethuel, H. Brezis, and F. Hélein. *Ginzburg-Landau vortices.* Vol. 13. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, MA, 1994, pp. xxviii+159.

[7] P. Bousquet, A. C. Ponce, and J. Van Schaftingen. "Density of bounded maps in Sobolev spaces into complete manifolds". In: *Ann. Mat. Pura Appl. (4)* 196.6 (2017), pp. 2261–2301.

[8] P. Bousquet, A. C. Ponce, and J. Van Schaftingen. "Strong density for higher order Sobolev spaces into compact manifolds". In: *J. Eur. Math. Soc. (JEMS)* 17.4 (2015), pp. 763–817.

[9] H. Brezis and L. Nirenberg. "Degree theory and BMO. I. Compact manifolds without boundaries". In: *Selecta Math. (N.S.)* 1.2 (1995), pp. 197–263.

[10] H. Brezis, J.-M. Coron, and E. H. Lieb. "Harmonic maps with defects". In: *Comm. Math. Phys.* 107.4 (1986), pp. 649–705.

[11] H. Brezis and L. Nirenberg. "Degree theory and BMO. II. Compact manifolds with boundaries". In: *Selecta Math. (N.S.)* 2.3 (1996). With an appendix by the authors and Petru Mironescu, pp. 309–368.

[12] G. Canevari and G. Orlandi. "Topological singular set of vector-valued maps. II: Γ-convergence for Ginzburg-Landau type functionals". In: *Arch. Ration. Mech. Anal.* 241.2 (2021), pp. 1065–1135.

[13] R. E. Castillo and H. Rafeiro. *An introductory course in Lebesgue spaces.* CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC. Springer, Cham, 2016, pp. xii+461.

[14] J. A. Clarkson. "Uniformly convex spaces". In: *Trans. Amer. Math. Soc.* 40.3 (1936), pp. 396–414.

[15] A. Convent and J. Van Schaftingen. "Intrinsic co-local weak derivatives and Sobolev spaces between manifolds". In: *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)* 16.1 (2016), pp. 97–128.

[16] E. Gagliardo. "Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili". In: *Rend. Sem. Mat. Univ. Padova* 27 (1957), pp. 284–305.

[17] M. Giaquinta and D. Mucci. *Maps into manifolds and currents: area and $W^{1,2}$-, $W^{1/2}$-, BV-energies.* Vol. 3. Centro di Ricerca Matematica Ennio De Giorgi (CRM) Series. Edizioni della Normale, Pisa, 2006, pp. xxii+391.

[18] M. Gromov. "Filling Riemannian manifolds". In: *J. Differential Geom.* 18.1 (1983), pp. 1–147.

[19] O. Hanner. "On the uniform convexity of L^p and l^p". In: *Ark. Mat.* 3 (1956), pp. 239–244.

[20] R. Hardt and F. Lin. "Mappings minimizing the L^p norm of the gradient". In: *Commun. Pure Appl. Math.* 40.5 (1987), pp. 555–588.

[21] R. Hardt and F. Lin. “Singularities for p-energy minimizing unit vectorfields on planar domains”. In: *Calc. Var. Partial Differential Equations* 3.3 (1995), pp. 311–341.

[22] R. L. Jerrard. “Lower bounds for generalized Ginzburg-Landau functionals”. In: *SIAM J. Math. Anal.* 30.4 (1999), pp. 721–746.

[23] J. M. Lee. *Introduction to Riemannian manifolds.* Vol. 176. Graduate Texts in Mathematics. Springer, Cham, 2018, pp. xiii+437.

[24] J. M. Lee. *Introduction to smooth manifolds.* Second. Vol. 218. Graduate Texts in Mathematics. Springer, New York, 2013, pp. xvi+708.

[25] A. Macq, M. Reberol, F. Henrotte, P.-A. Beaufort, A. Chemin, J.-F. Remacle, and J. Van Schaftingen. “Ginzburg-Landau energy and placement of singularities in generated cross fields”. In: *arXiv:2010.16381* (2020).

[26] J. Marcinkiewicz. “Sur l’interpolation d’opérations”. In: *CR Acad. Sci. Paris.* 208 (1939), pp. 1272–1273.

[27] P. Mironescu and J. Van Schaftingen. “Trace theory for Sobolev mappings into a manifold”. In: *Ann. Fac. Sci. Toulouse Math. (6)* 30.2 (2021), pp. 281–299.

[28] A. Monteil, R. Rodiac, and J. Van Schaftingen. “Ginzburg-Landau relaxation for harmonic maps on planar domains into a general compact vacuum manifold”. In: *Arch. Ration. Mech. Anal.* 242.2 (2021), pp. 875–935.

[29] A. Monteil, R. Rodiac, and J. Van Schaftingen. “Renormalised energies and renormalisable singular harmonic maps into a compact manifold on planar domains”. In: *Math. Ann.* 383.3-4 (2022), pp. 1061–1125.

[30] O. Müller. “A note on closed isometric embeddings”. In: *J. Math. Anal. Appl.* 349.1 (2009), pp. 297–298.

[31] J. Nash. “The imbedding problem for Riemannian manifolds”. In: *Ann. of Math. (2)* 63 (1956), pp. 20–63.

[32] P. Neff. “A geometrically exact Cosserat shell-model including size effects, avoiding degeneracy in the thin shell limit. I. Formal dimensional reduction for elastic plates and existence of minimizers for positive Cosserat couple modulus”. In: *Contin. Mech. Thermodyn.* 16.6 (2004), pp. 577–628.

[33] A. C. Ponce and J. Van Schaftingen. “Closure of smooth maps in $W^{1,p}(B^3; S^2)$”. In: *Differential Integral Equations* 22.9-10 (2009), pp. 881–900.

[34] P. M. Pu. “Some inequalities in certain nonorientable Riemannian manifolds”. In: *Pacific J. Math.* 2 (1952), pp. 55–71.

[35] E. Sandier. “Lower bounds for the energy of unit vector fields and applications”. In: *J. Funct. Anal.* 152.2 (1998), pp. 379–403.

[36] S. Serfaty and I. Tice. “Lorentz space estimates for the Ginzburg-Landau energy”. In: *J. Funct. Anal.* 254.3 (2008), pp. 773–825.

[37] D. Stern. “p-Harmonic maps to S^1 and stationary varifolds of codimension two”. In: *Calc. Var. Partial Differential Equations* 59.6 (2020), Paper No. 187, 47.

[38] J. Van Schaftingen. “Sobolev mappings into manifolds: nonlinear methods for approximation, extension and lifting problems”. In: *To appear* (2021).

[39] J. Van Schaftingen and B. Van Vaerenbergh. “Asymptotic behavior of minimizing p-harmonic maps when $p \nearrow 2$ in dimension 2”. In: *Calc. Var. Partial Differ. Equ.* 62.8 (2023). Id/No 229, p. 45.

[40] M. Willem. *Functional analysis: Fundamentals and applications.* Cornerstones. Birkhäuser/Springer, New York, 2013, pp. xiv+213.

Limiting behavior of minimizing p-harmonic maps in 3d as p goes to 2 with finite fundamental group

3

ABSTRACT. We study the limiting behavior of minimizing p-harmonic maps from a bounded Lipschitz domain $\Omega \subset \mathbb{R}^3$ to a compact connected Riemannian manifold without boundary and with finite fundamental group as $p \nearrow 2$. We prove that there exists a closed set S_* of finite length such that minimizing p-harmonic maps converge to a locally minimizing harmonic map in $\Omega \setminus S_*$. We prove that locally inside Ω the singular set S_* is a finite union of straight line segments, and it minimizes the mass in the appropriate class of admissible chains. Furthermore, we establish local and global estimates for the limiting singular harmonic map. Under additional assumptions, we prove that globally in $\overline{\Omega}$ the set S_* is a finite union of straight line segments, and it minimizes the mass in the appropriate class of admissible chains, which is defined by a given boundary datum and Ω.

3.1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^3$ with a smooth boundary $\partial\Omega$ and a compact connected smooth Riemannian manifold $\mathcal{N}$ without boundary and of finite dimension, which will be assumed, thanks to Nash's embedding theorem, to be isometrically embedded into the Euclidean space $\mathbb{R}^\nu$ for some $\nu \in \mathbb{N} \setminus \{0\}$, we are interested in the problem of finding a minimizing harmonic map with boundary datum $g : \partial\Omega \to \mathcal{N}$, that is, find a minimizer of the Dirichlet integral

$$\int_\Omega \frac{|Du|^2}{2}\,dx \tag{3.1.1}$$

This chapter originally appeared as B. Bulanyi, J. Van Schaftingen, and B. Van Vaerenbergh. "Limiting behavior of minimizing p-harmonic maps in 3d as p goes to 2 with finite fundamental group". English. In: *Arch. Ration. Mech. Anal.* 249.3 (2025). Id/No 29, p. 103

among all Sobolev mappings $u \in W^{1,2}(\Omega, \mathcal{N})$ that have g as a trace on $\partial\Omega$.

If there exists some $v \in W^{1,2}(\Omega, \mathcal{N})$ having g as a trace on $\partial\Omega$, standard methods in the calculus of variations show the existence of a minimizer $u \in W^{1,2}(\Omega, \mathcal{N})$ of the energy (3.1.1). Following the classical works of Schoen and Uhlenbeck, the map u is continuous outside a discrete set of points in Ω [53]; moreover, the solution u inherits further regularity of the boundary datum: for example, if $g \in C^{2,\alpha}(\partial\Omega, \mathcal{N})$, then u has the same regularity in a neighborhood of the boundary [54].

When the target manifold $\mathcal{N}$ is simply connected, or equivalently has a trivial fundamental group (*i.e.*, $\pi_1(\mathcal{N}) \simeq \{0\}$), the space of admissible traces is the fractional Sobolev space $W^{1/2,2}(\partial\Omega, \mathcal{N})$ [35], similarly to the trace theory of linear spaces [32], giving then rise to a nice theory of harmonic extension of mappings into $\mathcal{N}$.

When the manifold $\mathcal{N}$ is not simply connected, the admissible traces form a strict subset of $W^{1/2,2}(\partial\Omega, \mathcal{N})$ due to several obstruction phenomena. First, one has global topological obstructions created by the combination of the topologies of $\partial\Omega$ and $\mathcal{N}$ [12, Theorem 5]: if $\Omega \simeq B_1^2 \times \mathbb{S}^1$ is a solid torus, taking $\gamma \in C^\infty(\mathbb{S}^1, \mathcal{N})$, which is not homotopic to a constant, the function $g \in C^\infty(\partial\Omega, \mathcal{N})$ defined by $g(x', x'') = \gamma(x')$ for $(x', x'') \in \mathbb{S}^1 \times \mathbb{S}^1 \simeq \partial\Omega$ has no extension in $W^{1,2}(\Omega, \mathcal{N})$. Next, one has local topological obstructions with boundary data that behave like $g(x) = \gamma(x'/|x'|)$ [35, Section 6.3][12, Theorem 4]. Finally, there are some analytical obstructions coming from the infiniteness of $\pi_1(\mathcal{N})$ [8], and since the integrability exponent 2 is an integer, from the nontriviality of $\pi_1(\mathcal{N})$ [45, Theorem 1.10]; these obstructions arise as limits of juxtaposition of boundary data, which have arbitrarily large lower bounds on the extension energy.

These obstructions to trace extension disappear if one relaxes the problem and considers the problem of minimizing, for $1 < p < 2$, the p-energy

$$\int_\Omega \frac{|Du|^p}{p}\, dx.$$

Indeed, given $g \in W^{1/2,2}(\partial\Omega, \mathcal{N}) \subset W^{1-1/p,p}(\partial\Omega, \mathcal{N})$, there exists $v \in W^{1,p}(\Omega, \mathcal{N})$ such that $\operatorname{tr}_{\partial\Omega}(v) = g$. One way then to find a good replacement for solutions to the problem of minimizing harmonic maps is to define u_p to be a minimizing p-harmonic map under the boundary condition $u = g$ on $\partial\Omega$ and study the asymptotics of the solutions u_p of this p-harmonic relaxation as $p \nearrow 2$.

Such an approach has been developed successfully for planar domains $\Omega \subset \mathbb{R}^2$ when $\mathcal{N}$ is a circle [36] or a general compact Riemannian manifold [57], where u_p converges as $p \nearrow 2$ then to a harmonic map outside a finite set of singular points whose positions and topological charges minimize a suitable renormalized energy [10][47]. Similar results holds for p-energy minimizing maps from $\Omega \subseteq \mathbb{R}^m$ to $\mathbb{S}^{m-1}$ as $p \nearrow m$ [37].

In higher dimension, this problem has been studied for maps taking their value in the circle: Hardt and Lin [42, Section 3] have proved that the energy density of p-energy minimizing maps from an m-dimensional domain to a sphere $\mathbb{S}^n$ as $p \nearrow n < m$ converges to a limit measure whose support coincides with some area minimizing current of dimension $m - n$; Stern [55] has obtained the convergence of the energy density of p-energy minimizing maps from a three-dimensional manifold to the circle $\mathbb{S}^1$ as $p \nearrow 2$ to the weight measure of a stationary varifold.

In three dimensions, these asymptotics have been analyzed for maps from a Riemannian manifold into the circle $\mathbb{S}^1$ [42, 55].

Our first result is the following.

Theorem 3.1.1. *If $\Omega \subset \mathbb{R}^3$ is a bounded Lipschitz domain, $\pi_1(\mathcal{N})$ is finite, $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and if for each $n \in \mathbb{N}$, $u_n \in W^{1,p_n}(\Omega, \mathcal{N})$ is a minimizing p_n-harmonic map with boundary datum g, then*

$$\limsup_{n\to+\infty}(2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n}\, dx \leq C|g|^2_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)},$$

where $C = C(\partial\Omega, \mathcal{N}) > 0$. Furthermore, there exist a closed set $S_* \subset \overline{\Omega}$, at most a countable union of segments $(L_j)_{j\in J}$ lying in $\overline{\Omega}$, mappings $(\gamma_j)_{j\in J}$ in $C(\mathbb{S}^1, \mathcal{N})$, a map $u_* : \Omega \to \mathcal{N}$ such that $S_* \cap \Omega = \bigcup_{j\in J} L_j \cap \Omega$ and, up to a subsequence (not relabeled), the following assertions hold.

(i) *$u_n \to u_*$ almost everywhere in Ω.*

(ii) *For each $q \in [1,2)$, $u_* \in W^{1,q}_{\mathrm{loc}}(\Omega, \mathcal{N})$, $u_n \to u_*$ in $W^{1,q}_{\mathrm{loc}}(\Omega, \mathbb{R}^\nu)$ and*

$$\limsup_{n\to+\infty} \int_\Omega (\mathrm{dist}(x, \partial\Omega)|Du_n(x)|)^q\, dx \leq C|\Omega| + \frac{C}{2-q} \limsup_{n\to+\infty}(2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n}\, dx,$$

where $C = C(\mathrm{diam}(\Omega), \mathcal{N}) > 0$.

(iii) *Assume that there exists $r_0 = r_0(\Omega) > 0$ such that if $x \in \mathbb{R}^3 \setminus \Omega$ and $\mathrm{dist}(x, \partial\Omega) \leq r_0$, there exists a unique point $P(x) \in \partial\Omega$ such that $\mathrm{dist}(x, \partial\Omega) = |x - P(x)|$. If g is a C^1 map outside a finite set of points around each of which g is homotopically nontrivial and whose Euclidean norm of the gradient behaves like the distance*

to the power of -1 *close to these points, then for each* $q \in [1,2)$, $u_* \in W^{1,q}(\Omega, \mathcal{N})$, $u_n \to u_*$ *in* $W^{1,q}(\Omega, \mathbb{R}^\nu)$ *and*

$$\limsup_{n\to+\infty} \int_\Omega |Du_n|^q \, dx \le C + \frac{C}{2-q} \limsup_{n\to+\infty} (2-p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \, dx,$$

where $C = C(g, \Omega, \mathcal{N}) > 0$.

(iv) Any point of Ω *has a neighborhood that only intersects finitely many* L_j.

(v) $u_* \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$ *is a locally minimizing harmonic map in* $\Omega \setminus S_*$: *if* $K \Subset \Omega \setminus S_*$ *is an open set and* $v \in W^{1,2}(K, \mathbb{R}^\nu)$ *satisfies* $v - u_* \in W^{1,2}_0(K, \mathbb{R}^\nu)$, *then*

$$\int_K \frac{|Du_*|^2}{2} \, dx \le \int_K \frac{|Dv|^2}{2} \, dx.$$

Furthermore, there exists at most a countable and locally finite subset S_0 *of* $\Omega \backslash S_*$ *such that* $u_* \in C^\infty(\Omega \setminus (S_* \cup S_0), \mathcal{N})$.

(vi) The varifold $V_* = (\mathrm{Id}, A_*)_\# \Big(\sum_{j\in J} \mathcal{E}^{\mathrm{sg}}_2(\gamma_j) \mathcal{H}^1 \llcorner (L_j \cap \Omega)\Big)$ *supported by* $\bigcup_{j\in J} L_j \cap \Omega$ *and with a multiplicity* $\mathcal{E}^{\mathrm{sg}}_2(\gamma_j)$ *is a stationary varifold which is the limit of the rescaled stress-energy tensors and*

$$\begin{aligned} V_*(\Omega \times \mathrm{G}(3,1)) &= \sum_{j\in J} \mathcal{E}^{\mathrm{sg}}_2(\gamma_j) \mathcal{H}^1(L_j \cap \Omega) \\ &\le \liminf_{n\to+\infty} (2-p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \, dx, \end{aligned}$$

where $\mathrm{Id} : \mathbb{R}^3 \to \mathbb{R}^3$ *is the identity map,* $A_*(x) \in \mathbb{R}^3 \otimes \mathbb{R}^3$ *represents the orthogonal projection onto the approximate tangent plane* $T_x S_*$ *to* S_* *at* x *and* $\mathrm{G}(3,1)$ *is the Grassmann manifold.*

(vii) S_* *minimizes the mass among all admissible chains locally inside* Ω, *namely, locally inside* Ω *it solves a homotopical Plateau problem in codimension 2.*

(viii) If $\partial\Omega$ *is homeomorphic to* $\mathbb{S}^2$ *and* $g \in W^{1,2}(\partial\Omega, \mathcal{N})$, *then* $S_* = \emptyset$ *and* $u_* \in C^\infty(\Omega \setminus S_0, \mathcal{N})$ *is a minimizing harmonic extension of* g *to* Ω, *where* $S_0 \subset \Omega$ *is a discrete set.*

The singular energy $\mathcal{E}^{\mathrm{sg}}_2(\gamma)$, appearing in the definition of the varifold V_* in Theorem 3.1.1, is defined for $\gamma \in C(\mathbb{S}^1, \mathcal{N})$ as the infimum of

$\sum_{i=1}^{m} \int_{\mathbb{S}^1} \frac{|f_i'|^2}{2} \, d\mathcal{H}^1$ for $f_1, \dots, f_m \in C^1(\mathbb{S}^1, \mathcal{N})$ such that γ has a continuous extension to a domain consisting of the 2-disk B_1^2 from which m small 2-disks have been removed whose restriction on the boundary of the i-th disk is a rescaled version of the mapping f_i.

The finite assumption on the fundamental group in Theorem 3.1.1 makes it inapplicable to the case of the unit circle $\mathcal{N} = \mathbb{S}^1$; the framework of our results is thus completely disjoint from Stern's [55]. Among the situations covered by our theorem, we point out the case where $\mathcal{N}$ is the real projective plane $\mathbb{RP}^2$ motivated by liquid crystals, and the case where $\mathcal{N}$ is the space of attitudes of the cube $SO(3)/O$, where O is the octahedral group, which appears as framefields in meshing problems. We point out that the finiteness assumption on the fundamental group of the compact manifold $\mathcal{N}$ is equivalent to a compactness assumption on the universal covering $\widetilde{\mathcal{N}}$ of the manifold $\mathcal{N}$.

When $\mathcal{N}$ is the real projective plane $\mathbb{RP}^2$, Theorem 3.1.1 is a counterpart of Canevari's result for Landau-de Gennes models of nematic liquid crystals [22].

The strategy of our proof of Theorem 3.1.1 is inspired by Canevari's η-compactness approach for Landau-de Gennes [22]. Whereas Canevari works with a perturbation involving no derivatives and with a domain enlarged to a full linear Sobolev space, in our setting we have a perturbation involving the first-order derivatives defined on a nonlinear space with the same manifold constraint but enlarged by lower integrability requirements. This results in different constructions to deal with the different structures and different analytical arguments because the space in which the minimization is performed depends on the relaxation parameter p.

We also emphasize that in our results we obtain the uniform interior $W^{1,q}$ estimates and the local minimization property outside the singular set S_* for u_*, in contrast to a minimization result restricted to balls by Canevari, and we prove that locally inside Ω, S_* solves a homotopical Plateau problem in codimension 2.

At a more technical point, we could identify the stationary varifold structure of the limit of the stress-energy tensors directly through Arroyo-Rabasa, De Philippis, Hirsch and Rindler's work on rectifiability of measures satisfying some differential constraints [6].

It is also worth noting that the smaller the fractional seminorm of the boundary datum g, the closer to $\partial\Omega$ the singular set S_* of Theorem 3.1.1 is located (see Proposition 3.7.1).

Under additional assumptions we get global properties of the limit.

Theorem 3.1.2. *With the assumptions and notations of Theorem 3.1.1 if we assume moreover that Ω is of class C^2, $\overline{\Omega}$ is strongly convex at every* *point of $\partial\Omega$, g is a C^1 map outside a finite set of points $\{a_1, \dots, a_m\} \subset \partial\Omega$ around each of which g is homotopically nontrivial and whose Euclidean norm of the gradient behaves like the distance to the power of -1 close to these points and if for each $n \in \mathbb{N}$, Du_n is continuous on $\partial\Omega \setminus \{a_1, \dots, a_m\}$, then the following assertions hold.*

(i) *$S_* \cap \partial\Omega = \{a_1, \dots, a_m\}$ and S_* lies in the convex hull of $\{a_1, \dots, a_m\}$.*

(ii) *The set J is finite and $S_* = \bigcup_{j \in J} L_j$.*

(iii) *The quantity*

$$\sum_{j \in J} \mathcal{E}_2^{\mathrm{sg}}(\gamma_j)\mathcal{H}^1(L_j) \quad \textit{minimizes} \quad \sum_{i \in I} \mathcal{E}_2^{\mathrm{sg}}(f_i)\mathcal{H}^1(K_i)$$

among all configurations of closed straight line segments $(K_i)_{i \in I}$ and topological charges $(f_i)_{i \in I} \subset C(\mathbb{S}^1, \mathcal{N})$ satisfying the following conditions: I is finite; each K_i has a positive length; the interiors of the segments (i.e., the segments without their endpoints) are pairwise disjoint; $K = \bigcup_{i \in I} K_i$ is a compact subset of $\overline{\Omega}$ and $K \cap \partial\Omega$ is a finite set of points; K has no endpoints in Ω, namely, if K_i is a segment having an endpoint $x \in \Omega$, then there exists another segment K_j emanating from x; there exists a map $v \in C(\Omega \setminus (K \cup F), \mathcal{N})$, with F being a locally finite subset of $\Omega \setminus K$, such that $\mathrm{tr}_{\partial\Omega}(v) = g$ and v restricted to small circles transversal to K_i is, up to orientation, homotopic to f_i. Namely, S_* solves a homotopical Plateau problem in codimension 2.

The additional ingredient in the proof of Theorem 3.1.2 is the exploitation of the Pohozaev-type identity combined with White's maximum principle for minimizing varifolds [58].

Although our results and proof are written in the framework of domains of the Euclidean space $\mathbb{R}^3$, most of the analysis can be trandfered to compact 3-dimensional Riemannian manifolds. The adaptation of Theorem 3.1.1 should be completely straightforward, provided line segments are replaced by geodesics. Transfering Theorem 3.1.2 to manifolds would require a suitable use of strong convexity of the boundary of submanifolds.

Finally, we point out that if $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$ and $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$, then the singular set S_* is a minimal connection (see Definition 3.7.24 and Proposition 3.7.25). Thus, in this case, we exclude the appearance of closed loops (i.e., homeomorphic images of $\mathbb{S}^1$) in S_*. In fact, the singular sets of minimizing p_n-harmonic maps u_n could have disclination loops [26, Section 2.4], however, in the limit, as $n \to +\infty$, each of these loops, which is contractible in Ω, should become unstable, shrink and disappear. We also suspect that the singular sets of the corresponding minimizing p_n-harmonic maps converge, up to a subsequence, to the singular set S_* in the Hausdorff distance, but this question is beyond the scope of our paper.

3.2 Preliminaries

3.2.1 Conventions and Notation

Conventions: in this paper, we say that a value is positive if it is strictly greater than zero, and a value is nonnegative if it is greater than or equal to zero. Euclidean spaces are endowed with the Euclidean inner product $\langle \cdot, \cdot \rangle$ and the induced norm $|\cdot|$. A set will be called a domain whenever it is open and connected. Throughout this paper, Ω, unless otherwise stated, will denote a bounded (locally) Lipschitz domain in $\mathbb{R}^3$, and $\mathcal{N}$ will denote a finite-dimensional compact connected smooth Riemannian manifold without boundary and with finite fundamental group, which will be assumed, thanks to Nash's embedding theorem, to be isometrically embedded into the Euclidean space $\mathbb{R}^\nu$ for some $\nu \in \mathbb{N} \setminus \{0\}$. The Hausdorff measures, which we shall use, coincide in terms of normalization with the appropriate outer Lebesgue measures. We shall say that a property holds μ-*almost everywhere* (or simply *almost everywhere*) for a measure μ (abbreviated μ-a.e.) if it holds outside a set of zero μ-measure.

Notation: if A, B, M are real matrices, we denote the inner product (also known as Frobenius or Hilbert-Schmidt) between A and B by $A : B = \operatorname{tr}(A^{\mathrm{T}} B)$ and the Euclidean norm of M by $|M| = \sqrt{M : M}$. We denote by $B_r^l(x)$, $\overline{B}_r^l(x)$, and $\partial B_r^l(x)$, respectively, the open ball in $\mathbb{R}^l$, the closed ball in $\mathbb{R}^l$, and its boundary the $(l-1)$-sphere with center x and radius r, where $l \in \mathbb{N} \setminus \{0\}$. If the center is at the origin 0, we write B_r^l, $\overline{B}_r^l$ and ∂B_r^l the corresponding balls and the $(l-1)$-sphere. We shall sometimes write $\mathbb{S}^{m-1}$ instead of ∂B_1^m. We denote by $\operatorname{dist}(x, A)$ and $|A|$, respectively, the Euclidean distance from $x \in \mathbb{R}^l$ to $A \subset \mathbb{R}^l$, and the l-dimensional Lebesgue measure of A. We shall denote by $\mathcal{H}^l(A)$ the l-dimensional Hausdorff measure of A. If $U \subset \mathbb{R}^l$ is Lebesgue measurable, then for $p \in [1, +\infty) \cup \{+\infty\}$, $L^p(U)$ will denote the space consisting of all real measurable functions on U that are p^{th}-power integrable on U if $p \in [1, +\infty)$ and are essentially bounded if $p = +\infty$; $L^p(U; \mathbb{R}^k)$ is the respective space of functions with values in $\mathbb{R}^k$. If $p \in [1, +\infty)$ and $E \subset \mathbb{R}^m$ is a Borel-measurable set of Hausdorff dimension $l \leq m$, then $L^p(E, \mathcal{H}^l)$ is the space of real measurable functions on E that are p^{th}-power integrable on E with respect to the $\mathcal{H}^l$-measure; $L^p(E, \mathbb{R}^k, \mathcal{H}^l)$ is the respective space of functions with values in $\mathbb{R}^k$. By $L^1_{\mathrm{loc}}(U)$ we denote the space of functions u such that $u \in L^1(V)$ for all $V \Subset U$; if $E \subset \mathbb{R}^m$ is a Borel-measurable set of Hausdorff dimension $l \leq m$, $L^1_{\mathrm{loc}}(E, \mathbb{R}^k, \mathcal{H}^l)$ denote the space of measurable functions on E with values in $\mathbb{R}^k$ such that $u \in L^1(F, \mathbb{R}^k, \mathcal{H}^l)$ for all $F \Subset E$. The norm on $L^p(U)$ ($L^p(U; \mathbb{R}^k)$) is denoted by $\|\cdot\|_{L^p(U)}$ ($\|\cdot\|_{L^p(U;\mathbb{R}^k)}$) or $\|\cdot\|_p$ when appropriate. We use the standard notation for Sobolev spaces. If $L : \mathbb{R}^m \to \mathbb{R}^l$ is a linear map, $L[h] \in \mathbb{R}^l$ will denote the value of L at the vector $h \in \mathbb{R}^m$. For each $x \in \mathcal{N}$ and for each $r > 0$, we shall denote by $B^{\mathcal{N}}_r(x)$ the geodesic ball in $\mathcal{N}$ with center x and radius r, namely $B^{\mathcal{N}}_r(x) = \{y \in \mathcal{N} : d_{\mathcal{N}}(y, x) \leq r\}$, where $d_{\mathcal{N}} : \mathcal{N} \times \mathcal{N} \to [0, +\infty)$ is the geodesic distance in $\mathcal{N}$. If $U \subset \mathbb{R}^l$ is a bounded Lipschitz domain, then $\mathrm{tr}_{\partial U}$ will denote the trace operator (the reader may consult, for instance, [29, Section 4.3]). We denote by $|\cdot|_{W^{s,p}(E,\mathbb{R}^k)}$ (or $|\cdot|_{W^{s,p}(E)}$ when appropriate) the canonical seminorm on $W^{s,p}(E, \mathbb{R}^k)$: if E is a Borel-measurable set of Hausdorff dimension l, then $|u|^p_{W^{s,p}(E,\mathbb{R}^k)} = \int_E \int_E \frac{|u(x)-u(y)|^p}{|x-y|^{l+sp}} \, d\mathcal{H}^l(x) \, d\mathcal{H}^l(y)$. By $\#X$ we shall denote the cardinality of X.

3.2.2 Embedding and nearest point retraction

For each $\lambda \in (0, +\infty)$, we define the neighborhood

$$\mathcal{N}_\lambda = \{x \in \mathbb{R}^\nu : \mathrm{dist}(x, \mathcal{N}) < \lambda\}.$$

The next lemma describes the nearest point retraction of a neighborhood of $\mathcal{N}$ onto $\mathcal{N}$.

Lemma 3.2.1. *There exists $\lambda_{\mathcal{N}} > 0$ such that the nearest point retraction $\Pi_{\mathcal{N}} : \mathcal{N}_{\lambda_{\mathcal{N}}} \to \mathcal{N}$ characterized by*

$$|x - \Pi_{\mathcal{N}}(x)| = \mathrm{dist}(x, \mathcal{N})$$

is a well-defined and smooth map. Moreover, if the mappings $P^{\top}_{\mathcal{N}} : \mathcal{N} \to \mathrm{Lin}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ and $P^{\perp}_{\mathcal{N}} : \mathcal{N} \to \mathrm{Lin}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ are defined for each $x \in \mathcal{N}$ by setting $P^{\top}_{\mathcal{N}}(x)$ and $P^{\perp}_{\mathcal{N}}(x)$ as the orthogonal projections on $T_x\mathcal{N}$ and $(T_x\mathcal{N})^{\perp}$, identified as linear subspaces of $\mathbb{R}^\nu$, then for each $x \in \mathcal{N}_{\lambda_{\mathcal{N}}}$ and $v \in \mathbb{R}^\nu$,

$$\left(1 - \frac{\mathrm{dist}(x, \mathcal{N})}{\lambda_{\mathcal{N}}}\right) |D\Pi_{\mathcal{N}}(x)[v]|^2 \leq |P^{\top}_{\mathcal{N}}(\Pi_{\mathcal{N}}(x))[v]|^2 \leq C|D\Pi_{\mathcal{N}}(x)[v]|^2,$$

where $C = C(\mathcal{N}, \nu) > 0$. Furthermore, for each $x \in \mathcal{N}_{\lambda_{\mathcal{N}}} \setminus \mathcal{N}$,

$$|D\,\mathrm{dist}(x, \mathcal{N})| = 1.$$

Proof. For a proof, we refer to [46, Lemma 2.1]. □

3.2.3 Sobolev mappings into manifolds

Let $\mathcal{X}$ and $\mathcal{Y}$ be compact Riemannian manifolds. Assume that $\mathcal{X}$ and $\mathcal{Y}$ are isometrically embedded into Euclidean spaces, which is always possible by Nash's isometric embedding theorem [50]. Notice that these embeddings are bilipschitz. In fact, if $\mathcal{M}$ is a compact Riemannian manifold, which is isometrically embedded into a Euclidean space, then, clearly, the Euclidean distance between two points of $\mathcal{M}$ is less than or equal to the geodesic distance between these points, which is induced by the Riemannian metric on $\mathcal{M}$. On the other hand, there exists a constant $L = L(\mathcal{M}) > 0$ such that the geodesic distance on $\mathcal{M}$ is less than or equal to the Euclidean distance on $\mathcal{M}$ times L (the idea behind the proof is that each compact Riemannian manifold $\mathcal{M}$ has a positive normal injectivity radius, thus, one can construct a smooth nearest point retraction from a tubular neighborhood of $\mathcal{M}$ onto $\mathcal{M}$ (see Lemma 3.2.1) and appropriately use the fact that for each open cover of $\mathcal{M}$ there is a finite subcover). Thus, identifying $\mathcal{X}$ and $\mathcal{Y}$ to their embedding's images, for each $s \in (0,1)$ and $p \in [1,+\infty)$, we can define the fractional Sobolev space $W^{s,p}(\mathcal{X},\mathcal{Y})$ as

$$W^{s,p}(\mathcal{X},\mathcal{Y}) = \{u \in L^p(\mathcal{X},\mathbb{R}^k,\mathcal{H}^l) : \\ u(x) \in \mathcal{Y} \text{ for a.e. } x \in \mathcal{X} \text{ and } |u|_{W^{s,p}(\mathcal{X},\mathbb{R}^k)} < +\infty\}, \tag{3.2.1}$$

where $\mathbb{R}^k$ is the Euclidean space into which $\mathcal{Y}$ is embedded and $l \in \mathbb{N}\backslash\{0\}$ is the Hausdorff dimension of $\mathcal{X}$ (the reader may also consult [24, Section 3.1]). This definition does not depend on the choice of the isometric Euclidean embedding of $\mathcal{Y}$. We shall consider in $W^{1,p}(\mathcal{X},\mathbb{R}^k)$ its strongly closed subset

$$W^{1,p}(\mathcal{X},\mathcal{Y}) = \{u \in W^{1,p}(\mathcal{X},\mathbb{R}^k) : u(x) \in \mathcal{Y} \text{ for a.e. } x \in \mathcal{X}\}$$

This definition does not depend on the choice of the isometric Euclidean embedding of $\mathcal{Y}$ (see, for instance, [15, Proposition 2.1] and [39, Section 3.1]; for intrinsic definitions of Sobolev spaces for Riemannian manifolds, see [39, Section 3.2]).

For convenience, we recall the definition of weak differentiability of mappings defined on bilipschitz images of open sets.

Definition 3.2.2. Let $U \subset \mathbb{R}^l$ be open and $\Phi : E \to U$ be bilipschitz. A map $u \in L^1_{\mathrm{loc}}(E,\mathbb{R}^k,\mathcal{H}^l)$ is said to be weakly differentiable if the map $u \circ \Phi^{-1}$ is weakly differentiable.

Since the weak differentiability is preserved under bilipschitz maps (see [59, Theorem 2.2.2]), the above definition is independent of the bilipschitz parametrization of E. Taking into account that E is bilipschitz homeomorphic to an open set $U \subset \mathbb{R}^l$, through a bilipschitz homeomorphism $\Phi : E \to U$, E admits the approximate tangent plane $T_x E$ for $\mathcal{H}^l$-a.e. $x \in E$ (for the definition of the approximate tangent plane, see, for example, [31, Section 3.2.16] and [30, Definition 4.3]). Thus, we can define the notion of the tangential derivative of Φ for $\mathcal{H}^l$-a.e. $x \in E$, as the linear map from the tangent plane $T_x E$ to $\mathbb{R}^l$. If $u \in L^1_{\mathrm{loc}}(E, \mathbb{R}^k, \mathcal{H}^l)$ is weakly differentiable, the weak (tangential) derivative of u at $\mathcal{H}^l$-a.e. $x \in E$ is defined by $D_\top u(x) = D(u \circ \Phi^{-1})(\Phi(x)) \circ D_\top \Phi(x)$, where $D_\top \Phi(x)$ is the tangential derivative of Φ at x (which is defined for $\mathcal{H}^l$-a.e. $x \in E$).

We shall denote by $W^{1,1}_{\mathrm{loc}}(E, \mathbb{R}^k)$ the space of all weakly differentiable mappings on E with values in $\mathbb{R}^k$. If $p \in [1, +\infty)$ and E is a bilipschitz image of an open set in $\mathbb{R}^l$, then we define

$$W^{1,p}(E, \mathbb{R}^k) = \{u \in W^{1,1}_{\mathrm{loc}}(E, \mathbb{R}^k) : |u|, |D_\top u| \in L^p(E, \mathcal{H}^l)\}.$$

Next, we define Sobolev spaces of mappings on bilipschitz images of open sets into a compact Riemannian manifold.

Definition 3.2.3. Let $p \in [1, +\infty)$ and E be a bilipschitz image of an open set in $\mathbb{R}^l$ with $l \in \mathbb{N} \setminus \{0\}$. Let $\mathcal{Y}$ be a compact Riemannian manifold isometrically embedded into $\mathbb{R}^k$. Then we define the Sobolev space $W^{1,p}(E, \mathcal{Y})$ by

$$W^{1,p}(E, \mathcal{Y}) = \{u \in W^{1,p}(E, \mathbb{R}^k) : u(x) \in \mathcal{Y} \text{ for a.e. } x \in E\}.$$

3.2.4 Topological resolution of the boundary datum

Since $\mathcal{N}$ is a connected Riemannian manifold, it is path-connected and its fundamental group, namely $\pi_1(\mathcal{N}, x_0)$, where $x_0 \in \mathcal{N}$, is, up to isomorphism, independent of the choice of a basepoint x_0 (see [38, Proposition 1.5]) and denoted by $\pi_1(\mathcal{N})$. Moreover, there is a one-to-one correspondence between the set of conjugacy classes in $\pi_1(\mathcal{N})$ and the set $[\mathbb{S}^1, \mathcal{N}]$ of free homotopy classes of maps $\mathbb{S}^1 \to \mathcal{N}$, that is, homotopy class without any condition on some basepoints (see Exercise 6 in [38, Section 1.1, p. 38]). In general, there is no group structure on $[\mathbb{S}^1, \mathcal{N}]$ (there exist groups G with the functor $L : G \to LG$ sending G to its set of conjugacy classes, which cannot be lifted to a group-valued functor; for instance, one can find an inclusion of groups $H \to G$ implying an inclusion of conjugacy classes $LH \to LG$ such that the order of LH does not divide the order of LG). After all, hereinafter, when we talk about "homotopy", we are talking about *free* homotopy (with no conditions on base points). We shall denote by $\#[\mathbb{S}^1, \mathcal{N}]$ the number of equivalence classes of $[\mathbb{S}^1, \mathcal{N}]$.

Next, for convenience, we recall the definition of a bounded Lipschitz domain.

Definition 3.2.4. A bounded domain $U \subset \mathbb{R}^l$, $l \geq 2$ and its boundary ∂U are said to be (locally) Lipschitz if there exist a radius $r_{\partial U}$ and a constant $\delta > 0$ such that for every point $x \in \partial U$ and every radius $s \in (0, r_{\partial U})$, up to a rotation of coordinates, it holds

$$U \cap B^l_s(x) = \{(y', y_l) \in \mathbb{R}^{l-1} \times \mathbb{R} : y_l > \varphi(y')\} \cap B^l_s(x)$$

for some Lipschitz function $\varphi : \mathbb{R}^{l-1} \to \mathbb{R}$ satisfying $\|D\varphi\|_{L^\infty(\mathbb{R}^{l-1})} \leq \delta$.

Given $k \in \mathbb{N}\backslash\{0\}$ and a bounded Lipschitz domain $U \subset \mathbb{R}^2$, we denote by $\mathrm{Conf}_k U$ the configuration space of k ordered points of U, namely

$$\mathrm{Conf}_k U = \{(a_1, \dots, a_k) \in U^k : a_i \neq a_j \text{ if } i \neq j\}. \tag{3.2.2}$$

Given $(a_1, \dots, a_k) \in \mathrm{Conf}_k U$, we define the quantity

$$\begin{aligned}\bar{\varrho}(a_1, \dots, a_k) = \min\Bigg\{ &\left\{ \frac{|a_i - a_j|}{2} : i, j \in \{1, \dots, k\} \text{ and } i \neq j \right\} \\ &\cup \Big\{ \mathrm{dist}(a_i, \partial U) : i \in \{1, \dots, k\} \Big\} \Bigg\},\end{aligned} \tag{3.2.3}$$

so that if $\varrho \in (0, \bar{\varrho}(a_1, \dots, a_k))$, we have $\overline{B}^2_\varrho(a_i) \cap \overline{B}^2_\varrho(a_j) = \emptyset$ for each $i, j \in \{1, \dots, k\}$ such that $i \neq j$ and $\overline{B}^2_\varrho(a_i) \subset U$ for each $i \in \{1, \dots, k\}$, and hence the set $U \setminus \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i)$ is connected and has a Lipschitz boundary $\partial(U \setminus \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i)) = \partial U \cup \bigcup_{i=1}^k \partial B^2_\varrho(a_i)$.

Definition 3.2.5. We say that $(\gamma_1, \dots, \gamma_k) \in C(\mathbb{S}^1, \mathcal{N})^k$ is a *topological resolution* of $g \in C(\partial U, \mathcal{N})$ whenever there exist $(a_1, \dots, a_k) \in \mathrm{Conf}_k U$, $\varrho \in (0, \bar{\varrho}(a_1, \dots, a_k))$ and $u \in C(U \setminus \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i), \mathcal{N})$ such that $u|_{\partial U} = g$ and for each $i \in \{1, \dots, k\}$, $u(a_i + \varrho \cdot)|_{\mathbb{S}^1} = \gamma_i$.

The definition of topological resolution is independent of the order of the curves in the k-tuple $(\gamma_1, \dots, \gamma_k)$. Furthermore, if $(b_1, \dots, b_k) \in \mathrm{Conf}_k U$ and $r \in (0, \bar{\varrho}(b_1, \dots, b_k))$, then there exists a homeomorphism between $U \setminus \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i)$ and $U \setminus \bigcup_{i=1}^k \overline{B}^2_r(b_i)$, which shows that the statement of the previous definition is independent of the choice of the points and of the radius. Similarly, if for each $i \in \{1, \dots, k\}$ the map $\widetilde{\gamma}_i$ is freely homotopic to γ_i and if $\widetilde{g}$ is freely homotopic to g, then, by definition of homotopy and the fact that an annulus is homeomorphic to a finite cylinder, we have that $(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_k)$ is also a topological resolution of $\widetilde{g}$. Thus, the property of being a topological resolution is invariant under homotopies.

Definition 3.2.5 can be extended to the case when $g : \partial U \to \mathcal{N}$ is a map of vanishing mean oscillation, namely $g \in \mathrm{VMO}(\partial U, \mathcal{N})$ and $(\gamma_1, \dots, \gamma_k) \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})^k$ (see [18, 19]). Indeed, if $\Gamma \simeq \mathbb{S}^1$ is a closed curve, path-connected components of the space $\mathrm{VMO}(\Gamma, \mathcal{N})$ are known to be the closure in $\mathrm{VMO}(\Gamma, \mathcal{N})$ of path-connected components of $C(\Gamma, \mathcal{N})$ (see [18, Lemma A.23]). Since $W^{1/2,2}(\partial U, \mathcal{N}) \subset \mathrm{VMO}(\partial U, \mathcal{N})$, Definition 3.2.5 extends also to the case when $g \in W^{1/2,2}(\partial U, \mathcal{N})$.

Definition 3.2.6. We say that $(\gamma_1, \dots, \gamma_k) \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})^k$ is a *topological resolution* of a map $g \in \mathrm{VMO}(\partial U, \mathcal{N})$ whenever $(\gamma_1, \dots, \gamma_k)$ is homotopic in $\mathrm{VMO}(\mathbb{S}^1, \mathcal{N})^k$ to a *topological resolution* $(\widetilde{\gamma}_1, \dots, \widetilde{\gamma}_k)$ of $\widetilde{g} \in C(\partial U, \mathcal{N})$ which is homotopic to g in $\mathrm{VMO}(\partial U, \mathcal{N})$.

Since Definition 3.2.5 is invariant under homotopies, and continuous maps are homotopic in VMO if and only if they are homotopic through a continuous homotopy, Definition 3.2.6 generalizes Definition 3.2.5.

3.2.5 Singular p-energy

First, we define the minimal length in the homotopy class of a map $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ by

$$\begin{aligned}(3.2.4)\quad \lambda(\gamma) = \inf\Big\{ \int_{\mathbb{S}^1} |\widetilde{\gamma}'| \, d\mathcal{H}^1 :& \\ \widetilde{\gamma} \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ and } \gamma \text{ are homotopic in } \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})\Big\},&\end{aligned}$$

where $|\cdot|$ is the norm induced by the Riemannian metric $g_{\mathcal{N}}$ of the manifold $\mathcal{N}$ on each fiber of the tangent bundle $T\mathcal{N}$, and since $\mathcal{N}$ is isometrically embedded into $\mathbb{R}^\nu$, $|\cdot|$ is the Euclidean norm.

For each $p \in [1, +\infty)$, using the parametrization by arc length and Hölder's inequality if $p \in (1, +\infty)$, we have

$$\begin{aligned}\inf\Big\{ \int_{\mathbb{S}^1} \frac{|\widetilde{\gamma}'|^p}{p} \, d\mathcal{H}^1 : \widetilde{\gamma} \in C^1(\mathbb{S}^1, \mathcal{N})& \\ \text{and } \gamma \text{ are homotopic in } \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})\Big\} &= \frac{\lambda(\gamma)^p}{(2\pi)^{p-1} p}.\end{aligned}$$

In particular, if γ is a minimizing closed geodesic, then

$$\lambda(\gamma) = \int_{\mathbb{S}^1} |\gamma'| \, d\mathcal{H}^1 = \left((2\pi)^{p-1} p \int_{\mathbb{S}^1} \frac{|\gamma'|^p}{p} \, d\mathcal{H}^1 \right)^{\frac{1}{p}} = 2\pi \|\gamma'\|_\infty,$$

since $|\gamma'|$ is constant for a minimizing geodesic.

Recall that the *systole* of the compact manifold $\mathcal{N}$ is the length of the shortest closed nontrivial geodesic on $\mathcal{N}$, namely,

$$\begin{aligned}(3.2.5)\quad &\operatorname{sys}(\mathcal{N})\\ &\quad = \inf\{\lambda(\gamma) : \gamma \in C^1(\mathbb{S}^1, \mathcal{N}) \text{ is not homotopic to a constant}\}.\end{aligned}$$

If $[\mathbb{S}^1, \mathcal{N}] \simeq \{0\}$, we set $\operatorname{sys}(\mathcal{N}) = +\infty$. Notice that, if $[\mathbb{S}^1, \mathcal{N}] \not\simeq \{0\}$, then for each map $\gamma \in \operatorname{VMO}(\mathbb{S}^1, \mathcal{N})$, it holds $\lambda(\gamma) \in \{0\} \cup [\operatorname{sys}(\mathcal{N}), +\infty)$.

We now define the singular p-energy of a map $g \in \operatorname{VMO}(\partial U, \mathcal{N})$, where $U \subset \mathbb{R}^2$ is a bounded Lipschitz domain. This notion was previously introduced for $p = 2$ in [47] and generalized in [57] to $p \in [1, +\infty)$.

Definition 3.2.7. If $p \in [1, +\infty)$, $U \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $g \in \operatorname{VMO}(\partial U, \mathcal{N})$, we define the *singular p-energy* of g by

$$\begin{aligned}(3.2.6)\quad \mathcal{E}_p^{\mathrm{sg}}(g) = \inf\Bigg\{ &\sum_{i=1}^{k} \frac{\lambda(\gamma_i)^p}{(2\pi)^{p-1}p} :\\ &k \in \mathbb{N} \setminus \{0\} \text{ and } (\gamma_1, \dots, \gamma_k) \text{ is a topological resolution of } g \Bigg\}.\end{aligned}$$

Observe that $\mathcal{E}_p^{\mathrm{sg}}$ is invariant under homotopies and depends continuously on p. For each $g \in \operatorname{VMO}(\mathbb{S}^1, \mathbb{S}^1)$, $\mathcal{E}_p^{\mathrm{sg}}(g) = 2\pi|\deg(g)|^p/p$, where $\pi_1(\mathbb{S}^1) \simeq \mathbb{Z}$. If $\mathcal{N} = \mathbb{S}^2$, then $\pi_1(\mathcal{N})$ is trivial and for each $g \in \operatorname{VMO}(\mathbb{S}^1, \mathbb{S}^2)$, $\mathcal{E}_p^{\mathrm{sg}}(g) = 0$. If $\mathcal{N} = \{n \otimes n - \frac{1}{3}\mathrm{Id} : n \in \mathbb{S}^2\}$ and $g(t) = v(t) \otimes v(t) - \frac{1}{3}\mathrm{Id}$, where $v(t) = (\cos(t/2), \sin(t/2), 0) \in \mathbb{S}^2$ for each $t \in [0, 2\pi]$, then $\mathcal{N}$ is diffeomorphic to $\mathbb{RP}^2$, $g \in C^\infty(\mathbb{S}^1, \mathcal{N})$ and $\mathcal{E}_p^{\mathrm{sg}}(g) = 2^{\frac{2-p}{2}}\pi/p$ (see [22, Lemma 19]). The reader may consult [47, Section 9.3] for other interesting examples in the case where $p = 2$.

Lemma 3.2.8. *Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain. Then for each $g \in \operatorname{VMO}(\partial U, \mathcal{N})$, the function $p \in [1, +\infty) \mapsto \mathcal{E}_p^{\mathrm{sg}}(g)$ is locally bounded and locally Lipschitz continuous. In particular, if $\mathcal{N}$ is simply connected, then $\mathcal{E}_p^{\mathrm{sg}}(g) = 0$ for each $p \in [1, +\infty)$. Furthermore, the infimum in* (3.2.6) *is actually a minimum and the set $\{\mathcal{E}_p^{\mathrm{sg}}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}$ is finite.*

Proof. If $\mathcal{N}$ is simply connected, then each $\gamma \in \operatorname{VMO}(\partial U, \mathcal{N})$ is homotopic to a constant map, which yields that $\lambda(\gamma) = 0$ and hence $\mathcal{E}_p^{\mathrm{sg}}(g) = 0$.

Otherwise, $[\mathbb{S}^1, \mathcal{N}] \neq \{0\}$ and then $\operatorname{sys}(\mathcal{N}) \in (0, +\infty)$. This comes, since $\mathcal{N}$ is compact, and hence the length spectrum (i.e., the set of nonnegative real numbers $\{\lambda(\gamma) : \gamma \in \operatorname{VMO}(\mathbb{S}^1, \mathcal{N})\}$) is discrete (for a proof, see [46, Proposition 3.2]). Then, according to [57, Lemma 2.10],

$p \mapsto \mathcal{E}_p^{\mathrm{sg}}(g)$ is locally bounded and locally Lipschitz continuous. The fact that the infimum in (3.2.6) is a minimum comes from the discreteness of the length spectrum. Finally, since $\mathcal{E}_p^{\mathrm{sg}}$ is invariant under homotopies and $\#[\mathbb{S}^1, \mathcal{N}] < +\infty$ by our assumption on $\mathcal{N}$, the set $\{\mathcal{E}_p^{\mathrm{sg}}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}$ is finite. This completes our proof of Lemma 3.2.8. □

It is also worth recalling the notion of a *minimal topological resolution* (when $p = 2$, see [47, Definition 2.7]).

Definition 3.2.9. A topological resolution $(\gamma_1, \dots, \gamma_k)$ of g is said to be p-minimal if

$$\mathcal{E}_p^{\mathrm{sg}}(g) = \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^p}{(2\pi)^{p-1} p}$$

and $\lambda(\gamma_i) > 0$ for each $i \in \{1, \dots, k\}$.

3.2.6 Renormalized energy of renormalizable maps

We recall the definition of a renormalizable map (see [47, Definition 7.1]).

Definition 3.2.10. Given an open set $U \subset \mathbb{R}^2$, a map $u : U \to \mathcal{N}$ is said to be *renormalizable* whenever there exists a finite set $\mathcal{S} \subset U$ such that for each sufficiently small radius $\varrho > 0$, $u \in W^{1,2}(U \setminus \bigcup_{a \in \mathcal{S}} \overline{B}^2_\varrho(a), \mathcal{N})$ and its *renormalized* energy $\mathcal{E}_2^{\mathrm{ren}}(u)$ is finite, where

$$\begin{aligned} \text{(3.2.7)} \quad & \mathcal{E}_2^{\mathrm{ren}}(u) \\ & = \liminf_{\varrho \to 0+} \int_{U \setminus \bigcup_{a \in \mathcal{S}} \overline{B}^2_\varrho(a)} \frac{|Du|^2}{2}\,dx - \sum_{a \in \mathcal{S}} \frac{\lambda(\mathrm{tr}_{\partial B^2_\varrho(a)}(u))^2}{4\pi} \ln\left(\frac{1}{\varrho}\right). \end{aligned}$$

We denote by $W^{1,2}_{\mathrm{ren}}(U, \mathcal{N})$ the set of renormalizable maps.

For each renormalizable map $u : U \to \mathcal{N}$, there exists the *minimal* set $\mathcal{S}_0$ for which it holds $u \in W^{1,2}(U \setminus \bigcup_{a \in \mathcal{S}_0} \overline{B}^2_\varrho(a), \mathcal{N})$ for each small enough $\varrho > 0$. If $a \in U \setminus \mathcal{S}_0$, then $\lambda(\mathrm{tr}_{\partial B^2_\varrho(a)}(u)) = 0$ for $\varrho > 0$ small enough. Thus, the $\liminf$ defining $\mathcal{E}_2^{\mathrm{ren}}$ in (3.2.7) does not depend on the set $\mathcal{S}$. If $\mathcal{S}_0 = \emptyset$, then $u \in W^{1,2}(U, \mathcal{N})$ and $\mathcal{E}_2^{\mathrm{ren}}(u) = \int_U \frac{|Du|^2}{2}\,dx$. If $\mathcal{S}_0 = \{a_1, \dots, a_k\}$ with $k \in \mathbb{N} \setminus \{0\}$ and $(a_1, \dots, a_k) \in \mathrm{Conf}_k(U)$ (see (3.2.2)), then, according to [47, Lemma 2.11], the map

$$\begin{aligned} \text{(3.2.8)} \quad & \varrho \in (0, \bar{\varrho}(a_1, \dots, a_k)) \\ & \mapsto \int_{U \setminus \bigcup_{i=1}^k \overline{B}^2_\varrho(a_i)} \frac{|Du|^2}{2}\,dx - \sum_{i=1}^{k} \frac{\lambda(\mathrm{tr}_{\partial B^2_\varrho(a_i)}(u))^2}{4\pi} \ln\left(\frac{1}{\varrho}\right) \end{aligned}$$

is nonincreasing, where $\bar{\varrho}(a_1, \ldots, a_k) \in (0, +\infty)$ is defined in (3.2.3). The main ingredients of the proof of [47, Lemma 2.11] are the use of the additivity of the integral and the estimate of the energy on the annulus. The latter comes from the application of the special case of the coarea formula and the comparison of the angular energy taking into account the definition of the minimal length in the homotopy class of a map $\gamma \in \mathrm{VMO}(\mathbb{S}^1, \mathcal{N})$ (see (3.2.4)).

Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain, $u \in W^{1,2}_{\mathrm{ren}}(U, \mathcal{N})$ and $S_0 = \{a_1, \ldots, a_k\}$ for some $k \in \mathbb{N} \setminus \{0\}$, where $(a_1, \ldots, a_k) \in \mathrm{Conf}_k U$. By [47, Theorem 5.1] (see also [57, Corollary 5.13]), $Du \in L^{2,\infty}(U, \mathbb{R}^\nu \otimes \mathbb{R}^2) \subset L^p(U, \mathbb{R}^\nu \otimes \mathbb{R}^2)$ for each $p \in [1, 2)$ (see, for instance, [25, Theorem 5.9]), where $L^{2,\infty}(U, \mathbb{R}^\nu \otimes \mathbb{R}^2)$ denotes the Marcinkiewicz space or endpoint Lorentz space. In addition, we can define the p-renormalized energy of u by

$$\mathcal{E}^{\mathrm{ren}}_p(u) = \int_U \frac{|Du|^p}{p}\, dx - \sum_{i=1}^{k} \frac{\lambda(\mathrm{tr}_{\partial B^2_\varrho(a_i)}(u))^p}{(2-p)(2\pi)^{p-1}p}, \tag{3.2.9}$$

which is finite and tends to $\mathcal{E}^{\mathrm{ren}}_2(u)$ as $p \nearrow 2$.

Proposition 3.2.11. *Let $U \subset \mathbb{R}^2$ be a bounded Lipschitz domain. If $u \in W^{1,2}_{\mathrm{ren}}(U, \mathcal{N})$, then*

$$\mathcal{E}^{\mathrm{ren}}_p(u) \to \mathcal{E}^{\mathrm{ren}}_2(u) \tag{3.2.10}$$

as $p \nearrow 2$.

Proof. For a proof, see [57, Proposition 2.17]. □

Lemma 3.2.12. *Let $\gamma \in C^1(\mathbb{S}^1, \mathcal{N})$ be a minimizing geodesic in its (free) homotopy class. Then there exists a renormalizable map $v \in W^{1,2}_{\mathrm{ren}}(B^2_1, \mathcal{N})$ satisfying the following conditions.*

(i) $\mathrm{tr}_{\mathbb{S}^1}(v) = \gamma$. *If γ is a constant map, then v is the constant map.*

(ii) For each $p \in [1, 2)$, the following estimate holds

$$\int_{B^2_1} \frac{|Dv|^p}{p}\, dx = \mathcal{E}^{\mathrm{ren}}_p(v) + \frac{\mathcal{E}^{\mathrm{sg}}_p(\gamma)}{2-p}.$$

Proof. If γ is a constant map, then we define v to be the constant map taking the same value in $\mathcal{N}$ as γ. Clearly, in this case, the assertion (ii) holds.

If γ is not homotopic to a constant, we apply [47, Proposition 8.3]. A key step for the proof of [47, Proposition 8.3] is to use the estimate of [47, Lemma 2.11], which is based on the comparison of the angular energy. In particular, let $(\gamma_1, \dots, \gamma_k)$ be a 2-minimal topological resolution of γ (see Definition 3.2.9) such that for each $j \in \{1, \dots, k\}$, $\gamma_j \in C^1(\mathbb{S}^1, \mathcal{N})$ is a minimizing geodesic. Then, using [47, Proposition 8.3], together with (3.2.9), we deduce that there exists $v \in W^{1,2}_{\mathrm{ren}}(B_1^2, \mathcal{N})$ satisfying

$$\mathcal{E}_p^{\mathrm{ren}}(v) = \int_{B_1^2} \frac{|Dv|^p}{p}\, dx - \sum_{i=1}^{k} \frac{\lambda(\gamma_i)^p}{(2-p)(2\pi)^{p-1}p} = \int_{B_1^2} \frac{|Dv|^p}{p}\, dx - \frac{\mathcal{E}_p^{\mathrm{sg}}(\gamma)}{2-p},$$

which completes our proof of Lemma 3.2.12. □

3.2.7 Several extension results

We obtain an extension of the boundary datum with a *linear*-type estimate.

Lemma 3.2.13. *Let $m \in \mathbb{N} \setminus \{0, 1\}$, $p \in [1, 2]$, $\widetilde{g} \in W^{1,p}(\partial B_r^m(x_0), \widetilde{\mathcal{N}})$ and $g = \pi \circ \widetilde{g}$, where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the universal covering of $\mathcal{N}$. Then there exists a map $w \in W^{1,p}(B_r^m(x_0), \mathcal{N})$ such that $\mathrm{tr}_{\partial B_r^m(x_0)}(w) = g$ and*

$$\|Dw\|^p_{L^p(B_r^m(x_0))} \leq Cr\|D_\top g\|^p_{L^p(\partial B_r^m(x_0))}, \tag{3.2.11}$$

where $C = C(m, \mathcal{N}) > 0$. Moreover, $w \in W^{1,2}(B_r^m(x_0), \mathcal{N})$.

Remark 3.2.14. If $m \geq 3$, then w can be chosen as the homogeneous extension of g. In this case, (3.2.11) holds with the constant $C = \frac{1}{m-p} \leq 1$ and without assuming that $\pi_1(\mathcal{N})$ is finite and g has a lifting. It is also worth mentioning that for each $p \in (1, 2)$, the estimate (3.2.11) holds without assuming that g has a lifting, but the constant C in (3.2.11) then behaves like $C \sim \frac{1}{2-p}$.

Proof of Lemma 3.2.13. Since the manifold $\mathcal{N}$ is compact and its fundamental group $\pi_1(\mathcal{N})$ is finite, its covering space $\widetilde{\mathcal{N}}$ is compact. Notice that $g \in W^{1,p}(\partial B_r^m(x_0), \mathcal{N})$, since $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is a local isometry. According to Nash's embedding theorem [50], $\widetilde{\mathcal{N}}$ can be isometrically embedded into $\mathbb{R}^k$ for some $k \in \mathbb{N} \setminus \{0\}$. If $p = 1$, we define $w(x) := g(x_0 + r(x - x_0)/|x - x_0|)$ for $x \in B_r^m(x_0) \setminus \{x_0\}$ (see [53, Proposition 2.4]). Then $w \in W^{1,1}(B_r^m(x_0), \mathcal{N})$ and

$$\int_{B_r^m(x_0)} |Dw|\, dx = \frac{r}{m-1} \int_{\partial B_r^m(x_0)} |D_\top g|\, d\mathcal{H}^{m-1},$$

which proves (3.2.11) for $p = 1$. If $p \in (1, 2]$, according to [41, Section 6.9], $\widetilde{g}$ admits the extension $\widetilde{v} \in C^\infty(B_r^m(x_0), \mathbb{R}^k)$ satisfying the estimate

$$\|D\widetilde{v}\|^p_{L^p(B_r^m(x_0))} \leq C|\widetilde{g}|^p_{W^{1-1/p,p}(\partial B_r^m(x_0))}, \tag{3.2.12}$$

where $C = C(m) > 0$. The construction of the extension $\widetilde{v}$ is standard: since $\partial B_r^m(x_0)$ is smooth, using a partition of unity, the problem reduces to constructing for a given trace $u \in W^{1-1/p,p}(Q)$, where $Q = \{(x_1, x_2) \in \mathbb{R}^2 : |x_i| < 1,\ i = 1,2\}$, a real-valued function $U \in W^{1,p}(P)$, where $P = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : 0 < x_3 < 1,\ |x_i| < 1 - x_3\}$, by the convolution of u with a kernel φ satisfying [41, (2.5.1.1)]. Namely, first, extend u on $\mathbb{R}^2 \setminus Q$ by $u = 0$ and next define

$$U(x) = \frac{1}{x_3^2} \int_{|x'-y'|<x_3} \varphi\left(\frac{x'-y'}{x_3}\right) u(y')\, dy',$$

where $x = (x', x^3) \in \mathbb{R}^3$, $x_3 > 0$. Then, proceeding as in [35, Theorem 6.2], namely, using Sard's Theorem and choosing an appropriate locally Lipschitz retraction [35, Lemma 6.1], we define $\widetilde{w} \in W^{1,2}(B_r^m(x_0), \mathcal{N})$ (generally speaking, the statement of [35, Theorem 6.2] provides us with a $W^{1,p}$ extension, however, using the fact that $\pi_1(\widetilde{\mathcal{N}})$ is trivial, one observes that the locally Lipschitz retraction coming from [35, Lemma 6.1] belongs to $W^{1,q}_{\mathrm{loc}}(\mathbb{R}^k, \mathcal{N})$ for each $q \in [1,3)$, which, since $\widetilde{g} \in W^{1/2,2}(\partial B_r^m(x_0), \mathcal{N})$, implies that the extension coming from [35, Theorem 6.2] is actually a $W^{1,2}$ extension in this particular case) such that $\mathrm{tr}_{\partial B_r^m(x_0)}(\widetilde{w}) = \widetilde{g}$ and, in view of (3.2.12),

$$\begin{aligned}\|D\widetilde{w}\|^p_{L^p(B_r^m(x_0))} &\le C'|\widetilde{g}|^p_{W^{1-1/p,p}(\partial B_r^m(x_0))} \\ &\le C'' r \|D_\top \widetilde{g}\|^p_{L^p(\partial B_r^m(x_0))},\end{aligned} \tag{3.2.13}$$

where $C' = C'(m, \widetilde{\mathcal{N}}) > 0$, $C'' = C''(m, \widetilde{\mathcal{N}}) > 0$ and the last estimate comes from (A.8). Defining $w := \pi \circ \widetilde{w}$ and using (3.2.13), together with the fact that π is a local isometry, we deduce (3.2.11) for $p \in (1,2)$ (since the universal covering is unique up to a diffeomorphism, we can actually assume that C'' depends only on m and $\mathcal{N}$). This completes our proof of Lemma 3.2.13. □

The next lemma provides us with a *nonlinear*-type estimate on the extension of the boundary datum, in contrast to a *linear*-type estimate provided by Lemma 3.2.13.

Lemma 3.2.15. *Let $m \in \mathbb{N} \setminus \{0,1\}$, $p_0 \in (1,2)$, $p \in [p_0, 2)$, $\widetilde{g} \in W^{1,p}(\partial B_r^m(x_0), \widetilde{\mathcal{N}})$, $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ be the universal covering of $\mathcal{N}$ and $g = \pi \circ \widetilde{g}$. Then there exists a map $w \in W^{1,p}(B_r^m(x_0), \mathcal{N})$ such that $\mathrm{tr}_{\partial B_r^m(x_0)}(w) = g$ and*

$$\|Dw\|^p_{L^p(B_r^m(x_0))} \le C r^{\frac{m-1}{p}} \|D_\top g\|^{p-1}_{L^p(\partial B_r^m(x_0))}, \tag{3.2.14}$$

where $C = C(p_0, m, \mathcal{N}) > 0$.

Remark 3.2.16. The estimate (3.2.14) is based on the fact that $\pi_1(\mathcal{N})$ is finite, since in this case $\widetilde{\mathcal{N}}$ is compact (due to the compactness of $\mathcal{N}$, the manifold $\widetilde{\mathcal{N}}$ is compact if and only if $\pi_1(\mathcal{N})$ is finite). More precisely, to obtain (3.2.14), we have used that the diameter of the manifold $\widetilde{\mathcal{N}}$ (which is identified with its Euclidean embedding) is finite. We have used the constant p_0 to avoid the dependence of the constant C in (3.2.14) on p when p is close to 1 (see (A.1)). Notice also that if p is close to 2, the estimate (3.2.14) is valid without assuming that g has a lifting, but the constant C in (3.2.14) then behaves like $C \sim \frac{1}{2-p}$.

Proof of Lemma 3.2.15. Let $\widetilde{w} \in W^{1,2}(B^m_r(x_0), \widetilde{\mathcal{N}})$ be the map as in the proof of Lemma 3.2.13. Using (3.2.12) and (A.9), we deduce the estimate

$$\|D\widetilde{w}\|^p_{L^p(B^m_r(x_0))} \leq Cr^{\frac{m-1}{p}} \|D_\top \widetilde{g}\|^{p-1}_{L^p(\partial B^m_r(x_0))}, \tag{3.2.15}$$

where $C = C(p_0, m, \mathcal{N}) > 0$. Define $w = \pi \circ \widetilde{w}$. Since π is a local isometry, (3.2.15) implies (3.2.14), which concludes our proof of Lemma 3.2.15. □

For convenience, we recall the next proposition.

Lemma 3.2.17. *Let $m \in \mathbb{N} \setminus \{0, 1\}$, $p \in [1, +\infty)$, $u \in W^{1,p}(B^m_r(x_0), \mathcal{Y})$ and $v \in W^{1,p}(\partial B^m_r(x_0), \mathcal{Y})$, where $\mathcal{Y}$ is either a Euclidean space or a compact Riemannian manifold which is isometrically embedded into $\mathbb{R}^k$ for some $k \in \mathbb{N} \setminus \{0\}$. Then the following assertions hold.*

(i) For $\mathcal{H}^1$-a.e. $\varrho \in (0, r)$, $\operatorname{tr}_{\partial B^m_\varrho(x_0)}(u) = u|_{\partial B^m_\varrho(x_0)} \in W^{1,p}(\partial B^m_\varrho(x_0), \mathcal{Y})$.

(ii) Assume that $E \subset \partial B^m_r(x_0)$ is bilipschitz homeomorphic to an open bounded set in $\mathbb{R}^l$ with $l \leq m-1$. Then for $\mathcal{H}^{m(m-1)/2}$-a.e. $\omega \in SO(m)$, $\operatorname{tr}_{\omega(E)}(v) = v|_{\omega(E)} \in W^{1,p}(\omega(E), \mathcal{Y})$.

Proof. The proof follows using the appropriate coarea formula (see [29, Theorem 3.12] for the assertion (i) and [31, Theorem 3.2.48] for the assertion (ii)) and the fact that each map $u \in W^{1,p}(B^m_r(x_0), \mathbb{R}^k)$ and $v \in W^{1,p}(\partial B^m_r(x_0), \mathbb{R}^k)$ can be approximated by smooth maps in $C^\infty(B^m_r(x_0), \mathbb{R}^k)$ and $C^\infty(\partial B^m_r(x_0), \mathbb{R}^k)$, respectively. □

Lemma 3.2.18. *Let $p \in [1, 2)$ and $g \in W^{1,p}(\partial B^2_r(x_0), \mathcal{N})$. Then there exists $u \in W^{1,p}(B^2_r(x_0), \mathcal{N})$ such that $\operatorname{tr}_{\partial B^2_r(x_0)}(u) = g$ and*

$$\int_{B^2_r(x_0)} \frac{|Du|^p}{p}\, dx \leq Cr \int_{\partial B^2_r(x_0)} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^1 + \frac{\mathcal{E}^{\mathrm{sg}}_p(g) r^{2-p}}{2-p}, \tag{3.2.16}$$

where $C = C(\mathcal{N}) > 0$.

Proof. Up to scaling and translation, we can assume that $r = 1$ and $x_0 = 0$. First, if g were homotopic to a constant map, then we would have $\mathcal{E}_p^{\mathrm{sg}}(g) = 0$ and, according to Lemma 3.2.13, (3.2.16) would hold.

Thus, we can assume that g is not homotopic to a constant. Let $\gamma \in C^1(\mathbb{S}^1, \mathcal{N})$ be a minimizing geodesic in the (free) homotopy class of g. Hereinafter in this proof, C will denote a positive constant that can only depend on $\mathcal{N}$ and can be different from line to line. Let $v \in W^{1,2}_{\mathrm{ren}}(B_1^2, \mathcal{N})$ be the map of Lemma 3.2.12 such that $\operatorname{tr}_{\mathbb{S}^1}(v) = \gamma$. Define $u(x) = v(2x)$ for each $x \in B^2_{1/2}$. Using Lemma 3.2.12 (ii) and the fact that g is homotopic to γ, we get

$$\begin{aligned}(3.2.17)\quad \int_{B^2_{1/2}} \frac{|Du|^p}{p}\, dx &= 2^{p-2} \int_{B_1^2} \frac{|Dv|^p}{p}\, dx \\ &\le \mathcal{E}_p^{\mathrm{ren}}(v) + \frac{\mathcal{E}_p^{\mathrm{sg}}(g)}{2-p} \le C + \frac{\mathcal{E}_p^{\mathrm{sg}}(g)}{2-p},\end{aligned}$$

where the last estimate comes, since the value $\mathcal{E}_p^{\mathrm{ren}}(v)$ depends only on p and γ and tends to $\mathcal{E}_2^{\mathrm{ren}}(v)$ as $p \nearrow 2$ (see Proposition 3.2.11), and hence it is bounded by the constant depending only on $\mathcal{N}$, since there are finitely many free homotopy classes and we can fix a geodesic in each one.

Finally, we define u on the annulus $B_1^2 \setminus B^2_{1/2}$. Set $f(x) := \gamma(2x)$ for each $x \in \partial B^2_{1/2}$. In view of the Morrey embedding theorem, we can assume that $g \in C(\mathbb{S}^1, \mathcal{N})$. Then g is continuously homotopic to f. Restricting a continuous homotopy between the mappings g and f to the segment $[\sigma/2, \sigma] \subset \overline{B}_1^2 \backslash B^2_{1/2}$ for some $\sigma \in \mathbb{S}^1$, we obtain a map $h \in C([\sigma/2, \sigma], \mathcal{N})$ such that $h(\sigma/2) = f(\sigma/2)$ and $h(\sigma) = g(\sigma)$. Next, observe that $B_1^2 \backslash \overline{B}^2_{1/2}$ is bilipschitz homeomorphic to $(0,1)^2/\sim$, where $\sim$ is the equivalence relation identifying the sides $[(0,0),(0,1)]$ and $[(1,0),(1,1)]$. Thus, we obtain the map $w_0 \in C(\partial(0,1)^2, \mathcal{N})$ taking the values of g on $[(0,0),(1,0)]$, of f on $[(0,1),(1,1)]$ and of h on $[(0,0),(0,1)] \cup [(1,0),(1,1)]$. Then w_0 is continuously homotopic to a constant, and the classical theory of liftings says that there exists $\widetilde{w}_0 \in C(\partial(0,1)^2, \widetilde{\mathcal{N}})$ such that $w_0 = \pi \circ \widetilde{w}_0$, where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the universal covering map. Let $\widetilde{c} \in C^1([0,1], \widetilde{\mathcal{N}})$ be a minimizing geodesic joining $\widetilde{w}_0((0,0))$ and $\widetilde{w}_0((0,1))$. Since $\pi_1(\mathcal{N})$ is finite and $\mathcal{N}$ is compact, $\widetilde{\mathcal{N}}$ is compact and hence there exists $L = L(\mathcal{N}) > 0$ such that $\max\{\int_0^1 |\widetilde{c}'|^p\, dt : p \in [1,2]\} \le L$ (actually, L depends on $\widetilde{\mathcal{N}}$, but the universal covering is unique up to a diffeomorphism, and we can assume that L depends only on $\mathcal{N}$). Now we define the map $\widetilde{w} : \partial(0,1)^2 \to \widetilde{\mathcal{N}}$ taking the values of $\widetilde{w}_0$ on $[(0,0),(1,0)] \cup [(0,1),(1,1)]$ and of $\widetilde{c}$ on $[(0,0),(0,1)] \cup [(1,0),(1,1)]$. Define $w = \pi \circ \widetilde{w}$. Then w takes the values of g on $[(0,0),(1,0)]$, of $\pi \circ \widetilde{c}$ on $[(0,0),(0,1)] \cup [(1,0),(1,1)]$

and the values of f on $[(0,1),(1,1)]$. According to Lemma 3.2.13 with B_1^2 replaced by $(0,1)^2$ ($(0,1)^2$ is bilipschitz homeomorphic to B_1^2 (see [33])), there exists $W \in W^{1,p}((0,1)^2, \mathcal{N})$ such that

$$\begin{aligned}\int_{(0,1)^2} \frac{|DW|^p}{p}\, dx &\le C \int_{\partial(0,1)^2} \frac{|D_\top w|^p}{p}\, d\mathcal{H}^1 \\ &\le C\bigg(\int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^1 + \int_{\mathbb{S}^1} \frac{|D_\top \gamma|^p}{p}\, d\mathcal{H}^1 \\ &\qquad + \int_0^1 \frac{|(\pi \circ \widetilde{c})'|^p}{p}\, dt \bigg) \\ &\le C \int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^1 + C,\end{aligned}$$

where we have used that π is a local isometry and that $\int_0^1 |\widetilde{c}'|^p\, dt \le C$. Passing to the quotient $(0,1)^2 \to B_1^2 \setminus \overline{B}^2_{1/2}$, we obtain the map $\varphi \in W^{1,p}(B_1^2 \setminus \overline{B}^2_{1/2}, \mathcal{N})$ such that $\operatorname{tr}_{\mathbb{S}^1}(\varphi) = g$, $\operatorname{tr}_{\partial B^2_{1/2}}(\varphi) = f$ and

$$\int_{B_1^2 \setminus \overline{B}^2_{1/2}} \frac{|D\varphi|^p}{p}\, dx \le C \int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^1 + C. \tag{3.2.18}$$

Next, using (3.2.5) and if necessary Hölder's inequality, we have

$$\begin{aligned}\operatorname{sys}(\mathcal{N}) &\le \int_{\mathbb{S}^1} |D_\top g|\, d\mathcal{H}^1 \\ &\le (2\pi)^{1-\frac{1}{p}} \left(\int_{\mathbb{S}^1} |D_\top g|^p\, d\mathcal{H}^1 \right)^{\frac{1}{p}} = (2\pi)^{1-\frac{1}{p}} \|D_\top g\|_{L^p(\mathbb{S}^1)}.\end{aligned}$$

This, together with the facts that $p \in [1,2)$ and $\operatorname{sys}(\mathcal{N}) \in (0, +\infty)$ (since g is not nullhomotopic and $\mathcal{N}$ is compact), implies that

$$\int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^1 \ge \frac{1}{C}. \tag{3.2.19}$$

Define $u(x) = \varphi(x)$ for each $x \in B_1^2 \setminus \overline{B}^2_{1/2}$. Then $u \in W^{1,p}(B_1^2, \mathcal{N})$, $\operatorname{tr}_{\mathbb{S}^1}(u) = g$ and combining (3.2.17), (3.2.18) and (3.2.19), we deduce the following

$$\int_{B_1^2} \frac{|Du|^p}{p}\, dx \le C \int_{\mathbb{S}^1} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^1 + \frac{\mathcal{E}_p^{\mathrm{sg}}(g)}{2-p},$$

which completes our proof of Lemma 3.2.18. □

To lighten the notation, for every $x_0 \in \mathbb{R}^3$, $r > 0$ and $L > 0$, we define the cylinder and its lateral surface, namely

$$\Lambda_{r,L}(x_0) := B_r^2(x_0) \times (-L, L) \tag{3.2.20}$$

$$\Gamma_{r,L}(x_0) = \partial B_r^2(x_0) \times (-L, L). \tag{3.2.21}$$

Remark 3.2.19. The next construction is very reminiscent the one proposed by Rubinstein and Sternberg (see [52, Section 2]) in the case when the target manifold is the circle $\mathbb{S}^1$, which was extended by Brezis, Li, Mironescu and Nirenberg (see [16, Section 1]) to the case where the targed manifold is compact, connected and oriented. Let $\Gamma = \Gamma_{r,L}(x_0)$ for some $x_0 \in \mathbb{R}^3$, $r, L > 0$. Then, for each $g \in W^{1,2}(\Gamma, \mathcal{N})$, we can define the so-called "homotopy class" of g. Indeed, according to [54, Proposition, p. 267], there exists $(g_n)_{n\in\mathbb{N}} \subset C^\infty(\Gamma, \mathcal{N})$ such that $g_n \to g$ strongly in $W^{1,2}(\Gamma, \mathbb{R}^\nu)$. Then, using the coarea formula and the Sobolev embedding, we observe that for $\mathcal{H}^1$-a.e. $z \in (-L, L)$, $\operatorname{tr}_{\partial B_r^2(x_0)\times\{z\}}(g) = g|_{\partial B_r^2(x_0)\times\{z\}}$ is continuous, belongs to $W^{1,2}(\partial B_r^2(x_0) \times \{z\}, \mathcal{N})$ and

$$g_n|_{\partial B_r^2(x_0)\times\{z\}} \to g|_{\partial B_r^2(x_0)\times\{z\}}$$

uniformly. This implies that, for $\mathcal{H}^1$-a.e. $z_1, z_2 \in (-L, L)$, $g|_{\partial B_r^2(x_0)\times\{z_1\}}$ is continuously homotopic to $g|_{\partial B_r^2(x_0)\times\{z_2\}}$, and we can define the "homotopy class" of g, namely $[g]_\Gamma$, by setting $[g]_\Gamma := [g|_{\partial B_r^2(x_0)\times\{z_1\}}]$. To simplify our notation, hereinafter, we shall always write $[g]_\Gamma$ instead of $[g]_{\Gamma_{r,L}(x_0)}$.

Lemma 3.2.20. *Let $p \in [1, 2)$ and $g \in W^{1,2}(\Gamma_{r,L}(x_0), \mathcal{N})$. Then there exists $u \in W^{1,p}(\Lambda_{r,L}(x_0), \mathcal{N})$ such that $\operatorname{tr}_{\Gamma_{r,L}(x_0)}(u) = g$ and the following assertions hold.*

(i) For some $C = C(\mathcal{N}) > 0$, the following estimate holds

$$\begin{aligned}&\int_{\Lambda_{r,L}(x_0)} \frac{|Du|^p}{p}\,dx \\ &\qquad \le \frac{2L\mathcal{E}_p^{\mathrm{sg}}([g]_\Gamma) r^{2-p}}{2-p} + C\left(\frac{L^p}{r^{p-1}} + r\right)\int_{\Gamma_{r,L}(x_0)} \frac{|D_\top g|^p}{p}\,d\mathcal{H}^2.\end{aligned}$$

(ii) For each $z \in \{-L, L\}$, $\operatorname{tr}_{B_r^2(x_0)\times\{z\}}(u) \in W^{1,p}(B_r^2(x_0)\times\{z\}, \mathcal{N})$ and there exists a constant $C = C(\mathcal{N}) > 0$ such that

$$\begin{aligned}&\int_{B_r^2(x_0)\times\{z\}} \frac{|D\operatorname{tr}_{B_r^2(x_0)\times\{z\}}(u)|^p}{p}\,d\mathcal{H}^2 \\ &\quad \le \frac{\mathcal{E}_p^{\mathrm{sg}}([g]_\Gamma) r^{2-p}}{2-p} + C\left(\left(\frac{L}{r}\right)^{p-1} + \frac{r}{L}\right)\int_{\Gamma_{r,L}(x_0)} \frac{|D_\top g|^p}{p}\,d\mathcal{H}^2.\end{aligned}$$

Proof. Up to translation, we can assume that $x_0 = 0$. Denote $\Gamma = \Gamma_{r,L}$ and $\Lambda = \Lambda_{r,L}$. Using the coarea formula, we get some $z_0 \in (-L/2, L/2)$ such that $g|_{\partial B_r^2 \times \{z_0\}} \in W^{1,2}(\partial B_r^2 \times \{z_0\}, \mathcal{N})$ and

$$\|D_\top g\|^p_{L^p(\partial B_r^2 \times \{z_0\})} \leq \frac{2}{L} \|D_\top g\|^p_{L^p(\Gamma)}. \tag{3.2.22}$$

First, we define u on $(B_r^2 \backslash B_{r/2}^2) \times (-L, L)$. Let $f(\sigma, z) := g(r\sigma, z)$ for each $\sigma \in \mathbb{S}^1$ and $z \in (-L, L)$. For every $\varrho \in [r/2, r]$, $\sigma \in \mathbb{S}^1$ and $z \in (-L, L)$, we set

$$u(\varrho\sigma, z) := \begin{cases} f(\sigma, \xi(\varrho, z)) & \text{if } \varrho \in [\varrho_0(z), r], \\ f(\sigma, z_0) & \text{if } \varrho \in [r/2, \varrho_0(z)], \end{cases}$$

where

$$\xi(\varrho, z) := \begin{cases} z - \frac{2(r-\varrho)(L-z_0)}{r} & \text{if } z \in [z_0, L], \\ z + \frac{2(r-\varrho)(L+z_0)}{r} & \text{if } z \in [-L, z_0] \end{cases}$$

and

$$\varrho_0(z) := \begin{cases} r - \frac{r(z-z_0)}{2(L-z_0)} & \text{if } z \in [z_0, L], \\ r + \frac{r(z-z_0)}{2(L+z_0)} & \text{if } z \in [-L, z_0]. \end{cases}$$

Notice that $z \mapsto \varrho_0(z)$ is affine on $[-L, z_0]$ and on $[z_0, L]$. Also $\varrho_0(-L) = \varrho_0(L) = r/2$, $\varrho_0(z_0) = r$. Furthermore, $\xi(\varrho, z) \in (-L, L)$ if $z \in (-L, L)$ and $\varrho \in [\varrho_0(z), r]$. Performing the changes of variables, we have, firstly, the splitting

$$\|Du\|^p_{L^p((B_r^2 \backslash \overline{B}^2_{r/2}) \times (z_0, L))} = \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} := \int_{r/2}^{\varrho_0(z)} d\varrho \int_{\mathbb{S}^1} d\mathcal{H}^1(\sigma) \int_{z_0}^{L} \varrho^{1-p} \left|\frac{\partial f}{\partial \sigma}\right|^p (\sigma, z_0)\, dz$$

and

$$\mathrm{II} := \int_{\varrho_0(z)}^{r} d\varrho \int_{\mathbb{S}^1} d\mathcal{H}^1(\sigma) \int_{z_0}^{L} \varrho \Bigg(\left(\frac{4(L-z_0)^2}{r^2} + 1 \right) \left|\frac{\partial f}{\partial z}\right|^2 (\sigma, \xi(\varrho, z)) \\ + \frac{1}{\varrho^2} \left|\frac{\partial f}{\partial \sigma}\right|^2 (\sigma, \xi(\varrho, z)) \Bigg)^{\frac{p}{2}} dz.$$

Using the fact that $\frac{1-(1/2)^{2-p}}{2-p} \le \ln(2)$, the elementary inequality $(a+b)^{\frac{p}{2}} \le a^{\frac{p}{2}} + b^{\frac{p}{2}}$ for each $a, b \ge 0$ and (3.2.22), we estimate I by

$$
\begin{aligned}
\mathrm{I} &\le \int_{\mathbb{S}^1} \ln(2)(L-z_0)r^{2-p} \left|\frac{\partial f}{\partial \sigma}\right|^p (\sigma, z_0)\, d\mathcal{H}^1(\sigma) \\
&\le \ln(2)(L-z_0)r\|D_\top g\|^p_{L^p(\partial B^2_r \times \{z_0\})}
\end{aligned} \tag{3.2.23}
$$

and II by

$$
\begin{aligned}
\mathrm{II} \le{}& \frac{r}{2(L-z_0)} \int_{z_0}^{z} d\xi \int_{\mathbb{S}^1} d\mathcal{H}^1(\sigma) \int_{z_0}^{L} r\Bigg(\left(\frac{4(L-z_0)^2}{r^2}+1\right)\left|\frac{\partial f}{\partial z}\right|^2 (\sigma,\xi) \\
&+ \frac{4}{r^2}\left|\frac{\partial f}{\partial \sigma}\right|^2 (\sigma,\xi)\Bigg)^{\frac{p}{2}} dz \\
\le{}& \frac{r}{2}\left(\frac{4(L-z_0)^2}{r^2}+4\right)^{\frac{p}{2}} \times \\
& \int_{z_0}^{L} d\xi \int_{\mathbb{S}^1} r\left(\left|\frac{\partial f}{\partial z}\right|^2(\sigma,\xi) + \frac{1}{r^2}\left|\frac{\partial f}{\partial \sigma}\right|^2(\sigma,\xi)\right)^{\frac{p}{2}} d\mathcal{H}^1(\sigma) \\
\le{}& 2\left(\frac{(L-z_0)^p}{r^{p-1}} + r\right)\|D_\top g\|^p_{L^p(\partial B^2_r \times (z_0, L))},
\end{aligned}
$$

which, together with (3.2.23), implies that

$$
\|Du\|^p_{L^p((B^2_r \setminus \overline{B}^2_{r/2}) \times (z_0, L))} \le C\left(\frac{L^p}{r^{p-1}} + r\right)\|D_\top g\|^p_{L^p(\Gamma)}, \tag{3.2.24}
$$

where $C > 0$ is the constant that, hereinafter in this proof, can depend only on $\mathcal{N}$ and can be different from line to line. Similarly, we have

$$
\|Du\|^p_{L^p((B^2_r \setminus \overline{B}^2_{r/2}) \times (-L, z_0))} \le C\left(\frac{L^p}{r^{p-1}} + r\right)\|D_\top g\|^p_{L^p(\Gamma)}. \tag{3.2.25}
$$

Applying Lemma 3.2.18, we define u on $B^2_{r/2} \times \{z_0\}$, satisfying

$$
\begin{aligned}
&\int_{B^2_{r/2} \times \{z_0\}} \frac{|Du|^p}{p}\, d\mathcal{H}^2 \\
&\qquad \le \frac{\mathcal{E}^{\mathrm{sg}}_p([g]_\Gamma) r^{2-p}}{2-p} + Cr\int_{\partial B^2_r \times \{z_0\}} \frac{|D_\top u|^p}{p}\, d\mathcal{H}^1 \\
&\qquad \le \frac{\mathcal{E}^{\mathrm{sg}}_p([g]_\Gamma) r^{2-p}}{2-p} + \frac{Cr}{L}\int_{\Gamma} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2,
\end{aligned} \tag{3.2.26}
$$

where to obtain the last estimate, we have used (3.2.22). Since u restricted to $\partial B^2_{r/2} \times (-L, L)$ is independent of the z-variable, we can define $u(\varrho\sigma, z) = u(\varrho\sigma, z_0)$ for every $\varrho \in (0, r/2]$, $\sigma \in \mathbb{S}^1$ and $z \in (-L, L)$. At this point, u is defined on Λ and $u \in W^{1,p}(\Lambda, \mathcal{N})$. Observe that, integrating both sides of (3.2.26) with respect to $z \in (-L, L)$, we obtain that

$$\int_{B^2_{r/2}\times(-L,L)} \frac{|Du|^p}{p}\, dx \leq \frac{2L\mathcal{E}^{\mathrm{sg}}_p([g]_\Gamma) r^{2-p}}{2-p} + Cr \int_\Gamma \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2. \tag{3.2.27}$$

Combining (3.2.24) with (3.2.25) and (3.2.27), we complete the proof of the assertion (i).

Lastly, looking at the definition of u, one observes that $\mathrm{tr}_{B^2_r\times\{z\}}(u) \in W^{1,p}(B^2_r \times \{z\}, \mathcal{N})$ for each $z \in \{-L, L\}$. In particular, if $z = L$, then, setting $\mathrm{III} = \|D\,\mathrm{tr}_{B^2_r\times\{L\}}(u)\|^p_{L^p((B^2_r\setminus\overline{B}^2_{r/2})\times\{L\})}$, we deduce

$$\begin{aligned}
\mathrm{III} \leq& \int_{r/2}^r d\varrho \int_{\mathbb{S}^1} \varrho \Bigg(\frac{4(L-z_0)^2}{r^2} \left|\frac{\partial f}{\partial z}\right|^2 (\sigma, \xi(\varrho, L)) \\
&+ \frac{1}{\varrho^2}\left|\frac{\partial f}{\partial \sigma}\right|^2 (\sigma, \xi(\varrho, L)) \Bigg)^{\frac{p}{2}} d\mathcal{H}^1(\sigma) \\
\leq& \frac{r}{2(L-z_0)} \left(\frac{4(L-z_0)^2}{r^2} + 4 \right)^{\frac{p}{2}} \times \\
&\int_{z_0}^L d\xi \int_{\mathbb{S}^1} r \left(\left|\frac{\partial f}{\partial z}\right|^2 (\sigma, \xi) + \frac{1}{r^2}\left|\frac{\partial f}{\partial \sigma}\right|^2 (\sigma, \xi) \right)^{\frac{p}{2}} d\mathcal{H}^1(\sigma) \\
\leq& \frac{r}{2(L-z_0)} \left(\frac{4(L-z_0)^p}{r^p} + 4 \right) \int_\Gamma |D_\top g|^p\, d\mathcal{H}^2 \\
\leq& C\left(\left(\frac{L}{r}\right)^{p-1} + \frac{r}{L} \right) \int_\Gamma |D_\top g|^p\, d\mathcal{H}^2.
\end{aligned}$$

This, together with (3.2.26), proves the inequality of the assertion (ii) for $z = L$. Similarly, one obtains the same estimate for

$$\|D\,\mathrm{tr}_{B^2_r\times\{-L\}}(u)\|^p_{L^p(B^2_r\times\{-L\})},$$

which completes our proof of the assertion (ii) and of Lemma 3.2.20. □

3.2.8 Minimizing p-harmonic maps

Given an open bounded set $U \subset \mathbb{R}^3$, recall that a mapping $u_p \in W^{1,p}(U, \mathcal{N})$ is a p-minimizer in U whenever the following holds

$$\begin{aligned}(3.2.28)\quad &\int_U \frac{|Du_p|^p}{p}\,dx \\ &= \min\left\{\int_U \frac{|Du|^p}{p}\,dx : u \in W^{1,p}(U, \mathcal{N}),\ \ u - u_p \in W_0^{1,p}(U, \mathbb{R}^\nu)\right\}.\end{aligned}$$

In particular, if U has a Lipschitz boundary, then $u_p \in W^{1,p}(U, \mathcal{N})$ is a p-minimizer in U if and only if

$$\begin{aligned}(3.2.29)\quad &\int_U \frac{|Du_p|^p}{p}\,dx \\ &= \min\left\{\int_U \frac{|Du|^p}{p}\,dx : u \in W^{1,p}(U, \mathcal{N}),\ \ \mathrm{tr}_{\partial U}(u) = \mathrm{tr}_{\partial U}(u_p)\right\}.\end{aligned}$$

When appropriate, we shall simply say that u_p is a p-minimizer.

The next proposition says that if $p \in (1, 2)$, then for every p-minimizer in Ω which is equal on $\partial\Omega$ to some fixed mapping g lying in $W^{1/2,2}(\partial\Omega, \mathcal{N})$, the p-energy is bounded from above by $C((2-p)^{-1})$, where $C > 0$ is a constant depending only on g, $\partial\Omega$ and $\mathcal{N}$.

Proposition 3.2.21. *Let $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ and $p \in (1, 2)$. Then there exists a positive constant C depending only on $\partial\Omega$ and $\mathcal{N}$ such that if u_p is a p-minimizer with $\mathrm{tr}_{\partial\Omega}(u_p) = g$, then*

$$(2-p)\int_\Omega \frac{|Du_p|^p}{p}\,dx \le C|g|^p_{W^{1/2,2}(\partial\Omega, \mathbb{R}^\nu)}.$$

Proof. Inasmuch as $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, it holds $g \in W^{1-1/p,p}(\partial\Omega, \mathbb{R}^\nu)$ (the main ingredients in the proof are Hölder's inequality, the Sobolev inequality for fractional Sobolev spaces (the reader may consult [1, Theorem 7.57], [56, Section 3.3] and [14, Theorem 1]) and the fact that $\partial\Omega \times \partial\Omega = \bigcup_{i=1}^n \varphi_i(B_1^2) \times \varphi_i(B_1^2)$ for some $n \in \mathbb{N} \setminus \{0\}$, where φ_i is a bilipschitz map; see also [28, Proposition 2.1]) and

$$(3.2.30)\qquad |g|_{W^{1-1/p,p}(\partial\Omega, \mathbb{R}^\nu)} \le C|g|_{W^{1/2,2}(\partial\Omega, \mathbb{R}^\nu)}.$$

According to [35, Theorem 6.2], there exists $v \in W^{1,p}(\Omega, \mathcal{N})$ such that $\mathrm{tr}_{\partial\Omega}(v) = g$ and

$$(3.2.31)\qquad (2-p)\int_\Omega \frac{|Dv|^p}{p}\,dx \le C'|g|^p_{W^{1-1/p,p}(\partial\Omega, \mathbb{R}^\nu)}.$$

Next, using the direct method in the calculus of variations (notice that $p \in (1,2)$), we deduce that there exists a p-minimizer u_p such that $\mathrm{tr}_{\partial\Omega}(u_p) = g$. In view of the minimality of u_p and the estimates (3.2.31), (3.2.30), the following holds

$$(3.2.32)\quad (2-p)\int_\Omega \frac{|Du_p|^p}{p}\,dx \le C'|g|^p_{W^{1-1/p,p}(\partial\Omega,\mathbb{R}^\nu)} \le C''|g|^p_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)}$$

for some $C'' = C''(\partial\Omega,\mathcal{N}) > 0$. This completes our proof of Proposition 3.2.21. $\square$

We also recall the following fact. Let $u \in W^{1,p}(\Omega,\mathcal{N})$ be a p-minimizer. Fix an arbitrary $\xi \in C_c^\infty(\Omega,\mathbb{R}^3)$, which is not identically zero. Define $\varrho_0 = \min\{\mathrm{dist}(x,\partial\Omega) : x \in \mathrm{supp}(\xi)\}$. If $s \in (0, \varrho_0/\|\xi\|_\infty)$ is small enough, then the mapping $\varphi_s(x) = x + s\xi(x)$, $x \in \Omega$ is invertible. For each $s > 0$ small enough, define $u_s(x) = u \circ \varphi_s^{-1}(x)$ for each $x \in \Omega$. Since u minimizes the p-energy, $\frac{d}{ds}\int_\Omega |Du_s|^p\,dx\Big|_{s=0} = 0$. This, together with the facts that $D\varphi_s^{-1}\circ\varphi_s = (D\varphi_s)^{-1}$, $(D\varphi_s)^{-T} = \mathrm{Id} - s(D\xi)^T + O(s^2)$ and $\det(D\varphi_s) = 1 + s\,\mathrm{div}(\xi) + O(s^2)$, implies that the stress-energy tensor $T_u^p = \frac{|Du|^p}{p}\mathrm{Id} - \frac{Du\otimes Du}{|Du|^{2-p}}$ is (row-wise) divergence free. Namely, we have the following integral identity

$$(3.2.33)\quad \int_\Omega \sum_{i,j=1}^3 (|Du|^p\delta_{i,j} - p|Du|^{p-2}\langle D_i u, D_j u\rangle) D_i\xi_j\,dx = 0 \quad \forall \xi \in C_c^1(\Omega,\mathbb{R}^3),$$

where $\delta_{i,j} = 1$ if $i = j$ and $\delta_{i,j} = 0$ otherwise. A map satisfying (3.2.33) is called *p-stationary* or simply *stationary.*

3.3 Extension from triangulation

3.3.1 Lower bounds for the energy

We begin with the following simple estimate.

Lemma 3.3.1. *If $p \in [1,2]$, $\delta \in (0,+\infty)$ and $g \in W^{1,p}(\mathbb{S}^1,\mathbb{R}^k)$, then, defining $v(x) := g(x/|x|)$ for each $x \in B^2_{1+\delta}\setminus\overline{B}^2_1$, we have $v \in W^{1,p}(B^2_{1+\delta}\setminus\overline{B}^2_1,\mathbb{R}^k)$ and*

$$\int_{B^2_{1+\delta}\setminus\overline{B}^2_1} |Dv|^p\,dx \le \delta\int_{\mathbb{S}^1} |D_\top g|^p\,d\mathcal{H}^1.$$

Proof. If $p \in [1,2)$, then $v \in W^{1,p}(B^2_{1+\delta} \setminus \overline{B}^2_1, \mathbb{R}^k)$ and

$$\int_{B^2_{1+\delta}\setminus\overline{B}^2_1} |Dv|^p \, dx = \int_1^{1+\delta} \varrho^{1-p} \, d\varrho \int_{\mathbb{S}^1} |D_\top g|^p \, d\mathcal{H}^1 = \frac{(1+\delta)^{2-p}-1}{2-p} \int_{\mathbb{S}^1} |D_\top g|^p \, d\mathcal{H}^1. \tag{3.3.1}$$

Since $1 \le p \le 2$, the function $t \in [0,+\infty) \mapsto t^{2-p}$ is concave and hence $(1+\delta)^{2-p} \le 1 + (2-p)\delta$. This, together with (3.3.1), completes our proof of Lemma 3.3.1, since the case $p = 2$ is similar. □

Recall that if $p \in [1,2)$ and $\pi_1(\mathcal{N}) \not\simeq \{0\}$, then the set $C^\infty(B^2_r(x_0), \mathcal{N})$ is not dense in $W^{1,p}(B^2_r(x_0), \mathcal{N})$. In fact, in the latter case there exists a map $\gamma \in W^{1,p}(\partial B^2_r(x_0), \mathcal{N})$, which is not nullhomotopic. Defining $u(x) = \gamma(x_0 + r(x-x_0)/|x-x_0|)$ for each $x \in B^2_r(x_0)\setminus\{x_0\}$, we have $u \in W^{1,p}(B^2_r(x_0), \mathcal{N})$. Assuming that there exists a sequence of smooth maps $u_n : B^2_r(x_0) \to \mathcal{N}$ such that $u_n \to u$ in $W^{1,p}(B^2_r(x_0), \mathbb{R}^\nu)$, and using the coarea formula together with the Sobolev embedding theorem, we could find some $\varrho \in (0,r)$ such that $u_n|_{\partial B^2_\varrho(x_0)}$ converges uniformly to $\mathrm{tr}_{\partial B^2_\varrho(x_0)}(u) = u|_{\partial B^2_\varrho(x_0)}$. But this would lead to a contradiction, since $u_n|_{\partial B^2_\varrho(x_0)}$ is nullhomotopic contrary to $u|_{\partial B^2_\varrho(x_0)}$. This type of counterexample has its roots in the seminal work of Schoen and Uhlenbeck (see [54, Example]). Nonetheless, as was first observed by Bethuel and Zheng for the circle $\mathbb{S}^1$ (see [13, Theorem 4]) and generalized by Bethuel (see [9, Theorem 2]) to general compact Riemannian manifolds, Sobolev maps in $W^{1,p}(B^2_r(x_0), \mathcal{N})$ can be approximated by continuous maps outside a finite set of points in $B^2_r(x_0)$. For the reader's convenience, we recall Bethuel's theorem, slightly modified for our purposes.

Theorem 3.3.2. *Let $p \in [1,2)$. Then every map $u \in W^{1,p}(B^2_r(x_0), \mathcal{N}) \cap C(B^2_r(x_0)\setminus B^2_{r/2}(x_0), \mathcal{N})$ can be approximated in $W^{1,p}(B^2_r(x_0), \mathbb{R}^\nu)$ by maps $u_n \in W^{1,p}(B^2_r(x_0), \mathcal{N})$ which are continuous outside some finite set of points $A_n \subset B^2_{r/2}(x_0)$ such that $u_n = u$ on $B^2_r(x_0)\setminus B^2_{3r/4}(x_0)$ and $u_n|_{\partial B^2_\varrho(a)}$ is not nullhomotopic for each $a \in A_n$ and for each sufficiently small $\varrho > 0$ depending on n.*

Sketch of the proof. The proof consists of covering $B^2_{r/2}(x_0)$ with "good" and "bad" balls with respect to the p-energy of u, relying on an argument based on the coarea formula, and by changing u inside each good and bad ball from the covering by a smooth function in the case of a good ball and a smooth beyond the center of the ball in the case of a bad ball; the reader may consult [9, Theorem 2], where the balls are replaced by cubes. We only need to clarify that $u_n|_{\partial B^2_\varrho(a)}$ is not nullhomotopic for each $a \in A_n$ and for each sufficiently small $\varrho > 0$. In fact, otherwise, using the coarea formula, we could modify u_n inside a ball $B^2_\delta(a)$, where $\delta \in (\varrho/2, \varrho)$ for some sufficiently small $\varrho > 0$, with the quality of the approximation controlled by the energy of the original map on the ball $B^2_\varrho(a)$, so that the modified map u_n is continuous on $B^2_\varrho(a)$. $\square$

Proposition 3.3.3. *Let* $p \in (1,2)$, $u \in W^{1,p}(B^2_r(x_0), \mathcal{N})$ *and*

$$\mathrm{tr}_{\partial B^2_r(x_0)}(u) \in W^{1,p}(\partial B^2_r(x_0), \mathcal{N}).$$

Then the following estimate holds

$$\int_{B^2_r(x_0)} \frac{|Du|^p}{p}\,dx + r\int_{\partial B^2_r(x_0)} \frac{|D_\top \mathrm{tr}_{\partial B^2_r(x_0)}(u)|^p}{p}\,d\mathcal{H}^1 \geq \frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\mathrm{tr}_{\partial B^2_r(x_0)}(u))r^{2-p}}{2-p}.$$

Proof. Up to scaling and translation, we can assume that $r = 1$ and $x_0 = 0$. Since $p > 1$, in view of Morrey's embedding for Sobolev mappings, without loss of generality, we suppose that $\mathrm{tr}_{\mathbb{S}^1}(u) \in C(\mathbb{S}^1, \mathcal{N})$. Define the mapping $v : B^2_2 \to \mathcal{N}$ by

$$v(x) := \begin{cases} \mathrm{tr}_{\mathbb{S}^1}(u)\left(\frac{x}{|x|}\right) & \text{if } x \in B^2_2 \setminus B^2_1, \\ u(x) & \text{if } x \in B^2_1. \end{cases}$$

Then $v \in W^{1,p}(B^2_2, \mathcal{N}) \cap C(B^2_2 \setminus B^2_1, \mathcal{N})$. By Lemma 3.3.1 applied with $\delta = 1$, we obtain

$$\int_{B^2_2} |Dv|^p\,dx \leq \int_{B^2_1} |Du|^p\,dx + \int_{\mathbb{S}^1} |D_\top \mathrm{tr}_{\mathbb{S}^1}(u)|^p\,d\mathcal{H}^1. \tag{3.3.2}$$

By Theorem 3.3.2, there exists $(v_n)_{n\in\mathbb{N}} \subset W^{1,p}(B^2_2, \mathcal{N})$ such that $v_n \to v$ in $W^{1,p}(B^2_2, \mathbb{R}^\nu)$ and for each $n \in \mathbb{N}$, v_n is continuous in $B^2_2 \setminus A_n$, where $A_n \subset B^2_1$ is a finite set of points; $\mathrm{tr}_{\partial B^2_2}(v_n)$ is homotopic to $\mathrm{tr}_{\mathbb{S}^1}(u)$; for each point $a \in A_n$ and for each sufficiently small radius $\varrho > 0$, $v_n|_{\partial B^2_\varrho(a)}$ is homotopically nontrivial. Next, using [57, Corollary 4.4] with $\delta = 1 < \mathrm{dist}(a, \partial B^2_2)$ (in the statement of [57, Corollary 4.4], δ is assumed to satisfy the same assumptions as in [57, Proposition 4.3]) to $v_n \in W^{1,p}(B^2_2, \mathcal{N})$ with the set of singular points A_n, we get

$$\begin{aligned} \mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\mathrm{tr}_{\mathbb{S}^1}(u)) &= \mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\mathrm{tr}_{\partial B^2_2}(v_n)) \\ &\leq (2-p)\frac{(p-1)^{p-1}}{((2\pi)/p)^{2-p}} \int_{B^2_2} \frac{|Dv_n|^p}{p}\,dx \\ &\leq (2-p)\int_{B^2_2} \frac{|Dv_n|^p}{p}\,dx, \end{aligned} \tag{3.3.3}$$

since the singular p-energy is invariant under homotopies and $\operatorname{tr}_{\partial B_2^2}(v_n)$ is homotopic to $\operatorname{tr}_{\mathbb{S}^1}(u)$ and since for $1 < p < 2$, $\frac{(p-1)^{p-1}}{((2\pi)/p)^{2-p}} < 1$. $\square$

Remark 3.3.4. Adapting the proof of [57, Proposition 4.3], yields a variant of the latter, where the estimate depends on $\operatorname{sys}(\mathcal{N})$. Therefore, one can obtain a variant of Proposition 3.3.3 with the estimate depending on $\operatorname{sys}(\mathcal{N})$. More precisely, if

$$\int_{B_r^2(x_0)} |Du|^p \, dx + r \int_{\partial B_r^2(x_0)} |D_\top \operatorname{tr}_{\partial B_r^2(x_0)}(u)|^p \, d\mathcal{H}^1 > 0,$$

then one can obtain the estimate

$$\frac{2-p}{p} \left(\int_{B_r^2(x_0)} |Du|^p \, dx + r \int_{\partial B_r^2(x_0)} |D_\top \operatorname{tr}_{\partial B_r^2(x_0)}(u)|^p \, d\mathcal{H}^1 \right) \\ \geq C \mathcal{E}_p^{\mathrm{sg}}(\operatorname{tr}_{\partial B_r^2(x_0)}(u))^{p-1} r^{2-p},$$

where $C = \left(\frac{(\operatorname{sys}(\mathcal{N}))^p}{(2\pi)^{p-1}p}\right)^{2-p}$. This is an upper bound of $\mathcal{E}_p^{\mathrm{sg}}(\operatorname{tr}_{\mathbb{S}^1}(u))$ by the p-energy, which is in one instance better than the control in Proposition 3.3.3 of $\mathcal{E}_{p/(p-1)}^{\mathrm{sg}}(\operatorname{tr}_{\mathbb{S}^1}(u))$ by the p-energy. However, the estimate depends on $\operatorname{sys}(\mathcal{N})$. Since we prefer to avoid the use of estimates depending on the systole, we decided to keep the estimation of Proposition 3.3.3.

3.3.2 Lipschitz triangulations of $\mathbb{S}^2$

For a given integer $n \in \mathbb{N} \setminus \{0\}$, we shall call the uniform 1-dimensional grid of step $1/n$ in $\partial[-1,1]^3$ the set of all points $x \in \partial[-1,1]^3$ such that $nx_i \in \mathbb{Z}$ for all $i = 1, 2, 3$ except for at most one.

The uniform 1-dimensional grid of step $1/n$ gives the decomposition of $\partial[-1,1]^3$ into mutually disjoint sets (points, edges and two-dimensional cells), namely

$$\partial[-1,1]^3 = \bigcup_{i=0}^{2} \bigcup_{j=1}^{k_i} E_{i,j}, \tag{3.3.4}$$

where $\{E_{0,j}, j \in \{1, \dots, k_0\}\} \coloneqq ((1/n)\mathbb{Z}^3) \cap \partial[-1,1]^3$ and each set $E_{i,j}$ with $i \in \{1, 2\}$ is bilipschitz homeomorphic to $B^i_{1/n}$. It is worth noting that for each integer $l \in \mathbb{N} \setminus \{0\}$, the cube $(0,1)^l$ is bilipschitz homeomorphic to B^l_1 (see, for instance, [33]).

Let $\delta \in (0, 1]$ and $n = \lfloor 1/\delta \rfloor$, where $\lfloor \cdot \rfloor$ denotes the integer part. In view of (3.3.4), we obtain the following decomposition of $\mathbb{S}^2$ into mutually disjoint sets

$$\mathbb{S}^2 = \bigcup_{i=0}^{2} \bigcup_{j=1}^{k_i} F_{i,j} = \bigcup_{i=0}^{2} \bigcup_{j=1}^{k_i} \omega \circ \mathrm{P}_{\mathbb{S}^2}(E_{i,j}), \tag{3.3.5}$$

where $\mathrm{P}_{\mathbb{S}^2} : \mathbb{R}^3 \setminus \{0\} \to \mathbb{S}^2$ is the radial projection ($\mathrm{P}_{\mathbb{S}^2}(x) = x/|x|$, $x \in \mathbb{R}^3 \setminus \{0\}$), ω is an element of $SO(3)$, $\{F_{0,1}, \dots, F_{0,k_0}\}$ is the collection of points and for each $F_{i,j}$ with $i \in \{1, 2\}$, we have a bilipschitz homeomorphism

$$\Phi_{i,j} : F_{i,j} \to B^i_\delta \quad \text{with } \|D_\top \Phi_{i,j}\|_{L^\infty(F_{i,j})} + \|D(\Phi_{i,j})^{-1}\|_{L^\infty(B^i_\delta)} \leq A_1, \tag{3.3.6}$$

where $A_1 > 0$ is an absolute constant. The map $\Phi_{i,j}$ in (3.3.6) extends by continuity to a bilipschitz map between $\overline{F}_{i,j}$ and $\overline{B}^i_\delta$ with the same bilipschitz constant A_1 as in (3.3.6).

If (3.3.5) and (3.3.6) hold for some $\omega \in SO(3)$, we shall call the decomposition (3.3.5) a uniform Lipschitz *triangulation* (or simply a Lipschitz triangulation) of step δ of $\mathbb{S}^2$. The points $F_{0,1}, \dots, F_{0,k_0}$ will be called the vertices and the set $\bigcup_{i=0}^{1} \bigcup_{j=1}^{k_i} F_{i,j}$ the 1-skeleton.

To lighten the notation, we denote the collection of 1-skeletons of all Lipschitz triangulations of step δ of $\mathbb{S}^2$ by RT_δ, namely

$$\mathrm{RT}_\delta := \left\{ \bigcup_{i=0}^{1} \bigcup_{j=1}^{k_i} \omega \circ \mathrm{P}_{\mathbb{S}^2}(E_{i,j}) : \omega \in SO(3) \right\},$$

where the $E_{i,j}$'s come from (3.3.4) for $n = \lfloor 1/\delta \rfloor$. The importance of the rotation ω in the definition of RT_δ will be illustrated a little later, for example in Lemma 3.3.7.

Let us now define a Sobolev space on a finite union of bilipschitz images of d-cubes (or d-dimensional intervals), in particular, on the 1-skeleton of a uniform Lipschitz triangulation of the sphere $\mathbb{S}^2$.

Definition 3.3.5. Let $d, N \in \mathbb{N} \setminus \{0\}$. For each $p \in [1, +\infty)$ and for each set $\Sigma \subset \mathbb{R}^d$ being the closure of $\bigcup_{i=1}^{N} \Phi_i((a_{i1}, b_{i1}) \times ... \times (a_{id}, b_{id})) =: \bigcup_{i=1}^{N} \Delta^d_i$, where for each $i \in \{1, \dots, N\}$, $\Phi_i : (a_{i1}, b_{i1}) \times ... \times (a_{id}, b_{id}) \to \Delta^d_i$ is a bilipschitz homeomorphism and, in addition, $\Delta^d_i \cap \Delta^d_j = \emptyset$ if $i \neq j$, $j \in \{1, \dots, N\}$, we define $W^{1,p}(\Sigma, \mathbb{R}^k)$ by

$$W^{1,p}(\Sigma, \mathbb{R}^k) = \{u : \Sigma \to \mathbb{R}^k \text{ is Borel-measurable} : u|_{\Delta^d_i} \in W^{1,p}(\Delta^d_i, \mathbb{R}^k),$$
$$\mathrm{tr}_{\partial \Delta^d_i}(u|_{\Delta^d_i}) = u|_{\partial \Delta^d_i} \text{ for each } i \in \{1, \dots, N\}\},$$

where $\partial\Delta_i^d$ denotes the relative boundary of Δ_i^d. For $u \in W^{1,p}(\Sigma, \mathbb{R}^k)$, we set

$$\|u\|_{W^{1,p}(\Sigma,\mathbb{R}^k)} = \left(\sum_{i=1}^{N} \|u|_{\Delta_i^d}\|^p_{W^{1,p}(\Delta_i^d,\mathbb{R}^k)}\right)^{\frac{1}{p}}.$$

If $\mathcal{Y}$ is a compact Riemannian manifold which is isometrically embedded into $\mathbb{R}^k$, then $W^{1,p}(\Sigma, \mathcal{Y})$ is defined by

$$W^{1,p}(\Sigma, \mathcal{Y}) = \{u \in W^{1,p}(\Sigma, \mathbb{R}^k) : u \in \mathcal{Y} \text{ a.e. in } \Sigma\}.$$

Proposition 3.3.6. *Let $p \in [1,2]$, $\delta \in (0,1]$, $\Sigma \in \mathrm{RT}_\delta$ and $g \in W^{1,p}(\Sigma, \mathcal{N})$ (see Definition 3.3.5). Assume that for each 2-cell F from the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ, the map $g|_{\partial F}$ is homotopic to a constant, where ∂F is the relative boundary of F. Then there exists $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$, where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the universal covering of $\mathcal{N}$, such that $\mathrm{tr}_\Sigma(\pi \circ \widetilde{v}) = g$ and*

$$\int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, d\mathcal{H}^2 \le C\delta \int_\Sigma |D_\top g|^p \, d\mathcal{H}^1,$$

where $C = C(\mathcal{N}) > 0$.

Proof. We have $\mathbb{S}^2 = \bigcup F$ and $\Sigma = \bigcup \partial F$, where the unions are taken over all 2-cells from the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ, see (3.3.5).

Step 1. Fix an arbitrary 2-cell F from our Lipschitz triangulation of $\mathbb{S}^2$. We know that $g|_{\partial F}$ is homotopic to a constant. Let $\Phi : \overline{F} \to \overline{B}^2_\delta$ be the extension of the corresponding map coming from (3.3.6). Then, by the classical theory of continuous liftings, $g|_{\partial F} \circ \Phi^{-1}|_{\partial B^2_\delta}$ has a continuous lifting, which is also Sobolev; this, together with Lemma 3.2.13 and the estimate (3.3.6), implies that there exists $v \in W^{1,2}(F, \mathcal{N})$ such that $\mathrm{tr}_{\partial F}(v) = g|_{\partial F}$ and

$$\int_F |D_\top v|^p \, d\mathcal{H}^2 \le C\delta \int_{\partial F} |D_\top g|^p \, d\mathcal{H}^1, \tag{3.3.7}$$

where $C = C(\mathcal{N}) > 0$.

Step 2. Let $v \in L^\infty(\mathbb{S}^2, \mathcal{N})$ satisfy $v|_F = v_F$ for each 2-cell F from our Lipschitz triangulation of $\mathbb{S}^2$, where v_F is the mapping from *Step 1* corresponding to the cell F. Since $v_F \in W^{1,2}(F, \mathcal{N})$ and $\mathrm{tr}_{\partial F}(v_F) = g|_{\partial F}$, we have $v \in W^{1,2}(\mathbb{S}^2, \mathcal{N})$. In particular, there exists a map $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$ such that $v = \pi \circ \widetilde{v}$ (see, for instance, [11, Theorem 1]). Next, using (3.3.7) and the fact that each 1-cell of our Lipschitz triangulation of $\mathbb{S}^2$ lies in the relative boundary of exactly two 2-cells, since π is a local isometry, we get

$$\int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, d\mathcal{H}^2 = \int_{\mathbb{S}^2} |D_\top v|^p \, d\mathcal{H}^2$$
$$= \sum \int_F |D_\top v|^p \, d\mathcal{H}^2 \le 2C\delta \int_\Sigma |D_\top g|^p \, d\mathcal{H}^1,$$

where the sum is taken over all 2-cells F. This completes our proof of Proposition 3.3.6. $\square$

Lemma 3.3.7. *Let $p \in [1, +\infty)$ and $g \in W^{1,p}(\mathbb{S}^2, \mathcal{Y})$, where $\mathcal{Y}$ is either a Euclidean space or a compact Riemannian manifold. Let $\delta \in (0,1]$ and $\Sigma \in \mathrm{RT}_\delta$. Then for $\mathcal{H}^3$-a.e. $\omega \in SO(3)$, $\mathrm{tr}_{\omega(\Sigma)}(g) = g|_{\omega(\Sigma)} \in W^{1,p}(\omega(\Sigma), \mathcal{Y})$. Furthermore, there exists a rotation $\omega \in SO(3)$ such that we have $\mathrm{tr}_{\omega(\Sigma)}(g) = g|_{\omega(\Sigma)} \in W^{1,p}(\omega(\Sigma), \mathcal{Y})$ and*

$$\int_{\omega(\Sigma)} |D_\top g|^p \, d\mathcal{H}^1 \le \frac{A_2}{\delta} \int_{\mathbb{S}^2} |D_\top g|^p \, d\mathcal{H}^2,$$

where $A_2 > 0$ is an absolute constant.

Proof. Since $\Sigma \in \mathrm{RT}_\delta$, there exists $\omega \in SO(3)$ such that

$$\Sigma = \bigcup_{i=0}^{1} \bigcup_{j=1}^{k_i} \omega(E_{i,j}) = \bigcup_{i=0}^{1} \bigcup_{j=1}^{k_i} F_{i,j}$$

and hence $\Sigma = \bigcup_{j=1}^{k_1} \overline{F}_{1,j}$, where the $E_{i,j}$'s come from (3.3.4) for $n = \lfloor 1/\delta \rfloor$. Fix an arbitrary $j \in \{1, \ldots, k_0\}$. Let $\overline{F}_{1,j_1}$ and $\overline{F}_{1,j_2}$ be two edges emanating from $F_{0,j}$. Denote $V = \overline{F}_{1,j_1} \cup \overline{F}_{1,j_2}$. Then, in view of (3.3.6), there is a bilipschitz mapping $\Psi : \overline{B}^1_\delta \to V$. Denote $U = \Psi(B^1_\delta)$. Applying Lemma 3.2.17 (ii) with $v = g$ and $E = U$, we have $\mathrm{tr}_{\omega(U)}(g) = g|_{\omega(U)} \in W^{1,p}(\omega(U), \mathcal{Y})$ for $\mathcal{H}^3$-a.e. $\omega \in SO(3)$. Furthermore, for each $\omega \in SO(3)$ for which $g|_{\omega(U)} \in W^{1,p}(\omega(U), \mathcal{Y})$, there exists a unique mapping $h \in C(\omega(V), \mathcal{Y})$ such that $g|_{\omega(U)} = h$ $\mathcal{H}^1$-a.e. on $\omega(U)$, since $\omega(U)$ is bilipschitz homeomorphic to B^1_δ. Thus, successively applying Lemma 3.2.17 (ii), altogether we obtain that there exists $E \subset SO(3)$ such that $\mathcal{H}^3(E) = 0$ and the following holds. For each $\omega \in SO(3) \setminus E$ and $j \in \{1, \ldots, k_1\}$ we have $\mathrm{tr}_{\omega(F_{1,j})}(g) = g|_{\omega(F_{1,j})} \in W^{1,p}(\omega(F_{1,j}), \mathcal{Y})$ and there exists a unique mapping $h \in C(\omega(\Sigma), \mathcal{Y})$ such that $\mathrm{tr}_{\omega(\Sigma)}(g) = g|_{\omega(\Sigma)} = h$ $\mathcal{H}^1$-a.e. on $\omega(\Sigma)$, namely $g|_{\omega(\Sigma)} \in W^{1,p}(\omega(\Sigma), \mathcal{Y})$ (see Definition 3.3.5). Next, according to [31, Theorem 3.2.48],

$$\int_{O(3)} d\theta_3(\omega) \int_{\omega(\Sigma)} |D_\top g|^p \, d\mathcal{H}^1 = \frac{\mathcal{H}^1(\Sigma)}{\mathcal{H}^2(\mathbb{S}^2)} \int_{\mathbb{S}^2} |D_\top g|^p \, d\mathcal{H}^2, \tag{3.3.8}$$

where $\theta_3 = (\mathcal{H}^3(O(3)))^{-1}\mathcal{H}^3 \llcorner O(3)$ is the Haar measure of the orthogonal group $O(3)$ such that $\theta_3(O(3)) = 1$. Denoting

$$A = \left\{\omega \in O(3) : \int_{\omega(\Sigma)} |D_\top g|^p \, d\mathcal{H}^1 < \frac{3\mathcal{H}^1(\Sigma)}{\mathcal{H}^2(\mathbb{S}^2)} \int_{\mathbb{S}^2} |D_\top g|^p \, d\mathcal{H}^2 \right\},$$

we observe that $\theta_3(A) \geq 2/3$, otherwise (3.3.8) would not be true. Thus, $\theta_3(A \cap SO(3)) \geq 1/6$, since $\theta_3(O(3) \setminus SO(3)) = 1/2$. On the other hand, using (3.3.6), we get $\mathcal{H}^1(\Sigma) \leq C/\delta$, where $C > 0$ is an absolute constant. This completes our proof of Lemma 3.3.7. □

Lemma 3.3.8. *Let $p_0 \in (1, 2)$, $p \in [p_0, 2)$, $\delta \in (0, 1]$ and $\Sigma \in \mathrm{RT}_\delta$. Let F be a 2-cell of the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ. Then there exists a constant $\alpha_0 = \alpha_0(p_0, \mathcal{N}) > 0$ such that the following holds. If $u \in W^{1,p}(F, \mathcal{N})$, $\mathrm{tr}_{\partial F}(u) \in W^{1,p}(\partial F, \mathcal{N})$ and*

$$\int_F \frac{|D_\top u|^p}{p} \, d\mathcal{H}^2 + \delta \int_{\partial F} \frac{|D_\top \, \mathrm{tr}_{\partial F}(u)|^p}{p} \, d\mathcal{H}^1 \leq \frac{\alpha_0 \delta^{2-p}}{2-p}, \tag{3.3.9}$$

then $\mathrm{tr}_{\partial F}(u)$ is homotopic to a constant, where ∂F denotes the relative boundary of F.

Proof. Let $\Phi : F \to B^2_{\frac{\delta}{2}}$ be a bilipschitz mapping coming from (3.3.6). To lighten the notation, denote $v := u \circ \Phi^{-1}$. According to Definition 3.2.7 and (3.2.5), if a map $\gamma \in W^{1,p}(\partial B^2_{\frac{\delta}{2}}, \mathcal{N})$ is not homotopic to a constant, then

$$\frac{(\mathrm{sys}(\mathcal{N}))^{\frac{p}{p-1}}(p-1)}{(2\pi)^{\frac{1}{p-1}} p} \leq \mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma). \tag{3.3.10}$$

Let $\alpha_0 > 0$ be a constant, which will be defined a little later for the proof to work. Next, using Proposition 3.3.3, the chain rule for Sobolev mappings, [31, Theorem 3.2.5] and (3.3.9), we get

$$\begin{aligned} &\frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\mathrm{tr}_{\partial B^2_{\frac{\delta}{2}}}(v))\delta^{2-p}}{2-p} \\ &\leq \int_{B^2_{\frac{\delta}{2}}} \frac{|Dv|^p}{p} \, d\mathcal{H}^2 + \delta \int_{\partial B^2_{\frac{\delta}{2}}} \frac{|D_\top \, \mathrm{tr}_{\partial B^2_{\frac{\delta}{2}}}(v)|^p}{p} \, d\mathcal{H}^1 \leq \frac{C\alpha_0 \delta^{2-p}}{2-p}, \end{aligned} \tag{3.3.11}$$

where $C > 0$ is an absolute constant. Now we define

$$\alpha_0 = \min\left\{ \frac{(\mathrm{sys}(\mathcal{N}))^{\frac{p}{p-1}}(p-1)}{2C(2\pi)^{\frac{1}{p-1}} p} : p \in [p_0, 2] \right\}.$$

In view of the estimates (3.3.10) and (3.3.11), $\mathrm{tr}_{\partial B^2_{\frac{\delta}{2}}}(v)$ is homotopic to a constant, and hence $\mathrm{tr}_{\partial F}(u)$ is homotopic to a constant. This completes our proof of Lemma 3.3.8. □

The next lemma provides us with an extension to a 3-cell of given mappings on a 2-cell with appropriate energy control.

Lemma 3.3.9. *Let $p \in [1,2]$, $\delta \in (0,1]$ and for each $i \in \{0,1\}$, $g_i \in W^{1,p}((0,\delta)^2, \mathcal{N})$. If $\mathrm{tr}_{\partial(0,\delta)^2}(g_i) \in W^{1,p}(\partial(0,\delta)^2, \mathcal{N})$ and if*

$$\mathrm{tr}_{\partial(0,\delta)^2}(g_0) = \mathrm{tr}_{\partial(0,\delta)^2}(g_1), \tag{3.3.12}$$

then there exists $w \in W^{1,p}((0,\delta)^3, \mathcal{N})$ such that $\mathrm{tr}_{(0,\delta)^2\times\{i\delta\}}(w) = g_i$ for each $i \in \{0,1\}$ and

$$\begin{aligned} &\int_{(0,\delta)^3} |Dw|^p \, dx \\ &\quad \le A_3 \delta \sum_{i=0}^{1} \int_{(0,\delta)^2} |Dg_i|^p \, d\mathcal{H}^2 + A_3 \delta^2 \int_{\partial(0,\delta)^2} |D_\top \mathrm{tr}_{\partial(0,\delta)^2}(g_1)|^p \, d\mathcal{H}^1, \end{aligned} \tag{3.3.13}$$

where $A_3 > 0$ is an absolute constant.

Proof. Denote $E = \partial(0,\delta)^2 \times (0,\delta)$. We define w on E by $w|_E(x) = \mathrm{tr}_{\partial(0,\delta)^2}(g_1)(x')$, where $x = (x', x_3) \in E$. Since

$$\mathrm{tr}_{\partial(0,\delta)^2}(g_1) \in W^{1,p}(\partial(0,\delta)^2, \mathcal{N}),$$

we have $w|_E \in W^{1,p}(E, \mathcal{N})$. Using [31, Theorem 3.2.22 (3)] and changing the variables, one has

$$\int_E |Dw|^p \, d\mathcal{H}^2 = \delta \int_{\partial(0,\delta)^2} |D_\top \mathrm{tr}_{\partial(0,\delta)^2}(g_1)|^p \, d\mathcal{H}^1. \tag{3.3.14}$$

Next, we define the map $w|_{\partial(0,\delta)^3} : \partial(0,\delta)^3 \to \mathcal{N}$ by

$$w|_{\partial(0,\delta)^3}(x) = \begin{cases} g_1(x') & \text{if } x = (x', x_3) \in (0,\delta)^2 \times \{\delta\}, \\ \mathrm{tr}_{\partial(0,\delta)^2}(g_1)(x') & \text{if } x = (x', x_3) \in \partial(0,\delta)^2 \times \{\delta\}, \\ w|_E(x) & \text{if } x \in E, \\ \mathrm{tr}_{\partial(0,\delta)^2}(g_0)(x') & \text{if } x = (x', x_3) \in \partial(0,\delta)^2 \times \{0\}, \\ g_0(x') & \text{if } x = (x', x_3) \in (0,\delta)^2 \times \{0\}. \end{cases}$$

Then, in view of (3.3.12), we deduce that $w|_{\partial(0,\delta)^3} \in W^{1,p}(\partial(0,\delta)^3, \mathcal{N})$ and $\mathrm{tr}_{(0,\delta)^2\times\{i\delta\}}(w) = g_i$ for each $i \in \{0,1\}$. Let $\Psi : \overline{B}^3_\delta \to [0,\delta]^3$ be a bilipschitz homeomorphism with an absolute bilipschitz constant (see [33, Corollary 3]). Notice that $w|_{\partial(0,\delta)^3} \circ \Psi|_{\partial B^3_\delta} \in W^{1,p}(\partial B^3_\delta, \mathcal{N})$. Thus, defining $v(x) = w|_{\partial(0,\delta)^3} \circ \Psi(\delta x/|x|)$ for each $x \in B^3_\delta \setminus \{0\}$ and using that $p \in [1,2]$, we have $v \in W^{1,p}(B^3_\delta, \mathcal{N})$. Next, setting $\zeta(x) = x/|x|$ for $x \in$

$\mathbb{R}^3\backslash\{0\}$, using the special case of the coarea formula, [31, Theorem 3.2.5], the fact that $p \in [1,2]$ and the chain rule for Sobolev mappings, we obtain the following chain of estimates

$$\begin{aligned}\int_{B^3_\delta} &|Dv|^p \, dx \\ &= \int_0^\delta dr \int_{\partial B^3_r} \delta^p |D(w \circ \Psi)(\delta\zeta(\sigma)) \circ D\zeta(\sigma)|^p \, d\mathcal{H}^2(\sigma) \\ &= \int_0^\delta \left(\frac{r}{\delta}\right)^{2-p} dr \int_{\partial B^3_\delta} |D_\top (w \circ \Psi)|^p \, d\mathcal{H}^2 \\ &\leq \delta \int_{\partial B^3_\delta} |D_\top (w \circ \Psi)|^p \, d\mathcal{H}^2 \\ &\leq C\delta \int_{\partial(0,\delta)^3} |D_\top w|^p \, d\mathcal{H}^2,\end{aligned} \tag{3.3.15}$$

where $C > 0$ is a constant depending only on the bilipschitz constant of Ψ. Then, defining for each $x \in (0,\delta)^3 \setminus \{\Psi(0)\}$, $w(x) = v(\Psi^{-1}(x))$, we observe that $w \in W^{1,p}((0,\delta)^3, \mathcal{N})$, since $v \in W^{1,p}(B^3_\delta, \mathcal{N})$ and Ψ is bilipschitz. Using [31, Theorem 3.2.5], the fact that Ψ is bilipschitz and (3.3.15), we have

$$\int_{(0,\delta)^3} |Dw|^p \, dx \leq C' \int_{B^3_\delta} |Dv|^p \, dx \leq C''\delta \int_{\partial(0,\delta)^3} |D_\top w|^p \, d\mathcal{H}^2,$$

where $C'' > 0$ depends only on the bilipschitz constant of Ψ, which is an absolute constant. This, together with (3.3.14), yields (3.3.13) and completes our proof of Lemma 3.3.9. □

Using Lemma 3.3.9, we obtain the next Luckhaus-type lemma (see [44, Lemma 1]).

Lemma 3.3.10. *Let $p_0 \in (1,2)$, $p \in [p_0, 2)$, $\delta \in (0,1)$ and $g \in W^{1,p}(\mathbb{S}^2, \mathcal{N})$. Then there exists a constant $\alpha = \alpha(p_0, \mathcal{N}) > 0$ such that the following holds. Assume that*

$$\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \leq \frac{\alpha \delta^{2-p}}{2-p}. \tag{3.3.16}$$

Then there exist mappings $\varphi_\delta \in W^{1,p}(B^3_1 \setminus \overline{B}^3_{1-\delta}, \mathcal{N})$ and $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$, where $\pi : \widetilde{\mathcal{N}} \to \mathcal{N}$ is the universal covering of $\mathcal{N}$, such that

$$\operatorname{tr}_{\mathbb{S}^2}(\varphi_\delta) = g, \qquad \operatorname{tr}_{\mathbb{S}^2}(\varphi_\delta((1-\delta)\cdot)) = \pi \circ \widetilde{v}, \tag{3.3.17}$$

and the following estimates are satisfied

$$\int_{B_1^3 \setminus \overline{B}_{1-\delta}^3} |D\varphi_\delta|^p \, dx \leq C\delta \int_{\mathbb{S}^2} |D_\top g|^p \, d\mathcal{H}^2, \tag{3.3.18}$$

$$\int_{\mathbb{S}^2} |D_\top(\pi \circ \widetilde{v})|^p \, d\mathcal{H}^2 = \int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, d\mathcal{H}^2 \leq C \int_{\mathbb{S}^2} |D_\top g|^p \, d\mathcal{H}^2, \tag{3.3.19}$$

where $C = C(\mathcal{N}) > 0$.

Proof. Let $\alpha_0 = \alpha_0(p_0, \mathcal{N}) > 0$ be the constant given by Lemma 3.3.8. Define $\alpha = \alpha_0/2A_2$, where A_2 is the absolute constant of Lemma 3.3.7. Without loss of generality, we assume that $A_2 \geq 1$. Then α depends only on p_0 and $\mathcal{N}$. In view of Lemma 3.3.7, the estimate (3.3.16) and the definition of α, there exists $\Sigma \in \mathrm{RT}_\delta$ such that $\mathrm{tr}_\Sigma(g) = g|_\Sigma \in W^{1,p}(\Sigma, \mathcal{N})$,

$$\int_\Sigma |D_\top g|^p \, d\mathcal{H}^1 \leq \frac{A_2}{\delta} \int_{\mathbb{S}^2} |D_\top g|^p \, d\mathcal{H}^2 \tag{3.3.20}$$

and for each 2-cell F from the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ,

$$\int_F \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 + \delta \int_{\partial F} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^1 \leq 2A_2 \int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \leq \frac{\alpha_0 \delta^{2-p}}{2-p},$$

where ∂F denotes the relative boundary of F. This, according to Lemma 3.3.8, implies that $g|_{\partial F}$ is homotopically trivial for each 2-cell F from the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ. By Proposition 3.3.6, there exists $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$ such that $\mathrm{tr}_\Sigma(\pi \circ \widetilde{v}) = \mathrm{tr}_\Sigma(g)$ and

$$\int_{\mathbb{S}^2} |D_\top(\pi \circ \widetilde{v})|^p \, d\mathcal{H}^2 = \int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, d\mathcal{H}^2 \leq C\delta \int_\Sigma |D_\top g|^p \, d\mathcal{H}^1, \tag{3.3.21}$$

where $C = C(\mathcal{N}) > 0$. In (3.3.21) we used that π is a local isometry. Using (3.3.20) and (3.3.21), we get (3.3.19).

The Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ yields the decomposition of $B_1^3 \setminus \overline{B}_{1-\delta}^3$ into the cells

$$D_{i,j} = \left\{ x \in \mathbb{R}^3 : (1-\delta) < |x| < 1, \ \frac{x}{|x|} \in F_{i,j} \right\}, \tag{3.3.22}$$

where $i \in \{0, 1, 2\}$ and $j \in \{1, \ldots, k_i\}$ (see (3.3.5)). Notice that $D_{i,j}$ is of Hausdorff dimension $(i+1)$. Fix an arbitrary 3-cell $D := D_{2,j}$ from the above decomposition. Let F be the 2-cell of the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ such that $D = \{x \in \mathbb{R}^3 : (1-\delta) < |x| < 1, \ \frac{x}{|x|} \in F\}$.

Fix a bilipschitz homeomorphism $\Phi : [0,\delta]^3 \to \overline{D}$ with an absolute bilipschitz constant (see (3.3.6)) such that $\Phi((0,\delta)^2 \times \{0\}) = (1-\delta)F$ and $\Phi((0,\delta)^2 \times \{\delta\}) = F$. Applying Lemma 3.3.9 with g_0 and g_1 taking the values of $\pi \circ \widetilde{v} \circ (\frac{\Phi}{1-\delta})|_{(0,\delta)^2\times\{0\}}$ and $g \circ \Phi|_{(0,\delta)^2\times\{1\}}$, respectively, we obtain a mapping $\varphi_j \in W^{1,p}(D,\mathcal{N})$ satisfying the estimate

$$\int_D |D\varphi_j|^p \, dx \le C'\delta \left(\int_F |D_\top \widetilde{v}|^p \, d\mathcal{H}^2 + \int_F |D_\top g|^p \, d\mathcal{H}^2 \right) + C'\delta^2 \int_{\partial F} |D_\top g|^p \, d\mathcal{H}^1, \tag{3.3.23}$$

where $C' > 0$ is an absolute constant.

Let $\varphi_\delta \in L^\infty(B_1^3 \setminus \overline{B}_{1-\delta}^3, \mathcal{N})$ satisfy $\varphi_\delta|_{D_{2,j}} = \varphi_j \in W^{1,p}(D_{2,j},\mathcal{N})$ for each 3-cell $D_{2,j}$ from the decomposition (3.3.22). In view of our construction (see Lemma 3.3.9), we can ensure that the traces of the maps φ_j coincide on the mantles of the cells $D_{2,j}$ and hence $\varphi_\delta \in W^{1,p}(B_1^3 \setminus \overline{B}_{1-\delta}^3, \mathcal{N})$. Summing (3.3.23), where $D = D_{2,j}$ and $F = F_{2,j}$, over $j \in \{1,\dots,k_2\}$, taking into account that each 1-cell of the Lipschitz triangulation of $\mathbb{S}^2$ generated by Σ lies in the relative boundary of exactly two 2-cells from this triangulation, and also using (3.3.19) together with (3.3.20), we deduce the following chain of estimates

$$\begin{aligned}\int_{B_1^3\setminus\overline{B}_{1-\delta}^3} |D\varphi_\delta|^p \, dx &\le C'\delta \sum_{j=1}^{k_2} \left(\int_{F_{2,j}} |D_\top \widetilde{v}|^p \, d\mathcal{H}^2 + \int_{F_{2,j}} |D_\top g|^p \, d\mathcal{H}^2 \right) \\ &\quad + 2C'\delta^2 \int_\Sigma |D_\top g|^p \, d\mathcal{H}^1 \\ &\le C''\delta \int_{\mathbb{S}^2} |D_\top g|^p \, d\mathcal{H}^2,\end{aligned}$$

where $C'' = C''(\mathcal{N}) > 0$. This completes our proof of Lemma 3.3.10. □

We state now an η-extension with sublinear growth, which will turn out to be the key tool in the proof of our η-*compactness* lemma (see Lemma 3.3.12).

Lemma 3.3.11. *Let $p_0 \in (1,2)$ and $p \in [p_0,2)$. There exist constants $\eta_0, C > 0$, depending only on p_0 and $\mathcal{N}$, such that if $g \in W^{1,p}(\partial B_r^3, \mathcal{N})$ satisfies*

$$\int_{\partial B_r^3} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \le \frac{\eta_0 r^{2-p}}{2-p}, \tag{3.3.24}$$

then there exists $u \in W^{1,p}(B_r^3,\mathcal{N})$ such that $\operatorname{tr}_{\partial B_r^3}(u) = g$ and

$$\int_{B_r^3} \frac{|Du|^p}{p} \, dx \le C r^{\frac{2}{p}} \left(\int_{\partial B_r^3} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \right)^{1-\frac{1}{p}}. \tag{3.3.25}$$

Proof. By scaling, we can assume that $r = 1$. Letting $\delta \in (0, 1/2]$, which will be fixed later for the proof to work, we define

$$\eta_0 = \alpha\delta^{2-p}, \tag{3.3.26}$$

where $\alpha = \alpha(p_0, \mathcal{N}) > 0$ is the constant of Lemma 3.3.10. Then, by (3.3.24) and (3.3.26), (3.3.16) is satisfied in Lemma 3.3.10 applied with δ and g, which yields $\varphi_\delta \in W^{1,p}(B_1^3 \setminus \overline{B}_{1-\delta}^3, \mathcal{N})$ and $\widetilde{v} \in W^{1,2}(\mathbb{S}^2, \widetilde{\mathcal{N}})$, satisfying the condition (3.3.17) together with the estimates (3.3.18) and (3.3.19). Using Lemma 3.2.15 with the boundary datum $\pi \circ \widetilde{v}$, we obtain a map $w \in W^{1,p}(B_1^3, \mathcal{N})$ such that $\operatorname{tr}_{\mathbb{S}^2}(w) = \pi \circ \widetilde{v}$ and

$$\int_{B_1^3} |Dw|^p \, dx \le C \left(\int_{\mathbb{S}^2} |D_\top \widetilde{v}|^p \, d\mathcal{H}^2 \right)^{1-\frac{1}{p}} \le C' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \right)^{1-\frac{1}{p}}, \tag{3.3.27}$$

where $C = C(p_0, \mathcal{N}) > 0$ and $C' = C'(p_0, \mathcal{N}) > 0$. In the last estimate we used (3.3.19) under the condition that $p \in [p_0, 2)$. Now we define the mapping $u : B_1^3 \to \mathcal{N}$ by

$$u(x) = \begin{cases} \varphi_\delta(x) & \text{if } x \in B_1^3 \setminus \overline{B}_{1-\delta}^3, \\ \pi \circ \widetilde{v}\left(\dfrac{x}{1-\delta}\right) & \text{if } x \in \partial B_{1-\delta}^3, \\ w\left(\dfrac{x}{1-\delta}\right) & \text{if } x \in B_{1-\delta}^3. \end{cases}$$

Observe that $u \in W^{1,p}(B_1^3, \mathcal{N})$ and $\operatorname{tr}_{\mathbb{S}^2}(u) = g$ (see (3.3.17)). Using this, (3.3.18) and (3.3.27), we have

$$\int_{B_1^3} \frac{|Du|^p}{p} \, dx \le C' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \right)^{1-\frac{1}{p}} + C'\delta \int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2, \tag{3.3.28}$$

where we assume that C' is large enough so that (3.3.18) is valid with $C = C'$. Now we need to distinguish between two further cases.

Case 1: If $\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \le 1$, then $\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \le (\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2)^{1-\frac{1}{p}}$, which, in view of (3.3.28), yields

$$\int_{B_1^3} \frac{|Du|^p}{p} \, dx \le 2C' \left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \right)^{1-\frac{1}{p}}. \tag{3.3.29}$$

Case 2: When $\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 > 1$, we need to fix δ so that $\delta^{2-p} = \text{const}$ and $\delta \le 2 - p$. This holds for

$$\text{const} = \left(\frac{1}{e}\right)^{\frac{1}{e}} \quad \text{and} \quad \delta = \left(\frac{1}{e}\right)^{\frac{1}{e(2-p)}}, \tag{3.3.30}$$

since $\inf_{x\in(0,+\infty)} x^x = (1/e)^{1/e}$. This defines our δ and hence η_0 (see (3.3.26)). Using (3.3.24), (3.3.26), (3.3.28), (3.3.30) and the fact that $\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 > 1$, we obtain

$$\begin{aligned}\int_{B_1^3} \frac{|Du|^p}{p}\, dx &\le C'\left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2\right)^{1-\frac{1}{p}} + \frac{C'\alpha\delta^{3-p}}{2-p}\\ &\le C''\left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2\right)^{1-\frac{1}{p}} + C''\\ &\le 2C''\left(\int_{\mathbb{S}^2} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2\right)^{1-\frac{1}{p}},\end{aligned} \tag{3.3.31}$$

where $C'' = C''(p_0, \mathcal{N}) > 0$. The estimates (3.3.29) and (3.3.31) imply then, up to a scaling, (3.3.25). This completes our proof of Lemma 3.3.11. □

Now we prove our η-compactness lemma.

Lemma 3.3.12. *Let $\kappa \in (0,1)$, $p_0 \in (1,2)$, $p \in [p_0, 2)$ and let $E = \Psi(B_r^3) \subset \mathbb{R}^3$ be an open set, where $\Psi : B_r^3 \to \mathbb{R}^3$ is a bilipschitz homeomorphism onto its image with a bilipschitz constant $L \ge 1$. There exist $\eta, C > 0$, depending only on κ, p_0, L and $\mathcal{N}$, such that if $u_p \in W^{1,p}(E, \mathcal{N})$ is a p-minimizer and satisfies*

$$\int_E \frac{|Du_p|^p}{p}\, dx \le \frac{\eta r^{3-p}}{2-p}, \tag{3.3.32}$$

then

$$\int_{\Psi(B_{\kappa r}^3)} \frac{|Du_p|^p}{p}\, dx \le C r^{3-p}. \tag{3.3.33}$$

Remark 3.3.13. Let us point out that the dependence on p in Lem. 3.3.12 is essential, since it is used in Lemma 3.5.2.

Proof of Lemma 3.3.12. We define $v_p = u_p \circ \Psi$, so that, by the chain rule, we have $v_p \in W^{1,p}(B_r^3, \mathcal{N})$ and

$$\int_{B_r^3} \frac{|Dv_p|^p}{p}\, dx \le L^{3+p} \int_E \frac{|Du_p|^p}{p}\, dx. \tag{3.3.34}$$

Let $\eta_0 > 0$ be the constant of Lemma 3.3.11. We define

$$G_p = \Bigg\{ \varrho \in (\kappa r, r) : \mathrm{tr}_{\partial B_\varrho^3}(v_p) = v_p|_{\partial B_\varrho^3} \in W^{1,p}(\partial B_\varrho^3, \mathcal{N}) \\ \text{and } \int_{\partial B_\varrho^3} \frac{|Dv_p|^p}{p}\, d\mathcal{H}^2 \le \frac{\eta_0 (\kappa r)^{2-p}}{2-p} \Bigg\}.$$

In view of (3.3.34) and (3.3.32),

$$\mathcal{H}^1((\kappa r, r)\backslash G_p) \leq \frac{2-p}{\eta_0(\kappa r)^{2-p}} \int_{B_r^3} \frac{|Dv_p|^p}{p}\,dx \leq \frac{L^{3+p}\eta r}{\eta_0 \kappa^{2-p}} \leq \frac{L^5 \eta r}{\eta_0 \kappa}. \tag{3.3.35}$$

Choosing

$$\eta := \frac{\eta_0 \kappa (1-\kappa)}{2L^5}$$

and using (3.3.35), we obtain that

$$\mathcal{H}^1(G_p) \geq \frac{(1-\kappa)r}{2}. \tag{3.3.36}$$

If $\varrho \in G_p$, then

$$\int_{\partial B_\varrho^3} \frac{|Dv_p|^p}{p}\,d\mathcal{H}^2 \leq \frac{\eta_0(\kappa r)^{2-p}}{2-p} \leq \frac{\eta_0 \varrho^{2-p}}{2-p}.$$

Thus, letting $v_p^\varrho : B_\varrho^3 \to \mathcal{N}$ be the extension of $v_p|_{\partial B_\varrho^3}$ given by Lemma 3.3.11, we have by the p-minimizing property and the bilipschitz invariance,

$$\begin{aligned} L^{-3-p} & \int_{B_\varrho^3} \frac{|Dv_p|^p}{p}\,dx \\ & \leq \int_{\Psi(B_\varrho^3)} \frac{|Du_p|^p}{p}\,dx \leq \int_{\Psi(B_\varrho^3)} \frac{|D(v_p^\varrho \circ \Psi^{-1})|^p}{p}\,dx \\ & \leq L^{3+p} \int_{B_\varrho^3} \frac{|Dv_p^\varrho|^p}{p}\,dx \\ & \leq C L^{3+p} \varrho^{\frac{2}{p}} \left(\int_{\partial B_\varrho^3} \frac{|Dv_p|^p}{p}\,d\mathcal{H}^2 \right)^{1-\frac{1}{p}} \end{aligned} \tag{3.3.37}$$

for each $\varrho \in G_p$, where $C = C(p_0, \mathcal{N}) > 0$. We define now the function $F : [\kappa r, r] \to [0, \infty)$ by

$$F(\varrho) = \int_{B_\varrho^3} \frac{|Dv_p|^p}{p}\,dx.$$

Without loss of generality, we assume that $F(\kappa r) > 0$, because otherwise (3.3.33) follows. The function F is absolutely continuous and for every $\varrho \in G_p$, we have by (3.3.37),

$$F'(\varrho) = \int_{\partial B_\varrho^3} \frac{|Dv_p|^p}{p}\,d\mathcal{H}^2 \geq c \frac{F(\varrho)^{\frac{p}{p-1}}}{\varrho^{\frac{2}{p-1}}},$$

and thus

$$\begin{aligned}\frac{p-1}{F(\kappa r)^{\frac{1}{p-1}}} &\geq \frac{p-1}{F(\kappa r)^{\frac{1}{p-1}}} - \frac{p-1}{F(r)^{\frac{1}{p-1}}} \\ &= \int_{\kappa r}^{r} \frac{F'(\varrho)}{F(\varrho)^{\frac{p}{p-1}}}\, d\varrho \\ &\geq c \int_{G_\varrho} \frac{d\varrho}{\varrho^{\frac{2}{p-1}}} \\ &\geq c \frac{\mathcal{H}^1(G_\varrho)}{r^{\frac{2}{p-1}}} \geq \frac{c(1-\kappa)}{2r^{\frac{3-p}{p-1}}},\end{aligned}$$

where $c = c(p_0, L, \mathcal{N}) > 0$ and the last estimate comes from (3.3.36). Then the estimate (3.3.33) follows through a bilipschitz change of variable, completing the proof of Lemma 3.3.12. □

3.4 Convergence to a harmonic map

The following variant (Lemma 3.4.1) of the Luckhaus lemma (see [44, Lemma 1], [43, Lemma 3]) is our key tool in the proof of Proposition 3.4.2 of the fact that a weakly convergent sequence of minimizing p-harmonic maps converges to a minimizing 2-harmonic map when $p \nearrow 2$. It says that if two $W^{1,p}$ mappings from a compact 2-dimensional Riemannian manifold $\mathcal{M}$ without boundary into $\mathcal{N}$ are close in L^p, then we can find a map w in $\mathcal{M} \times (0,T)$, having these two mappings as boundary values, such that w does not leave a tubular neighborhood of $\mathcal{N}$ and such that its $W^{1,p}$ norm is of order T.

Lemma 3.4.1. *Let $\mathcal{M}$ be a compact Riemannian manifold of dimension 2 without boundary and $p \in (1,2]$. Then there exist positive constants C, T_0, depending only on $\mathcal{M}$, such that for every $u, v \in W^{1,p}(\mathcal{M},\mathcal{N})$ and $T \in (0,T_0)$ there exists a map $w \in W^{1,p}(\mathcal{M}\times(0,T),\mathbb{R}^\nu)$ satisfying*

(i) $\operatorname{tr}_{\mathcal{M}\times\{0\}}(w) = u, \;\; \operatorname{tr}_{\mathcal{M}\times\{T\}}(w) = v$;

(ii) $$\int_{\mathcal{M}\times(0,T)} |Dw|^p \, d\mathcal{H}^2\, dt \leq CT \int_{\mathcal{M}} \left(|D_\top u|^p + |D_\top v|^p + \frac{|u-v|^p}{T^p} \right) d\mathcal{H}^2;$$

(iii) we have

$$\begin{aligned}\operatorname{esssup}_{\mathcal{M}\times(0,T)} \operatorname{dist}(w,\mathcal{N}) &\le C\|u-v\|_{L^\infty(\mathcal{M})}\\ &\le CT^{1-\frac{2}{p}}\left(\int_{\mathcal{M}}\left(|D_\top u|^p+|D_\top v|^p+\frac{|u-v|^p}{T^p}\right)d\mathcal{H}^2\right)^{\frac{1}{p}\left(1-\frac{1}{2p}\right)}\\ &\qquad\qquad\times\left(\int_{\mathcal{M}}\frac{|u-v|^p}{T^p}\,d\mathcal{H}^2\right)^{\frac{1}{p}\frac{1}{2p}}.\end{aligned}$$

Proof. For a proof, we refer the reader to [43, Lemma 3] (see also [44, Lemma 1] for the original Luckhaus lemma, where $\mathcal{M}=\mathbb{S}^2$). Notice that the inductive assumption in [43, Lemma 3] may not be satisfied. However, if one slightly modifies the Luckhaus proof, namely, one defines $w(x,t)=\frac{t}{T}v(x)+\left(1-\frac{t}{T}\right)u(x)$ for x belonging to the closure of the l-skeleton of $\mathcal{M}$ in the case when either $[p]<p$ and $l=[p]$ or $[p]=p$ and $l=[p-1]$, then the inductive assumption in [43, Lemma 3] is satisfied, and the proof works as written (observe that Luckhaus defines w as above for x belonging to the closure of the l-skeleton of $\mathcal{M}$ in the case when $l=[p-1]$). It is also possible to reduce the dependence of the constant C in [43, Lemma 3] on $\dim(\mathcal{N})$. $\square$

Proposition 3.4.2. *Let $U\subset\mathbb{R}^3$ be an open set. Let $(p_n)_{n\in\mathbb{N}}\subset[1,2)$, $u_n\in W^{1,p_n}(U,\mathcal{N})$ and $p_n\nearrow 2$ as $n\to+\infty$. If for each open set $K\Subset U$ and each $n\in\mathbb{N}$, u_n is a p_n-minimizer in K and if*

$$\sup_{n\in\mathbb{N}}\int_K\frac{|Du_n|^{p_n}}{p_n}\,dx\le C, \tag{3.4.1}$$

where $C=C(K)>0$, then there exists $u_*\in W^{1,2}_{\mathrm{loc}}(U,\mathcal{N})$ such that, up to a subsequence (not relabeled), the following assertions hold.

(i) u_n converges to u_* weakly in $W^{1,p}_{\mathrm{loc}}(U,\mathbb{R}^\nu)$ for all $p\in(1,2)$ and a.e. in U.

(ii) For each open set $K\Subset U$,

$$\int_K\frac{|Du_n|^{p_n}}{p_n}\,dx\to\int_K\frac{|Du_*|^2}{2}\,dx$$

as $n\to+\infty$. Furthermore, $\int_K|Du_n-Du_*|^{p_n}\,dx\to0$ and $Du_n\to Du_*$ a.e. in U as $n\to+\infty$.

(iii) If $K\Subset U$ is open, then u_* is a 2-minimizer in K. Furthermore, there exists at most a countable and locally finite subset S_0 of U such that $u_*\in C^\infty(U\setminus S_0,\mathcal{N})$.

Proof. Fix an open set $K \Subset U$. Since $\overline{K} \subset U$ is compact, for any cover by open balls, $\overline{K}$ admits a finite subcover. Then there exists an open set V such that $\overline{K} \subset V$, $\overline{V} \subset U$ and $\overline{V}$ is a compact smooth Riemannian manifold of dimension 3 with boundary ∂V (possibly disconnected, but with a finite number of connected components). Then ∂V is a 2-dimensional compact smooth Riemannian manifold without boundary. According to Lemma 3.2.1 with $\mathcal{N}$ replaced by ∂V, there exists $\lambda_{\partial V} > 0$ such that the nearest point retraction $\Pi_{\partial V} : \partial V_{\lambda_{\partial V}} \to \partial V$ is a well-defined and smooth mapping, where $\partial V_{\lambda_{\partial V}} = \{x \in \overline{V} : \operatorname{dist}(x, \partial V) \leq \lambda_{\partial V}\}$. Choose $\lambda \in (0, \lambda_{\partial V})$ so that $\overline{K} \subset V \setminus \overline{A}_\lambda$, where $A_\lambda = \{x \in V : \operatorname{dist}(x, \partial V) < \lambda\}$.

If $p_n \in (p, 2)$ is large enough, then, applying Young's inequality, we have

$$\int_V \frac{|Du_n|^p}{p}\,dx \leq \int_V \frac{|Du_n|^{p_n}}{p_n}\,dx + \Big(\frac{1}{p} - \frac{1}{p_n}\Big)|V| \tag{3.4.2}$$

so that

$$\limsup_{n\to+\infty} \int_V \frac{|Du_n|^p}{p}\,dx < \infty \tag{3.4.3}$$

(see (3.4.1)). Since $\mathcal{N}$ is compact, $(u_n)_{n\in\mathbb{N}}$ is bounded in $L^\infty(U, \mathbb{R}^\nu)$, which, together with (3.4.3), implies that for each $p \in (1,2)$, $(u_n)_{n\in\mathbb{N}}$ is bounded in $W^{1,p}(V, \mathbb{R}^\nu)$. Thus, there exists a map $u_* : V \to \mathbb{R}^\nu$ such that, up to a subsequence (still denoted by n), for each $p \in (1,2)$,

$$u_n \rightharpoonup u_* \text{ weakly in } W^{1,p}(V, \mathbb{R}^\nu) \qquad \text{and} \qquad u_n \to u_* \text{ a.e. in } V. \tag{3.4.4}$$

In particular, $u_* \in \mathcal{N}$ a.e. in V.

By the weak convergence, (3.4.2) and the fact that $p_n \nearrow 2$ as $n \to +\infty$, for each $p \in (1,2)$ we have

$$\begin{aligned}\int_V \frac{|Du_*|^p}{p}\,dx &\leq \liminf_{n\to+\infty} \int_V \frac{|Du_n|^p}{p}\,dx \\ &\leq \liminf_{n\to+\infty} \int_V \frac{|Du_n|^{p_n}}{p_n}\,dx + \Big(\frac{1}{p} - \frac{1}{2}\Big)|V|.\end{aligned} \tag{3.4.5}$$

Letting $p \nearrow 2$ in (3.4.5) and using Fatou's lemma, we obtain the estimate

$$\int_V \frac{|Du_*|^2}{2}\,dx \leq \liminf_{n\to+\infty} \int_V \frac{|Du_n|^{p_n}}{p_n}\,dx, \tag{3.4.6}$$

which, together with (3.4.1), implies that $u_* \in W^{1,2}(V, \mathcal{N})$. We prove that u_* is a 2-minimizer in K and that, up to a subsequence (not relabeled),

$$\int_K \frac{|Du_n|^{p_n}}{p_n}\,dx \to \int_K \frac{|Du_*|^2}{2}\,dx \tag{3.4.7}$$

and $\|Du_n - Du_*\|^{p_n}_{L^{p_n}(K,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \to 0$ as $n \to +\infty$.

Applying [31, Theorem 3.2.22 (3)] with $W = A_\lambda$, $Z = [0,\lambda]$ and $f : W \to Z$ defined by $f(x) = |x - \Pi_{\partial V}(x)| = \operatorname{dist}(x, \partial V)$, and using also the fact that $|Df(x)| = 1$ for each $x \in A_\lambda$ (see Lemma 3.2.1), Fatou's lemma and (3.4.1), we obtain

$$\int_0^\lambda d\varrho \liminf_{n\to+\infty} \int_{f^{-1}(\{\varrho\})} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \le \liminf_{n\to+\infty} \int_{A_\lambda} \frac{|Du_n|^{p_n}}{p_n}\, dx \le C.$$

This implies that

$$\mathcal{H}^1\Big(\Big\{\varrho \in (0,\lambda) : \liminf_{n\to+\infty} \int_{f^{-1}(\{\varrho\})} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \ge \frac{4C}{\lambda}\Big\}\Big) \le \frac{\lambda}{4}.$$

So there exists $\varrho \in (0, \lambda/2)$ such that, up to a subsequence (not relabeled), we know that for each sufficiently large $n \in \mathbb{N}$,

$$\begin{aligned} &\operatorname{tr}_{f^{-1}(\{\varrho\})}(u_n) = u_n|_{f^{-1}(\{\varrho\})} \in W^{1,p_n}(f^{-1}(\{\varrho\}), \mathcal{N}), \\ &\qquad\qquad \int_{f^{-1}(\{\varrho\})} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 < \frac{4C}{\lambda} \end{aligned} \tag{3.4.8}$$

and $\operatorname{tr}_{f^{-1}(\{\varrho\})}(u_*) = u_*|_{f^{-1}(\{\varrho\})}$. Fix such $\varrho \in (0, \lambda/2)$ and an arbitrary $p \in (1,2)$. Now we define $\mathcal{M} = f^{-1}(\{\varrho\})$. Notice that $\mathcal{M}$ is a 2-dimensional compact smooth Riemannian manifold without boundary. In view of (3.4.4) and the trace theorem, we have $u_n|_{\mathcal{M}} \rightharpoonup u_*|_{\mathcal{M}}$ weakly in $W^{1-1/p,p}(\mathcal{M}, \mathbb{R}^\nu)$ (we use that a bounded linear operator maps a weakly convergent sequence to a weakly convergent one). Then, by the compact embedding, up to a subsequence (not relabeled),

$$u_n|_{\mathcal{M}} \to u_*|_{\mathcal{M}} \text{ strongly in } L^q(\mathcal{M}, \mathbb{R}^\nu) \tag{3.4.9}$$

for each $q \in \left[1, \frac{2p}{2-p}\right)$. Since $u_n|_{\mathcal{M}} \in W^{1,p_n}(\mathcal{M}, \mathcal{N})$, (3.4.9) implies that $u_*(x) \in \mathcal{N}$ for $\mathcal{H}^2$-a.e. $x \in \mathcal{M}$. Using (3.4.8) and (3.4.9), we deduce that, up to a subsequence (not relabeled),

$$u_n|_{\mathcal{M}} \rightharpoonup u_*|_{\mathcal{M}} \text{ weakly in } W^{1,p}(\mathcal{M}, \mathbb{R}^\nu).$$

Then, by the weak convergence, Young's inequality and (3.4.8), we have

$$\begin{aligned} \int_{\mathcal{M}} \frac{|D_\top u_*|^p}{p}\, d\mathcal{H}^2 &\le \liminf_{n\to+\infty} \int_{\mathcal{M}} \frac{|D_\top u_n|^p}{p}\, d\mathcal{H}^2 \\ &\le \liminf_{n\to+\infty} \Big(\int_{\mathcal{M}} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 + \Big(\frac{1}{p} - \frac{1}{p_n}\Big)\mathcal{H}^2(\mathcal{M})\Big) \\ &\le \frac{4C}{\lambda} + \Big(\frac{1}{p} - \frac{1}{2}\Big)\mathcal{H}^2(\mathcal{M}), \end{aligned} \tag{3.4.10}$$

where we have also used that $|D_\top u_n| \le |Du_n|$. This implies that $u_*|_{\mathcal{M}} \in W^{1,2}(\mathcal{M},\mathcal{N})$, because $p \in (1,2)$ was arbitrarily chosen and we already know that $u_*(x) \in \mathcal{N}$ for $\mathcal{H}^2$-a.e. $x \in \mathcal{M}$. Moreover, $u_* \in L^\infty(\mathcal{M},\mathbb{R}^\nu)$. Also, (3.4.10) yields

$$\int_{\mathcal{M}} \frac{|D_\top u_*|^2}{2}\, d\mathcal{H}^2 \le \lim_{p\nearrow 2} \int_{\mathcal{M}} \frac{|D_\top u_*|^p}{p}\, d\mathcal{H}^2 \le \frac{4C}{\lambda}. \tag{3.4.11}$$

Next, we want to apply Lemma 3.4.1. Let $T_0 = T_0(\mathcal{M}) > 0$ be the constant of Lemma 3.4.1. Choose $T \in (0, T_0)$ sufficiently small and let $\Phi : A_\lambda \setminus A_\varrho \to \mathcal{M} \times [0, \lambda - \varrho)$ be defined by

$$\Phi(x) = (\Pi_{\mathcal{M}}(x), \operatorname{dist}(x,\mathcal{M})).$$

By Lemma 3.2.1, there exists $\lambda_{\mathcal{N}} > 0$ such that the nearest point retraction $\Pi_{\mathcal{N}} : \mathcal{N}_{\lambda_{\mathcal{N}}} \to \mathcal{N}$, where $\mathcal{N}_{\lambda_{\mathcal{N}}} = \{x \in \mathbb{R}^\nu : \operatorname{dist}(x,\mathcal{N}) < \lambda_{\mathcal{N}}\}$, is well defined and smooth. Setting $s_n = \|u_n - u_*\|_{L^{p_n}(\mathcal{M},\mathbb{R}^\nu)}$ (without loss of generality, we assume that for each $n \in \mathbb{N}$ large enough, $s_n > 0$) and using (3.4.8), (3.4.9) and (3.4.11), we observe that $s_n \to 0$ and for each sufficiently large $n \in \mathbb{N}$,

$$\int_{\mathcal{M}} \left(|D_\top u_n|^{p_n} + |D_\top u_*|^{p_n} + \frac{|u_n - u_*|^{p_n}}{s_n^{p_n/2}} \right) d\mathcal{H}^2 \le C', \tag{3.4.12}$$

where C' is a positive constant independent of n. Applying Lemma 3.4.1 with $u = u_n|_{\mathcal{M}}$, $v = u_*|_{\mathcal{M}}$ and $T = s_n^{1/2}$, we obtain a map $w_n \in W^{1,p_n}(\mathcal{M} \times (0, s_n^{1/2}), \mathbb{R}^\nu)$ interpolating between $u_n|_{\mathcal{M}}$ and $u_*|_{\mathcal{M}}$. According to Lemma 3.4.1 (iii), (3.4.12) and the definition of s_n, if $n \in \mathbb{N}$ is large enough, we have $\operatorname{dist}(w_n(x,t),\mathcal{N}) < \lambda_{\mathcal{N}}$ for a.e. $(x,t) \in \mathcal{M} \times (0, s_n^{1/2})$. Thus, for each $n \in \mathbb{N}$ large enough, we can define the map

$$\varphi_n(y) = \Pi_{\mathcal{N}}(w_n(\Phi(y)))$$

for each $y \in \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}))$. Then we have the following: $\varphi_n \in W^{1,p_n}(\Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2})), \mathcal{N})$; $\operatorname{tr}_{\mathcal{M}}(\varphi_n) = u_n|_{\mathcal{M}}$, $\operatorname{tr}_{\Phi^{-1}(\mathcal{M}\times\{s_n^{1/2}\})}(\varphi_n) = u_*|_{\mathcal{M}} \circ \Phi_n \circ \Phi|_{\Phi^{-1}(\mathcal{M}\times\{s_n^{1/2}\})}$, where $\Phi_n : \mathcal{M} \times [s_n^{1/2}, T]$ is defined by $\Phi_n(x,t) = \Phi^{-1}(x, t - s_n^{1/2})$ and we have used that $\Phi^{-1}(x,0) = x$; in view of Lemma 3.4.1 (ii) and (3.4.12),

$$\int_{\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}))} \frac{|D\varphi_n|^{p_n}}{p_n}\, dy \le C'' s_n^{1/2}, \tag{3.4.13}$$

where $C'' > 0$ is a constant independent of n, but depending on the bilipschitz constant of Φ.

Now let $E = V \setminus \overline{A}_\varrho$. According to our construction, E is a finite union of bounded Lipschitz domains, and E contains K. Fix an arbitrary $w_* \in W^{1,2}(E, \mathcal{N})$ with $\operatorname{tr}_{\mathcal{M}}(w_*) = \operatorname{tr}_{\mathcal{M}}(u_*)$. Let $\zeta \in C_c([0, +\infty))$ be the Lipschitz map such that $\zeta = 1$ on $[0, 1]$, $\zeta(s) = 2 - s$ for $s \in [1, 2]$, $\zeta = 0$ on $[2, +\infty]$. Hence $\|\zeta'\|_{L^\infty([0,+\infty))} \leq 1$. Next, for each sufficiently large $n \in \mathbb{N}$, define $w_n : E \to \mathcal{N}$ by

$$w_n(y) = \begin{cases} \varphi_n(y) & \text{if } y \in \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}]), \\ w_*(\Psi_n(y)) & \text{if } y \in E \setminus \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}]), \end{cases} \tag{3.4.14}$$

where $\Psi_n : E \setminus \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}]) \to E$ is the bilipschitz map defined by

$$\Psi_n(y) = \left(1 - \zeta\left(\frac{\operatorname{dist}(y, \mathcal{M})}{s_n^{1/3}}\right)\right) y + \zeta\left(\frac{\operatorname{dist}(y, \mathcal{M})}{s_n^{1/3}}\right) \Phi_n(\Phi(y)).$$

Notice that: $\operatorname{tr}_{\Phi^{-1}(\mathcal{M} \times \{s_n^{1/2}\})}(w_* \circ \Psi_n) = u_*|_{\mathcal{M}} \circ \Phi_n \circ \Phi|_{\Phi^{-1}(\mathcal{M} \times \{s_n^{1/2}\})}$; if $y \in E \setminus \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}])$ and $\operatorname{dist}(y, \mathcal{M}) \leq s_n^{1/3}$, then we have $\Psi_n(y) = \Phi_n(\Phi(y))$; if $y \in E \setminus \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}])$ and $\operatorname{dist}(y, \mathcal{M}) \geq 2s_n^{1/3}$, then $\Psi_n(y) = y$; for each $y \in E$, if $n \in \mathbb{N}$ is large enough, then $\Psi_n(y)$ is defined and $\Psi_n(y) \to y$ as $n \to +\infty$; for a.e. $y \in E \setminus \Phi(\mathcal{M} \times (0, s_n^{1/2}])$ and for each $n \in \mathbb{N}$ large enough,

$$\begin{aligned} &|D\Psi_n(y) - \operatorname{Id}| \\ &\quad = \left| \zeta'\left(\frac{\operatorname{dist}(y, \mathcal{M})}{s_n^{1/3}}\right) \frac{D \operatorname{dist}(y, \mathcal{M})}{s_n^{1/3}} \otimes (\Phi_n(\Phi(y)) - y) \right. \\ &\qquad \left. + \zeta\left(\frac{\operatorname{dist}(y, \mathcal{M})}{s_n^{1/3}}\right) (D\Phi_n(\Phi(y)) \circ D\Phi(y) - \operatorname{Id}) \right| \\ &\quad \leq L s_n^{1/6}, \end{aligned} \tag{3.4.15}$$

where $L > 0$ is a constant independent of n, but depending on the bilipschitz constant of Φ. In the above computations we have used that $|\zeta'| \leq 1$, $|D \operatorname{dist}(y, \mathcal{M})| = 1$ and $|a \otimes b| \leq |a||b|$ for $a, b \in \mathbb{R}^3$. In particular, when $n \in \mathbb{N}$ is large enough, $|D\Psi_n(y) - \operatorname{Id}| < 1/2$. It is worth noting that, according to the definition of Ψ_n and [30, 4.8 (5)], Ψ_n is continuously differentiable on the three regions of $E \setminus \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}])$, where $\operatorname{dist}(\cdot, \mathcal{M}) < s_n^{1/3}$, $\operatorname{dist}(\cdot, \mathcal{M}) \in (s_n^{1/3}, 2s_n^{1/3})$, and $\operatorname{dist}(\cdot, \mathcal{M}) > 2s_n^{1/3}$, respectively. Then, using the inverse function theorem for Ψ_n on each of these three regions (namely, on compact subsets of these regions), and taking into account the definition of Ψ_n (namely, the injectivity of Ψ_n), one observes that Ψ_n admits the inverse Ψ_n^{-1} such that Ψ_n^{-1} is a.e. differentiable on E and $D\Psi_n^{-1}$ is sufficiently close to the identity a.e. on E. Furthermore, Ψ_n^{-1} is absolutely continuous on each segment lying in the closure of one of the images of Ψ_n of the three regions of $E \setminus \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}])$ considered above. Thus, Ψ_n is bilipschitz on $E \setminus \Phi^{-1}(\mathcal{M} \times (0, s_n^{1/2}])$, as we mentioned earlier. It is also worth noting that for a.e. $y \in E$, if n is large enough, then $D\Psi_n(y)$ is defined, $D(w_* \circ \Psi_n)(y) = Dw_*(\Psi_n(y)) \circ D\Psi_n(y)$ and $D\Psi_n(y) \to \mathrm{Id}$, $D(w_* \circ \Psi_n)(y) \to Dw_*(y)$ as $n \to +\infty$. In addition, if n is large enough, then for a.e. $y \in E$,

$$\begin{aligned}
&\frac{|D(w_* \circ \Psi_n)(y) 1_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])}(y)|^{p_n}}{p_n} \\
&\leq 2^{p_n} \frac{|Dw_*(\Psi_n(y)) 1_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])}(y)|^{p_n}}{p_n} \\
&\leq 2^{p_n} \left(\frac{|Dw_*(\Psi_n(y)) 1_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])}(y)|^2}{2} + \frac{1}{p_n} - \frac{1}{2} \right) \leq h(y),
\end{aligned}$$

where we have used (3.4.15), the fact that $|\mathrm{Id}| = \sqrt{3}$, Young's inequality and since $|Dw_*|^2 \in L^1(E)$, there exists $h \in L^1(E)$ such that the last estimate holds.

Taking into account the above observations and using the Lebesgue dominated convergence theorem, we deduce that

$$\int_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])} \frac{|Dw_n|^{p_n}}{p_n}\, dy \to \int_E \frac{|Dw_*|^2}{2}\, dy \tag{3.4.16}$$

as $n \to +\infty$. The mapping w_n is a competitor for u_n in E, and hence, in view of (3.4.14),

$$\begin{aligned}
&\int_E \frac{|Du_n|^{p_n}}{p_n}\, dy \\
&\quad \leq \int_{\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}))} \frac{|D\varphi_n|^{p_n}}{p_n}\, dy + \int_{E\setminus\Phi^{-1}(\mathcal{M}\times(0,s_n^{1/2}])} \frac{|Dw_n|^{p_n}}{p_n}\, dy.
\end{aligned} \tag{3.4.17}$$

Using (3.4.6) with V replaced by E (which is possible thanks to (3.4.4) and proceeding as in (3.4.5)), together with (3.4.13), (3.4.16) and (3.4.17), we obtain

$$\begin{aligned}
\int_E \frac{|Du_*|^2}{2}\, dy &= \lim_{p \nearrow 2} \int_E \frac{|Du_*|^p}{p}\, dy \\
&\leq \liminf_{n\to+\infty} \int_E \frac{|Du_n|^{p_n}}{p_n}\, dy \\
&\leq \limsup_{n\to+\infty} \int_E \frac{|Du_n|^{p_n}}{p_n}\, dy \leq \int_E \frac{|Dw_*|^2}{2}\, dy.
\end{aligned}$$

This implies that $\int_E \frac{|Du_n|^{p_n}}{p_n}\,dy \to \int_E \frac{|Du_*|^2}{2}\,dy$ and u_* is a 2-minimizer in E. In particular, according to Lemma 3.4.3 below, it holds

$$\|Du_n - Du_*\|^{p_n}_{L^{p_n}(E,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \to 0,$$

which proves (3.4.7).

Since u_* is a 2-minimizer in E and $K \Subset E$ is open, u_* is a 2-minimizer in K and, according to [53, Theorem II], there exists a finite set $S_K \subset K$ such that $u_* \in C^\infty(K \setminus S_K, \mathcal{N})$.

In order to reach the conclusion, we use a diagonal argument and rely on the fact that one can write $U = \bigcup_{m\in\mathbb{N}} K_m$, with $K_m \subset K_{m+1}$ so that if $K \Subset U$, then $K \subset K_m$ for some $m \in \mathbb{N}$. This completes our proof of Proposition 3.4.2. □

Lemma 3.4.3. *Let $U \subset \mathbb{R}^3$ be open and bounded, $u_* \in W^{1,2}(U,\mathbb{R}^\nu)$, $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_n}(U,\mathbb{R}^\nu)$. If $u_n \rightharpoonup u_*$ weakly in $W^{1,p}(U,\mathbb{R}^\nu)$ for all $p \in (1,2)$ and $\int_U |Du_n|^{p_n}\,dx \to \int_U |Du_*|^2\,dx$ as $n \to +\infty$, then $\int_U |Du_n - Du_*|^{p_n}\,dx \to 0$ as $n \to +\infty$.*

Proof. The desired convergence comes by using Hanner's inequality, see [57, Lemma 6.3]. To apply [57, Lemma 6.3], we only need to prove that

$$4\|Du_*\|^2_{L^2(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \le \liminf_{n\to+\infty} \|Du_n + Du_*\|^{p_n}_{L^{p_n}(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)}. \tag{3.4.18}$$

Let $p \in (1,2)$ be arbitrary. Notice that $Du_n + Du_* \rightharpoonup 2Du_*$ weakly in $L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)$ and hence

$$\liminf_{n\to+\infty} \|Du_n + Du_*\|_{L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \ge 2\|Du_*\|_{L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)}. \tag{3.4.19}$$

Using Hölder's inequality and (3.4.19), we obtain

$$\begin{aligned} &\liminf_{n\to+\infty} \|Du_n + Du_*\|^{p_n}_{L^{p_n}(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \\ &\qquad \ge \liminf_{n\to+\infty} |U|^{1-\frac{p_n}{p}} \|Du_n + Du_*\|^{p_n}_{L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \\ &\qquad \ge 4|U|^{1-\frac{2}{p}} \|Du_*\|^2_{L^p(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)}. \end{aligned} \tag{3.4.20}$$

Letting $p \nearrow 2$ and using the continuity of the L^p-norm, in view of (3.4.20), we obtain (3.4.18), which completes our proof of Lemma 3.4.3. □

3.5 The singular set

Let $p \in (1,2)$ and $u_p \in W^{1,p}(\Omega,\mathcal{N})$ be a p-minimizer in Ω. For each Borel-measurable set $E \in \mathcal{B}(\overline{\Omega})$, we define the positive Radon measure μ_p at E by

$$\mu_p(E) := (2-p)\int_E \frac{|Du_p|^p}{p}\,dx.$$

Let $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers in Ω. We assume that for each $n \in \mathbb{N}$ large enough, setting $\mu_n = \mu_{p_n}$, it holds

$$\mu_n(\overline{\Omega}) \leq C_0, \tag{3.5.1}$$

where $C_0 > 0$ is a constant independent of n. According to Proposition 3.2.21, the condition (3.5.1) is satisfied whenever for each $n \in \mathbb{N}$ large enough, $\operatorname{tr}_{\partial\Omega}(u_n) = g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$. Next, in view of (3.5.1), using the Banach-Alaoglu theorem, we obtain a positive Radon measure $\mu_* \in (C(\overline{\Omega}))'$ such that, up to a subsequence (not relabeled),

$$\mu_n \overset{*}{\rightharpoonup} \mu_* \text{ weakly* in } (C(\overline{\Omega}))'. \tag{3.5.2}$$

We denote the support of μ_* by $S_* \subset \overline{\Omega}$.

Remark 3.5.1. If the measure μ_* is independent of the subsequence, the entire sequence $(\mu_n)_{n\in\mathbb{N}}$ weakly* converges to μ_* as $n \to +\infty$, since in the latter case each subsequence of $(\mu_n)_{n\in\mathbb{N}}$ contains a subsequence weakly* converging to μ_*. In general, the limit measure μ_* does depend on the subsequence. To see this, we consider the projective plane $\mathcal{N} = \mathbb{RP}^2$, whose universal covering $\pi\colon \mathbb{S}^2 \to \mathbb{RP}^2$ identifies antipodal points (see e.g. [38, Example 0.4]), so that in particular $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}/2\mathbb{Z}$. Defining first $\hat{g}\colon \mathbb{S}^2 \to \mathbb{S}^1$ by

$$\hat{g}(x) = \frac{(x_1x_2, x_3)}{(x_1^2x_2^2 + x_3^2)^{1/2}},$$

one checks that $\hat{g}(x_1,x_2,x_3) = \hat{g}(x_2,x_1,x_3) = -\hat{g}(x_1,-x_2,-x_3)$, that $\hat{g}$ has degree 1 around small circles around $(1,0,0)$ and $(-1,0,0)$ and degree 1 around small circles around $(0,1,0)$ and $(0,-1,0)$ and that $\hat{g} \in W^{1/2,2}(\mathbb{S}^2,\mathbb{S}^1)$. We define $g = \tau \circ \hat{g}$, where the map $\tau\colon \mathbb{S}^1 \to \mathbb{RP}^2$ defined by the condition that $\tau(\cos\theta, \sin\theta) = \pi(\cos\frac{\theta}{2}, \sin\frac{\theta}{2}, 0)$. We have $g \in W^{1/2,2}(\mathbb{S}^2, \mathbb{RP}^2)$; since τ is not homotopic to a constant, g is topologically nontrivial in small circles around the same points as $\hat{g}$; we have $g(x_1,x_2,x_3) = g(x_2,x_1,x_3) = \sigma(g(x_1,-x_2,x_3))$, where $\sigma\colon \mathbb{RP}^2 \to \mathbb{RP}^2$ is the isometry characterized by $\sigma(\pi(y_1,y_2,y_3)) = \pi(-y_1,-y_2,y_3)$, so that $\sigma(\tau(-z)) = \tau(z)$. In particular, if $\rho(x_1,x_2,x_3) = (x_1,-x_2,x_3)$ and if $u_p \in W^{1,p}(\mathbb{B}^3,\mathbb{RP}^2)$ is a p-minimizing map having g as trace, $\sigma \circ u_p \circ \rho$ is also. Letting $p \to 2$, by Proposition 3.7.25, the corresponding limiting measure μ_* has as support either $[(1,0,0),(0,1,0)] \cup [(-1,0,0),(0,-1,0)]$ or $[(1,0,0),(0,-1,0)] \cup [(-1,0,0),(0,1,0)]$. By the symmetry it follows that if μ_* is limiting measure, then its pushforward $\rho_*\mu_*$ is also a limiting measure. Since none of the possible supports is invariant under ρ we conclude that μ_* is not unique.

Lemma 3.5.2. *Let $\eta > 0$ be the constant of Lemma 3.3.12, where $\kappa = 1/2$, $p_0 = 3/2$, $\Psi(x) = x + x_0$. Assume that $B^3_r(x_0) \subset \Omega$ and $\mu_*(\overline{B}^3_r(x_0)) < \eta r$. Then $\mu_*(B^3_{r/2}(x_0)) = 0$.*

Proof. In this proof, every ball is centered at point x_0. It is a well-known fact (see, for instance, [29, Section 1.9]) that the weak* convergence of the Borel measures μ_n is equivalent to the following two inequalities

$$\mu_*(F) \geq \limsup_{n\to+\infty} \mu_n(F), \quad \mu_*(G) \leq \liminf_{n\to+\infty} \mu_n(G) \tag{3.5.3}$$

whenever $F \subset \overline{\Omega}$ is closed and $G \subset \overline{\Omega}$ is open. Notice that this is where we take advantage of working with $(C(\overline{\Omega}))'$ instead of $(C_0(\Omega))'$. Using (3.5.3) and the facts that $\mu_*(\overline{B}^3_r) < \eta r$ and $\mu_n(\overline{B}^3_r) = \mu_n(B^3_r)$, we obtain the following estimate

$$\eta r > \limsup_{n\to+\infty} \mu_n(B^3_r).$$

This estimate, together with the fact that $\eta r^{3-p_n} \to \eta r$ as $n \to +\infty$, implies that for all sufficiently large $n \in \mathbb{N}$, it holds

$$\mu_n(B^3_r) < \eta r^{3-p_n}.$$

But then Lemma 3.3.12 says that for all sufficiently large $n \in \mathbb{N}$,

$$\mu_n(B^3_{r/2}) \leq (2 - p_n) C r^{3-p_n}, \tag{3.5.4}$$

where $C > 0$ is a constant independent of n. Letting n tend to $+\infty$ in (3.5.4) and using (3.5.3), we complete our proof of Lemma 3.5.2. $\square$

The following monotonicity of the p-energy holds, which will be used later.

Lemma 3.5.3. *Let $p \in [1, 3)$, $x_0 \in \Omega$ and $0 < \varrho < r < \operatorname{dist}(x_0, \partial\Omega)$. Let $u_p \in W^{1,p}(\Omega, \mathcal{N})$ be a p-minimizer. Then*

$$\varrho^{p-3} \int_{B^3_\varrho(x_0)} \frac{|Du_p|^p}{p}\, dx \leq r^{p-3} \int_{B^3_r(x_0)} \frac{|Du_p|^p}{p}\, dx.$$

Proof. For a proof, the reader may consult Section 4 in [35]. $\square$

Corollary 3.5.4. Let $x_0 \in \Omega$. Then the function

$$r \in (0, \operatorname{dist}(x_0, \partial\Omega)) \mapsto \frac{\mu_*(\overline{B}^3_r(x_0))}{r}$$

is nondecreasing.

Proof of Corollary 3.5.4. Let $0 < \varrho < r < \operatorname{dist}(x_0, \partial\Omega)$. In view of Lemma 3.5.3, for each $n \in \mathbb{N}$ and for an arbitrary $\varepsilon \in (0, r - \varrho)$,

$$(\varrho + \varepsilon)^{p_n - 3}\mu_n(B^3_{\varrho+\varepsilon}(x_0)) \le r^{p_n - 3}\mu_n(\overline{B}^3_r(x_0)).$$

Letting n tend to $+\infty$ and using (3.5.3), we obtain

$$(\varrho + \varepsilon)^{-1}\mu_*(B^3_{\varrho+\varepsilon}(x_0)) \le r^{-1}\mu_*(\overline{B}^3_r(x_0)).$$

Then, letting $\varepsilon \searrow 0$ and using the fact that $\mu_*(\overline{\Omega}) < +\infty$, we conclude our proof of Corollary 3.5.4. $\square$

Thanks to Corollary 3.5.4, for each $x_0 \in \Omega$, the 1-dimensional density

$$\Theta_1(\mu_*, x_0) := \lim_{r\to 0+} \frac{\mu_*(\overline{B}^3_r(x_0))}{2r} = \lim_{r\to 0+} \frac{\mu_*(B^3_r(x_0))}{2r} \tag{3.5.5}$$

exists and finite. In order to lighten the notation, we shall simply write $\Theta_1(x_0)$ instead of $\Theta_1(\mu_*, x_0)$ when appropriate.

Proposition 3.5.5. *Let $\eta > 0$ be the constant of Lemma 3.3.12, where $\kappa = 1/2$, $p_0 = 3/2$ and $\Psi(x) = x + x_0$ whenever $E = B^3_r(x_0)$. Then it holds*

$$\Omega \cap S_* = \{x \in \Omega : \Theta_1(x) > 0\} = \Big\{x \in \Omega : \Theta_1(x) \ge \frac{\eta}{2}\Big\}.$$

Proof. First, assume by contradiction that for some $x \in \Omega$, $\Theta_1(x) = a \in (0, \eta/2)$. Fix

$$\varepsilon \in \Big(0, \min\Big\{\frac{a}{2}, \frac{\eta}{2} - a\Big\}\Big).$$

Then there exists $\delta > 0$ such that for all $r \in (0, \delta)$,

$$0 < a - \varepsilon < \frac{\mu_*(\overline{B}^3_r(x))}{2r} < \frac{\eta}{2}.$$

However, Lemma 3.5.2 says that $\mu_*(B^3_{r/2}(x)) = 0$, which leads to a contradiction and proves that

$$\{x \in \Omega : \Theta_1(x) > 0\} = \{x \in \Omega : \Theta_1(x) \ge \eta/2\}.$$

On the other hand, by the definition of the support of a measure, $x \in \Omega \cap S_*$ if and only if for all sufficiently small $r > 0$, $\mu_*(\overline{B}^3_r(x)) > 0$, which holds, according to Lemma 3.5.2, if and only if for all sufficiently small $r > 0$, $\mu_*(\overline{B}^3_r(x)) \ge \eta r$. The last holds if and only if $\Theta_1(x) \ge \eta/2$. This proves that $\Omega \cap S_* = \{x \in \Omega : \Theta_1(x) \ge \eta/2\}$ and completes our proof of Proposition 3.5.5. $\square$

Proposition 3.5.6. *Let $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers in Ω satisfying* (3.5.1). *Then there exists a mapping $u_* \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$ such that, up to a subsequence (not relabeled), $u_n \to u_*$ a.e. in Ω and for which the conclusions of Proposition 3.4.2 apply with $U = \Omega \setminus S_*$.*

Remark 3.5.7. For interior quantitative estimates of the weak gradient of u_*, the reader may consult Proposition 3.5.23.

Proof of Proposition 3.5.6. Let $K \Subset \Omega \setminus S_*$ be open. Using (3.5.2), a standard covering argument and Lemma 3.3.12, we conclude that there exists a positive constant C, possibly depending only on K, Ω and $\mathcal{N}$, such that, up to a subsequence (not relabeled), for each $n \in \mathbb{N}$,

$$\int_K \frac{|Du_n|^{p_n}}{p_n}\, dx \le C.$$

Applying a diagonal argument, we observe that, up to a subsequence (not relabeled), the condition (3.4.1) of Proposition 3.4.2 is satisfied, where $U = \Omega \setminus S_*$, for our sequence of p_n-minimizers. Thus, the conclusions of Proposition 3.4.2 apply, yielding the desired mapping u_*. This completes our proof of Proposition 3.5.6. $\square$

3.5.1 The limit measure is the weight measure of the stationary varifold

Recall that a set $E \subset \mathbb{R}^3$ is said to be countably $\mathcal{H}^1$-rectifiable if there exist countably many Lipschitz functions $f_i : \mathbb{R}^1 \to \mathbb{R}^3$ such that

$$\mathcal{H}^1\Big(E \setminus \bigcup_{i=0}^{+\infty} f_i(\mathbb{R}^1)\Big) = 0.$$

Let $\mathrm{G}(3,1)$ be the Grassmann manifold, which is the space of 1-dimensional subspaces of $\mathbb{R}^3$. Following Allard (see [2]), for each $A \in \mathrm{G}(3,1)$, we shall also use "A" to denote the orthogonal projection of $\mathbb{R}^3$ onto A. That is, $\mathrm{G}(3,1) = \{A \in M_3(\mathbb{R}) : A \circ A = A,\ A^{\mathrm{T}} = A \text{ and } \mathrm{Im}(A) = A\}$, where $M_3(\mathbb{R})$ denotes the space of real 3×3 matrices. We also define $\mathrm{G}_3(\Omega) = \Omega \times \mathrm{G}(3,1)$. Recall that V is said to be a 1-dimensional varifold in Ω if V is a positive Radon measure on $\mathrm{G}_3(\Omega)$.

Let $S \subset \Omega$ be countably $\mathcal{H}^1$-rectifiable and $\mu \in (C(\overline{\Omega}))'$ be a positive Radon measure satisfying $\mu \llcorner \Omega = \theta \mathcal{H}^1 \llcorner S$ for some nonnegative $\theta \in L^1(\Omega)$. Rectifiability gives that for $\mathcal{H}^1$-a.e. $x \in S$ (or, equivalently, for μ-a.e. $x \in \Omega$), there exists a unique approximate tangent plane $T_x S \in \mathrm{G}(3,1)$ to S at x (see, for instance, [4, Theorem 2.83 (i)]). For μ-a.e. $x \in \Omega$, we denote by $A(x) \in \mathrm{G}(3,1)$ the orthogonal projection onto T_xS. We define the varifold V, which is naturally associated to μ, as the pushforward measure $V = (\mathrm{Id}, A)_{\#}\mu \llcorner \Omega$. Thus, for each Borel-measurable set $E \in \mathcal{B}(\mathrm{G}_3(\Omega))$, $V(E) = \mu(\{x \in \Omega : (x, A(x)) \in E\})$. The varifold V is said to be stationary (see Section 4.2 in [2]) if its first variation vanishes, namely

$$\int_\Omega A(x) : D\xi(x)\, d\mu(x) = 0 \quad \forall \xi \in C^1_c(\Omega, \mathbb{R}^3).$$

Proposition 3.5.8. *The set $S_* \cap \Omega$ is countably $\mathcal{H}^1$-rectifiable and $\mathcal{H}^1(S_* \cap \Omega) < +\infty$. The measure $V_* = (\mathrm{Id}, A_*)_{\#}\mu_* \llcorner \Omega$ is a stationary 1-dimensional rectifiable varifold in Ω, where for μ_*-a.e. $x \in \Omega$, $A_*(x)$ represents the orthogonal projection onto the approximate tangent plane T_xS_* to S_* at x. Furthermore, it holds $\mu_* \llcorner \Omega(dx) = \Theta_1(x)\mathcal{H}^1 \llcorner (S_* \cap \Omega)(dx)$, where the density Θ_1 is defined in* (3.5.5).

Proof. Let $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_n}(\Omega, \mathcal{N})$, where $(p_n)_{n\in\mathbb{N}} \subset (3/2, 2)$, be a sequence of p_n-minimizers satisfying (3.5.2). For each $n \in \mathbb{N}$, define the map $A^{p_n} : \Omega \to M_3(\mathbb{R})$ by

$$A^{p_n}_{i,j} = \frac{2-p_n}{p_n}\left(|Du_n|^{p_n}\delta_{i,j} - p_n|Du_n|^{p_n-2}\langle D_iu_n, D_ju_n\rangle\right).$$

Then observe that

$$(A^{p_n})^{\mathrm{T}} = A^{p_n}, \quad \mathrm{tr}(A^{p_n}) = (3-p_n)\frac{d\mu_n}{dx} \geq \frac{d\mu_n}{dx} \tag{3.5.6}$$

and

$$|A^{p_n}| = \sqrt{A^{p_n} : A^{p_n}} = \sqrt{3 - 2p_n + p_n^2}\,\frac{d\mu_n}{dx} \leq \sqrt{3}\frac{d\mu_n}{dx}, \tag{3.5.7}$$

where $\frac{d\mu_n}{dx} = (2-p_n)\frac{|Du_n|^{p_n}}{p_n}$. For each $v \in \mathbb{S}^2$,

$$\begin{aligned}\sum_{i,j=1}^{3}\langle A^{p_n}_{i,j}v_i, v_j\rangle \\ = \frac{2-p_n}{p_n}\left(|Du_n|^{p_n} - p_n|Du_n|^{p_n-2}\left|\sum_{i=1}^{3} v_iD_iu_n\right|^2\right) \leq \frac{d\mu_n}{dx},\end{aligned} \tag{3.5.8}$$

which implies that all eigenvalues of A^{p_n} are less than or equal to $\frac{d\mu_n}{dx}$. Furthermore, by (3.2.33),

$$\int_\Omega A^{p_n}(x) : D\xi(x)\, dx = 0 \quad \forall \xi \in C^1_c(\Omega, \mathbb{R}^3). \tag{3.5.9}$$

In view of (3.5.1) and (3.5.7), there exists $\widetilde{A} \in (C_0(\Omega, M_3(\mathbb{R})))'$ such that, up to a subsequence (not relabeled), $A^{p_n}\,dx \overset{*}{\rightharpoonup} \widetilde{A}$ weakly* in $(C_0(\Omega, M_3(\mathbb{R})))'$. By (3.5.2) and (3.5.7),

$$|\widetilde{A}| \le \sqrt{3}\mu_* \llcorner \Omega, \tag{3.5.10}$$

which implies that $\widetilde{A}$ is absolutely continuous with respect to $\mu_* \llcorner \Omega$. Then, by the Radon-Nikodým theorem, there exists $F \in L^1(\Omega, M_3(\mathbb{R}), \mu_*)$ such that $\widetilde{A} = F\mu_* \llcorner \Omega$ in $(C_0(\Omega, M_3(\mathbb{R})))'$. Letting n tend to $+\infty$ and taking into account (3.5.6), (3.5.8) and (3.5.9), we observe that for μ_*-a.e. $x \in \Omega$, $(F(x))^{\mathrm{T}} = F(x)$, $\mathrm{tr}(F(x)) \ge 1$, the eigenvalues of $F(x)$ are less than or equal to 1 and

$$\int_\Omega F(x) : D\xi(x)\, d\mu_*(x) = 0 \quad \forall \xi \in C^1_c(\Omega, \mathbb{R}^3). \tag{3.5.11}$$

According to [6, Proposition 3.1] and [6, Lemma 2.3], the identity (3.5.11) implies that there exists an $\mathcal{H}^1$-rectifiable set $R \subset \Omega$ and a map $\lambda \in L^\infty(R, M_3(\mathbb{R}))$ satisfying the following conditions. For $\mathcal{H}^1$-a.e. $x_0 \in R$, $|\lambda(x_0)| = 1$, $\lambda(x_0) \in \mathrm{G}(3,1)$ is the orthogonal projection matrix onto $T_{x_0}R$ such that $\widetilde{A}\llcorner\{\Theta^*_1(|\widetilde{A}|) > 0\}(dx) = \Theta^*_1(|\widetilde{A}|, x)\lambda(x)\mathcal{H}^1\llcorner R(dx)$, where $T_{x_0}R$ is the approximate tangent plane to R at x_0 and $\Theta^*_1(|\widetilde{A}|, x_0) = \limsup_{r\to 0+} \frac{|\widetilde{A}|(B^3_r(x_0))}{2r}$. Then, for μ_*-a.e. $x \in \Omega$, two eigenvalues of $F(x)$ are zeros. Since for μ_*-a.e. $x \in \Omega$, $\mathrm{tr}(F(x)) \ge 1$ and the eigenvalues of $F(x)$ are less than or equal to 1, we deduce that the eigenvalues of $F(x)$ (up to a permutation) are $(1,0,0)$ and $|F(x)| = 1$ for μ_*-a.e. $x \in \Omega$. Using this, together with (3.5.5) and Proposition 3.5.5, we observe that $|\widetilde{A}| = \mu_* \llcorner \Omega$, $\Theta^*_1(|\widetilde{A}|) = \Theta_1(\mu_*)$ and $R = S_* \cap \Omega$. In particular, for μ_*-a.e. $x \in \Omega$, $F(x) = A_*(x)$ and $V_*(dA, dx) = \delta_{A_*(x)}(dA) \otimes \mu_*(dx) = \delta_{A_*(x)}(dA) \otimes \Theta_1(\mu_*, x)\mathcal{H}^1 \llcorner (S_* \cap \Omega)(dx)$, which, together with (3.5.11), implies that V_* is a 1-dimensional stationary rectifiable varifold (see [2, 27]). This completes our proof of Proposition 3.5.8. □

Remark 3.5.9. Our proof of Proposition 3.5.8, based on [6, Proposition 3.1], differs from Canevari's proof for the three-dimensional Landau-de Gennes model [22, Proposition 55]. His strategy consists in first showing that the set $S_* \cap \Omega$ is countably $\mathcal{H}^1$-rectifiable using the Moore-Preiss rectifiability theorem (see [48, Theorem 3.5], [51, Theorem 5.3]); next, using the Ambrosio-Soner construction (see [5, Lemma 3.9]), he proves that the Radon-Nikodým derivative of the limit tensor has the eigenvalues (up to a permutation) $(1,0,0)$, and concludes that the naturally associated one-dimensional varifold is stationary; the rectifiability of the latter then follows from Rectifiability Theorem in [2, Section 5.5].

Our more direct strategy of proof could be adapted to work in Canevari's setting, and his strategy of proof would provide another pathway to Proposition 3.5.8.

3.5.2 Energy bounds on a cylinder

Let us define the notion of a uniform Lipschitz triangulation on the lateral surface of a (finite) cylinder $\Lambda_{1,L} := B_1^2 \times (-L, L)$. Let $L \geq 1$ and $h \in (0,1)$. Set $n := \lfloor 1/h \rfloor \geq 1$. For each $k \in [0, n-1]$, we define the line segment $S_k = [(e^{\frac{2\pi i k}{n}}, -L), (e^{\frac{2\pi i k}{n}}, L)] \subset \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3$ and, for $j \in [0, 2\lfloor L \rfloor n]$ the points

$$a_j^k := \left(e^{\frac{2\pi i k}{n}}, -L + \frac{Lj}{\lfloor L \rfloor n} \right) \in \mathbb{C} \times \mathbb{R} \simeq \mathbb{R}^3.$$

For each $k \in [0, n-1]$ and $j \in [0, 2\lfloor L \rfloor n - 1]$, let γ_j^k be the union of two geodesics (*a geodesic cross*) on $\overline{\Gamma}_{1,L}$, where one of which connecting a_j^k with a_{j+1}^{k+1} and the other connecting a_{j+1}^k with a_j^{k+1}. By $c_{k,z}$, where $z \in \{-L, L\}$, we denote the arc on $\mathbb{S}^1 \times \{z\}$ connecting $a_{2\lfloor L \rfloor n}^k$ with $a_{2\lfloor L \rfloor n}^{k+1}$ if $z = L$ and a_0^k with a_0^{k+1} if $z = -L$. Observe that the union of all the line segments S_k, all the geodesic crosses γ_j^k and all the arcs $c_{k,z}$ provides us with the decomposition of $\overline{\Gamma}_{1,L}$ into mutually disjoint 0, 1 and 2-dimensional cells. Hereinafter, we denote by $SO(2)$ the subgroup of $SO(3)$ of rotations with respect to the axis $\{0\} \times \mathbb{R}$, and whose action on the plane $\mathbb{R}^2 \times \{0\}$ coincides with the action of the orthogonal group $SO(2)$ on the plane. Thus, for each $\omega \in SO(2)$, we obtain a decomposition of $\overline{\Gamma}_{1,L}$ into mutually disjoint 0, 1 and 2-dimensional cells, namely

$$\overline{\Gamma}_{1,L} = \bigcup_{i=0}^{2} \bigcup_{j=1}^{k_i} \omega(E_{i,j}) = \bigcup_{i=0}^{2} \bigcup_{j=1}^{k_i} F_{i,j}, \tag{3.5.12}$$

where $\{F_{0,1}, \dots, F_{0,k_0}\}$ is the set of points and for each $i \in \{1, 2\}$, there exists a bilipschitz homeomorphism

(3.5.13)

$$\Phi_{i,j} : F_{i,j} \to B_h^i \text{ with } \|D_\top \Phi_{i,j}\|_{L^\infty(F_{i,j})} + \|D_\top (\Phi_{i,j})^{-1}\|_{L^\infty(B_h^i)} \leq A_4,$$

where $A_4 > 0$ is an absolute constant; see Figure 3.1. The map $\Phi_{i,j}$ in (3.5.13) extends by continuity to a bilipschitz map between $\overline{F}_{i,j}$ and $\overline{B}_h^i$ with the same bilipschitz constant A_4 as in (3.5.13).

The decomposition (3.5.12) we shall call a uniform Lipschitz (or simply a Lipschitz) triangulation of $\overline{\Gamma}_{1,L}$ of step h with the 1-skeleton $\Sigma = \bigcup_{i=0}^{1} \bigcup_{j=0}^{k_i} F_{i,j}$.

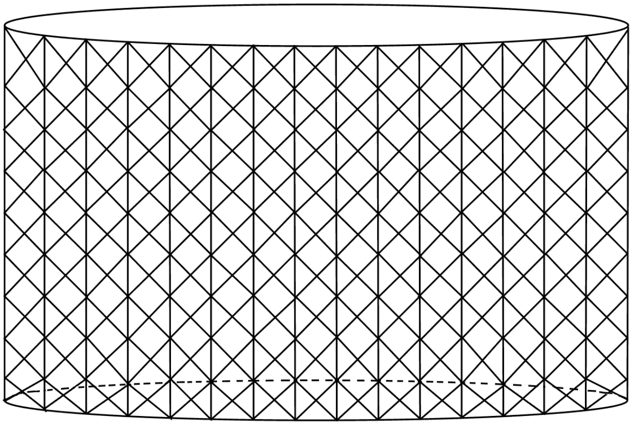

Figure 3.1: A uniform Lipschitz triangulation of the lateral surface of a cylinder.

To lighten the notation, we denote the collection of 1-skeletons of all Lipschitz triangulations of $\overline{\Gamma}_{1,L}$ of step h by $\mathrm{RC}_{L,h}$, namely

$$\mathrm{RC}_{L,h} := \left\{ \bigcup_{i=0}^{1} \bigcup_{j=1}^{k_i} \omega(E_{i,j}) : \omega \in SO(2) \right\},$$

where $\bigcup_{i=0}^{1} \bigcup_{j=1}^{k_i} E_{i,j}$ is the union of all the line segments S_k, all the geodesic crosses γ_j^k and all the arcs $c_{k,z}$.

It is worth noting that Lemmas 3.5.10, 3.5.11, 3.5.12 are the counterparts of Lemmas 3.3.7, 3.3.8, 3.3.10 on the lateral surface of a cylinder. A consequence of Lemma 3.5.12 is Corollary 3.5.13, which provides us with the key arguments that will be used in the proof of Proposition 3.5.14. The latter gives us the estimates of the p-energy on a cylinder, which will be used to prove the fact that the limit measure has a finite set of values.

Lemma 3.5.10. *Let* $p \in [1, +\infty)$, $L \geq 1$, $h \in (0, 1)$ *and* $\mathcal{Y}$ *be a compact Riemannian manifold or a Euclidean space. Let* $g \in W^{1,p}(\Gamma_{1,L}, \mathcal{Y})$ *and* $\Sigma \in \mathrm{RC}_{L,h}$*. Then for* $\mathcal{H}^1$*-a.e.* $\omega \in SO(2)$ *we have* $\mathrm{tr}_{\omega(\Sigma \cap \Gamma_{1,L})}(g) = g|_{\omega(\Sigma \cap \Gamma_{1,L})} \in W^{1,p}(\omega(\Sigma \cap \Gamma_{1,L}), \mathcal{Y})$*. Furthermore, there exists* $\omega \in SO(2)$ *such that* $\mathrm{tr}_{\omega(\Sigma \cap \Gamma_{1,L})}(g) = g|_{\omega(\Sigma \cap \Gamma_{1,L})} \in W^{1,p}(\omega(\Sigma \cap \Gamma_{1,L}), \mathcal{Y})$ *and*

$$\int_{\omega(\Sigma \cap \Gamma_{1,L})} |D_{\top} g|^p \, d\mathcal{H}^1 \leq \frac{A_5}{h} \int_{\Gamma_{1,L}} |D_{\top} g|^p \, d\mathcal{H}^2,$$

where $A_5 > 0$ *is an absolute constant.*

Proof. For a proof, we refer to [34, Lemma 3.4]. The main ingredient is the integralgeometric formula

$$\int_{SO(2)} d\omega \int_{\omega(\Sigma\cap\Gamma_{1,L})} |D_\top g|^p \, d\mathcal{H}^1 = \frac{\mathcal{H}^1(\Sigma\cap\Gamma_{1,L})}{\mathcal{H}^2(\Gamma_{1,L})} \int_{\Gamma_{1,L}} |D_\top g|^p \, d\mathcal{H}^2$$

(for the integralgeometric formulas, the reader may consult [31, 20]; see also [34, Lemma 3.3]). □

Lemma 3.5.11. *Let $p_0 \in (1,2)$, $p \in [p_0, 2)$, $L \geq 1$ and $h \in (0,1)$. Then there exists a constant $\kappa_0 = \kappa_0(p_0, \mathcal{N}) > 0$ such that the following holds. Let $\Sigma \in \mathrm{RC}_{L,h}$ and F be a 2-cell of the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ. Let $g \in W^{1,p}(F, \mathcal{N})$ and $\mathrm{tr}_{\partial F}(g) \in W^{1,p}(\partial F, \mathcal{N})$, where ∂F denotes the relative boundary of F. If*

$$\int_F \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 + h \int_{\partial F} \frac{|D_\top \, \mathrm{tr}_{\partial F}(g)|^p}{p} \, d\mathcal{H}^1 \leq \frac{\kappa_0 h^{2-p}}{2-p},$$

then $\mathrm{tr}_{\partial F}(g)$ is nullhomotopic.

Proof. The proof follows by reproducing the proof of Lemma 3.3.8 with minor modifications. Essentially, it is enough to replace the 2-dimensional cell in the proof of Lemma 3.3.8 by the 2-cell F of Lemma 3.5.11. □

Lemma 3.5.12. *Let $p_0 \in (1,2)$, $p \in [p_0, 2)$, $L_0 \geq 2$, $h \in (0, 1/2]$ and $g \in W^{1,p}(\Gamma_{1,L_0}, \mathcal{N})$, where Γ_{1,L_0} is defined in (3.2.20). Then there exists $\kappa = \kappa(p_0, \mathcal{N}) > 0$ such that the following holds. Assume that*

$$\int_{\Gamma_{1,L_0}} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \leq \frac{\kappa h^{2-p}}{2-p}. \tag{3.5.14}$$

There exist $z_- \in (-L_0 + h, -L_0 + 2h)$, $z_+ \in (L_0 - 2h, L_0 - h)$, $\varphi_h \in W^{1,p}((B_1^2 \setminus \overline{B}_{1-h}^2) \times (z_-, z_+), \mathcal{N})$ such that the following assertions hold.

(i) $\mathrm{tr}_{\mathbb{S}^1\times(z_-,z_+)}(\varphi_h) = g|_{\mathbb{S}^1\times(z_-,z_+)}$,
$v_h := \mathrm{tr}_{\partial B_{1-h}^2\times(z_-,z_+)}(\varphi_h) \in W^{1,2}(\partial B_{1-h}^2 \times (z_-, z_+), \mathcal{N})$.

(ii) For each $z \in \{z_-, z_+\}$, $\mathrm{tr}_{(B_1^2\setminus\overline{B}_{1-h}^2)\times\{z\}}(\varphi_h) \in W^{1,p}((B_1^2 \setminus \overline{B}_{1-h}^2) \times \{z\}, \mathcal{N})$ and

$$\int_{(B_1^2\setminus\overline{B}_{1-h}^2)\times\{z\}} |D_\top \, \mathrm{tr}_{(B_1^2\setminus\overline{B}_{1-h}^2)\times\{z\}}(\varphi_h)|^p \, d\mathcal{H}^2 \leq A_6 \int_{\Gamma_{1,L_0}} |D_\top g|^p \, d\mathcal{H}^2,$$

where $A_6 > 0$ is an absolute constant.

(iii) There exists a constant $C = C(\mathcal{N}) > 0$ such that

$$\int_{\partial B^2_{1-h}\times(z_-,z_+)} |D_\top v_h|^p \, d\mathcal{H}^2 \leq C \int_{\Gamma_{1,L_0}} |D_\top g|^p \, d\mathcal{H}^2 \tag{3.5.15}$$

and

$$\int_{(B^2_1\setminus\overline{B}^2_{1-h})\times(z_-,z_+)} |D\varphi_h|^p \, dx \leq Ch \int_{\Gamma_{1,L_0}} |D_\top g|^p \, d\mathcal{H}^2. \tag{3.5.16}$$

Proof. Using the coarea formula, we find $z_- \in (-L_0 + h, -L_0 + 2h)$ and $z_+ \in (L_0 - 2h, L_0 - h)$ such that for each $z \in \{z_-, z_+\}$, $\operatorname{tr}_{\mathbb{S}^1\times\{z\}}(g) = g|_{\mathbb{S}^1\times\{z\}} \in W^{1,p}(\mathbb{S}^1 \times \{z\}, \mathcal{N})$ and

$$\int_{\mathbb{S}^1\times\{z\}} |D_\top g|^p \, d\mathcal{H}^1 \leq \frac{2}{h} \int_{\Gamma_{1,L_0}} |D_\top g|^p \, d\mathcal{H}^2. \tag{3.5.17}$$

Define $L := (z_+ - z_-)/2$, $\mathrm{e} := (0, 0, (z_+ + z_-)/2)$ and $u(y) := g(\mathrm{e} + y)$ for each $y \in \Gamma_{1,L}$. Let $\kappa_0 = \kappa_0(p_0, \mathcal{N}) > 0$ be the constant given by Lemma 3.5.11. Define $\kappa = \kappa_0/2A_5$, where $A_5 > 0$ is the absolute constant of Lemma 3.5.10. Without loss of generality, we assume that $A_5 \geq 2$. Then κ depends only on p_0 and $\mathcal{N}$. Using Lemma 3.5.10, the estimates (3.5.14), (3.5.17) and taking into account the definition of κ, we obtain $\Sigma \in \mathrm{RC}_{L,h}$ such that $\operatorname{tr}_\Sigma(u) \in W^{1,p}(\Sigma, \mathcal{N})$, $\operatorname{tr}_{\Sigma\cap\Gamma_{1,L}}(u) = u|_{\Sigma\cap\Gamma_{1,L}} \in W^{1,p}(\Sigma \cap \Gamma_{1,L}, \mathcal{N})$,

$$\int_{\Sigma\cap\Gamma_{1,L}} |D_\top u|^p \, d\mathcal{H}^1 \leq \frac{A_5}{h} \int_{\Gamma_{1,L}} |D_\top u|^p \, d\mathcal{H}^2 \tag{3.5.18}$$

and for each 2-cell F from the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ,

$$\begin{aligned}\int_F \frac{|D_\top u|^p}{p} \, d\mathcal{H}^2 + h \int_{\partial F} \frac{|D_\top \operatorname{tr}_{\partial F}(u)|^p}{p} \, d\mathcal{H}^1& \\ \leq 2A_5 \int_{\Gamma_{1,L_0}} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 &\leq \frac{\kappa_0 h^{2-p}}{2-p},\end{aligned}$$

where ∂F denotes the relative boundary of F. This, according to Lemma 3.5.11, implies that $\operatorname{tr}_{\partial F}(u)$ is nullhomotopic for each 2-cell F from the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ. Proceeding similarly as in Proposition 3.3.6 (changing $\mathbb{S}^2$ to $\overline{\Gamma}_{1,L}$ and the geometry of cells, namely changing "squares" to "squares or triangles" which are all bilipschitz homeomorphic to B_h^2), we obtain $w_h \in W^{1,2}(\Gamma_{1,L}, \mathcal{N})$ such that $\mathrm{tr}_\Sigma(w_h) = \mathrm{tr}_\Sigma(u)$, $\mathrm{tr}_{\Sigma\cap\Gamma_{1,L}}(w_h) = w_h|_{\Sigma\cap\Gamma_{1,L}} = u|_{\Sigma\cap\Gamma_{1,L}}$ and

$$\int_{\Gamma_{1,L}} |D_\top w_h|^p \, d\mathcal{H}^2 \le Ch \int_\Sigma |D_\top u|^p \, d\mathcal{H}^1, \tag{3.5.19}$$

where $C = C(\mathcal{N}) > 0$. Using (3.5.17), (3.5.18) and (3.5.19), we obtain the estimate

$$\int_{\Gamma_{1,L}} |D_\top w_h|^p \, d\mathcal{H}^2 \le C' \int_{\Gamma_{1,L_0}} |D_\top g|^p \, d\mathcal{H}^2,$$

where $C' = C'(\mathcal{N}) > 0$. Now we define $v_h(x) = w_h((x_1/(1-h), x_2/(1-h), x_3) - \mathrm{e})$ for $x \in \partial B_{1-h}^2 \times (z_-, z_+)$. In view of the last estimate, we get (3.5.15).

From the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ we obtain the decomposition of $(B_1^2 \setminus \overline{B}_{1-h}^2) \times (z_-, z_+)$ into the cells

$$D_{i,j} = \left\{ x = (x', x_3) \in \mathbb{R}^3 : (1-h) < |x'| < 1, \ \left(\frac{x'}{|x'|}, x_3 \right) \in (F_{i,j} + \mathrm{e}) \right\}, \tag{3.5.20}$$

where $D_{i,j}$ is of Hausdorff dimension $(i+1)$. Fix an arbitrary 3-cell $D_{2,j}$ from the above decomposition. Let $F_{2,j}$ be the 2-cell of the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ such that $D_{2,j} = \{x \in \mathbb{R}^3 : (1-h) < |x'| < 1, \ (\frac{x'}{|x'|}, x_3) \in (F_{2,j} + \mathrm{e})\}$. Let $\Phi_j : [0,h]^3 \to \overline{D}_{2,j}$ be a bilipschitz homeomorphism with an absolute bilipschitz constant (see (3.5.13)) such that $\Phi_j((0,h)^2 \times \{0\}) = \{x \in \mathbb{R}^3 : |x'| = 1-h, \ (\frac{x'}{|x'|}, x_3) \in (F_{2,j} + \mathrm{e})\}$ and $\Phi_j((0,h)^2 \times \{1\}) = F_{2,j} + \mathrm{e}$. Applying Lemma 3.3.9 with g_0 and g_1 taking the values of $v_h \circ \Phi_j|_{(0,h)^2\times\{0\}}$ and $g \circ \Phi_j|_{(0,h)^2\times\{1\}}$, respectively, and changing the variables, we obtain a map $\varphi_j \in W^{1,p}(D_{2,j}, \mathcal{N})$ satisfying the estimate

$$\begin{aligned} \int_{D_{2,j}} |D\varphi_j|^p \, dx &\le C''h \left(\int_{\Phi_j((0,h)^2\times\{0\})} |D_\top v_h|^p \, d\mathcal{H}^2 + \int_{F_{2,j}+\mathrm{e}} |D_\top g|^p \, d\mathcal{H}^2 \right) \\ &\qquad + C''h^2 \int_{\partial F_{2,j}+\mathrm{e}} |D_\top g|^p \, d\mathcal{H}^1, \end{aligned} \tag{3.5.21}$$

where $C'' > 0$ is an absolute constant.

Let $\varphi_h \in L^\infty((B_1^2 \setminus \overline{B}_{1-h}^2) \times (z_-, z_+), \mathcal{N})$ satisfy $\varphi_h|_{D_{2,j}} = \varphi_j \in W^{1,p}(D_{2,j}, \mathcal{N})$ for each 3-cell $D_{2,j}$ from the decomposition (3.5.20). In view of our construction (see Lemma 3.3.9), we can ensure that the traces of the maps φ_j coincide on the mantles of the 3-cells $D_{2,j}$. Thus, we deduce that $\varphi_h \in W^{1,p}((B_1^2 \setminus \overline{B}_{1-h}^2) \times (z_-, z_+), \mathcal{N})$. Observe that (i) and (ii) are satisfied, where (ii) comes from the fact that

$$\int_{(B_1^2\setminus\overline{B}_{1-h}^2)\times\{z\}} |D_\top \operatorname{tr}_{(B_1^2\setminus\overline{B}_{1-h}^2)\times\{z\}}(\varphi_h)|^p \, d\mathcal{H}^2$$
$$= \int_{1-h}^{1} r^{1-p}\, dr \int_{\mathbb{S}^1\times\{z\}} |D_\top g|^p \, d\mathcal{H}^1$$
$$\leq 2h \int_{\mathbb{S}^1\times\{z\}} |D_\top g|^p \, d\mathcal{H}^1$$

for each $z \in \{z_-, z_+\}$ and (3.5.17). Furthermore, summing (3.5.21) over $j \in \{1, \ldots, k_2\}$, taking into account that each 1-cell of the Lipschitz triangulation of $\overline{\Gamma}_{1,L}$ generated by Σ lies in the relative boundary of at most two 2-cells from this triangulation, and also using (3.5.15) together with (3.5.17) and (3.5.18), we deduce the following chain of estimates

$$\int_{(B_1^2\setminus\overline{B}_{1-h}^2)\times(z_-,z_+)} |D\varphi_h|^p \, dx$$
$$\leq C''h\left(\int_{\partial B_{1-h}^2\times(z_-,z_+)} |D_\top v_h|^p \, d\mathcal{H}^2 + \int_{\mathbb{S}^1\times(z_-,z_+)} |D_\top g|^p \, d\mathcal{H}^2\right)$$
$$+ 2C''h^2 \int_{\Sigma+\mathrm{e}} |D_\top g|^p \, d\mathcal{H}^1$$

$$\leq C'''h \int_{\Gamma_{1,L_0}} |D_\top g|^p \, d\mathcal{H}^2,$$

where $C''' = C'''(\mathcal{N}) > 0$. This completes our proof of (iii) and hence of Lemma 3.5.12. □

Corollary 3.5.13. Let

$$p_0 \in (1,2), p \in [p_0, 2), \delta \in (0, 1/2] \text{ and } g \in W^{1,p}(\Gamma_{\delta r,r}, \mathcal{N}),$$

where $\Gamma_{\delta r,r}$ is defined in (3.2.20). Then there exists $\tau = \tau(p_0, \mathcal{N}) > 0$ such that the following holds. Assume that

$$\int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p} \, d\mathcal{H}^2 \leq \frac{\tau(\delta r)^{2-p}}{2-p}. \tag{3.5.22}$$

Then there exist points $z_- \in (-r + \delta r/2, -r + \delta r)$, $z_+ \in (r - \delta r, r - \delta r/2)$ and a mapping $\varphi_\delta \in W^{1,p}((B_{\delta r}^2 \setminus \overline{B}_{\delta r/2}^2) \times (z_-, z_+), \mathcal{N})$ such that the following assertions hold.

(i) $\operatorname{tr}_{\partial B^2_{\delta r}\times I}(\varphi_\delta) = g|_{\partial B^2_{\delta r}\times I}$, $v_\delta := \operatorname{tr}_{\partial B^2_{\delta r/2}\times I}(\varphi_\delta) \in W^{1,2}(\partial B^2_{\delta r/2} \times I, \mathcal{N})$, where $I = (z_-, z_+)$.

(ii) For each $z \in \{z_-, z_+\}$, $\operatorname{tr}_{(B^2_{\delta r}\setminus \overline{B}^2_{\delta r/2})\times\{z\}}(\varphi_\delta) \in W^{1,p}((B^2_{\delta r} \setminus \overline{B}^2_{\delta r/2}) \times \{z\}, \mathcal{N})$ and

$$\int_{(B^2_{\delta r}\setminus \overline{B}^2_{\delta r/2})\times\{z\}} \frac{|D_\top \operatorname{tr}_{(B^2_{\delta r}\setminus \overline{B}^2_{\delta r/2})\times\{z\}}(\varphi_\delta)|^p}{p}\, d\mathcal{H}^2 \leq A_6 \int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2.$$

(iii) There exists a constant $C = C(\mathcal{N}) > 0$ such that

$$\int_{\partial B^2_{\delta r/2}\times(z_-,z_+)} \frac{|D_\top v_\delta|^p}{p}\, d\mathcal{H}^2 \leq C \int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2$$

and

$$\int_{(B^2_{\delta r}\setminus \overline{B}^2_{\delta r/2})\times(z_-,z_+)} \frac{|D\varphi_\delta|^p}{p}\, dx \leq C\delta r \int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2.$$

Proof of Corollary 3.5.13. Define $f(x) := g(\delta r x)$ for each $x \in \Gamma_{1,1/\delta}$. Let $\kappa = \kappa(p_0, \mathcal{N}) > 0$ be the constant of Lemma 3.5.12. Fix $\tau \in (0, \kappa/2]$. Then, in view of (3.5.22), it holds

$$\int_{\Gamma_{1,1/\delta}} \frac{|D_\top f|^p}{p}\, d\mathcal{H}^2 \leq \frac{\tau}{2-p} \leq \frac{\kappa}{2(2-p)} < \frac{\kappa(1/2)^{2-p}}{2-p}.$$

This allows us to apply Lemma 3.5.12 with $L_0 = 1/\delta$, $h = 1/2$ for the map $f \in W^{1,p}(\Gamma_{1,1/\delta}, \mathcal{N})$. Thus, we obtain points $\xi_- \in (-1/\delta + 1/2, -1/\delta + 1)$, $\xi_+ \in (1/\delta - 1, 1/\delta - 1/2)$ and maps $w \in W^{1,2}(\partial B^2_{1/2} \times (\xi_-, \xi_+), \mathcal{N})$, $\psi \in W^{1,p}((B^2_1 \setminus \overline{B}^2_{1/2}) \times (\xi_-, \xi_+), \mathcal{N})$ such that the assertions (i)-(iii) of Lemma 3.5.12 hold for z_-, z_+, v_h, φ_h replaced by ξ_-, ξ_+, w, ψ, respectively. Scaling back to $\Gamma_{\delta r,r}$, namely, defining $z_- = \delta r \xi_-$, $z_+ = \delta r \xi_+$, $v_\delta(\cdot) = w(\cdot/\delta r)$ and $\varphi_\delta(\cdot) = \psi(\cdot/\delta r)$, we complete our proof of Corollary 3.5.13. $\square$

Proposition 3.5.14. *Let $p_0 \in (1,2)$, $p \in [p_0, 2)$, $\delta \in (0, 1/2]$ and $g \in W^{1,p}(\partial\Lambda_{\delta r,r}, \mathcal{N})$, where $\Lambda_{\delta r,r}$ and $\Gamma_{\delta r,r}$ are defined in* (3.2.20)*. Let $\tau = \tau(p_0, \mathcal{N}) > 0$ be the constant of Corollary 3.5.13. Assume that*

$$\int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 \leq \frac{\tau(\delta r)^{2-p}}{2-p}.$$

Let u be a p-minimizer in $\Lambda_{\delta r,r}$ with $\mathrm{tr}_{\partial\Lambda_{\delta r,r}}(u) = g$. Then there exists $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ such that the following estimates hold

$$\begin{aligned}\int_{\Lambda_{\delta r,r}} \frac{|Du|^p}{p}\,dx \le \frac{2\mathcal{E}_p^{\mathrm{sg}}(\gamma) r^{3-p}}{2-p} &+ C\delta^{1-p} r \int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\,d\mathcal{H}^2 \\ &+ C\delta r \biggl(\int_{B^2_{\delta r}\times\{-r\}} \frac{|D_\top g|^p}{p}\,d\mathcal{H}^2 \\ &+ \int_{B^2_{\delta r}\times\{r\}} \frac{|D_\top g|^p}{p}\,d\mathcal{H}^2\biggr) + \frac{C(\delta r)^{3-p}}{2-p}\end{aligned} \tag{3.5.23}$$

and

$$\begin{aligned}(2-2\delta)\frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)\delta^{2-p}r^{3-p}}{2-p} - C\delta r \int_{\Gamma_{\delta r,r}} \frac{|D_\top g|^p}{p}\,d\mathcal{H}^2 \\ \le \int_{\Lambda_{\delta r,r}} \frac{|Du|^p}{p}\,dx\end{aligned} \tag{3.5.24}$$

where $C = C(\mathcal{N}) > 0$.

Remark 3.5.15. The singular p-energies $\mathcal{E}^{\mathrm{sg}}_p(\gamma)$ and $\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)$ appearing in (3.5.23) and (3.5.24), respectively, are not the same in general (see Definition 3.2.7).

Before proving Proposition 3.5.14, let us consider the case where $r = 1$, $\gamma \in C^1(\mathbb{S}^1, \mathcal{N})$ and $g \in W^{1,p}(\partial\Lambda_{\delta,1}, \mathcal{N})$ is defined on the closure of the lateral surface $\Gamma_{\delta,1}$ (see (3.2.20)) as $g(\sigma, z) = \gamma(\sigma/\delta)$ for each point $(\sigma, z) \in \partial B^2_\delta \times [-1, 1]$ and on the ball $B^2_\delta \times \{z\}$ with $z \in \{-1, 1\}$, as the extension of the mapping $g(\cdot, z) \in C^1(\partial B^2_\delta \times \{z\}, \mathcal{N})$ coming from Lemma 3.2.18, so that g is of class C^1 on $\Gamma_{\delta,1}$ and its homotopy class on $\Gamma_{\delta,1}$ (see Remark 3.2.19) is given by $[\gamma] \in [\mathbb{S}^1, \mathcal{N}]$. The upper bound (3.5.23) for the p-energy on $\Lambda_{\delta,1}$ of a minimizer $u \in W^{1,p}(\Lambda_{\delta,1}, \mathcal{N})$ with $\mathrm{tr}_{\partial\Lambda_{\delta,1}}(u) = g$ comes from Lemma 3.2.20 (i), while the lower bound (3.5.24) for the p-energy of u on $\Lambda_{\delta,1}$ is a consequence of the application of the coarea formula and Proposition 3.3.3.

Proof of Proposition 3.5.14. By scaling, it is enough to prove Proposition 3.5.14 in the case when $r = 1$. So we assume that $r = 1$. In view of the conditions of Proposition 3.5.14, we can apply Corollary 3.5.13 to g in $\Gamma_{\delta,1}$. Let points $z_- \in (-1 + \delta/2, -1 + \delta)$, $z_+ \in (1 - \delta, 1 - \delta/2)$ and maps $v_\delta \in W^{1,2}(\partial B^2_{\delta/2} \times (z_-, z_+), \mathcal{N})$, $\varphi_\delta \in W^{1,p}((B^2_\delta \setminus \overline{B}^2_{\delta/2}) \times (z_-, z_+), \mathcal{N})$ be given by Corollary 3.5.13 applied to our $g \in W^{1,p}(\Gamma_{\delta,1}, \mathcal{N})$. According to Remark 3.2.19, v_δ has a well-defined "homotopy class" on

$\partial B^2_{\delta/2} \times (z_-, z_+)$. Thus, there exists $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ such that for $\mathcal{H}^1$-a.e. $z \in (z_-, z_+)$, $[v_\delta|_{\partial B^2_{\delta/2}\times\{z\}}] = [\gamma]$. This defines our γ. Hereinafter in this proof, C will denote a positive constant that can depend only on $\mathcal{N}$ and can be different from line to line.

Step 1. We prove the upper bound (3.5.23). For convenience, in order to construct a competitor for u, we define the following subdomains of $\Lambda_{\delta,1}$:

$$\Lambda^-_\delta := B^2_\delta \times (-1, z_-), \qquad \Lambda^0_\delta := B^2_{\delta/2} \times (z_-, z_+),$$
$$E_\delta := (B^2_\delta \setminus \overline{B}^2_{\delta/2}) \times (z_-, z_+), \qquad \Lambda^+_\delta := B^2_\delta \times (z_+, 1).$$

Let us construct a competitor w for u, namely $w \in W^{1,p}(\Lambda_{\delta,1}, \mathcal{N})$ with $\mathrm{tr}_{\partial\Lambda_{\delta,1}}(w) = g$. We define w in Λ^0_δ by using Lemma 3.2.20 (i), namely we obtain a map $w|_{\Lambda^0_\delta} \in W^{1,p}(\Lambda^0_\delta, \mathcal{N})$ such that

$$\begin{aligned}(3.5.25)\quad \int_{\Lambda^0_\delta} \frac{|Dw|^p}{p}\,dx \le{}& \frac{(z_+ - z_-)\mathcal{E}^{\mathrm{sg}}_p(\gamma)\delta^{2-p}}{(2-p)2^{2-p}}\\ &+ C\left(\frac{(z_+ - z_-)^p}{2\delta^{p-1}} + \frac{\delta}{2}\right)\int_{\partial B^2_{\delta/2}\times(z_-,z_+)} \frac{|D_\top v_\delta|^p}{p}\,d\mathcal{H}^2\\ &\le \frac{2\mathcal{E}^{\mathrm{sg}}_p(\gamma)}{2-p} + \frac{C}{\delta^{p-1}}\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\,d\mathcal{H}^2,\end{aligned}$$

where we have also used that $(z_+ - z_-) \le 2$, $\delta \in (0, 1/2]$, $p \in (1,2)$ and Corollary 3.5.13 (iii). Next, we define the mapping $g^+ : \partial\Lambda^+_\delta \to \mathcal{N}$ by

$$g^+(x) := \begin{cases} g(x) & \text{if } x \in \partial\Lambda^+_\delta \cap \partial\Lambda_{\delta,1},\\ \mathrm{tr}_{(B^2_\delta\setminus\overline{B}^2_{\delta/2})\times\{z_+\}}(\varphi_\delta)(x) & \text{if } x \in (B^2_\delta \setminus \overline{B}^2_{\delta/2}) \times \{z_+\},\\ \mathrm{tr}_{\overline{B}^2_{\delta/2}\times\{z_+\}}(w)(x) & \text{if } x \in \overline{B}^2_{\delta/2} \times \{z_+\}.\end{cases}$$

By Lemma 3.2.20 (ii), $g^+|_{B^2_{\delta/2}\times\{z_+\}} \in W^{1,p}(B^2_{\delta/2}\times\{z_+\}, \mathcal{N})$. On the other hand, according to Corollary 3.5.13 (ii), $g^+|_{(B^2_\delta\setminus\overline{B}^2_{\delta/2})\times\{z_+\}} \in W^{1,p}((B^2_\delta \setminus \overline{B}^2_{\delta/2}) \times \{z_+\}, \mathcal{N})$. Thus, taking into account that the traces of g and φ_δ coincide on $\partial B^2_\delta \times \{z_+\}$, and the traces of φ_δ and w coincide on $\partial B^2_{\delta/2} \times \{z_+\}$, we conclude that $g^+ \in W^{1,p}(\partial\Lambda^+_\delta, \mathcal{N})$. In particular, using

Lemma 3.2.20 (ii), we obtain

$$\begin{aligned}
(3.5.26)\quad &\int_{\partial\Lambda_\delta^+} \frac{|D_\top g^+|^p}{p}\, d\mathcal{H}^2 \\
&\leq \int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 + \int_{B_\delta^2\times\{1\}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 + \frac{\mathcal{E}_p^{\mathrm{sg}}(\gamma)\delta^{2-p}}{(2-p)2^{2-p}} \\
&\quad + C\left(\left(\frac{z_+ - z_-}{\delta}\right)^{p-1} + \frac{\delta}{z_+ - z_-}\right)\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 \\
&\leq \frac{C}{\delta^{p-1}}\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 + \int_{B_\delta^2\times\{1\}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 + \frac{C\delta^{2-p}}{2-p},
\end{aligned}$$

where the last estimate comes since $(z_+ - z_-) \leq 2$, $\delta \in (0, 1/2]$, $p \in (1, 2)$ and $\#[\mathbb{S}^1, \mathcal{N}] < +\infty$ (hence $\mathcal{E}_p^{\mathrm{sg}}(\gamma) \leq C$). Notice that $\overline{\Lambda}_\delta^+$ is bilipschitz homeomorphic to $\overline{B}_\delta^3$, namely, there exists a bilipschitz mapping $\Phi : \overline{\Lambda}_\delta^+ \to \overline{B}_\delta^3$ with an absolute bilipschitz constant. We define

$$w|_{\Lambda_\delta^+}(x) := g^+\left(\Phi^{-1}\left(\frac{\delta\Phi(x)}{|\Phi(x)|}\right)\right) \qquad \text{for } x \in \Lambda_\delta^+ \setminus \{\Phi^{-1}(0)\}.$$

Then, using (3.5.26), we obtain

$$\begin{aligned}
&(3.5.27)\\
&\int_{\Lambda_\delta^+} \frac{|Dw|^p}{p}\, dx \leq C\delta\int_{\partial\Lambda_\delta^+} \frac{|D_\top g^+|^p}{p}\, d\mathcal{H}^2 \\
&\leq C\delta^{2-p}\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 + C\delta\int_{B_\delta^2\times\{1\}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 + \frac{C\delta^{3-p}}{2-p}.
\end{aligned}$$

We carry out a similar construction on the set Λ_δ^- and obtain a map $w|_{\Lambda_\delta^-} \in W^{1,p}(\Lambda_\delta^-, \mathcal{N})$ satisfying the same type of energy control as in (3.5.27). In E_δ, we define $w|_{E_\delta} = \varphi_\delta$. By Corollary 3.5.13 (iii),

$$(3.5.28)\qquad \int_{E_\delta} \frac{|Dw|^p}{p}\, dx \leq C\delta\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2.$$

Altogether, we have constructed the map $w \in W^{1,p}(\Lambda_{\delta,1}, \mathcal{N})$ with $\mathrm{tr}_{\partial\Lambda_{\delta,1}}(w) = g$ (we refer to Corollary 3.5.13 i). Since u is a p-minimizer in $\Lambda_{\delta,1}$ with $\mathrm{tr}_{\partial\Lambda_{\delta,1}}(u) = g$ and, in view of (3.5.25), (3.5.27), (3.5.28), the facts that $\delta \in (0, 1/2]$ and $p \in (1, 2)$, we have

$$\begin{aligned}
\int_{\Lambda_{\delta,1}} \frac{|Du|^p}{p}\, dx &\leq \int_{\Lambda_{\delta,1}} \frac{|Dw|^p}{p}\, dx \leq \frac{2\mathcal{E}_p^{\mathrm{sg}}(\gamma)}{2-p} + \frac{C}{\delta^{p-1}}\int_{\Gamma_{\delta,1}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 \\
&\quad + C\delta\left(\int_{B_\delta^2\times\{-1\}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2 + \int_{B_\delta^2\times\{1\}} \frac{|D_\top g|^p}{p}\, d\mathcal{H}^2\right) + \frac{C\delta^{3-p}}{2-p},
\end{aligned}$$

which yields (3.5.23).

Step 2. We prove the lower bound (3.5.24). For each $(\varrho,\sigma,z)\in[0,\delta)\times\mathbb{S}^1\times(z_-,z_+)$, define the map $\psi: B^2_\delta\times(z_-,z_+)$ by

$$\psi(\varrho\sigma,z):=\begin{cases}u(2\varrho\sigma,z) & \text{if } \varrho\in[0,\delta/2) \text{ and } z\in(z_-,z_+),\\ \varphi_\delta((3\delta/2-\varrho)\sigma,z) & \text{if } \varrho\in[\delta/2,\delta) \text{ and } z\in(z_-,z_+).\end{cases}$$

Then $\psi\in W^{1,p}(B^2_\delta\times(z_-,z_+),\mathcal{N})$ and $\operatorname{tr}_{\partial B^2_\delta\times(z_-,z_+)}(\psi)=v_\delta(\cdot/2,\cdot)\in W^{1,2}(\partial B^2_\delta\times(z_-,z_+),\mathcal{N})$. By the definition of ψ and Corollary 3.5.13 (iii),

(3.5.29)
$$\begin{aligned}\int_{B^2_\delta\times(z_-,z_+)}\frac{|D\psi|^p}{p}\,dx &\le \int_{B^2_\delta\times(z_-,z_+)}\frac{|Du|^p}{p}\,dx+\int_{E_\delta}\frac{|D\varphi_\delta|^p}{p}\,dx\\ &\le \int_{B^2_\delta\times(z_-,z_+)}\frac{|Du|^p}{p}\,dx+C\delta\int_{\Gamma_{\delta,1}}\frac{|D_\top g|^p}{p}\,d\mathcal{H}^2.\end{aligned}$$

Using the coarea formula and that

$$\operatorname{tr}_{\partial B^2_\delta\times(z_-,z_+)}(\psi)=v_\delta(\cdot/2,\cdot)\in W^{1,2}(\partial B^2_\delta\times(z_-,z_+),\mathcal{N}),$$

we observe that $\operatorname{tr}_{\partial B^2_\delta\times\{z\}}(\psi)=\psi|_{\partial B^2_\delta\times\{z\}}\in W^{1,2}(\partial B^2_\delta\times\{z\},\mathcal{N})$ and $[\psi|_{\partial B^2_\delta\times\{z\}}]=[\gamma]$ for $\mathcal{H}^1$-a.e. $z\in(z_-,z_+)$. Then, applying Proposition 3.3.3, for $\mathcal{H}^1$-a.e. $z\in(z_-,z_+)$, we obtain

$$\int_{B^2_\delta\times\{z\}}\frac{|D\psi|^p}{p}\,d\mathcal{H}^2+\delta\int_{\partial B^2_\delta\times\{z\}}\frac{|D_\top\psi|^p}{p}\,d\mathcal{H}^1\ge\frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)\delta^{2-p}}{2-p}.$$

Integrating both sides of the above inequality over (z_-,z_+) with respect to dz, we get

(3.5.30)
$$\begin{aligned}&\int_{B^2_\delta\times(z_-,z_+)}\frac{|D\psi|^p}{p}\,dx\\ &\ge\frac{(z_+-z_-)\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)\delta^{2-p}}{2-p}-\frac{\delta}{2^{p-1}}\int_{\partial B^2_{\delta/2}\times(z_-,z_+)}\frac{|Dv_\delta|^p}{p}\,d\mathcal{H}^2\\ &\ge(2-2\delta)\frac{\mathcal{E}^{\mathrm{sg}}_{p/(p-1)}(\gamma)\delta^{2-p}}{2-p}-C\delta\int_{\Gamma_{\delta,1}}\frac{|D_\top g|^p}{p}\,d\mathcal{H}^2,\end{aligned}$$

where we have also used Corollary 3.5.13 (iii). Combining (3.5.29) and (3.5.30), we obtain (3.5.24). This completes our proof of Prop. 3.5.14. □

3.5.3 The limit measure has the density with a finite set of values

Recall that μ_* is the positive Radon measure on $\overline{\Omega}$ defined at the beginning of Section 3.5 and its support (*i.e.*, $\operatorname{supp}(\mu_*)$) is the set S_*. We prove the following

Proposition 3.5.16. *For $\mathcal{H}^1$-a.e. $x \in S_* \cap \Omega$, we have $\Theta_1(\mu_*, x) \in \{\mathcal{E}_2^{\mathrm{sg}}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\} \setminus \{0\}$.*

Proof. According to Proposition 3.5.8 and [4, Theorem 2.83 (i)], μ_* admits an approximate tangent space with multiplicity $\Theta_1(x_0)$ for $\mathcal{H}^1$-a.e. $x_0 \in S_* \cap \Omega$. Namely, for $\mathcal{H}^1$-a.e. $x_0 \in S_* \cap \Omega$ (or, equivalently, for μ_*-a.e. $x_0 \in \Omega$), there exists a unique approximate tangent plane $T_{x_0}S_* \in \mathrm{G}(3,1)$ to S_* at x_0 and

$$\mu_{x_0,r} = T_{\#}^{x_0,r}\mu_* \llcorner \Omega \overset{*}{\rightharpoonup} \mu_0 = \Theta_1(x_0)\mathcal{H}^1 \llcorner T_{x_0}S_* \quad \text{weakly* in } (C_0(\mathbb{R}^3))' \text{ as } r \searrow 0, \tag{3.5.31}$$

where $T^{x_0,r}(x) := (x - x_0)/r$. Up to rotation and translation, to lighten the notation, we shall assume that $x_0 = 0$ and $T_{x_0}S_* = \{(0,0)\} \times \mathbb{R}$. Recall that $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and for each $n \in \mathbb{N}$ large enough,

$$\int_\Omega \frac{|Du_n|^{p_n}}{p_n}\, dx \le \frac{C_0}{2 - p_n}, \tag{3.5.32}$$

where $C_0 > 0$ is a constant independent of n (see (3.5.1)).

Let $\delta \in (0, 1/2)$ be fairly small, which we shall carefully choose later for the proof to work. Let $\tau > 0$ be the constant of Corollary 3.5.13, where $p_0 = 3/2$. Let $\eta > 0$ be the constant of Lemma 3.3.12, where $\kappa = 1/2$, $p_0 = 3/2$ and Ψ is a bilipschitz homeomorphism between a ball and a cube with an absolute bilipschitz constant (see, for instance, [33, Corollary 3]). We fix some $\varepsilon \in (0, \min\{\eta, \tau\})$. Next, since $\mu_0((\overline{B}^2_{3\delta/2} \setminus B^2_{\delta/2}) \times [-2,2]) = 0$, in view of the weak* convergence (see (3.5.31)), there exists $r_0 = r_0(\varepsilon, \delta) \in (0, \min\{1, \operatorname{dist}(x_0, \partial\Omega)/4\})$ such that

$$\mu_*((\overline{B}^2_{3\delta r/2} \setminus B^2_{\delta r/2}) \times [-2r, 2r]) < \frac{\varepsilon\delta r}{100} \quad \text{for each } r \in (0, r_0]. \tag{3.5.33}$$

Taking into account (3.5.2) and (3.5.3), the estimate (3.5.33), together with the Fatou lemma, implies that for a fixed $r \in (0, r_0]$,

$$\int_{\delta r/2}^{3\delta r/2} d\varrho \liminf_{n\to+\infty} (2 - p_n) \int_{\Gamma_{\varrho,r}} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 < \frac{\varepsilon\delta r}{100}. \tag{3.5.34}$$

Due to (3.5.34), there exist $\varrho \in (3\delta r/4, 5\delta r/4)$ and $n_0 = n_0(\delta, r, \varepsilon) \in \mathbb{N}$ such that, up to a subsequence (not relabeled), for each $n \geq n_0$, $\mathrm{tr}_{\Gamma_{\varrho,r}}(u_n) = u_n|_{\Gamma_{\varrho,r}} \in W^{1,p_n}(\Gamma_{\varrho,r}, \mathcal{N})$ (see (3.2.20)) and

$$\int_{\Gamma_{\varrho,r}} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 < \frac{\varepsilon \varrho^{2-p_n}}{2-p_n}, \tag{3.5.35}$$

where we have used that $\varrho^{2-p_n} \to 1$ as $n \to +\infty$. To simplify the notation, without loss of generality, we assume that $\varrho = \delta r$. Notice that $\Lambda_{\delta r, 5r/4} \subset B^3_{\sqrt{2}r}$, where $\sqrt{2}r < d_0 := \mathrm{dist}(x_0, \partial\Omega)/2$ (since it holds $r \leq r_0 < d_0/2$). Thus, using the monotonicity of the p-energy (see Lemma 3.5.3) and (3.5.32), for each $n \in \mathbb{N}$ large enough, we get

$$\begin{aligned}
\int_{\Lambda_{\delta r, 5r/4}} \frac{|Du_n|^{p_n}}{p_n}\, dx &\leq \left(\frac{\sqrt{2}r}{d_0}\right)^{3-p_n} \int_{B^3_{d_0}} \frac{|Du_n|^{p_n}}{p_n}\, dx \\
&\leq \left(\frac{\sqrt{2}r}{d_0}\right)^{3-p_n} \frac{C_0}{2-p_n}.
\end{aligned}$$

Using again the Fatou lemma, it holds

$$\int_{-5r/4}^{5r/4} dz \liminf_{n\to+\infty} (2-p_n) \int_{B^2_{\delta r}\times\{z\}} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \leq \frac{C_0\sqrt{2}r}{d_0}.$$

This implies that there exist $z^r_- \in (-5r/4, -3r/4)$ and $z^r_+ \in (3r/4, 5r/4)$ such that, up to a subsequence (still denoted by the same index), for each $z \in \{z^r_-, z^r_+\}$ we have the following $\mathrm{tr}_{B^2_{\delta r}\times\{z\}}(u_n) = u_n|_{B^2_{\delta r}\times\{z\}} \in W^{1,p_n}(B^2_{\delta r} \times \{z\}, \mathcal{N})$ and for each $n \in \mathbb{N}$ large enough,

$$\int_{B^2_{\delta r}\times\{z\}} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \leq \frac{C_1}{2-p_n}, \tag{3.5.36}$$

where $C_1 = C_1(C_0, d_0) > 0$. Without loss of generality, to lighten the notation, we shall assume that $z^r_- = -r$ and $z^r_+ = r$. Thus, $\mathrm{tr}_{\partial\Lambda_{\delta r,r}}(u_n) = u_n|_{\partial\Lambda_{\delta r,r}} \in W^{1,p_n}(\partial\Lambda_{\delta r,r}, \mathcal{N})$. From (3.5.35) and (3.5.36) we obtain

$$\int_{\Gamma_{\delta r,r}} \frac{|D_\top \mathrm{tr}_{\Gamma_{\delta r,r}}(u_n)|^{p_n}}{p_n}\, d\mathcal{H}^2 \leq \frac{\varepsilon(\delta r)^{2-p_n}}{2-p_n} \tag{3.5.37}$$

and

$$\int_{B^2_{\delta r}\times\{z\}} \frac{|D_\top \mathrm{tr}_{B^2_{\delta r}\times\{z\}}(u_n)|^{p_n}}{p_n}\, d\mathcal{H}^2 \leq \frac{C_1}{2-p_n} \tag{3.5.38}$$

for each $z \in \{-r, r\}$ and $n \in \mathbb{N}$ large enough. Thus, for each sufficiently large $n \in \mathbb{N}$, u_n fulfills the conditions of Proposition 3.5.14 (recall that $0 < \varepsilon < \tau$). Using Proposition 3.5.14 and (3.5.37), we obtain $\gamma_n \in [\mathbb{S}^1, \mathcal{N}]$ such that the following holds. Firstly,

$$\begin{aligned} (3.5.39) \quad &\int_{\Lambda_{\delta r, r}} \frac{|Du_n|^{p_n}}{p_n}\, dx \\ &\leq \frac{2\mathcal{E}^{\mathrm{sg}}_{p_n}(\gamma_n) r^{3-p_n}}{2-p_n} + \frac{C\varepsilon\delta^{3-2p_n} r^{3-p_n}}{2-p_n} + \frac{C\delta r}{2-p_n} + \frac{C(\delta r)^{3-p_n}}{2-p_n} \end{aligned}$$

for each sufficiently large $n \in \mathbb{N}$, where $C = C(C_1, \mathcal{N}) > 0$. Secondly,

(3.5.40)
$$(2-2\delta)\frac{\mathcal{E}^{\mathrm{sg}}_{p_n/(p_n-1)}(\gamma_n)\delta^{2-p_n} r^{3-p_n}}{2-p_n} - \frac{C'\varepsilon(\delta r)^{3-p_n}}{2-p_n} \leq \int_{\Lambda_{\delta r, r}} \frac{|Du_n|^{p_n}}{p_n}\, dx$$

for each sufficiently large $n \in \mathbb{N}$, where $C' = C'(\mathcal{N}) > 0$.

Now, applying Lemma 3.5.2, due to (3.5.33), we can cover the compact set $\partial B^2_{\delta r} \times [-r, r]$ by a finite number of small cubes (which are bilipschitz homeomorphic to balls) being subsets of $\Omega \setminus S_*$ and obtain that the set

$$E_{\delta,r} = (\overline{B}^2_{101\delta r/100} \setminus B^2_{99\delta r/100}) \times [-r - \delta r/100, r + \delta r/100]$$

is a subset of $\Omega \setminus S_*$. Denote by $E^0_{\delta,r}$ the interior of $E_{\delta,r}$. Then, by Proposition 3.5.6, $u_n \rightharpoonup u_*$ weakly in $W^{1,p}(E^0_{\delta,r}, \mathbb{R}^\nu)$ for all $p \in (1,2)$. Furthermore, there exists a finite set $\{x_1, \ldots, x_l\} \subset E^0_{\delta,r}$ such that $u_* \in C^\infty(E^0_{\delta,r} \setminus \{x_1, \ldots, x_l\}, \mathcal{N})$. By the weak convergence and the Sobolev embedding, without loss of generality, $u_n|_{\partial B^2_{\delta r}\times\{z\}}$ converges uniformly to $u_*|_{\partial B^2_{\delta r}\times\{z\}} \in C(\partial B^2_{\delta r} \times \{z\}, \mathcal{N})$ for each $z \in \{-r, r\}$.

We claim that for each $n \in \mathbb{N}$ large enough, $[\gamma_n] = [u_*|_{\partial B^2_{\delta r}\times\{-r\}}] = [u_*|_{\partial B^2_{\delta r}\times\{r\}}]$. Indeed, a close inspection of the proofs of Lemma 3.5.12 and Corollary 3.5.13 shows that the mapping $v_{\delta,n} \in W^{1,2}(\partial B^2_{\delta r/2}\times (z^n_-, z^n_+), \mathcal{N})$, coming from Corollary 3.5.13, has a well-defined homotopy class on the surface $\partial B^2_{\delta r/2} \times (z^n_-, z^n_+)$ and for each $z \in \{z^n_-, z^n_+\}$, $[v_{\delta,n}|_{\partial B^2_{\delta r/2}\times\{z\}}] = [\gamma_n]$, where $z^n_- \in (-r + \delta r/2, -r + \delta r)$ and $z^n_+ \in (r - \delta r, r - \delta r/2)$ (the reader may consult the arguments given immediately above the estimate (3.5.19), take into account Remark 3.2.19, and observe that the trace on $\mathbb{S}^1 \times \{L\}$ of the map $w_h \in W^{1,2}(\Gamma_{1,L}, \mathcal{N})$ constructed in the proof of Proposition 3.5.14 admits a continuous representative). We can actually extend $\varphi_{\delta,n}$ on $(B^2_{\delta r} \setminus \overline{B}^2_{\delta r/2}) \times (-r, r)$ and $v_{\delta,n}$ on $\partial B^2_{\delta r/2} \times (-r, r)$, where $\varphi_{\delta,n}$ comes from Corollary 3.5.13, so that these extensions have the following properties: $\varphi_{\delta,n} \in W^{1,p}((B^2_{\delta r} \setminus \overline{B}^2_{\delta r/2}) \times (-r, r), \mathcal{N})$, $v_{\delta,n} \in W^{1,2}(\partial B^2_{\delta r/2} \times (-r, r), \mathcal{N})$,

$\mathrm{tr}_{\partial B^2_{\delta r/2}\times(-r,r)}(\varphi_{\delta,n}) = v_{\delta,n}$ (the proof consists in constructing the map on the union of surfaces $(\partial B^2_{\delta r/2}\times(-r,z^n_-))\cup(\partial B^2_{\delta r/2}\times(z^n_+,r))$ by similar way as was constructed $v_{\delta,n}$ on $\partial B^2_{\delta r/2}\times(z^n_-,z^n_+)$, namely, by choosing appropriate triangulations on the sets $\partial B^2_{\delta r}\times(-r,z^n_-)$, $\partial B^2_{\delta r}\times(z^n_+,r)$ and projecting them onto $\partial B^2_{\delta r/2}\times(-r,z^n_-)$, $\partial B^2_{\delta r/2}\times(z^n_+,r)$, respectively, and then proceed as in the construction of the maps $v_{\delta,n}$, $\varphi_{\delta,n}$; the traces of the obtained extensions of $\varphi_{\delta,n}$ and $v_{\delta,n}$ will coincide with the traces of $\varphi_{\delta,n}$ and $v_{\delta,n}$ on $\partial B^2_{\delta r}\times\{z^n_-\}$, $\partial B^2_{\delta r}\times\{z^n_+\}$ and $\partial B^2_{\delta r/2}\times\{z^n_-\}$, $\partial B^2_{\delta r/2}\times\{z^n_+\}$, respectively, and hence we obtain a $W^{1,p}$ extension of $\varphi_{\delta,n}$ on $(B^2_{\delta r}\setminus\overline{B}^2_{\delta r/2})\times(-r,r)$ and a $W^{1,2}$ extension of $v_{\delta,n}$ on $\partial B^2_{\delta r/2}\times(-r,r)$). Since for each $z\in\{-r,r\}$, $\mathrm{tr}_{\partial B^2_{\delta r/2}\times\{z\}}(v_{\delta,n})$ is homotopic to $u_n|_{\partial B^2_{\delta r}\times\{z\}}$, we deduce that for each $n\in\mathbb{N}$ large enough, $[\gamma_n]=[u_*|_{\partial B^2_{\delta r}\times\{z\}}]$. This proves our claim.

Thus, there exists $\gamma\in[\mathbb{S}^1,\mathcal{N}]$ such that for each $n\in\mathbb{N}$ large enough, $[\gamma_n]=[\gamma]$. Next, multiplying both sides of (3.5.39) and (3.5.40) by $(2-p_n)$ and then passing to the limit when n tend to $+\infty$, we obtain

$$(3.5.41)\quad r(2-2\delta)\mathcal{E}^{\mathrm{sg}}_2(\gamma) - C'\varepsilon\delta r \le \mu_*(\Lambda_{\delta r,r}) \le 2r\mathcal{E}^{\mathrm{sg}}_2(\gamma) + \frac{C\varepsilon r}{\delta} + 2C\delta r,$$

where we have used Lemma 3.2.8. Now we choose $\delta=\varepsilon^{1/2}$ and hence $\delta\searrow 0$ as $\varepsilon\searrow 0$. Dividing both sides of (3.5.41) by r and letting $r\searrow 0$ and then $\varepsilon\searrow 0$, we obtain

$$2\mathcal{E}^{\mathrm{sg}}_2(\gamma)\le\Theta_1(x_0)\mathcal{H}^1(\{(0,0)\}\times[-1,1])\le 2\mathcal{E}^{\mathrm{sg}}_2(\gamma).$$

This implies that $\Theta_1(x_0)=\mathcal{E}^{\mathrm{sg}}_2(\gamma)$. Since $x_0\in S_*\cap\Omega$, by Proposition 3.5.5, $\Theta_1(x_0)>0$ and hence $\mathcal{E}^{\mathrm{sg}}_2(\gamma)>0$. This completes our proof of Proposition 3.5.16. □

According to Lemma 3.2.8 and Proposition 3.5.16, the density of the one-dimensional varifold V_* coming from Proposition 3.5.8, which was *naturally* associated with $\mu_*\llcorner\Omega$, is bounded from below $\|V_*\|$-a.e. in Ω (*i.e.*, μ_*-a.e. in Ω or, equivalently, $\mathcal{H}^1$-a.e. in $S_*\cap\Omega$) and has a finite set of values. Thus, due to Allard and Almgren's theorem [3, Theorem, p. 89], we obtain the following structure for $S_*\cap\Omega$.

Proposition 3.5.17. *Any compact set $K\subset\Omega$ such that $S_*\cap K\neq\emptyset$ has an open neighborhood $U\Subset\Omega$ with a Lipschitz boundary and a finite number of connected components such that $S_*\cap\overline{U}$ is a union of a finite number of closed straight line segments, each of which has a positive length, and the interiors of these segments (i.e., the segments without their endpoints) are pairwise disjoint. If K is connected, then U is* *connected; otherwise, U may have several connected components, but their closures are pairwise disjoint. Furthermore, $S_* \cap \partial U$ is a finite set of points such that for each point $x \in S_* \cap \partial U$ there is exactly one segment lying in $S_* \cap \overline{U}$ and emanating from x.*

Proof. If $\pi_1(\mathcal{N}) = \{0\}$, then by Proposition 3.5.16 and Proposition 3.5.5, it holds $S_* \cap \Omega = \emptyset$ and the proof is trivial.

Assume that $\pi_1(\mathcal{N}) \neq \{0\}$. According to the definition of the varifold V_* (see Proposition 3.5.8), the weight of V_* (or, also the projection of V_* on Ω), defined by $\|V_*\|(E) = V_*(E \times \mathrm{G}(3,1))$ for each $E \in \mathcal{B}(\Omega)$, is equal to $\mu_* \llcorner \Omega = \Theta_1 \mathcal{H}^1 \llcorner (S_* \cap \Omega)$. Then, by Proposition 3.5.5, there exists $c = c(\mathcal{N}) > 0$ such that for each $x \in S_* \cap \Omega$,

$$\begin{aligned}(3.5.42)\quad \Theta_1(\|V_*\|, x) &:= \lim_{r \to 0+} \frac{\|V_*\|(B_r^3(x))}{2r} \\ &= \lim_{r \to 0+} \frac{\mu_*(B_r^3(x))}{2r} = \Theta_1(x) \geq c.\end{aligned}$$

In view of Proposition 3.5.8 and (3.5.42), we can apply [3, Theorem, p. 89] and obtain that there exists a countable family $\mathcal{P}$ of bounded open intervals (*i.e.*, segments without endpoints) contained in $S_* \cap \Omega$ such that $V_* = \sum\{\Theta_1(I)|I| : I \in \mathcal{P}\}$, where $\Theta_1(I)$ is the unique member of the range of Θ_1 restricted to the interval I and $|I| \in (C_0(\mathrm{G}_3(\Omega)))'$ is a finite positive Radon measure defined by $|I| = \delta_{\mathrm{e}\otimes\mathrm{e}}(dT) \otimes \mathcal{H}^1 \llcorner I(dx)$, where e is the unit direction vector of I. Taking into account Lemma 3.2.8 and Proposition 3.5.16, we have that $\{\Theta_1(I) : I \in \mathcal{P}\} \subset \{\mathcal{E}_2^{\mathrm{sg}}(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}$ is a finite set of values. Furthermore, the proof of [3, Theorem, p. 89] says that for each $x \in S_* \cap \Omega$, there exists a radius $r > 0$ such that $S_* \cap \Omega \cap \overline{B}_r^3(x)$ is a finite union of radii of $\overline{B}_r^3(x)$, that is, closed straight line segments joining x with a point on $\partial B_r^3(x)$. Let $K \subset \Omega$ be compact and satisfy $K \cap S_* \neq \emptyset$. Then there exists a finite collection of open balls $B_{2r_1}^3(x_1), \ldots, B_{2r_l}^3(x_l)$ whose closures are contained in Ω, centered at S_* and the following holds: $K \cap S_* \subset \bigcup_{i=1}^l B_{r_i}^3(x_i)$; $S_* \cap \overline{B}_{2r_i}^3(x_i)$ is a finite union of radii of $\overline{B}_{2r_i}^3(x_i)$ for each $i \in \{1, \ldots, l\}$ (at least two radii are contained in each of these covering balls, because otherwise we would obtain a contradiction with the stationarity of V_*). Now fix an arbitrary $i \in \{1, \ldots, l\}$. If for some $j \in \{1, \ldots, l\}$, $j \neq i$ a radius L_j of $\overline{B}_{r_j}^3(x_j)$ (*i.e.*, a closed segment joining x_j with a point on $\partial B_{r_j}^3(x_j)$) lying in $S_* \cap \overline{B}_{r_j}^3(x_j)$ intersects $\overline{B}_{r_i}^3(x_i)$, then, by construction of our covering, there exists a radius L_i of $\overline{B}_{r_i}^3(x_i)$ and a straight line containing both L_i and L_j (the union of these radii is a straight line segment). Covering the rest of K with a finite number of chains of sufficiently small balls, one can appropriately choose a neighborhood U of K satisfying the conditions of Proposition 3.5.17 by passing from ball to ball in each chain from the resulting cover and successively constructing ∂U. This completes our proof of Proposition 3.5.17. □

As a byproduct of the proof of Proposition 3.5.16, taking into account Proposition 3.5.17, we obtain the following

Corollary 3.5.18. Let $L \subset S_*$ be a closed straight line segment. Let x be a point lying in the relative interior of L and $D \subset \Omega$ be a closed 2-disk (which is an embedded submanifold of $\mathbb{R}^3$) with center x such that $D \cap S_* = \{x\}$ and $\partial D \cap S_0 = \emptyset$, where S_0 is at most a countable and locally finite subset of $\Omega \backslash S_*$ coming from Proposition 3.4.2 (iii), which applies to u_* in $\Omega \backslash S_*$ (see Proposition 3.5.6). Then the homotopy class of $u_*|_{\partial D}$ is nontrivial and for each point y in the relative interior of L, $\Theta_1(\mu_*, y) = \mathcal{E}_2^{\mathrm{sg}}(u_*|_{\partial D})$.

Proof of Corollary 3.5.18. Fix an arbitrary point y lying in the relative interior of L. According to Proposition 3.5.6 and Proposition 3.5.17, we can find an open set $U \Subset \Omega$ such that $x, y \in U$, $D \subset U$ and $u_* \in W^{1,2}_{\mathrm{loc}}(U \backslash L, \mathcal{N}) \cap C^\infty(U \backslash (L \cup S_0), \mathcal{N})$. Since S_0 is locally finite, there exists a closed 2-disk $D_y \subset U$ such that $D_y \cap S_* = \{y\}$, $\partial D_y \cap S_0 = \emptyset$ and $u_*|_{\partial D}$ is continuously homotopic to $u_*|_{\partial D_y}$. Then $\mathcal{E}_2^{\mathrm{sg}}(u_*|_{\partial D}) = \mathcal{E}_2^{\mathrm{sg}}(u_*|_{\partial D_y})$. On the other hand, as a byproduct of the proof of Proposition 3.5.16, $\Theta_1(y) = \mathcal{E}_2^{\mathrm{sg}}(u_*|_{\partial D})$, which completes our proof of Corollary 3.5.18. □

The stationarity of V_* and Proposition 3.5.17 yield the following observation.

Corollary 3.5.19. Let $x_0 \in S_* \cap \Omega$. Then, according to Proposition 3.5.17, there exists a radius $r > 0$ such that $\overline{B}^3_r(x_0) \subset \Omega$ and $S_* \cap \overline{B}^3_r(x_0) = \bigcup_{i=1}^l L_i$, where $l \in \mathbb{N} \setminus \{0\}$ and L_i is a closed straight line segment emanating from x_0. Let $v_i \in \mathbb{S}^2$ be the direction vector of L_i pointing outward from x_0. Then the stationarity of V_* yields that the weighted sum of the v_i's is zero, namely

$$\sum_{i=1}^{l} \lambda_i v_i = 0, \tag{3.5.43}$$

where $\lambda_i = \mathcal{E}_2^{\mathrm{sg}}(u_*|_{\partial D_i}) > 0$ with D_i being a closed 2-disk (which is an embedded submanifold of $\mathbb{R}^3$) centered inside L_i and lying in $B^3_r(x_0)$ on the 2-plane orthogonal to L_i such that $D_i \cap S_*$ is the singleton, $\partial D_i \cap S_0 = \emptyset$, where S_0 is at most a countable and locally finite subset of $\Omega \backslash S_*$ coming from Proposition 3.4.2 (iii), which applies to u_* in $\Omega \backslash S_*$ (see Proposition 3.5.6).

Proof of Corollary 3.5.19. Fix an arbitrary $\chi \in C^1_c(B^3_r(x_0), \mathbb{R}^3)$. Since the first variation of V_* vanishes, where $V_*(dT, dx) = \delta_{A_*(x)}(dT) \otimes \Theta_1(x)\mathcal{H}^1 \llcorner (S_* \cap \Omega)(dx)$ with $A_*(x)$ being the orthogonal projection matrix onto the approximate tangent plane $T_x S_*$ to S_* at x, we have

$$0 = \int_\Omega A_*(x) : D\chi(x)\, d\mu_*(x) = \int_{S_* \cap B^3_r(x_0)} \Theta_1(x) A_*(x) : D\chi(x)\, d\mathcal{H}^1(x). \tag{3.5.44}$$

Next, using that $S_* \cap \overline{B}^3_r(x_0) = \bigcup_{i=1}^l L_i$, $\mu_* \llcorner B^3_r(x_0) = \sum_{i=1}^l \lambda_i \mathcal{H}^1 \llcorner L_i$ and $A_*|_{L_i} = v_i \otimes v_i$, in view of (3.5.44), we get

$$\begin{aligned} 0 &= \sum_{i=1}^l \lambda_i \int_{L_i} v_i \otimes v_i : D\chi(x)\, d\mathcal{H}^1(x) \\ &= \sum_{i=1}^l \lambda_i \int_0^{\mathcal{H}^1(L_i)} \frac{d}{dt} \langle \chi(x_0 + tv_i), v_i \rangle\, dt \\ &= -\sum_{i=1}^l \langle \chi(x_0), \lambda_i v_i \rangle. \end{aligned} \tag{3.5.45}$$

Since $\chi \in C^1_c(B^3_r(x_0), \mathbb{R}^3)$ was arbitrarily chosen, (3.5.45) yields (3.5.43), which completes our proof of Corollary 3.5.19. □

Remark 3.5.20. We obtain several direct consequences of Corollary 3.5.19.

(i) For each $x_0 \in S_* \cap \Omega$, at least two segments emanate from x_0, because otherwise x_0 would be an endpoint for some segment lying in $S_* \cap \Omega$, but this would contradict (3.5.43).

(ii) If locally around x_0 the set $S_* \cap \Omega$ is a union of exactly two segments emanating from x_0, then the angle between them is π, since a smaller angle would contradict (3.5.43).

(iii) If exactly three segments emanate from x_0, then their direction vectors lie in the same two-dimensional plane.

(iv) If exactly four segments emanate from x_0, then $\lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + \lambda_4 v_4 = 0$, which yields a system of equations. Firstly, $|\lambda_1 v_1 + \lambda_2 v_2|^2 = |\lambda_3 v_3 + \lambda_4 v_4|^2$ and hence

$$\lambda_1^2 + \lambda_2^2 + 2\lambda_1\lambda_2 \langle v_1, v_2 \rangle = \lambda_3^2 + \lambda_4^2 + 2\lambda_3\lambda_4 \langle v_3, v_4 \rangle.$$

Secondly, for each $j \in \{1, \dots, 4\}$,

$$\lambda_j + \sum_{i=1,\, i \neq j}^4 \lambda_i \langle v_i, v_j \rangle = 0,$$

since $|v_j| = 1$. Thirdly, for each $j \in \{1, \dots, 4\}$,

$$|\lambda_j v_j|^2 = |\sum_{i=1,\, i\neq j}^{4} \lambda_i v_i|^2$$

and hence

$$\lambda_j^2 = \sum_{i=1,\, i\neq j}^{4} \lambda_i^2 + 2 \sum_{i,\, k\in(\{1,\dots,4\}\setminus\{j\})\cap\{i<k\}} \lambda_i \lambda_k \langle v_i, v_k \rangle.$$

If $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4$ in (3.5.43), then, up to relabeling, $v_1 + v_2 = -v_3 - v_4$, and there exists a double cone with apex 0 such that v_1, v_2 lie on the upper nappe and v_3, v_4 lie on the lower nappe of the cone, respectively. Furthermore, the angle between v_1 and v_2 is the same as between v_3 and v_4, which is twice the angle of the double cone.

Moreover, in [22, Proposition 2] it was proved that in the case when $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$ (for instance, $\mathcal{N} = SO(n)$ or $\mathcal{N} = \mathbb{RP}^{n-1}$ for $n \in \mathbb{N}$, $n \geq 3$), then only an even number of segments can emanate from $x \in S_* \cap \Omega$. We prove that in the case when $\pi_1(\mathcal{N}) = \mathbb{Z}/2\mathbb{Z}$, there cannot be branching points inside Ω (see Corollary 3.6.8).

3.5.4 Interior estimates and $W^{1,q}_{\mathrm{loc}}(\Omega)$ compactness for p-minimizers when $q < 2$

We prove that the q-energy of p_n-minimizers with $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$ and $p_n \nearrow 2$ as $n \to +\infty$ is locally bounded inside Ω for each $q \in [1, 2)$ under the condition (3.5.1).

For the reader's convenience, let us recall the construction of dyadic cubes in $\mathbb{R}^3$ (see, for instance, [7]).

Definition 3.5.21. Let $[0, 1)^3$ be the reference cube, $j \in \mathbb{Z}$ and $x \in \mathbb{Z}^3$. Then define the dyadic cube of generation j with lower left corner $2^{-j}x$ by

$$Q_{j,x} := 2^{-j}(x + [0, 1)^3) = \{y \in \mathbb{R}^3 : 2^j y - x \in [0, 1)^3\},$$

the set of generation j dyadic cubes by $\mathscr{D}_j = \{Q_{j,x} : x \in \mathbb{Z}^3\}$ and the set of all dyadic cubes by

$$\mathscr{D} := \bigcup_{j\in\mathbb{Z}} \mathscr{D}_j = \{Q_{j,x} : j \in \mathbb{Z} \text{ and } x \in \mathbb{Z}^3\}.$$

For each $k \in \mathbb{N} \setminus \{0\}$, we also define the enlarged cubes

$$kQ_{j,x} := \left\{ y \in \mathbb{R}^3 : 2^j y - x \in \left[-\frac{k-1}{2}, \frac{k+1}{2} \right)^3 \right\}.$$

According to [7, Theorem 2.2.3 (ii)], for each $j \in \mathbb{Z}$ and $x \in \mathbb{Z}^3$, there exists a unique $y \in \mathbb{Z}^3$ such that $Q_{j,x} \subset Q_{j-1,y}$. The cube $Q_{j-1,y}$ will be called the *parent* of $Q_{j,x}$. For each $j \in \mathbb{Z}$ and $y \in \mathbb{Z}^3$, the cubes $Q_{j+1,x}$ for which $Q_{j,y}$ is the parent are called the *children* of $Q_{j,y}$.

The distance from $U \subset \mathbb{R}^3$ to $V \subset \mathbb{R}^3$ is given by

$$\operatorname{dist}(U, V) := \inf\{|x - y| : x \in U, \ \ y \in V\}.$$

Proposition 3.5.22. *Let $V \subset \mathbb{R}^3$ be a bounded open set. Let C_1, C_2 be positive constants, $p \in (1, 2]$ and $f \in L^p(V)$. Assume that for each dyadic cube Q_{j,x_0} such that $2Q_{j,x_0} \subset V$ and*

$$C_1 \int_{2Q_{j,x_0}} |f|^p \, dx \leq 2^{-j(3-p)} \tag{3.5.46}$$

it holds

$$\int_{Q_{j,x_0}} |f|^p \, dx \leq C_2 2^{-j(3-p)}. \tag{3.5.47}$$

Then for each $q \in [1, p)$, $f \in L^q(V)$ and

$$\begin{aligned} &\int_V (\operatorname{dist}(x, \partial V)|f(x)|)^q \, dx \\ &\qquad \leq 10^4 C_2^{\frac{q}{p}} |V| + 10^4 \frac{C_1 C_2^{\frac{q}{p}}}{p - q} \int_V (\operatorname{dist}(x, \partial V)|f(x)|)^p \, dx. \end{aligned} \tag{3.5.48}$$

Proof. According to the Whitney covering theorem for dyadic cubes, namely [7, Theorem 2.3.1], there exists a collection of mutually disjoint dyadic cubes $\mathcal{F} = \{Q_i\}_{i \in I}$, where I is countable, such that

$$\frac{1}{30} \operatorname{dist}(Q_i, \partial V) \leq \operatorname{diam}(Q_i) \leq \frac{1}{10} \operatorname{dist}(Q_i, \partial V) \tag{3.5.49}$$

and

$$V = \bigcup_{i \in I} Q_i. \tag{3.5.50}$$

Since V is bounded, there exists (the minimal) $j_0 \in \mathbb{Z}$ such that for some $x_0 \in \mathbb{Z}^3$, $Q_{j_0,x_0} \in \mathcal{F}$ and for each $x \in \mathbb{Z}^3$ and $j \in (-\infty, j_0) \cap \mathbb{Z}$, $Q_{j,x} \notin \mathcal{F}$. For each $j \in [j_0, +\infty) \cap \mathbb{Z}$, define $\mathcal{F}_j := \mathscr{D}_j \cap \mathcal{F}$. Since V is bounded, $\mathcal{F}_j$ is finite. Notice also that for each $Q \in \mathcal{F}$, $3Q \subset V$ (this comes from (3.5.49), see [7, Proposition 2.3.3 (i)]). We say that a dyadic cube $Q \in \mathcal{F}_j$ is *good* if (3.5.46) holds with Q_{j,x_0} replaced by Q, and it is *bad* otherwise. Let us now define the collection of all good cubes of $\mathcal{F}_j$ by $\mathcal{F}_{j,0}$ and their union by $G_{j,0}$. For each bad cube $Q \in \mathcal{F}_j$, we consider the children of Q. Similarly, we say that a child Q' of Q is good if (3.5.46) holds with Q_{j,x_0} replaced by Q', and it is bad otherwise. We define the collection of all good children of bad cubes of $\mathcal{F}_j$ by $\mathcal{F}_{j,1}$ and their union by $G_{j,1}$. Next, we select good children of bad children of bad cubes in $\mathcal{F}_j$ and denote this collection by $\mathcal{F}_{j,2}$ and their union by $G_{j,2}$. Denote the number of bad cubes in $\mathcal{F}_j$ by $N_{j,0}$. The number of bad children of bad cubes of $\mathcal{F}_j$ denote by $N_{j,1}$ and the number of bad children of bad children of bad cubes of $\mathcal{F}_j$ denote by $N_{j,2}$. Repeating this procedure, we obtain sequences $(\mathcal{F}_{j,n})_{n\in\mathbb{N}}$, $(G_{j,n})_{n\in\mathbb{N}}$ and $(N_{j,n})_{n\in\mathbb{N}}$. Define $G_j := \bigcup_{n\in\mathbb{N}} G_{j,n}$. We claim that G_j has full measure in $\bigcup_{Q\in\mathcal{F}_j} Q$. Indeed, by [7, Theorem 2.2.3], for each $y \in \mathbb{R}^3$, there exists a sequence of dyadic cubes $(Q_{k,y_k})_{k\in\mathbb{N}}$ such that $\{y\} = \bigcap_{k\in\mathbb{N}} Q_{k,y_k}$. If $y \in \bigcup_{Q\in\mathcal{F}_j} Q \setminus G_j$, then, by construction, for each $k \in \mathbb{N}$ large enough, $2Q_{k,y_k} \subset B^3_{2^{-k+1}}(y) \subset V$ and

$$C_1 \int_{B^3_{2^{-k+1}}(y)} |f|^p \, dx \geq 2^{-k(3-p)},$$

which implies that

$$\liminf_{k\to+\infty} \frac{1}{|B^3_{2^{-k+1}}|} \int_{B^3_{2^{-k+1}}(y)} |f|^p \, dx = +\infty.$$

Thus, in view of the Lebesgue differentiation theorem (see [29, Theorem 1.32]) and the fact that $f \in L^p(V)$, $|\bigcup_{Q\in\mathcal{F}_j} Q \setminus G_j| = 0$. This proves our claim. On the other hand, observe that for each $n \in \mathbb{N}$,

$$\#\mathcal{F}_{j,n+1} \leq 2^3 N_{j,n} \tag{3.5.51}$$

and, in view of (3.5.46),

$$N_{j,n} \leq 3^3 2^{(j+n)(3-p)} C_1 \int_{G_j} |f|^p \, dx, \tag{3.5.52}$$

where we have also used that if $Q \in \mathscr{D}_j$, then there exists exactly $3^3 - 1$ other dyadic cubes $Q' \in \mathscr{D}_j$ such that $2Q \cap 2Q' \neq \emptyset$, and that G_j has full measure in $\bigcup_{Q\in\mathcal{F}_j} Q$. Combining (3.5.51) with (3.5.52), for each $n \in \mathbb{N}$, we get

$$\#\mathcal{F}_{j,n+1} \leq 6^3 2^{(j+n)(3-p)} C_1 \int_{G_j} |f|^p \, dx. \tag{3.5.53}$$

It is also worth noting that

$$\#\mathcal{F}_{j,0} \leq 2^{3j} |G_j|. \tag{3.5.54}$$

Applying Hölder's inequality and (3.5.47), for each $Q \in \mathfrak{F}_{j,n}$, we obtain

$$\begin{aligned}\int_Q |f|^q\,dx &\le |Q|^{1-\frac{q}{p}} \left(\int_Q |f|^p\,dx\right)^{\frac{q}{p}} \\ &\le C_2^{\frac{q}{p}} 2^{-3(j+n)\left(1-\frac{q}{p}\right)} 2^{-(j+n)\left(\frac{3q}{p}-q\right)} \le C_2^{\frac{q}{p}} 2^{-(j+n)(3-q)}.\end{aligned}$$

Using this, (3.5.53) and (3.5.54), we deduce that

$$\begin{aligned}\int_{G_j} |f|^q\,dx &= \sum_{n=0}^{+\infty} \int_{G_{j,n}} |f|^q\,dx \\ &= \sum_{Q\in\mathfrak{F}_{j,0}} \int_Q |f|^q\,dx + \sum_{n=0}^{+\infty} \sum_{Q\in\mathfrak{F}_{j,n+1}} \int_Q |f|^q\,dx \\ &\le \#\mathfrak{F}_{j,0} C_2^{\frac{q}{p}} 2^{-j(3-q)} + \sum_{n=0}^{+\infty} \#\mathfrak{F}_{j,n+1} C_2^{\frac{q}{p}} 2^{-(j+n+1)(3-q)} \\ &\le C_2^{\frac{q}{p}} 2^{jq}|G_j| + 3^3 2^q C_1 C_2^{\frac{q}{p}} \sum_{n=0}^{+\infty} 2^{-(j+n)(p-q)} \int_{G_j} |f|^p\,dx \\ &\le C_2^{\frac{q}{p}} 2^{jq}|G_j| + 3^3 2^{jq+p} \frac{C_1 C_2^{\frac{q}{p}}}{\ln(2)(p-q)} \int_{G_j} 2^{-jp}|f|^p\,dx,\end{aligned}$$

where, in the last estimate, we have used that $2^{p-q}-1 \ge \ln(2)(p-q)$. Hence

$$\int_{G_j} 2^{-jq}|f|^q\,dx \le C_2^{\frac{q}{p}}|G_j| + 3^3 2^p \frac{C_1 C_2^{\frac{q}{p}}}{\ln(2)(p-q)} \int_{G_j} 2^{-jp}|f|^p\,dx. \tag{3.5.55}$$

Since $1 \le q < p \le 2$, from (3.5.49), (3.5.55) and the fact that G_j has full measure in $\bigcup_{Q\in\mathfrak{F}_j} Q$, we obtain

$$\begin{aligned}&\int_{G_j} (\operatorname{dist}(x,\partial V)|f(x)|)^q\,dx \\ &\qquad\le 10^4 C_2^{\frac{q}{p}}|G_j| + 10^4 \frac{C_1 C_2^{\frac{q}{p}}}{p-q} \int_{G_j} (\operatorname{dist}(x,\partial V)|f(x)|)^p\,dx.\end{aligned}$$

Summing both sides of the above inequality over $j \in [j_0,+\infty)\cap\mathbb{Z}$ and taking into account that $\bigcup_{j=j_0}^{+\infty} G_j$ has full measure in V (this comes from

(3.5.50) and the fact that G_j has full measure in $\bigcup_{Q\in\mathcal{F}_j} Q$), yields

$$\int_V (\operatorname{dist}(x,\partial V)|f(x)|)^q\,dx \\ \leq 10^4 C_2^{\frac{q}{p}}|V| + 10^4\frac{C_1 C_2^{\frac{q}{p}}}{p-q}\int_V (\operatorname{dist}(x,\partial V)|f(x)|)^p\,dx.$$

This completes our proof of Proposition 3.5.22. □

As a consequence of Proposition 3.5.22, we obtain an improvement of Proposition 3.5.6.

Proposition 3.5.23. *Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers in Ω. Then the following assertions hold.*

(i) There exists $C = C(\operatorname{diam}(\Omega),\mathcal{N}) > 0$ such that for each $q\in[1,2)$,

$$\limsup_{n\to+\infty}\int_\Omega (\operatorname{dist}(x,\partial\Omega)|Du_n(x)|)^q\,dx \\ \leq C|\Omega| + \frac{C}{2-q}\limsup_{n\to+\infty}(2-p_n)\int_\Omega \frac{|Du_n|^{p_n}}{p_n}\,dx.$$

(ii) Let $(u_n)_{n\in\mathbb{N}}$ satisfy (3.5.1) and let u_* be given by Proposition 3.5.6. Then for each $q\in[1,2)$, $u_*\in W^{1,q}_{\mathrm{loc}}(\Omega,\mathcal{N})$ and, up to a subsequence (not relabeled), $u_n\to u_*$ in $W^{1,q}_{\mathrm{loc}}(\Omega,\mathbb{R}^\nu)$ and

$$\int_\Omega (\operatorname{dist}(x,\partial\Omega)|Du_*(x)|)^q\,dx \\ \leq C|\Omega| + \frac{C}{2-q}\limsup_{n\to+\infty}(2-p_n)\int_\Omega \frac{|Du_n|^{p_n}}{p_n}\,dx, \tag{3.5.56}$$

where $C = C(\operatorname{diam}(\Omega),\mathcal{N}) > 0$.

Proof. Let $\eta, C > 0$ be the constants of Lemma 3.3.12, where $\kappa = 1/2$, $p_0 = 3/2$ and Ψ comes from [33, Corollary 3]. Observe that if Q is a dyadic cube with barycenter x_Q and sidelength $r > 0$, then, denoting its interior by Q°, we have $Q^\circ = \Psi(B^3_{r/2}) + x_Q$, $2Q^\circ = \Psi(B^3_r) + x_Q$ and the bilipschitz constant of the map $\Psi + x_Q$ is the bilipschitz constant of Ψ. Thus, applying Proposition 3.5.22 with $V = \Omega$, $p = p_n$, $f = |Du_n|$, $C_2 = C$, $C_1 = \frac{2-p_n}{\eta}$ and taking into account that $\operatorname{dist}(\cdot,\partial\Omega)\leq\operatorname{diam}(\Omega)$ everywhere in Ω, for each $q\in[1,2)$ we get

$$\limsup_{n\to+\infty}\int_\Omega (\operatorname{dist}(x,\partial\Omega)|Du_n(x)|)^q\,dx \\ \leq C'|\Omega| + \frac{C'}{2-q}\limsup_{n\to+\infty}(2-p_n)\int_\Omega \frac{|Du_n|^{p_n}}{p_n}\,dx,$$

where $C' = C'(\operatorname{diam}(\Omega), \mathcal{N}) > 0$. This yields (i).

Now we prove (ii). Let $q \in [1, 2)$ and $\mathcal{F} = \{Q_i\}_{i\in I}$ be a Whitney covering of $\Omega\backslash S_*$ by dyadic cubes such that $\Omega\backslash S_* = \bigcup_{i\in I} Q_i$ and $2Q_i \subset \Omega\backslash S_*$ for each $i \in I$ (see, for example, [7, Theorem 2.3.1]). Notice that I is countable. Using the fact that $|S_*| = 0$ (see Proposition 3.5.8), the weak convergence (see Proposition 3.4.2 (i)), together with the convexity, and also using Fatou's lemma, up to a subsequence (not relabeled), one has

$$\begin{aligned}\int_\Omega (\operatorname{dist}(x, \partial\Omega)|Du_*(x)|)^q\, dx &= \sum_{i\in I} \int_{Q_i} (\operatorname{dist}(x, \partial\Omega)|Du_*(x)|)^q\, dx \\ &\le \sum_{i\in I} \liminf_{n\to+\infty} \int_{Q_i} (\operatorname{dist}(x, \partial\Omega)|Du_n(x)|)^q\, dx \\ &\le \liminf_{n\to+\infty} \sum_{i\in I} \int_{Q_i} (\operatorname{dist}(x, \partial\Omega)|Du_n(x)|)^q\, dx \\ &= \liminf_{n\to+\infty} \int_\Omega (\operatorname{dist}(x, \partial\Omega)|Du_n(x)|)^q\, dx.\end{aligned}$$

This, in view of (i) and (3.5.1), implies (3.5.56) and the fact that $u_* \in W^{1,q}_{\text{loc}}(\Omega, \mathcal{N})$ for all $q \in [1, 2)$. Next, let $q_0 \in (q, 2)$ and $U \Subset \Omega$ be an open set. Fix a sequence $(K_n)_{n\in\mathbb{N}}$ of compact sets such that $K_n \Subset U\backslash S_*$, $K_n \subset K_{n+1}$ and $\bigcup_{n\in\mathbb{N}} K_n = U \setminus S_*$. Applying Hölder's inequality and using the fact that $|Du_n - Du_*|^{q_0} \le 2^{q_0-1}(|Du_n|^{q_0} + |Du_*|^{q_0})$ (by convexity), for each $n \in \mathbb{N}$ large enough we obtain

$$\begin{aligned}&\int_U |Du_n - Du_*|^q\, dx \\ &\qquad \le |U \setminus K_n|^{1-\frac{q}{q_0}} 2^{q-\frac{q}{q_0}} \left(\int_{U\setminus K_n} (|Du_n|^{q_0} + |Du_*|^{q_0})\, dx \right)^{\frac{q}{q_0}} \\ &\qquad\qquad + |K_n|^{1-\frac{q}{p_n}} \left(\int_{K_n} |Du_n - Du_*|^{p_n}\, dx \right)^{\frac{q}{p_n}}.\end{aligned}$$

By carefully choosing K_n and letting n tend to $+\infty$, in view of Proposition 3.4.2 (ii), up to a subsequence (not relabeled), we can ensure that the last two terms tend to 0 as $n \to +\infty$, and hence $u_n \to u_*$ in $W^{1,q}(U, \mathbb{R}^\nu)$ as $n \to +\infty$. Using a diagonal argument, we deduce (ii), which completes our proof of Proposition 3.5.23. $\square$

3.6 Interior minimality of the singular set

In this section, we prove some properties of local minimality inside Ω of the positive Radon measure $\mu_* \in (C(\overline{\Omega}))'$ and its support S_*, defined at the beginning of Section 5.

First, we define the notion of an *admissible chain* for a bounded Lipschitz domain $U \subset \mathbb{R}^3$ and a boundary datum $g \in W^{1/2,2}(\partial U, \mathcal{N})$.

Definition 3.6.1. Given a bounded Lipschitz domain $U \subset \mathbb{R}^3$ and a map $g \in W^{1/2,2}(\partial U, \mathcal{N})$, we say that a closed set $S \subset \overline{U}$ is an *admissible chain* for U and g if the following properties are satisfied.

P.1 S is the union of a finite number of closed straight line segments, each of which has a positive length, and the interiors of these segments (*i.e.*, the segments without their endpoints) are pairwise disjoint.

P.2 $S \cap \partial U$ is a finite set of points.

P.3 S has no endpoints in U, namely, if $L \subset S$ is a segment having an endpoint $x \in U$, then there exists another segment $L' \subset S$ emanating from x.

P.4 There exists a map $u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N})$ such that $\mathrm{tr}_{\partial U}(u) = g$.

The set of all admissible chains for U and g will be denoted by $\mathscr{C}(U, g)$.

Remark 3.6.2. Assume that there exists $u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N})$ satisfying (4). Then there exists $v \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N}) \cap C(U \setminus (S \cup S_v), \mathcal{N})$ such that $\mathrm{tr}_{\partial U}(v) = g$, where S_v is at most a countable and locally finite subset of $U \setminus S$ (see [53, Theorem II]). Let L be a segment in the chain S and x_1, x_2 be interior points of L such that $x_1 \neq x_2$. Since $v \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N})$, using the coarea formula, we can find a sufficiently small radius $r > 0$ such that the following holds. There exists $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ such that for $\mathcal{H}^1$-a.e. $x \in [x_1, x_2]$, letting ∂D to be the circle with center x and radius r lying in the 2-plane orthogonal to L, we have $\mathrm{tr}_{\partial D}(v) = v|_{\partial D} \in W^{1,2}(\partial D, \mathcal{N})$, $v|_{\partial D}$ is homotopic to γ and $\mathcal{E}_2^{\mathrm{sg}}(v|_{\partial D}) = \mathcal{E}_2^{\mathrm{sg}}(\gamma)$. On the other hand, since $v \in C(U \setminus (S \cup S_v), \mathcal{N})$ and S_v is locally finite, we can choose x_1 and x_2 to be arbitrarily close to the endpoints of L and repeat the previous procedure for a smaller radius $\varrho \in (0, r)$, but without changing $\gamma \in [\mathbb{S}^1, \mathcal{N}]$. Thus, we can define the *mass* of S with respect to the extension v of g as the sum $\sum_{i=1}^{l} \lambda_i \mathcal{H}^1(L_i)$, where $\bigcup_{i=1}^{l} L_i = S$ is the union of the segments lying in S and $\lambda_i = \mathcal{E}_2^{\mathrm{sg}}(\gamma_i)$, where $\gamma_i \in [\mathbb{S}^1, \mathcal{N}]$ is defined for each L_i with respect to v. Denote this mass by $\mathbb{M}(v, g, S)$. It is worth noting that for different extensions $u_1 \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N}) \cap C(U \setminus (S \cup S_{u_1}), \mathcal{N})$ and $u_2 \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N}) \cap C(U \setminus (S \cup S_{u_2}), \mathcal{N})$ of g satisfying the property (4) of Definition 3.6.1, we can obtain different masses of S with respect to u_1 and u_2, however, the following

$$\inf\{\mathbb{M}(u, g, S) : u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N}) \cap C(U \setminus (S \cup S_u), \mathcal{N}),\ \mathrm{tr}_{\partial U}(u) = g\} \tag{3.6.1}$$

is achieved. Indeed, if $u_n \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N})$ is a minimizing sequence for (3.6.1), then, since $\#[\mathbb{S}^1, \mathcal{N}] < +\infty$, the set $\{\mathcal{E}^{\mathrm{sg}}_2(\gamma) : \gamma \in [\mathbb{S}^1, \mathcal{N}]\}$ is finite, and hence $\{\mathbb{M}(u_n, g, S) : n \in \mathbb{N}\}$ is finite. Thus, for each sufficiently large $n \in \mathbb{N}$, $\mathbb{M}(u_n, g, S)$ is equal to the infimum in (3.6.1), and each u_n, for sufficiently large $n \in \mathbb{N}$, realizes the infimum in (3.6.1).

Next, using Remark 3.6.2, we define the notion of the *mass* of an admissible chain.

Definition 3.6.3. Given a bounded Lipschitz domain $U \subset \mathbb{R}^3$ and a map $g \in W^{1/2,2}(\partial U, \mathcal{N})$, if $S \in \mathscr{C}(U, g)$, we define its mass by

$$\mathbb{M}(g, S) = \mathbb{M}(u, g, S),$$

where $u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N}) \cap C(U \setminus (S \cup S_u), \mathcal{N})$ realizes the infimum in (3.6.1) for g and S.

When appropriate, we shall simply write $\mathbb{M}(S)$ instead of $\mathbb{M}(g, S)$. Notice that the definition of $\mathbb{M}(g, S)$ also depends on $\mathcal{N}$, but we decided not to mention it explicitly to simplify the notation.

Now we prove the interior minimality of S_*.

Proposition 3.6.4. *Let $u_* \in W^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$ and $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$ be given by Proposition 3.5.6. Then for any domain $K \Subset \Omega$ such that $\overline{K} \cap S_* \neq \emptyset$, there exists a Lipschitz domain $U \Subset \Omega$ containing K such that $\mathrm{tr}_{\partial U}(u_*) = u_*|_{\partial U} \in W^{1/2,2}(\partial U, \mathcal{N})$, $S_* \cap \overline{U} \in \mathscr{C}(U, u_*|_{\partial U})$ and for each $S \in \mathscr{C}(U, u_*|_{\partial U})$,*

$$\mu_*(S_* \cap U) = \mathbb{M}(u_*|_{\partial U}, S_* \cap \overline{U}) \leq \mathbb{M}(u_*|_{\partial U}, S). \tag{3.6.2}$$

Remark 3.6.5. Proposition 3.6.4 allows the choice of the domain U to be arbitrarily close to K.

Proof of Proposition 3.6.4. According to Proposition 3.5.17, since $\overline{K}$ is connected, there exists a Lipschitz domain $V \Subset \Omega$ containing $\overline{K}$ such that $S_* \cap \overline{V}$ is a union of a finite number of closed straight line segments, each of which has a positive length, and the interiors of these segments are pairwise disjoint. Also $S_* \cap \partial V$ is a finite set of points such that for each point $x \in S_* \cap \partial V$ there is exactly one segment lying in $S_* \cap \overline{V}$ and emanating from x. Changing, if necessary, ∂V locally around each point of $S_* \cap \partial V$, we can assume that for each $x \in S_* \cap \partial V$, ∂V coincides with its tangent plane at x in a small neighborhood of x, and the segment lying in $S_* \cap \overline{V}$ and emanating from x is orthogonal to this tangent plane.

Step 1. We define U. Fix $\lambda > 0$ small enough such that for each $t \in (0, \lambda]$, all the above properties prescribed for V concerning S_* and K hold also for $V_t = \{y \in V : \operatorname{dist}(y, \partial V) > t\}$, which is a Lipschitz domain. Applying [31, Theorem 3.2.22 (3)] with $W = V \setminus \overline{V}_\lambda$, $Z = (0, \lambda)$ and $f : W \to Z$ defined by $f(y) = \operatorname{dist}(y, \partial V)$, and using also the fact that $|Df(y)| = 1$ for a.e. $y \in W$, Fatou's lemma, (3.5.1) and Proposition 3.5.23 (ii), we obtain

$$\begin{aligned} (3.6.3) \quad \int_0^\lambda d\varrho \liminf_{n\to+\infty} \int_{f^{-1}(\{\varrho\})} & (|u_n - u_*|^q + |Du_n - Du_*|^q)\, d\mathcal{H}^2 \\ & \le \liminf_{n\to+\infty} \int_W (|u_n - u_*|^q + |Du_n - Du_*|^q)\, dy = 0 \end{aligned}$$

for each $q \in (1, 2)$ and

$$\begin{aligned} \int_0^\lambda d\varrho \liminf_{n\to+\infty} (2 - p_n) \int_{f^{-1}(\{\varrho\})} & (|Du_n|^{p_n} + |Du_*|^{p_n})\, d\mathcal{H}^2 \\ & \le \liminf_{n\to+\infty} (2 - p_n) \int_W (|Du_n|^{p_n} + |Du_*|^{p_n})\, dy \le C', \end{aligned}$$

where $C' = C'(\operatorname{dist}(V, \Omega), \operatorname{diam}(\Omega), C_0, \mathcal{N}) > 0$ and $C_0 > 0$ is the constant coming from (3.5.1). On the other hand, using Proposition 3.5.6 (thanks to which we know that, up to a subsequence (not relabeled), $u_n \to u_*$ in $L^p(\Omega, \mathbb{R}^\nu)$ for all $p \in [1, +\infty)$ (see Proposition 3.4.2 (i))) and the continuity of the L^p-norm, we deduce that $\|u_n - u_*\|^{p_n}_{L^{p_n}(W, \mathbb{R}^\nu)} \to 0$ as $n \to +\infty$ and hence

$$\lim_{n\to+\infty} \int_0^\lambda d\varrho \int_{f^{-1}(\{\varrho\})} |u_n - u_*|^{p_n}\, d\mathcal{H}^2 = 0.$$

Then there exists $t \in (0, \lambda)$ such that for $U = V_t$, up to a subsequence (not relabeled), for each $n \in \mathbb{N}$ large enough: $\operatorname{tr}_{\partial U}(u_n) = u_n|_{\partial U}, \operatorname{tr}_{\partial U}(u_*) = u_*|_{\partial U} \in W^{1,p_n}(\partial U, \mathcal{N}) \subset W^{1/2,2}(\partial U, \mathcal{N})$ and

$$(3.6.4) \qquad \int_{\partial U} (|Du_n|^{p_n} + |Du_*|^{p_n})\, d\mathcal{H}^2 \le \frac{C''}{2 - p_n},$$

where $C'' = C''(t, \operatorname{dist}(V, \Omega), \operatorname{diam}(\Omega), C_0, \mathcal{N}) > 0$;

$$(3.6.5) \qquad \|u_n|_{\partial U} - u_*|_{\partial U}\|^{p_n}_{L^{p_n}(\partial U, \mathbb{R}^\nu)} \to 0$$

as $n \to +\infty$;

$$(3.6.6) \qquad \|u_n|_{\partial U} - u_*|_{\partial U}\|_{L^\infty(\partial U; \mathbb{R}^\nu)} \to 0$$

as $n \to +\infty$, in view of (3.6.3) and Morrey's embedding for Sobolev mappings. Since t and V uniquely determine U, we can assume that

C'' actually depends on $U, \operatorname{diam}(\Omega)$, C_0 and $\mathcal{N}$. Thus, we have defined U. According to our construction and Proposition 3.5.6, the properties (1)-(4) of Definition 3.6.1 are fulfilled for U, $u_*|_{\partial U}$ and $S_* \cap \overline{U}$. Therefore, $S_* \cap \overline{U} \in \mathscr{C}(U, u_*|_{\partial U})$.

Step 2. Given $S \in \mathscr{C}(U, u_*|_{\partial U})$, we construct a sequence of mappings $w_n \in W^{1,p_n}(U, \mathcal{N})$ such that $\operatorname{tr}_{\partial U}(w_n) = u_*|_{\partial U}$ and

$$\limsup_{n\to+\infty}(2-p_n)\int_U \frac{|Dw_n|^{p_n}}{p_n}\,dy \leq \mathbb{M}(S). \tag{3.6.7}$$

Consider a map $u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N})$ minimizing the expression in (3.6.1), where $g = u_*|_{\partial U}$. Let $\varepsilon \in (0,1)$ and $\delta \in (0,1)$ be small enough and δ be fairly small with respect to ε. Then, using the coarea formula and taking into account that $u \in W^{1,2}_{\mathrm{loc}}(U \setminus S, \mathcal{N})$, we obtain $\delta' \in (3\delta/4, 5\delta/4)$ and $\varepsilon' \in (3\varepsilon/4, 5\varepsilon/4)$ such that, defining $S_{\delta'} = \{y \in U : \operatorname{dist}(y, S) \leq \delta'\}$, we have the following. Firstly, $\operatorname{tr}_{\partial S_{\delta'} \cap U}(u) = u|_{\partial S_{\delta'} \cap U} \in W^{1,2}(\partial S_{\delta'} \cap U, \mathcal{N})$. Secondly, for each branching point $x \in S \cap \overline{U}$ of S (if such a point exists), $\operatorname{tr}_{(\partial B^3_{\varepsilon'}(x)\setminus S_{\delta'})\cap U}(u) = u|_{(\partial B^3_{\varepsilon'}(x)\setminus S_{\delta'})\cap U} \in W^{1,2}((\partial B^3_{\varepsilon'}(x) \setminus S_{\delta'}) \cap U, \mathcal{N})$, where $(\partial B^3_{\varepsilon'}(x) \setminus S_{\delta'}) \cap U = \partial B^3_{\varepsilon'}(x) \setminus S_{\delta'}$ if $x \in U$ (we recall that δ is fairly small with respect to ε and the structure of S is given by Definition 3.6.1). To lighten the notation, we assume that $\delta' = \delta$ and $\varepsilon' = \varepsilon$. Let L be a segment of S. Up to rotation and translation, we can assume that $L = \{(0,0)\} \times [-h, h]$, where $h = \mathcal{H}^1(L)/2 > 0$. Let $\Lambda := B^2_\delta \times (z(0,0,-h), z(0,0,h))$, where $z(0,0,-h) := -h + \sqrt{\varepsilon^2 - \delta^2}$ if $(0,0,-h)$ is a branching point of S (*i.e.*, several segments of S emanate from this point), $z(0,0,h) := h - \sqrt{\varepsilon^2 - \delta^2}$ if $(0,0,h)$ is a branching point of S, $z(0,0,-h) := -h + \kappa\delta$ if $(0,0,-h) \in \partial U$ is an endpoint of S (*i.e.*, only one segment of S emanates from this point, namely L), $z(0,0,h) = h - \kappa\delta$ if $(0,0,h) \in \partial U$ is an endpoint of S, where $\kappa = \kappa(S) > 0$ will be defined later for the proof to work. Hereinafter in this proof, C will denote a positive constant that is independent of n, ε and δ and can be different from line to line. We define w_n in Λ by using Lemma 3.2.20 (i), namely we obtain a map $w_n|_\Lambda \in W^{1,p_n}(\Lambda, \mathcal{N})$ such that

$$\int_\Lambda \frac{|Dw_n|^{p_n}}{p_n}\,dy \leq \frac{\mathcal{H}^1(L)\mathcal{E}^{\mathrm{sg}}_{p_n}(\gamma)\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}}\int_\Gamma \frac{|D_\top u|^{p_n}}{p_n}\,d\mathcal{H}^2, \tag{3.6.8}$$

where Γ is the lateral surface of Λ and $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ is homotopic to $u|_{\partial D} \in C(\partial D, \mathcal{N})$, where ∂D is the circle being the boundary of a 2-disk orthogonal to L with fairly small radius and whose center belongs to the interior of L. We have used that $\Gamma \subset (\partial S_\delta \cap U)$ and $u|_{\partial S_\delta \cap U} \in W^{1,2}(\partial S_\delta \cap U, \mathcal{N})$.

We need to distinguish between two further cases.

Case 1. Let $x \in \partial U$ be an endpoint of S (if it exists). Then, taking into account the properties (1) and (2) of Definition 3.6.1 and the fact that for each $x \in S \cap \partial U$, ∂U coincides with its tangent plane at x in a small neighborhood of x, we can find $\kappa = \kappa(S) > 0$ such that for each endpoint $y \in \partial U$ of S, if L is a segment of S emanating from y, then the 2-disk orthogonal to L with center $\xi \in L$ such that $|\xi - y| = \kappa\delta$ lies entirely in U. Now let L denote the segment of S emanating from x. Up to rotation and translation, $L = \{(0,0)\} \times [-h, h]$ and $x = (0,0,h)$, where $h := \mathcal{H}^1(L)/2 > 0$. Let $\Lambda_\delta := \{y = (y_1, y_2, t) \in U : \operatorname{dist}(y, L) < \delta$ and $t \in (h - \kappa\delta, h)\}$. Then $\overline{\Lambda}_\delta$ is bilipschitz homeomorphic to $\overline{B}^3_\delta$ with a bilipschitz constant depending on κ. Thus, there exists a bilipschitz map $\Psi : \overline{\Lambda}_\delta \to \overline{B}^3_\delta$. We have constructed w_n on $\Lambda = B^2_\delta \times (z(0,0,-h), h - \kappa\delta)$. By Lemma 3.2.20 (ii), $\operatorname{tr}_{B^2_\delta \times \{h-\kappa\delta\}}(w_n) \in W^{1,p_n}(B^2_\delta \times \{h - \kappa\delta\}, \mathcal{N})$ and

$$
\begin{aligned}
(3.6.9)\quad \int_{B^2_\delta \times \{h-\kappa\delta\}} & \frac{|D \operatorname{tr}_{B^2_\delta \times \{h-\kappa\delta\}}(w_n)|^{p_n}}{p_n}\, d\mathcal{H}^2 \\
&\le \frac{\mathcal{E}^{\mathrm{sg}}_{p_n}(\gamma)\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}} \int_\Gamma \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2 \\
&\le \frac{C\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}} \int_\Gamma \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2,
\end{aligned}
$$

where $\gamma \in [\mathbb{S}^1, \mathcal{N}]$ comes from (3.6.8), Γ is the lateral surface of Λ, and we have used that $\#[\mathbb{S}^1, \mathcal{N}] < +\infty$. Now we define the mapping $g_n : \partial\Lambda_\delta \to \mathcal{N}$ by

$$
g_n(y) = \begin{cases} u_*|_{\partial U}(y) & \text{if } y \in \partial\Lambda_\delta \cap \partial U, \\ u|_{\partial S_\delta \cap U}(y) & \text{if } y \in \partial S_\delta \cap U, \\ \operatorname{tr}_{B^2_\delta \times \{h-\kappa\delta\}}(w_n) & \text{if } y \in B^2_\delta \times \{h - \kappa\delta\}. \end{cases}
$$

Then, since the traces of $u_*|_{\partial U}$ and $u|_{\partial S_\delta \cap U}$ coincide on $\partial S_\delta \cap \partial U$, and the traces of $u|_{\partial S_\delta \cap U}$ and $\operatorname{tr}_{B^2_\delta \times \{h-\kappa\delta\}}(w_n)$ coincide on $\partial B^2_\delta \times \{h - \kappa\delta\}$, we deduce that $g_n \in W^{1,p_n}(\partial\Lambda_\delta, \mathcal{N})$. Using (3.6.4), (3.6.9) and the fact that $\delta \in (0,1)$ is fairly small, we obtain

$$
\begin{aligned}
(3.6.10)\quad \int_{\partial\Lambda_\delta} & \frac{|D_\top g_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \\
&\le \frac{C}{2-p_n} + \frac{C\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}} \int_{\partial S_\delta \cap U} \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2.
\end{aligned}
$$

Next, we define $w_n|_{\Lambda_\delta}(y) = g_n(\Psi^{-1}(\delta\Psi(y)/|\Psi(y)|))$ for $y \in \Lambda_\delta \setminus \{\Psi^{-1}(0)\}$. Then, using (3.6.10), we deduce the following

$$\int_{\Lambda_\delta} \frac{|Dw_n|^{p_n}}{p_n}\, dy \le C\delta \int_{\partial\Lambda_\delta} \frac{|D_\top g_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \tag{3.6.11}$$

$$\le \frac{C\delta}{2-p_n} + \frac{C\delta^{3-p_n}}{2-p_n} + C\delta^{2-p_n} \int_{\partial S_\delta \cap U} \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2.$$

Case 2. Let x be a branching point of S (if it exists). Then, according to our construction, $u|_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U} \in W^{1,2}((\partial B^3_\varepsilon(x) \setminus S_\delta) \cap U, \mathcal{N})$. Furthermore, $(\partial B^3_\varepsilon(x) \setminus S_\delta) \cap U = \partial B^3_\varepsilon(x) \setminus S_\delta$ if $x \in U$. Let L_i, $i \in \{1, \dots, s\}$ be the segments of S emanating from x. We fill the holes in $(\partial B^3_\varepsilon(x) \setminus S_\delta) \cap U$ with the closures of the open 2-disks D_i such that D_i is orthogonal to L_i, has radius δ, and its center lies on L_i at a distance of $\sqrt{\varepsilon^2 - \delta^2}$ from the point x. Thus, $\mathcal{M} = (\partial U \cap \overline{B}^3_\varepsilon(x)) \cup ((\partial B^3_\varepsilon(x) \setminus S_\delta) \cap U) \cup \bigcup_{i=1}^{l} \overline{D}_i$ is a compact 2-manifold without boundary, which is bilipschitz homeomorphic to ∂B^3_ε. Furthermore, according to our construction, for each $i \in \{1, \dots, l\}$, $\mathrm{tr}_{D_i}(w_n) \in W^{1,p_n}(D_i, \mathcal{N})$. Since the traces of $u_*|_{\partial U}$ and $u|_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U}$ coincide on $\partial U \cap \partial B^3_\varepsilon(x)$, and the traces of $u|_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U}$ and $\mathrm{tr}_{D_i}(w_n)$ coincide on ∂D_i, defining $v_n : \mathcal{M} \to \mathcal{N}$ by

$$v_n(y) = \begin{cases} u_*|_{\partial U}(y) & \text{if } y \in \partial U \cap \overline{B}^3_\varepsilon(x), \\ u|_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U}(y) & \text{if } y \in (\partial B^3_\varepsilon(x) \setminus S_\delta) \cap U, \\ \mathrm{tr}_{\overline{D}_i}(w_n)(y) & \text{if } y \in \overline{D}_i, \end{cases}$$

we observe that $v_n \in W^{1,p_n}(\mathcal{M}, \mathcal{N})$. Next, let E be an open bounded set whose boundary is $\mathcal{M}$. Let $\Phi : \overline{E} \to \overline{B}^3_\varepsilon$ be a bilipschitz homeomorphism. We define $w_n|_E(y) = v_n(\Phi^{-1}(\varepsilon\Phi(y)/|\Phi(y)|))$ for $y \in E \setminus \{\Phi^{-1}(0)\}$. Then, using (3.6.4), (3.6.9) and Definition 3.6.1, we obtain

$$\begin{aligned}
&\int_E \frac{|Dw_n|^{p_n}}{p_n}\, dy \le C\varepsilon \int_{\mathcal{M}} \frac{|D_\top v_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \qquad (3.6.12)\\
&\le C\varepsilon \int_{\partial U \cap B^3_\varepsilon(x)} \frac{|D_\top u_*|^{p_n}}{p_n}\, d\mathcal{H}^2 + C\varepsilon \int_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U} \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2 \\
&\quad + C\varepsilon \sum_{i=1}^{l} \int_{D_i} \frac{|D_\top \mathrm{tr}_{D_i}(w_n)|^{p_n}}{p_n}\, d\mathcal{H}^2 \\
&\le \frac{C\varepsilon}{2-p_n} + C\varepsilon \int_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U} \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2 + \frac{C\varepsilon\delta^{2-p_n}}{2-p_n} \\
&\quad + \frac{C\varepsilon}{\delta^{p_n-1}} \int_{\partial S_\delta \cap U} \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2.
\end{aligned}$$

Finally, we define w_n for each y in U lying outside the set $E_{\varepsilon,\delta}$, which is the union of S_δ with the union of $\overline{B}^3_\varepsilon(x) \cap U$ over all branching points x of S, by $w_n(y) = u(y)$. Since the traces of the piecewise definitions of w_n coincide, $w_n \in W^{1,p_n}(U, \mathcal{N})$. Next, gathering together (3.6.8), (3.6.11), (3.6.12) and using the facts that $\varepsilon, \delta \in (0,1)$ are sufficiently small, δ is fairly small with respect to ε, $(p_n)_{n\in\mathbb{N}} \subset [1,2)$ and $p_n \nearrow 2$ as $n \to +\infty$, S consists of a finite union of segments (and hence it has a finite number of branching points and endpoints), for all sufficiently large $n \in \mathbb{N}$, we deduce the following

$$\begin{aligned}\int_U \frac{|Dw_n|^{p_n}}{p_n}\, dy &\\ \leq \frac{\mathbb{M}(S)\delta^{2-p_n}}{2-p_n} &+ \frac{C\varepsilon}{2-p_n} + C\varepsilon \sum_{x\in P} \int_{(\partial B^3_\varepsilon(x)\setminus S_\delta)\cap U} \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2 \\ &+ \frac{C\varepsilon\delta^{2-p_n}}{2-p_n} + \frac{C}{\delta^{p_n-1}} \int_{\partial S_\delta \cap U} \frac{|D_\top u|^{p_n}}{p_n}\, d\mathcal{H}^2 + \int_{U\setminus E_{\varepsilon,\delta}} \frac{|Du|^{p_n}}{p_n}\, dy,\end{aligned}$$

where P is the set of branching points of S. Multiplying both sides of the above inequality by $(2-p_n)$, then letting n tend to $+\infty$ and using the continuity of the L^p-norm and the facts that $u \in W^{1,2}(\partial S_\delta \cap U, \mathcal{N})$, $u \in W^{1,2}((\partial B^3_\varepsilon(x) \setminus S_\delta) \cap U, \mathcal{N})$ for each branching point x of S, $u \in W^{1,2}(U \setminus E_{\varepsilon,\delta}, \mathcal{N})$, we deduce that

$$\limsup_{n\to+\infty} (2-p_n) \int_U \frac{|Dw_n|^{p_n}}{p_n}\, dy \leq \mathbb{M}(S) + C\varepsilon.$$

Since $\varepsilon > 0$ was chosen arbitrarily small and C is independent of ε, the last estimate implies (3.6.7).

Step 3. We prove (3.6.2). Let the sequence of maps $w_n \in W^{1,p_n}(U, \mathcal{N})$ be as in *Step 2.* Using Lemma 3.4.1, we construct a competitor for the minimizer u_n by interpolating between u_n and w_n. In view of (3.6.5) and the fact that $\mathrm{tr}_{\partial U}(w_n) = u_*|_{\partial U}$,

$$\|u_n|_{\partial U} - \mathrm{tr}_{\partial U}(w_n)\|^{p_n}_{L^{p_n}(\partial U, \mathbb{R}^\nu)} \to 0 \tag{3.6.13}$$

as $n \to +\infty$. Let $T_0 = T_0(\partial U) > 0$ be the constant of Lemma 3.4.1. Choose $T' \in (0, T_0)$ sufficiently small and $\Phi : \partial U \times [0, T'] \to \overline{E}_{r_0}$ such that Φ is bilipschitz onto its image, $\Phi(\sigma, 0) = \sigma$, $D_\top\Phi(\sigma, 0)$ is an isometry for $\sigma \in \partial U$, where $E_{r_0} = \{x \in U : \mathrm{dist}(x, \partial U) < r_0\}$ for some fairly small constant $r_0 > 0$ depending on T'. According to Lemma 3.2.1, there exists a constant $\lambda_\mathcal{N} > 0$ such that the nearest point retraction $\Pi_\mathcal{N} : \mathcal{N}_{\lambda_\mathcal{N}} \to \mathcal{N}$

is well defined and smooth, where $\mathcal{N}_{\lambda_{\mathcal{N}}} = \{x \in \mathbb{R}^\nu : \operatorname{dist}(x, \mathcal{N}) < \lambda_{\mathcal{N}}\}$. Setting $s_n = \|u_n|_{\partial U} - \operatorname{tr}_{\partial U}(w_n)\|_{L^{p_n}(\partial U, \mathbb{R}^\nu)}$ and using (3.6.4) and (3.6.13), we observe that $s_n \to 0$ and for each sufficiently large $n \in \mathbb{N}$,

$$\int_{\partial U} \left(|D_\top u_n|^{p_n} + |D_\top \operatorname{tr}_{\partial U}(w_n)|^{p_n} + \frac{|u_n - u_*|^{p_n}}{s_n^{p_n/2}} \right) d\mathcal{H}^2 \leq \frac{C}{2 - p_n}. \tag{3.6.14}$$

Then, by Lemma 3.4.1, applied with $u = u_n|_{\partial U}$, $v = \operatorname{tr}_{\partial U}(w_n) = u_*|_{\partial U}$ and $T = s_n^{1/2}$, there exists a mapping $\xi_n \in W^{1,p_n}(\partial U \times (0, s_n^{1/2}), \mathbb{R}^\nu)$ interpolating between $u_n|_{\partial U}$ and $\operatorname{tr}_{\partial U}(w_n)$. By (3.6.6), we deduce that

$$\operatorname{esssup}_{(\sigma,t) \in \partial U \times (0, s_n^{1/2})} \operatorname{dist}(\xi_n(\sigma, t), \mathcal{N}) \leq \|u_n|_{\partial U} - u_*|_{\partial U}\|_{L^\infty(\partial U, \mathbb{R}^\nu)} \to 0$$

as $n \to +\infty$. This implies that for each $n \in \mathbb{N}$ large enough, it holds

$$\operatorname{dist}(\xi_n(\sigma, t), \mathcal{N}) < \frac{\lambda_{\mathcal{N}}}{10} \tag{3.6.15}$$

for a.e. $(\sigma, t) \in \partial U \times (0, s_n^{1/2})$.

In view of (3.6.15) and Lemma 3.2.1, for each $n \in \mathbb{N}$ large enough we can define the mapping $\varphi_n(y) := \Pi_{\mathcal{N}}(\xi_n(\Phi^{-1}(y)))$ for each $y \in \Phi(\partial U \times (0, s_n^{1/2}))$. Then the following assertions hold. First, $\varphi_n \in W^{1,p_n}(\Phi(\partial U \times (0, s_n^{1/2})), \mathcal{N})$. Second,

$$\operatorname{tr}_{\partial U}(\varphi_n) = u_n|_{\partial U}, \;\; \operatorname{tr}_{\Phi(\partial U \times \{s_n^{1/2}\})}(\varphi_n) = u_*|_{\partial U} \circ \Phi_n \circ \Phi^{-1}|_{\Phi(\partial U \times \{s_n^{1/2}\})},$$

where $\Phi_n : \partial U \times [s_n^{1/2}, T']$ is defined by $\Phi_n(\sigma, t) := \Phi(\sigma, t - s_n^{1/2})$ and we have used the fact that $\Phi(\sigma, 0) = \sigma$. Thirdly, using [31, Theorem 3.2.5], the fact that Ψ is bilipschitz, Lemma 3.4.1 (ii), where $T = s_n^{1/2}$, and (3.6.14), we have the following

$$\begin{aligned}
&(2 - p_n) \int_{\Phi(\partial U \times (0, s_n^{1/2}))} \frac{|D\varphi_n|^{p_n}}{p_n} \, dy \\
&\leq C(2 - p_n) \int_{\partial U \times (0, s_n^{1/2})} \frac{|D\xi_n|^{p_n}}{p_n} \, d\mathcal{H}^2 \, dt \\
&\leq C(2 - p_n) s_n^{1/2} \times \\
&\quad \int_{\partial U} \left(|D_\top u_n|^{p_n} + |D_\top \operatorname{tr}_{\partial U}(w_n)|^{p_n} + \frac{|u_n - u_*|^{p_n}}{s_n^{p_n/2}} \right) d\mathcal{H}^2 \\
&\leq C s_n^{1/2}.
\end{aligned} \tag{3.6.16}$$

Let $\zeta \in C_c([0,+\infty))$ be the Lipschitz map such that $\zeta = 1$ on $[0,1]$, $\zeta(s) = 2 - s$ for $s \in [1,2]$, $\zeta = 0$ on $[2,+\infty]$. Hence $\|\zeta'\|_{L^\infty([0,+\infty))} \le 1$. Next, for each sufficiently large $n \in \mathbb{N}$, define $\psi_n : U \to \mathcal{N}$ by

(3.6.17)
$$\psi_n(y) := \begin{cases} \varphi_n(y) & \text{if } y \in \Phi(\partial U \times (0, s_n^{1/2})), \\ u_*|_{\partial U} \circ \Phi_n \circ \Phi^{-1}|_{\Phi(\partial U \times \{s_n^{1/2}\})}(y) & \text{if } y \in \Phi(\partial U \times \{s_n^{1/2}\}), \\ w_n(\Psi_n(y)) & \text{if } y \in U \setminus \Phi(\partial U \times (0, s_n^{1/2}]), \end{cases}$$

where $\Psi_n : U \setminus \Phi(\partial U \times (0, s_n^{1/2}]) \to U$ is the map defined by

$$\Psi_n(y) := \left(1 - \zeta\left(\frac{\operatorname{dist}(y, \partial U)}{s_n^{1/3}}\right)\right) y + \zeta\left(\frac{\operatorname{dist}(y, \partial U)}{s_n^{1/3}}\right) \Phi_n(\Phi^{-1}(y)).$$

We note that $\psi_n \in W^{1,p_n}(U, \mathcal{N})$, $\operatorname{tr}_{\partial U}(\psi_n) = u_n|_{\partial U}$, Ψ_n is bilipschitz, $|D\Psi_n - \operatorname{Id}| = O(s_n^{1/6})$ a.e. in $U \setminus \Phi(\partial U \times (0, s_n^{1/2}])$ (for more details, see the proof of Proposition 3.4.2). Thus, ψ_n is a competitor for u_n. Then, using (3.6.16), (3.6.17), [31, Theorem 3.2.5] and (3.6.7), we obtain

$$\begin{aligned} &\liminf_{n\to+\infty} (2 - p_n) \int_U \frac{|Du_n|^{p_n}}{p_n}\, dy \\ &\le \liminf_{n\to+\infty} (2 - p_n) \int_U \frac{|D\psi_n|^{p_n}}{p_n}\, dy \\ &\le \limsup_{n\to+\infty} \left(C s_n^{1/2} + (2 - p_n)(1 + o(s_n^{1/10})) \int_U \frac{|Dw_n|^{p_n}}{p_n}\, dy \right) \\ &\le \mathbb{M}(S). \end{aligned} \tag{3.6.18}$$

On the other hand, using (3.5.3), the fact that U is open, Corollary 3.5.18 and the geometry of $S_* \cap \overline{U}$ (see the beginning of the proof and *Step 1*), we deduce that

$$\mu_*(S_* \cap U) = \mathbb{M}(S_* \cap \overline{U}) \le \liminf_{n\to+\infty} (2 - p_n) \int_U \frac{|Du_n|^{p_n}}{p_n}\, dy. \tag{3.6.19}$$

Combining (3.6.18) with (3.6.19), yields (3.6.2). This completes our proof of Proposition 3.6.4. □

We shall apply Proposition 3.6.4 to prove that in the case when $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$ the set S_* has no branching points inside Ω. To this end, we need the next extension result. The following statement is similar to [23, Lemma 3].

Proposition 3.6.6. *Let $\Omega \subset \mathbb{R}^3$ be a bounded strictly convex domain, $\pi_1(\mathcal{N})$ be Abelian (and endowed with the law $+$) and $a_1, \dots, a_k \subset \overline{\Omega}$ be distinct points. Let $g \in C(\partial\Omega \setminus \{a_1, \dots, a_k\}, \mathcal{N})$ and $\alpha : \{a_1, \dots, a_k\}^2 \to \pi_1(\mathcal{N})$ satisfy the following conditions:*

(i) $\alpha(a_i, a_j) = -\alpha(a_j, a_i)$, *so that, in particular,* $\alpha(a_i, a_i) = 0$*;*

(ii) if $a_i \in \partial\Omega$*, then* $\sum_{j=1}^{k} \alpha(a_i, a_j)$ *is the homotopy class of* g *around* a_i*;*

(iii) if $a_i \in \Omega$*, then* $\sum_{j=1}^{k} \alpha(a_i, a_j) = 0$.

Then there exists $u \in C(\Omega \setminus (\Sigma \cup F), \mathcal{N})$*, where* F *is finite and* $\Sigma = \bigcup\{[a_i, a_j] : \alpha(a_i, a_j) \neq 0\}$*, and for every* $i, j \in \{1, \dots, k\}$*,* u *is homotopic to* $\alpha(a_i, a_j)$ *on small circles transversal to* $[a_i, a_j]$*.*

The proof of Proposition 3.6.6 will use the following two-dimensional construction.

Proposition 3.6.7. *Let* $U \subset \mathbb{R}^2$ *be a bounded strictly convex domain,* $\pi_1(\mathcal{N})$ *be Abelian and* $a_1, \dots, a_k \subset U$ *be distinct points. Assume that* $h \in C(\partial U, \mathcal{N})$ *and* $\gamma \colon \{a_1, \dots, a_k\} \to \pi_1(\mathcal{N})$ *satisfy the condition that* $\sum_{i=1}^{k} \gamma(a_i)$ *is the homotopy class of* h *on* ∂U*. Then there exists* $u \in C(\overline{U} \setminus \{a_1, \dots, a_k\}, \mathcal{N})$*, such that* $u|_{\partial U} = h$ *and* u *is homotopic to* $\gamma(a_i)$ *on small circles transversal around* a_i*.*

Proof. We consider a set L of segments in $\mathbb{R}^2$ such that $\{a_1, \dots, a_k\} \subset L$ and $U \setminus L$ is simply connected. For $r > 0$ small enough, the disks $B^2_{2r}(a_1), \dots, B^2_{2r}(a_k)$ are pairwise disjoint. We fix u on $\partial B^2_r(a_i)$ so that $u|_{\partial B^2_r(a_i)}$ has the prescribed homotopy class $\gamma(a_i)$. We extend then this map u to $(U \cap L) \setminus \bigcup_{i=1}^{k} \overline{B}^2_r(a_i)$ by connectedness of $\mathcal{N}$. Since $V = U \setminus (L \cup \bigcup_{i=1}^{k} \overline{B}^2_r(a_i))$ is simply connected and by our assumption on the homotopy classes, u has a trivial homotopy on ∂V. Thus, we can extend u to $\overline{V}$ in such a way that $u = h$ on ∂U. Finally, u is extended homogeneously on the set $B^2_r(a_i) \setminus \{a_i\}$ for each $i \in \{1, \dots, k\}$. □

Proof of Proposition 3.6.6. As a first step, given a plane $P \subset \mathbb{R}^3$ that is transversal to Σ and does not contain F, the map g can be extended to $(\partial\Omega \cup (P \cap \Omega)) \setminus \Sigma$. Indeed, the homotopy class of g on $\partial\Omega$ is equal to the sum of the homotopy classes on small circles around $\{a_1, \dots, a_k\} \cap \partial\Omega$ on one side of P. By the conditions (ii) and (iii), the homotopy class of g on $\partial\Omega$ is equal to the sum of the homotopy classes on small circles around $\Sigma \cap P$. By Proposition 3.6.7, g can be extended to $P \cap \Omega \setminus \Sigma$ with the prescribed homotopy on small circles around points of $\Sigma \cap P$. Reiterating this construction finitely many times, the problem is reduced to the case where either $k = 0$ and one can perform a homogeneous extension with respect to any interior point, or the set Σ is starshaped with respect to some point in $\overline{\Omega}$ and we perform then a homogeneous extension with respect to that point. □

Corollary 3.6.8. Let $\pi_1(\mathcal{N}) = \mathbb{Z}/2\mathbb{Z}$. Then S_* has no branching points inside Ω.

Proof of Corollary 3.6.8. Assume by contradiction that S_* has a branching point $x_0 \in \Omega$. Then, according to Remark 3.5.20, locally around x_0, S_* is a finite union of an even number of segments emanating from x_0. Fix two segments from this union that form the smallest angle among all possible pairs. Denote these segments by L_1 and L_2. Then there exists a truncated cone $K \subset \Omega$ such that $\overline{K} \cap S_* = \overline{K} \cap (L_1 \cup L_2)$ and $\partial K \cap (L_1 \cup L_2) = \{x_1, x_2, y_1, y_2\}$, where x_1, y_1 and x_2, y_2 are the endpoints of $L_1 \cap \overline{K}$, $L_2 \cap \overline{K}$, respectively. Furthermore, x_1, x_2 belong to the smaller circular base of K and y_1, y_2 belong to the bigger circular base of K. Choosing K so that its smaller circular base is fairly close to x_0 and using the triangle inequality, we can ensure that $|x_1 - x_2| + |y_1 - y_2| < |x_1 - y_1| + |x_2 - y_2|$. This, together with Propositions 3.6.4, 3.6.6, yields a contradiction, since the set U of Proposition 3.6.4, which contains K, can be chosen to be convex and arbitrarily close to K (for instance, we can choose $U = \{x \in \mathbb{R}^3 : \operatorname{dist}(x, K) < \varepsilon\}$ for some sufficiently small $\varepsilon \in (0, 1)$). Therefore, S_* has no branching points inside Ω, which completes our proof of Corollary 3.6.8. □

3.7 Analysis up to the boundary

3.7.1 Preliminary results

If the fractional seminorm of the boundary datum is fairly small, we may have a concentration of the p-energy as $p \nearrow 2$ only near the boundary of the domain. Namely, the smaller the fractional seminorm of the boundary datum, the closer to $\partial\Omega$ the singular set is located.

Proposition 3.7.1. *Let $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$, $r_0 \in (0, 1)$, $\Omega_{r_0} := \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > r_0\} \neq \emptyset$, $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_n}(\Omega, \mathcal{N})$ be a sequence of p_n-minimizers with $\operatorname{tr}_{\partial\Omega}(u_n) = g$. Then there exists $\varepsilon = \varepsilon(r_0, \partial\Omega, \mathcal{N}) > 0$ such that the following holds. Assume that $|g|_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)} \leq \varepsilon$. Then $S_* \cap \Omega_{r_0} = \emptyset$.*

Proof. Fix arbitrary $x_0 \in \Omega_{r_0}$ and $r \in (0, r_0)$. Let $\eta > 0$ be the constant of Lemma 3.3.12, where $\kappa = 1/2$, $p_0 = 3/2$ and $\Psi(x) = x + x_0$ whenever $E = B^3_r(x_0)$. Let $\varepsilon > 0$ be small enough so that for each $n \in \mathbb{N}$,

$$(2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n}\, dx \leq \eta r_0^{3/2}, \tag{3.7.1}$$

which is possible in view of Proposition 3.2.21 and the assumption $|g|_{W^{1/2,2}(\partial\Omega,\mathbb{R}^\nu)} \leq \varepsilon$. Then, applying Lemma 3.5.3 and using (3.7.1), for each fairly small $\delta > 0$ we have

$$
\begin{aligned}
(2-p_n)\int_{B^3_{r+\delta}(x_0)} \frac{|Du_n|^{p_n}}{p_n}\,dx \\
&\le (2-p_n)\left(\frac{r+\delta}{r_0}\right)^{3-p_n}\int_{B^3_{r_0}(x_0)} \frac{|Du_n|^{p_n}}{p_n}\,dx \\
&\le (2-p_n)\left(\frac{r+\delta}{r_0}\right)^{3-p_n}\int_{\Omega} \frac{|Du_n|^{p_n}}{p_n}\,dx \le \eta r_0^{3/2-3+p_n}(r+\delta)^{3-p_n}.
\end{aligned}
$$

Letting n tend to $+\infty$, we have $\mu_*(\overline{B}^3_r(x_0)) \le \mu_*(B^3_{r+\delta}(x_0)) \le \eta r_0^{1/2}(r+\delta)$ (see (3.5.3)). Letting δ tend to $0+$, yields $\mu_*(\overline{B}^3_r(x_0)) < \eta r$, since $r_0 \in (0,1)$. Then, according to Lemma 3.5.2, $\mu_*(B^3_{r/2}(x_0)) = 0$ and hence $B^3_{r/2}(x_0) \subset \Omega \setminus S_*$. Since $x_0 \in \Omega_{r_0}$ was arbitrary, $S_* \cap \Omega_{r_0} = \emptyset$, which completes our proof of Proposition 3.7.1. $\square$

The next proposition says that if the boundary of the domain is homeomorphic to $\mathbb{S}^2$ and the boundary datum is regular enough, then no singular set arises in the limit.

Proposition 3.7.2. *Let $\partial\Omega$ be homeomorphic to $\mathbb{S}^2$, $g \in W^{1,2}(\partial\Omega, \mathcal{N})$, $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}} \subset W^{1,p_n}(\Omega,\mathcal{N})$ be a sequence of p_n-minimizers such that for each $n \in \mathbb{N}$, $\mathrm{tr}_{\partial\Omega}(u_n) = g$. Then $S_* = \emptyset$ and the map u_* coming from Proposition 3.5.6 is a minimizing harmonic extension of g to Ω. Furthermore, $u_* \in C^\infty(\Omega \setminus S_0, \mathcal{N})$, where $S_0 \subset \Omega$ is a discrete set.*

Remark 3.7.3. In Proposition 3.7.2, the topology of Ω is simple, namely $\Omega \simeq B^3_1$. However, the singular set S_* does not occur in the limit in some cases even if the topology of Ω is not simple. For example, if $\Omega \simeq B^2_1 \times \mathbb{S}^1$ is a solid torus and $\gamma \in C^\infty(\mathbb{S}^1, \mathcal{N})$ is nullhomotopic, we can define $g \in C^\infty(\partial\Omega, \mathcal{N})$ by $g(x', x'') = \gamma(x')$ for each $(x', x'') \in \partial\Omega \simeq \mathbb{S}^1 \times \mathbb{S}^1$. Then g admits a smooth extension lying in $W^{1,2}(\Omega, \mathcal{N})$ (this comes, for instance, from the proof of [12, Theorem 2]).

Proof of Proposition 3.7.2. Since $\Omega \subset \mathbb{R}^3$ is a bounded (locally) Lipschitz domain and $\partial\Omega$ is homeomorphic to $\mathbb{S}^2$, there exists a bilipschitz homeomorphism $\Phi : \overline{\Omega} \to \overline{B}^3_1$. Let $h = g \circ \Phi^{-1}|_{\mathbb{S}^2}$. Then we have $h \in W^{1,2}(\mathbb{S}^2, \mathcal{N})$. Next, defining, as it was done in [53, Proposition 2.4], $u(x) = h(x/|x|)$ for $x \in B^3_1 \setminus \{0\}$, we have $u \in W^{1,2}(B^3_1, \mathcal{N})$ and $\mathrm{tr}_{\mathbb{S}^2}(u) = h$. Since Φ is bilipschitz and $u \circ \Phi$ is a competitor for u_n, we deduce that

$$\begin{aligned}(3.7.2)\quad \mu_n(\overline{\Omega}) &\le C(2-p_n)\int_{B_1^3}\frac{|Du|^{p_n}}{p_n}\,dx\\ &= \frac{C(2-p_n)}{3-p_n}\int_{\mathbb{S}^2}\frac{|D_\top h|^{p_n}}{p_n}\,d\mathcal{H}^2 \to 0\end{aligned}$$

as $n \to +\infty$, where $C > 0$ is a constant depending only on the bilipschitz constant of Φ, and we have used that $\mu_n(\Omega) = \mu_n(\overline{\Omega})$. Thus, $\mu_*(\overline{\Omega}) = 0$ (here we again take advantage of considering the μ_n's as elements of $(C(\overline{\Omega}))'$ instead of $(C_0(\Omega))'$, because any constant belongs to $C(\overline{\Omega})$, and hence the weak* convergence preserves mass, namely, since $\mu_n \overset{*}{\rightharpoonup} \mu_*$ weakly* in $(C(\overline{\Omega}))'$, it holds $\mu_n(\overline{\Omega}) \to \mu_*(\overline{\Omega})$). Therefore, the support of μ_* is the empty set, that is $S_* = \emptyset$. On the other hand, in view of (3.7.2), the condition (3.4.1) of Proposition 3.4.2 is valid for the sequence $(u_n)_{n\in\mathbb{N}}$ and the set $K = \Omega$. Thus, the conclusions of Proposition 3.4.2 apply for some mapping $u_* \in W^{1,2}(\Omega, \mathcal{N})$, which comes from Proposition 3.5.6. In particular, u_* is a minimizing harmonic extension of g to Ω. Then, according to [53, Theorem II], $u_* \in C^\infty(\Omega \setminus S_0, \mathcal{N})$, where $S_0 \subset \Omega$ is a discrete set, which completes our proof of Proposition 3.7.2. □

As an immediate consequence of Lemma 3.3.11, we have its counterpart on a half-ball. Recall that $\mathbb{R}^3_+ = \{(x', x_3) \in \mathbb{R}^3 : x_3 \in (0, +\infty)\}$.

Lemma 3.7.4. *Let $p_0 \in (1,2)$, $p \in [p_0, 2)$. There exist constants $\eta_0, C > 0$, depending only on p_0 and $\mathcal{N}$, such that if $g \in W^{1,p}(\partial(B_r^3 \cap \mathbb{R}^3_+), \mathcal{N})$ satisfies*

$$\int_{\partial(B_r^3\cap\mathbb{R}^3_+)}\frac{|D_\top g|^p}{p}\,d\mathcal{H}^2 \le \frac{\eta_0 r^{2-p}}{2-p},$$

then there exists $u \in W^{1,p}(B_r^3 \cap \mathbb{R}^3_+, \mathcal{N})$ such that $\operatorname{tr}_{\partial(B_r^3\cap\mathbb{R}^3_+)}(u) = g$ and

$$\int_{B_r^3\cap\mathbb{R}^3_+}\frac{|Du|^p}{p}\,dx \le C r^{\frac{2}{p}}\left(\int_{\partial(B_r^3\cap\mathbb{R}^3_+)}\frac{|D_\top g|^p}{p}\,d\mathcal{H}^2\right)^{1-\frac{1}{p}}.$$

Next, we prove the η-compactness lemma up to the boundary of Ω, which is the counterpart of Lemma 3.3.12 on the boundary.

Proposition 3.7.5. *Let $\kappa \in (0,1)$, $p_0 \in (1,2)$, $p \in [p_0, 2)$ and let $U = \Psi(B_r^3 \cap \mathbb{R}^3_+, \mathcal{N})$ be an open set, where $\Psi : \overline{B_r^3} \cap \overline{\mathbb{R}^3_+} \to \mathbb{R}^3$ is a bilipschitz homeomorphism onto its image $\overline{U}$ with a bilipschitz constant $L \ge 1$. There exist $\eta, C > 0$, depending only on κ, p_0, L and $\mathcal{N}$, such that* *if* $u_p \in W^{1,p}(U, \mathcal{N})$ *is a* p*-minimizer and if*

$$\int_U \frac{|Du_p|^p}{p}\,dx + r\int_{\Psi(B_r^3\cap\mathbb{R}^2\times\{0\})} \frac{|D_\top \operatorname{tr}_{\Psi(B_r^3\cap\mathbb{R}^2\times\{0\})}(u_p)|^p}{p}\,d\mathcal{H}^2 \le \frac{\eta r^{3-p}}{2-p}, \tag{3.7.3}$$

then

$$\int_{\Psi(B_{\kappa r}^3\cap\mathbb{R}^3_+)} \frac{|Du_p|^p}{p}\,dx \le Cr^{3-p}. \tag{3.7.4}$$

Proof. Given the mapping $v_p := u_p\circ\Psi \in W^{1,p}(B_r^3\cap\mathbb{R}^3_+, \mathcal{N})$ satisfying

$$\int_{B_r^3\cap\mathbb{R}^3_+} \frac{|Dv_p|^p}{p}\,dx \le L^{3+p}\int_U \frac{|Du_p|^p}{p}\,dx \tag{3.7.5}$$

and the constant $\eta_0 = \eta_0(p_0, \mathcal{N}) > 0$ of Lemma 3.7.4, we define the set

$$G_p = \Big\{\varrho \in (\kappa r, r) : \operatorname{tr}_{\partial(B_\varrho^3\cap\mathbb{R}^3_+)}(v_p) = v_p|_{\partial(B_\varrho^3\cap\mathbb{R}^3_+)} \in W^{1,p}(\partial(B_\varrho^3\cap\mathbb{R}^3_+), \mathcal{N}) \text{ and } \int_{\partial B_\varrho^3\cap\mathbb{R}^3_+} \frac{|Dv_p|^p}{p}\,d\mathcal{H}^2 \le \frac{\eta_0(\kappa r)^{2-p}}{2(2-p)}\Big\}. \tag{3.7.6}$$

As a consequence of (3.7.3), (3.7.5) and (3.7.6),

$$\mathcal{H}^1((\kappa r, r)\setminus G_p) \le \frac{2(2-p)}{\eta_0(\kappa r)^{2-p}}\int_{B_r^3\cap\mathbb{R}^3_+} \frac{|Dv_p|^p}{p}\,dx \le \frac{2(2-p)L^{3+p}}{\eta_0(\kappa r)^{2-p}}\int_U \frac{|Du_p|^p}{p}\,dx \le \frac{(1-\kappa)r}{2},$$

provided we choose η satisfying

$$\eta \le \frac{\eta_0\kappa(1-\kappa)}{4L^5}; \tag{3.7.7}$$

equivalently we have then

$$\mathcal{H}^1(G_p) \ge \frac{(1-\kappa)r}{2}. \tag{3.7.8}$$

Fix η satisfying (3.7.7). For every $\varrho \in G_p$ the following holds

$$\begin{aligned}\int_{\partial(B^3_\varrho\cap\mathbb{R}^3_+)} \frac{|D_\top v_p|^p}{p}\,d\mathcal{H}^2 &= \int_{\partial B^3_\varrho\cap\mathbb{R}^3_+} \frac{|D_\top v_p|^p}{p}\,d\mathcal{H}^2 \\ &\quad + \int_{B^3_\varrho\cap\mathbb{R}^2\times\{0\}} \frac{|D_\top v_p|^p}{p}\,d\mathcal{H}^2 \\ &\le \frac{\eta_0(\kappa r)^{2-p}}{2(2-p)} + \frac{\eta r^{2-p}L^{2+p}}{2-p} \le \frac{\eta_0\varrho^{2-p}}{2-p},\end{aligned}$$

where we have used (3.7.3), the fact that $|D_\top v_p| \le |Dv_p|$ and (3.7.7). Under this condition, according to Lemma 3.7.4,

$$\int_{B^3_\varrho\cap\mathbb{R}^3_+} \frac{|Dv_p|^p}{p}\,dx \le C\varrho^{\frac{2}{p}}\left(\int_{\partial(B^3_\varrho\cap\mathbb{R}^3_+)} \frac{|D_\top v_p|^p}{p}\,d\mathcal{H}^2\right)^{1-\frac{1}{p}}. \tag{3.7.9}$$

To obtain (3.7.4), we conclude as in the proof of Lemma 3.3.12. □

The next corollary is a direct consequence of Proposition 3.7.5 and Definition 3.2.4, so we omit its proof.

Corollary 3.7.6. Let $\kappa \in (0,1)$ and $g \in W^{1,2}(E,\mathcal{N})$, where E is a relatively open subset of $\partial\Omega$. Let $r_{\partial\Omega} > 0$ be the number coming from Definition 3.2.4, which is applied with $U = \Omega$. Then there exist constants $p_1 = p_1(\partial\Omega, \mathcal{N}, \|D_\top g\|_{L^2(E,\mathbb{R}^\nu\otimes\mathbb{R}^3)}) \in [3/2, 2)$, $\eta = \eta(\kappa, \partial\Omega, \mathcal{N}) > 0$ and $C = C(\kappa, \partial\Omega, \mathcal{N}) > 0$ such that the following holds. Let $p \in [p_1, 2)$ and $u_p \in W^{1,p}(\Omega, \mathcal{N})$ be a p-minimizer such that $\mathrm{tr}_E(u_p) = g$. Then for each $r \in (0, r_{\partial\Omega})$ and $x_0 \in E$ such that $B^3_r(x_0) \cap \partial\Omega \subset E$ and

$$\int_{B^3_r(x_0)\cap\Omega} \frac{|Du_p|^p}{p}\,dx \le \frac{\eta r^{3-p}}{2-p},$$

it holds

$$\int_{B^3_{\kappa r}(x_0)\cap\Omega} \frac{|Du_p|^p}{p}\,dx \le Cr^{3-p}.$$

Corollary 3.7.7. Let E be a relatively open subset of $\partial\Omega$. Assume that (3.5.1) holds and that for all $n \in \mathbb{N}$ large enough, $\mathrm{tr}_E(u_n) = g \in W^{1,2}(E,\mathcal{N})$. Then there exists a closed set $S_* \subset \overline{\Omega}$ and a mapping $u_* \in W^{1,2}_{\mathrm{loc}}((\Omega\cup E)\setminus S_*, \mathcal{N})$ satisfying the assertions of Proposition 3.4.2 and the assertion (ii) of Proposition 3.5.23 such that $\mathrm{tr}_E(u_*) = g$ and, up to a subsequence (not relabeled),

$$u_n \rightharpoonup u_* \text{ weakly in } W^{1,p}_{\mathrm{loc}}((\Omega\cup E)\setminus S_*, \mathbb{R}^\nu) \text{ for all } p \in (1,2).$$

Proof of Corollary 3.7.7. The proof of the assertion (ii) of Proposition 3.5.23 remains unchanged. One only needs to take Corollary 3.7.6 into account in the proof of Proposition 3.4.2. □

3.7.2 Global estimates and $W^{1,q}$ compactness

It is well known that the set $C^\infty(\partial\Omega, \mathcal{N})$ is dense in $W^{1/2,2}(\partial\Omega, \mathcal{N})$ if and only if $\pi_1(\mathcal{N}) \simeq \{0\}$ (we refer the reader to [17, Theorem 4]). Although $C^\infty(\partial\Omega, \mathcal{N})$ is not dense in $W^{1/2,2}(\partial\Omega, \mathcal{N})$ if $\pi_1(\mathcal{N})$ is nontrivial, the set of smooth maps outside a finite set of points whose Euclidean norm of the gradient behaves like the distance to the power of -1 close to these points is dense in $W^{1/2,2}(\partial\Omega, \mathcal{N})$ (see [49, Theorem 1] and [17, Theorem 3]). Namely, letting $\mathcal{R}_0^1(\partial\Omega, \mathcal{N})$ to be the class of maps $\varphi \in C^1(\partial\Omega \setminus \{a_1, \dots, a_l\}, \mathcal{N})$ for some $a_1, \dots, a_l \in \partial\Omega$ such that

$$|D_\top \varphi(x)| \leq \frac{C}{\operatorname{dist}(x, \bigcup_{i=1}^l a_i)}, \tag{3.7.10}$$

where C is a positive constant independent of x, we have

$$\overline{\mathcal{R}_0^1(\partial\Omega, \mathcal{N})}^{W^{1/2,2}} = W^{1/2,2}(\partial\Omega, \mathcal{N}).$$

Hereinafter, for each $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N})$, we shall denote by $S(g)$ the singular set of g, namely the set of points $a \in \partial\Omega$ such that for each sufficiently small radius $\varrho > 0$, $g|_{\partial B_\varrho^{\partial\Omega}(a)}$ is not nullhomotopic, where $B_\varrho^{\partial\Omega}(a)$ is the geodesic ball in $\partial\Omega$ with center a and radius ϱ such that $B_\varrho^{\partial\Omega}(a)$ is far enough from $S(g)\setminus\{a\}$.

It is worth noting that for each $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N})$, there exists the *minimal* set of points $\{a_1, \dots, a_l\} \subset \partial\Omega$ such that $g \in C^1(\partial\Omega\setminus\{a_1, \dots, a_l\}, \mathcal{N})$. Moreover, $S(g) \subset \{a_1, \dots, a_l\}$ and the inclusion can be strict.

Lemma 3.7.8. *Let $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N})$ and $\{a_1, \dots, a_l\} \subset \partial\Omega$ be the minimal set of different points such that $g \in C^1(\partial\Omega \setminus \{a_1, \dots, a_l\}, \mathcal{N})$. Then for each $q \in [1, 2)$, $g \in W^{1,q}(\partial\Omega, \mathcal{N})$ and*

$$\int_{\partial\Omega} |D_\top g|^q \, d\mathcal{H}^2 \leq \frac{C}{2-q},$$

where $C > 0$ is a constant depending only on $\partial\Omega$, the set $\{a_1, \dots, a_l\}$ and the constant of the estimate (3.7.10) *applied with $\varphi = g$.*

Proof. Fix an arbitrary $q \in [1, 2)$. Define

$$\bar{\varrho}_{\partial\Omega}(a_1, \dots, a_l) = \min\left\{\frac{|a_i - a_j|}{2} : i, j \in \{1, \dots, l\} \text{ and } i \neq j\right\}$$

so that if $\varrho \in (0, \bar{\varrho}_{\partial\Omega}(a_1, \dots, a_l))$, then $\overline{B}_\varrho^3(a_i) \cap \overline{B}_\varrho^3(a_j) = \emptyset$ for each $i, j \in \{1, \dots, l\}$ such that $i \neq j$. Then for each $i \in \{1, \dots, l\}$ and for each sufficiently small $\varrho \in (0, \bar{\varrho}_{\partial\Omega}(a_1, \dots, a_l))$,

$$\int_{B_\varrho^3(a_i)\cap\partial\Omega} |D_\top g|^q \, d\mathcal{H}^2 \leq C\int_0^\varrho \frac{dt}{t^{1-q}} \leq \frac{C}{2-q}, \tag{3.7.11}$$

where $C > 0$ is a constant depending only on $\partial\Omega$, $\bar{\varrho}_{\partial\Omega}(a_1, \ldots, a_l)$ and the constant of the estimate (3.7.10) applied with $\varphi = g$. On the other hand, in view of (3.7.10), if $\varrho > 0$ and $x \in \partial\Omega \setminus \bigcup_{i=1}^{l} \overline{B}^3_\varrho(a_i)$, then $|D_\top g(x)| \leq C/\varrho$. Taking this and (3.7.11) into account, we complete our proof of Lemma 3.7.8. $\square$

Assuming that $g \in \mathcal{R}^1_0(\partial\Omega, \mathcal{N})$, we obtain the next improvement of Proposition 3.5.23.

Proposition 3.7.9. *Let $g \in \mathcal{R}^1_0(\partial\Omega, \mathcal{N})$ be such that $g \in C^1(\partial\Omega \setminus S(g), \mathcal{N})$. Let $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers such that $\operatorname{tr}_{\partial\Omega}(u_n) = g$ for each $n \in \mathbb{N}$. Let u_* be a map given by Proposition 3.5.6. Assume that there exists $r_0 > 0$ such that if $x \in \mathbb{R}^3 \setminus \Omega$ and $\operatorname{dist}(x, \partial\Omega) \leq r_0$, there exists a unique point $P(x) \in \partial\Omega$ such that $\operatorname{dist}(x, \partial\Omega) = |x - P(x)|$. Then the following assertions hold.*

(i) There exists $C = C(g, \Omega, \mathcal{N}) > 0$ such that for each $q \in [1, 2)$,

$$\limsup_{n\to+\infty} \int_\Omega |Du_n|^q \, dx \leq C + \frac{C}{2-q} \limsup_{n\to+\infty} (2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \, dx.$$

(ii) For each $q \in [1, 2)$, $u_* \in W^{1,q}(\Omega, \mathcal{N})$ and, up to a subsequence (not relabeled), $u_n \to u_*$ in $W^{1,q}(\Omega, \mathbb{R}^\nu)$ and for some $C = C(g, \Omega, \mathcal{N}) > 0$,

$$\int_\Omega |Du_*|^q \, dx \leq C + \frac{C}{2-q} \limsup_{n\to+\infty} (2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n} \, dx.$$

Remark 3.7.10. If Ω is of class C^2 or is convex, then it has the *unique nearest point property*, namely, the assumption of Proposition 3.7.9 applies (for more details, we refer to [30], [40]). It is also worth noting that the constants in the assertions (i), (ii) depend on $S(g)$ and the constant coming from (3.7.10), where $\varphi = g$. Since the latter constant depends only on g, we can assume that the constants in (i), (ii) depend only on g, Ω and $\mathcal{N}$.

Proof of Proposition 3.7.9. Fix $R > 0$ large enough so that $\Omega \Subset B^3_R$ and for each $x \in \Omega$ the distance from x to ∂B^3_R is bounded from below by 1. Without loss of generality, we assume that r_0 is small enough with respect to R and that P is a 2-Lipschitz map (the reader may consult [30, Theorem 4.8 (8)]). To lighten the notation, we denote $W = \{x \in B^3_R : \operatorname{dist}(x, \Omega) \leq r_0\}$ and $V = \{x \in B^3_R \setminus \Omega : \operatorname{dist}(x, \partial\Omega) \leq r_0\}$. For each $n \in \mathbb{N}$, define the map $f_n : B^3_R \to [0, +\infty)$ by $f_n = |Du_n|$ in Ω,

$f_n = |D(g \circ P)|$ in V and $f_n = 0$ in $B^3_R \backslash W$. Since P is 2-Lipschitz and $g \in \mathcal{R}^1_0(\partial\Omega, \mathcal{N})$, using Lemma 3.7.8, we observe that

$$(2-p_n)\int_{B^3_R\backslash\Omega} f_n^{p_n}\,dx = (2-p_n)\int_V f_n^{p_n}\,dx \leq C, \tag{3.7.12}$$

where $C = C(g, \Omega, \mathcal{N}) > 0$ (here C depends on r_0, $S(g)$ and the constant coming from (3.7.10), where $\varphi = g$, but r_0 depends only on Ω and the latter constant depends only on g). Then the proof of the assertion (i) follows by reproducing the proof of the assertion (i) of Proposition 3.5.23 with minor modifications, namely, applying Proposition 3.5.22 with $f = f_n$, $p = p_n$, $V = B^3_R$, $C_1 = \frac{2-p_n}{\eta}$ and $C_2 = C$, where $n \in \mathbb{N}$ is large enough and $\eta, C > 0$ depend only on g, Ω, $\mathcal{N}$ and will be fixed later for the proof to work. Let us fix an arbitrary dyadic cube Q such that $2Q \subset B^3_R$. Let $r > 0$ be the sidelength of Q. To apply Proposition 3.5.22 with the above parameters, we need to prove the following: if $n \in \mathbb{N}$ is large enough and

$$\int_{2Q} f_n^{p_n}\,dx \leq \frac{\eta r^{3-p_n}}{2-p_n}, \tag{3.7.13}$$

then

$$\int_Q f_n^{p_n}\,dx \leq C r^{3-p_n}. \tag{3.7.14}$$

First, notice the following. Fix an arbitrary $x_0 \in S(g)$. Let $B^{\partial\Omega}_\varrho(x_0)$ be the geodesic ball in $\partial\Omega$ with center x_0 and radius $\varrho > 0$ such that every point in $S(g)\backslash\{x_0\}$ is far enough from $B^{\partial\Omega}_\varrho(x_0)$. Then, using Proposition 3.3.3 and the facts that $p_n \in [3/2, 2)$ for each $n \in \mathbb{N}$ large enough and $B^{\partial\Omega}_\varrho(x_0)$ is bilipschitz homeomorphic to $B^2_\varrho(x_0)$, one has

$$\int_{B^{\partial\Omega}_\varrho(x_0)} |D_\top g|^{p_n}\,d\mathcal{H}^2 + \varrho\int_{\partial B^{\partial\Omega}_\varrho(x_0)} |D_\top(g|_{\partial B^{\partial\Omega}_\varrho(x_0)})|^{p_n}\,d\mathcal{H}^1 \geq \frac{\mathcal{E}^{\mathrm{sg}}_{p_n/(p_n-1)}(g|_{\partial B^{\partial\Omega}_\varrho(x_0)})\varrho^{2-p_n}}{C_0(2-p_n)},$$

where $\partial B^{\partial\Omega}_\varrho(x_0)$ denotes the relative boundary of $B^{\partial\Omega}_\varrho(x_0)$, $C_0 = C_0(\Omega) > 0$ and we have used that $\mathrm{tr}_{\partial B^{\partial\Omega}_\varrho(x_0)}(g) = g|_{\partial B^{\partial\Omega}_\varrho(x_0)}$. Then, proceeding as in the proof of Lemma 3.3.8, we deduce that

$$\int_{B^{\partial\Omega}_\varrho(x_0)} |D_\top g|^{p_n}\,d\mathcal{H}^2 + \varrho\int_{\partial B^{\partial\Omega}_\varrho(x_0)} |D_\top(g|_{\partial B^{\partial\Omega}_\varrho(x_0)})|^{p_n}\,d\mathcal{H}^1 \geq \frac{2\varepsilon_0\varrho^{2-p_n}}{2-p_n}, \tag{3.7.15}$$

for each $n \in \mathbb{N}$ large enough, where $\varepsilon_0 = \varepsilon_0(C_0, \mathcal{N}) > 0$. Since C_0 depends only on Ω, we can assume that ε_0 depends only on Ω and $\mathcal{N}$. Also, since $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N})$, we have the estimate

$$\varrho \int_{\partial B_\varrho^{\partial\Omega}(x_0)} |D_\top(g|_{\partial B_\varrho^{\partial\Omega}(x_0)})|^{p_n} \, d\mathcal{H}^1 \leq C' \varrho \mathcal{H}^1(\partial B_\varrho^{\partial\Omega}(x_0)),$$

where $C' > 0$ is a constant depending only on the constant coming from (3.7.10), which depends only on g. This, since $p_n \nearrow 2$ as $n \to +\infty$, implies that for each $n \in \mathbb{N}$ large enough (depending possibly only on g, Ω and $\mathcal{N}$),

$$\varrho \int_{\partial B_\varrho^{\partial\Omega}(x_0)} |D_\top(g|_{\partial B_\varrho^{\partial\Omega}(x_0)})|^{p_n} \, d\mathcal{H}^1 \leq \frac{\varepsilon_0 \varrho^{2-p_n}}{2 - p_n}. \tag{3.7.16}$$

Combining (3.7.15) and (3.7.16), we get the next estimate

$$\int_{B_\varrho^{\partial\Omega}(x_0)} |D_\top g|^{p_n} \, d\mathcal{H}^2 \geq \frac{\varepsilon_0 \varrho^{2-p_n}}{2 - p_n} \tag{3.7.17}$$

for each $n \in \mathbb{N}$ large enough. On the other hand, if $E \subset \partial\Omega$ is Borel and $\operatorname{dist}(E, S(g)) \geq \varrho > 0$, then, using (3.7.10), we obtain

$$\int_E |D_\top g|^{p_n} \, d\mathcal{H}^2 \leq \frac{C' \mathcal{H}^2(E)}{\varrho^{p_n}}. \tag{3.7.18}$$

Next, observe that, since Ω is a bounded Lipschitz domain having the unique nearest point property in V, there exist fairly small constants $0 < a < \delta < 1/2$ and a number $N_0 \in \mathbb{N} \setminus \{0\}$, depending only on Ω, such that the following holds. We need to distinguish between two further cases.

Case 1: $Q \subset W$. If $\operatorname{dist}(P(Q \cap V), S(g)) \geq \delta r$, then we proceed as follows. Applying (3.7.18) with $E = P(Q \cap V)$, we have

$$\int_{Q \cap V} f_n^{p_n} \, dx \leq Lr \int_{P(Q \cap V)} |D_\top g|^{p_n} \, d\mathcal{H}^2 \leq C'' r^{3-p_n}, \tag{3.7.19}$$

where $L = L(P) > 0$ and $C'' = C''(g, \Omega, \mathcal{N}) > 0$. Next, there exists at most N_0 balls $B_{ar}^3(x_1), \dots, B_{ar}^3(x_k)$ such that $Q \cap \Omega \subset \bigcup_{i=1}^k B_{ar}^3(x_i) \subset 2Q$ and for each $i \in \{1, \dots, k\}$, either $B_{9ar/8}^3(x_i) \subset \Omega$ or $x_i \in \partial\Omega$ and $\operatorname{dist}(B_{9ar/8}^3(x_i), S(g)) \geq \delta r/2$. We shall apply the corresponding η-compactness argument for each ball $B_{9ar/8}^3(x_i)$, namely Lemma 3.3.12 in the case when $B_{9ar/8}^3(x_i) \subset \Omega$ and Proposition 3.7.5 otherwise. Let

$\eta, C''' > 0$ be constants that are valid for both Lemma 3.3.12 and Proposition 3.7.5, where $\kappa = 8/9$, $p_0 = 3/2$ and the corresponding bilipschitz homeomorphism whose bilipschitz constant can depend only on Ω. Thus, we can assume that η, C''' depend only on Ω and $\mathcal{N}$. Assume that

$$\int_{2Q} f_n^{p_n}\,dx \leq \frac{\eta r^{3-p_n}}{2(2-p_n)}. \tag{3.7.20}$$

In the case when $B^3_{9ar/8}(x_i) \subset \Omega$, by Lemma 3.3.12 and (3.7.20), we immediately obtain the estimate

$$\int_{B^3_{ar}(x_i)} |Du_n|^{p_n}\,dx \leq C''' r^{3-p_n}. \tag{3.7.21}$$

Let now $x_i \in \partial\Omega$. Using (3.7.19) and the fact that $p_n \nearrow 2$, we observe that

$$\frac{9ar}{8}\int_{B^3_{\frac{9ar}{8}}(x_i)\cap\partial\Omega} |D_\top g|^{p_n}\,d\mathcal{H}^2 \leq \frac{\eta r^{3-p_n}}{2(2-p_n)} \tag{3.7.22}$$

for each $n \in \mathbb{N}$ large enough depending only on g, Ω and $\mathcal{N}$. Thus, in view of (3.7.20) and (3.7.22),

$$\int_{B^3_{\frac{9ar}{8}}(x_i)\cap\Omega} |Du_n|^{p_n}\,dx + \frac{9ar}{8}\int_{B^3_{\frac{9ar}{8}}(x_i)\cap\partial\Omega} |D_\top g|^{p_n}\,d\mathcal{H}^2 \leq \frac{\eta r^{3-p_n}}{2-p_n},$$

for each $n \in \mathbb{N}$ large enough, which according to Proposition 3.7.5, implies that

$$\int_{B^3_{ar}(x_i)} |Du_n|^{p_n}\,dx \leq C''' r^{3-p_n}. \tag{3.7.23}$$

Combining (3.7.21) and (3.7.23), we get

$$\int_{Q\cap\Omega} f_n^{p_n}\,dx \leq N_0 C''' r^{3-p_n} \tag{3.7.24}$$

for each $n \in \mathbb{N}$ large enough. Taking into account the estimates (3.7.19) and (3.7.24), we observe that (3.7.13) implies (3.7.14), where $C = C'' + N_0 C'''$, which can depend only on g, Ω and $\mathcal{N}$.

If $\operatorname{dist}(P(Q\cap V), S(g)) < \delta r$, then there exists $x_0 \in S(g)$ such that $2Q \cap V$ contains a cylinder $U \simeq B^{\partial\Omega}_{ar}(x_0) \times (0, ar)$ such that, in view of (3.7.17) and the definition of f_n in V, it holds

$$\int_{2Q} f_n^{p_n}\,dx \geq \int_U f_n^{p_n}\,dx \geq \frac{\varepsilon_0 (a^{10} r)^{3-p_n}}{2-p_n} \tag{3.7.25}$$

for each $n \in \mathbb{N}$ large enough. Thus, taking η small enough with respect to $\varepsilon_0 a^{15}$ so that Proposition 3.7.5 applies with 2η, under the condition that $\int_{2Q} f_n^{p_n}\,dx \le \frac{\eta r^{3-p_n}}{2-p_n}$, we exclude the situation where $\operatorname{dist}(P(Q \cap V), S(g)) < \delta r$.

Case 2: $Q \cap (B_R^3 \setminus W) \neq \emptyset$. If $Q \subset B_R^3 \setminus W$, then, clearly, (3.7.14) holds, since $f_n = 0$ on Q by definition. Otherwise, we repeat the strategy of the *Case 1* for $Q \cap W$ to conclude that for each $n \in \mathbb{N}$ large enough, (3.7.13) implies (3.7.14) for some $\eta, C > 0$ depending only on g, Ω and $\mathcal{N}$.

After all, we have showed that there exist constants $\eta, C > 0$ depending only on g, Ω and $\mathcal{N}$ such that (3.7.13) implies (3.7.14) for each $n \in \mathbb{N}$ large enough depending on g, Ω and $\mathcal{N}$. Therefore, we can apply Proposition 3.5.22 with $f = f_n$, $V = B_R^3$, $C_1 = \frac{2-p_n}{\eta}$ and $C_2 = C$ for each $n \in \mathbb{N}$ large enough. Taking into account (3.7.12) and the fact that $f_n = 0$ in $B_R^3 \setminus W$, we obtain (i).

The assertion (ii) then comes from a diagonal argument, proceeding by the same way as in the proof of Proposition 3.5.23 (ii). This completes our proof of Proposition 3.7.9. □

3.7.3 Proof of Theorem 3.1.1

Now we prove our first main theorem.

Proof of Theorem 3.1.1. The proof follows from Propositions 3.5.6, 3.5.8, 3.5.17, 3.5.23, 3.6.4, 3.7.2, 3.7.9 and Corollary 3.5.18. □

3.7.4 Boundary repulsion property

In this subsection, we derive the Pohozaev-type identity for p-minimizers and prove that the varifold associated to the limit of the stress-energy tensors of p-minimizers as $p \nearrow 2$ is *minimizing area to first order* in the Brian White sense (see [58]). As a consequence, we prove that if $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$, the boundary datum g is an element of $\mathcal{R}_0^1(\partial\Omega, \mathcal{N})$ and $g \in C^1(\partial\Omega \setminus S(g), \mathcal{N})$, then $S_* \cap \partial\Omega = S(g)$, where S_* is the energy concentration set supporting the varifold and $S(g)$ is the singular set of g.

We begin by establishing the boundary stress-energy tensor identity. For convenience, if $U \subset \mathbb{R}^3$ is open, we define for $u \in W^{1,p}(U, \mathbb{R}^\nu)$ the map $T_u^p : U \to \mathbb{R}^3 \otimes \mathbb{R}^3$ by

$$T_u^p = \frac{|Du|^p}{p}\mathrm{Id} - \frac{Du \otimes Du}{|Du|^{2-p}}, \tag{3.7.26}$$

where $Du \otimes Du = (Du)^{\mathrm{T}} Du$. If U is a bounded Lipschitz domain and u is a minimizing p-harmonic map in U, we have $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(U, \mathbb{R}^3)$ (see (3.2.33)).

Proposition 3.7.11. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in [1, +\infty)$. If $u \in W^{1,p}(\Omega \cap B_r^3(x_0), \mathbb{R}^\nu)$ satisfies $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(\Omega \cap B_r^3(x_0), \mathbb{R}^3)$ and if Du is continuous at every point of $\partial\Omega \cap B_r^3(x_0)$, then for each $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$,*

$$\int_{\Omega \cap B_r^3(x_0)} T_u^p : D\xi \, dx = \int_{\partial\Omega \cap B_r^3(x_0)} \langle \nu, T_u^p[\xi] \rangle \, d\mathcal{H}^2, \tag{3.7.27}$$

where $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ stands for the outward pointing unit normal vector field to $\partial\Omega$.

Proof. To lighten the notation, denote $B = B_r^3(x_0)$. Observe that $\partial\Omega$ is a 2-dimensional compact Riemannian manifold of class C^2 without boundary. Hence there exists $r_0 = r_0(\partial\Omega) > 0$ such that the function $f(x) = \operatorname{dist}(x, \partial\Omega)$, $x \in \Omega$ is of class C^1 in $V = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) < r_0\}$ (see [30, Theorem 4.12]) and, furthermore, $|Df(x)| = 1$ for $x \in V$ (see [30, 4.8 (5), 4.8 (3)]). Let us fix a sufficiently small $\varepsilon > 0$ and define

$$g_\varepsilon(t) = \begin{cases} 1 & \text{if} \quad t \in [\varepsilon, +\infty) \\ \frac{t}{\varepsilon} & \text{if} \quad t \in [0, \varepsilon]. \end{cases}$$

Since g_ε is a Lipschitz function on $[0, +\infty)$, for each $\xi \in W_0^{1,\infty}(B, \mathbb{R}^3)$, it is clear that the function $\varphi_\varepsilon(x) \coloneqq g_\varepsilon(f(x))\xi(x)$ is an element of $W_0^{1,\infty}(\Omega \cap B, \mathbb{R}^3)$. Since $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(\Omega \cap B, \mathbb{R}^3)$, $u \in W^{1,p}(\Omega \cap B, \mathbb{R}^\nu)$ and Du is continuous at every point of $\partial\Omega \cap B$, by approximation, we have $\int_{\Omega \cap B} T_u^p : D\xi \, dx = 0$ for each $\xi \in W_0^{1,\infty}(\Omega \cap B, \mathbb{R}^3)$. Thus, using the function φ_ε as a test function in the weak formulation of $\operatorname{div}(T_u^p) = 0$, we get

$$\begin{aligned} 0 &= \int_{\Omega \cap B} T_u^p : g_\varepsilon(f) D\xi \, dx + \int_{\Omega \cap B} \langle D(g_\varepsilon(f)), T_u^p[\xi] \rangle \, dx \\ &= \int_{\Omega \cap B} T_u^p : g_\varepsilon(f) D\xi \, dx + \int_{\Omega \cap B} \langle g_\varepsilon'(f) Df, T_u^p[\xi] \rangle \, dx. \end{aligned}$$

Letting $\varepsilon \searrow 0$ and using the Lebesgue dominated convergence theorem, we have

$$\int_{\Omega \cap B} T_u^p : g_\varepsilon(f) D\xi \, dx \to \int_{\Omega \cap B} T_u^p : D\xi \, dx. \tag{3.7.28}$$

On the other hand, using the coarea formula (see [31, Thm 3.2.22 (3)]), the fact that $|Df| = 1$ in V and changing the variables, we get

$$\begin{aligned}
&\int_{\Omega\cap B} \langle g_\varepsilon'(f)Df, T_u^p[\xi]\rangle\, dx \qquad (3.7.29)\\
&= \frac{1}{\varepsilon}\int_0^\varepsilon dt \int_{\{f=t\}\cap B} \langle Df(\sigma), T_u^p[\xi](\sigma)\rangle\, d\mathcal{H}^2(\sigma)\\
&= \frac{1}{\varepsilon}\int_0^\varepsilon dt \int_{\partial\Omega\cap B} \langle Df(y-t\nu(y)), T_u^p[\xi](y)\rangle\, d\mathcal{H}^2(y) + o(1)_{\varepsilon\searrow 0}\\
&\to -\int_{\partial\Omega\cap B} \langle \nu, T_u^p[\xi]\rangle\, d\mathcal{H}^2,
\end{aligned}$$

as $\varepsilon \searrow 0$. In fact, since $u \in W^{1,p}(\Omega\cap B, \mathbb{R}^\nu)$, Du is continuous at every point of $\partial\Omega\cap B$, $\xi \in C_c^1(B,\mathbb{R}^3)$ and $Df(y-t\nu(y)) \to -\nu(y)$ as $t\searrow 0$ for each $y \in \partial\Omega$, the function

$$r \in (0,a] \mapsto \Psi(r) := \int_0^a dt \int_{\partial\Omega\cap B} \langle Df(y-t\nu(y)), T_u^p[\xi](y)\rangle\, d\mathcal{H}^2(y),$$

where $a > 0$ is fairly small, has the right derivative at 0 equal to $\Psi'(0+) = -\int_{\partial\Omega\cap B}\langle \nu, T_u^p[\xi]\rangle\, d\mathcal{H}^2$. Combining (3.7.28) and (3.7.29), we deduce the desired stress-energy identity, which completes our proof of Proposition 3.7.11. □

Now we deduce the normal boundary estimate.

Proposition 3.7.12. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in (1,2]$. Let $K \Subset \partial\Omega\cap B_r^3(x_0)$ be relatively open. If $u \in W^{1,p}(\Omega\cap B_r^3(x_0),\mathbb{R}^\nu)$ satisfies $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(\Omega\cap B_r^3(x_0),\mathbb{R}^3)$ and if Du is continuous at every point of $\partial\Omega\cap B_r^3(x_0)$, then*

$$\begin{aligned}
(3.7.30)\quad &\int_K |Du|^p\, d\mathcal{H}^2\\
&\le \frac{C}{p-1}\left(\int_{\Omega\cap B_r^3(x_0)} \frac{|Du|^p}{p}\, dx + \int_{\partial\Omega\cap B_r^3(x_0)} \frac{|D_\top u|^p}{p}\, d\mathcal{H}^2\right),
\end{aligned}$$

where $C = C(K) > 0$.

To prove Proposition 3.7.12, we need the next estimate.

Lemma 3.7.13. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in [1,2]$. Let $u \in W^{1,p}(\Omega\cap B_r^3(x_0),\mathbb{R}^\nu)$ and Du*

be continuous at every point of $\partial\Omega \cap B_r^3(x_0)$. Then the following estimate holds

$$(p-1)|D_\perp u|^p \leq -p\langle \nu, T_u^p[\nu]\rangle + (3-p)|D_\top u|^p \tag{3.7.31}$$

on $\partial\Omega \cap B_r^3(x_0)$, where $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ stands for the outward pointing unit normal vector field to $\partial\Omega$.

Proof. Applying successively Young's inequality for products, the definition of the stress-energy tensor T_u^p and the subadditivity of the function $t \in (0,\infty) \mapsto t^{p/2}$, we have

$$\begin{aligned}
|D_\perp u|^p &= |D_\perp u|^p |Du|^{\frac{p(p-2)}{2}} |Du|^{\frac{p(2-p)}{2}} \\
&\leq \frac{p}{2}|D_\perp u|^2 |Du|^{p-2} + \frac{2-p}{2}|Du|^p \\
&= -\frac{p}{2}\left(\frac{|Du|^p}{p} - \frac{|D_\perp u|^2}{|Du|^{2-p}}\right) + \frac{3-p}{2}|Du|^p \\
&= -\frac{p}{2}\langle \nu, T_u^p[\nu]\rangle + \frac{3-p}{2}(|D_\top u|^2 + |D_\perp u|^2)^{\frac{p}{2}} \\
&\leq -\frac{p}{2}\langle \nu, T_u^p[\nu]\rangle + \frac{3-p}{2}(|D_\top u|^p + |D_\perp u|^p),
\end{aligned}$$

which yields (3.7.31) by rearranging the terms. □

Proof of Proposition 3.7.12. Let $\theta \in C^1(\partial\Omega, \mathbb{R})$ be such that $\theta = 1$ on K, $0 \leq \theta \leq 1$ and $\theta = 0$ in $\partial\Omega \setminus B_r^3(x_0)$. Fix $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$ such that $\xi = \theta\nu$ on $\partial\Omega$. Applying Proposition 3.7.11 and using the Cauchy-Schwarz inequality, and also the facts that $|\mathrm{Id}| = \sqrt{3}$, $|Du \otimes Du| = |Du|^2$, we get

$$\begin{aligned}
&\left|\int_{\partial\Omega\cap B_r^3(x_0)} \theta\langle \nu, T_u^p[\nu]\rangle \, d\mathcal{H}^2\right| \\
&= \left|\int_{\Omega\cap B_r^3(x_0)} T_u^p : D\xi \, dx\right| \\
&= \left|\int_{\Omega\cap B_r^3(x_0)} \left(\frac{|Du|^p}{p}\mathrm{Id} - \frac{Du \otimes Du}{|Du|^{2-p}}\right) : D\xi \, dx\right| \\
&\leq \int_{\Omega\cap B_r^3(x_0)} \left(\frac{\sqrt{3}}{p} + 1\right) |Du|^p |D\xi| \, dx \\
&\leq (2+\sqrt{3})\|D\xi\|_{L^\infty(B_r^3(x_0), \mathbb{R}^3\otimes\mathbb{R}^3)} \int_{\Omega\cap B_r^3(x_0)} \frac{|Du|^p}{p} \, dx.
\end{aligned} \tag{3.7.32}$$

Setting $E = \{x \in \partial\Omega \cap B^3_r(x_0) : (p-1)|D_\perp u(x)|^2 \geq |D_\top u(x)|^2\}$, observe that $-\langle \nu, T^p_u[\nu]\rangle \geq 0$ on E and hence

$$-\frac{p}{p-1}\int_{K\cap E} \langle \nu, T^p_u[\nu]\rangle \, d\mathcal{H}^2 \leq -\frac{p}{p-1}\int_E \theta \langle \nu, T^p_u[\nu]\rangle \, d\mathcal{H}^2, \tag{3.7.33}$$

where we have used that $K \Subset \partial\Omega \cap B^3_r(x_0)$, $\theta = 1$ on K and $0 \leq \theta \leq 1$. On the other hand, on $(\partial\Omega \cap B^3_r(x_0)) \setminus E$ it holds

$$\begin{aligned} 0 \leq \langle \nu, T^p_u[\nu]\rangle &= |Du|^{p-2}\left(\frac{|D_\top u|^2 + |D_\perp u|^2}{p} - |D_\perp u|^2\right) \\ &\leq |D_\top u|^{p-2}\left(\frac{|D_\top u|^2 + (p-1)|D_\perp u|^2}{p}\right) \leq \frac{2}{p}|D_\top u|^p. \end{aligned} \tag{3.7.34}$$

Hereinafter in this proof, C denotes an absolute positive constant that can be different from line to line. Using Lemma 3.7.13, the estimates (3.7.32), (3.7.33) and (3.7.34), we deduce the following chain of estimates

$$\int_K |Du|^p \, d\mathcal{H}^2 \leq \int_K (|D_\top u|^p + |D_\perp u|^p) \, d\mathcal{H}^2$$

$$\begin{aligned} &\leq -\frac{p}{p-1}\int_K \langle \nu, T^p_u[\nu]\rangle \, d\mathcal{H}^2 + \frac{C}{p-1}\int_K \frac{|D_\top u|^p}{p} \, d\mathcal{H}^2 \\ &\leq -\frac{p}{p-1}\int_{\partial\Omega\cap B^3_r(x_0)} \theta\langle \nu, T^p_u[\nu]\rangle \, d\mathcal{H}^2 \\ &\quad + \frac{C}{p-1}\int_{\partial\Omega\cap B^3_r(x_0)} \frac{|D_\top u|^p}{p} \, d\mathcal{H}^2 \\ &\leq \frac{C}{p-1}\left(\|D\xi\|_\infty \int_{\Omega\cap B^3_r(x_0)} \frac{|Du|^p}{p} \, dx + \int_{\partial\Omega\cap B^3_r(x_0)} \frac{|D_\top u|^p}{p} \, d\mathcal{H}^2\right), \end{aligned}$$

which completes our proof of Proposition 3.7.12. □

Next, we analyze the boundary and inward perturbations for the limit stress-energy tensor, and we prove that the set supporting the associated varifold to this tensor is *minimizing area to first order* in the Brian White sense. First, we analyze the boundary perturbations.

Proposition 3.7.14. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in [1, +\infty)$. If $u \in W^{1,p}(\Omega \cap B^3_r(x_0), \mathbb{R}^\nu)$ satisfies $\operatorname{div}(T^p_u) = 0$ in $\mathscr{D}'(\Omega \cap B^3_r(x_0), \mathbb{R}^3)$ and if Du is continuous at every point of $\partial\Omega \cap B^3_r(x_0)$, then for each $\xi \in C^1_c(B^3_r(x_0), \mathbb{R}^3)$,*

$$\begin{aligned} &\left|\int_{\Omega\cap B^3_r(x_0)} T^p_u : D\xi \, dx\right| \\ &\qquad \leq \left(1 + \frac{1}{p}\right) \|\xi\|_{L^\infty(\partial\Omega\cap B^3_r(x_0), \mathbb{R}^3)} \int_{\partial\Omega\cap B^3_r(x_0)} |Du|^p \, d\mathcal{H}^2. \end{aligned} \tag{3.7.35}$$

Proof. Let $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ be the outward pointing unit normal vector field to $\partial\Omega$ and let $\xi \in C^1_c(B^3_r(x_0), \mathbb{R}^3)$. Applying the triangle and the Cauchy-Schwarz inequalities, we have

$$|\langle \nu, T^p_u[\xi]\rangle| \leq (1 + 1/p)|Du|^p|\xi|$$

on $\partial\Omega \cap B^3_r(x_0)$. Using this estimate and Proposition 3.7.11, we complete our proof of Proposition 3.7.14. □

Proposition 3.7.15. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small, $(p_n)_{n\in\mathbb{N}} \subset [1, 2)$ and $p_n \nearrow 2$ as $n \to +\infty$. Let $(u_n)_{n\in\mathbb{N}}$ be a sequence of mappings such that $u_n \in W^{1,p_n}(\Omega \cap B^3_r(x_0), \mathbb{R}^\nu)$, Du_n be continuous at every point of $\partial\Omega \cap B^3_r(x_0)$ and $\operatorname{div}(T^{p_n}_{u_n}) = 0$ in $\mathscr{D}'(\Omega \cap B^3_r(x_0), \mathbb{R}^3)$. Assume also that $(2 - p_n)T^{p_n}_{u_n} \overset{*}{\rightharpoonup} T_*$ weakly* in $(C(\overline{\Omega \cap B^3_r(x_0)}, M_3(\mathbb{R})))'$. If*

$$\limsup_{n\to+\infty}(2 - p_n)\int_{\Omega\cap B^3_r(x_0)} \frac{|Du_n|^{p_n}}{p_n}\, dx < +\infty \tag{3.7.36}$$

and

$$\limsup_{n\to+\infty}(2 - p_n)\int_{\partial\Omega\cap B^3_r(x_0)} \frac{|D_\top u_n|^{p_n}}{p_n}\, d\mathcal{H}^2 < +\infty, \tag{3.7.37}$$

then there exists a constant $C > 0$ such that for each $\xi \in C^1_c(B^3_r(x_0), \mathbb{R}^3)$ we have

$$\left|\int_{\Omega\cap B^3_r(x_0)} T_* : D\xi\, dx\right| \leq C\|\xi\|_{L^\infty(\partial\Omega\cap B^3_r(x_0),\mathbb{R}^3)}. \tag{3.7.38}$$

Proof. By the weak* convergence and the convexity, and also using Proposition 3.7.12, Proposition 3.7.14 and the estimates (3.7.36), (3.7.37), we obtain the following chain of estimates

$$\begin{aligned}
&\left|\int_{\Omega\cap B^3_r(x_0)} T_* : D\xi\, dx\right| \leq \liminf_{n\to+\infty}\left|(2 - p_n)\int_{\Omega\cap B^3_r(x_0)} T^{p_n}_{u_n} : D\xi\, dx\right| \\
&\leq \limsup_{n\to+\infty} 2\|\xi\|_{L^\infty(\partial\Omega\cap B^3_r(x_0),\mathbb{R}^3)}(2 - p_n)\int_{\partial\Omega\cap B^3_r(x_0)} |Du_n|^{p_n}\, d\mathcal{H}^2 \\
&\leq \limsup_{n\to+\infty} C'\|\xi\|_{L^\infty(\partial\Omega\cap B^3_r(x_0),\mathbb{R}^3)}(2 - p_n)\times \\
&\quad\left(\int_{\Omega\cap B^3_r(x_0)} \frac{|Du_n|^{p_n}}{p_n}\, dx + \int_{\partial\Omega\cap B^3_r(x_0)} \frac{|D_\top u_n|^{p_n}}{p_n}\, d\mathcal{H}^2\right) \\
&\leq C\|\xi\|_{L^\infty(\partial\Omega\cap B^3_r(x_0),\mathbb{R}^3)},
\end{aligned}$$

where $C', C > 0$ are constants independent of ξ. This leads to (3.7.38) and thereby completes our proof of Proposition 3.7.15. □

Now we analyze the inward perturbations.

Proposition 3.7.16. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$, $r > 0$ be fairly small and $p \in [1,2]$. If $u \in W^{1,p}(\Omega \cap B_r^3(x_0), \mathbb{R}^\nu)$ satisfies $\operatorname{div}(T_u^p) = 0$ in $\mathscr{D}'(\Omega \cap B_r^3(x_0), \mathbb{R}^3)$ and if Du is continuous at every point of $\partial\Omega \cap B_r^3(x_0)$, then for each $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$ such that $\langle \xi, \nu \rangle \leq 0$ on $\partial\Omega \cap B_r^3(x_0)$, where $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ denotes the outward pointing unit normal vector field to $\partial\Omega$, we have*

$$\begin{aligned}
&\int_{\Omega \cap B_r^3(x_0)} T_u^p : D\xi \, dx \\
&\qquad \geq -\|\xi\|_{L^\infty(U,\mathbb{R}^3)} \int_U \left((3-p)\frac{|D_\top u|^p}{p} + |D_\perp u|^{p-1}|D_\top u| \right) d\mathcal{H}^2,
\end{aligned}$$

where $U = \partial\Omega \cap B_r^3(x_0)$.

Proof. Using Lemma 3.7.13 and the fact that $\langle \xi, \nu \rangle \leq 0$ on U, we deduce that

$$\begin{aligned}
\langle \nu, T_u^p[\xi] \rangle &= \langle \nu, \xi \rangle \langle \nu, T_u^p[\nu] \rangle - |Du|^{p-2} \langle Du[\xi - \langle \nu, \xi \rangle \nu], Du[\nu] \rangle \\
&\geq \langle \nu, \xi \rangle \left(\frac{1-p}{p} \right) |D_\perp u|^p + \langle \nu, \xi \rangle \left(\frac{3-p}{p} \right) |D_\top u|^p \\
&\quad - \|\xi\|_{L^\infty(U,\mathbb{R}^3)} |D_\perp u|^{p-1} |D_\top u| \\
&\geq -\|\xi\|_{L^\infty(U,\mathbb{R}^3)} \left(\frac{3-p}{p} \right) |D_\top u|^p - \|\xi\|_{L^\infty(U,\mathbb{R}^3)} |D_\perp u|^{p-1} |D_\top u|.
\end{aligned}$$

The conclusion then follows from Proposition 3.7.11. $\square$

The next proposition says that the set supporting the varifold associated to the tensor T_* of Proposition 3.7.15 is *minimizing area to first order* in the Brian White sense (see [58]).

Proposition 3.7.17. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2, $x_0 \in \partial\Omega$ and $r > 0$ be fairly small. Let $(p_n)_{n\in\mathbb{N}}$, $(u_n)_{n\in\mathbb{N}}$ and T_* be as in Proposition 3.7.15. Let $\nu \in C^1(\partial\Omega, \mathbb{S}^2)$ be the outward pointing unit normal vector field to $\partial\Omega$. Assume that*

$$\lim_{n \to +\infty} (2 - p_n) \int_{\partial\Omega \cap B_r^3(x_0)} \frac{|D_\top u_n|^{p_n}}{p_n} \, d\mathcal{H}^2 = 0. \tag{3.7.39}$$

Then for each $\xi \in C_c^1(B_r^3(x_0), \mathbb{R}^3)$ such that $\langle \xi, \nu \rangle \leq 0$ on $\partial\Omega \cap B_r^3(x_0)$,

$$\int_{\Omega \cap B_r^3(x_0)} T_* : D\xi \, dx \geq 0.$$

Proof. Set $U = \partial\Omega \cap B^3_r(x_0)$. Taking into account (3.7.36), (3.7.37) and Proposition 3.7.12, we get

$$A := \limsup_{n\to+\infty}(2-p_n)\int_U |Du_n|^{p_n}\, d\mathcal{H}^2 < +\infty. \tag{3.7.40}$$

Next, using Proposition 3.7.16, Hölder's inequality, (3.7.40) and (3.7.39), we deduce that

$$\begin{aligned}\int_{\Omega\cap B^3_r(x_0)} T_* : D\xi\, dx &\geq -\|\xi\|_{L^\infty(U,\mathbb{R}^3)} \limsup_{n\to+\infty}(2-p_n)\int_U \frac{|D_\top u_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \\ &\quad -\|\xi\|_{L^\infty(U,\mathbb{R}^3)} \limsup_{n\to+\infty}(2-p_n)\|Du_n\|^{1-\frac{1}{p_n}}_{L^{p_n}(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)}\|D_\top u_n\|_{L^{p_n}(U,\mathbb{R}^\nu\otimes\mathbb{R}^3)} \\ &\geq -\|\xi\|_{L^\infty(U,\mathbb{R}^3)} A^{\frac{1}{2}}\left(\limsup_{n\to+\infty}(2-p_n)\int_U \frac{|D_\top u_n|^{p_n}}{p_n}\, d\mathcal{H}^2\right)^{\frac{1}{2}} = 0,\end{aligned}$$

which completes our proof of Proposition 3.7.17. $\square$

We shall say that $\overline{\Omega}$ is *strongly convex* at a point $x_0 \in \partial\Omega$ whenever the two principal curvatures of $\partial\Omega$ at x_0, with respect to the inward pointing unit normal vector to $\partial\Omega$ at x_0, are positive. If Ω is convex and $\partial\Omega$ has the second fundamental form that is positive semidefinite with respect to the inward pointing unit normal vector field to $\partial\Omega$, then $\overline{\Omega}$ is strongly convex at $x_0 \in \partial\Omega$ whenever the Gaussian curvature of $\partial\Omega$ is positive at x_0.

As a result of our analysis in this subsection, we obtain the next boundary repulsion property.

Theorem 3.7.18. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class C^2 such that $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$. Let $g \in \mathcal{R}^1_0(\partial\Omega,\mathcal{N})$ be such that $g \in C^1(\partial\Omega \setminus S(g),\mathcal{N})$. Let $(p_n)_{n\in\mathbb{N}} \subset [1,2)$, $p_n \nearrow 2$ as $n \to +\infty$ and $(u_n)_{n\in\mathbb{N}}$ be a sequence of p_n-minimizers in Ω such that for each $n \in \mathbb{N}$, $\mathrm{tr}_{\partial\Omega}(u_n) = g$. Assume that for each $n \in \mathbb{N}$, Du_n is continuous on $\partial\Omega \setminus S(g)$. Then there exists $\mu_* \in (C(\overline{\Omega}))'$ such that, up to a subsequence,* (3.5.2) *holds. Moreover, S_* lies in the convex hull of $S(g)$ and $S_* \cap \partial\Omega = S(g)$, where S_* is the support of μ_*.*

Remark 3.7.19. According to [35], every p-minimizer u in Ω is locally $C^{1,\alpha}$ regular in $\Omega\backslash S(u)$ for some relatively closed subset $S(u)$ of Ω which has Hausdorff dimension at most 1 when $p \in (1,2]$. Furthermore, p-minimizers with smooth boundary values on smooth domains are Hölder continues near the boundary and, in view of [54], it is reasonable to assume that they are C^1 regular near the boundary. In Theorem 3.7.18, we assume that the gradient of a p-minimizer is continuous on a portion of the boundary, where the boundary datum is smooth. This assumption is significantly less demanding than the assumption that the minimizer is C^1 regular outside the union of its singular set with the singular set of the boundary datum.

Proof of Theorem 3.7.18. Using Proposition 3.2.21, the fact that $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N}) \subset W^{1/2,2}(\partial\Omega, \mathcal{N})$ and the Banach-Alaoglu theorem, we obtain $\mu_* \in (C(\overline{\Omega}))'$ such that, up to a subsequence, (3.5.2) holds. Fix an arbitrary point $x_0 \in \partial\Omega \setminus S(g)$ and a fairly small $r > 0$ such that $\overline{B}_r^3(x_0) \cap S(g) = \emptyset$. In view of Proposition 3.2.21, the fact that $g \in W^{1/2,2}(\partial\Omega, \mathcal{N})$ and Lemma 3.7.8, (3.7.36) and (3.7.37) hold. Then, taking into account (3.5.7), Proposition 3.2.21 and using the Banach-Alaoglu theorem, we deduce that there exists $T_* \in (C_0(\overline{\Omega \cap B_r^3(x_0)}, M_3(\mathbb{R})))'$ such that, up to a subsequence (still denoted by the same index), $(2-p_n)T_{u_n}^{p_n} \overset{*}{\rightharpoonup} T_*$ weakly* in $(C_0(\overline{\Omega \cap B_r^3(x_0)}, M_3(\mathbb{R})))'$. Keeping the same notation, let us define $T_*(\cdot) = T_*(\cdot \cap \overline{\Omega} \cap B_r^3(x_0)) \in (C_0(B_r^3(x_0), M_3(\mathbb{R})))'$. By Proposition 3.7.15, $\operatorname{div}(T_*) \in (C_0(B_r^3(x_0), \mathbb{R}^3))'$. Then, according to [6, Proposition 3.1], there exists an $\mathcal{H}^1$-rectifiable set $R \subset B_r^3(x_0) \cap \overline{\Omega}$ and a map $\lambda \in L^\infty(R, M_3(\mathbb{R}))$ satisfying the following conditions. For $\mathcal{H}^1$-a.e. $x_0 \in R$, $|\lambda(x_0)| = 1$, $\lambda(x_0) \in \mathrm{G}(3,1)$ is the orthogonal projection matrix onto $T_{x_0}R$ such that $T_* \llcorner \{\Theta_1^*(|T_*|) > 0\}(dx) = \Theta_1^*(|T_*|, x)\lambda(x)\mathcal{H}^1 \llcorner R(dx)$, where $T_{x_0}R$ is the approximate tangent plane to R at x_0. One has that $|T_*| = \mu_*$, $\Theta_1^*(|T_*|) = \Theta_1(\mu_*)$ and $R = S_* \cap B_r^3(x_0)$ (we refer the reader to the proof of Proposition 3.5.8). In particular, we deduce that $V_*(dA, dx) = \delta_{\lambda_*(x)}(dA) \otimes |T_*|(dx)$ is a 1-dimensional rectifiable varifold (see [2, 27]). Since $g \in C^1(\partial\Omega \cap B_r^3(x_0), \mathcal{N})$, $(u_n)_{n\in\mathbb{N}}$ fulfill the condition (3.7.39) of Proposition 3.7.17, which says that V_* is *minimizing area to first order* in $(\Omega \cap B_r^3(x_0)) \cup (\partial\Omega \cap B_r^3(x_0))$ in the Brian White sense (see [58]). Then, by [58, Theorem 1], $x_0 \notin S_*$. Thus, $S_* \cap \partial\Omega \subset S(g)$. Denote by E the convex hull of $S(g)$. By contradiction, assume that $S_* \cap (\Omega \setminus E) \neq \emptyset$. Then, since $S_* \cap \partial\Omega \subset S(g)$, the set M of points $x_0 \in S_*$ such that $\operatorname{dist}(x_0, E) = \max\{\operatorname{dist}(x, E) : x \in S_*\}$ is a subset of $\Omega \setminus E$. Fix a point $x_0 \in M$ such that the segments of S_* emanating from x_0 do not lie in the same plane. Then these segments lie in the same half-space (lying under a 2-plane parallel to some face of E) and form a vertex, which cannot happen in view of the stationarity of the varifold coming from Proposition 3.5.8 (see Corollary 3.5.19). Therefore, $S_* \subset E$.

Next, let us prove that $S(g) \subset S_* \cap \partial\Omega$. Using Corollary 3.7.7 and the fact that $S_* \cap \partial\Omega \subset S(g)$, for all fairly small $\delta > 0$, we have $\mathrm{tr}_{\partial\Omega \setminus S_g^\delta}(u_*) = g|_{S_g^\delta}$, where u_* is a map coming from Corollary 3.7.7 and $S_g^\delta = \{x \in \partial\Omega : \operatorname{dist}(x, S(g)) \geq \delta\}$. For each $a \in S(g)$, denoting by $\partial B_\varrho^{\partial\Omega}(a)$ the boundary of the geodesic ball in $\partial\Omega$ with center a and radius $\varrho > 0$, we know that $u_*|_{\partial B_\varrho^{\partial\Omega}(a)} = g|_{\partial B_\varrho^{\partial\Omega}(a)}$ is not nullhomotopic for each sufficiently small $\varrho > 0$. This, together with the fact that $S_* \subset E$ and Corollary 3.5.18, implies that $a \in S_* \cap \partial\Omega$ and hence $S(g) \subset S_* \cap \partial\Omega$. Therefore, $S_* \cap \partial\Omega = S(g)$, which completes our proof of Theorem 3.7.18. □

3.7.5 Structure of the singular set and the limit measure

Theorem 3.7.20. *Let Ω, g, $S(g)$, $(u_n)_{n\in\mathbb{N}}$, μ_* and S_* be as in Theorem 3.7.18. Let u_* be the map of Proposition 3.5.6 being an element of $W^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$. Then S_* is a finite union of finite chains of segments connecting the points of $S(g)$ and lying in the convex hull of $S(g)$. More precisely, there exists a finite set of segments $L_1, \dots, L_n \subset \overline{\Omega}$ such that $S_* = \bigcup_{i=1}^n L_i$ and $u_* \in C^\infty(\Omega \setminus (\bigcup_{i=1}^n L_i \cup S_0), \mathcal{N})$, where S_0 is a locally finite subset of $\Omega \setminus \bigcup_{i=1}^n L_i$. If x is a point inside L_i, then $\Theta_1(\mu_*, x) = \mathcal{E}_2^{\mathrm{sg}}([u_*|_{\partial D}]) =: \lambda_i$, where D is a closed 2-disk in Ω such that $D \cap S_* = \{x\}$, $\partial D \cap S_0 = \emptyset$ and x is an interior point of D. If x is a branching point of S_*, namely there exists $L_{i_1}, \dots, L_{i_k}$ emanating from x, then $\sum_{j=1}^k \lambda_{i_j} v_{i_j} = 0$, where v_{i_j} is the unit direction vector of L_{i_j} pointing from x. Furthermore, $\mu_* = \sum_{i=1}^n \lambda_i \mathcal{H}^1 \llcorner L_i$. Finally, for each $i \in \{1, \dots, n\}$, each endpoint of L_i is either a branching point of S_* lying in Ω or it belongs to $S(g)$. If x is an endpoint of some L_i lying in $S(g)$, then $[g|_{\partial D_x}] = [u_*|_{\partial D}]$, where D_x is the geodesic disk on $\partial\Omega$ with center x and lying far from $S(g) \setminus \{x\}$, and D is a closed 2-disk in Ω centered inside L_i with ∂D lying outside the convex hull of $S(g)$ and outside of S_0.*

Proof. Denote by E the convex hull of $S(g)$. The fact that $S_* \subset E$ comes from Theorem 3.7.18. Fix an arbitrary $x_0 \in S(g)$. Further in this proof, the center of each ball and sphere is at x_0. In view of (3.5.1), for each $r_0 > 0$,

$$\limsup_{n\to+\infty} (2 - p_n) \int_{B_{r_0}^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, dx < +\infty.$$

Then, up to extracting a subsequence (not relabeled) and using the coarea formula, together with the Fatou lemma, we deduce that $\mathrm{tr}_{\partial B_r^3 \cap \Omega}(u_n) = u_n|_{\partial B_r^3 \cap \Omega} \in W^{1,p_n}(\partial B_r^3 \cap \Omega, \mathcal{N})$ and

$$\limsup_{n\to+\infty} (2 - p_n) \int_{\partial B_r^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, d\mathcal{H}^2 < +\infty$$

for $\mathcal{H}^1$-a.e. $r \in (0, r_0]$. Let $r_0 > 0$ be small enough so that $\overline{B}_{r_0}^3$ does not contain any point of $S(g)$ except of x_0. Furthermore, by virtue of Proposition 3.5.17 and the fact that $S_* \subset E$, we can also assume that

$\partial B^3_{r_0}$ does not contain any branching point of S_*. Next, we recall that $\mu_* \llcorner \Omega(dx) = \Theta_1(x)\mathcal{H}^1 \llcorner (S_* \cap \Omega)(dx)$, where the density Θ_1 is defined in (3.5.5). For each $r \in (0, r_0]$, define $M(r) = \int_{S_* \cap \partial B^3_r} \Theta_1(x)\, d\mathcal{H}^0(x)$, where $\mathcal{H}^0$ denotes the 0-dimensional Hausdorff measure in $\mathbb{R}^3$, namely, for a discrete set of points $X \subset \mathbb{R}^3$, $\mathcal{H}^0(X) = \#X$ (the cardinality of X). In view of Lemma 3.2.8, Proposition 3.5.16, Proposition 3.5.17 and the fact that $S_* \subset E$, M is a bounded from below piecewise constant function with a discrete set of values. Indeed, there exists an at most countable collection of intervals $\{(a_i, b_i] : i \in \mathbb{N} \cap [0, i_0]\}$, where $i_0 \in \mathbb{N} \cup \{+\infty\}$, such that $b_0 = r_0$, $(0, r_0] = \bigcup_{i=0}^{i_0}(a_i, b_i]$ and for each $i \in \mathbb{N} \cap [0, i_0]$: M is constant on (a_i, b_i); $a_i = b_{i+1}$; for each $r \in (a_i, b_i)$, ∂B^3_r does not meet any branching point of S_*; $\partial B^3_{a_i}$ meets a branching point of S_*. By definition of r_0, M is constant on $(a_0, b_0]$. Notice that, by the upper semicontinuity of Θ_1 (see, for instance, [3, 2(8)]) and Corollary 3.5.18, for each $i \in \mathbb{N} \cap [0, i_0]$ and $x \in (a_i, b_i) \cup (a_{i+1}, b_{i+1})$, $M(x) \leq M(b_i)$. Also, taking into account Proposition 3.5.17 and the facts that $S_* \subset E$, $E \cap \partial\Omega = S(g)$ (since $\overline{\Omega}$ is strongly convex at every point of $\partial\Omega$), we observe that $\mu_*(\partial(B^3_r \cap \Omega)) = 0$ for each $r \in (0, r_0]$. This, together with the weak* convergence of μ_n to μ_* (see (3.5.2)), implies that for each $i \in \mathbb{N} \cap [0, i_0]$,

$$\begin{aligned}\mu_n((B^3_{b_i} \setminus B^3_{a_i}) \cap \Omega) &- \mu_*((B^3_{b_i} \setminus B^3_{a_i}) \cap \Omega)\\ &= \int_{a_i}^{b_i} dr \left((2 - p_n) \int_{\partial B^3_r \cap \Omega} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 - M(r) \right) \to 0\end{aligned}$$

as $n \to +\infty$, where we have used the coarea formula. Applying the Fatou lemma, up to a subsequence (not relabeled), we get

$$\int_{a_i}^{b_i} dr \left(\limsup_{n \to +\infty} (2 - p_n) \int_{\partial B^3_r \cap \Omega} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 - M(r) \right) \leq 0. \tag{3.7.41}$$

Fix an arbitrary $i \in \mathbb{N} \cap [0, i_0]$. Then either for $\mathcal{H}^1$-a.e. $r \in [a_i, b_i]$,

$$\limsup_{n \to +\infty} (2 - p_n) \int_{\partial B^3_r \cap \Omega} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 \geq M(r), \tag{3.7.42}$$

or there exists a set $I \subset [a_i, b_i]$ of positive $\mathcal{H}^1$-measure such that for each $r \in I$, (3.7.42) does not hold. In the case when (3.7.42) holds for $\mathcal{H}^1$-a.e. $r \in [a_i, b_i]$, according to (3.7.41), we have

$$\limsup_{n \to +\infty} (2 - p_n) \int_{\partial B^3_r \cap \Omega} \frac{|Du_n|^{p_n}}{p_n}\, d\mathcal{H}^2 = M(r)$$

for $\mathcal{H}^1$-a.e. $r \in [a_i, b_i]$.

Thus, there exists $r \in (a_i, b_i)$ such that $\mathrm{tr}_{\partial B_r^3 \cap \Omega}(u_n) = u_n|_{\partial B_r^3 \cap \Omega} \in W^{1,p_n}(\partial B_r^3 \cap \Omega, \mathcal{N})$ for each n and

$$\limsup_{n \to +\infty} (2 - p_n) \int_{\partial B_r^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, d\mathcal{H}^2 \leq M(r). \tag{3.7.43}$$

Next, we construct a competitor v_n for u_n. Define $v_n = u_n$ in $\Omega \setminus B_r^3$. For each x in the cone $K = \{x \in B_r^3 \cap \Omega : (x_0 + r(x - x_0)/|x - x_0|) \in \partial B_r^3 \cap \overline{\Omega}\}$, set

$$v_n(x) = \mathrm{tr}_{\partial B_r^3 \cap \overline{\Omega}}(u_n) \left(x_0 + \frac{r(x - x_0)}{|x - x_0|} \right). \tag{3.7.44}$$

In the region $V = (B_r^3 \cap \Omega) \setminus K$, we define v_n by homotopy, using [12, Theorem 2]. More precisely, fix a point $x \in V$ and consider a 2-plane passing through x, which is orthogonal to the axis of the cone K. The intersection of this 2-plane with V is an annulus whose boundary consists of two curves $\gamma_1 \subset \partial\Omega$ and $\gamma_2 \subset \partial K$. By definition, $g|_{\gamma_1}$ is continuously homotopic to $v_n|_{\gamma_2}$. Then, by [12, Theorem 2], on this annulus there exists a $W^{1,2}$ map with values in $\mathcal{N}$ and with boundary values $g|_{\gamma_1}$ and $v_n|_{\gamma_2}$ on γ_1 and γ_2, respectively. Observe that the set of values of $v_n|_{\gamma_2}$ coincides with the set of values of $g|_{\partial B_r^3 \cap \partial\Omega}$. Continuing this process, namely filling V, we obtain a map $v_n|_V$ such that $v_n|_V \in W^{1,2}(V \setminus \overline{B}_\delta^3, \mathcal{N})$ for each $\delta \in (0, r)$ and

$$\limsup_{n \to +\infty} (2 - p_n) \int_V \frac{|D(v_n|_V)|^{p_n}}{p_n} \, dx = 0, \tag{3.7.45}$$

which is possible due to the fact that $g \in \mathcal{R}_0^1(\partial\Omega, \mathcal{N})$ (see Lemma 3.7.8), since the p_n-energy of $v_n|_V$ in $V \cap B_\delta^3$ is of order $\delta/(2 - p_n)$ for $\delta \in (0, r)$. Thus, we have constructed a competitor v_n for u_n.

Next, again by the weak* convergence of μ_n to μ_* and the fact that $\mu_*(\partial(B_r^3 \cap \Omega)) = 0$, we have

$$\int_0^r M(\varrho) \, d\varrho = \mu_*(B_r^3 \cap \Omega) = \lim_{n \to +\infty} (2 - p_n) \int_{B_r^3 \cap \Omega} \frac{|Du_n|^{p_n}}{p_n} \, dx. \tag{3.7.46}$$

Since v_n is a competitor for u_n, $v_n = u_n$ in $\Omega \setminus B_r^3$, using (3.7.43), (3.7.44), (3.7.45) and (3.7.46), we deduce that

$$\begin{aligned} \int_0^r M(\varrho) \, d\varrho &\leq \limsup_{n \to +\infty} (2 - p_n) \int_K \frac{|Dv_n|^{p_n}}{p_n} \, dx \\ &\leq \limsup_{n \to +\infty} \int_0^r \varrho^{2 - p_n} \, d\varrho \, (2 - p_n) \int_{\partial B_r^3 \cap \Omega} \frac{|D_\top u_n|^{p_n}}{p_n} \, d\mathcal{H}^2 \\ &\leq r M(r), \end{aligned} \tag{3.7.47}$$

where we have also used that $p_n \nearrow 2$ and $|D_\top u_n| \leq |Du_n|$. We can choose $i \in \mathbb{N} \cap [0, i_0]$ and $r > 0$ so that, apart from (3.7.47), $M(r)$ is the minimal value of M on $(0, r_0]$. In fact, since M is bounded from below and has a discrete set of values, it achieves its minimum at every point of the interval (a_i, b_i) for some $i \in \mathbb{N} \cap [0, i_0]$ (notice that it can achieve the minimum at a_i or b_i, but in this case it achieves its minimum at every point of (a_i, b_i), since $M(x) \leq \min\{M(a_i), M(b_i)\}$ as was observed earlier). Thus, taking such $i \in \mathbb{N} \cap [0, i_0]$ and $r \in (a_i, b_i)$, it holds $M = M(r)$ on $(0, r]$, because otherwise (3.7.47) would not hold. Then, since M has a discrete set of values, there exists a bounded set $J \subset \mathbb{N}$ such that for each $\varrho \in (0, r]$, the number of branching points of $S_* \cap \partial B^3_\varrho$ belongs to J. This, together with the stationarity of the associated varifold V_* (see Corollary 3.5.19), implies that the only possible situation describing $S_* \cap \overline{B}^3_r$ is when $S_* \cap \overline{B}^3_r$ is the union of closed straight line segments connecting the points of $S_* \cap \partial B^3_r$ with x_0 and whose relative interiors are mutually disjoint.

We complete our proof of Theorem 3.7.20 using Prop. 3.5.6, 3.5.17, Cor. 3.5.18, 3.5.19, 3.7.7 and Thm 3.7.18. $\square$

The next corollary is a direct consequence of Theorem 3.7.20, so we omit its proof.

Corollary 3.7.21. If the set $S(g)$ of Theorem 3.7.20 lies in a 2-plane, then S_* is the segment connecting the points of $S(g)$ if $S(g)$ has only two points, is contained in the triangle with vertices from $S(g)$ if $S(g)$ has only three points and is contained in the polygon with vertices from $S(g)$ if $S(g)$ has more than four points.

We provide a revised version of Corollary 3.7.21 for the case when $S(g)$ has only two points, also without proof.

Corollary 3.7.22. Let Ω, g and $(u_n)_{n\in\mathbb{N}}$ be as in Theorem 3.7.18. Let $S(g) = \{a, b\}$. Then $S_* = [a, b]$ and for each $x \in (a, b)$, $\Theta_1(\mu_*, x) = \mathcal{E}^{\mathrm{sg}}_2([g|_{\partial D_a}]) = \mathcal{E}^{\mathrm{sg}}_2([g|_{\partial D_b}])$, where for $z \in \{a, b\}$, D_z is a geodesic (closed) disk in $\partial\Omega$ with center z, not containing b if $z = a$ and a if $z = b$. Furthermore, $\mu_* = \mathcal{E}^{\mathrm{sg}}_2([g|_{\partial D_a}])\mathcal{H}^1 \llcorner [a, b]$.

3.7.6 Minimality of the singular set

Now we prove the global minimality of S_*.

Proposition 3.7.23. *Let Ω, g, $S(g)$, $(u_n)_{n\in\mathbb{N}}$, S_* and μ_* be as in Theorem 3.7.18. Let u_* be the map of Proposition 3.5.6 being an element of $W^{1,2}_{\mathrm{loc}}(\Omega \setminus S_*, \mathcal{N})$. Then $S_* \in \mathscr{C}(\Omega, g)$ and for each $S \in \mathscr{C}(\Omega, g)$,*

$$\mu_*(S_*) = \mathbb{M}(g, S_*) \leq \mathbb{M}(g, S).$$

Proof. According to Theorem 3.7.20, $S_* \in \mathscr{C}(\Omega, g)$ (see Definition 3.6.1). Let $S \in \mathscr{C}(\Omega, g)$. By reproducing the *Step 2* of the proof of Proposition 3.6.4, we obtain a sequence of competitors $v_n \in W^{1,p_n}(\Omega, \mathcal{N})$ for u_n such that

$$\limsup_{n \to +\infty} (2 - p_n) \int_\Omega \frac{|Dv_n|^{p_n}}{p_n}\, dx \le \mathbb{M}(g, S), \tag{3.7.48}$$

such reproduction is possible thanks to Lemma 3.7.8. By Theorem 3.7.20 and Definition 3.6.3, $\mu_*(S_*) = \mathbb{M}(g, S_*)$. Using this, (3.5.2), the fact that v_n is a competitor for u_n and (3.7.48), we obtain

$$\begin{aligned} \mathbb{M}(g, S_*) = \lim_{n \to +\infty} (2 - p_n) \int_\Omega \frac{|Du_n|^{p_n}}{p_n}\, dx \\ &\le \limsup_{n \to +\infty} (2 - p_n) \int_\Omega \frac{|Dv_n|^{p_n}}{p_n}\, dx \le \mathbb{M}(g, S), \end{aligned}$$

which completes our proof of Proposition 3.7.23. □

For convenience, we provide the definition of a *minimal connection.*

Definition 3.7.24. A minimal connection of a set $\{a_1, \ldots, a_{2n}\} \subset \mathbb{R}^3$ is a union S of mutually disjoint closed segments, each of which connects two points from $\{a_1, \ldots, a_{2n}\}$ so that the sum of the lengths of these segments is minimal among all possible such connections.

Proposition 3.7.25. *Let $\Omega, g, S(g), S_*$ be as in Theorem 3.7.18, $S(g) \neq \emptyset$ and $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$. Then $S(g)$ has an even number of elements and S_* is a minimal connection of $S(g)$ in the sense of Definition 3.7.24.*

Proof. First, notice that $\#S(g)$ is even, because otherwise, using the fact that $\pi_1(\mathcal{N}) \simeq \mathbb{Z}/2\mathbb{Z}$, we would obtain a map which is not nullhomotopic, but having zero degree, which is a contradiction. Next, in view of Corollary 3.6.8 and Theorem 3.7.20, S_* is a finite union of segments whose relative interiors are mutually disjoint, and each of these segments connects two points from $S(g)$. Let S be a union of mutually disjoint closed segments, each of which connects two points from $S(g)$. Then, according to Proposition 3.6.6 and Definition 3.6.1, $S \in \mathscr{C}(\Omega, g)$. By Proposition 3.7.23,

$$\mathbb{M}(g, S_*) \le \mathbb{M}(g, S).$$

Since $\pi_1(\mathcal{N})$ has only one nontrivial element, the above estimate implies that $S_* \in \mathscr{C}(\Omega, g)$ is a minimal connection of $S(g)$ in the sense of Definition 3.7.24. This completes our proof of Proposition 3.7.25. □

3.7.7 Proof of Theorem 3.1.2

Finally, we prove our second main theorem.

Proof of Theorem 3.1.2. The proof follows from Propositions 3.7.9, 3.7.23 and Theorem 3.7.20. □

A Auxiliary results

We provide a proof of the following lemma leading to Corollary A.2, which was used in the proofs of Lemmas 3.2.13, 3.2.15.

Lemma A.1. *Let $m \in \mathbb{N} \setminus \{0,1\}$, $p \in (1,+\infty)$ and $U \subset \mathbb{R}^{m-1}$ be open and convex. Let $\varphi : U \to \mathbb{R}^m$ be a bilipschitz mapping and $E = \varphi(U)$. Then there exists a constant $C = C(m) > 0$ such that for every $u \in W^{1,p}(E, \mathbb{R}^k)$,*

$$|u|^p_{W^{1-1/p,p}(E)} \leq \frac{2^p}{p-1} L^{4m-5+2p} C \|u\|_{L^p(E)} \, \|D_\top u\|^{p-1}_{L^p(E)} \tag{A.1}$$

and

$$\begin{aligned} |u|^p_{W^{1-1/p,p}(E)} \leq{} & 2^p L^{2m-3+p} C \Big(\frac{2}{\operatorname{diam}(U)}\Big)^{p-1} \|u\|^p_{L^p(E)} \\ & + L^{4m-5+2p} C \Big(\frac{\operatorname{diam}(U)}{2}\Big) \|D_\top u\|^p_{L^p(E)}, \end{aligned} \tag{A.2}$$

where $|u|^p_{W^{1-1/p,p}(E)}$ denotes the integral

$$\int_E \int_E \frac{|u(y) - u(x)|^p}{|y-x|^{m-2+p}} \, d\mathcal{H}^{m-1}(y) \, d\mathcal{H}^{m-1}(x)$$

and $L > 0$ is a bilipschitz constant of φ.

Proof. Let $\|D_\top u\|_{L^p(E)} > 0$, because otherwise (A.1) and (A.2) are satisfied. According to [31, Theorem 3.2.5, p. 244], we have

$$\begin{aligned} & |u|^p_{W^{1-1/p,p}(E)} \\ & = \int_U \int_U \frac{|u(\varphi(y)) - u(\varphi(x))|^p}{|\varphi(y) - \varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x)) \, dy \, dx, \end{aligned} \tag{A.3}$$

where J_{m-1} denotes the $(m-1)$-dimensional Jacobian (see, for instance, Section 3.2.1, p. 241 in [31]). We define for each $\delta > 0$ the sets

$$W_\delta = \{(x,y) \in U^2 : |y-x| \geq \delta\} \quad \text{and} \quad F_\delta = \{(x,y) \in U^2 : |y-x| < \delta\}.$$

Then $U^2 = W_\delta \cup F_\delta$. Hereinafter in this proof, C denotes a positive constant that can only depend on m and can be different from line to line. We first prove (A.1). For each $\delta > 0$,

$$
\begin{aligned}
&\iint_{W_\delta} \frac{|u(\varphi(y)) - u(\varphi(x))|^p}{|\varphi(y) - \varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))\, dy\, dx \qquad \text{(A.4)}\\
&\le 2^{p-1} \iint_{W_\delta} \frac{|u(\varphi(y))|^p + |u(\varphi(x))|^p}{|\varphi(y) - \varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))\, dy\, dx\\
&\le 2^{p-1} L^{m-2+p} \iint_{W_\delta} \frac{|u(\varphi(y))|^p + |u(\varphi(x))|^p}{|y - x|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))\, dy\, dx\\
&\le 2^p L^{2m-3+p} \int_E |u|^p\, d\mathcal{H}^{m-1} \int_{\mathbb{R}^{m-1} \setminus B^{m-1}_\delta} \frac{dz}{|z|^{m-2+p}}\\
&= \frac{2^p C L^{2m-3+p}}{(p-1)\delta^{p-1}} \int_E |u|^p\, d\mathcal{H}^{m-1},
\end{aligned}
$$

where we have used that φ is bilipschitz with the bilipschitz constant L and [31, Theorem 3.2.5]. Next, using Jensen's inequality, Fubini's theorem and the fact that φ is bilipschitz with the bilipschitz constant L, we have

$$
\begin{aligned}
&\iint_{F_\delta} \frac{|u(\varphi(y)) - u(\varphi(x))|^p}{|\varphi(y) - \varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))\, dy\, dx \qquad \text{(A.5)}\\
&= \iint_{F_\delta} \left| \int_0^1 \frac{\partial}{\partial t} u(\varphi((1-t)x + ty))\, dt \right|^p \frac{J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))}{|\varphi(y) - \varphi(x)|^{m-2+p}}\, dy\, dx\\
&\le L^{4m-5+2p} \int_0^1 dt\\
&\quad \left(\iint_{F_\delta} \frac{|D_\top u(\varphi((1-t)x + ty))|^p |y - x|^p}{|y - x|^{m-2+p}} J_{m-1}(\varphi((1-t)x + ty))\, dy\, dx \right)\\
&\le L^{4m-5+2p} \int_0^1 dt\\
&\quad \left(\iint_{F_\delta} \frac{|D_\top u(\varphi((1-t)x + ty))|^p}{|y - x|^{m-2}} J_{m-1}(\varphi((1-t)x + ty))\, dy\, dx \right).
\end{aligned}
$$

Performing the change of variable $z = (1-t)x + ty$ in the integral with respect to x, we have

$$
\begin{aligned}
\text{(A.6)} \quad &\iint_{F_\delta} \frac{|D_\top u(\varphi((1-t)x + ty))|^p}{|y - x|^{m-2}} J_{m-1}(\varphi((1-t)x + ty))\, dy\, dx\\
&= \frac{1}{1-t} \iint_{F_{(1-t)\delta}} \frac{|D_\top u(\varphi(z))|^p}{|y - z|^{m-2}} J_{m-1}(\varphi(z))\, dy\, dz.
\end{aligned}
$$

Therefore, integrating (A.6) with respect to $t \in [0,1]$, we get

(A.7)
$$\begin{aligned}
&\int_0^1 \left(\iint_{F_\delta} \frac{|D_\top u(\varphi((1-t)x+ty))|^p}{|y-x|^{m-2}} J_{m-1}(\varphi((1-t)x+ty))\,dy\,dx \right) dt \\
&\quad = \int_0^1 \frac{1}{1-t} \iint_{F_{(1-t)\delta}} \frac{|D_\top u(\varphi(z))|^p}{|y-z|^{m-2}} J_{m-1}(\varphi(z))\,dy\,dz\,dt \\
&\quad \le \int_E |D_\top u|^p \,d\mathcal{H}^{m-1} \int_0^1 \frac{1}{1-t} \int_{B^{m-1}_{(1-t)\delta}} \frac{1}{|\xi|^{m-2}}\,d\xi\,dt \\
&\quad \le C\delta \int_E |D_\top u|^p \,d\mathcal{H}^{m-1}.
\end{aligned}$$

Then, taking $\delta > 0$ such that

$$\delta = \frac{\|u\|_{L^p(E)}}{\|D_\top u\|_{L^p(E)}},$$

we obtain (A.1) from (A.3), (A.4), (A.5) and (A.7).

Next, we prove (A.2). Let us now choose $\delta = \operatorname{diam}(U)/2$. Then, changing $\int_{\mathbb{R}^{m-1}\setminus B^{m-1}_\delta} \frac{dz}{|z|^{m-2+p}}$ to $\int_{B^{m-1}_{2\delta}\setminus B^{m-1}_\delta} \frac{dz}{|z|^{m-2+p}}$ and using that $\frac{1-(1/2)^{p-1}}{p-1} \le \ln(2)$ in (A.4), we obtain

$$\begin{aligned}
&\iint_{W_\delta} \frac{|u(\varphi(y))-u(\varphi(x))|^p}{|\varphi(y)-\varphi(x)|^{m-2+p}} J_{m-1}(\varphi(y)) J_{m-1}(\varphi(x))\,dy\,dx \\
&\qquad\qquad \le 2^p \ln(2) L^{2m-3+p} C \left(\frac{1}{\delta}\right)^{p-1} \int_E |u|^p \,d\mathcal{H}^{m-1},
\end{aligned}$$

which, together with (A.7), implies (A.2). This completes our proof of Lemma A.1. $\square$

Corollary A.2. Let $p \in (1,2]$. Then there exists a constant $C = C(m) > 0$ such that for every $u \in W^{1,p}(\partial B^m_r(x_0), \mathbb{R}^k)$,

$$|u|^p_{W^{1-1/p,p}(\partial B^m_r(x_0))} \le Cr \|D_\top u\|^p_{L^p(\partial B^m_r(x_0))}. \tag{A.8}$$

Furthermore, if $\mathcal{Y}$ is a compact Riemannian manifold, which is isometrically embedded into $\mathbb{R}^k$, $p \in [p_0, 2]$ for some $p_0 \in (1,2]$ and $u \in W^{1,p}(\partial B^m_r(x_0), \mathcal{Y})$, then there exists a constant $C = C(p_0, m, \mathcal{Y}) > 0$ such that

$$|u|^p_{W^{1-1/p,p}(\partial B^m_r(x_0))} \le Cr^{\frac{m-1}{p}} \|D_\top u\|^{p-1}_{L^p(\partial B^m_r(x_0))}. \tag{A.9}$$

Proof of Corollary A.2. First, let us prove (A.8). By scaling and translation, it is enough to show that for each $u \in W^{1,p}(\mathbb{S}^{m-1}, \mathbb{R}^k)$,

$$|u|^p_{W^{1-1/p,p}(\mathbb{S}^{m-1})} \leq C\|D_\top u\|^p_{L^p(\mathbb{S}^{m-1})}. \tag{A.10}$$

Recall that $m \geq 2$ and for each $j \in \{1, \dots, m\}$, define $E_{2j-1} = \{x \in \mathbb{S}^{m-1} : x_j < 3/4\}$ and $E_{2j} = \{x \in \mathbb{S}^{m-1} : x_j > -3/4\}$. For every $x \in \mathbb{S}^{m-1}$, there is at most one $i \in \{1, \dots, 2m\}$ such that $x \notin E_i$. Thus, $\mathbb{S}^{m-1} \times \mathbb{S}^{m-1} = \bigcup_{i=1}^{2m} E_i \times E_i$. Moreover, the E_i's are bilipschitz images of the unit ball B_1^{m-1}.

Hereinafter in this proof, C will denote a positive constant that can depend only on m and can be different from line to line. Then, in view of (A.2) with $U = B_1^{m-1}$, we have

$$|u|^p_{W^{1-1/p,p}(E_i)} \leq C(\|u\|^p_{L^p(E_i)} + \|D_\top u\|^p_{L^p(E_i)}). \tag{A.11}$$

Applying (A.11) with $v = u - \frac{1}{\mathcal{H}^{m-1}(E_i)} \int_{E_i} u \, d\mathcal{H}^{m-1}$ and using the Poincaré inequality for v (observe that the constant in the Poincaré inequality can be chosen independently of $p \in (1, 2]$), we get the estimate $|u|^p_{W^{1-1/p,p}(E_i)} \leq C\|D_\top u\|^p_{L^p(E_i)}$. Then, summing over i, we obtain (A.10), which implies (A.8).

Next, we prove (A.9). To this end, we show that if $p \in [p_0, 2]$ and $u \in W^{1,p}(\partial B_r^m(x_0), \mathcal{Y})$, then

$$|u|^p_{W^{1-1/p,p}(\partial B_r^m(x_0))} \leq C'\|u\|_{L^p(\partial B_r^m(x_0))}\|D_\top u\|^{p-1}_{L^p(\partial B_r^m(x_0))}, \tag{A.12}$$

where $C' = C'(p_0, m) > 0$. Notice that if (A.12) would hold, then it would be "scale-invariant". Thus, it is enough to prove (A.12) in the case when $\partial B_r^m(x_0) = \mathbb{S}^{m-1}$. Applying Lemma A.1 with $E = E_i$ as above for each $i \in \{1, \dots, 2m\}$ and then summing over i, we get

$$|u|^p_{W^{1-1/p,p}(\mathbb{S}^{m-1})} \leq \frac{C}{p_0 - 1}\|u\|_{L^p(\mathbb{S}^{m-1})}\|D_\top u\|^{p-1}_{L^p(\mathbb{S}^{m-1})}. \tag{A.13}$$

Setting $C' = C/(p_0 - 1) > 0$, (A.13) yields (A.12). The inequality (A.9) is a direct consequence of (A.12) and the fact that $\mathcal{Y}$ is a compact Riemannian manifold, which is isometrically embedded into $\mathbb{R}^k$. This completes our proof of Corollary A.2. $\square$

Bibliography of Chapter 3

[1] R. A. Adams. *Sobolev spaces.* Academic Press. 1975.

[2] W. K. Allard. "On the first variation of a varifold". In: *Ann. Math.* 95.3 (1972), pp. 417–491.

[3] W. K. Allard and F. J. Almgren Jr. “The structure of stationary one dimensional varifolds with positive density”. In: *Invent. Math.* 34.2 (1976), pp. 83–97.

[4] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] L. Ambrosio and H. M. Soner. “A measure theoretic approach to higher codimension mean curvature flows”. In: *Ann. Scuola Norm. Sup. Pisa CI. Sci* 25.4 (1997), pp. 27–49.

[6] A. Arroyo-Rabasa, G. De Philippis, J. Hirsch, and F. Rindler. “Dimensional estimates and rectifiability for measures satisfying linear PDE constraints”. In: *Geom. Funct. Anal.* 29.3 (2019), pp. 639–658.

[7] P. Auscher. *Real Harmonic Analysis.* ANU Press, 2012.

[8] F. Bethuel. “A new obstruction to the extension problem for Sobolev maps between manifolds”. In: *J. Fixed Point Theory Appl.* 15.1 (2014), pp. 155–183.

[9] F. Bethuel. “The approximation problem for Sobolev maps between two manifolds”. In: *Acta Math.* 167 (1991), pp. 153–206.

[10] F. Bethuel, H. Brezis, and F. Hélein. *Ginzburg-Landau vortices.* Vol. 13. Progress in Nonlinear Differential Equations and their Applications. Birkhäuser Boston, Inc., Boston, Mass., 1994, pp. xxviii+159.

[11] F. Bethuel and D. Chiron. *Some questions related to the lifting problem in Sobolev spaces.* In: Perspectives in Nonlinear Partial Differential Equations, Contemp. Math., 446. Amer. Math. Soc., Providence, RI, 2007, pp. 125–152.

[12] F. Bethuel and F. Demengel. “Extensions for Sobolev mappings between manifolds”. In: *Calc. Var. Partial Differential Equations* 3 (1995), pp. 475–491.

[13] F. Bethuel and X. M. Zheng. “Density of smooth functions between two manifolds in Sobolev spaces”. In: *J. Funct. Anal.* 80.1 (1988), pp. 60–75.

[14] J. Bourgain, H. Brezis, and P. Mironescu. “Limiting embedding theorems for $W^{s,p}$ when $s \uparrow 1$ and applications”. In: *J. Anal. Math.* 87 (2002), pp. 77–101.

[15] P. Bousquet, A. C. Ponce, and J. Van Schaftingen. “Density of bounded maps in Sobolev spaces into complete manifolds”. In: *Ann. Mat. Pura Appl.* 196.6 (2017), pp. 2261–2301.

[16] H. Brezis, Y. Li, P. Mironescu, and L. Nirenberg. “Degree and Sobolev spaces”. In: *Topol. Methods Nonlinear Anal.* 13.2 (1999), pp. 181–190.

[17] H. Brezis and P. Mironescu. “Density in $W^{s,p}(\Omega, \mathcal{N})$”. In: *J. Funct. Anal.* 269 (2015), pp. 2045–2109.

[18] H. Brezis and L. Nirenberg. “Degree theory and BMO. I: Compact manifolds without boundaries”. In: *Selecta Math. (N.S.)* 1.2 (1995), pp. 197–263.

[19] H. Brezis and L. Nirenberg. “Degree theory and BMO. II: Compact manifolds with boundaries, with an appendix by the authors and Petru Mironescu”. In: *Selecta Math. (N.S.)* 2.3 (1996), pp. 309–368.

[20] J. E. Brothers. “Integral geometry in homogeneous spaces”. In: *Trans. Amer. Math. Soc.* 124.3 (1966), pp. 480–517.

[21] B. Bulanyi, J. Van Schaftingen, and B. Van Vaerenbergh. “Limiting behavior of minimizing p-harmonic maps in 3d as p goes to 2 with finite fundamental group”. In: *Arch. Ration. Mech. Anal.* 249.3 (2025). Id/No 29, p. 103.

[22] G. Canevari. “Line defects in the small elastic constant limit of a three-dimensional Landau-de Gennes model”. In: *Arch. Ration. Mech. Anal.* 223 (2017), pp. 591–676.

[23] G. Canevari and G. Orlandi. "Topological singular set of vector-valued maps, II: Γ-convergence for Ginzburg-Landau type functionals". In: *Arch. Ration. Mech. Anal.* 241.2 (2021), pp. 1065–1135.

[24] M. Caselli, E. Florit-Simon, and J. Serra. "Fractional Sobolev spaces on Riemannian manifolds". In: *Math. Ann.* (2024).

[25] R. E. Castillo and H. Rafeiro. *An introductory course in Lebesgue spaces*. CMS Books in Mathematics. Springer, Cham, 2016.

[26] S. Chandrasekhar and G. S. Ranganath. "The structure and energetics of defects in liquid crystals". In: *Adv. Phys.* 35.6 (1986), pp. 507–596.

[27] G. De Philippis, A. De Rosa, and F. Ghiraldin. "Rectifiability of varifolds with locally bounded first variation with respect to anisotropic surface energies". In: *Comm. Pure Appl. Math.* 71.6 (2018), pp. 1123–1148.

[28] E. Di Nezza, G. Palatucci, and E. Valdinoci. "Hitchhiker's guide to the fractional Sobolev spaces". In: *Bull. Sci. Math.* 136.5 (2012), pp. 521–573.

[29] L. C. Evans and R. F. Gariepy. *Measure theory and fine properties of functions*. Revised. Textbooks in Mathematics. CRC Press, Boca Raton, FL, 2015.

[30] H. Federer. "Curvature measures". In: *Trans. Amer. Math. Soc.* 93 (1959), pp. 418–491.

[31] H. Federer. *Geometric measure theory*. Die Grundlehren der mathematischen Wissenschaften, Band 153. Springer-Verlag New York Inc., New York, 1969.

[32] E. Gagliardo. "Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili". In: *Rend. Sem. Mat. Univ. Padova* 27 (1957), pp. 284–305.

[33] J. A. Griepentrog, W. Höppner, H.-C. Kaiser, and J. Rehberg. "A bilipschitz continuous, volume preserving map from the unit ball onto a cube". In: *Note Mat.* 28.1 (2008), pp. 177–193.

[34] F. Hang and F.-H. Lin. "Topology of Sobolev mappings, II". In: *Acta Math.* 191 (2003), pp. 55–107.

[35] R. Hardt and F.-H. Lin. "Mappings minimizing the L^p norm of the gradient". In: *Comm. Pure Appl. Math.* 40 (1987), pp. 555–588.

[36] R. Hardt and F.-H. Lin. "Singularities for p-energy minimizing unit vectorfields on planar domains". In: *Calc. Var. Partial Differential Equations* 3.3 (1995), pp. 311–341.

[37] R. Hardt, F. Lin, and C. Wang. "Singularities of p-energy minimizing maps". In: *Comm. Pure Appl. Math.* 50.5 (1997), pp. 399–447.

[38] A. Hatcher. *Algebraic Topology*. Cambridge University Press, Cambridge, 2002.

[39] F. Hélein and J. C. Wood. "Harmonic maps: Dedicated to the memory of James Eells". In: *Handbook of Global Analysis*. Amsterdam: Elsevier, 2008, pp. 417–491.

[40] S. Krantz and H. Parks. "Distance to C^k hypersurfaces". In: *J. Differential Equations* 40 (1981), pp. 116–120.

[41] A. Kufner, O. John, and S. Fučik. *Function Spaces*. Noordhoff, Leyden, 1977.

[42] F.-H. Lin. "Topology of Sobolev mappings and associated variational problems". In: *Surveys in geometric analysis and relativity*. Vol. 20. Adv. Lect. Math. (ALM). International Press, 2011, pp. 363–393.

[43] S. Luckhaus. "Convergence of minimizers for the p-Dirichlet integral". In: *Math. Z.* 213 (1993), pp. 449–456.

[44] S. Luckhaus. "Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold". In: *Indiana Univ. Math. J.* 37.2 (1988), pp. 349–367.

[45] P. Mironescu and J. Van Schaftingen. "Trace theory for Sobolev mappings into a manifold". In: *Ann. Fac. Sci. Toulouse Math. (6)* 30.2 (2021), pp. 281–299.

[46] A. Monteil, R. Rodiac, and J. Van Schaftingen. "Ginzburg–Landau relaxation for harmonic maps on planar domains into a general compact vacuum manifold". In: *Arch. Ration. Mech. Anal.* 242 (2021), pp. 875–935.

[47] A. Monteil, R. Rodiac, and J. Van Schaftingen. "Renormalised energies and renormalisable singular harmonic maps into a compact manifold on planar domains". In: *Math. Ann.* 383 (2022), pp. 1061–1125.

[48] E. F. Moore. "Density ratios and (ϕ,1) rectifiability in n-space". In: *Trans. Amer. Math. Soc.* 69.2 (1950), pp. 324–334.

[49] D. Mucci. "Strong density results in trace spaces of maps between manifolds". In: *Manuscripta Math.* 128.4 (2009), pp. 421–441.

[50] J. Nash. "The imbedding problem for Riemannian manifolds". In: *Ann. Math.* 63.1 (1956), pp. 20–63.

[51] D. Preiss. "Geometry of measures in $\mathbb{R}^n$: Distribution, rectifiability, and densities". In: *Ann. Math.* 125.3 (1987), pp. 537–643.

[52] J. Rubinstein and P. Sternberg. "Homotopy classification of minimizers of the Ginzburg-Landau energy and the existence of permanent currents". In: *Comm. Math. Phys.* 179.1 (1996), pp. 257–263.

[53] R. Schoen and K. Uhlenbeck. "A regularity theory for harmonic maps". In: *J. Differ. Geom.* 17 (1982), pp. 307–335.

[54] R. Schoen and K. Uhlenbeck. "Boundary regularity and the Dirichlet problem for harmonic maps". In: *J. Differ. Geom.* 18.2 (1983), pp. 253–268.

[55] D. Stern. "p-Harmonic maps to $\mathbb{S}^1$ and stationary varifolds of codimension two". In: *Calc. Var. Partial Differential Equations* 59.6 (2020), Paper No. 187, 47 pp.

[56] H. Triebel. *Theory of function spaces.* Monographs in Mathematics. Birkhäuser, Basel, 1983.

[57] J. Van Schaftingen and B. Van Vaerenbergh. "Asymptotic behavior of minimizing p-harmonic maps when $p \nearrow 2$ in dimension 2". In: *Calc. Var. Partial Differential Equations* 62.8 (2023), Paper No. 229, 45 pp.

[58] B. White. "The maximum principle for minimal varieties of arbitrary codimension". In: *Comm. Anal. Geom.* 18.3 (2010), pp. 421–432.

[59] W. P. Ziemer. *Weakly differentiable functions. Sobolev spaces and functions of bounded variation.* Vol. 120. Graduate Texts in Mathematics. Springer-Verlag, New York, 1989.

Universality of renormalisable mappings in two dimensions: the case of polar convex integrands

4

Abstract. We establish universality of the renormalised energy for mappings from a planar domain to a compact manifold, by approximating subquadratic polar convex functionals of the form $\int_\Omega f(|\mathrm{D}u|)\,\mathrm{d}x$. The analysis relies on the condition that the vortex map $x/|x|$ has finite energy and that $t \mapsto f(\sqrt{t})$ is concave. We derive the leading order asymptotics and provide a detailed description of the convergence of $\mathrm{W}^{1,1}$–almost minimizers, leading to a characterization of second-order asymptotics. At the core of the method, we prove a ball merging construction (following Jerrard and Sandier's approach) for a general class of convex integrands. We therefore generalize the approximation by p–harmonic mappings when $p \nearrow 2$ and can also cover linearly growing functionals, including those of area-type.

4.1 Introduction

Given a bounded Lipschitz planar domain $\Omega \subset \mathbb{R}^2$, a compact Riemannian manifold $\mathcal{N} \subset \mathbb{R}^\nu$ ($\nu \in \mathbb{N}\setminus\{0\}$), and boundary datum $g \in \mathrm{W}^{1/2,2}(\partial\Omega;\mathcal{N})$, a *minimising harmonic map* is a mapping $u \in \mathrm{W}^{1,2}(\Omega;\mathcal{N}) \doteq \mathrm{W}^{1,2}(\Omega;\mathbb{R}^\nu) \cap \{\text{for a.e. } x \in \Omega, u(x) \in \mathcal{N}\}$ solving the minimisation problem

$$\inf\left\{\int_\Omega \frac{|\mathrm{D}u|^2}{2}\,\mathrm{d}x : \begin{array}{c} u \in \mathrm{W}^{1,2}(\Omega;\mathcal{N}) \\ \operatorname{tr}_{\partial\Omega} u = g \end{array}\right\}. \tag{4.1.1}$$

If $\mathrm{W}^{1,2}_g(\Omega;\mathcal{N}) \doteq \mathrm{W}^{1,2}(\Omega;\mathcal{N}) \cap \{\operatorname{tr}_{\partial\Omega} u = g\}$ contains at least one map, the existence of harmonic mappings in this Dirichlet class can be shown

This chapter originally appeared as C. Irving and B. Van Vaerenbergh. *Universality of renormalisable mappings in two dimensions: the case of polar convex integrands.* Preprint, arXiv:2411.17520 [math.AP] (2024). 2024

by the Direct Method of the Calculus of Variations. However, due to topological obstructions, there may exist $g \in \mathrm{W}^{1/2,2}(\partial\Omega;\mathcal{N})$ such that $\mathrm{W}^{1,2}_g(\Omega;\mathcal{N}) = \varnothing$. In particular, such a g exists when $\mathcal{N}$ is not simply connected, or equivalently the fundamental group $\pi_1(\mathcal{N})$ is non-trivial; see for instance [5, 42].

While $\mathrm{W}^{1,2}_g(\Omega;\mathcal{N})$ may be empty, if we relax our admissible class we can find singular mappings $u\colon \Omega \to \mathcal{N}$ attaining the boundary condition. Thus we may seek to understand (4.1.1) in the obstructed case through a suitable sequence of approximate problems which we can solve, by analysing the behaviour of these approximate solutions in the singular limit. Moreover, we seek to obtain a *singular renormalisable harmonic map* in the limit; a notion introduced in [31, 32].

More precisely, we consider a sequence $(f_n)_n$ of Young functions $f_n : \mathbb{R}_+ \to \mathbb{R}$ (referred to as integrands in the sequel) for which the *vortex energy*

$$\mathcal{V}(f_n) \doteq 2\int_0^1 f_n\Big(\frac{1}{r}\Big) r \,\mathrm{d}r = \frac{1}{\pi}\int_{\mathbb{B}^2(0)} f_n(|\mathrm{D}u_{\mathrm{V}}(x)|)\,\mathrm{d}x \quad \text{is finite,} \tag{4.1.2}$$

where $u_{\mathrm{V}}(x) = \frac{x}{|x|}$ is the vortex map. Condition (4.1.2) ensures the set

$$\mathrm{A}_g(f_n) \doteq \Big\{u \in \mathrm{W}^{1,1}(\Omega;\mathcal{N}) : \int_\Omega f_n(|\mathrm{D}u|)\,\mathrm{d}x \text{ is finite},\quad \mathrm{tr}_{\partial\Omega}\, u = g\Big\} \tag{4.1.3}$$

is non-empty (see Lemma 4.2.15) and therefore

$$\inf_{v \in \mathrm{A}_g(f_n)} \int_\Omega f_n(|\mathrm{D}v|)\,\mathrm{d}x \text{ is finite,} \tag{4.1.4}$$

so we obtain a sequence of well-posed minimisation problems. To recover the Dirichlet case (4.1.1), we consider general families $(f_n)_n$ satisfying Definition 4.2.1, which will include

$$\text{(a)}\, f_n(z) = \frac{|z|^{p_n}}{p_n} \quad \text{where } p_n \nearrow 2 \ (p\text{–energy}),$$

$$\text{(b)}\, f_n(z) = \frac{\sqrt{1+\delta_n^2|z|^2}-1}{\delta_n^2} \quad \text{where } \delta_n \searrow 0 \text{ (rescaled area functional)},$$

$$\text{(c)}\, f_n(z) = \begin{cases} \frac{1}{2}|z|^2 \text{ if } |z| < \kappa_n, \\ \kappa_n(|z| - \kappa_n/2) \text{ if } |z| \geq \kappa_n \end{cases} \quad \text{where } \kappa_n \to \infty \text{ (trunc. energy)},$$

$$\text{(d)}\, f_n(z) = \frac{|z|^2}{2(1+\eta_n|\log|z||)^2} \quad \text{where } \eta_n \searrow 0, \ \ (\mathrm{L}\log^{-2}\mathrm{L}\text{–energy}).$$

For such integrands we will show the following prototype theorem:

Theorem 4.1.1. *Let $(f_n)_n$ be as in Definition 4.2.1, which includes any of* (a)–(d)*. Then for any $g \in \mathrm{W}^{1/2,2}(\partial\Omega;\mathcal{N})$ we have*

$$\lim_{n\to\infty} \frac{1}{\mathcal{V}(f_n)} \inf\left\{ \int_\Omega f_n(|\mathrm{D}v|)\,\mathrm{d}x : \begin{matrix} v \in \mathrm{W}^{1,1}(\Omega;\mathcal{N}) \\ \operatorname{tr}_{\partial\Omega} v = g \end{matrix} \right\} = \mathrm{E}^{1,2}_{\mathrm{sg}}([g]). \tag{4.1.5}$$

Here $\mathrm{E}^{1,2}_{\mathrm{sg}}([g])$ is the singular energy; cf. Definition 4.3.4.

Since $\lim_{n\to\infty} \mathcal{V}(f_n) = \infty$ by Fatou's lemma, Theorem 4.1.1 quantifies the rate of blow-up of (4.1.4) up to first order by the singular energy, which is a homotopy invariant quantity that does not depend on the choice of the sequence $(f_n)_n$. When $\Omega = \mathbb{B}^2 \subset \mathbb{R}^2$ is the unit ball and $\mathcal{N} = \mathbb{S}^1$, we have $\mathrm{E}^{1,2}_{\mathrm{sg}}([g]) = \pi|\deg(g)|$, where $\deg(g)$ is the topological degree of g viewed as a map $\partial\mathbb{B} = \mathbb{S}^1 \to \mathbb{S}^1$.

Moreover we prove a stronger result; namely we are able to completely characterise the next leading order of the asymptotic expansion in n, and establish convergence results for asymptotically minimising sequences.

Theorem 4.1.2. *Under the assumptions of Theorem 4.1.1, suppose $(u_n)_n \subset \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ is an asymptotically minimising sequence in the sense that $\operatorname{tr}_{\partial\Omega} u_n = g$ for all n and*

$$\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \inf\left\{ \int_\Omega f_n(|\mathrm{D}v|)\,\mathrm{d}x : \begin{matrix} v \in \mathrm{W}^{1,1}(\Omega;\mathcal{N}) \\ \operatorname{tr}_{\partial\Omega} v = g \end{matrix} \right\} \xrightarrow{n\to\infty} 0. \tag{4.1.6}$$

Then, passing to a subsequence, u_n converges strongly in $\mathrm{W}^{1,1}$ to a limit map $u_* \in \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ with $\operatorname{tr}_{\partial\Omega} u_* = g$ satisfying

$$u_* \in \mathrm{W}^{1,2}\Big(\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\rho);\mathcal{N}\Big) \quad \textit{for all } \rho > 0, \tag{4.1.7}$$

where $\kappa \in \mathbb{N}$ and $a_1,\dots,a_\kappa \in \Omega$. Moreover, for all $\rho > 0$ sufficiently small,

(i) *u_* is a minimizing harmonic map on $\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}^2(a_i,\rho)$ with respect to its own boundary conditions,*

(ii) $$\int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}^2(a_i,\rho)} f_n(|\mathrm{D}u_* - \mathrm{D}u_n|)\,\mathrm{d}x \xrightarrow{n\to\infty} 0,$$

(iii) *u_* is renormalisable in that the renormalised energy*

$$\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) = \lim_{\rho\to 0}\left[\int_{\Omega\setminus\bigcup_{i=1}^{k}\mathbb{B}^2(a_i,\rho)} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}x - \mathrm{E}^{1,2}_{\mathrm{sg}}([g]) \log\frac{1}{\rho} \right] \tag{4.1.8}$$

exists and is finite, and we have

$$\lim_{n\to\infty}\left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] = \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*,a_i])_{i=1}^{\kappa}, \tag{4.1.9}$$

where $\mathrm{H}([u_*,a_i])_{i=1}^{\kappa}$ *is defined in* (4.4.2).

The goal of this paper is to identify general conditions for sequences of integrands $(f_n)_n$, covering the four examples (a)–(d), such that Theorems 4.1.1 and 4.1.2 hold, working in the spirit of Γ-convergence (also known as variational convergence). These results show that *the renormalised energy is universal*: regardless of the sequence of approximating integrands $(f_n)_n$, the limiting mappings have finite renormalised energy (4.1.8). This universality suggests that the manifold constrained harmonic extension of a map $g : \partial\Omega \to \mathcal{N}$ should be defined through the minimization of the renormalised energy $\mathrm{E}^{1,2}_{\mathrm{ren}}(u)$; this quantity was introduced in [32]. Under certain assumptions on the domain, the manifold and the boundary data, we can explicitly compute the manifold constrained harmonic extension:

Theorem 4.1.3. *Suppose* $\Omega = \mathbb{B}^2$ *is the unit ball centred at the origin,* $\mathcal{N} = \mathbb{S}^1$ *and* $g : \partial\mathbb{B} = \mathbb{S}^1 \to \mathbb{S}^1$ *is defined by* $g(x) \doteq x$ *for all* $x \in \partial\mathbb{B}$*, and let* $(f_n)_n$ *satisfy Definition 4.2.1 (which includes* (a)–(d)*). Then for any asymptotically minimizing sequence* $(u_n)_n \subset \mathrm{W}^{1,1}(\mathbb{B},\mathbb{S}^1)$ *satisfying* $\mathrm{tr}_{\partial\mathbb{B}}\, u_n = g$ *for all* n *and* (4.1.6)*, we have*

$$u_n \to u_{\mathrm{V}}(x) = \frac{x}{|x|} \quad \textit{strongly in } \mathrm{W}^{1,1}(\mathbb{B},\mathbb{S}^1). \tag{4.1.10}$$

Theorem 4.1.3 asserts *universality of the vortex map* u_{V} and can be extended to other manifolds and boundary data, *cf.* Theorem 4.7.1. We underline that in Theorem 4.1.3 the full sequence u_n converges instead of some subsequence.

The assumptions we use are detailed in Section 4.2, but we will consider convex polar integrands of the form $f_n(|z|)$, with $f_n : t \in \mathbb{R}_+ \to \mathbb{R}_+$; here, and throughout the paper, we will use $|\cdot|$ to denote the Euclidean / Frobenius norm. We impose that the sequence $f_n(t)$ converges pointwise to $t^2/2$, that each f_n has finite vortex energy $\mathcal{V}(f_n)$ (see (4.1.2)), and that $t \in \mathbb{R}_+ \mapsto f_n \circ \sqrt{t}$ is concave. These assumptions covers the case of p-harmonic mappings, which was studied by Hardt and Lin in [20] for $\mathcal{N} = \mathbb{S}^1$, and for general $\mathcal{N}$ in [43]. Our assumption allows for a much more general class of integrands however, including linear growth functionals such as (b), and integrands which are not strictly convex such as (c), which is $|z|^2/2$ truncated at $z = \kappa$ (see Definition 4.2.9).

The family (b) is motivated by the observation that the Dirichlet energy can be obtained from the expansion of the area functional, namely

$$\sqrt{1+|z|^2} = 1 + \frac{|z|^2}{2} + O(|z|^4), \quad z \to 0. \tag{4.1.11}$$

Note that this integrand arises by considering the area of the graph of $|u|$; that is $\mathcal{H}^2(\{(x,|u(x)|) : x \in \Omega\})$. To capture the second term in the expansion (4.1.11), we consider the change of variables $u_\delta(x) = u(\delta x)$ for $u \in \mathrm{W}^{1,1}(\mathbb{B}^2; \mathcal{N})$ which satisfies

$$\int_{\mathbb{B}^2(0,1/\delta)} \sqrt{1+|\mathrm{D}u_\delta|^2} - 1 \,\mathrm{d}x = \int_{\mathbb{B}^2} \frac{\sqrt{1+\delta^2|\mathrm{D}u|^2} - 1}{\delta^2} \,\mathrm{d}x, \tag{4.1.12}$$

which leads us to consider f_n defined as in (b), which converges pointwise to $\frac{1}{2}|z|^2$ as $\delta_n \searrow 0$. This approximation has recently appeared in [34] as a relaxation to the problem of quadrilateral and hexagonal meshing, formulated as a minimisation problem over suitable fiber bundles. This approximation was also used in [36, §A.3(2) p.185] to prove the Pólya-Szegő inequality for $p = 2$.

One could sacrifice the convexity of the integrands and the intrinsic character of the problem by using a Ginzburg-Landau approach, where for every $\varepsilon > 0$ one considers

$$\inf\left\{\int_\Omega \frac{|\mathrm{D}u|^2}{2} + \frac{F(u)}{\varepsilon^2} \,\mathrm{d}x : u \in \mathrm{W}^{1,2}_g(\Omega; \mathbb{R}^\nu)\right\}. \tag{4.1.13}$$

Here $F(u)$ behaves as $\operatorname{dist}(u; \mathcal{N})^2$ near $\mathcal{N}$ (see [31, (1.4), §2] for detailed assumptions on F), and $\mathcal{N}$ is a connected compact submanifold of $\mathbb{R}^\nu$. Results similar to Theorems 4.1.1 and 4.1.2 can be obtained in that setting (see [31, Thm. 7.1, 7.3]), and this relaxation has also been applied to the theory of meshing; see for instance [4]. However, this approach is extrinsic since $F(u)$ depends on an isometric embedding $\mathcal{N} \subset \mathbb{R}^\nu$, which always exists [33], but is highly non-unique. Moreover, $F(u)$ is typically non-convex, as the classical case attests [7] for $\mathcal{N} = \mathbb{S}^1$, where $F(u) = (|u|^2 - 1)^2/4$. The extrinsic behaviour is recorded in the obtained renormalised limit; see the Q-term in [31, Thm. 7.3(vi)].

Innovations. While many prior works focused on specific approximations such as the p-energy and the Ginzburg Landau relaxation, we initiate a systematic study of functionals $\mathcal{F}_n$ approximating the Dirichlet energy along which this asymptotic analysis can be carried out. Our assumptions on the integrands $(f_n)_n$ is sufficiently general to allow for linearly growing integrands such as (b) and (c), thus leading us to work in the framework of L^1 convergence.

The key results of our analysis are contained in the compactness theorem (Theorem 4.5.1), valid for sequences whose energy in bounded in a renormalised sense. The result crucially relies on a lower bound to control the singularities (see Proposition 4.4.7) based on the classical merging ball procedure of Jerrard [24] and Sandier [39]. While the a-priori bounds allows us to extract a subsequence of $(\mathrm{D}u_n)_n$ which converges weakly* (in the sense of measures), by separate asymptotic estimates of the sequence near and away from the singularities, we can improve this convergence to hold weakly in L^1 (see Proposition 4.5.2 and Theorem 4.5.1(vi)). Moreover, in the case of asymptotically minimising sequences, in Theorem 4.6.1 we can improve the convergence of gradients to hold strongly L^1, by exploiting the uniform convexity of the limiting L^2 scale (see Proposition 4.6.2).

Future directions. Our claim of the universality of the renormalized energy naturally raises the question of whether more general classes of integrands can be considered (*e.g.* to allow for non-polar integrands with an (x, u)-dependence), or even discrete versions (see [15] for $\mathcal{N} = \mathbb{S}^1$). Further approximations based variants of Bourgain-Brézis-Mironescu-type formulas [9] can also be considered, as discussed in [41] for $\mathcal{N} = \mathbb{S}^1$. Additionally, our analysis is confined to mappings in $\mathrm{W}^{1,1}$; however for linearly growing integrands we are naturally lead to consider minimisers in BV (see Section 4.2.4 for a further discussion).

Structure of the paper. Assumptions on the sequence of approximating integrands $(f_n)_n$ are described in Section 4.2. The singular energy and the renormalised energies are detailed in Section 4.3. In Section 4.4, we present key ingredients needed for the formal Γ-convergence analysis; namely an upper bound corresponding to the Γ-limsup, and a merging ball construction which is used to obtain asymptotic lower bounds. The main compactness theorem is proven in Section 4.5, which relies on compactness results for unconstrained mappings and for traces, and corresponds to the Γ-liminf analysis. In Section 4.6, we prove an extended version of Theorem 4.1.2 (see Theorem 4.6.1). Finally, in Section 4.7, we demonstrate that in the specific case where $\Omega = \mathbb{B}^2(0, 1) \subset \mathbb{R}^2$ is the unit ball, $\mathcal{N} = \mathbb{S}^1$ and $g(z) = z$, the entire sequence $(u_n)_n$ in Theorem 4.1.2 converges to the vortex map $x/|x|$ asserting the *universality of the vortex map.*

4.2 Growth structures for the Calculus of Variations

In this section we describe the set of assumptions we impose on our integrands f_n (namely in Definition 4.2.1), and record some examples and useful properties thereof. We will write $\mathbb{R}_+ = [0, \infty)$ in what follows.

4.2.1 Assumptions

Let $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a *Young function*, that is a convex non-decreasing function such that $f(0) = 0$.

We will impose that f is subquadratic; either in the sense of the *natural sense* that

$$(\mathrm{Dec})_2^{\mathrm{n}} \qquad t \mapsto \frac{1}{t^2} f(t) \quad \text{is non-increasing on } [t_0, \infty),$$

or in the *controlled sense* that

$$(\mathrm{Dec})_2^{\mathrm{c}} \qquad t \mapsto \frac{1}{t} f'_+(t) \quad \text{is non-increasing on } [t_0, \infty),$$

where $t_0 \geq 0$. The $(\mathrm{Dec})_2^{\mathrm{n}}$ is well–known in the Orlicz literature, see for instance [22, §2.1]. One may readily verify by elementary calculus that $(\mathrm{Dec})_2^{\mathrm{c}}$ is equivalent to

$$t \mapsto f(\sqrt{t}) \quad \text{is concave on } [t_0, \infty). \tag{4.2.1}$$

Furthermore $(\mathrm{Dec})_2^{\mathrm{c}}$ implies $(\mathrm{Dec})_2^{\mathrm{n}}$ by (4.2.22), but is in general is a stronger condition, see Example 4.2.7 below.

We will also impose the integrability condition

$$(\mathrm{vInt}) \qquad \mathcal{V}(f) = 2\int_0^1 f\left(\frac{1}{r}\right) r \, \mathrm{d}r = 2\int_1^\infty \frac{f(s)}{s^2} \frac{\mathrm{d}s}{s} < \infty,$$

where the equality follows from the change of variable $s = \frac{1}{r}$. The factor 2 is present to match the singular energy in the asymptotic expansion; see Theorem 4.1.1. We call $\mathcal{V}(f)$ the *vortex energy* of f; see Proposition 4.2.2 below for consequences of this condition.

Definition 4.2.1. *A* sequence of approximating integrands *is a family* $(f_n)_n$ *of Young functions* $f_n : \mathbb{R}_+ \to \mathbb{R}$ *satisfying the following:*

(a) *There is* $t_0 \geq 0$ *such that each* f_n *satisfies* $(\mathrm{Dec})_2^{\mathrm{c}}$ *with* t_0 *uniform in* n*,*

(b) *each* f_n *satisfies* (vInt)*,*

(c) *the sequence* $(f_n)_n$ *converges pointwise to* $\frac{1}{2}t^2$*, that is*

$$\lim_{n\to\infty} f_n(t) = \frac{1}{2}t^2 \quad \textit{for all } t \geq 0. \tag{4.2.2}$$

Note that combining $(\mathrm{Dec})_2^{\mathrm{n}}$ and (vInt) gives

$$\lim_{t\to+\infty}\frac{f(t)}{t^2}=0, \tag{4.2.3}$$

so we infer that f satisfies a subquadratic growth condition. However $(\mathrm{Dec})_2^{\mathrm{n}}$ is a stronger condition than 4.2.3, as illustrated by Example 4.2.6 below.

Proposition 4.2.2. *Let $\Omega\subset\mathbb{R}^2$ be a domain and $\mathbb{B}(0,1)\subset\mathbb{R}^2$ be the unit ball. Then for a Young function $f:\mathbb{R}_+\to\mathbb{R}$, the following conditions are equivalent:*

(i) *the integrand f satisfies* (vInt)*; that is, $\mathcal{V}(f)$ is finite*

(ii) *For the vortex map $u_{\mathrm{V}}(x)=x/|x|$ we have $\int_{\mathbb{B}(0,1)} f(|\mathrm{D}u_{\mathrm{V}}|)\,\mathrm{d}x$ is finite,*

(iii) *for every $m\in\mathrm{L}^{2,\infty}(\Omega)$, we have $\int_\Omega f(|m(x)|)\,\mathrm{d}x$ is finite.*

Moreover if any of the above holds, for each non-zero measurable $m:\Omega\to\mathbb{R}_+$ we have the estimate

$$\int_\Omega f\circ m\,\mathrm{d}x\le 2\|m\|^2_{\mathrm{L}^{2,\infty}(\Omega)}\int_{\|m\|_{\mathrm{L}^{2,\infty}(\Omega)}/\sqrt{\mathcal{L}^2(\Omega)}}^{\infty}\frac{f(s)}{s^3}\,\mathrm{d}s. \tag{4.2.4}$$

Definition 4.2.3. *Let $\Omega\subset\mathbb{R}^2$ be open, the* weak-L^2 space $\mathrm{L}^{2,\infty}(\Omega)$ *is the space of measurable functions $u\colon\Omega\to\mathbb{R}$ such that*

$$\|f\|_{\mathrm{L}^{2,\infty}(\Omega)}\doteq\left[\sup_{t>0}t^2\,\mathcal{L}^2\left(\{x\in\Omega:|f(x)|>t\}\right)\right]^{\frac12}<\infty, \tag{4.2.5}$$

where $\mathcal{L}^2$ denotes the Lebesgue measure on $\mathbb{R}^2$.

Other names for weak-L^2 space are *Marcinkiewicz space* [30] or *Lorentz space* [28]. Here, $\mathcal{L}^2$ denotes the 2-dimensional Lebesgue measure. We point out that, despite the notation, $\|\cdot\|_{\mathrm{L}^{2,\infty}(\Omega)}$ is not a norm as the triangle inequality does not hold. However, $\|m_1+m_2\|_{\mathrm{L}^{2,\infty}(\Omega)}\le 2\big(\|m_1\|_{\mathrm{L}^{2,\infty}(\Omega)}+\|m_2\|_{\mathrm{L}^{2,\infty}(\Omega)}\big)$.

Example 4.2.4. The prototypical example of a mapping u for which $|\mathrm{D}u|\in\mathrm{L}^{2,\infty}(\mathbb{B}_1)$ but $|\mathrm{D}u|\notin\mathrm{L}^2(\mathbb{B}_1)$ is the *vortex map* $u_{\mathrm{V}}\colon\mathbb{B}_1\to\mathbb{S}^1$, which is defined as

$$u_{\mathrm{V}}(x)=\frac{x}{|x|}. \tag{4.2.6}$$

Indeed since $|\mathrm{D}u_{\mathrm{V}}|(x) = \frac{1}{|x|}$, we have $\|\mathrm{D}u_{\mathrm{V}}\|_{\mathrm{L}^{2,\infty}(\mathbb{B}_1)} = \sqrt{\pi} < \infty$, but $\|\mathrm{D}u_{\mathrm{V}}\|^2_{\mathrm{L}^2(\mathbb{B}_1)} = 2\pi \int_0^1 \mathrm{d}r/r = \infty$, where we used polar coordinates to compute the L^2-norm. For a general manifold $\mathcal{N}$, we can produce similar examples by taking an immersed curve $\gamma\colon \mathbb{S}^1 \to \mathcal{N}$ and considering $\gamma(x/|x|)$.

Proof of Proposition 4.2.2. The equivalence between (ii) and (i) follows from integrating in polar coordinates and since $|\mathrm{D}u_{\mathrm{V}}|(x) = 1/|x|$; indeed

$$\int_{\mathbb{B}(0,1)} f(|\mathrm{D}u_{\mathrm{V}}|) = 2\pi \int_0^1 f\left(\frac{1}{r}\right) r \,\mathrm{d}r = \pi \mathcal{V}(f). \tag{4.2.7}$$

Since $1/|x| \in \mathrm{L}^{2,\infty}(\Omega)$ by Example 4.2.4, we have (iii) implies (ii). To show that (i) implies (iii), it suffices to prove the estimate (4.2.4). Let $m \in \mathrm{L}^{2,\infty}(\Omega)$ be non-negative, and set $M \doteq \|m\|_{\mathrm{L}^{2,\infty}}/\sqrt{\mathcal{L}^2(\Omega)} > 0$. Since $s \mapsto f(s)/s^3$ is continuous on $(0,\infty)$ and (vInt) holds, we have $\int_M^\infty f(s)/s^3 \,\mathrm{d}s < \infty$. Then by the layer cake representation (see [44, Cor. 2.2.34]) and by a change of variables we have,

$$\begin{aligned}\int_\Omega f \circ m \,\mathrm{d}x = \int_0^\infty \mathcal{L}^2(\{x \in \Omega : f \circ m(x) > t\}) \,\mathrm{d}t \\ = \int_0^\infty f'(t)\, \mathcal{L}^2(\{x \in \Omega : m(x) > t\}) \,\mathrm{d}t.\end{aligned}$$

Note that since f is convex, we have f is locally Lipschitz, so this change of variables is justified (see for instance [16, Thm. 3.9, 6.7]). We can split this integral as

$$\begin{aligned}\int_\Omega f \circ m &\le \mathcal{L}^2(\Omega) \int_0^M f'(t) \,\mathrm{d}t + \|m\|^2_{\mathrm{L}^{2,\infty}(\Omega)} \int_r^\infty \frac{f'(t)}{t^2} \,\mathrm{d}t \\ &= \mathcal{L}^2(\Omega) f(M) + \|m\|^2_{\mathrm{L}^{2,\infty}(\Omega)} \left[2 \int_M^\infty \frac{f(s)}{s^3} \,\mathrm{d}s - \frac{f(M)}{M^2} \right].\end{aligned}$$

By our choice of M, we obtain (4.2.4) and hence (iii). This concludes the proof. □

4.2.2 Some examples

Example 4.2.5. We detail here explicitly the vortex energy for the examples from the introduction. For the p-case (a), we compute, for $f_p(t) = |t|^p/p$,

$$\mathcal{V}(f_p) = \frac{2}{(2-p)p} = \frac{1}{2-p} + O(1) \quad \text{as } p \nearrow 2$$

so that Theorem 4.1.1 reads

$$\lim_{p\nearrow 2}(2-p)\inf_{v\in \mathrm{W}^{1,p}_g(\Omega;\mathcal{N})}\int_\Omega \frac{|\mathrm{D}v|^p}{p}\,\mathrm{d}x = \mathrm{E}^{1,2}_{\mathrm{sg}}([g]). \tag{4.2.8}$$

For the rescaled area functional (b), taking $f_\delta(t) = (\sqrt{1+\delta^2 t^2}-1)/\delta^2$ we can compute

$$\mathcal{V}(f_\delta) = 2\int_0^1 \frac{\sqrt{1+\delta^2 r^{-2}}-1}{\delta^2}\,r\,\mathrm{d}r = f_\delta(1) + \operatorname{arcsinh}(\frac{1}{\delta}) = \log\frac{1}{\delta} + O(1)$$

as $\delta \searrow 0$, where $\operatorname{arcsinh}(t) = \log(t+\sqrt{1+t^2})$. In that case Theorem 4.1.1 reads as

$$\lim_{\delta\searrow 0}\frac{1}{\log\frac{1}{\delta}}\inf_{\mathrm{W}^{1,1}_g(\Omega;\mathcal{N})}\int_\Omega \frac{\sqrt{1+\delta^2|\mathrm{D}v|^2}-1}{\delta^2}\,\mathrm{d}x = \mathrm{E}^{1,2}_{\mathrm{sg}}([g]). \tag{4.2.9}$$

For the truncated functional (c), namely $f_\kappa(z) = \frac{|z|^2}{2}\chi_{\{|z|<\kappa\}} + \kappa(|z| - \frac{\kappa}{2})\chi_{\{|z|>\kappa\}}$, for $\kappa > 1$,

$$\mathcal{V}(f_\kappa) = \int_{\frac{1}{\kappa}}^1 \frac{\mathrm{d}t}{t} + 2\kappa\int_0^{\frac{1}{\kappa}}\left(\frac{1}{t}-\frac{\kappa}{2}\right)t\,\mathrm{d}t = \log\kappa + \frac{7}{4}.$$

For (d) we can more generally consider $f_\eta(t) = \frac{t^2}{2(1+\eta|\log t|)^\alpha}$ with $\eta > 0$, for $\alpha > 1$ fixed. By computing $f_\eta'(t)$ we see that each f_η is a Young function satisfying $(\mathrm{Dec})^{\mathrm{c}}_2$ with $t_0 = 0$, and by a change of variables $s = -\eta\log r$ we can also compute

$$\mathcal{V}(f_\eta) = \int_0^1 \frac{\mathrm{d}r}{r(1+\eta|\log r|)^\alpha} = \frac{1}{\eta}\int_0^\infty \frac{\mathrm{d}s}{(1+s)^\alpha} = \frac{1}{(\alpha-1)\eta}.$$

We will also describe two counterexamples relating to the (vInt), $(\mathrm{Dec})^{\mathrm{n}}_2$ and $(\mathrm{Dec})^{\mathrm{c}}_2$ conditions.

Example 4.2.6. The following is a variant of [29, §11, Eg. 6]. For $p \in (1,2)$ and $\beta > 0$ to be determined, consider

$$f(t) = \frac{1}{p}t^p\left(1+\frac{1}{\beta}\sin(p\log t)\right) \quad t > 0. \tag{4.2.10}$$

Then for the choice $\beta \geq 5$, we have f is a Young function such that $\frac{1}{p}t^p \leq f(t) \leq \frac{1}{p}(1+\frac{1}{\beta})t^p$. Now we can compute

$$f'(t) = t^{p-1}\left(1+\frac{\sqrt{2}}{p}\sin(p\log t + \pi/4)\right), \tag{4.2.11}$$

and so

$$\frac{t_k f'(t_k)}{f(t_k)} = p\left(1 + \frac{1}{\sqrt{\beta^2 - 1}}\right) \tag{4.2.12}$$
$$\text{where } t_k = \exp\left(\frac{1}{p}((2k+1)\pi - \arcsin(1/\beta))\right)$$

for all $k \in \mathbb{N}$ with $k \geq 1$. Since $(\mathrm{Dec})_2^{\mathfrak{n}}$ is equivalent to $\frac{tf'(t)}{f(t)} \leq 2$ holding for sufficiency large $t > 0$, we see this condition is violated for $\beta = 5$ and $p > \frac{4}{23}(12 - \sqrt{6}) \simeq 1.66\ldots$, thereby giving examples of Young functions satisfying (vInt) and (4.2.3), but not $(\mathrm{Dec})_2^{\mathfrak{n}}$.

To obtain a sequence of such functions f_n converging to $\frac{1}{2}t^2$, we can choose $p_n = \frac{2n+1}{n+1}$ and $\beta_n = \sqrt{n^2 - 1}$ for $n \geq 6$.

Example 4.2.7. To construct a family of integrands satisfying $(\mathrm{Dec})_2^{\mathfrak{n}}$ but not $(\mathrm{Dec})_2^{\mathfrak{c}}$, we will first define a function $g\colon \mathbb{R}_+ \to \mathbb{R}_+$ via

$$g(t) = g_p(t) \doteq t^{p-2} + \sum_{k=3}^{\infty} a_k\, h\left(\frac{t-k}{r_k}\right), \tag{4.2.13}$$

where $p \in (1, 2)$ is sufficiently close to 2, the "spike function" $h\colon \mathbb{R} \to \mathbb{R}$ is defined via

$$h(t) = \begin{cases} t & \text{if } 0 \leq t < \frac{1}{2}, \\ 1 - t & \text{if } \frac{1}{2} \leq t \leq 1, \\ 0 & \text{otherwise}, \end{cases} \tag{4.2.14}$$

and we set $a_k = \frac{2(2-p)}{k^2}$, $r_k = k^{1-p}$ for all $k \in \mathbb{N}$, $k \geq 3$. We then claim the family

$$f(t) = f_p(t) \doteq \int_0^t g_p(s)s\,\mathrm{d}s \tag{4.2.15}$$

is a sequence of Young functions satisfying $(\mathrm{Dec})_2^{\mathfrak{n}}$, (vInt), but does not satisfy $(\mathrm{Dec})_2^{\mathfrak{c}}$ on $[t_0, \infty)$ for any $t_0 \geq 0$. Note that setting $a_k = 0$ for all k would recover the p-energy, so as $p \nearrow 2$ we have f_p converges pointwise to $\frac{1}{2}t^2$.

To establish this claim, note that since g is non-negative we have f is increasing, and we can compute

$$\begin{aligned} f''_+(t) &= g(t) + tg'_+(t) \\ &= (p-1)t^{p-2} + \sum_{k=1}^{\infty} \frac{a_k t}{r_k}\left(\mathbb{1}_{[k,k+r_k/2)}(t) - \mathbb{1}_{[k+r_k/2,k+r_k)}(t)\right). \end{aligned} \tag{4.2.16}$$

Here f''_+ denotes the right derivative of f'; see Lemma 4.2.8. To show this is non-negative, it suffices to check the endpoint cases $t = k + r_k$, which rearranges to give $\frac{p-1}{2-p} \geq 2(1 + k^{-p})^{3-p}$. By checking the $k = 3$ case, this holds for all $p \in (p_0, 2)$ with $p_0 \simeq 1.706\ldots$ Hence, for this range of p, we have f is a Young function. To see that f does not satisfy $(\mathrm{Dec})^{\mathrm{c}}_2$, we note that $\frac{f'(t)}{t} = g(t)$ by construction, and since $g'_+(k) = \frac{2-p}{k^{3-p}} > 0$ for each $k \geq 3$, it follows that g is not non-increasing.

To verify $(\mathrm{Dec})^{\mathrm{n}}_2$, observe this is equivalent to showing that $g(t) = \frac{f'(t)}{t} \leq \frac{2f(t)}{t^2}$. Moreover, by integrating by parts we can compute

$$f(t) = \int_0^t g(s)s \,\mathrm{d}s = \frac{1}{2}t^2 g(t) - \frac{1}{2}\int_0^t g'_+(s)s^2 \,\mathrm{d}s, \tag{4.2.17}$$

so rearranging we see that $(\mathrm{Dec})^{\mathrm{n}}_2$ is equivalent to showing that

$$\int_0^t g'_+(s)s^2 \,\mathrm{d}s \leq 0$$

for all $t \geq 0$. To compute this integral, we first calculate the contribution of each of the "spike" terms. For this we have

$$\int_k^{k+r_k} \frac{\mathrm{d}}{\mathrm{d}s}\left(a_k\, h\left(\frac{s-k}{r_k}\right)\right) s^2 \,\mathrm{d}s = -a_k\left(\frac{1}{2}kr_k + \frac{1}{4}r_k^2\right) \leq 0, \tag{4.2.18}$$

so the contribution over a full spike is negative. Hence for any $t \in (k + r_k, k+1)$ we see that $\int_0^t g'_+(s)s^2 \,\mathrm{d}s \leq -\frac{(2-p)}{p}k^p < 0$, and if $t \in (k, k+r_k/2)$, then

$$\begin{aligned} &\int_0^t g'_+(s)s^2 \,\mathrm{d}s \\ &\qquad \leq -\frac{2-p}{p}\,k^p + \frac{a_k}{r_k}\int_k^{k+r_k/2} s^2 \,\mathrm{d}s \leq -(2-p)\left(\frac{1}{p}k^p - 2\right), \end{aligned} \tag{4.2.19}$$

noting that $r_k \leq 1$, where the last line is non-negative since $k \geq 3$. The case $t \in (k + r_k/2, k + r_k)$ follows since the relevant terms are decreasing, thereby verifying $(\mathrm{Dec})^{\mathrm{n}}_2$.

Finally to show (vInt) holds, we can compute $\int_k^{k+r_k} a_k\, h\left(\frac{t-k}{r_k}\right) \mathrm{d}t \leq \frac{2-p}{2}$ for each k, and hence we can estimate $f(t) \leq t^p + \frac{1}{2}(t+1)$ for all $t \geq 0$. Thus it follows that $\mathcal{V}(f)$ is finite, verifying all of the claimed properties of f for all $p_0 < p < 2$.

4.2.3 Convexity properties

Lemma 4.2.8. *Let f be a convex function on $\mathbb{R}_+$. Then the right derivative*

$$f'_+(t) \doteq \lim_{h \searrow 0} \frac{f(t+h) - f(t)}{h} \tag{4.2.20}$$

exists for all $t \geq 0$ and defines a non-decreasing and right-continuous function on $\mathbb{R}_+$.

We refer to [38, Thm. 24.1] for a proof. The proof of this assertion relies on the *slope inequality* for convex functions

$$\frac{f(s) - f(r)}{s - r} \leq \frac{f(t) - f(r)}{t - r} \leq \frac{f(t) - f(s)}{t - s} \tag{4.2.21}$$

valid for all $0 \leq r < s < t$. From this we also infer that

$$f'_+(s)(t-s) \leq f(t) - f(s) \leq f'_+(t)(t-s) \quad \text{for all } s \leq t. \tag{4.2.22}$$

Analogously we can also define left derivatives f'_-, which will be non-decreasing and left-continuous on $(0, \infty)$. However, we have opted for right-derivatives to ensure $f'_+(t)$ is well-defined at $t = 0$.

Definition 4.2.9. *Let f be a Young function, then for $\kappa > 0$ define the* κ-truncation $T_\kappa f$ *of f as*

$$T_\kappa f(t) = \begin{cases} f(t) & \text{if } 0 \leq t \leq \kappa, \\ f'_+(\kappa)(t-\kappa) + f(\kappa) & \text{if } t > \kappa. \end{cases} \tag{4.2.23}$$

which exists for all $\kappa \geq 0$.

Observe that $T_\kappa f$ is itself a Young function. If f satisfies $(\mathrm{Dec})_2^{\mathrm{n}}$ or $(\mathrm{Dec})_2^{\mathrm{c}}$, then $T_\kappa f$ also does with the same t_0. We also have $T_\kappa f(t) \leq f(t)$ for all $t \geq 0$ and $T_\kappa f$ converges pointwise monotonically to f as $\kappa \to \infty$. We also note that $T_\kappa f$ always satisfies (vInt) and $\mathcal{V}(T_\kappa f) \leq \mathcal{V}(f)$.

Lemma 4.2.10. *Let $(f_n)_n$ be a sequence of Young functions converging pointwise to $f(t) = t^2/2$ on $\mathbb{R}_+$. Then, as $n \to \infty$,*

(a) *f_n converges locally uniformly to f on $\mathbb{R}_+$;* i.e. *uniformly on $[0, T]$ for each $T > 0$,*

(b) *$f'_{n,+}(t)$ converges pointwise to $f'(t) = t$ on $\mathbb{R}_+$.*

Proof of Lemma 4.2.10. Assertion (a) is proven in [38, Thm. 10.8]. For (b) we can use (4.2.22) to show that

$$\frac{f_n(s)-f_n(r)}{s-r} \leq f'_{n,+}(s) \leq \frac{f_n(t)-f_n(s)}{t-s} \tag{4.2.24}$$

for all $0 \leq r < s < t$. Sending $n \to \infty$ followed by $r, t \to s$ establishes the result. □

Remark 4.2.11. An alternative way to truncate f is to define

$$\tilde{T}_L f(t) = \inf\{f(s) + L|s-t| \colon s \in \mathbb{R}_+\} \tag{4.2.25}$$

for some $L > 0$. One can show that the choice $L = f'_+(\kappa)$ gives $\tilde{T}_L f = T_\kappa f$; indeed using (4.2.22) one shows that $\tilde{T}_L f(t) = f(t)$ for all $t \in [0, \kappa]$, and if $t > \kappa$ one has the infimum in (4.2.25) is attained with the choice $s = \kappa$.

Lemma 4.2.12. *Let f be a Young function satisfying* $(\mathrm{Dec})_2^{\mathrm{n}}$. *Then*

$$0 \leq f(t) \leq f(\max\{1, t_0\})(1+t^2) \quad \textit{for all } t \geq 0. \tag{4.2.26}$$

In particular, for a family $(f_n)_n$ as in Definition 4.2.1, we can choose $C > 0$ independently of n such that

$$0 \leq f_n(t) \leq C(1+t^2) \quad \textit{for all } t \geq 0. \tag{4.2.27}$$

Proof of Lemma 4.2.12. We can assume that the $(\mathrm{Dec})_2^{\mathrm{n}}$–condition holds with $t_0 \geq 1$. Then for $t \geq t_0$ we have

$$0 \leq f(t) \leq \frac{f(t_0)}{t_0^2} t^2 \leq f(t_0) t^2. \tag{4.2.28}$$

When $0 \leq t \leq t_0$, since f is non-decreasing we have $0 \leq f(t) \leq f(t_0)$, from which the result follows. For a family (f_n), since $\lim_{n\to\infty} f_n(t_0) = \frac{1}{2}t_0^2$ we can choose $C > 0$ such that (4.2.27) holds uniformly in n. □

The following is used in the proof of the upper bound (Proposition 4.4.1), whose proof crucially relies the stronger $(\mathrm{Dec})_2^{\mathrm{c}}$ condition.

Lemma 4.2.13. *Suppose f is a Young function satisfying* $(\mathrm{Dec})_2^{\mathrm{c}}$ *for some $t_0 \geq 0$. Then for all $\sigma, \rho > 0$ we have*

$$f(\sigma) - f(\rho) \leq \frac{f'_+(\max\{\rho, t_0\})}{2\max\{\rho, t_0\}}(\sigma^2 - \rho^2) + f(t_0) + f'_+(t_0)t_0. \tag{4.2.29}$$

Proof of Lemma 4.2.13. Define $g(t) = f(\sqrt{t})$, which is concave on $[t_0, \infty)$ by the $(\mathrm{Dec})_2^{\mathrm{c}}$ condition. By a variant of (4.2.22) for concave functions, for $\sigma, \rho > t_0$ we have

$$g(\sigma^2) - g(\rho^2) \le g'_+(\rho^2)(\sigma^2 - \rho^2). \tag{4.2.30}$$

That is,

$$f(\sigma) - f(\rho) \le \frac{f'_+(\rho)}{2\rho}(\sigma^2 - \rho^2) \tag{4.2.31}$$

holds. For the remaining cases, note that if $\rho \le t_0 \le \sigma$, then by (4.2.31) with t_0 in place of σ we have

$$\begin{aligned} &f(\sigma) - f(\rho) \\ &\quad\le \frac{f'_+(t_0)}{2t_0}(\sigma^2 - t_0^2) + f(t_0) - f(\rho) \le \frac{f'_+(t_0)}{2t_0}(\sigma^2 - \rho^2) + f(t_0), \end{aligned} \tag{4.2.32}$$

whereas if $\sigma \le t_0 \le \rho$ applying (4.2.31) similarly gives

$$\begin{aligned} &f(\sigma) - f(\rho) \\ &\quad\le f(\sigma) - f(t_0) + \frac{f'_+(\rho)}{2\rho}(t_0^2 - \rho^2) \le \frac{f'_+(\rho)}{2\rho}(\sigma^2 - \rho^2) + \frac{1}{2} f'_+(t_0) t_0. \end{aligned} \tag{4.2.33}$$

Finally if both $\rho, \sigma \le t_0$ we simply bound

$$f(\sigma) - f(\rho) \le f(t_0) \le \frac{f'_+(t_0)}{2t_0}(\sigma^2 - \rho^2) + f(t_0) \quad \text{if } \sigma \ge \rho, \tag{4.2.34}$$

$$f(\sigma) - f(\rho) \le 0 \le \frac{f'_+(t_0)}{2t_0}(\sigma^2 - \rho^2) + \frac{1}{2} f'_+(t_0) t_0 \quad \text{if } \sigma < \rho, \tag{4.2.35}$$

thereby verifying (4.2.29) in all cases. □

Example 4.2.14. Suppose $(f_n)_n$ is a sequence of approximating integrands as in Definition 4.2.1, except that each f_n only satisfies $(\mathrm{Dec})_2^{\mathrm{n}}$ in place of $(\mathrm{Dec})_2^{\mathrm{c}}$. Then we can define a modified family $(\bar{f}_n)_n$ by

$$\bar{f}_n(t) = \int_0^t \frac{2 f_n(s)}{s} \,\mathrm{d}s \quad \text{for all } t \ge 0, \tag{4.2.36}$$

which satisfies Definition 4.2.1, including $(\mathrm{Dec})_2^{\mathrm{c}}$, and that $f_n \le \bar{f}_n$ pointwise using $(\mathrm{Dec})_2^{\mathrm{n}}$. Moreover, integrating by parts gives

$$\mathcal{V}(\bar{f}_n) = \int_1^\infty \bar{f}_n(s) \,\frac{\mathrm{d}s}{s^3} = \mathcal{V}(f_n) + \frac{1}{2}\bar{f}_n(1). \tag{4.2.37}$$

This construction can be used to prove Theorem 4.1.1 assuming only $(\mathrm{Dec})_2^{\mathrm{n}}$, which we will sketch in Remark 4.6.5.

4.2.4 Existence of minima

We will briefly discuss the existence of minima for our approximate problems, using the Direct Method. Here $\Omega \subset \mathbb{R}^2$ will be a bounded Lipschitz domain, and $\mathcal{N}$ will be a compact Riemannian manifold embedded in $\mathbb{R}^\nu$; we will make this precise in Section 4.3. Compactness of $\mathcal{N}$ implies that there exists a constant $C > 0$ depending only on $\mathcal{N}$ such that any measurable map $u : \Omega \to \mathcal{N}$ satisfies $\|u\|_{\mathrm{L}^\infty(\Omega)} \leq C$; we will therefore never insist on the L^p–convergence of mappings and will express our results in terms of L^1–convergence.

4.2.4.1 Existence of competitors

Lemma 4.2.15. *Let* $g \in \mathrm{W}^{1/2,2}(\partial\Omega; \mathcal{N})$. *If the vortex energy* $\mathcal{V}(f)$ *is finite, the admissible class* $\mathrm{A}_g(f)$ *in* (4.1.3) *is non-empty.*

Two proofs of Lemma 4.2.15 are available. One is established in [43, Prop. 2.19] by showing that there exists a renormalisable map (see Proposition 4.3.6) realizing the constraints. Since renormalisable mappings have gradient in $\mathrm{L}^{2,\infty}$ by [32, Thm. 5.1], combining this with Lemma 4.2.2 we infer the result. Alternatively, we can use the fact that any map $g \in \mathrm{W}^{1/2,2}(\partial\Omega; \mathcal{N})$ can be extended into a map whose gradient is $\mathrm{L}^{2,\infty}(\Omega)$, with a non-linear estimate relating the $\mathrm{L}^{2,\infty}$-norm and the $\mathrm{W}^{1/2,2}(\partial\Omega; \mathcal{N})$-norm; see [14, 35].

4.2.4.2 The Direct Method

Since f is assumed to be convex, we have $t \mapsto \frac{f(t)}{t}$ is non-decreasing on $[0, \infty)$. Hence we either have f is *asymptotically linear* in that

$$f^\infty(1) \doteq \lim_{t\to\infty} \frac{f(t)}{t} \quad \text{exists and is finite,} \tag{4.2.38}$$

or f is *asymptotically superlinear* in that

$$\lim_{t\to\infty} \frac{f(t)}{t} = \infty. \tag{4.2.39}$$

In the asymptotically linear case, the Direct Method naturally leads us to consider generalised minimisers in the class BV of functions of bounded variation; this requires us to give a meaning to

$$\mathcal{F}(u, \Omega) \doteq \int_\Omega f(|\mathrm{D}u|)\,\mathrm{d}x \tag{4.2.40}$$

for mappings $u \in \mathrm{BV}(\Omega;\mathcal{N}) \doteq \mathrm{BV}(\Omega;\mathbb{R}^\nu) \cap \{u(x) \in \mathcal{N} \text{ a.e. in } \Omega\}$. This will be done through the $\mathrm{L}^1(\Omega;\mathbb{R}^\nu)$-*relaxation*

$$\begin{aligned}(4.2.41)\quad &\overline{\mathcal{F}}(u,\Omega)\\ &= \inf\left\{ \varliminf_{n\to\infty} \mathcal{F}(u_n,\Omega) : u_n \in \mathrm{W}^{1,1}_g(\Omega;\mathbb{R}^\nu),\ u_n \to u \text{ strongly in } \mathrm{L}^1(\Omega)\right\}.\end{aligned}$$

Here $\mathrm{W}^{1,1}_g(\Omega;\mathcal{N})$ denotes the space of functions $u \in \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ such that $\mathrm{tr}_{\partial\Omega}\, u = g$, which corresponds to the admissible class (4.1.3) with the choice $f(z) = |z|$.

Proposition 4.2.16. *Let f be a Young function and $g \in \mathrm{W}^{1/2,2}(\Omega;\mathcal{N})$.*

(a) *If* (4.2.38) *holds, then $\overline{\mathcal{F}}$ admits a minimiser $u \in \mathrm{BV}(\Omega;\mathcal{N})$.*

(b) *If* (4.2.39) *holds, then $\mathcal{F}$ admits a minimiser $u \in \mathrm{W}^{1,1}_g(\Omega;\mathcal{N})$.*

To treat both cases at once, we rely a following representation formula, see Lemma 4.2.17. We provide a proof here of Proposition 4.2.16 and Lemma 4.2.17, although results characterizing lower-semicontinous relaxations appear to be already known, see *e.g.* [2, 1].

Lemma 4.2.17. *Let f be a Young function which is asymptotically linear in that* (4.2.38) *holds. Then the $L^1(\Omega;\mathbb{R}^\nu)$-relaxation $\overline{\mathcal{F}}(\cdot,\Omega)$ admits the measure representation*

$$\begin{aligned}(4.2.42)\quad &\overline{\mathcal{F}}(u,\Omega)\\ &= \int_\Omega f(|\mathrm{D^a}u|)\,\mathrm{d}x + f^\infty(1)|\mathrm{D^s}u|(\Omega) + f^\infty(1)\int_{\partial\Omega} |\mathrm{tr}_{\partial\Omega}\, u - g|\,\mathrm{d}\mathcal{H}^1\end{aligned}$$

for $u \in \mathrm{BV}(\Omega;\mathcal{N})$, where $\mathrm{D}u = \mathrm{D^a}u\,\mathcal{L}^2 + \mathrm{D^s}u$ is the Lebesgue decomposition of the gradient. Moreover, this relaxation is weakly lower semicontinuous in $\mathrm{BV}(\Omega;\mathcal{N})$.*

Proof of Lemma 4.2.17. Denote the right-hand side of (4.2.42) by $\widetilde{\mathcal{F}}(\cdot,\Omega)$, which by [18, Thm. 3] is weakly* sequentially lower-semicontinuous in $\mathrm{BV}(\Omega;\mathbb{R}^\nu)$, noting the recession function is given by $f^\infty(|z|) = f^\infty(1)|z|$. Since $\widetilde{\mathcal{F}}(v,\Omega) = \mathcal{F}(v,\Omega)$ for all $v \in \mathrm{W}^{1,1}_g(\Omega;\mathbb{R}^\nu)$, from the lower-semicontinuity it follows that $\widetilde{\mathcal{F}}(u,\Omega) \le \overline{\mathcal{F}}(u,\Omega)$ for all $u \in \mathrm{BV}(\Omega;\mathbb{R}^\nu)$. Moreover, if $u \in \mathrm{BV}(\Omega;\mathcal{N})$, letting $\widetilde{\Omega} \Supset \Omega$ be a Lipschitz domain and $G \in \mathrm{W}^{1,2}(\widetilde{\Omega} \setminus \overline{\Omega};\mathbb{R}^\nu)$ be an extension of g, by [8, Lem. B.2] there exists a sequence $U_k \in \mathrm{W}^{1,2}(\widetilde{\Omega};\mathbb{R}^\nu)$ such that $U_k = G$ on $\widetilde{\Omega}\setminus\overline{\Omega}$ and U_k converges BV-strictly to

$$(4.2.43)\qquad U(x) = \begin{cases} u(x) & \text{if } x \in \Omega,\\ G(x) & \text{if } x \in \widetilde{\Omega} \setminus \overline{\Omega};\end{cases}$$

that is $U_k \to U$ strongly in L^1 and $|\mathrm{D}U_k|(\widetilde{\Omega}) \to |\mathrm{D}U|(\widetilde{\Omega})$. Then by Reshetnyak's theorem (see *e.g.* [37, Thm. 10.3]) we infer the restrictions $u_k = U_k|_\Omega \in \mathrm{W}^{1,2}_g(\Omega;\mathbb{R}^\nu)$ satisfies

$$\lim_{k\to\infty} \int_\Omega f(|\mathrm{D}u_n|)\,\mathrm{d}x = \widetilde{\mathcal{F}}(u,\Omega), \tag{4.2.44}$$

so the converse inequality follows, thereby establishing that $\widetilde{\mathcal{F}} = \overline{\mathcal{F}}$. □

Proof of Proposition 4.2.16. If (4.2.38) holds, then by Lemma 4.2.17 we have $\overline{\mathcal{F}}$ is weakly* sequentially lower semicontinuous on $\mathrm{BV}(\Omega;\mathcal{N})$, so by the Direct Method we can infer the existence of a minimiser. Indeed, since (4.1.3) is non-empty by Lemma 4.2.15, we can take a minimising sequence $(u_n)_n$, and assume f is not identically zero (else every admissible map is a minimiser). Then, there is some $t_0 \geq 0$ such that $f'(t_0) > 0$, and so $f(t) \geq f'(t_0)t$ for all $t \geq t_0$, so it follows that $(|\mathrm{D}u_n|)_n$ is bounded in $\mathrm{L}^1(\Omega)$. Therefore, we can pass to a weakly* convergent subsequence $u_{n_k} \overset{*}{\rightharpoonup} u$ in $\mathrm{BV}(\Omega;\mathcal{N})$, and by lower-semicontinuity of $\overline{\mathcal{F}}$ it follows that u is a minimiser.

In the asymptotically superlinear case where (4.2.39) holds, for $\kappa > 0$ we consider the truncations $T_\kappa f$ from Definition 4.2.9, noting that $T_\kappa f^\infty(1) = f'_+(\kappa)$. Then if we take a minimising sequence $(u_n)_n$ for $\mathcal{F}$ in $\mathrm{A}_g(f)$, then analogously to the asymptotically linear case above, we obtain a weak*-limit $u \in \mathrm{BV}(\Omega;\mathcal{N})$. By lower semicontinuity applied to the relaxation $\overline{\mathcal{F}_\kappa}$ associated to $T_\kappa f$, we have

$$\begin{aligned}\overline{\mathcal{F}_\kappa}(u,\Omega) &= \int_\Omega T_\kappa f(|\mathrm{D}^{\mathrm{a}}u|)\,\mathrm{d}x \\ &\quad + f'_+(\kappa)|\mathrm{D}^{\mathrm{s}}u|(\Omega) + f'_+(\kappa)\int_{\partial\Omega} |\mathrm{tr}_{\partial\Omega}\, u - g|\,\mathrm{d}\mathcal{H}^1 \\ &\leq \varliminf_{n\to\infty} \int_\Omega T_\kappa f(|\mathrm{D}u_n|)\,\mathrm{d}x \leq \varliminf_{n\to\infty} \mathcal{F}(u_n,\Omega) < \infty.\end{aligned} \tag{4.2.45}$$

By (4.2.39) and (4.2.22) we have $f'_+(\kappa) \to \infty$ as $\kappa \to \infty$, so it follows that $|\mathrm{D}^{\mathrm{s}}u|(\Omega) = 0$ and that $\mathrm{tr}_{\partial\Omega}\, u = g$ on $\partial\Omega$. Therefore $u \in \mathrm{W}^{1,1}(\Omega;\mathcal{N})$, and by the monotone convergence theorem we have

$$\mathcal{F}(u,\Omega) = \lim_{\kappa\to\infty} \int_\Omega T_\kappa f(|\mathrm{D}u|)\,\mathrm{d}x \leq \varliminf_{n\to\infty} \mathcal{F}(u_n,\Omega). \tag{4.2.46}$$

Since $(u_n)_n$ was a minimising sequence, it follows that u is a minimiser of $\mathcal{F}$. □

Remark 4.2.18. Under (4.2.38), the minima u in $\mathrm{BV}(\Omega;\mathcal{N})$ obtained in Proposition 4.2.16(a) do not necessarily satisfy $\mathrm{tr}_{\partial\Omega}\, u = g$, where we consider the interior trace in the sense of *e.g.*, [16, Thm. 5.6]. This may arise due to potential jumps across the boundary, which is penalised by the term $f^\infty(1)\int_{\partial\Omega}|\mathrm{tr}_{\partial\Omega}\, u - g|\,\mathrm{d}\mathcal{H}^1$ in the relaxation $\overline{\mathcal{F}}(u)$. For this reason, in this setting our analysis will be confined to almost minimising sequences in $\mathrm{W}^{1,1}(\Omega;\mathcal{N})$ (see Theorem 4.1.2). In the superlinear case (4.2.39) the trace attainment is not an issue, as the proof of Proposition 4.2.16(b) shows.

4.3 Energies and topological quantities

In this section, we introduce scalar quantities that measure the non trivial-ness of the free homotopy class of a map $\partial\Omega \to \mathcal{N}$ as well as the renormalised energy, which the limiting energy maps will minimize.

Throughout the whole text, the domain Ω is assumed to be an open connected bounded set of $\mathbb{R}^2$ with Lipschitz boundary. Concerning the manifold $\mathcal{N}$, our setting covers the case where $\mathcal{N}$ is a smooth compact connected Riemannian manifold which is assumed to be isometrically embedded in the Euclidean space $\mathbb{R}^\nu$ (with canonical metric) for some $\nu \in \mathbb{N}$. Isometric embedding is not a restriction by Nash embedding theorem [33].

We will also denote by $\mathbb{B}(a,\rho)$ the closed ball in $\mathbb{R}^2$ centred at $a \in \mathbb{R}^2$ with radius $\rho > 0$, and we will sometimes denote the boundary by $\mathbb{S}^1(a,\rho) \doteq \partial\mathbb{B}(a,\rho)$. The unit circle will be denoted by $\mathbb{S}^1 = \mathbb{S}^1(0,1)$.

4.3.1 Topological quantities

4.3.1.1 Minimal length of the free homotopy class

A curve a vanishing mean oscillation $\gamma \in \mathrm{VMO}(\mathbb{S}^1;\mathcal{N})$ can be given a free homotopy class that extends the notion in the continuous category (see [12]). In the sequel all the homotopy are assumed to be free. Every $\mathrm{VMO}(\mathbb{S}^1;\mathcal{N})$-homotopy class contains a smooth length minimising geodesic in its homotopy class; indeed any $\mathrm{VMO}(\mathbb{S}^1;\mathcal{N})$ curve is VMO-homotopic to a smooth curve by [12, Cor. 4], so the claim follows from the classical result for the smooth category (see *e.g.* [26, Prop. 6.28]). In particular, the following infimum is reached by such a geodesic.

Definition 4.3.1. *Given* $\gamma \in \mathrm{VMO}(\mathbb{S}^1;\mathcal{N})$*, denote by* $[\gamma]$ *its* $\mathrm{VMO}(\mathbb{S}^1;\mathcal{N})$*-homotopy class. The* minimal length of the free homotopy class $[\gamma]$ *is*

$$\lambda([\gamma]) = \inf\left\{\int_{\mathbb{S}^1}|\tilde\gamma'| : \tilde\gamma \in \dot{\mathrm{W}}^{1,1}(\mathbb{S}^1,\mathcal{N}) \text{ and } \tilde\gamma \text{ is freely homotopic to } \gamma\right\}.$$

The *systole* of the manifold $\mathcal{N}$ is the length of the shortest closed non-trivial geodesic on $\mathcal{N}$, namely,

$$\operatorname{sys}(\mathcal{N}) \doteq \inf\{\lambda([\gamma]) : \gamma \in \operatorname{VMO}(\mathbb{S}^1, \mathcal{N}) \text{ is not freely homotopic to a constant}\}. \tag{4.3.1}$$

The *length spectrum* of the manifold $\mathcal{N}$ is the set

$$\{\lambda([\gamma]) : \gamma \in \operatorname{VMO}(\mathbb{S}^1, \mathcal{N})\} \subset \{0\} \cup [\operatorname{sys}(\mathcal{N}), +\infty)$$

Lemma 4.3.2. *The length spectrum is a discrete set of the real line.*

In particular, we have $\operatorname{sys}(\mathcal{N}) > 0$. We refer to [31, Prop. 3.2] for a proof; note that compactness of $\mathcal{N}$ is crucial for Lemma 4.3.2 to hold.

If $u \in \mathrm{W}^{1,2}(\mathbb{B}(a,\sigma) \setminus \mathbb{B}(a,\rho); \mathcal{N})$ for some $a \in \mathbb{R}^2$, $\sigma > 0$ and all $\rho \in (0,\sigma)$, the trace $\operatorname{tr}_{\mathbb{S}^1(a,\rho)} u \in \mathrm{W}^{1/2,2}(\mathbb{S}^1(a,\rho); \mathcal{N}) \subset \operatorname{VMO}(\mathbb{S}^1(a,\rho); \mathcal{N})$ has a well defined homotopy class which is independent of ρ. Hence, minimal length of the free homotopy class of u at a is defined by

$$\lambda([u,a]) \doteq \lambda(\operatorname{tr}_{\mathbb{S}^1(a,\rho)} u). \tag{4.3.2}$$

4.3.1.2 Singular energy

We will use the following variant of the notion of the topological resolution [32, Def. 2.2].

Definition 4.3.3. *A finite family of closed curves $\gamma_i \in \operatorname{VMO}(\mathbb{S}^1; \mathcal{N})$, $i = 1, \dots, k$, with $k \in \mathbb{N}$ is a* topological resolution *of the map g whenever there exist k non-intersecting closed balls $\mathbb{B}(a_i; \rho_i) \subset \Omega$ with $\rho_i > 0$ and a map $u \in \mathrm{W}^{1,2}(\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i; \rho_i); \mathcal{N})$ such that $\operatorname{tr}_{\partial\Omega} u$ and g are homotopic in $\operatorname{VMO}(\partial\Omega; \mathcal{N})$ and $\operatorname{tr}_{\mathbb{S}^1} u(a_i + \rho\cdot)$ and γ_i are homotopic in $\operatorname{VMO}(\mathbb{S}^1; \mathcal{N})$ for $i = 1, \dots, k$.*

Definition 4.3.4. *Let $g \in \operatorname{VMO}(\partial\Omega; \mathcal{N})$, the* singular energy *of g is*

$$\mathrm{E}^{1,2}_{\mathrm{sg}}([g]) = \inf\left\{\sum_{i=1}^k \frac{\lambda([\gamma_i])^2}{4\pi} : \begin{array}{c} k \in \mathbb{N}, \gamma_1, \dots, \gamma_k \in \operatorname{VMO}(\mathbb{S}^1; \mathcal{N}) \\ \textit{is a topological resolution of } g \end{array}\right\}. \tag{4.3.3}$$

Observe that (see [32, Prop. 2.6]) the singular energy is *subadditive* in the sense that, if $(\gamma_i)_{i=1}^k$ is a topological resolution of g, we have

$$\mathrm{E}^{1,2}_{\mathrm{sg}}([g]) \leq \sum_{i=1}^k \mathrm{E}^{1,2}_{\mathrm{sg}}([\gamma_i]). \tag{4.3.4}$$

A *minimal* topological resolution of g is topological resolution that reach the infimum (4.3.3). A curve $\gamma \in \mathrm{VMO}(\mathbb{S}^1;\mathcal{N})$ is said *atomic* if $\mathrm{E}^{1,2}_{\mathrm{sg}}([\gamma]) = \lambda([\gamma])^2/4\pi$. Often we will write $\mathrm{E}^{1,2}_{\mathrm{sg}}(g)$ instead of $\mathrm{E}^{1,2}_{\mathrm{sg}}([g])$.

A similar construction to (4.3.2) can be done to define $\mathrm{E}^{1,2}_{\mathrm{sg}}([u,a])$ if $u \in \bigcup_{\rho>0} \mathrm{W}^{1,2}(\mathbb{B}(a,\sigma) \setminus \mathbb{B}(a,\sigma);\mathcal{N})$ for some $\sigma > 0$.

4.3.2 Renormalisable mappings

In our analysis we will consider maps that are $\mathrm{W}^{1,2}$-regular away from at most finitely many points, for which we introduce the following space.

Definition 4.3.5. *Given $\delta > 0$, let $\mathrm{R}^{1,2}_{\delta}(\Omega;\mathcal{N})$ denote the space of mappings $u \in \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ for which there is $k \in \mathbb{N}$ and points $a_1,\dots,a_k \in \Omega$ satisfying* $\operatorname{dist}(\{a_i\}_{i=1}^k, \partial\Omega) \geq \delta$ *such that*

$$u \in \mathrm{W}^{1,2}\left(\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i,\rho);\mathcal{N}\right) \text{ for all } \rho > 0. \tag{4.3.5}$$

The points $a_1,\dots,a_k$ will be referred to as the singularities *of u. We also define* $\mathrm{R}^{1,2}(\Omega;\mathcal{N}) = \bigcup_{\delta>0} \mathrm{R}^{1,2}_{\delta}(\Omega;\mathcal{N})$.

We will later show in Lemma 4.5.10 that this class in dense in $\mathrm{W}^{1,1}(\Omega;\mathcal{N})$. Also given finitely many points $a_1,\dots,a_k \in \Omega$, we will introduce the notation

$$\rho_\Omega(a_1,\dots,a_k) \doteq \min\{\operatorname{dist}(a_i,\partial\Omega), |a_i - a_j| : 1 \leq i < j \leq k\} > 0. \tag{4.3.6}$$

We also record the class of renormalisable mappings introduced in [32, Def. 7.1], which further specifies the behaviour of the mappings at the singular points. The following is proven in [43, Prop. 2.16] (see also [32, Prop. 7.2]).

Proposition 4.3.6. *Let $u \in \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ and let $a_1,\dots,a_k \in \Omega$ be $k \in \mathbb{N}$ distinct points. Then the following are equivalent:*

(i) *The renormalised energy*

$$\mathrm{E}^{1,2}_{\mathrm{ren}}(u) = \lim_{\rho\to 0}\left[\int_{\Omega\setminus\bigcup_{i=1}^k \mathbb{B}(a_i,\rho)} \frac{|\mathrm{D}u|^2}{2}\,\mathrm{d}x - \sum_{i=1}^{k} \frac{\lambda([u,a_i])^2}{4\pi}\log\frac{1}{\rho}\right] \tag{4.3.7}$$

exists as a limit and is finite.

(ii) *For each $0 < \rho < \rho_\Omega(a_1,\dots,a_k)$, $|\mathrm{D}u| \in \mathrm{L}^2(\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i,\rho))$ and the map*

$$r \mapsto \int_{\mathbb{S}^1(a_i,r)} \frac{|\mathrm{D}u|^2}{2}\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])^2}{4\pi r} \tag{4.3.8}$$

is non-negative and integrable on $(0, \rho_\Omega(a_1, \dots, a_k))$.

In any case for all $0 < \rho < \rho_\Omega(a_1, \dots, a_k)$,

$$\begin{aligned}(4.3.9)\quad &\mathrm{E}^{1,2}_{\mathrm{ren}}(u) + \sum_{i=1}^{k} \frac{\lambda([u,a_i])^2}{4\pi} \log\frac{1}{\rho} \\ &= \int_{\Omega \setminus \bigcup_{i=1}^{k} \mathbb{B}(a_i,\rho)} \frac{|\mathrm{D}u|^2}{2}\,\mathrm{d}x + \sum_{i=1}^{k} \int_0^\rho \left[\int_{\mathbb{S}^1(a_i,r)} \frac{|\mathrm{D}u|^2}{2}\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])^2}{4\pi r}\right] \mathrm{d}r.\end{aligned}$$

Definition 4.3.7. *A map* $u \in \mathrm{W}^{1,1}(\Omega; \mathcal{N})$ *is said to be* renormalisable *if it satisfies any of the equivalent conditions in Proposition 4.3.6; that is* $u \in \mathrm{R}^{1,2}(\Omega;\mathcal{N})$ *and* (4.3.7) *is finite. We denote the space of all such mappings by* $\mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$.

Prototypical examples of such mappings are *vortex-type maps* taking the form $u_\gamma(x) = \gamma(x/|x|)$, where $\gamma \colon \mathbb{S}^1 \to \mathcal{N}$ is a minimising geodesic in its free homotopy class (*i.e.* $\lambda([\gamma]) = \int_{\mathbb{S}^1} |\dot\gamma|\,\mathrm{d}\mathcal{H}^1$). In this case, we have $u_\gamma \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\mathbb{B}(0,1);\mathcal{N})$ with $\mathrm{E}^{1,2}_{\mathrm{ren}}(u_\gamma) = 0$.

Later in Section 4.5, we will also use the notation $\mathrm{E}^{1,2}_{\mathrm{ren}}(u;\Omega)$ if we wish to specify the domain. In particular, note that

$$\begin{aligned}(4.3.10)\quad &\mathrm{E}^{1,2}_{\mathrm{ren}}(u;\mathbb{B}(a_i,\rho)) + \frac{\lambda([u,a_i])^2}{4\pi}\log\frac{1}{\rho} \\ &= \int_0^\rho \left[\int_{\partial\mathbb{B}(a_i,r)} \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])^2}{4\pi r}\right]\mathrm{d}r\end{aligned}$$

for each $i = 1, \dots, k$, and $\rho < \rho_\Omega(a_1, \dots, a_k)$.

In [32, Prop. 7.2(iii)] it is shown that a property of renormalised maps is that for each $i \in \{1, \dots, k\}$, there exists a sequence $(\rho_\ell)_{\ell\in\mathbb{N}}$ converging to zero such that the sequence $(\mathrm{tr}_{\mathbb{S}^1} u(a_i + \rho_\ell \cdot))_{\ell\in\mathbb{N}}$ converges strongly to a geodesic $\gamma_i \in \mathrm{C}^1(\mathbb{S}^1;\mathcal{N})$ in $\mathrm{W}^{1,2}(\mathbb{S}^1;\mathcal{N})$. We show in the next Example 4.3.8 that the selection of a sequence is in general necessary. However, the geodesics obtained along different subsequences lie in the same homotopy class (and moreover the same synharmony class; *cf.* [32, Def. 3.2, Prop. 7.2(iv)]) (see also (4.7.1) in Section 4.7).

Example 4.3.8 (Non-uniqueness of the limiting geodesic)**.** Let $\mathbb{B} = \mathbb{B}(0,1) \subset \mathbb{R}^2 \cong \mathbb{C}$ be the unit disc, and consider polar coordinates $(r,\theta) \mapsto z = re^{i\theta}$. Then define $u \colon \mathbb{B} \to \mathbb{S}^1$ by

$$(4.3.11)\qquad u(re^{i\theta}) = e^{i\theta + i\eta(r)}, \quad \text{where } \eta(r) = \log\log\left(1 + \frac{1}{r}\right).$$

Then since $|\mathrm{D}u|^2 = |\eta'(r)|^2 + \frac{1}{r^2}$, for $\rho \in (0,1)$ we have

$$(4.3.12)\quad \int_{\mathbb{B}\setminus\mathbb{B}(0,\rho)} \frac{1}{2}|\mathrm{D}u|^2 \, \mathrm{d}z - \pi \log\frac{1}{\rho} = \pi \int_\rho^1 r|\eta'(r)|^2 + \frac{1}{r}\, \mathrm{d}r - \pi\log\frac{1}{\rho},$$

which remains bounded as $\rho \to 0$ since $r|\eta'(r)|^2 \in \mathrm{L}^1((0,1))$, thereby defining a map in $\mathrm{W}^{1,2}_{\mathrm{ren}}(\mathbb{B}, \mathbb{S}^1)$. However if choose subsequences $\rho_k, \tilde{\rho}_k \searrow 0$ satisfying

$$(4.3.13)\qquad \log\log\left(1 + \frac{1}{\rho_k}\right) = 2k\pi, \quad \log\log\left(1+\frac{1}{\tilde{\rho}_k}\right) = 2k\pi - \frac{\pi}{2}$$

for all k, we see that

$$(4.3.14)\qquad \mathrm{tr}_{\mathbb{S}^1} u(\rho_n \cdot) \to \mathrm{id}_{\mathbb{S}^1}, \quad \mathrm{tr}_{\mathbb{S}^1} u(\tilde{\rho}_n \cdot) \to e^{-i}\,\mathrm{id}_{\mathbb{S}^1},$$

exhibiting non-uniqueness of the limiting geodesic.

We can also construct a similar example in the case of the flat torus $\mathcal{N} = \mathbb{S}^1 \times \mathbb{S}^1 \subset \mathbb{C}^2$, by defining $v(re^{i\theta}) = (e^{i\theta}, e^{i\eta(r)})$ with η as above. An analogous argument shows that $v \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\mathbb{B}; \mathcal{N})$, and along the same subsequences $\rho_k, \tilde{\rho}_k$, we get two geodesics with disjoint images, namely $(\mathrm{id}_{\mathbb{S}^1}, 1)$ and $(\mathrm{id}_{\mathbb{S}^1}, -1)$.

4.4 Asymptotic bounds

In this section, we derive bounds for our formal Γ-convergence analysis, starting with an upper bound in Proposition 4.4.1. Then, through a merging ball process, a lower bound for each fixed n is established in Proposition 4.4.7, which allows us to establish asymptotic lower bounds detailed in Proposition 4.4.10.

4.4.1 Upper bound

We will show the renormalised limit along a fixed map is controlled by the renormalised energy, which corresponds to the Γ-limsup in the Γ-convergence analysis.

Proposition 4.4.1. *Let $u \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega; \mathcal{N})$ have $k \in \mathbb{N}$ distinct singularities $a_1, \ldots, a_k \in \Omega$. Then if $(f_n)_n$ is a family of approximating integrands (as in Definition 4.2.1), we have*

$$(4.4.1)\quad \lim_{n\to\infty} \left[\int_\Omega f_n(|\mathrm{D}u|)\,\mathrm{d}x - \mathcal{V}(f_n) \sum_{i=1}^k \frac{\lambda([u,a_i])^2}{4\pi} \right] = \mathrm{E}^{1,2}_{\mathrm{ren}}(u) + \mathrm{H}([u,a_i])_{i=1}^{\kappa},$$

where we define

$$\mathrm{H}([u,a_i])_{i=1}^{\kappa} \doteq \sum_{i=1}^{k} \frac{\lambda([u,a_i])^2}{4\pi} \log \frac{2\pi}{\lambda([u,a_i])}. \tag{4.4.2}$$

We will record the following elementary calculation as we will need it several times later. Note that it also shows how the $\mathrm{H}([u,a_i])_{i=1}^{\kappa}$ term arises.

Lemma 4.4.2. *Let $(f_n)_n$ be as in Definition 4.2.1. Then for all $\lambda, \rho > 0$, we have*

$$\lim_{n\to\infty} \left[\int_0^\rho 2\pi r f_n\left(\frac{\lambda}{2\pi r}\right) \mathrm{d}r - \mathcal{V}(f_n)\frac{\lambda^2}{4\pi} \right] = \frac{\lambda^2}{4\pi} \log \frac{2\pi\rho}{\lambda^2}. \tag{4.4.3}$$

Proof of Lemma 4.4.2. For each n, change of variables $r \mapsto \frac{2\pi r}{\lambda}$ in the first integral gives

$$\int_0^\rho 2\pi r f_n\left(\frac{\lambda}{2\pi r}\right) \mathrm{d}r - \mathcal{V}(f_n)\frac{\lambda^2}{4\pi} = \frac{\lambda^2}{2\pi} \int_1^{\frac{2\pi\rho}{\lambda}} f_n\left(\frac{1}{r}\right) r \, \mathrm{d}r, \tag{4.4.4}$$

and sending $n \to \infty$ using the dominated convergence theorem we have

$$\lim_{n\to\infty} \int_1^{\frac{2\pi\rho}{\lambda}} f_n\left(\frac{1}{r}\right) r\,\mathrm{d}r = \frac{1}{2}\int_1^{\frac{2\pi\rho}{\lambda}} \frac{\mathrm{d}r}{r} = \frac{1}{2}\log\frac{2\pi\rho}{\lambda}, \tag{4.4.5}$$

thereby establishing the result. $\square$

Remark 4.4.3. In the case of the p-energy $f_p(t) = \frac{t^p}{p}$ (case (a)), it was shown in [43, Prop. 3.1] that

$$\overline{\lim_{p\nearrow 2}} \left[\int_\Omega \frac{|\mathrm{D}u|^p}{p} - \frac{1}{2-p} \sum_{i=1}^{\kappa} \frac{\lambda([u,a_i])^2}{4\pi} \right] \le \mathrm{E}^{1,2}_{\mathrm{ren}}(u) + \sum_{i=1}^{k} \frac{\lambda([u,a_i])^2}{4\pi} \left[\frac{1}{2} + \log\left(\frac{2\pi}{\lambda([u,a_i])} \right) \right].$$

This is consistent with Proposition 4.4.1 above, and the discrepancy in the remainder term arises because $\mathcal{V}(f_p) = \frac{2}{p} \cdot \frac{1}{2-p} = \frac{1}{2-p} + \frac{1}{2} + o(1)$ as $p \nearrow 2$.

Proof of Proposition 4.4.1. By Lemmas 4.2.12 and 4.2.13, there exists $C_1, C_2 > 0$ such that

$$|f_n(t)| \le C_1(1+t^2), \tag{4.4.6}$$

$$f_n(t) - f_n(s) \le \frac{f'_{n,+}(\max\{s,t_0\})}{2\max\{s,t_0\}}(t^2 - s^2) + C_2, \tag{4.4.7}$$

for all n and $s,t \geq 0$, where $t_0 \geq 1$ is as in the $(\mathrm{Dec})_2^{\mathrm{c}}$ condition, chosen uniformly in n. Fixing $\rho \in (0, \rho_\Omega(a_1, \ldots, a_k)$, we will first show that

$$\lim_{n\to\infty} \int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u|) = \int_{\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i,\rho)} \frac{|\mathrm{D}u|^2}{2}. \tag{4.4.8}$$

This follows by the dominated convergence theorem; since $|\mathrm{D}u| \in \mathrm{L}^2(\Omega \setminus \bigcup_{i=1}^k \mathbb{B}(a_i,\rho))$ by Proposition 4.3.6, using (4.4.6) we obtain a uniform majorant.

We now consider the contribution in $\mathbb{B}(a_i, r)$ for each $i = 1, \ldots, k$. Using polar coordinates, we can write

$$\begin{aligned} &\int_{\mathbb{B}(a_i,r)} f_n(|\mathrm{D}u|)\,\mathrm{d}x - \mathcal{V}(f_n)\frac{\lambda([u,a_i])^2}{4\pi} \\ &\quad = \int_0^\rho \int_{\mathbb{S}^1(a_i,r)} f_n(|\mathrm{D}u|) - f_n\left(\frac{\lambda([u,a_i])}{2\pi r}\right)\mathrm{d}\mathcal{H}^1\,\mathrm{d}r \\ &\qquad + \int_0^\rho 2\pi r f_n\left(\frac{\lambda([u,a_i])}{2\pi r}\right)\mathrm{d}r - \frac{\lambda([u,a_i])^2}{2\pi}\int_0^1 f_n(1/r) r\,\mathrm{d}r \\ &\quad \doteq \mathrm{I}(n) + \mathrm{II}(n). \end{aligned} \tag{4.4.9}$$

For the first term, we will show that

$$\lim_{n\to\infty} \mathrm{I}(n) = \int_0^\rho \int_{\mathbb{S}^1(a_i,r)} \frac{|\mathrm{D}u|^2}{2}\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])^2}{4\pi r}\,\mathrm{d}r \tag{4.4.10}$$

To see this, observe that for almost all $r \in (0,1)$ such that $|\mathrm{D}u| \in \mathrm{L}^2(\partial\mathbb{B}(a_i,r))$, we have

$$\begin{aligned} \lim_{n\to\infty} \int_{\mathbb{S}^1(a_i,r)} f_n(|\mathrm{D}u|) - f_n\left(\frac{\lambda([u,a_i])}{2\pi r}\right)\mathrm{d}r \\ = \int_{\mathbb{S}^1(a_i,r)} \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])}{4\pi r}, \end{aligned} \tag{4.4.11}$$

and using (4.4.7) we have

$$\begin{aligned} 0 &\leq \int_{\mathbb{S}^1(a_i,r)} f_n(|\mathrm{D}u|) - f_n\left(\frac{\lambda([u,a_i])}{2\pi r}\right)\mathrm{d}r \\ &\leq \frac{f'_{n,+}(\max\{\lambda([u,a_i])/(2\pi r), t_0\})}{\max\{\lambda([u,a_i])/(2\pi r), t_0\}} \int_{\mathbb{S}^1(a_i,r)} |\mathrm{D}u|^2 \\ &\quad - \frac{\lambda([u,a_i])^2}{4\pi^2 r^2}\,\mathrm{d}\mathcal{H}^1 + 2\pi C_2 r \\ &\leq \left(\sup_{n\in\mathbb{N}} \frac{f'_{n,+}(t_0)}{t_0}\right)\left[\int_{\mathbb{S}^1(a_i,r)} \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u,a_i])}{4\pi r}\right] + 2\pi C_2 r, \end{aligned} \tag{4.4.12}$$

using $(\mathrm{Dec})_2^{\mathrm{c}}$ and non-negativity of the integral in the last line, noting the supremum is finite by Lemma 4.2.10(b). Thus, we have a uniform upper bound which is integrable on $(0,\rho)$ by Proposition 4.3.6(ii), from which (4.4.10) follows by the dominated convergence theorem. For the second term, we use Lemma 4.4.2, which gives

$$\begin{aligned} \lim_{n\to\infty} \mathrm{II}(n) &= \frac{\lambda([u,a_i])^2}{2\pi} \lim_{n\to\infty} \int_1^{\frac{2\pi\rho}{\lambda([u,a_i])}} f_n(1/r) r \,\mathrm{d}r \\ &= \frac{\lambda([u,a_i])^2}{2\pi} \int_1^{\frac{2\pi\rho}{\lambda([u,a_i])}} \frac{1}{2r} \,\mathrm{d}r = \frac{1}{2} \log \frac{\lambda([u,a_i])}{2\pi\rho}. \end{aligned} \tag{4.4.13}$$

Combining (4.4.10) and (4.4.13) in (4.4.9), the result follows. □

Note that, instead of using $(\mathrm{Dec})_2^{\mathrm{c}}$ to establish (4.4.12), one could have assumed that there exists $A, B > 0$ such that for all $s, t \in \mathbb{R}_+$ and all $n \in \mathbb{N}$, $f_n(t) - f_n(s) \leq A\left(t^2 - s^2\right) + B$.

4.4.2 Ball merging construction

In this section, we will consider a Young function f satisfying $(\mathrm{Dec})_2^{\mathrm{n}}$ and (vInt), and establish lower bounds for the functional $\int_\Omega f(|\mathrm{D}u|)\,\mathrm{d}x$ which will capture the singularities of u. For this we start with a fundamental lower bound for annuli, which we denote by

$$\mathbb{A}(a,\rho,\sigma) \doteq \{x \in \mathbb{R}^2 : \rho \leq |x-a| \leq \sigma\} = \mathbb{B}(a,\rho) \setminus \mathbb{B}^\circ(a,\sigma)$$

where $a \in \mathbb{R}^2$ is the centre and the endpoints are given by $0 \leq \rho \leq \sigma$, recalling that balls are always chosen to be closed.

Lemma 4.4.4. *Let $u \in \mathrm{W}^{1,2}(\mathbb{A}(a,\rho,\sigma);\mathcal{N})$ with $a \in \mathbb{R}^2$ and $0 \leq \rho < \sigma < \frac{\mathrm{sys}(\mathcal{N})}{2\pi t_0}$ (with no restriction if $t_0 = 0$). If $E \doteq \lambda(\mathrm{tr}_{\partial\mathbb{B}_\sigma(a)} u(a+\sigma\,\cdot))^2/4\pi$ is non-zero, we have*

$$\int_{\mathbb{A}(a,\rho,\sigma)} f(|\mathrm{D}u|)\,\mathrm{d}x \geq E\left(\Lambda\left(\frac{\sigma}{E}\right) - \Lambda\left(\frac{\rho}{E}\right)\right), \tag{4.4.14}$$

where

$$\Lambda(t) = \Lambda_f(t) \doteq \frac{1}{2}\mathrm{sys}(\mathcal{N})^2 \int_0^t f\left(\frac{2}{\mathrm{sys}(\mathcal{N})s}\right) s\,\mathrm{d}s, \tag{4.4.15}$$

which is finite by (vInt).

The quantity $\Lambda_f(\cdot)$ plays a similar role to the quantity $\Lambda_\varepsilon(\cdot)$ considered by JERRARD in [24, §3] in the context of Ginzburg-Landau functionals. Also setting $c_0 \doteq \frac{1}{2}\mathrm{sys}(\mathcal{N})$, we have

$$\Lambda(t) = 2c_0^2 \int_0^t f\left(\frac{1}{c_0 s}\right) s \,\mathrm{d}s. \tag{4.4.16}$$

We will often use Lemma 4.4.4 with $\rho = 0$, however the general form is also used for instance in (4.4.35). The result also holds when $E = 0$ if we interpret the right hand side of (4.4.14) to be zero in that case.

Proof of Lemma 4.4.4. By converting to radial coordinates and using Jensen's inequality, we have

$$\begin{aligned}\int_{\mathbb{A}(a,\rho,\sigma)} f(|\mathrm{D}u|)\,\mathrm{d}x &= 2\pi \int_\rho^\sigma \int_{\mathbb{S}^1} f\left(|\mathrm{D}u(a+r\omega)|\right)\,\mathrm{d}\mathcal{H}^1(\omega)\,r\,\mathrm{d}r \\ &\geq 2\pi \int_\rho^\sigma f\left(\frac{\lambda}{2\pi r}\right) r\,\mathrm{d}r,\end{aligned} \tag{4.4.17}$$

where $\lambda = \lambda([u,a])$ and where we have used that $u \in \mathrm{W}^{1,2}(\mathbb{A}(a,\rho,\sigma);\mathcal{N})$ in this annulus, so $\lambda = \lambda(\mathrm{tr}_{\mathbb{S}^1} u(a + r\,\cdot))$ for all $r \in [\rho,\sigma]$. Note that, since $\lambda \neq 0$, we have $\lambda \geq \mathrm{sys}(\mathcal{N})$. Then making the substitution $s = r/E$ we write

$$\begin{aligned}\int_{\mathbb{A}(a,\rho,\sigma)} f(|\mathrm{D}u|)\,\mathrm{d}x &\geq 2\pi \int_\rho^\sigma f\left(\frac{\sqrt{E}}{\sqrt{\pi}r}\right) r\,\mathrm{d}r \\ &= 2\pi E^2 \int_{\frac{\rho}{E}}^{\frac{\sigma}{E}} f\left(\frac{1}{\sqrt{\pi E}s}\right) s\,\mathrm{d}s \\ &\geq \frac{\mathrm{sys}(\mathcal{N})^2}{2} E \int_{\frac{\rho}{E}}^{\frac{\sigma}{E}} f\left(\frac{2}{\mathrm{sys}(\mathcal{N})}\frac{1}{s}\right) s\,\mathrm{d}s \\ &= E\left(\Lambda(\frac{\sigma}{E}) - \Lambda(\frac{\rho}{E})\right)\end{aligned} \tag{4.4.18}$$

where we used $(\mathrm{Dec})_2^{\mathrm{n}}$ noting that $\frac{1}{\sqrt{\pi E}s} \geq \frac{\mathrm{sys}(\mathcal{N})}{2\pi\sigma} \geq t_0$ for all $s \leq \frac{\sigma}{E}$, by the upper bound on σ. $\square$

Lemma 4.4.5. *Let f be a Young function, and let $\Lambda(t)$ be as in* (4.4.15). *Then for $0 \leq \rho < \sigma$, we have*

$$E \mapsto E\left(\Lambda\left(\frac{\sigma}{E}\right) - \Lambda\left(\frac{\rho}{E}\right)\right) \tag{4.4.19}$$

is non-decreasing in $E > 0$.

Lemma 4.4.5 is analogous to the mean-value property observed by JERRARD [24, Prop. 3.1, (3.3)], and by taking $\rho = 0$, we see that $t \mapsto \frac{1}{t}\,\Lambda(t)$ is also non-increasing in t. In addition, by combining Lemmas 4.4.4 and 4.4.5 we infer that (4.4.14) holds for any $E \in [0, \lambda(\mathrm{tr}_{\mathbb{S}^1}\, u(a+\sigma\,\cdot))^2/4\pi]$.

Proof of Lemma 4.4.5. For $E_1 < E_2$ we use the change of variable $r = \frac{E_1}{E_2} s$ in

$$\begin{aligned} E_1\left(\Lambda\left(\frac{\sigma}{E}\right) - \Lambda\left(\frac{\rho}{E}\right)\right) &= 2c_0^2 E_1 \int_{\rho/E_1}^{\sigma/E_1} f\left(\frac{1}{c_0 s}\right) s \,\mathrm{d}s \\ &= 2c_0^2 E_2 \int_{\rho/E_2}^{\sigma/E_2} f\left(\frac{E_1}{E_2}\frac{1}{c_0 r}\right)\frac{E_2}{E_1} r \,\mathrm{d}r \qquad (4.4.20) \\ &\leq 2c_0^2 E_2 \int_{\rho/E_2}^{\sigma/E_2} f\left(\frac{1}{c_0 r}\right) r \,\mathrm{d}r, \end{aligned}$$

where we used the fact that $f(st) \leq s f(t)$ with $s = \frac{E_1}{E_2} \in (0,1)$ in the last line, which follows from convexity of f and the fact that $f(0) = 0$. $\square$

We record the following ball merging construction, which can be found in [24, Lem. 3.1] (see also [39, Fig. 1]).

Lemma 4.4.6 (Merging lemma)**.** *Given any finite collection $\mathcal{B}$ of closed balls in $\mathbb{R}^2$, we can associate a pairwise disjoint collection of balls $\widetilde{\mathcal{B}}$ such that $\bigcup \mathcal{B} \subset \bigcup \widetilde{\mathcal{B}}$ and $\sum_{\mathbb{B}\in\mathcal{B}} r(\mathbb{B}) = \sum_{\tilde{\mathbb{B}}\in\widetilde{\mathcal{B}}} r(\tilde{\mathbb{B}})$. Moreover, the construction is such that for each $\tilde{\mathbb{B}} \in \widetilde{\mathcal{B}}$, there exist finitely many $\mathbb{B}_1, \dots, \mathbb{B}_k$ such that $\bigcup_{i=1}^k \mathbb{B}_i \subset \tilde{\mathbb{B}}$ and $\sum_{i=1}^k r(\mathbb{B}_i) = r(\tilde{\mathbb{B}})$. In particular, if $k = 1$ we have $\mathbb{B}_1 = \tilde{\mathbb{B}}$.*

Informally, Lemma 4.4.6 provides a way to systematically "disjointify" a given collection of balls, by merging any two balls which intersect. In the sequel, we will say a ball $\tilde{\mathbb{B}}$ can be *obtained by merging* balls $\mathbb{B}_1, \dots, \mathbb{B}_k$ if we have $\mathbb{B}_i \subset \mathbb{B}$ for all $i = 1, \dots, m$, and $r(\mathbb{B}) = \sum_{i=1}^m r(\mathbb{B}_i)$.

Proof of Lemma 4.4.6. We inductively merge any two balls $\mathbb{B}_1 = \mathbb{B}_{\sigma_1}(a_1)$ and $\mathbb{B}_2 = \mathbb{B}_{\sigma_2}(a_2)$ which intersect, by replacing them by $\mathbb{B}_\sigma(a)$, where $a = \frac{\sigma_1 a_1 + \sigma_2 a_2}{\sigma_1+\sigma_2}$ and $\sigma = \sigma_1 + \sigma_2$. We repeat this procedure for any two intersecting balls until they are all pairwise disjoint. $\square$

Proposition 4.4.7 (Ball merging for $\mathrm{E}^{1,2}_{\mathrm{sg}}$)**.** *Let f be a Young function satisfying* $(\mathrm{Dec})^{\mathrm{n}}_2$ *and* (vInt)*, and let $\Lambda(t)$ be as in* (4.4.15)*. For $\delta > 0$, let $u \in \mathrm{R}^{1,2}_\delta(\Omega; \mathcal{N})$ have singularities at distinct $a_1, \dots, a_k \in \Omega$ with $k \geq 1$, and set $\eta_0 = \min\left\{\frac{\mathrm{sys}(\mathcal{N})}{2\pi t_0}, \delta\right\} > 0$.*

Then for each $\eta \in (0, \eta_0]$ there exists a finite collection $\mathcal{B}(\eta)$ of disjoint closed balls in Ω whose sum of radii is η, such that $\bigcup \mathcal{B}(\eta)$ contains $\{a_i\}_{i=1}^k$ and that

$$\left(\sum_{\mathbb{B} \in \mathcal{B}(\eta)} \mathrm{E}_{\mathrm{sg}}^{1,2}(\mathrm{tr}_{\partial \mathbb{B}}\, u) \right) \Lambda_f \left(\frac{\eta}{\sum_{\mathbb{B} \in \mathcal{B}(\eta)} \mathrm{E}_{\mathrm{sg}}^{1,2}(\mathrm{tr}_{\partial \mathbb{B}}\, u)} \right) \leq \int_{\bigcup \mathcal{B}(\eta)} f(|\mathrm{D}u|)\, \mathrm{d}x, \tag{4.4.21}$$

understanding the left-hand side to be zero when u has no singularities in Ω.

Note that for the collection $\mathcal{B}(\eta)$ obtained by Proposition 4.4.7, $(\mathrm{tr}_{\partial \mathbb{B}}\, u)_{\mathbb{B} \in \mathcal{B}(\eta)}$ is topological resolution of $\mathrm{tr}_{\partial \Omega}\, u$ for each η, since

$$\{a_i\}_{i=1,\dots,k} \subset \bigcup \mathcal{B}(\eta)$$

and the balls are disjoint. The key step in proving Proposition 4.4.7 will be to show the energy inequality (4.4.21) is preserved under a suitable merging process; we will record this as a separate lemma.

Lemma 4.4.8. *Suppose $(\sigma_i)_{i=1}^k$, $(E_i)_{i=1}^k$ are positive numbers and let*

$$0 < E_\Sigma \leq \sum_{i=1}^k E_i, \quad \sigma_\Sigma = \sum_{i=1}^k \sigma_i. \tag{4.4.22}$$

Suppose there is a parameter $T > 0$ such that

$$\sigma_i \leq E_i T \quad \textit{for all } 1 \leq i \leq k, \tag{4.4.23}$$

$$\sigma_\Sigma \geq E_\Sigma T, \tag{4.4.24}$$

then we have

$$\sum_{i=1}^k E_i \Lambda \left(\frac{\sigma_i}{E_i} \right) \geq E_\Sigma \Lambda \left(\frac{\sigma}{E_\Sigma} \right). \tag{4.4.25}$$

In the proof of Lemma 4.4.8, we use Lemma 4.4.5 with $\rho = 0$ twice.

Proof of Lemma 4.4.8. Since (4.4.23) gives $\frac{\sigma_i}{E_i} \geq T$ for all i, and since $E \mapsto \Lambda_f(\sigma/E)$ is non-decreasing by Lemma 4.4.5 (applied with $\rho = 0$), we have

$$\sum_{i=1}^k E_i \Lambda_f \left(\frac{\sigma_i}{E_i} \right) \geq \sum_{i=1}^k \frac{\sigma_i}{T} \Lambda_f (T) = \frac{\sigma_\Sigma}{T} \Lambda_f (T), \tag{4.4.26}$$

where the last equality follows by $(4.4.22)_2$. Using Lemma 4.4.5 again with (4.4.24), we see that

$$\frac{\sigma_\Sigma}{T}\Lambda_f(T) \geq E_\Sigma \Lambda_f\left(\frac{\sigma_\Sigma}{E_\Sigma}\right), \tag{4.4.27}$$

which we combine with (4.4.26) to conclude. □

Proof of Proposition 4.4.7. Fix $\eta \in (0, \eta_0]$. Given a parameter $t > 0$, we will inductively define two collections of closed balls $\mathcal{B}^+(t), \mathcal{B}^-(t)$ which satisfy the following properties for $t > 0$:

(i) The collections $\mathcal{B}^\pm(t)$ are both finite collections of balls, such that balls in $\mathcal{B}^+(t)$ are pairwise disjoint, and balls in $\mathcal{B}^-(t)$ have pairwise disjoint interiors. That is,

$$\begin{aligned} \overline{\mathbb{B}}_1 \cap \overline{\mathbb{B}}_2 &= \varnothing \quad \text{for all } \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{B}^+(t), \\ \mathbb{B}_1^\circ \cap \mathbb{B}_2^\circ &= \varnothing \quad \text{for all } \mathbb{B}_1, \mathbb{B}_2 \in \mathcal{B}^-(t). \end{aligned}$$

(ii) The collections $(c_{\mathbb{B}}, \operatorname{tr}_{\partial\mathbb{B}} u)_{\mathbb{B}\in\mathcal{B}^+(t)}$ and $(c_{\mathbb{B}}, \operatorname{tr}_{\partial\mathbb{B}} u)_{\mathbb{B}\in\mathcal{B}^-(t)}$ are topological resolutions of $\operatorname{tr}_{\partial\Omega} u$, where $c_{\mathbb{B}}$ denotes the centre of the ball $\mathbb{B}$. We will say, by abuse of language, that both $\mathcal{B}^+(t), \mathcal{B}^-(t)$ are topological resolutions of u in Ω. In particular,

$$\sum_{\mathbb{B}\in\mathcal{B}^\pm(t)} \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\mathbb{B}} u) \geq \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega} u) \quad \text{for all } t > 0.$$

Here we use the notation $\mathcal{B}^\pm(t)$ to mean the property should hold with both $\mathcal{B}^+(t)$ and $\mathcal{B}^-(t)$ in place of $\mathcal{B}^\pm(t)$.

(iii) The collection $\mathcal{B}^+(t)$ can be obtained by applying Lemma 4.4.6 to $\mathcal{B}^-(t)$. That is, to each $\tilde{\mathbb{B}} \in \mathcal{B}^+(t)$ there exists $\mathbb{B}_1, \ldots, \mathbb{B}_m \in \mathcal{B}^-(t)$ such that $\tilde{\mathbb{B}}$ can be obtained by merging $\mathbb{B}_1, \ldots, \mathbb{B}_m$.

(iv) We have the radius bounds

$$\begin{aligned} r(\mathbb{B}) &\leq \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\mathbb{B}} u)\, t \quad \text{for all } \mathbb{B} \in \mathcal{B}^-(t), \\ r(\mathbb{B}) &\geq \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\mathbb{B}} u)\, t \quad \text{for all } \mathbb{B} \in \mathcal{B}^+(t), \end{aligned}$$

where $r(\mathbb{B})$ denotes the radius of the ball $\mathbb{B}$.

(v) We have the lower bound

$$\int_{\mathbb{B}} f(|\mathrm{D}u|)\,\mathrm{d}x \geq \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\mathbb{B}} u)\, \Lambda_f\left(\frac{r(\mathbb{B})}{\mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\mathbb{B}} u)}\right) \quad \text{for each } \mathbb{B} \in \mathcal{B}^\pm(t).$$

(vi) The sum of radii satisfies

$$r(\mathcal{B}^{\pm}(t)) \doteq \sum_{\mathbb{B}\in\mathcal{B}^{\pm}(t)} r(\mathbb{B}) \leq \eta.$$

We will describe this process in Steps 1–4. The idea will be to "expand" the balls with respect to the parameter t, where we consider concentric balls $\mathbb{B}(t)$ following the expansion law

$$r(\mathbb{B}(t)) = \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}(t)}\, u)\, t. \tag{4.4.28}$$

We cannot do this indefinitely without having balls intersect however, in which case we will need to merge balls using Lemma 4.4.6. To ensure the lower bound (v) is preserved under this process, we use Lemma 4.4.8 on specific balls; for this reason, we have two collections of balls $\mathcal{B}^{\pm}(t)$ at each time-step satisfying (iv).

In what follows we will abbreviate

$$\mathrm{E}^{1,2}_{\mathrm{sg}}(a,\rho) \doteq \mathrm{E}^{1,2}_{\mathrm{sg}}([\mathrm{tr}_{\partial\mathbb{B}(a,\rho)}\, u]), \tag{4.4.29}$$

and also put $\mathrm{E}^{1,2}_{\mathrm{sg}}(a) \doteq \mathrm{E}^{1,2}_{\mathrm{sg}}([u,a])$, where $[u,a]$ denotes a limiting geodesic of u at a.

Step 1

The initial parametrisation.

For $t > 0$ sufficiently small, we define $\mathcal{B}^{+}(t)$ and $\mathcal{B}^{-}(t)$ to be equal as the collection of balls $\mathbb{B}(a_i, \rho_i(t))$ for each $1 \leq i \leq k$, where

$$\rho_i(t) = \mathrm{E}^{1,2}_{\mathrm{sg}}(a_i)t. \tag{4.4.30}$$

Here $t > 0$ is chosen so these balls do not intersect; that is $t < t_1$ with

$$t_1 = \sup\{t > 0 : \mathcal{B}^{+}(t) \text{ is pairwise disjoint}\}, \tag{4.4.31}$$

noting that $t_1 > 0$ since the starting points $a_1, \ldots, a_k$ were distinct. This defines $\mathcal{B}^{\pm}(t)$ for all $0 < t < t_1$, which by definition satisfies (ii)–(iv). Since u is only singular at each $a_i \in \mathbb{B}(a_i, \rho_i(t))$, by Lemma 4.4.4 we have

$$\int_{\mathbb{B}(a_i,\rho_i(t))} f(|\mathrm{D}u|)\,\mathrm{d}x \geq \mathrm{E}^{1,2}_{\mathrm{sg}}(a_i,\rho_i(t))\Lambda_f\left(\frac{\rho_i(t)}{\mathrm{E}^{1,2}_{\mathrm{sg}}(a_i,\rho_i(t))}\right), \tag{4.4.32}$$

establishing (v). Lastly, if we have that (vi) is satisfied with equality we will terminate the process and proceed to Step 5; otherwise, our collection $\mathcal{B}^{\pm}(t)$ satisfies all the claimed properties for $0 < t < t_1$, and so we can move to Step 2.

Step 2

Evolving and frozen balls.

We will split $\mathcal{B}^{\pm}(t)$ further into disjoint unions

$$\mathcal{B}^{+}(t) = \mathcal{B}^{+}_{\mathrm{F}}(t) \cup \mathcal{B}^{+}_{\mathrm{G}}(t), \quad \mathcal{B}^{-}(t) = \mathcal{B}^{-}_{\mathrm{F}}(t) \cup \mathcal{B}^{-}_{\mathrm{G}}(t) \tag{4.4.33}$$

distinguishing between the "frozen" and "growing" balls respectively. Note that none of the balls are initially frozen, so we define $\mathcal{B}^{+}_{\mathrm{G}}(t) = \mathcal{B}^{-}_{\mathrm{G}}(t) \doteq \mathcal{B}^{+}(t)$ and $\mathcal{B}^{+}_{\mathrm{G}}(t) = \mathcal{B}^{-}_{\mathrm{G}}(t) = \varnothing$ for all $t \in (0, t_1)$.

We will inductively show there are finitely many $0 = t_0 < t_1 < \cdots$ such that

(a) $\mathcal{B}^{+}_{\mathrm{G}}(t) = \mathcal{B}^{-}_{\mathrm{G}}(t)$ for $t \in (t_{j-1}, t_j)$ and each ball $\mathbb{B} = \mathbb{B}(a, \sigma) \in \mathcal{B}^{+}_{\mathrm{G}}(t_{j-1})$ evolves as $\mathbb{B}(t)$ with respect to the expansion law (4.4.28); that is, $\mathbb{B}(t) = \mathbb{B}(a, \sigma(t))$ where $\sigma(t) = \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}}\, u)t$. In particular, balls in $\mathcal{B}^{\pm}_{\mathrm{G}}$ satisfy (iv) with equality.

(b) both $\mathcal{B}^{\pm}_{\mathrm{F}}(t)$ are constant on each (t_{j-1}, t_j) and $\mathcal{B}^{+}_{\mathrm{F}}(t)$ can be obtained by applying Lemma 4.4.6 to $\mathcal{B}^{-}_{\mathrm{F}}(t)$. Moreover, we require that balls $\mathbb{B} \in \mathcal{B}^{+}_{\mathrm{F}}(t)$ satisfy (iv) with a strict inequality.

In Step 1, we verified this holds up to $t \leq t_1$ defined in (4.4.31). Given $\mathcal{B}^{\pm}(t_{j-1})$, we can define $\mathcal{B}^{\pm}(t)$ via (a) and (b) until we reach $t \doteq t_j > t_{j-1}$ such that one of the following holds.

Case 1 (collision): We have $\mathcal{B}^{+}(t)$ is no longer pairwise disjoint, in which case we define (4.4.34) as below and move to Step 3.

Case 2 (saturation): There is $\mathbb{B} \in \mathcal{B}^{+}_{\mathrm{F}}(t)$ which satisfies (iv) with equality, in which case we define (4.4.34) as below and move to Step 4.

Case 3 (termination): We have $r(\mathcal{B}^{+}(t)) = \eta$, in which we terminate the process and move to Step 5.

If either Case 1 or 2 occurs, at this time step $t = t_j$ we let

$$\mathcal{B}^{\pm}(t_j-) = \mathcal{B}^{\pm}_{\mathrm{F}}(t_j-) \cup \mathcal{B}^{\pm}_{\mathrm{G}}(t_j-) \tag{4.4.34}$$

be the collections obtained by applying (a) and (b) at this time step, which we will suitably modify in the subsequent steps. If multiple cases occur at once, we apply the corresponding steps in order. This procedure is shown in Figure 4.1.

We claim that if $\mathcal{B}^{\pm}(t)$ satisfies (a), (b) for all $t \in (t_{j-1}, t_j)$, then the claimed properties (i)–(vi) hold for all t in this range. Indeed, property (i) holds for $t < t_j$ by Case 1 of t_j and by the fact that the only expanding balls are those in $\mathcal{B}^{+}_{\mathrm{G}}(t) = \mathcal{B}^{-}_{\mathrm{G}}(t)$. Similarly Case 2 of t_j ensures (iv) holds provided $t < t_j$, noting the inequality for balls in $\mathcal{B}^{-}_{\mathrm{F}}(t)$ weakens as t is increased. Also since $\bigcup \mathcal{B}^{\pm}(t)$ still contains the initial singularities

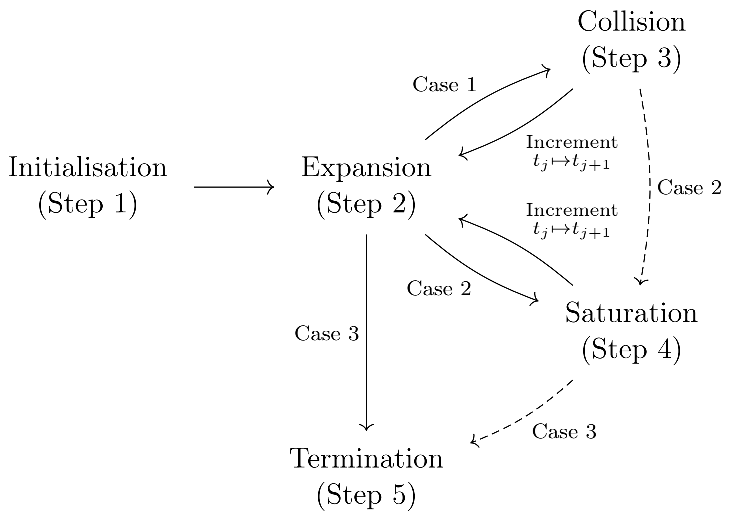

Figure 4.1: Flowchart of the inductive process

$a_1, \ldots, a_k$, $\mathcal{B}^{\pm}(t)$ remains topological resolutions of u in Ω and satisfies (ii). By induction hypothesis (iii) holds at $t = t_{j-1}$, and hence is preserved for $t \in (t_{j-1}, t_j)$ since there are no new intersections. Also Case 3 ensures (vi) holds.

For (v), this is satisfied by balls in $\mathcal{B}^{\pm}_{\mathrm{F}}(t)$ by inductive assumption, so we only need to check the balls in $\mathcal{B}^{\pm}_{\mathrm{G}}(t)$. For this let $\mathbb{B}(t) = \mathbb{B}(a, \sigma(t)) \in \mathcal{B}^{+}_{\mathrm{G}}(t) = \mathcal{B}^{-}_{\mathrm{G}}(t)$ with $\sigma(t) = \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}}\, u)t$, which corresponds to $\mathbb{B} = \mathbb{B}(a, \sigma(t_{j-1})) \in \mathcal{B}^{+}_{\mathrm{G}}(t_{j-1})$. Then by applying Lemma 4.4.4 (annular lower bound) and Lemma 4.4.5 (the mean value property), we can estimate

$$
\begin{aligned}
\int_{\mathbb{B}(a,\sigma(t))} f(|\mathrm{D}u|)\,\mathrm{d}x &\geq \int_{\mathbb{B}(a,\sigma(t))\setminus\mathbb{B}(a,\sigma(t_{j-1}))} f(|\mathrm{D}u|)\,\mathrm{d}x && (4.4.35)\\
&\quad + \int_{\mathbb{B}(a,\sigma(t_{j-1}))} f(|\mathrm{D}u|)\,\mathrm{d}x && (4.4.36)\\
&\geq \mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t)) \left(\Lambda_f \left(\frac{\sigma(t)}{\mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t))} \right) - \Lambda_f \left(\frac{\sigma(t_{j-1})}{\mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t))} \right) \right)\\
&\quad + \mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t_{j-1})) \Lambda_f \left(\frac{\sigma(t)}{\mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t_{j-1}))} \right)\\
&= \mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t)) \Lambda_f \left(\frac{\sigma(t)}{\mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t))} \right),
\end{aligned}
$$

since $\mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t)) = \mathrm{E}^{1,2}_{\mathrm{sg}}(a,\sigma(t_{j-1}))$. This verifies (v), so we infer that (i)–(vi) holds for all $t \in (t_{j-1}, t_j)$.

Step 3

Ball merging I (Collision).

Suppose we are in Case 1 of the previous step, so $\mathcal{B}^+(t_j-)$ is not pairwise disjoint. We will apply Lemma 4.4.6 to $\mathcal{B}^+(t_j-)$ and let $\mathcal{B}^+_{\mathrm{New}}(t_j)$ denote the collection of *new* balls obtained in this process, *i.e.* those which strictly contains one of the existing balls.

Then, each $\mathbb{B} \in \mathcal{B}^+_{\mathrm{New}}(t_j)$ arises from $\mathbb{B}^{\mathrm{F}}_1, \dots, \mathbb{B}^{\mathrm{F}}_m \in \mathcal{B}^+_{\mathrm{F}}(t_j-)$ and $\mathbb{B}^{\mathrm{G}}_1, \dots, \mathbb{B}^{\mathrm{G}}_\ell \in \mathcal{B}^+_{\mathrm{G}}(t_j-)$ which are contained in $\mathbb{B}$. Note that by (iii), there also exists $\widetilde{\mathbb{B}}^{\mathrm{F}}_1, \dots, \widetilde{\mathbb{B}}^{\mathrm{F}}_{\tilde{m}} \in \mathcal{B}^-_{\mathrm{F}}(t_j-)$ from which we can obtain $\mathbb{B}^{\mathrm{F}}_1, \dots, \mathbb{B}^{\mathrm{F}}_m$ by merging. Then, since by (iv) we have

$$r(\mathbb{B}^{\mathrm{F}}_i) \geq \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}^{\mathrm{F}}_i} u) t_j \quad \text{for all } i = 1, \dots, m, \tag{4.4.37}$$

$$r(\widetilde{\mathbb{B}}^{\mathrm{F}}_i) \leq \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \widetilde{\mathbb{B}}^{\mathrm{F}}_i} u) t_j \quad \text{for all } i = 1, \dots, \tilde{m}, \tag{4.4.38}$$

$$r(\mathbb{B}^{\mathrm{G}}_i) = \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}^{\mathrm{G}}_i} u) t_j \quad \text{for all } i = 1, \dots, \ell, \tag{4.4.39}$$

it follows from (4.4.38) and (4.4.39) that

$$\begin{aligned} r(\mathbb{B}) &= \sum_{i=1}^m r(\mathbb{B}^{\mathrm{F}}_i) + \sum_{i=1}^\ell r(\mathbb{B}^{\mathrm{G}}_i) \\ &= \sum_{i=1}^{\tilde{m}} r(\widetilde{\mathbb{B}}^{\mathrm{F}}_i) + \sum_{i=1}^\ell r(\mathbb{B}^{\mathrm{G}}_i) \\ &\geq t_j \sum_{i=1}^m \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}^{\mathrm{F}}_i} u) + t_j \sum_{i=1}^\ell \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}^{\mathrm{G}}_i} u) \\ &\geq \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}} u) t_j, \end{aligned} \tag{4.4.40}$$

where we used the subadditivity of the singular energy (namely (4.3.4)) in the last line. Then since $\{\mathbb{B}^{\mathrm{F}}_i\}_{i=1}^m$, $\{\mathbb{B}^{\mathrm{G}}_i\}_{i=1}^\ell$ satisfies (v) by inductive assumption, by Lemma 4.4.8 using (4.4.37) and (4.4.39), we have

$$\begin{aligned} \int_{\mathbb{B}} f(|\mathrm{D}u|)\, \mathrm{d}x &\geq \sum_{i=1}^m \int_{\mathbb{B}^{\mathrm{F}}_i} f(|\mathrm{D}u|)\, \mathrm{d}x + \sum_{i=1}^\ell \int_{\mathbb{B}^{\mathrm{G}}_i} f(|\mathrm{D}u|)\, \mathrm{d}x \\ &\geq \sum_{i=1}^m \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}^{\mathrm{F}}_i} u) \Lambda_f \left(\frac{r(\mathbb{B}^{\mathrm{F}}_i)}{\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}^{\mathrm{F}}_i} u)} \right) \\ &\qquad + \sum_{i=1}^\ell \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}^{\mathrm{G}}_i} u) \Lambda_f \left(\frac{r(\mathbb{B}^{\mathrm{G}}_i)}{\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}^{\mathrm{G}}_i} u)} \right) \\ &\geq \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}} u) \Lambda_f \left(\frac{r(\mathbb{B})}{\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial \mathbb{B}} u)} \right), \end{aligned} \tag{4.4.41}$$

thereby establishing (v) for $\mathbb{B}$.

We then define $\mathcal{B}^{\pm}_{\mathrm{F}}(t_j)$ and $\mathcal{B}^{\pm}_{\mathrm{G}}(t_j)$ as $\mathcal{B}^{\pm}_{\mathrm{F}}(t_j-)$ and $\mathcal{B}^{\pm}_{\mathrm{G}}(t_j-)$ respectively, except we make the following modifications corresponding to each $\mathbb{B} \in \mathcal{B}^{+}_{\mathrm{New}}(t_j)$:

- We remove $\mathbb{B}^{\mathrm{F}}_1, \ldots, \mathbb{B}^{\mathrm{F}}_m$ from $\mathcal{B}^{+}_{\mathrm{F}}(t_j)$ and instead put $\mathbb{B} \in \mathcal{B}^{+}_{\mathrm{F}}(t_j)$.
- We remove $\mathbb{B}^{\mathrm{G}}_1, \ldots, \mathbb{B}^{\mathrm{G}}_\ell$ from $\mathcal{B}^{\pm}_{\mathrm{G}}(t_j)$ and put them in $\mathcal{B}^{-}_{\mathrm{F}}(t_j)$.

The above shows that (iv) and (v) hold, and (iii) holds since each $\mathbb{B} \in \mathcal{B}^{+}_{\mathrm{new}}(t_j)$ can be obtained by merging $\widetilde{\mathbb{B}}^{\mathrm{F}}_1, \ldots, \widetilde{\mathbb{B}}^{\mathrm{F}}_{\tilde{m}}, \mathbb{B}^{\mathrm{G}}_1, \ldots \mathbb{B}^{\mathrm{G}}_\ell$, which now lie in $\mathcal{B}^{-}_{\mathrm{F}}(t_j)$. The remaining properties are a routine check.

Thus we have defined $\mathcal{B}^{\pm}(t_j)$. Now if (4.4.40) holds with equality for any of the $\mathbb{B} \in \mathcal{B}^{+}_{\mathrm{F}}(t_j)$, since Case 2 would be violated, we rename our new collections as $\mathcal{B}^{\pm}(t_j-)$ and move to Step 4. Otherwise, we return to Step 2 to continue the inductive process.

Step 4

Ball merging II (Saturation).

Suppose there is $\mathbb{B} \in \mathcal{B}^{+}_{\mathrm{F}}(t_j-)$ satisfying (iv) with equality, that is

$$r(\mathbb{B}) = \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}}\, u)t_j. \tag{4.4.42}$$

If this holds, let $\mathbb{B}_1, \ldots, \mathbb{B}_\ell \in \mathcal{B}^{-}_{\mathrm{F}}(t_j-)$ be the balls which merge to give $\mathbb{B}$, which exist by (iii). We then define $\mathcal{B}^{\pm}(t_j)$ as $\mathcal{B}^{\pm}(t_j-)$ except that for any such ball $\mathbb{B}$ we make the following modifications:

- We move $\mathbb{B}$ from $\mathcal{B}^{+}_{\mathrm{F}}(t_j)$ to $\mathcal{B}^{+}_{\mathrm{G}}(t_j)$.
- We remove $\mathbb{B}_1, \ldots, \mathbb{B}_\ell$ from $\mathcal{B}^{-}_{\mathrm{F}}(t_j)$ and add $\mathbb{B}$ to $\mathcal{B}^{-}_{\mathrm{G}}(t_j)$.

We show that (i)–(vi) holds. Since we apply Step 3 first if both Cases 1 and 2 occur, (i) holds true. Since we have replaced $\mathbb{B}_1, \ldots, \mathbb{B}_\ell$ by the merged ball $\mathbb{B}$ whenever $\mathbb{B}$ satisfies (4.4.42), items (ii), (iii) and (iv) still hold. For (v), we use in order that $\bigcup_{i=1}^{\ell} \mathbb{B}_i \subset \mathbb{B}$, (v) and (iv) holds for each $\mathbb{B}_i \in \mathcal{B}^{-}_{\mathrm{F}}(t_j-)$, Lemma 4.4.5, that $r(\mathbb{B}) = \sum_{i=1}^{\ell} r(\mathbb{B}_i)$ and (4.4.42) to bound

$$\begin{aligned}
\int_{\mathbb{B}} f(|\mathrm{D}u|)\,\mathrm{d}x &\geq \sum_{i=1}^{\ell} \int_{\mathbb{B}_i} f(|\mathrm{D}u|)\,\mathrm{d}x \\
&\geq \sum_{i=1}^{\ell} \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}_i} u)\,\Lambda_f\left(\frac{r(\mathbb{B}_i)}{\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}_i} u)}\right) \\
&\geq \sum_{i=1}^{\ell} \frac{r(\mathbb{B}_i)}{t_j}\Lambda_f(t_j) \\
&= \frac{r(\mathbb{B})}{t_j}\Lambda_f(t_j) \\
&= \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}} u)\,\Lambda_f\left(\frac{r(\mathbb{B})}{\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}} u)}\right).
\end{aligned}$$

Step 5

Termination.

Observe the sum of radii $r(\mathcal{B}^-(t))$ is continuous and non-decreasing in t, and by (iv) satisfies

$$r(\mathcal{B}^-(t)) \geq \sum_{\mathbb{B}\in\mathcal{B}(t)} \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}} u)t \geq \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega} u)t, \tag{4.4.43}$$

where we used the subadditivity inequality (4.3.4) of the singular energy. Since the right hand side is unbounded as $t \to \infty$, there exists $t = T > 0$ where

$$r(\mathcal{B}(T)) = \eta, \tag{4.4.44}$$

and hence we must eventually reach Case 3 in the iteration of Step 2. When this occurs, by construction the collection $\mathcal{B} \doteq \mathcal{B}^-(T)$ is a collection of disjoint balls $\{\mathbb{B}(c_i,\sigma_i)\}_{i=1}^N$ for which

$$\int_{\mathbb{B}(c_i,\sigma_i)} f(|\mathrm{D}u|)\,\mathrm{d}x \geq \mathrm{E}^{1,2}_{\mathrm{sg}}(c_i,\sigma_i)\Lambda_f\left(\frac{\sigma_i}{\mathrm{E}^{1,2}_{\mathrm{sg}}(c_i,\sigma_i)}\right), \tag{4.4.45}$$

by (v) and

$$\sigma_i \leq \mathrm{E}^{1,2}_{\mathrm{sg}}(c_i,\sigma_i)T \tag{4.4.46}$$

by (iv). Moreover we have

$$\begin{aligned}
\eta = r(\mathcal{B}^-(T)) &= r(\mathcal{B}^+(T)) \\
&\geq \sum_{\mathbb{B}\in\mathcal{B}^+(T)} \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}} u)T \geq \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega} u)T,
\end{aligned} \tag{4.4.47}$$

so applying Lemma 4.4.8 we deduce that

$$\begin{aligned}(4.4.48)\quad &\int_{\bigcup \mathcal{B}} f(|\mathrm{D}u|)\,\mathrm{d}x \\ &\qquad \geq \left(\sum_{\mathbb{B}\in\mathcal{B}} \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}}\, u)\right)\Lambda_f\left(\frac{\eta}{\sum_{\mathbb{B}\in\mathcal{B}} \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}}\, u)}\right),\end{aligned}$$

establishing our desired lower bound for the collection $\mathcal{B}$. □

Remark 4.4.9. The proof of Proposition (4.4.7) moreover shows that, analogously to what is proven in [43, Prop. 4.3], for each $\eta \in (0, \eta_0]$ there exists a collection $\mathcal{S}$ of circles $\mathbb{S}^1(a,\rho) = \partial\mathbb{B}(a,\rho)$ (understanding $\mathbb{S}^1(c,0) = \{c\}$) for which the following holds.

(a) We have $\bigcup \mathcal{S}$ is a disjoint union of finitely many annuli; that is, there exists $N \in \mathbb{N}$ and centres $c_j \in \Omega$, radii $0 \leq \underline{r}_j \leq \overline{r}_j \leq \eta$ for each $1 \leq j \leq N$ such that $\mathbb{S}^1(c_j, r) \in \mathcal{S}$ for each $\underline{r}_j \leq r < \overline{r}_j$.

(b) for each $\varepsilon \in (0,\eta]$ there exists finitely many $\{\mathbb{S}^1(c_{j_i}, \rho_i)\}_{i=1}^{\ell} \subset \mathcal{S}$ such that $\sum_{i=1}^{\ell} \rho_i = \varepsilon$ and $\{a_i\}_{i=1}^{\kappa} \subset \bigcup_{i=1}^{\ell} \mathbb{B}(c_{j_i}, \rho_i)$. If $\varepsilon = \eta$, this collection corresponds precisely to $\{\partial\mathbb{B} : \mathbb{B} \in \mathcal{B}(\eta)\} \subset \mathcal{S}$.

(c) For each $1 \leq j \leq N$ we have the localised estimate

$$\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}(c_j,r)}\, u)\Lambda_f\left(\frac{r}{\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}(c_j,r)}\, u)}\right) \leq \int_{\mathbb{B}(c_j,r)} f(|\mathrm{D}u|)\,\mathrm{d}x$$

for each $r \in [\underline{r}_i, \overline{r}_i)$.

This can be seen by defining

$$\mathcal{S} = \{\partial\mathbb{B} : \mathbb{B} \in \mathcal{B}^{+}_{\mathrm{G}}(t),\ 0 \leq t \leq T\}, \tag{4.4.49}$$

and verifying that the claimed properties follow from the items (i)–(vi) in the proof of Proposition 4.4.7. We will omit the details, as we will not need this in our subsequent analysis.

4.4.3 Asymptotic lower bounds

We will show how this merging ball construction can be used to obtain bounds for (almost-)minimising sequences in the class $\mathrm{R}^{1,2}_{\delta}(\Omega;\mathcal{N})$. For this we will first establish a general lower bound valid for all sequences, namely (4.4.50), and then we will prove refined bounds along sequences attaining this bound in the sense of (4.4.51).

Proposition 4.4.10. *Let $g \in \mathrm{W}^{1/2,2}(\partial\Omega;\mathcal{N})$ and $\delta > 0$, and suppose $(u_n)_n \subset \mathrm{R}^{1,2}_{\delta}(\Omega;\mathcal{N})$ is a sequence such that $\mathrm{tr}_{\partial\Omega} u_n = g$ for all n. Then given a sequence $f_n\colon \mathbb{R}_+ \to \mathbb{R}$ of approximating integrands, the following holds:*

(i) *We always have*

$$\mathrm{E}^{1,2}_{\mathrm{sg}}([g]) \leq \varliminf_{n\to\infty} \frac{1}{\mathcal{V}(f_n)} \int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x, \tag{4.4.50}$$

where the right-hand side may in general be infinite.

(ii) *Assume that*

$$\varlimsup_{n\to\infty} \left[\int_\Omega f_n(|\mathrm{D}u_n|) - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] \text{ is finite.} \tag{4.4.51}$$

Then for each $\eta \in (0,\eta_0)$ and $n \in \mathbb{N}$, there exists a finite collection of balls $\mathcal{B}^{\mathrm{Top}}_n(\eta)$ whose sum of radii is at most η, $(\mathrm{tr}_{\partial\mathbb{B}} u_n)_{\mathbb{B}\in\mathcal{B}^{\mathrm{Top}}_n(\eta)}$ forms a topological resolution of g, we have $\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}} u) > 0$ for all $\mathbb{B} \in \mathcal{B}^{\mathrm{Top}}_n(\eta)$, and the following asymptotic lower bound

$$\begin{aligned}\varlimsup_{n\to\infty} \sum_{\mathbb{B}\in\mathcal{B}^{\mathrm{Top}}_n(\eta)} &\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}} u_n) \\ &\leq \varlimsup_{n\to\infty} \mathcal{V}(f_n)^{-1} \int_{\bigcup \mathcal{B}^{\mathrm{Top}}_n(\eta)} f_n(|\mathrm{D}u_n|)\end{aligned} \tag{4.4.52}$$

holds. Moreover we have the remainder estimate

$$\begin{aligned}&\varlimsup_{n\to\infty} \int_{\Omega\setminus\bigcup \mathcal{B}^{\mathrm{Top}}_n(\eta)} f_n(|\mathrm{D}u_n|) \\ &\leq \varlimsup_{n\to\infty} \left[\int_\Omega f_n(|\mathrm{D}u_n|) - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] + \mathrm{E}^{1,2}_{\mathrm{sg}}(g) \log \frac{\mathrm{sys}(\mathcal{N})\mathrm{E}^{1,2}_{\mathrm{sg}}(g)}{2\eta}.\end{aligned} \tag{4.4.53}$$

Under the assumptions of (ii), note that (4.4.51) implies that

$$\mathrm{E}^{1,2}_{\mathrm{sg}}([g]) = \lim_{n\to\infty} \frac{1}{\mathcal{V}(f_n)} \int_\Omega f_n(|\mathrm{D}u_n|), \tag{4.4.54}$$

so (i) holds with equality. We also point out that $\mathcal{B}^{\mathrm{Top}}_n(\eta)$ is not the collection of balls $\mathcal{B}_n(\eta)$ obtained by applying Proposition 4.4.7 to u_n and f_n, as we will discard any balls carrying zero singular energy (*i.e.* balls $\mathbb{B}$ for which $\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}} u_n) = 0$).

Lemma 4.4.11. *For a family of approximating integrands $(f_n)_n$, we have*

$$\lim_{n\to\infty} [\Lambda_{f_n}(t) - \mathcal{V}(f_n)] = \log \frac{\operatorname{sys}(\mathcal{N})t}{2} \quad \text{for all } t > 0. \tag{4.4.55}$$

where $\Lambda_{f_n}(t)$ was defined in (4.4.15)*. In particular,*

$$\lim_{n\to\infty} \frac{\Lambda_n(t)}{\mathcal{V}(f_n)} = 1. \tag{4.4.56}$$

This result follows by applying Lemma 4.4.2, with $\rho = t$ and $\lambda = \frac{4\pi}{\operatorname{sys}(\mathcal{N})}$.

Proof of Proposition 4.4.10. For (i), we set

$$\mathrm{L}_1 \doteq \varlimsup_{n\to\infty} \mathcal{V}(f_n)^{-1} \int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x, \tag{4.4.57}$$

which we assume is finite, since (4.4.50) is vacuously true otherwise. Let $\eta_0 = \min\left\{\frac{\operatorname{sys}(\mathcal{N})}{2\pi t_0}, \delta\right\}$, then by Proposition 4.4.7 applied to u_n and f_n, there exists a collection $\mathcal{B}_n(\eta)$ of disjoint closed balls in Ω whose sum of radii is η, such that $\bigcup \mathcal{B}_n(\eta)$ contains singular points of u_n and that

$$\mathrm{E}(u_n, \mathcal{B}_n(\eta))\, \Lambda_{f_n}\Big(\frac{\eta}{\mathrm{E}(u_n, \mathcal{B}_n(\eta))}\Big) \le \int_{\bigcup \mathcal{B}_n(\eta)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x, \tag{4.4.58}$$

where we abbreviate

$$\mathrm{E}(u_n, \mathcal{B}_n(\eta)) \doteq \sum_{\mathbb{B}\in\mathcal{B}_n(\eta)} \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\mathbb{B}} u_n). \tag{4.4.59}$$

Let $0 \le \mathrm{M} < \varlimsup_{n\to\infty} \mathrm{E}(u_n, \mathcal{B}_n(\eta))$, so that we have $\mathrm{M} \le \mathrm{E}(u_n, \mathcal{B}_n(\eta))$ for infinitely many n. Thus for any such n, combining (4.4.58) with the mean value property of Λ_{f_n} (Lemma 4.4.5) we have

$$\mathrm{M}\,\Lambda_{f_n}(\eta/\mathrm{M}) \le \int_{\bigcup \mathcal{B}_n(\eta)} f_n(|\mathrm{D}u_n|). \tag{4.4.60}$$

Multiplying both sides by $\mathcal{V}(f_n)^{-1}$ we obtain, by (4.4.56) from Lemma 4.4.11,

$$\mathrm{M} = \lim_{n\to\infty} \mathcal{V}(f_n)^{-1}\mathrm{M}\,\Lambda_{f_n}(\eta/\mathrm{M}) \le \varlimsup_{n\to\infty} \mathcal{V}(f_n)^{-1} \int_{\bigcup \mathcal{B}_n(\eta)} f_n(|\mathrm{D}u_n|) \le \mathrm{L}_1.$$

Now increasing M to equal $\varlimsup_{n\to\infty} \mathrm{E}(u_n, \mathcal{B}_n(\eta))$ we infer that

$$\varlimsup_{n\to\infty} \sum_{\mathbb{B}\in\mathcal{B}_n(\eta)} \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\mathbb{B}} u_n) \le \mathrm{L}_1, \tag{4.4.61}$$

from which we deduce (4.4.52), and combining this with the subadditivity property (4.3.4) of the singular energy, (4.4.50) also follows. We now let

$$\mathcal{B}_n^{\mathrm{Top}}(\eta) \doteq \{\mathbb{B} \in \mathcal{B}_n(\eta) : \mathrm{E}_{\mathrm{sg}}^{1,2}(\mathrm{tr}_{\partial\mathbb{B}}\tilde{u}_n) > 0\}. \tag{4.4.62}$$

Then (4.4.52) and (4.4.58) remains satisfied with $\mathcal{B}_n^{\mathrm{Top}}(\eta)$ in place of $\mathcal{B}_n(\eta)$, since we have a smaller collection of balls. Also since for any $\mathbb{B} \in \mathcal{B}_n(\eta) \setminus \mathcal{B}_n^{\mathrm{Top}}(\eta)$ we have $\mathrm{E}_{\mathrm{sg}}^{1,2}(\mathrm{tr}_{\partial\mathbb{B}}\, u_n) = 0$, it follows that $\mathrm{tr}_{\partial\mathbb{B}}\, u_n$ is null-homotopic in $\mathcal{N}$, and hence we can find $\tilde{u}_{n,\mathbb{B}} \in \mathrm{W}^{1,2}(\mathbb{B}, \mathcal{N})$ satisfying $\mathrm{tr}_{\partial\mathbb{B}}\, u_n = \mathrm{tr}_{\partial\mathbb{B}}\, \tilde{u}_{n,\mathbb{B}}$. Then we can define $\tilde{u}_n \in \mathrm{W}^{1,2}(\Omega \setminus \bigcup \mathcal{B}_n^{\mathrm{Top}}(\eta); \mathcal{N})$ by setting $\tilde{u}_n = \tilde{u}_{u,\mathbb{B}}$ in each $\mathbb{B} \in \mathcal{B}_n(\eta) \setminus \mathbb{B}_n^{\mathrm{Top}}(\eta)$, and setting $\tilde{u}_n = u_n$ otherwise. Using this construction, it follows that $(\mathrm{tr}_{\partial\mathbb{B}}\, u_n)_{\mathbb{B}\in\mathcal{B}_n^{\mathrm{Top}}(\eta)}$ remains a topological resolution of g.

For (ii) we assume that

$$\mathrm{L}_2 \doteq \varlimsup_{n\to\infty} \left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathrm{E}_{\mathrm{sg}}^{1,2}(g)\right] \tag{4.4.63}$$

is finite, and dividing both sides by $\mathcal{V}(f_n)$ we see that

$$\lim_{n\to\infty} \frac{1}{\mathcal{V}(f_n)} \left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathrm{E}_{\mathrm{sg}}^{1,2}(g)\right] = \lim_{n\to\infty} \frac{\mathrm{L}_2}{\mathcal{V}(f_n)} = 0. \tag{4.4.64}$$

In particular we can apply (i), so for $\eta \in (0, \eta_0)$ let $\mathcal{B}_n^{\mathrm{Top}}(\eta)$ be the collection of balls as constructed above, noting that (4.4.58) remains satisfied. Then by subadditivity of the singular energy and Lemma 4.4.5,

$$\begin{aligned}\mathrm{E}_{\mathrm{sg}}^{1,2}(g)\Lambda_{f_n}\left(\frac{\eta}{\mathrm{E}_{\mathrm{sg}}^{1,2}(g)}\right) &\\ \leq \mathrm{E}(u_n, \mathcal{B}_n^{\mathrm{Top}}(\eta))\Lambda_{f_n}&\left(\frac{\eta}{\mathrm{E}(u_n, \mathcal{B}_n^{\mathrm{Top}}(\eta))}\right)\\ &\leq \int_{\bigcup \mathcal{B}_n^{\mathrm{Top}}(\eta)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x\end{aligned} \tag{4.4.65}$$

holds for all n. Since by 4.4.11 we have

$$\lim_{n\to\infty} \left[\Lambda_{f_n}\left(\frac{\eta}{\mathrm{E}_{\mathrm{sg}}^{1,2}(g)}\right) - \mathcal{V}(f_n)\right] = \log\left(\frac{\mathrm{sys}(\mathcal{N})\mathrm{E}_{\mathrm{sg}}^{1,2}(g)}{2\eta}\right), \tag{4.4.66}$$

it follows that

$$
\begin{aligned}
&\varlimsup_{n\to\infty} \int_{\Omega\setminus\bigcup \mathcal{B}_n^{\mathrm{Top}}(\eta)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x \\
&\le \varlimsup_{n\to\infty} \left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\right] \\
&\quad - \varliminf_{n\to\infty} \left[\int_{\bigcup \mathcal{B}_n^{\mathrm{Top}}(\eta)} f_n(|\mathrm{D}u_n|) - \mathrm{E}^{1,2}_{\mathrm{sg}}(g)\Lambda_{f_n}\left(\eta/\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right)\right] \\
&\quad + \mathrm{E}^{1,2}_{\mathrm{sg}}(g)\lim_{n\to\infty}\left[\Lambda_{f_n}\left(\eta/\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right) - \mathcal{V}(f_n)\right] \\
&\le \mathrm{L}_2 + \mathrm{E}^{1,2}_{\mathrm{sg}}(g)\log\left(\frac{\mathrm{sys}(\mathcal{N})\mathrm{E}^{1,2}_{\mathrm{sg}}(g)}{2\eta}\right),
\end{aligned}
$$

where we used (4.4.65) to bound the second term by zero, thereby establishing the result. $\square$

4.5 Compactness results

4.5.1 Compactness theorem

Using the asymptotic lower bounds established in Proposition 4.4.10, we can infer compactness for sequences $(u_n)_n$ satisfying the renormalised bound (4.5.1). More precisely, we will show the following.

Theorem 4.5.1. *Let $g \in \mathrm{W}^{1/2,2}(\partial\Omega;\mathcal{N})$ and $(f_n)_n$ be a sequence of approximating integrands. Given a sequence $(u_n)_n \subset \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ satisfying $\mathrm{tr}_{\partial\Omega}\, u_n = g$ for all $n \in \mathbb{N}$, assume that*

$$
\mathrm{L} \doteq \varliminf_{n\to\infty}\left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] \quad \textit{is finite.} \tag{4.5.1}
$$

Then there exists an unrelabelled subsequence and a limit map $u_* \in \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ satisfying $\mathrm{tr}_{\partial\Omega}\, u_* = g$ and for which the following holds:

(i) *The limit map $u \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ with $\kappa \in \mathbb{N}$ distinct singular points $a_1,\dots,a_\kappa \in \Omega$, such that for each $0 < \rho < \rho_\Omega(a_1,\dots,a_\kappa)$, $(\mathrm{tr}_{\mathbb{S}^1}\, u_*(a_i + \rho\,\cdot))_{i=1}^\kappa$ is a minimal topological resolution of g, in that*

$$
\mathrm{E}^{1,2}_{\mathrm{sg}}(g) = \sum_{i=1}^{\kappa} \frac{\lambda([u_*, a_i])^2}{4\pi}.
$$

In particular, $\kappa \le 4\pi \mathrm{E}^{1,2}_{\mathrm{sg}}(g)/\mathrm{sys}(\mathcal{N})^2$.

(ii) *The renormalised energy can be controlled:*

$$
\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^\kappa \le \mathrm{L}.
$$

(iii) *For* $0 < \rho < \rho_\Omega(a_1, \ldots, a_\kappa)$,

$$\varliminf_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^\kappa \mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x \geq \int_{\Omega\setminus\bigcup_{i=1}^\kappa \mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}x,$$

$$\varlimsup_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^\kappa \mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x \leq \int_{\Omega\setminus\bigcup_{i=1}^\kappa \mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}x + \left(\mathrm{L} - \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) - \mathrm{H}([u_*, a_i])_{i=1}^\kappa\right).$$

(iv) *For* $0 < \rho < \rho_\Omega(a_1, \ldots, a_\kappa)$ *and all* $i = 1, \ldots, \kappa$,

$$\begin{aligned}
&\varliminf_{n\to\infty} \left[\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \frac{\lambda([u_*, a_i])^2}{4\pi}\mathcal{V}(f_n)\right] \\
&\qquad \geq \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \mathbb{B}(a_i, \rho)) + \frac{\lambda([u_*, a_i])^2}{4\pi} \log \frac{2\pi}{\lambda([u_*, a_i])^2} \\
&\varlimsup_{n\to\infty} \left[\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \frac{\lambda([u_*, a_i])^2}{4\pi}\mathcal{V}(f_n)\right] \\
&\qquad \leq \mathrm{L} - \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) - \mathrm{H}([u_*, a_i])_{i=1}^\kappa + \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \mathbb{B}(a_i, \rho)) \\
&\qquad\quad + \frac{\lambda([u_*, a_i])^2}{4\pi} \log \frac{2\pi}{\lambda([u_*, a_i])}.
\end{aligned}$$

recalling the localised renormalised energy on balls was defined in (4.3.10).

(v) *We have narrow convergence of measures*

$$\frac{f_n(|\mathrm{D}u_n|)}{\mathcal{V}(f_n)}\mathcal{L}^2 \llcorner \Omega \rightharpoonup \sum_{i=1}^\kappa \frac{\lambda([u_*, a_i])^2}{4\pi}\delta_{a_i},$$

in that the measures converge weakly and the total mass converges.*

(vi) *For* $0 < \rho < \rho_\Omega(a_1, \ldots, a_\kappa)$ *and each* $i = 1, \ldots, \kappa$ *we have*

$$\begin{aligned}
&\varliminf_{n\to\infty} \int_{\mathbb{B}(a_i,\rho)} |\mathrm{D}u_n|\,\mathrm{d}x \\
&\quad \leq \frac{2\pi\rho}{\lambda([u_*, a_i])}\Bigg[\mathrm{L} - \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) - \mathrm{H}([u_*, a_i])_{i=1}^\kappa + \lambda([u_*, a_i])^2 \\
&\qquad\qquad + \int_0^\rho \int_{\partial\mathbb{B}(a_i,r)} \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r}\,\mathrm{d}r\Bigg].
\end{aligned}$$

(vii) *The sequence* $(\mathrm{D}u_n)_n$ *is equi-integrable and converges weakly to* $\mathrm{D}u_*$ *in* $\mathrm{L}^1(\Omega)$.

Observe that if equality is attained in (ii), that is, we have

$$\lim_{n\to\infty}\left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] = \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa}, \tag{4.5.2}$$

then the upper and lower limits in (iii) and (iv) coincide. We will analyse this equality case in further detail in Theorem 4.6.1. Additionally as the proof will reveal, this result also applies in the case $\mathrm{L} = -\infty$; more precisely such a possibility is ruled out by (ii).

We will present the proof of Theorem 4.5.1 in Section 4.5.4. For this we will need results concerning weak L^1-compactness and strong convergence of traces, which we first collect in Sections 4.5.2 and 4.5.3 respectively.

4.5.2 Weak convergence of bounded sequences

Our first ingredient will be a compactness result away from the singularities. More precisely, we prove a weak compactness result in L^1 under the presence of the uniform bound (4.5.3).

Proposition 4.5.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, and $f_n : \mathbb{R}_+ \to \mathbb{R}$ be a sequence of Young functions converging pointwise to $\frac{1}{2}t^2$. Let $u_n \in \mathrm{W}^{1,1}(\Omega; \mathbb{R}^\nu)$. Assume that*

$$\mathrm{M} \doteq \varliminf_{n\to\infty}\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x \quad \textit{is finite.} \tag{4.5.3}$$

Then up to an unrelabelled subsequence, the following holds.

(a) *We have*

$$\varlimsup_{n\to\infty}\int_A |\mathrm{D}u_n|\,\mathrm{d}x \le \sqrt{2\mathrm{M}\mathcal{L}^2(A)} \quad \textit{for all } A \subset \Omega \textit{ measurable,} \tag{4.5.4}$$

and $(|\mathrm{D}u_n|)_n$ is uniformly integrable.

(b) *If we additionally have u_n converges strongly to a limit map u in $\mathrm{L}^1(\Omega;\mathbb{R}^\nu)$, then $u \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^\nu)$, $\mathrm{D}u_n$ converges weakly to $\mathrm{D}u$ in $\mathrm{L}^1(\Omega)$, and*

$$\int_\Omega \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}x \le \varliminf_{n\to\infty}\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x. \tag{4.5.5}$$

Furthermore, if $u_n(x) \in \mathcal{N}$ for all n and almost every $x \in \Omega$, then the limit u satisfies $u(x) \in \mathcal{N}$ for almost every $x \in \Omega$.

Proposition 4.5.2 is a general compactness result for Young functions, we do not assume $(\mathrm{Dec})_2^c$ nor (vInt). Moreover while we stated Proposition 4.5.2 in $\mathbb{R}^2$, an inspection of the proof reveals that the argument applies more generally in $\mathbb{R}^d$, and also in $\mathrm{L}^1(\mathbb{S}^1;\mathcal{H}^1)$. We will use the latter case in the proof of Theorem 4.5.1, and we will omit the details of the necessary modifications.

Remark 4.5.3 (Weak and weak* convergence). We recall the difference between weak* and weak convergence for our uses. We say a sequence of functions $(U_n)_n \subset \mathrm{L}^1(\Omega)$ converges weakly in L^1 to a limit map $U \in \mathrm{L}^1(\Omega)$ if

$$\lim_{n\to\infty} \int_\Omega U_n\phi = \int_\Omega U\phi \quad \text{for all } \phi \in \mathrm{L}^\infty(\Omega). \tag{4.5.6}$$

We say a sequence $(U_n)_n \subset \mathrm{L}^1(\Omega)$ converges weakly* as measures to $U \in \mathrm{L}^1(\Omega)$ if (4.5.6) only holds for all $\phi \in \mathrm{C}_0(\Omega)$. Note that for mappings $(u_n)_n$ in $\mathrm{W}^{1,1}$, weak* convergence of $(\mathrm{D}u_n)_n$ corresponds to weak* convergence of (u_n) in BV.

The following variant of the Dunford-Pettis theorem will be of use, see [3, Thm. 2.4.5].

Lemma 4.5.4. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, and consider a sequence of mappings $(u_n)_n \subset \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ converging strongly in $\mathrm{L}^1(\Omega)$. Then $(\mathrm{D}u_n)_n$ converges weakly in L^1 if and only if $\{|\mathrm{D}u_n| : n \in \mathbb{N}\}$ is uniformly integrable.*

Classically, the Dunford-Pettis theorem asserts that weak compactness of a sequence $(U_n)_n$ in $\mathrm{L}^1(\Omega)$ is equivalent to uniform integrability of this family. Recall that equi-integrability and uniform integrability are synonymous. In our setting we apply this with $U_n = \mathrm{D}u_n$, noting the limiting map is necessarily unique by strong convergence of $(u_n)_n$ in $\mathrm{L}^1(\Omega)$.

In the proof of Proposition 4.5.2, we will use the following result (Lemma 4.5.5), which allows us to convert "asymptotic" uniform integrability into the well-known statement of uniform integrability. While we will state the result for the Lebesgue measure in $\mathbb{R}^2$, we note the following also holds in more general measure spaces.

Lemma 4.5.5. *For a sequence of functions $(U_n)_n \subset \mathrm{L}^1(\Omega)$, the following are equivalent:*

(i) *the sequence (U_n) is uniformly integrable; that is for all $\varepsilon > 0$ there exists $\delta > 0$ such that for every measurable $A \subset \Omega$ such that $\mathcal{L}^2(A) \leq \delta$, we have $\sup_n \int_A |U_n| \,\mathrm{d}x \leq \epsilon$,*

(ii) *for all* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that for every measurable* $A \subset \Omega$ *such that* $\mathcal{L}^2(A) \leq \delta$*, we have* $\varlimsup_{n\to\infty} \int_A |U_n| \,\mathrm{d}x \leq \epsilon$.

This result follows by applying the Vitali-Hahn-Saks Theorem (see *e.g.* [17, Thm. 2.53]), considering a countable family of measurable sets on which the superior limit in (ii) is a limit. The Vitali-Hahn-Saks is usually proven using Baire's category theorem, and it also proves Lemma 4.5.5 directly along similar lines as follows: Given $\varepsilon > 0$, let $\delta > 0$ as in (ii) and consider the complete metric space $\mathcal{A}_\delta$ of measurable subsets $A \subset \Omega$ satisfying $\mathcal{L}^2(A) \leq \delta$, with respect to the metric $\mathcal{L}^2(A \triangle B) = \int_\Omega |\chi_A - \chi_B| \,\mathrm{d}\mathcal{L}^2$. Then consider the closed subsets

$$\mathcal{M}(m) = \bigcap_{n\geq m} \left\{ A \in \mathcal{A}_\delta : \int_A |U_n| \,\mathrm{d}x \leq 2\epsilon \right\}, \tag{4.5.7}$$

which by (ii) satisfies $\bigcup_m \mathcal{M}(m) = \mathcal{A}_\delta$. Therefore by Baire's theorem, there is some $\mathcal{M}(m_0)$ with non-empty interior, and hence contains an r-ball for some $r > 0$. One can then verify that setting $\delta_0 = \min\{\delta, r\}$, we have $\mathcal{L}^2(A) \leq \delta_0$ implies that $\sup_{n\geq m_0} \int_A U_n \,\mathrm{d}x < 2\varepsilon$. Since the finite collection $\{U_1, \ldots, U_{m_0-1}\}$ is uniformly integrable, shrinking δ_0 further if necessary (i) follows.

Proof of Proposition 4.5.2. By passing to a subsequence, we can assume that

$$\mathrm{M} = \varlimsup_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u_n|) \,\mathrm{d}x < \infty. \tag{4.5.8}$$

We will first show (a), namely that (4.5.4) holds. For this let $A \subset \Omega$ be measurable, and assume that $\mathcal{L}^2(A) > 0$ since there is otherwise nothing to show. Then Jensen's inequality gives

$$\mathrm{M} \geq \mathcal{L}^2(A) \varlimsup_{n\to\infty} f_n\Big(\frac{1}{\mathcal{L}^2(A)} \int_A |\mathrm{D}u_n| \,\mathrm{d}x \Big).$$

Let

$$0 < \mathrm{L} < \frac{1}{\mathcal{L}^2(A)} \varlimsup_{n\to\infty} \int_A |\mathrm{D}u_n| \,\mathrm{d}x, \tag{4.5.9}$$

assuming this limit inferior is positive, as otherwise (4.5.4) holds since the left-hand side is zero. Then passing to a further subsequence where the limit supremum is attained, we can assume that $\mathrm{L}\,\mathcal{L}^2(A) \leq \int_A |\mathrm{D}u_n| \,\mathrm{d}x$

for all n sufficiency large. For such n, since each $f_n(t)$ is non-decreasing, $\mathrm{M} \geq \mathcal{L}^2(A)\,\overline{\lim}_{n\to\infty} f_n(\mathrm{L}) = \mathcal{L}^2(A)\mathrm{L}^2/2$, and hence that

$$\mathrm{M} \geq \frac{\mathcal{L}^2(A)}{2}\Big(\overline{\lim_{n\to\infty}}\int_A |\mathrm{D}u_n|\,\mathrm{d}x\Big)^2, \tag{4.5.10}$$

which rearranges to give (4.5.4). Therefore we have $(|\mathrm{D}u_n|)_n$ is uniformly integrable by Lemma 4.5.5, thereby establishing (a).

Now for (b), by the Dunford-Pettis theorem (see Lemma 4.5.4) we have the strong L^1 convergence and uniform integrability of the gradients implies that $\mathrm{D}u_n$ converges weakly to $\mathrm{D}u$ in $\mathrm{L}^1(\Omega)$. To show (4.5.5), let $(\kappa_m)_m \subset (0,\infty)$ be any sequence such that $\kappa_m \nearrow \infty$. Then by Lemma 4.2.10(b) we have $\lim_{n\to\infty} f'_{n,+}(\kappa_m) \to \kappa_m$ for each $m \in \mathbb{N}$, and writing $f(t) = \frac{1}{2}t^2$ we consider the truncations $T_{\kappa_m} f_n$ and $T_{\kappa_m} f$ from Definition 4.2.9. Note that for each m, $T_{\kappa_m} f$ is the pointwise limit of $T_{\kappa_m} f_n(t)$ as $n \to \infty$, and moreover this limit is uniform on $[0,\kappa_m]$ by Lemma 4.2.10(a). Also for $t > \kappa_m$ we have

$$\begin{aligned} &|T_{\kappa_m} f_n(t) - T_{\kappa_m} f(t)| \\ &\qquad \leq |f'_{n,+}(\kappa_m) - \kappa_m||t| + \left|f_n(\kappa_m) - f'_{n,+}(\kappa_m)\kappa_m + \frac{1}{2}\kappa_m^2\right|, \end{aligned} \tag{4.5.11}$$

where both coefficients vanish as $n \to \infty$. Hence setting $A_{m,n} = \{x \in \Omega : |\mathrm{D}u_n|(x) \leq \kappa_m\}$, for any m we have

$$\begin{aligned} \mathrm{M} &= \overline{\lim_{n\to\infty}}\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x \\ &\geq \overline{\lim_{n\to\infty}}\int_\Omega T_{\kappa_m} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x \\ &\geq \overline{\lim_{n\to\infty}}\int_\Omega T_{\kappa_m} f(|\mathrm{D}u_n|)\,\mathrm{d}x - \overline{\lim_{n\to\infty}}\left|\int_\Omega T_{\kappa_m} f_n(|\mathrm{D}u_n|) - T_{\kappa_m} f(|\mathrm{D}u_n|)\,\mathrm{d}x\right| \\ &\geq \overline{\lim_{n\to\infty}}\int_\Omega T_{\kappa_m} f(|\mathrm{D}u_n|)\,\mathrm{d}x - \overline{\lim_{n\to\infty}}\int_{A_{m,n}}\left|f_n(|\mathrm{D}u_n|) - \frac{1}{2}|\mathrm{D}u_n|^2\right|\,\mathrm{d}x \\ &\quad - \overline{\lim_{n\to\infty}}|f'_{n,+}(\kappa_m) - \kappa_m|\int_{\Omega\setminus A_{m,n}} |\mathrm{D}u_n|\,\mathrm{d}x \\ &\quad - \overline{\lim_{n\to\infty}}\left|f_n(\kappa_m) - f'_{n,+}(\kappa_m)\kappa_m + \frac{1}{2}\kappa_m^2\right|\mathcal{L}^2(\Omega\setminus A_{m,n}) \\ &= \overline{\lim_{n\to\infty}}\int_\Omega T_{\kappa_m} f(|\mathrm{D}u_n|)\,\mathrm{d}x. \end{aligned}$$

Note that in the last line, we used the L^1-boundedness of $|\mathrm{D}u_n|$ to show that $\int_{\Omega\setminus A_{m,n}} |\mathrm{D}u|\,\mathrm{d}x$ is uniformly bounded. Since $\mathrm{D}u_n \rightharpoonup \mathrm{D}u$ weakly in

L^1, using the lower-semicontinuity from Lemma 4.2.17 we have

$$\mathrm{M} \geq \varlimsup_{n\to\infty} \int_\Omega T_{\kappa_m} f(|\mathrm{D}u_n|)\,\mathrm{d}x \geq \int_\Omega T_{\kappa_m} f(|\mathrm{D}u|)\,\mathrm{d}x, \tag{4.5.12}$$

so sending $m \to \infty$ and applying the monotone convergence theorem we have

$$\begin{aligned} \varliminf_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x &= \mathrm{M} \\ &\geq \lim_{m\to\infty} \int_\Omega T_{\kappa_m} f(|\mathrm{D}u|)\,\mathrm{d}x = \int_\Omega \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}x, \end{aligned} \tag{4.5.13}$$

Finally since $(u_n)_n$ converges strongly in L^1, passing to a subsequence we can assume u_n converges almost everywhere to u in Ω. In particular, if for all n we have $u_n(x) \in \mathcal{N}$ for almost every $x \in \Omega$, then the same holds for the limit map u. □

4.5.3 Strong convergence of traces

As the example $(\chi_{\mathbb{B}(0,1-1/n)})_n$ illustrates, the trace operator $\mathrm{tr}_{\partial\mathbb{B}(0,1)} : \mathrm{BV}(\mathbb{R}^2) \to \mathrm{L}^1(\mathbb{S}^1)$ is not weakly* continuous. We therefore need refined convergence results of traces, which we establish in this section.

Proposition 4.5.6. *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz domain, $(f_n)_n$ a sequence of approximating integrands, and $(u_n)_n \subset \mathrm{W}^{1,1}(\Omega;\mathbb{R}^\nu)$ be a sequence converging weakly* in* BV *to $u \in \mathrm{W}^{1,2}(\Omega;\mathbb{R}^\nu)$. If*

$$\varlimsup_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u_n|) \quad \textit{is finite,} \tag{4.5.14}$$

then passing to a subsequence we have $\mathrm{tr}_{\partial\Omega} u_n \to \mathrm{tr}_{\partial\Omega} u$ strongly in $\mathrm{L}^1(\partial\Omega;\mathbb{R}^\nu)$. Moreover

$$\varlimsup_{n\to\infty} \int_{\partial\Omega} f_n(|\operatorname{tr}_{\partial\Omega} u_n|)\,\mathrm{d}\mathcal{H}^1 \ \textit{is finite} \tag{4.5.15}$$

and

$$\int_{\partial\Omega} \frac{|\mathrm{tr}_{\partial\Omega} u|^2}{2}\,\mathrm{d}\mathcal{H}^1 \leq \varliminf_{n\to\infty} \int_{\partial\Omega} f_n(|\operatorname{tr}_{\partial\Omega} u_n|)\,\mathrm{d}\mathcal{H}^1. \tag{4.5.16}$$

Lemma 4.5.7. *Let $\Omega \subset \mathbb{R}^2$ be convex and $f\colon \mathbb{R}_+ \to \mathbb{R}$ be a Young function satisfying $(\mathrm{Dec})_2^{\mathrm{n}}$. Then there exists $C > 0$ such that for all $v \in \mathrm{W}^{1,1}(\Omega;\mathbb{R}^\nu)$,*

$$\int_{\partial\Omega} f(|v|)\,\mathrm{d}\mathcal{H}^1 \leq C \int_\Omega 1 + f(|v|) + f(|\mathrm{D}v|)\,\mathrm{d}x.$$

Moreover if $(f_n)_n$ is a sequence of approximating integrands, then the constant $C > 0$ may be chosen to be independent of n.

We will present a proof relying on Gagliardo's L^1-trace theorem.

Proof of Lemma 4.5.7. By considering $|v|$ in place of v, we may assume that $v \geq 0$. Then by the chain rule and Lemma 4.5.8 below, we have

$$\mathrm{D}(f(v)) = f'_+(v)\mathrm{D}v \leq c_1(1 + f(|v|) + f(|\mathrm{D}v|)), \tag{4.5.17}$$

so $f(|v|) \in \mathrm{W}^{1,1}(\Omega)$. For a sequence of approximating integrands, Lemma 4.5.8 also asserts that this constant $c_1 > 0$ can be chosen independently of n. Hence by Gagliardo's trace theorem (see *e.g.* [27, Thm. 18.18]) there exists $c_2 > 0$, depending only on Ω, such that

$$\begin{aligned} \int_{\partial\Omega} & f(\mathrm{tr}_{\partial\Omega}\, v)\,\mathrm{d}\mathcal{H}^1 \\ &\leq c_2 \int_\Omega f(v) + |\mathrm{D}(f(v))|\,\mathrm{d}x \\ &\leq c_2(1 + c_1)\int_\Omega 1 + f(|v|) + f(|\mathrm{D}u|)\,\mathrm{d}x, \end{aligned} \tag{4.5.18}$$

as required. □

Lemma 4.5.8. *If $f : \mathbb{R}_+ \to \mathbb{R}$ is a Young function satisfying* $(\mathrm{Dec})_2^{\mathrm{n}}$, *there is $C > 0$ such that*

$$f'_+(s)t \leq C(1 + f(s) + f(t)) \quad \textit{for all } s, t \geq 0. \tag{4.5.19}$$

Moreover if (f_n) is a sequence of approximating integrands, the constant $C > 0$ may be chosen to be independent of n.

Proof of Lemma 4.5.8. Denoting the convex conjugate of f as $f^*(s) = \sup\{st - f(t) : t > 0\}$,

$$f'_+(s)t \leq f^*(f'_+(s)) + f(t) \tag{4.5.20}$$

holds for all $s, t \geq 0$. By (4.2.22) we know that $f'_+(t) \leq f(2t)/t$, and by [25, (2.10)] we have $f^*(f(t)/t) \leq f(t)$. Hence, combining these two inequalities gives $f^*(f'_+(t)) \leq f^*(f(2t)/t) \leq 2f(2t)$. By $(\mathrm{Dec})_2^{\mathrm{n}}$ we have $f(2t) \leq 4f(t)$ for all $t \geq t_0$, and for $t < t_0$ we have $f(2t) \leq f(2t_0)$, thereby giving $f(2t) \leq C(1 + f(t))$ for all $t \geq 0$. Observe that for an approximating family $(f_n)_n$, since $\sup_{n\in\mathbb{N}} f_n(2t_0)$ is bounded, this constant can be chosen uniformly in n. □

Proof of Proposition 4.5.6. Define

$$\mathrm{L} = \varlimsup_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u_n|), \tag{4.5.21}$$

which by (4.5.14) is finite. By [27, Thm. 18.21(iii)] there is $C > 0$ for which

(4.5.22)
$$\int_{\partial\Omega} |\mathrm{tr}_{\partial\Omega}\, u - \mathrm{tr}_{\partial\Omega}\, u_n| \,\mathrm{d}\mathcal{H}^1 \leq \frac{C}{\varepsilon}\int_{A_\varepsilon} |u - u_n| \,\mathrm{d}x + C|\mathrm{D}u - \mathrm{D}u_n|(A_\varepsilon),$$

where $A_\varepsilon = \{x \in \Omega : \mathrm{dist}(x, \partial\Omega) < \varepsilon\}$. Applying (4.5.4) from Proposition 4.5.2 and using that $u_n \to u$ strongly in $\mathrm{L}^1(\Omega; \mathbb{R}^\nu)$ (for instance by [16, Thm. 5.5]), we can send $n \to \infty$ in (4.5.22) to obtain

$$\varlimsup_{n\to\infty} \int_{\partial\Omega} |\mathrm{tr}_{\partial\Omega}\, u - \mathrm{tr}_{\partial\Omega}\, u_n| \,\mathrm{d}\mathcal{H}^1 \leq 2C\sqrt{2\mathrm{L}\mathcal{L}^2(A_\varepsilon)}, \tag{4.5.23}$$

which vanishes as $\varepsilon \to 0$. In particular, passing to a subsequence we have $\mathrm{tr}_{\partial\Omega}\, u_n \to \mathrm{tr}_{\partial\Omega}\, u$ pointwise $\mathcal{H}^1$-almost everywhere, from which (4.5.16) follows by Fatou's lemma. Finally since $\mathcal{N}$ is compact, $\sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{L}^\infty(\Omega)}$ is finite (see Section 4.2.4), so by Lemma 4.5.7 we can estimate

(4.5.24)
$$\varlimsup_{n\to\infty} \int_{\partial\Omega} f_n(|\mathrm{tr}_{\partial\Omega}\, u_n|) \,\mathrm{d}\mathcal{H}^1 \leq C(1 + \sup_{n\in\mathbb{N}} \|u_n\|_{\mathrm{L}^\infty(\Omega)})\mathcal{L}^2(\Omega) + C\,\mathrm{L},$$

establishing(4.5.15). □

The following Lemma 4.5.9 will be of use in the proof of Theorem 4.5.1, in order to associate a well-defined homotopy class to each singularity of the limiting map. We will use the notation D_τ for the tangential trace on any circle $\mathbb{S}^1(a, r) = \partial\mathbb{B}(a, r)$.

Lemma 4.5.9. *Let $a \in \mathbb{R}^2$ and $r > 0$. Suppose $(u_n)_n \subset \mathrm{W}^{1,1}(\partial\mathbb{B}(a, r); \mathcal{N})$ and $(f_n)_n$ is a sequence of approximating integrands such that*

$$\varlimsup_{n\to\infty} \int_{\partial\mathbb{B}(a,r)} f_n(|\mathrm{D}_\tau u_n|) \,\mathrm{d}\mathcal{H}^1 \quad \textit{is finite,} \tag{4.5.25}$$

and $\mathrm{tr}_{\partial\mathbb{B}(a,r)}\, u_n \to \mathrm{tr}_{\partial\mathbb{B}(a,r)}\, u$ strongly in $\mathrm{L}^1(\partial\mathbb{B}(a, r); \mathcal{N})$ to some limit $u \in \mathrm{W}^{1,1}(\partial\mathbb{B}(a, r); \mathcal{N})$. Then we have u_n and u are homotopic on $\partial\mathbb{B}(a, r)$ for all n sufficiently large, and in particular $\lambda(\mathrm{tr}_{\partial\mathbb{B}(a,r)}\, u_n) = \lambda(\mathrm{tr}_{\partial\mathbb{B}(a,r)}\, u)$ and $\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}(a,r)}\, u_n) = \mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\mathbb{B}(a,r)}\, u)$ for all such n.

Proof of Lemma 4.5.9. It suffices to show the traces of u_n and u on $\partial\mathbb{B}(a, r)$ are homotopic for sufficiently large n, which will imply the equality of the minimal lengths and singular energies. Taking the truncations $T_\kappa f_n$ of f_n, by (4.5.25) there is $M > 0$ such that

$$\int_{\partial\mathbb{B}(a,r)} T_\kappa f_n(|\mathrm{D}_\tau u_n|) \,\mathrm{d}x \leq \int_{\partial\mathbb{B}(a,r)} f_n(|\mathrm{D}_\tau u_n|) \,\mathrm{d}x \leq M \tag{4.5.26}$$

for all $n \in \mathbb{N}$ and $\kappa > 0$. Since the traces $\operatorname{tr}_{\partial\mathbb{B}(a,r)} u_n$, $\operatorname{tr}_{\partial\mathbb{B}(a,r)} u$ are assumed to lie in $\mathrm{W}^{1,1}(\partial\mathbb{B}(a,r);\mathcal{N})$, they admit a continuous representative which we denote simply by u_n, u respectively. By parametrising $\partial\mathbb{B}(a,r)$ by the map $\theta \mapsto a + (r\cos\theta, r\sin\theta)$, we mollify in θ to define $u_{n,\varepsilon} \doteq u_n * \rho_\varepsilon$, which satisfies

$$\|u_{n,\varepsilon} - u_\varepsilon\|_{L^\infty(\partial\mathbb{B}(a,r))} \leq \varepsilon^{-1}\|u_n - u\|_{L^1(\partial\mathbb{B}(a,r))}. \tag{4.5.27}$$

We will first estimate $|u_{n,\varepsilon} - u_n|$ pointwise. For this let $x, y \in \partial\mathbb{B}(a,r)$, and let $S_{x,y} \subset \partial\mathbb{B}(a,r)$ denote an arc connecting x and y with minimal length. Then by the fundamental theorem of calculus we have

$$\begin{aligned}
|u_n(x) - u_n(y)| &\leq \int_{S_{x,y}} |\mathrm{D}_\tau u_n| \,\mathrm{d}\mathcal{H}^1 \\
&= \int_{S_{x,y}} \chi_{\{|\mathrm{D}_\tau u_n| \geq \kappa\}} |\mathrm{D}_\tau u_n| \,\mathrm{d}\mathcal{H}^1 + \int_{S_{x,y}} \chi_{\{|\mathrm{D}_\tau u_n| < \kappa\}} |\mathrm{D}_\tau u_n| \,\mathrm{d}\mathcal{H}^1 \\
&\leq \frac{1}{f_{n,+}'(\kappa)} \int_{S_{x,y}} T_\kappa f_n(|\mathrm{D}_\tau u_n|) \,\mathrm{d}\mathcal{H}^1 \\
&\quad + \mathcal{H}^1(S_{x,y}) f_n^{-1}\left(\frac{1}{\mathcal{H}^1(S_{x,y})} \int_{S_{x,y}} \chi_{\{|\mathrm{D}_\tau u_n| < \kappa\}} f_n(|\mathrm{D}_\tau u_n|) \,\mathrm{d}\mathcal{H}^1\right) \\
&\leq \frac{M}{f_{n,+}'(\kappa)} + \mathcal{H}^1(S_{x,y}) f_n^{-1}\left(\frac{M}{\mathcal{H}^1(S_{x,y})}\right).
\end{aligned}$$

Then noting that

$$|x - y| \leq \mathcal{H}^1(S_{x,y}) = \operatorname{dist}_{\mathbb{S}^1}(x,y) = 2r\arcsin\left(|x-y|/2r\right) \leq 2|x-y|,$$

we have

$$\|u_{n,\varepsilon} - u_n\|_{\mathrm{L}^\infty(\partial\mathbb{B}(a,r))} \leq \frac{M}{f_{n,+}'(\kappa)} + 2\varepsilon f_n^{-1}\left(\frac{M}{\varepsilon}\right). \tag{4.5.28}$$

Since $u_n \to u$ almost everywhere on $\partial\mathbb{B}(a,r)$, passing to the limit in this estimate (noting by Lemma 4.2.10 that $f_{n,+}'(\kappa) \to \kappa$ and $f_n^{-1}(t) \to \sqrt{2t}$ as $n \to \infty$) we have

$$\|u_\varepsilon - u\|_{L^\infty(\partial\mathbb{B}(a,r))} \leq M/\kappa + \sqrt{8M\varepsilon}. \tag{4.5.29}$$

Now consider the tubular neighbourhood projection $\Pi_{\mathcal{N}} \colon \mathcal{U} \to \mathcal{N}$, so there is $\delta > 0$ such that $\mathcal{N} + \mathbb{B}(0,\delta) \subset \mathcal{U}$. We then define $\tilde{H}_n(t,x) = (1-t)u_n(x) + tu(x)$, and we will show that $\tilde{H}_n$ maps into $\mathcal{U}$ for n sufficiently large. Indeed for $(t,x) \in [0,1] \times \partial\mathbb{B}(a,r)$ we have by (4.5.27), (4.5.28) and (4.5.29),

$$
\begin{aligned}
&\operatorname{dist}(\tilde{H}_n(x,t);\mathcal{N}) \le |\tilde{H}(t,x) - u_n(x)| \\
&\le |u_n - u_{n,\varepsilon}|(x) + |u_{n,\varepsilon} - u_\varepsilon|(x) + |u_\varepsilon - u|(x) \\
&\le \frac{M}{f'_{n,+}(\kappa)} + 2\varepsilon f_n^{-1}\left(\frac{M}{\varepsilon}\right) + \varepsilon^{-2}\|u_n - u\|_{L^1(\partial\mathbb{B}(a,r))} + \frac{M}{\kappa} + \sqrt{8M\varepsilon}.
\end{aligned}
$$

Now we choose $\varepsilon > 0$ sufficiently small so that $\sqrt{8M\varepsilon} < \delta/9$. We also take $\kappa > 9M/\delta$, then for n sufficiently large we have

$$
\frac{M}{f'_{n,+}(\kappa)} + 2\varepsilon f_n^{-1}\left(\frac{M}{\varepsilon}\right) \le \frac{2M}{\kappa} + 2\sqrt{8M\varepsilon} < \frac{4\delta}{9}, \tag{4.5.30}
$$

and since $u_n \to u$ in L^1 on $\partial\mathbb{B}(a,r)$, increasing n if necessary we have a $\varepsilon^{-1}\|u_n - u\|_{L^1(\partial\mathbb{B}(a,r))} < \delta/3$. Combining our estimates, we have

$$
\operatorname{dist}(\tilde{H}(t,x);\mathcal{N}) \le \frac{4\delta}{9} + \varepsilon^{-1}\|u_n - u\|_{\mathrm{L}^1(\partial\mathbb{B}(a,r)} + \frac{M}{\kappa} + \sqrt{8M\varepsilon} < \delta \tag{4.5.31}
$$

for all $(t,x) \in [0,1] \times \partial\mathbb{B}(a,r)$, $\operatorname{dist}(\tilde{H}(t,x);\mathcal{N}) < \delta$ for all $(t,x) \in [0,1] \times \partial\mathbb{B}(a,r)$. Thus we see that $H_n(t,x) = \Pi_{\mathcal{N}} \circ \tilde{H}_n(t,x)$ is a well-defined homotopy between u_n and u for sufficiently large n, thereby establishing the result. □

4.5.4 Constrained compactness

Proof of Theorem 4.5.1. We will break up the proof into a series of steps. Steps 1–4 are devoted to obtaining a limiting map u_* which is renormalisable in Ω, along with (i) and (ii). This involves approximating $(u_n)_n$ by mappings in the $\mathrm{R}^{1,2}$-class (see Definition 4.3.5) in Step 1, to which we can apply Proposition 4.4.10, and from which a weakly*-convergent limit can be extracted in Step 2. It is then shown in Step 3 that the limiting map has singularities forming a minimal topological resolution of g, and furthermore in Step 4 it is shown that u_* is renormalisable and that singularities are bounded away from $\partial\Omega$. In Step 5, we prove the asymptotic estimates (iii) and (iv) for $f_n(|\mathrm{D}u_n|)$, from which we can infer convergence of the defect measures, namely (v), which we establish in Step 6. Finally in Step 7, we use (iv) to prove the L^1-estimates (vi), which is used to establish weak L^1 convergence (vii).

4.5.4.1 Step 1: Extension and approximation

Lemma 4.5.10. *Fix an open bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, a compact Riemannian manifold $\mathcal{N}$ and a map $u \in \mathrm{W}^{1,1}(\Omega;\mathcal{N})$. Assume* *that there exists an open set* $\omega \subset\subset \Omega$ *with* $u \in \mathrm{W}^{1,2}(\Omega \setminus \bar{\omega}, \mathcal{N})$. *Then, for every* $\varepsilon > 0$ *and every* $\theta \in (0,1)$, *setting*

$$\omega_\theta \doteq \{x \in \Omega : \mathrm{dist}(\omega, x) < \theta \ \mathrm{dist}(\omega, \Omega)\},$$

there exists a map $v \in \mathrm{W}^{1,1}(\Omega, \mathcal{N})$ *such that*

(i) *for some points* $\{a_i\}_{i=1,\dots,k} \subset \omega_\theta$ $(k \in \mathbb{N})$ *such that* $v \in C^\infty(\Omega \setminus \{a_i\}_{i=1,\dots,k}, \mathcal{N})$,

(ii) $\|u - v\|_{\mathrm{W}^{1,1}(\Omega)} \leq \varepsilon$,

(iii) $\mathrm{tr}_{\partial\Omega} u$ *and* $\mathrm{tr}_{\partial\Omega} v$ *are homotopic.*

This follows by adapting the arguments in [6, Thm. 2], noting that in the regions where u lies in $\mathrm{W}^{1,2}$, we can locally approximate by composing mollifications with a nearest point projection to $\mathcal{N}$ (see [40, §4]). This can be done up to the boundary by working with a local extension, which exists by [32, Lem. 5.5]. Note that the analogous result for $p \in (1,2)$ was proven in [43, Prop. 4.2]; in this case one does not need to assume the compactness of $\mathcal{N}$.

Since $g \in \mathrm{W}^{1/2,2}(\partial\Omega; \mathcal{N})$, by [32, Lem. 5.5] there exists a Lipschitz domain $\Omega' \subset \mathbb{R}^2$ such that $\Omega \Subset \Omega'$ and a map $U \in \mathrm{W}^{1,2}(\Omega' \setminus \Omega; \mathcal{N})$ such that $\mathrm{tr}_{\partial\Omega} U = g$ and such that $\mathrm{E}^{1,2}_{\mathrm{sg}}(\mathrm{tr}_{\partial\Omega'} U) = \mathrm{E}^{1,2}_{\mathrm{sg}}(g)$. For each n, we define

$$\bar{u}_n = \begin{cases} u_n & \text{on } \Omega \\ U & \text{on } \Omega' \setminus \Omega, \end{cases} \tag{4.5.32}$$

then each $\bar{u}_n \in \mathrm{W}^{1,1}(\Omega'; \mathcal{N})$, using [27, Thm. 18.1(ii)] to verify the weak differentiability. Observe by the dominated convergence theorem and Lemma 4.2.12 (as in the proof of Proposition 4.4.1) that,

$$\lim_{n\to\infty} \int_{\Omega'\setminus\Omega} f_n(|\mathrm{D}\bar{u}_n|) = \lim_{n\to\infty} \int_{\Omega'\setminus\Omega} f_n(|\mathrm{D}U|) = \int_{\Omega'\setminus\Omega} \frac{|\mathrm{D}U|^2}{2}. \tag{4.5.33}$$

Now recalling the κ-truncation operator from Definition 4.2.9, noting that $T_\kappa f_n \leq f_n$ for all n and $T_\kappa f_n \to f_n$ pointwise as $\kappa \to \infty$. Thus we can find a sequence $(\kappa_n)_n$ such that

$$\lim_{n\to\infty} (\mathcal{V}(f_n) - \mathcal{V}(T_{\kappa_n} f_n)) = 0 \tag{4.5.34}$$

$$\lim_{n\to\infty} \int_{\Omega'} |f_n(\mathrm{D}\bar{u}_n) - T_{\kappa_n} f_n(\mathrm{D}\bar{u}_n)| \,\mathrm{d}x = 0. \tag{4.5.35}$$

Then combining with (4.5.1) and (4.5.33) we deduce that

$$\varlimsup_{n\to\infty}\left[\int_{\Omega'} T_{\kappa_n} f_n(|\mathrm{D}\bar{u}_n|) - \mathcal{V}(T_{\kappa_n} f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] = \mathrm{L} + \int_{\Omega'\setminus\Omega} \frac{1}{2}|\mathrm{D}U|^2\,\mathrm{d}x, \tag{4.5.36}$$

recalling L was defined in(4.5.1). In particular,

$$\varlimsup_{n\to\infty} \mathcal{V}(T_{\kappa_n} f_n)^{-1} \int_{\Omega'} T_{\kappa_n} f_n(|\mathrm{D}\bar{u}_n|) = \mathrm{E}^{1,2}_{\mathrm{sg}}(g). \tag{4.5.37}$$

Now putting $\delta \doteq \operatorname{dist}(\Omega, \partial\Omega') > 0$, for each $n \in \mathbb{N}$ by Lemma 4.5.10 we can find a sequence $(u_{n,m})_m \subset \mathrm{R}^{1,2}_{\delta}(\Omega'; \mathcal{N})$ such that $u_{n,m} \to \bar{u}_n$ strongly in $\mathrm{W}^{1,1}(\Omega'; \mathcal{N})$ as $m \to \infty$ and such that $\mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega'} u_{n,m}) = \mathrm{E}^{1,2}_{\mathrm{sg}}(g)$ for all m. Then for each n, by passing to a subsequence in m if necessary, by the partial converse of the dominated convergence theorem (see for instance [11, Thm. 4.9]), we can find $g_n \in \mathrm{L}^1(\Omega', \mathbb{R})$ non-negative such that $|\mathrm{D}u_{n,m}| \le g_n$ for all m.

Therefore, by the linear growing behaviour of $T_{\kappa_n} f_n$ and the dominated convergence theorem (using the majorant g_n),

$$\lim_{m\to\infty} \int_{\Omega} |T_{\kappa_n} f_n(|\mathrm{D}u_{n,m}|) - T_{\kappa_n} f_n(|\mathrm{D}u_n|)|\,\mathrm{d}x = 0 \quad \text{for all } n.$$

Thus we choose a sequence $(m_n)_n$ such that

$$\lim_{n\to\infty} \int_{\Omega} |T_{\kappa_n} f_n(|\mathrm{D}u_{n,m_n}|) - T_{\kappa_n} f_n(|\mathrm{D}\bar{u}_n|)|\,\mathrm{d}x = 0, \tag{4.5.38}$$

so we have

$$\varlimsup_{n\to\infty}\left[\int_{\Omega'} T_{\kappa_n} f_n(|\mathrm{D}u_{n,m_n}|) - \mathcal{V}(T_{\kappa_n} f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] = \mathrm{L} + \int_{\Omega'\setminus\Omega} \frac{1}{2}|\mathrm{D}U|^2\,\mathrm{d}x. \tag{4.5.39}$$

We can also ensure that $|\mathrm{D}\bar{u}_n - \mathrm{D}u_{n,m_n}| \to 0$ strongly in $\mathrm{L}^1(\Omega')$. Hence by applying Proposition 4.4.10(ii) to $(u_{n,m_n})_n$ and $(T_{\kappa_n} f_n)_n$, for each $\eta \in (0, \min\{\frac{\operatorname{sys}(\mathcal{N})}{2\pi t_0}, \delta\})$ there exists a collection $\mathcal{B}^{\mathrm{Top}}_n(\eta)$ such that putting

$$\mathrm{E}(u_{n,m_n}, \mathcal{B}^{\mathrm{Top}}_n(\eta)) \doteq \sum_{\mathbb{B}\in\mathcal{B}^{\mathrm{Top}}_n(\eta)} \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\mathbb{B}} u_{n,m_n}) \tag{4.5.40}$$

we have

$$\mathrm{E}(u_{n,m_n}, \mathcal{B}^{\mathrm{Top}}_n(\eta))\Lambda_{T_{\kappa_n} f_n}\left(\frac{\eta}{\mathrm{E}(u_{n,m_n}, \mathcal{B}^{\mathrm{Top}}_n(\eta))}\right) \le \int_{\bigcup \mathcal{B}^{\mathrm{Top}}_n(\eta)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x, \tag{4.5.41}$$

$$\text{(4.5.42)}\quad \varlimsup_{n\to\infty} \mathrm{E}(u_{n,m_n}, \mathcal{B}_n^{\mathrm{Top}}(\eta)) \leq \varlimsup_{n\to\infty} \mathcal{V}(T_{\kappa_n} f_n)^{-1} \int_{\bigcup \mathcal{B}_n(\eta)} f_n(|\mathrm{D}u_{n,m_n}|),$$

and combining with (4.5.39),

$$\text{(4.5.43)}\quad \varlimsup_{n\to\infty} \int_{\Omega'\setminus\bigcup \mathcal{B}_n^{\mathrm{Top}}(\eta)} T_{\kappa_n} f_n(|\mathrm{D}u_{n,m_n}|) \leq \mathrm{L} + \int_{\Omega'\setminus\Omega} \frac{1}{2}|\mathrm{D}U|^2 \,\mathrm{d}x + \mathrm{E}_{\mathrm{sg}}^{1,2}(g) \log \frac{\operatorname{sys}(\mathcal{N})\mathrm{E}_{\mathrm{sg}}^{1,2}(g)}{2\eta}.$$

We also have $\mathrm{E}_{\mathrm{sg}}^{1,2}(\operatorname{tr}_{\partial B} u_{n,m_n}) > 0$ for each $\mathbb{B} \in \mathcal{B}_n^{\mathrm{Top}}(\eta)$.

4.5.4.2 Step 2: Existence of a limiting map

To simplify notation we will denote

$$\text{(4.5.44)}\qquad \tilde{u}_n \doteq u_{n,m_n} \in \mathrm{R}_\delta^{1,2}(\Omega'; \mathcal{N}), \quad G_n(t) \doteq T_{\kappa_n} f_n(t).$$

For each $\mathbb{B} \in \mathcal{B}_n^{\mathrm{Top}}(\eta_n)$, we have $\mathrm{E}_{\mathrm{sg}}^{1,2}(\operatorname{tr}_{\partial\mathbb{B}} u_n) > \frac{\operatorname{sys}(\mathcal{N})^2}{4\pi}$ since it is non-zero (using the systole defined in (4.3.1)). Combining this with (4.5.42) we can bound the number of balls as

$$\text{(4.5.45)}\quad \varlimsup_{n\to\infty} \#\mathcal{B}_n^{\mathrm{Top}}(\eta) \leq \frac{4\pi}{\operatorname{sys}(\mathcal{N})^2} \varlimsup_{n\to\infty} \mathcal{V}(G_n)^{-1} \int_{\bigcup \mathcal{B}_n(\eta)} G_n(|\mathrm{D}\tilde{u}_n|) = \frac{4\pi \mathrm{E}_{\mathrm{sg}}^{1,2}(g)}{\operatorname{sys}(\mathcal{N})^2}.$$

Hence by passing to a subsequence in n (which depends on $\eta > 0$), we can assume $\mathcal{B}_n^{\mathrm{Top}}(\eta)$ converges to a limit collection $\mathcal{B}^{\mathrm{Top}}(\eta)$ in the following sense: we have $\#\mathcal{B}_n^{\mathrm{Top}}(\eta) \equiv \kappa_1$ is constant and writing $\mathcal{B}_n^{\mathrm{Top}}(\eta) = \{\mathbb{B}(a_i^n, r_i^n)\}_{i=1}^{\kappa_1}$, we have $a_i^n \to \tilde{a}_i$ and $r_i^n \to \tilde{r}_i$ for all $1 \leq i \leq \kappa_1$. The limit collection $\mathcal{B}_*^{\mathrm{Top}}(\eta) = \{\mathbb{B}(a_i, r_i)\}_{i=1}^{\kappa}$ is then defined as these balls $\mathbb{B}(\tilde{a}_i, \tilde{r}_i)$, discarding any balls with $\tilde{r}_i = 0$, so in particular $\kappa \leq \kappa_1$. Note that if $\tilde{a}_i = \tilde{a}_j$ for some $i \neq j$, the disjointness of $\mathcal{B}_n^{\mathrm{Top}}(\eta)$ implies that either $\tilde{r}_i^n \to 0$ or $\tilde{r}_j^n \to 0$; thus we infer the centre points $\{a_i\}_{i=1}^{\kappa}$ are distinct, and that the limiting collection has pairwise disjoint interiors.

Now let $(\eta_m)_m$ be a positive sequence such that $\eta_m \searrow 0$, then applying the above to each η_m combined with a Cantor diagonal argument, we can find a common subsequence in n such that for each m, $\mathcal{B}_n^{\mathrm{Top}}(\eta_m)$ converges to $\mathcal{B}_*^{\mathrm{Top}}(\eta_m)$ in the above sense. Passing to a subsequence in m, we have the centres of $\mathcal{B}_*^{\mathrm{Top}}(\eta_m)$ converges to a set of finitely many points $\{a_i\}_{i=1}^{\kappa} \subset \overline{\Omega}$, such that $\kappa \leq \frac{4\pi \mathrm{E}_{\mathrm{sg}}^{1,2}(g)}{\operatorname{sys}(\mathcal{N})^2}$ by (4.5.45).

We will let $\bar{\rho}_{\Omega'} \doteq \rho_{\Omega'}(a_1, \ldots, a_\kappa)$ be the non-intersection radius as defined in (4.3.6). Then let $0 < \varepsilon < \bar{\rho}_{\Omega'}$, and choose $m_0 \geq 0$ such that $\eta_m < \varepsilon/2$ and $\bigcup \mathcal{B}_*^{\mathrm{Top}}(\eta_m) \subset \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \varepsilon/2)$ whenever $m \geq m_0$. Fixing any such n, the above convergence we have $\bigcup \mathcal{B}_n^{\mathrm{Top}}(\eta_m) \subset \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \varepsilon)$ for all n sufficiently large. Thus from (4.5.43), for all $m \geq m_0$ we have,

$$\overline{\lim_{n\to\infty}} \int_{\Omega' \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\epsilon)} G_n(|\mathrm{D}\tilde{u}_n|) \leq \tilde{\mathrm{L}}_2 + \mathrm{E}_{\mathrm{sg}}^{1,2}(g) \log \frac{1}{\eta_m}, \tag{4.5.46}$$

where we set

$$\tilde{\mathrm{L}}_2 \doteq \mathrm{L} + \mathrm{E}_{\mathrm{sg}}^{1,2}(g) \log \frac{\mathrm{sys}(\mathcal{N}) \mathrm{E}_{\mathrm{sg}}^{1,2}(g)}{2} + \int_{\Omega' \setminus \Omega} \frac{1}{2} |\mathrm{D}U|^2 \,\mathrm{d}x \tag{4.5.47}$$

Therefore by Proposition 4.5.2(a) and a Cantor diagonal argument in m, we have $(|\mathrm{D}\tilde{u}_n|)_n$ is bounded in each $\mathrm{L}^1(\Omega' \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \varepsilon))$. Moreover, since $\tilde{u}_n$ takes values in the compact manifold $\mathcal{N}$, the sequence is bounded in $\mathrm{W}^{1,1}(\Omega' \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \varepsilon))$. Hence by taking a further diagonal sequence, we obtain a limit map u_* such that passing to a subsequence, $\tilde{u}_n$ converges weakly* in BV to u_* in $\Omega' \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \varepsilon)$. Moreover weak* convergence in BV implies strong L^1 convergence of $\tilde{u}_n$, so Proposition 4.5.2(b) we also have the estimate

$$\begin{aligned} \int_{\Omega' \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\varepsilon)} \frac{1}{2} |\mathrm{D}u_*|^2 \,\mathrm{d}x & \\ \leq \overline{\lim_{n\to\infty}} \int_{\Omega' \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\varepsilon)} G_n(|\mathrm{D}\tilde{u}_n|) \,\mathrm{d}x & \\ \leq \tilde{\mathrm{L}}_2 + \mathrm{E}_{\mathrm{sg}}^{1,2}(g) \log \frac{1}{\eta_m} & \end{aligned} \tag{4.5.48}$$

using (4.5.46). Since this holds for all $\varepsilon > 0$ sufficiently small, this defines a map in $\mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega' \setminus \{a_i\}_{i=1}^{\kappa}; \mathcal{N})$, noting the L^1 convergence ensures the limit map u_* takes values in $\mathcal{N}$ for almost every $x \in \Omega$. Moreover since $\mathrm{D}\bar{u}_n - \mathrm{D}\tilde{u}_n \to 0$ strongly in L^1 and $\bar{u}_n|_{\Omega' \setminus \Omega} \equiv U$ for all n, it follows that $u_*|_{\Omega' \setminus \Omega} \equiv U$.

Now combining (4.5.48) with (4.5.35) and (4.5.38), we have

$$\begin{aligned} & \overline{\lim_{n\to\infty}} \int_{\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\varepsilon)} f_n(|\mathrm{D}u_n|) \,\mathrm{d}x \\ & = \overline{\lim_{n\to\infty}} \int_{\Omega' \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\varepsilon)} G_n(|\mathrm{D}\tilde{u}_n|) \,\mathrm{d}x \\ & \quad - \int_{\Omega' \setminus \left(\Omega \cup \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\varepsilon)\right)} \frac{1}{2} |\mathrm{D}U|^2 \,\mathrm{d}x \\ & \leq \tilde{\mathrm{L}}_2 + \log \frac{1}{\eta_m} + \int_{(\Omega' \setminus \Omega) \cap \left(\bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\varepsilon)\right)} \frac{1}{2} |\mathrm{D}U|^2 \,\mathrm{d}x. \end{aligned} \tag{4.5.49}$$

4.5.4.3 Step 3: Minimal topological resolution

To show the singularities of $\tilde{u}_n$ are transferred to the limiting map u_*, we will rely on Lemma 4.5.9.

Given $0 < \varepsilon < \bar{\rho}_{\Omega'}$, by converting to polar coordinates and applying Fatou's lemma, we have

$$\int_{\varepsilon}^{\bar{\rho}_{\Omega'}} \varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a_i,r)} G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}\mathcal{H}^1\,\mathrm{d}r \leq \varliminf_{n\to\infty} \int_{\mathbb{B}(a_i,\bar{\rho}_{\Omega'})\setminus\mathbb{B}(a_i,\varepsilon)} G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}x \tag{4.5.50}$$

is finite for each $i = 1, \ldots, \kappa$ by (4.5.49). Thus for almost every $r \in (0, \bar{\rho}_{\Omega'})$ we have

$$\varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a_i,r)} G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}\mathcal{H}^1 \quad \text{is finite for each } i = 1, \ldots, \kappa. \tag{4.5.51}$$

By Proposition 4.5.6, there exists a subsequence $\tilde{u}_{n_k}$ (depending on r) attaining the limit inferior and along which $\operatorname{tr}_{\partial\mathbb{B}(a_i,r)} \tilde{u}_{n_k} \to \operatorname{tr}_{\partial\mathbb{B}(a_i,r)} u_*$ in $\mathrm{L}^1(\partial\mathbb{B}(a_i,r);\mathcal{N})$. By a variant of Proposition 4.5.2, we deduce that for almost every $r \in (0, \bar{\rho})$ and each $i = 1, \ldots, \kappa$,

$$\int_{\partial\mathbb{B}(a_i,r)} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}\mathcal{H}^1 \leq \varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a_i,r)} G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}\mathcal{H}^1. \tag{4.5.52}$$

Now applying Lemma 4.5.9 to $\operatorname{tr}_{\partial\mathbb{B}(a,r)} \tilde{u}_{n_k}$ and G_{n_k} we have

$$\lambda(\operatorname{tr}_{\partial\mathbb{B}(a_i,r)} \tilde{u}_{n_k}) = \lambda(\operatorname{tr}_{\partial\mathbb{B}(a_i,r)} u_*) \tag{4.5.53}$$

for all $1 \leq i \leq \kappa$ and k sufficiently large.

Given $\varepsilon > 0$ such that (4.5.51) holds with $r = \varepsilon$, along the subsequence such that (4.5.53) holds, for k sufficiently large by Lemma 4.4.4 we have

$$\sum_{i=1}^{\kappa} \frac{\lambda(\operatorname{tr}_{\partial\mathbb{B}(a_i,\varepsilon)} u_*)^2}{4\pi} \Lambda_{G_{n_k}}\left(\frac{4\pi\varepsilon}{\lambda(\operatorname{tr}_{\partial\mathbb{B}(a_i,\varepsilon)} u_*)^2}\right) \leq \int_{\bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\varepsilon)} G_{n_k}(|\mathrm{D}\tilde{u}_{n_k}|)\,\mathrm{d}x. \tag{4.5.54}$$

Dividing both sides by $\mathcal{V}(G_{n_k})$ and sending $k \to \infty$, using Lemma 4.4.11

and (4.5.39) we have

$$\begin{aligned}
&\sum_{i=1}^{\kappa} \frac{\lambda(\operatorname{tr}_{\partial\mathbb{B}(a_i,\varepsilon)} u_*)^2}{4\pi} \\
&= \sum_{i=1}^{\kappa} \frac{\lambda(\operatorname{tr}_{\partial\mathbb{B}(a_i,\varepsilon)} u_*)^2}{4\pi} \lim_{k\to\infty} \mathcal{V}(G_{n_k})^{-1}\Lambda_{G_{n_k}}\left(\frac{4\pi\varepsilon}{\lambda(\operatorname{tr}_{\partial\mathbb{B}(a_i,\varepsilon)} u_*)^2}\right) \\
&\leq \varlimsup_{n\to\infty} \frac{1}{\mathcal{V}(G_n)} \int_\Omega G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}x = \mathrm{E}^{1,2}_{\mathrm{sg}}(g).
\end{aligned}$$

By subadditivity of the singular energy (namely (4.3.4)) and noting that $u_* \in \mathrm{W}^{1,2}_{\mathrm{loc}}(\Omega' \setminus \{a_i\}_{i=1}^{\kappa}; \mathcal{N})$, the above holds with equality for all $\varepsilon \in (0, \bar{\rho}_{\Omega'})$. Thus

$$\begin{aligned}
\text{(4.5.55)}\quad &\sum_{i=1}^{\kappa} \frac{\lambda([u_*, a_i])^2}{4\pi} \\
&= \sum_{i=1}^{\kappa} \frac{\lambda(\operatorname{tr}_{\partial\mathbb{B}(a_i,\varepsilon)} u_*)^2}{4\pi} = \mathrm{E}^{1,2}_{\mathrm{sg}}(g) = \mathrm{E}^{1,2}_{\mathrm{sg}}(\operatorname{tr}_{\partial\Omega'} U),
\end{aligned}$$

and so $(\operatorname{tr}_{\mathbb{S}^1}(a_i+\varepsilon\,\cdot))_{i=1}^{\kappa}$ forms a minimal topological resolution for $\operatorname{tr}_{\partial\Omega'} U$ in Ω' for all $\varepsilon \in (0, \bar{\rho}_{\Omega'})$ sufficiently small. Since $\operatorname{tr}_{\partial\Omega} u_* = g$, *if* each $a_i \in \Omega$, then for $\varepsilon > 0$ we will also have that $(\operatorname{tr}_{\mathbb{S}^1}(a_i + \varepsilon\,\cdot))_{i=1}^{\kappa}$ forms a minimal topological resolution for g in Ω.

4.5.4.4 Step 4: Limit map is renormalisable

Next, we will show that the limit map is renormalisable in Ω'. By (4.5.52), for almost all $r \in (0, \bar{\rho}_{\Omega'})$ we have

$$\begin{aligned}
\text{(4.5.56)}\quad &\int_{\partial\mathbb{B}(a_i,r)} \frac{1}{2}|\mathrm{D}u_*|^2\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r} \\
&\leq \varliminf_{n\to\infty} \int_{\partial\mathbb{B}(a_i,r)} G_n(|\mathrm{D}\tilde{u}_n|) - G_n\left(\frac{\lambda([u_*, a_i])}{2\pi r}\right)\mathrm{d}\mathcal{H}^1,
\end{aligned}$$

where we used the dominated convergence theorem and Lemma 4.2.12 to pass to the limit in the second term. Now for $0 < \rho < \bar{\rho}_{\Omega'}$, integrating (4.5.56) over $r \in (0, \rho)$, summing over $i = 1, \ldots, \kappa$ and applying Fatou's lemma, we have

$$
\begin{aligned}
&\sum_{i=1}^{\kappa}\int_0^{\rho}\left[\int_{\partial\mathbb{B}(a_i,r)}\frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}\mathcal{H}^1-\frac{\lambda([u_*,a_i])^2}{4\pi r}\right]\mathrm{d}r \qquad (4.5.57)\\
&\le\sum_{i=1}^{\kappa}\int_0^{\rho}\varliminf_{n\to\infty}\left[\int_{\partial\mathbb{B}(a_i,r)}G_n(|\mathrm{D}\tilde{u}_n|)-G_n\left(\frac{\lambda([u_*,a_i])}{2\pi r}\right)\mathrm{d}\mathcal{H}^1\right]\mathrm{d}r\\
&\le\varliminf_{n\to\infty}\left[\int_{\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)}G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}x\right.\\
&\qquad\left.-\sum_{i=1}^{\kappa}\frac{\lambda([u_*,a_i])^2}{4\pi}\int_0^{\frac{2\pi\rho}{\lambda([u_*,a_i])}}G_n(1/r)r\,\mathrm{d}r\right]\\
&=\varliminf_{n\to\infty}\left[\int_{\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)}G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}x-\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\mathcal{V}(G_n)\right]\\
&\qquad-\sum_{i=1}^{\kappa}\frac{\lambda([u_*,a_i])^2}{4\pi}\log\frac{2\pi\rho}{\lambda([u_*,a_i])}.
\end{aligned}
$$

where we used Lemma 4.4.2 and (4.5.55) in the last line, which is moreover finite by (4.5.39). Hence u_* verifies property (ii) of Proposition 4.3.6, from which we infer that $u_*\in\mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega';\mathcal{N})$.

To show the singular points a_i actually lie in Ω, we will make use of the following lemma from [32, Lem. 6.2].

Lemma 4.5.11. *Let $\Omega\subset\mathbb{R}^2$ be an open set, $a\in\mathbb{R}^2$ and $0<\sigma<\tau$. If $u\in\mathrm{W}^{1,2}(\mathbb{B}(a,\tau)\setminus\mathbb{B}(a,\sigma);\mathcal{N})$ such that $\lambda(\operatorname{tr}_{\partial\mathbb{B}(a,\tau)}u)>0$, then*

$$\Gamma^{\sigma}_{\tau}(a,\mathrm{D}u)\frac{\lambda(\operatorname{tr}_{\partial\mathbb{B}(a,\tau)}u)^2}{4\pi\nu^{\sigma}_{\tau}(a)}\log\frac{\tau}{\sigma}\le\int_{\Omega\cap\mathbb{B}(a,\tau)\setminus\mathbb{B}(a,\sigma)}\frac{|\mathrm{D}u|^2}{2},$$

where we have set

$$\Gamma^{\sigma}_{\tau}(a,\mathrm{D}u)\doteq\left[1-\frac{\sqrt{2\pi}}{\lambda(\operatorname{tr}_{\partial\mathbb{B}(a,\tau)}u)}\Big(\frac{1}{\log\frac{\tau}{\sigma}}\int_{\mathbb{B}(a,\tau)\setminus(\Omega\cup\mathbb{B}(a,\sigma))}|\mathrm{D}u|^2\Big)^{\frac12}\right]^2$$

and

$$\nu^{\sigma}_{\tau}(a)\doteq\frac{1}{2\pi\log\frac{\tau}{\sigma}}\int_{\Omega\cap\mathbb{B}(a,\tau)\setminus\mathbb{B}(a,\sigma)}\frac{\mathrm{d}x}{|x-a|^2}\le 1. \qquad (4.5.58)$$

This will be used in conjunction with the following observation from [32, §6].

Lemma 4.5.12. *Let $\Omega\subset\mathbb{R}^2$ be a bounded Lipschitz domain. Let $a\in\mathbb{R}^2$. We have $\lim_{\sigma\to 0}\nu^{\sigma}_{\tau}(a)=1$ if and only if $a\in\Omega$.*

Proof of Lemma 4.5.12. Observe that for any $\tau_1, \tau_2 > 0$ we have

$$\lim_{\sigma\to 0}(\nu^\sigma_{\tau_1}(a) - \nu^\sigma_{\tau_2}(a)) = 0,$$

so if $a \in \Omega$ or $a \notin \overline{\Omega}$, by taking τ sufficiently small so $\mathbb{B}(a,\tau) \subset \Omega$ or $\mathbb{B}(a,\tau) \subset \mathbb{R}^2 \setminus \overline{\Omega}$, we have $\lim_{\sigma\to 0}\nu^\sigma_\tau(a) = 1$ or 0 respectively. If $a \in \partial\Omega$, by the exterior cone condition (see [19, Thm. 1.2.2.2]) there exists $\sigma_0 > 0$, $v \in \mathbb{S}^1$ and $\theta \in (0, \pi/2)$ such that

$$\begin{aligned}(4.5.59)\quad & C_{\sigma_0}(a,\theta,v) \\ &\doteq \{a + w : w \in \mathbb{B}(0,\sigma_0), |\measuredangle(v,w)| < \theta/2\} \subset \Omega \setminus \mathbb{B}(a,\sigma_0),\end{aligned}$$

where $\measuredangle(v,w) = \arccos{(v|w)}/(|v||w|)$ denotes the angle between the vectors v and w in $\mathbb{R}^2$. Then we have

$$\begin{aligned}(4.5.60)\quad & \overline{\lim_{\sigma\to 0}}\,\nu^\sigma_\tau(a) \\ &\le \frac{1}{2\pi\log(\sigma_0/\sigma)}\int_{\mathbb{B}(a,\sigma_0)\setminus(\mathbb{B}(a,\sigma)\cup C_{\sigma_0}(a,v,\theta))} \frac{\mathrm{d}x}{|x-a|^2} = 1 - \frac{\theta}{2\pi} < 1,\end{aligned}$$

thereby establishing the result. □

By (4.5.57), for $0 < \sigma < \rho < \bar{\rho}_{\Omega'}$ and each $1 \le i \le \kappa$

$$\begin{aligned}(4.5.61)\quad & \frac{\lambda([u_*, a_i])^2}{4\pi}\log\frac{\rho}{\sigma} \\ &\le \int_{\mathbb{B}(a_i,\rho)\setminus\mathbb{B}(a_i,\sigma)} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}x \le \frac{\lambda([u_*, a_i])^2}{4\pi}\log\frac{\rho}{\sigma} + C_\rho,\end{aligned}$$

where C_ρ is an upper bound for (4.5.57), and the lower bound follows by converting to polar coordinates and recalling the definition of $\lambda([u_*, a_i])$. From this we deduce that

$$(4.5.62)\qquad \lim_{\sigma\to 0}\frac{1}{\log(\rho/\sigma)}\int_{\mathbb{B}(a_i,\rho)\setminus\mathbb{B}(a_i,\sigma)} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}x = \frac{\lambda([u_*, a_i])^2}{4\pi}.$$

Moreover since $u_*|_{\Omega'\setminus\Omega} \equiv U \in \mathrm{W}^{1,2}(\Omega' \setminus \Omega; \mathcal{N})$, we have

$$(4.5.63)\qquad \lim_{\sigma\to 0}\frac{1}{\log(\rho/\sigma)}\int_{\mathbb{B}(a_i,\rho)\setminus(\Omega\cup\mathbb{B}(a_i,\sigma))} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}x = 0,$$

so in particular, $\lim_{\sigma\to 0}\Gamma^\sigma_\rho(a_i, \mathrm{D}u_*) = 1$. Combining (4.5.62) and (4.5.63) we obtain

$$(4.5.64)\qquad \lim_{\sigma\to 0}\frac{1}{\log(\rho/\sigma)}\int_{\Omega\cap\mathbb{B}(a_i,\rho)\setminus\mathbb{B}(a_i,\sigma)} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}x = \frac{\lambda([u_*, a_i])^2}{4\pi},$$

so by Lemma 4.5.11 we have

$$
\begin{aligned}
(4.5.65)\quad & \varliminf_{\sigma\to 0} \nu_\rho^\sigma(a_i) \\
&\geq \lim_{\sigma\to 0} (\Gamma_\rho^\sigma(a_i, \mathrm{D}u_*))^{-1} \frac{4\pi}{\lambda([u_*, a_i])^2} \int_{\Omega\cap\mathbb{B}(a_i,\rho)\setminus\mathbb{B}(a_i,\sigma)} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}x = 1
\end{aligned}
$$

for each $i = 1, \ldots, \kappa$. It follows that $\lim_{\sigma\to 0} \nu_\rho^\sigma(a_i) = 1$, and hence each $a_i \in \Omega$ by Lemma 4.5.12. Therefore the restriction of u_* to Ω is also a renormalisable map, establishing (i).

We can also compute the renormalised energy of u_* in Ω using (4.3.9), (4.5.55) and (4.5.57) as, for any $\rho < \rho_\Omega(a_1, \ldots, a_\kappa)$,

$$
\begin{aligned}
&\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \Omega) \\
&= \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}x - \sum_{i=1}^{\kappa} \frac{\lambda([u_*, a_i])^2}{4\pi} \log\frac{1}{\rho} \\
&\quad + \sum_{i=1}^{\kappa} \int_0^\rho \left[\int_{\partial\mathbb{B}(a_i,r)} \frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r}\right] \mathrm{d}r \\
&\leq \varliminf_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} G_n(\mathrm{D}\tilde{u}_n)\,\mathrm{d}x - \sum_{i=1}^{\kappa} \frac{\lambda([u_*, a_i])^2}{4\pi} \log\frac{2\pi\rho}{\lambda([u_*, a_i])} \\
&\quad + \varliminf_{n\to\infty} \left[\int_{\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}x - \mathrm{E}^{1,2}_{\mathrm{sg}}(g)\mathcal{V}(G_n)\right] - \mathrm{E}^{1,2}_{\mathrm{sg}}(g)\log\frac{1}{\rho} \\
&\leq \varliminf_{n\to\infty} \left[\int_{\Omega} G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}x - \mathrm{E}^{1,2}_{\mathrm{sg}}(g)\mathcal{V}(G_n)\right] \\
&\quad - \sum_{i=1}^{\kappa} \frac{\lambda([u_*, a_i])^2}{4\pi} \log\frac{2\pi}{\lambda([u_*, a_i])} \\
&\leq \varliminf_{n\to\infty} \left[\int_{\Omega} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathrm{E}^{1,2}_{\mathrm{sg}}(g)\mathcal{V}(f_n)\right] \\
&\quad - \sum_{i=1}^{\kappa} \frac{\lambda([u_*, a_i])^2}{4\pi} \log\frac{2\pi}{\lambda([u_*, a_i])}.
\end{aligned}
$$

where the last line follows by (4.5.39), thereby establishing (ii).

4.5.4.5 Step 5: Localized asymptotic estimates

For $0 < \rho < \bar{\rho}_\Omega \doteq \rho_\Omega(a_1, \ldots, a_\kappa)$, by integrating (4.5.56) over $(0, \rho)$ and applying Fatou's lemma we have

$$\begin{aligned}
&\varliminf_{n\to\infty} \left[\int_{\mathbb{B}(a_i,\rho)} G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}x - 2\pi \int_0^\rho G_n\left(\frac{\lambda([u_*,a_i])}{2\pi r}\right) r\,\mathrm{d}r\right] \\
&\geq \int_0^\rho \left[\int_{\partial\mathbb{B}(a_i,r)} \frac{1}{2}|\mathrm{D}u_*|^2\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*,a_i])^2}{4\pi r}\right] \mathrm{d}r \\
&= \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \mathbb{B}(a_i,\rho)) + \frac{\lambda([u_*,a_i])^2}{4\pi} \log\frac{1}{\rho},
\end{aligned} \tag{4.5.66}$$

recalling the localised renormalised energy was defined in (4.3.10). Then Lemma 4.4.2 gives

$$\begin{aligned}
\lim_{n\to\infty} \left[2\pi \int_0^\rho G_n\left(\frac{\lambda([u_*,a_i])^2}{2\pi r}\right) r\,\mathrm{d}r - \frac{\lambda([u_*,a_i])^2}{4\pi}\mathcal{V}(G_n)\right] \\
= \frac{\lambda([u_*,a_i])^2}{4\pi} \log\frac{2\pi\rho}{\lambda([u_*,a_i])},
\end{aligned} \tag{4.5.67}$$

which we combine with (4.5.34), (4.5.35) and (4.5.38) to obtain

$$\begin{aligned}
&\varliminf_{n\to\infty} \left[\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \frac{\lambda([u_*,a_i])^2}{4\pi}\mathcal{V}(f_n)\right] \\
&\qquad = \varliminf_{n\to\infty} \left[\int_{\mathbb{B}(a_i,\rho)} G_n(|\mathrm{D}\tilde{u}_n|)\,\mathrm{d}x - \frac{\lambda([u_*,a_i])^2}{4\pi}\mathcal{V}(G_n)\right] \\
&\qquad \geq \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \mathbb{B}(a_i,\rho)) + \frac{\lambda([u_*,a_i])^2}{4\pi} \log\frac{2\pi}{\lambda([u_*,a_i])}.
\end{aligned} \tag{4.5.68}$$

This establishes the lower bound in (iv). Using this we can estimate

$$\begin{aligned}
&\varlimsup_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^\kappa \mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x \\
&\leq \mathrm{L} - \sum_{i=1}^\kappa \varliminf_{n\to\infty} \left[\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \frac{\lambda([u_*,a_i])^2}{4\pi}\mathcal{V}(f_n)\right] \\
&\leq \mathrm{L} - \sum_{i=1}^\kappa \frac{\lambda([u_*,a_i])^2}{4\pi} \log\frac{2\pi}{\lambda([u_*,a_i])} - \sum_{i=1}^\kappa \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \mathbb{B}(a_i,\rho)) \\
&= \int_{\Omega\setminus\bigcup_{i=1}^\kappa \mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u_*|^2\,\mathrm{d}x \\
&\quad + \left(\mathrm{L} - \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) - \mathrm{H}([u_*,a_i])_{i=1}^\kappa\right),
\end{aligned}$$

where we used (4.3.9) to write

$$\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) = \sum_{i=1}^{\kappa} \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \mathbb{B}(a_i,\rho)) + \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u_*|^2 \,\mathrm{d}x. \tag{4.5.69}$$

By Proposition 4.5.2 we also have the lower bound

$$\int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u_*|^2 \,\mathrm{d}x \le \varliminf_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_*|)\,\mathrm{d}x, \tag{4.5.70}$$

which establishes (iii). To obtain asymptotic upper bounds on $\mathbb{B}(a_i,\rho)$, we use (4.5.69), (4.5.70) and (4.5.68) applied for each $j \neq i$ to estimate

$$\begin{aligned}
&\varlimsup_{n\to\infty} \left[\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \frac{\lambda([u_*,a_i])^2}{4\pi}\mathcal{V}(f_n)\right] \\
&\quad\le \mathrm{L} - \sum_{j\neq i} \varliminf_{n\to\infty} \left[\int_{\mathbb{B}(a_j,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \frac{\lambda([u_*,a_j])^2}{4\pi}\mathcal{V}(f_n)\right] \\
&\qquad - \varliminf_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_*|)\,\mathrm{d}x \\
&\quad\le \mathrm{L} - \sum_{j\neq i} \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*,\mathbb{B}(a_j,\rho)) - \sum_{j\neq i} \frac{\lambda([u_*,a_j])^2}{4\pi} \log\frac{2\pi}{\lambda([u_*,a_j])} \\
&\qquad - \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u_*|^2\,\mathrm{d}x \\
&\quad= \mathrm{L} - \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) - \mathrm{H}([u_*,a_i]) \\
&\qquad + \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*;\mathbb{B}(a_i,\rho)) + \frac{\lambda([u_*,a_i])^2}{4\pi}\log\frac{2\pi}{\lambda([u_*,a_i])},
\end{aligned}$$

thereby establishing (iv).

4.5.4.6 Step 6: Strict measure convergence

Multiplying the inequalities in (iv) by $\mathcal{V}(f_n)^{-1}$, for each $0<\rho<\bar{\rho}_\Omega$ and $i=1,\dots,\kappa$ we have

$$\lim_{n\to\infty} \frac{1}{\mathcal{V}(f_n)} \int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x = \frac{\lambda([u_*,a_i])^2}{4\pi}. \tag{4.5.71}$$

Also by (4.5.1) and (i) we know that

$$\lim_{n\to\infty} \frac{1}{\mathcal{V}(f_n)} \int_{\Omega} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x = \mathrm{E}^{1,2}_{\mathrm{sg}}(g) = \sum_{i=1}^{\kappa} \frac{\lambda([u_*,a_i])^2}{4\pi}. \tag{4.5.72}$$

Hence if we define the measures

$$\mu_n \doteq \frac{f_n(|\mathrm{D}u_n|)}{\mathcal{V}(f_n)}\mathcal{L}^2 \llcorner \Omega, \tag{4.5.73}$$

by (4.5.71), for all $0<\rho<\bar{\rho}_\Omega$ and $i=1,\dots,\kappa$ we have

$$\lim_{n\to\infty}\mu_n(\mathbb{B}(a_i,\rho)) = \frac{\lambda([u_*,a_i])^2}{4\pi}, \tag{4.5.74}$$

whereas by (4.5.72) we have

$$\lim_{n\to\infty}\mu_n(\Omega) = \sum_{i=1}^{\kappa}\frac{\lambda([u_*,a_i])^2}{4\pi}. \tag{4.5.75}$$

Hence we infer that μ_n converges narrowly to a measure μ that is supported on $\{a_i\}_{i=1}^{\kappa}$, taking the form

$$\mu = \sum_{i=1}^{\kappa}\frac{\lambda([u_*,a_i])^2}{4\pi}\delta_{a_i}, \tag{4.5.76}$$

thereby establishing (v).

4.5.4.7 Step 7 : Bounds in L^1

For $0<\rho<\rho_\Omega(a_1,\dots,a_\kappa)$ and each $i=1,\dots,\kappa$, set

$$\mathrm{L}_{i,\rho} \doteq \varlimsup_{n\to\infty}\left[\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \frac{\lambda([u_*,a_i])^2}{4\pi}\mathcal{V}(f_n)\right], \tag{4.5.77}$$

which we know is finite by (iv). Then by convexity of f_n, using (4.2.22) we have
(4.5.78)

$$f_n(|\mathrm{D}u_n|) - f_n\Big(\frac{\lambda([u_*,a_i])}{2\pi r}\Big) \geq f'_{n,+}\Big(\frac{\lambda([u_*,a_i])}{2\pi r}\Big)\Big(|\mathrm{D}u_n| - \frac{\lambda([u_*,a_i])}{2\pi r}\Big)$$

for all $x\in\mathbb{B}(a_i,\rho)$ with $r=|x|$. Then integrating (4.5.78) over $x\in\mathbb{B}(a_i,\rho)$,

$$\begin{aligned}
&\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|) - 2\pi\int_0^\rho f_n\Big(\frac{\lambda([u_*,a_i])}{2\pi r}\Big) r\,\mathrm{d}r \\
&\quad\geq \int_0^\rho f'_{n,+}\Big(\frac{\lambda([u_*,a_i])}{2\pi r}\Big)\Big(\int_{\partial\mathbb{B}(a_i,r)}|\mathrm{D}u_n|\,\mathrm{d}\mathcal{H}^1 - \lambda([u_*,a_i])\Big)\,\mathrm{d}r \\
&\quad\geq f'_{n,+}\Big(\frac{\lambda([u_*,a_i])}{2\pi\rho}\Big)\int_0^\rho \Big(\int_{\partial\mathbb{B}(a_i,r)}|\mathrm{D}u_n|\,\mathrm{d}\mathcal{H}^1 - \lambda([u_*,a_i])\Big)\,\mathrm{d}r
\end{aligned}$$

since each $f'_{n,+}(t)$ is non-decreasing in t. Noting that $\lim_{n\to\infty} f'_{n,+}(t) = t$ by Lemma 4.2.10(b), taking the limit in n,

$$\begin{aligned}
&\frac{\lambda([u_*, a_i])}{2\pi\rho} \varlimsup_{n\to\infty} \int_{\mathbb{B}(a_i,\rho)} |\mathrm{D}u_n| \,\mathrm{d}x \\
&= \mathrm{L}_{i,\rho} + \frac{\lambda([u, a_i])^2}{2\pi} \\
&\quad + \varlimsup_{n\to\infty} \left[\frac{\lambda([u_*, a_i])^2}{4\pi} \mathcal{V}(f_n) - 2\pi \int_0^\rho f_n\Big(\frac{\lambda([u_*, a_i])}{2\pi r}\Big) r \,\mathrm{d}r \right] \\
&= \mathrm{L}_{i,\rho} + \frac{\lambda([u, a_i])^2}{2\pi} + \frac{\lambda([u_*, a_i])^2}{4\pi} \log \frac{\lambda([u_*, a_i])^2}{4\pi\rho}
\end{aligned}$$

using Lemma 4.4.2 in the last line. Then using the estimate (iv),

$$\begin{aligned}
\text{(4.5.79)}\quad & \mathrm{L}_{i,\rho} + \frac{\lambda([u_*, a_i])^2}{4\pi} \log \frac{\lambda([u_*, a_i])^2}{4\pi\rho} \\
&\le \mathrm{D} + \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \mathbb{B}(a_i, \rho)) + \frac{\lambda([u_*, a_i])^2}{4\pi} \log \frac{1}{\rho} \\
&= \mathrm{D} + \int_0^\rho \int_{\partial\mathbb{B}(a_i,r)} \frac{1}{2} |\mathrm{D}u_*|^2 \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r} \,\mathrm{d}r,
\end{aligned}$$

where we have set $\mathrm{D} \doteq \mathrm{L} - \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) - \mathrm{H}([u_*, a_i])$. Therefore we have

$$\begin{aligned}
\text{(4.5.80)}\quad & \varliminf_{n\to\infty} \int_{\mathbb{B}(a_i,\rho)} |\mathrm{D}u_n| \,\mathrm{d}x \le \frac{2\pi\rho}{\lambda([u_*, a_i])} \times \\
&\left[\mathrm{D} + \lambda([u_*, a_i])^2 + \int_0^\rho \int_{\partial\mathbb{B}(a_i,r)} \frac{1}{2} |\mathrm{D}u_*|^2 \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r} \,\mathrm{d}r \right],
\end{aligned}$$

establishing (vi).

To show equi-integrability, for each $i = 1, \ldots, \kappa$ we will set

$$\begin{aligned}
\text{(4.5.81)}\quad \mathrm{M}_i \doteq \frac{2\pi}{\lambda([u_*, a_i])} \Bigg[& \mathrm{D} + \lambda([u_*, a_i])^2 \\
& + \int_0^{\bar{\rho}_\Omega} \int_{\partial\mathbb{B}(a_i,r)} \frac{1}{2} |\mathrm{D}u_*|^2 \,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*, a_i])^2}{4\pi r} \,\mathrm{d}r \Bigg].
\end{aligned}$$

We will assume $\kappa \ge 1$ in what follows, as otherwise the result follows from Proposition 4.5.2(a). Then recall from (iii), using (4.3.9) for the renormalised energy, that for $0 < \rho < \bar{\rho}_\Omega$ we have

$$\varlimsup_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x \leq \mathrm{D} + \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u_*|^2\,\mathrm{d}x = \mathrm{L} - \mathrm{H}([u_*,a_i])_{i=1}^{\kappa} + \mathrm{E}_{\mathrm{sg}}^{1,2}(g)\log\frac{1}{\rho}. \tag{4.5.82}$$

Combining with (4.5.4) from Proposition 4.5.2(a) , for each $0 < \rho < \bar{\rho}_\Omega$ and any measurable subset $A \subset \Omega \setminus \bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)$ we have

$$\varlimsup_{n\to\infty} \int_A |\mathrm{D}u_n|\,\mathrm{d}x \leq \left(\mathrm{L} - \mathrm{H}([u_*,a_i])_{i=1}^{\kappa} + \mathrm{E}_{\mathrm{sg}}^{1,2}(g)\log\frac{1}{\rho}\right)^{\frac{1}{2}}\sqrt{2\mathcal{L}^2(A)}. \tag{4.5.83}$$

Let $\varepsilon > 0$, and choose $\rho > 0$ such that $\mathrm{M}_i\rho < \frac{\varepsilon}{2\kappa}$ for all i. For this choice of ρ we then take $\delta > 0$ such that

$$\left(\mathrm{L} - \mathrm{H}([u_*,a_i])_{i=1}^{\kappa} + \mathrm{E}_{\mathrm{sg}}^{1,2}(g)\log\frac{1}{\rho_0}\right)^{\frac{1}{2}}\sqrt{2\delta} < \frac{\varepsilon}{2}. \tag{4.5.84}$$

Then given $A \subset \Omega$ measurable such that $\mathcal{L}^2(A) \leq \delta$, using (4.5.80), (4.5.83) we have

$$\varlimsup_{n\to\infty} \int_A |\mathrm{D}u_n|\ \mathrm{d}x \leq \varlimsup_{n\to\infty} \int_{A\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho_0)} |\mathrm{D}u_n|\ \mathrm{d}x + \sum_{i=1}^{\kappa} \varlimsup_{n\to\infty} \int_{\mathbb{B}(a_i,\rho)} |\mathrm{D}u_n|\ \mathrm{d}x < \varepsilon. \tag{4.5.85}$$

Thus by Lemma 4.5.5 we have $(\mathrm{D}u_n)_n$ is uniformly integrable, from which weak L^1-convergence also follows by the Dunford–Pettis theorem (Lemma 4.5.4), establishing (vii).

This concludes the proof of Theorem 4.5.1. □

4.6 Convergence of almost minimisers

4.6.1 Convergence theorem

We will show the convergence of Theorem 4.5.1 can be improved for sequences of almost minimisers in the sense of (4.6.2) below. This will also imply Theorem 4.1.2 from the introduction.

Theorem 4.6.1. *Given an approximating family* $(f_n)_n$*, we let* $g \in \mathrm{W}^{1/2,2}(\partial\Omega;\mathcal{N})$ *and define*

$$\mathrm{I}_n = \inf\left\{\int_\Omega f_n(|\mathrm{D}v|)\,\mathrm{d}x : \begin{array}{c} v \in \mathrm{W}^{1,1}(\Omega;\mathcal{N}) \\ \operatorname{tr}_{\partial\Omega} v = g \end{array}\right\}. \tag{4.6.1}$$

Then suppose $(u_n)_n \subset \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ *is an asymptotically minimising in the sense that* $\operatorname{tr}_{\partial\Omega} u_n = g$ *for all* n *and*

$$\lim_{n\to\infty}\left(\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathrm{I}_n\right) = 0. \tag{4.6.2}$$

Then passing to a subsequence, we have $u_n \to u_*$ *strongly in* $\mathrm{L}^1(\Omega)$ *to a limit map* $u_* \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ *satisfying* $\operatorname{tr}_{\partial\Omega} u_* = g$ *with* $\kappa \in \mathbb{N}$ *singularities* $a_1,\dots,a_\kappa \in \Omega$*, such that the following holds:*

(i) $\displaystyle\lim_{n\to\infty}\left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] = \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*,a_i])_{i=1}^{\kappa}.$

(ii) *The limit map satisfies*

$$\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*,a_i])_{i=1}^{\kappa} = \inf\left\{\mathrm{E}^{1,2}_{\mathrm{ren}}(\bar u) + \mathrm{H}([\bar u,\bar a_i])_{i=1}^{\bar\kappa}\right\},$$

where the infimum is taken over all $\bar u \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ *with* $\operatorname{tr}_{\partial\Omega}\bar u = g$*, where* $\bar u$ *has* $\bar\kappa \in \mathbb{N}$ *singularities* $\bar a_1,\dots,\bar a_{\bar\kappa}$*, forming a minimal topological resolution in that* $\mathrm{E}^{1,2}_{\mathrm{sg}}(g) = \sum_{i=1}^{\bar\kappa}\frac{\lambda([\bar u,\bar a_i])^2}{4\pi}$.

(iii) $\displaystyle\lim_{n\to\infty}\int_\Omega f_n(|\mathrm{D}u_*|) - f_n(|\mathrm{D}u_n|)\,\mathrm{d}x = 0.$

(iv) $\mathrm{D}u_n \to \mathrm{D}u_*$ *strongly in* $\mathrm{L}^1(\Omega)$ *and pointwise almost everywhere.*

(v) *For each* $\rho < \rho_\Omega(a_1,\dots,a_\kappa)$ *we have,*

$$\lim_{n\to\infty}\int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_* - \mathrm{D}u_n|)\,\mathrm{d}x = 0$$

(vi) *For each* $\rho < \rho_\Omega(a_1,\dots,a_\kappa)$ *and each* $i = 1,\dots,\kappa$ *,*

$$\lim_{n\to\infty}\left[\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|) - f_n\left(\frac{\lambda([u_*,a_i])}{2\pi|x-a_i|}\right)\mathrm{d}x\right] = \int_0^\rho\int_{\partial\mathbb{B}(a_i,r)}\frac{|\mathrm{D}u_*|^2}{2}\,\mathrm{d}\mathcal{H}^1 - \frac{\lambda([u_*,a_i])^2}{4\pi r}\,\mathrm{d}r.$$

By (ii), if $v \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ also has κ singularities at $b_1,\dots,b_\kappa \in \Omega$ such that $[u_*,a_i] = [v,b_i]$ for all $1 \le i \le \kappa$ (*i.e.* the limiting geodesics are homotopic), then $\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) \le \mathrm{E}^{1,2}_{\mathrm{ren}}(v)$. Therefore we have u_* is a *minimal renormalisable singular harmonic map* in the sense of [32, Def. 7.8]. In particular, by a direct comparison argument, for all $\rho > 0$ sufficiently small we have u_* is a minimising harmonic map in $\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\rho)$ with respect to its own boundary conditions, thereby proving Theorem 4.1.2(i).

4.6.2 Asymptotic uniform convexity

The classical notion of uniform convexity of a normed space (see *e.g.* [44, Def. 5.2.2]) implies that weak convergence, combined with the convergence of the norm, implies strong convergence (known as Radon–Riesz property). In this section we will prove an asymptotic version, valid along sequences of approximating integrands $(f_n)_n$, by exploiting the uniform convexity of the limiting space L^2. More precisely, we will establish the following.

Proposition 4.6.2. *Consider a sequence of mappings $u_n \in \mathrm{W}^{1,1}(\Omega;\mathcal{N})$ and a sequence of approximating integrands $(f_n)_n$. Assume that $(\mathrm{D}u_n)_n$ converges weakly* to $\mathrm{D}u$, and that*

$$\int_\Omega \frac{1}{2}|\mathrm{D}u|^2 \,\mathrm{d}x = \lim_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x. \tag{4.6.3}$$

Then

$$\lim_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u_n - \mathrm{D}u|)\,\mathrm{d}x = 0. \tag{4.6.4}$$

In particular, $\mathrm{D}u_n \to \mathrm{D}u$ in L^1.

Here we will work at the level of gradients for the convenience of the reader, however one could more generally with a sequence $(U_n)_n \subset \mathrm{L}^1(\Omega)$ in place of $(\mathrm{D}u_n)_n$. We will adapt the strategy described in [10], and as a key step we will establish almost everywhere convergence of the gradients, which we specify as a separate lemma.

Lemma 4.6.3. *Under the assumptions of Proposition 4.6.2, up to a subsequence, $\mathrm{D}u_n$ converges pointwise almost everywhere to $\mathrm{D}u$ in Ω.*

We rely on the following result (Lemma 4.6.4) concerning sequences in $\mathbb{R}^d$.

Lemma 4.6.4. *Consider a sequence of Young function $(f_n)_n$ converging to $|z|^2/2$. Suppose $a \in \mathbb{R}$ and $(b_n)_n \subset \mathbb{R}^d$ is a sequence satisfying*

$$\frac{1}{2} f_n(|a|) + \frac{1}{2} f_n(|b_n|) - f_n\Big(\Big|\frac{a+b_n}{2}\Big|\Big) \xrightarrow{n\to\infty} 0.$$

Then $\lim_{n\to\infty} b_n = a$.

Proof of Lemma 4.6.4. It suffices to prove that the sequence $(b_n)_n$ is bounded. Indeed if this holds, for any convergent subsequence $b_{n_k} \to b$, since f_n converges locally uniformly to $t^2/2$ by Lemma 4.2.10, we have

$$0 = \frac{1}{4}|a|^2 + \frac{1}{4}|b|^2 - \frac{1}{2}\left|\frac{a+b}{2}\right|^2 = \frac{1}{8}|b-a|^2,$$

which implies that $b = a$. Therefore every subsequence has a further subsequence converging to a, from which it follows that $b_n \to a$.

To show the boundedness of $(b_n)_n$, we suppose otherwise. Then, passing to a subsequence, we can assume that $|b_n|$ is monotone and divergent. Letting

$$g_n(t) = \frac{1}{2}f_n(|a|) + \frac{1}{2}f_n(t) - f_n\left(\frac{|a|+t}{2}\right),$$

since f_n is increasing and convex we have

$$0 \leq g_n(|b_n|) \leq \frac{1}{2}f_n(|a|) + \frac{1}{2}f_n(|b|) - f_n\left(\left|\frac{a+b_n}{2}\right|\right)$$

and so $\lim_{n\to\infty} g_n(|b_n|) \to 0$. On the other hand we have g_n is non-decreasing by noting that

$$g_n'(t) = \frac{1}{2}\left(f'(b) - f_n'\left(\frac{|a|+t}{2}\right)\right) \geq 0$$

for almost all $t \geq 0$, where we used that $t \mapsto f_+'(t)$ is right-continuous and non-decreasing, and that g is absolutely continuous since f is (see *e.g.* [27, Thm. 3.20, Ex. 3.21]). Now fixing m sufficiently large such that $|b_m| > |a|$, which exists by unboundedness of $(b_n)_n$, we have

$$0 = \lim_{n\to\infty} g_n(|b_n|) \geq \lim_{n\to\infty} g_n(|b_m|) = \frac{1}{4}|a|^2 + \frac{1}{4}|b_m|^2 - \frac{1}{2}\left(\frac{|a|+|b_m|}{2}\right)^2 \geq 0.$$

Since $|a| \neq |b_m|$, we have the right-hand side is strictly positive, thereby contradicting the unboundedness of $(b_n)_n$. □

Proof of Lemma 4.6.3. Applying Proposition 4.5.2 to $(\mathrm{D}u_n + \mathrm{D}u)/2$, we have (4.5.5) reads as

$$\int_\Omega \frac{1}{2}|\mathrm{D}u|^2 \,\mathrm{d}x \leq \varliminf_{n\to\infty} \int_\Omega f_n\Big(\Big|\frac{\mathrm{D}u_n + \mathrm{D}u}{2}\Big|\Big)\,\mathrm{d}x, \tag{4.6.5}$$

and by the dominated convergence theorem and Lemma 4.2.12 we also have

$$\lim_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u|)\,\mathrm{d}x = \int_\Omega \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}x. \tag{4.6.6}$$

For each n, on Ω we define

$$g_n \doteq \frac{1}{2} f_n(|\mathrm{D}u|) + \frac{1}{2} f_n(|\mathrm{D}u_n|) - f_n\Big(\Big|\frac{\mathrm{D}u_n + \mathrm{D}u}{2}\Big|\Big).$$

By convexity of f_n we have $g_n \geq 0$, and so we have

$$\begin{aligned}
\varlimsup_{n\to\infty} \int_\Omega g_n \, \mathrm{d}x &\leq \frac{1}{2} \varlimsup_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u|)\, \mathrm{d}x + \frac{1}{2} \varlimsup_{n\to\infty} \int_\Omega f_n(|\mathrm{D}u_n|)\, \mathrm{d}x \\
&\quad - \varliminf_{n\to\infty} \int_\Omega f_n\Big(\Big|\frac{\mathrm{D}u_n + \mathrm{D}u}{2}\Big|\Big)\, \mathrm{d}x \\
&\leq \left(\frac{1}{2} + \frac{1}{2} - 1\right) \int_\Omega \frac{1}{2} |\mathrm{D}u|^2 \, \mathrm{d}x = 0
\end{aligned}$$

where we used (4.6.3), (4.6.5) and (4.6.6). We deduce that $g_n \to 0$ in $\mathrm{L}^1(\Omega)$, and hence (using *e.g.* [44, Prop. 4.2.10]), up to a subsequence we have $g_n \to 0$ almost everywhere. Then by Lemma 4.6.4, we infer that $\mathrm{D}u_n \to \mathrm{D}u$ almost everywhere. □

Proof of Proposition 4.6.2. By Proposition 4.6.3, passing to a subsequence we have $\mathrm{D}u_n \to \mathrm{D}u$ almost everywhere. Then by Scheffé's lemma (asserting that pointwise convergence and convergence of the norm implies strong L^1 convergence, see *e.g.* [44, Prop. 2.4.6]) we have

$$\lim_{n\to\infty} \int_\Omega \Big| f_n(|\mathrm{D}u_n|) - \frac{1}{2} |\mathrm{D}u|^2 \Big| \, \mathrm{d}x = 0. \tag{4.6.7}$$

In particular, $(f_n(|\mathrm{D}u_n|))_n$ is equi-integrable. By $(\mathrm{Dec})_2^{\mathfrak{n}}$ we have the almost-doubling estimate

$$f_n(2t) \leq f_n(2t_0) + 4f(t) \tag{4.6.8}$$

for all n and $t \geq 0$, so we obtain the pointwise estimate

$$|f_n(|\mathrm{D}u - \mathrm{D}u_n|)| \leq 2 f_n(2t_0) + 4\left(f_n(|\mathrm{D}u|) + f_n(|\mathrm{D}u_n|)\right). \tag{4.6.9}$$

Since $f_n(|\mathrm{D}u|) \to \frac{1}{2}|\mathrm{D}u|^2$ strongly in $\mathrm{L}^1(\Omega)$ and $f_n(2t_0) \to 2t_0^2$ as $n \to \infty$, we infer the left hand side is equi-integrable. Therefore (4.6.4) follows by Vitali's convergence theorem (see *e.g.* [44, Thm. 3.1.12]), from which convergence in L^1 follows by Jensen's inequality. □

4.6.3 Strong convergence of almost minimizers — Proof of Theorem 4.6.1

Proof of Theorem 4.6.1. Let $\bar{u} \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega; \mathcal{N})$ be such that $\mathrm{tr}_{\partial\Omega}\, \bar{u} = g$, which always exists by [32, Prop. 2.4, Prop. 8.1], and suppose it has $\bar{\kappa}$

singularities at $\bar{a}_1, \ldots, \bar{a}_{\bar{\kappa}}$ such that $\mathrm{E}^{1,2}_{\mathrm{sg}}(g) = \sum_{i=1}^{\bar{\kappa}} \frac{\lambda([\bar{u},\bar{a}_i])^2}{4\pi}$. Then by the upper bound (Proposition 4.4.1) we have

$$\begin{aligned}\overline{\lim_{n\to\infty}} \left(\mathrm{I}_n - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right) &\leq \left[\lim_{n\to\infty} \int_\Omega f_n(|\mathrm{D}\bar{u}|) - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] \\ &= \mathrm{E}^{1,2}_{\mathrm{ren}}(\bar{u}) + \mathrm{H}([\bar{u},\bar{a}_i])_{i=1}^{\bar{\kappa}},\end{aligned} \tag{4.6.10}$$

noting $\bar{u}$ is admissible for the minimisation problem (4.6.1) for each n. Therefore if $u_n \in \mathrm{W}^{1,1}_g(\Omega;\mathcal{N})$ satisfies (4.6.2), it follows that

$$\begin{aligned}\overline{\lim_{n\to\infty}} &\left[\int_\Omega f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] \\ &\qquad\qquad \leq \mathrm{E}^{1,2}_{\mathrm{ren}}(\bar{u}) + \mathrm{H}([\bar{u},\bar{a}_i])_{i=1}^{\bar{\kappa}}.\end{aligned} \tag{4.6.11}$$

Therefore by Theorem 4.5.1, noting (4.5.1) is satisfied, by passing to a subsequence we infer that u_n converges to a limiting map $u_* \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ with κ singularities $a_1, \ldots, a_\kappa$, which by Theorem 4.5.1(ii) satisfies

$$\underline{\lim_{n\to\infty}} \left[\int_\Omega f_n(|\mathrm{D}u_n|) - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] \geq \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*,a_i])_{i=1}^{\kappa}, \tag{4.6.12}$$

and hence that

$$\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*,a_i])_{i=1}^{\kappa} \leq \mathrm{E}^{1,2}_{\mathrm{ren}}(\bar{u}) + \mathrm{H}([\bar{u},\bar{a}_i])_{i=1}^{\bar{\kappa}}. \tag{4.6.13}$$

While we selected a particular competitor, the same argument goes through for any $\bar{u} \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ satisfying $\mathrm{tr}_{\partial\Omega}\,\bar{u} = g$, thereby establishing (ii). Moreover, if we take $\bar{u} = u_*$ in (4.6.11), combining with (4.6.12) we deduce (i). For (iii) we combine (i) and Proposition 4.4.1 applied with u_* which gives

$$\begin{aligned}\lim_{n\to\infty} &\left[\int_\Omega f(|\mathrm{D}u_*|) - f_n(|\mathrm{D}u_n|)\,\mathrm{d}x\right] \\ &= \lim_{n\to\infty}\left[\int_\Omega f(|\mathrm{D}u_*|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] \\ &\qquad - \lim_{n\to\infty}\left[\int_\Omega f(|\mathrm{D}u_n|)\,\mathrm{d}x - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right]\end{aligned} \tag{4.6.14}$$

which is zero, as required.

Now, combining (i) and Theorem 4.5.1(iii), for all $\rho \in (0, \rho_\Omega(a_1,\ldots,a_\kappa))$ we have

$$\lim_{n\to\infty} \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x = \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}x. \tag{4.6.15}$$

Hence applying Proposition 4.6.2 in $\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \rho)$ we infer both (v) and that $\mathrm{D}u_n \to \mathrm{D}u$ strongly in $\mathrm{L}^1(\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i, \rho))$. In particular, by applying this to $\rho_k \searrow 0$ and passing to a diagonal subsequence, we deduce that $\mathrm{D}u_n \to \mathrm{D}u$ almost everywhere. Since $(\mathrm{D}u_n)_n$ is equi-integrable in Ω by Theorem 4.5.1(vii), strong L^1 convergence in Ω follows, thereby establishing (iv).

Finally combining (i) and Theorem 4.5.1(iv) we have for $0 < \rho < \rho_\Omega(a_1, \dots, a_\kappa)$ and $i = 1, \dots, \kappa$,

$$\begin{aligned} \lim_{n\to\infty} &\left[\int_{\mathbb{B}(a_i,\rho)} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \frac{\lambda([u_*, a_i])^2}{4\pi}\mathcal{V}(f_n)\right] \\ &= \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*; \mathbb{B}(a_i, \rho)) + \frac{\lambda([u_*, a_i])^2}{4\pi} \log \frac{2\pi}{\lambda([u_*, a_i])}, \end{aligned} \tag{4.6.16}$$

which together with Lemma 4.4.2 gives (vi). □

Remark 4.6.5. We will sketch how the above proof can be adapted to prove Theorem 4.1.1 in the absence of the $(\mathrm{Dec})_2^{\mathfrak{c}}$ condition. Let $(\bar{f}_n)_n$ be the modified family from Example 4.2.14, and let $\bar{\mathrm{I}}_n$ as in (4.6.1) with $\bar{f}_n$ in place of f_n. Then for any $\bar{u} \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega; \mathcal{N})$ such that $\operatorname{tr}_{\partial\Omega} \bar{u} = g$ whose $\bar{\kappa} \in \mathbb{N}$ singularities $\bar{a}_1, \cdots, \bar{a}_{\bar{\kappa}}$ forms a minimal topological resolution of g in Ω, by Proposition 4.4.1 we have

$$\lim_{n\to\infty} \left[\int_\Omega \bar{f}_n(|\mathrm{D}\bar{u}|)\,\mathrm{d}x - \mathcal{V}(\bar{f}_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] = \mathrm{E}^{1,2}_{\mathrm{ren}}(\bar{u}) + \mathrm{H}([\bar{u}, \bar{a}_i])_{i=1}^{\bar{\kappa}}. \tag{4.6.17}$$

Therefore since $\mathrm{I}_n \le \bar{\mathrm{I}}_n$ for all n, using (4.2.37) we have

$$\begin{aligned} \varlimsup_{n\to\infty} &\left[\mathrm{I}_n - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] \\ &\le \varlimsup_{n\to\infty} \left[\bar{\mathrm{I}}_n - \mathcal{V}(\bar{f}_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] + \frac{1}{2}\mathrm{E}^{1,2}_{\mathrm{sg}}(g) \lim_{n\to\infty} \bar{f}_n(1) \\ &\qquad \le \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa} + \frac{1}{4}\mathrm{E}^{1,2}_{\mathrm{sg}}(g), \end{aligned} \tag{4.6.18}$$

which in particular is finite. Now taking a sequence $(u_n)_n$ satisfying (4.6.2), we can apply Theorem 4.5.1 with the family $(f_n)_n$, where it is sufficient to assume $(\mathrm{Dec})_2^{\mathfrak{n}}$. In particular using Theorem 4.5.1(ii) we obtain a limit map $u_* \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega; \mathcal{N})$ such that $\operatorname{tr}_{\partial\Omega} u_* = g$ with $\kappa \in \mathbb{N}$ singularities $a_1, \cdots, a_\kappa \in \Omega$ such that

$$\varliminf_{n\to\infty} \left[\mathrm{I}_n - \mathcal{V}(f_n)\mathrm{E}^{1,2}_{\mathrm{sg}}(g)\right] \ge \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa}. \tag{4.6.19}$$

Hence combining (4.6.18) and (4.6.19) we deduce that

$$\lim_{n\to\infty} \frac{1}{\mathcal{V}(f_n)} \mathrm{I}_n = \mathrm{E}^{1,2}_{\mathrm{sg}}(g), \tag{4.6.20}$$

thereby proving Theorem 4.1.1.

We point out that this argument does not allow us to relax the $(\mathrm{Dec})_2^{\mathrm{c}}$ assumption in Theorem 4.1.2; for this (4.2.37) is insufficient, we would instead need an approximating sequence $(\tilde{f}_n)_n$ satisfying $f_n \leq \tilde{f}_n$ for all n and such that $\lim_{n\to\infty} \big[\mathcal{V}(\tilde{f}_n) - \mathcal{V}(f_n)\big] = 0$.

4.7 Universality of the vortex map

In this section we will show, under strong conditions on the boundary datum and domain, that the full sequence of minimisers $(u_n)_n$ converges to a unique minimiser of vortex type. Since the obtained limit map is independent of the choice of approximating integrands $(f_n)_n$, this shows mappings of type $g(\frac{x}{|x|})$ are in some sense *universal*. An analogous result for the Ginzburg-Landau approximation was obtained in [31, Thm. 8.1], and we will employ a similar strategy of proof.

We use the synharmonic distance between two maps

$$g, g' \in \mathrm{W}^{1/2,2}(\mathbb{S}^1; \mathcal{N}),$$

which is defined as

$$\begin{aligned} \mathrm{dist}_{\mathrm{synh}}(g, g') &\doteq \\ &\inf\left\{ \int_{\mathbb{S}^1\times[0,T]} \left(\frac{|\mathrm{D}u|^2}{2} - \frac{|\gamma|^2}{8\pi^2}\right) \mathrm{d}\mathcal{H}^1 : \begin{matrix} T \in (0,+\infty), & \mathrm{tr}_{\mathbb{S}^1\times\{0\}}\, u = g \\ u \in \mathrm{W}^{1,2}(\mathbb{S}^1\times[0,T];\mathcal{N}) & \mathrm{tr}_{\mathbb{S}^1\times\{T\}}\, u = g' \end{matrix} \text{ on } \mathbb{S}^1 \right\} \end{aligned} \tag{4.7.1}$$

which was introduced in [32, Def. 3.2]. Note this infimum is empty if g and g' are not homotopic, in which case we have $\mathrm{dist}_{\mathrm{synh}}(g, g') = \infty$.

Theorem 4.7.1. *Let $g \in \mathrm{C}^1(\mathbb{S}^1; \mathcal{N})$ be a minimising geodesic satisfying the following:*

(a) If $(\gamma_1, \ldots, \gamma_k)$ is a minimal topological resolution of g, then $k = 1$ and γ_1 is homotopic to g.

(b) If $g' \in \mathrm{W}^{1/2,2}(\mathbb{S}^1; \mathcal{N})$ is homotopic to g, then $\mathrm{dist}_{\mathrm{synh}}(g, g') = 0$.

Let $(f_n)_n$ *be a sequence of approximating integrands, and*

$$(u_n)_n \subset \mathrm{W}^{1,1}(\mathbb{B};\mathcal{N})$$

be an asymptotically minimising sequence in that $\operatorname{tr}_{\mathbb{S}^1} u_n = g$ *for all* n *and*

$$\int_{\mathbb{B}_1} f_n(|\mathrm{D}u_n|)\,\mathrm{d}x - \inf\left\{\int_{\mathbb{B}_1} f_n(|\mathrm{D}v|)\,\mathrm{d}x : \begin{array}{c} v \in \mathrm{W}^{1,1}(\mathbb{B}_1;\mathcal{N}) \\ \operatorname{tr}_{\partial\Omega} v = g \end{array}\right\} \xrightarrow{n\to\infty} 0. \tag{4.7.2}$$

Then we have $u_n \to u_*$ *in* $\mathrm{W}^{1,1}(\mathbb{B};\mathcal{N})$ *as* $n \to \infty$*, where* $u_*(x) = g(\frac{x}{|x|})$*.*

Recall that $\mathbb{B}$ denotes the closed unit ball in $\mathbb{R}^2$. Assumption (a) implies the atomicity (defined above in Definition 4.3.4) of the geodesic g, however it is in general a stronger condition, as [32, §9.3.5] illustrates.

In the case $\mathcal{N} = \mathbb{S}^1$, up to reparametrisation there are two minimising geodesics; namely $g(z) = z$ and $\bar{g}(z) = \bar{z}$, where we identify $z \in \mathbb{S}^1 \subset \mathbb{C}$. Since g and $\bar{g}$ lie in different homotopy classes, and since any reparametrisation of g or $\bar{g}$ only differ by a rotation, by [32, Prop. 3.8] both mappings satisfy (a) and (b). Therefore Theorem 4.7.1 applies, and in particular we infer the universality of the vortex map $u_{\mathrm{V}}(x) = \frac{x}{|x|}$, proving Theorem 4.1.3. In the specific case of $f_n(z) = {}^{|z|^{p_n}}/_{p_n}$, where $p_n \nearrow 2$, this behaviour is expected because the vortex map is the unique p-minimizer for every $p \in (1,2)$; see [21] and [13, Thm. 13.6].

Theorem 4.7.1 will largely follow from properties of the geometric energy introduced in [32, (2.11)], which we recall is defined as follows: given a bounded Lipschitz domain $\Omega \subset \mathbb{R}^2$, an integer $k \in \mathbb{N} \setminus \{0\}$, distinct points $a_1, \dots, a_k \in \Omega$, a radius $\rho \in (0, \bar{\rho}(a_1, \dots, a_k))$, a map $g \in \mathrm{W}^{1/2,2}(\partial\Omega, \mathcal{N})$ and a topological resolution $\gamma_1, \dots, \gamma_k \in \mathrm{W}^{1/2,2}(\mathbb{S}^1, \mathcal{N})$ of g, the *geometrical renormalised energy* is given by:

$$\begin{aligned} &\mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_k}(a_1, \dots, a_k) \\ &\doteq \liminf_{\rho\searrow 0}\left\{\int_{\Omega\setminus\bigcup_{i=1}^k \mathbb{B}_\rho(a_i)} \frac{|\mathrm{D}u|^2}{2}\,\mathrm{d}x - \sum_{i=1}^k \frac{\lambda(\gamma_i)^2}{4\pi^2}\log\frac{1}{\rho}\right. \\ &\qquad\left. : \begin{array}{c} u \in \mathrm{W}^{1,2}\big(\Omega \setminus \bigcup_{i=1}^k \mathbb{B}_\rho(a_i);\mathcal{N}\big), \operatorname{tr}_{\partial\Omega} u = g \\ \operatorname{tr}_{\mathbb{S}^1} u(a_i + \rho\,\cdot) = \gamma_i \quad \text{for all } i = 1, \dots, k \end{array}\right\}. \end{aligned} \tag{4.7.3}$$

We will also need two auxiliary results; the first will be a consequence of Theorem 4.6.1(ii).

Lemma 4.7.2. *Under the assumptions of Theorem 4.6.1, the limit map* u_* *satisfies*

$$\begin{aligned}(4.7.4)\quad \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa} \\ = \inf\{\mathcal{E}^{\mathrm{geom}}_{g,\bar\gamma_1,\dots,\bar\gamma_{\bar\kappa}}(\bar a_1,\dots,\bar a_{\bar\kappa}) + \mathrm{H}([\bar\gamma_i])_{i=1}^{\bar\kappa}\}\end{aligned}$$

where the infimum is taken over all $\bar\kappa \in \mathbb{N}$, $\bar a_1,\dots,\bar a_{\bar\kappa} \in \Omega$ *distinct and minimal topological resolutions* $\bar\gamma_1,\dots,\bar\gamma_{\bar\kappa} \in \mathrm{C}^1(\mathbb{S}^1;\mathcal{N})$ *of* g*. Moreover, the infimum is attained in that there exists* $\gamma_1,\dots,\gamma_\kappa \in \mathrm{C}^1(\mathbb{S}^1;\mathcal{N})$ *such that*

$$(4.7.5)\qquad \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) = \mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_\kappa}(a_1,\dots,a_\kappa).$$

Proof of Lemma 4.7.2. Given any $\bar u \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ with $\bar\kappa$ singularities at $\bar a_1,\dots,\bar a_{\bar\kappa}$ with singularities $\bar\gamma_i$ at each $\bar a_i$, by [32, Prop. 7.2(v)] we have $\mathrm{E}^{1,2}_{\mathrm{ren}}(\bar u) \geq \mathcal{E}^{\mathrm{geom}}_{g,\bar\gamma_1,\dots,\bar\gamma_{\bar\kappa}}(\bar a_1,\dots,\bar a_{\bar\kappa})$. Thus combining with Theorem 4.6.1(ii) we obtain the lower bound.

Conversely let $u_* \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ be a limiting map from Theorem 4.6.1, and suppose u_* has $\kappa \in \mathbb{N}$ singularities at $a_1,\dots,a_\kappa$ with limiting geodesics $\gamma_1,\dots,\gamma_\kappa$, in that they can be obtained as the limit of suitable subsequences of $\mathrm{tr}_{\partial\mathbb{B}(a_i,\rho_\ell)}\, u$ as $\rho_\ell \searrow 0$ (see *e.g.* [32, Prop. 7.2(iii)]). Then for each $0 < \rho < \rho_\Omega(a_1,\dots,a_\kappa)$, there exists $u_\rho \in \mathrm{W}^{1,2}(\Omega \setminus \bigcup_{i=1}^{\kappa} \mathbb{B}(a_i,\rho);\mathcal{N})$ minimising $\int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u|^2\,\mathrm{d}x$ subject to $\mathrm{tr}_{\partial\Omega}\, u_\rho = g$ and $\mathrm{tr}_{\mathbb{S}^1}\, u_\rho(a_i + \rho\,\cdot) = \gamma_i$ for each $i = 1,\dots,\kappa$. We then extend u_ρ to Ω by setting $u_\rho = \gamma_i(\frac{\cdot - a_i}{\rho})$ in each $\mathbb{B}(a_i,\rho)$, which defines a map $u_\rho \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\Omega;\mathcal{N})$ such that

$$(4.7.6)\qquad \mathrm{E}^{1,2}_{\mathrm{ren}}(u_\rho) = \int_{\Omega\setminus\bigcup_{i=1}^{\kappa}\mathbb{B}(a_i,\rho)} \frac{1}{2}|\mathrm{D}u_\rho|^2 - \mathrm{E}^{1,2}_{\mathrm{sg}}(g)\log\frac{1}{\rho},$$

noting the contributions on each $\mathbb{B}(a_i,\rho)$ vanish, since each γ_i is a minimising geodesic. Hence using Theorem 4.6.1(ii) to compare u_* with u_ρ sending $\rho \searrow 0$, using (4.7.3), (4.7.6), we have

$$\begin{aligned}(4.7.7)\quad \mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa} \\ \leq \lim_{\rho\to 0} \mathrm{E}^{1,2}_{\mathrm{ren}}(u_\rho) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa} \\ = \mathcal{E}^{\mathrm{geom}}_{g,\gamma_1,\dots,\gamma_\kappa}(a_1,\dots,a_\kappa) + \mathrm{H}([u_*, a_i])_{i=1}^{\kappa}.\end{aligned}$$

This establishes the reverse inequality, and that equality is attained with $(\gamma_1,\dots,\gamma_\kappa)$. ☐

Lemma 4.7.3. *Given the unit ball* $\mathbb{B} \subset \mathbb{R}^2$*, let* $g \in \mathrm{C}^1(\partial\mathbb{B},\mathcal{N})$ *be a minimising geodesic.*

(i) *If* γ *is a minimising geodesic homotopic to* g *such that* $\operatorname{dist}_{\mathrm{synh}}(g,\gamma) = 0$, *then* $\mathcal{E}^{\mathrm{geom}}_{g,g}(a) = \mathcal{E}^{\mathrm{geom}}_{g,\gamma}(a)$ *for all* $a \in \mathbb{B}^{\circ}$.

(ii) *We have* $\mathcal{E}^{\mathrm{geom}}_{g,g}(0) \leq \mathcal{E}^{\mathrm{geom}}_{g,g}(a)$ *for all* $a \in \mathbb{B}^{\circ}$, *with equality if and only if* $a = 0$.

(iii) *If* $v \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\mathbb{B};\mathcal{N})$ *satisfies* $\mathrm{E}^{1,2}_{\mathrm{ren}}(v,\mathbb{B}) = 0$, *then* $v(x) = g(\frac{x}{|x|})$.

Proof of Lemma 4.7.3. We have (i) follows from [32, Prop. 3.10], and (ii) follows from [32, Thm. 10.1] (combined with [7, §VIII.4]). Item (iii) is follows from the proof of [31, Thm. 8.1]; here one observes that $\mathrm{E}^{1,2}_{\mathrm{ren}}(v) = 0$ implies that $\partial_r u \equiv 0$ on $\mathbb{B}$, from which the claimed form follows. □

Proof of Theorem 4.7.1. By Theorem 4.6.1 we know $(u_n)_n$ admits a subsequence converging to some $u_* \in \mathrm{W}^{1,2}_{\mathrm{ren}}(\mathbb{B},\mathcal{N})$, so it suffices to show this limit map is uniquely determined as $g(\frac{x}{|x|})$. By assumption (a), we have u_* has only singularity at some $a \in \mathbb{B}^{\circ}$, and let $\gamma \in \mathrm{C}^1(\mathbb{S}^1;\mathcal{N})$ be homotopic to g such that $\mathrm{E}^{1,2}_{\mathrm{ren}}(u_*) = \mathcal{E}^{\mathrm{geom}}_{g,\gamma}(a)$, which exists by Lemma 4.7.2.

Now by assumption (b) we have $\operatorname{dist}_{\mathrm{synh}}(g,\gamma) = 0$ so by Lemma 4.7.3(i) we have $\mathcal{E}^{\mathrm{geom}}_{g,g}(x) = \mathcal{E}^{\mathrm{geom}}_{g,\gamma}(x)$ for all $x \in \mathbb{B}^{\circ}$, and since Lemma 4.7.2 asserts this is minimised at $x = a$, by Lemma 4.7.3(ii) we have $a = 0$. Using $g(\frac{x}{|x|})$ as a competitor, we see that $\mathcal{E}^{\mathrm{geom}}_{g,g}(0) = 0$, so by Lemma 4.7.3(iii) we deduce that $u_*(x) = g(\frac{x}{|x|})$. □

Bibliography of Chapter 4

[1] E. Acerbi and G. Dal Maso. "New lower semicontinuity results for polyconvex integrals". In: *Calc. Var. Partial Differ. Equ.* 2.3 (1994), pp. 329–371.

[2] L. Ambrosio and G. Dal Maso. "On the relaxation in $BV(\Omega;\mathbb{R}^m)$ of quasi-convex integrals". In: *J. Funct. Anal.* 109.1 (1992), pp. 76–97.

[3] H. Attouch, G. Buttazzo, and G. Michaille. *Variational analysis in Sobolev and BV spaces: Applications to PDEs and optimization*. Second. Vol. 17. MOS-SIAM Series on Optimization. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA; Mathematical Optimization Society, Philadelphia, PA, 2014, pp. xii+793.

[4] P.-A. Beaufort, J. Lambrechts, F. Henrotte, C. Geuzaine, and J.-F. Remacle. "Computing cross fields : A PDE approach based on the Ginzburg-Landau theory". In: *Procedia Engineering* 203 (2017). 26th International Meshing Roundtable, IMR26, 18-21 September 2017, Barcelona, Spain, pp. 219–231.

[5] F. Bethuel and F. Demengel. "Extensions for Sobolev mappings between manifolds". In: *Calc. Var. Partial Differ. Equ.* 3.4 (1995), pp. 475–491.

[6] F. Bethuel. "The approximation problem for Sobolev maps between two manifolds". In: *Acta Math.* 167 (1991), pp. 153–206.

[7] F. Bethuel, H. Brezis, and F. Hélein. *Ginzburg-Landau vortices.* Reprint of the 1994 hardback edition. Mod. Birkhäuser Class. Cham: Birkhäuser Springer, 2017.

[8] M. Bildhauer. *Convex Variational Problems.* Vol. 1818. Lecture Notes in Mathematics. Berlin, Heidelberg: Springer Berlin Heidelberg, 2003.

[9] J. Bourgain, H. Brezis, and P. Mironescu. "Another look at Sobolev spaces". In: *Optimal control and partial differential equations.* IOS, Amsterdam, 2001, pp. 439–455.

[10] H. Brezis. "Convergence in $\mathscr{D}'$ and in L^1 under strict convexity". In: *Boundary value problems for partial differential equations and applications.* Vol. 29. RMA Res. Notes Appl. Math. Masson, Paris, 1993, pp. 43–52.

[11] H. Brezis. *Functional Analysis, Sobolev Spaces and Partial Differential Equations.* New York, NY: Springer New York, 2011.

[12] H. Brezis and L. Nirenberg. "Degree Theory and BMO; Part I: Compact Manifolds without Boundaries". In: *Sel. Math.* 1.2 (Sept. 1995), pp. 197–263.

[13] H. Brezis and P. Mironescu. *Sobolev maps to the circle. From the perspective of analysis, geometry, and topology.* Vol. 96. Prog. Nonlinear Differ. Equ. Appl. New York, NY: Birkhäuser, 2021.

[14] B. Bulanyi and J. Van Schaftingen. "Singular extension of critical Sobolev mappings under an exponential weak-type estimate". In: *J. Func. Anal.* 288.1 (Jan. 2025).

[15] G. Canevari and A. Segatti. "Defects in nematic shells: a Γ-convergence discrete-to-continuum approach". In: *Arch. Ration. Mech. Anal.* 229.1 (2018), pp. 125–186.

[16] L. C. Evans and R. F. Gariepy. *Measure Theory and Fine Properties of Functions.* Chapman and Hall/CRC, 2015.

[17] I. Fonseca and G. Leoni. *Modern Methods in the Calculus of Variations: L^p Spaces.* Springer Monographs in Mathematics. New York: Springer, 2007.

[18] C. Goffman and J. Serrin. "Sublinear Functions of Measures and Variational Integrals". In: *Duke Math. J.* 31.1 (Mar. 1964), pp. 159–178.

[19] P. Grisvard. *Elliptic Problems in Nonsmooth Domains.* Society for Industrial and Applied Mathematics, Jan. 2011.

[20] R. Hardt and F.-H. Lin. "Mappings Minimizing the L^p Norm of the Gradient". In: *Comm. Pure Appl. Math.* 40.5 (1987), pp. 555–588.

[21] R. Hardt, F. Lin, and C. Wang. "The p-energy minimality of $x/|x|$". In: *Commun. Anal. Geom.* 6.1 (1998), pp. 141–152.

[22] P. Harjulehto and P. Hästö. *Orlicz Spaces and Generalized Orlicz Spaces.* Cham: Springer International Publishing, 2019.

[23] C. Irving and B. Van Vaerenbergh. *Universality of renormalisable mappings in two dimensions: the case of polar convex integrands.* Preprint, arXiv:2411.17520 [math.AP] (2024). 2024.

[24] R. L. Jerrard. "Lower Bounds for Generalized Ginzburg–Landau Functionals". In: *SIAM J. Math. Anal.* 30.4 (Jan. 1999), pp. 721–746.

[25] M. A. Krasnosel'skii and Y. B. Rutickii. *Convex Functions and Orlicz Spaces.* P. Noordhoff, 1961.

[26] J. M. Lee. *Introduction to Riemannian manifolds.* Vol. 176. Graduate Texts in Mathematics. Springer, Cham, 2018, pp. xiii+437.

[27] G. Leoni. *A First Course in Sobolev Spaces.* 2nd ed. Graduate Studies in Mathematics 181. Providence: AMS, 2017.

[28] G. G. Lorentz. "Some new functional spaces". In: *Ann. of Math. (2)* 51 (1950), pp. 37–55.

[29] L. Maligranda. *Orlicz Spaces and Interpolation.* Seminars in Mathematics 5. Departamento de Matemática, Universidade Estadual de Campinas, 1989.

[30] J. Marcinkiewicz. "Sur l'interpolation d'opérations". In: *CR Acad. Sci. Paris.* 208 (1939), pp. 1272–1273.

[31] A. Monteil, R. Rodiac, and J. Van Schaftingen. "Ginzburg–Landau Relaxation for Harmonic Maps on Planar Domains into a General Compact Vacuum Manifold". In: *Arch Rational Mech Anal* 242.2 (Nov. 2021), pp. 875–935.

[32] A. Monteil, R. Rodiac, and J. Van Schaftingen. "Renormalised Energies and Renormalisable Singular Harmonic Maps into a Compact Manifold on Planar Domains". In: *Math. Ann.* 383.3 (Aug. 2022), pp. 1061–1125.

[33] J. Nash. "The imbedding problem for Riemannian manifolds". In: *Ann. Math. (2)* 63 (1956), pp. 20–63.

[34] D. Palmer, A. Chern, and J. Solomon. "Lifting Directional Fields to Minimal Sections". In: *ACM Transactions on Graphics (TOG)* 43.4 (2024), pp. 1–20.

[35] M. Petrache and J. Van Schaftingen. "Controlled singular extension of critical trace Sobolev maps from spheres to compact manifolds". In: *Int. Math. Res. Not.* 2017.12 (2017), pp. 3647–3683.

[36] G. Pólya and G. Szegö. *Isoperimetric Inequalities in Mathematical Physics.* Annals of Mathematics Studies 27. Princeton, NJ: Princeton University Press, 1951.

[37] F. Rindler. *Calculus of variations.* Universitext. Cham: Springer, 2018.

[38] R. T. Rockafellar. *Convex analysis.* Vol. No. 28. Princeton Mathematical Series. Princeton University Press, Princeton, NJ, 1970, pp. xviii+451.

[39] E. Sandier. "Lower Bounds for the Energy of Unit Vector Fields and Applications". In: *J. Func. Anal.* 152.2 (Feb. 1998), pp. 379–403.

[40] R. Schoen and K. Uhlenbeck. "Boundary Regularity and the Dirichlet Problem for Harmonic Maps". In: *J. Diff. Geom.* 18.2 (Jan. 1983), pp. 253–268.

[41] M. Solci. "Nonlocal-interaction vortices". In: *SIAM J. Math. Anal.* 56.3 (2024), pp. 3430–3451.

[42] J. Van Schaftingen. "The extension of traces for Sobolev mappings between manifolds". In: *arXiv preprint arXiv:2403.18738* (2024).

[43] J. Van Schaftingen and B. Van Vaerenbergh. "Asymptotic behavior of minimizing p-harmonic maps when $p \nearrow 2$ in dimension 2". In: *Calc. Var. Partial Differ. Equ.* 62.8 (2023). Id/No 229, p. 45.

[44] M. Willem. *Functional analysis: Fundamentals and applications.* Cornerstones. Birkhäuser/Springer, New York, 2013, pp. xiv+213.

Extensions of traces for Sobolev mappings into manifolds at the endpoint $p = 1$

5

Abstract. We give direct proofs and constructions of the trace and extension theorems for Sobolev mappings in $\mathrm{W}^{1,1}(M, N)$, where M is Riemannian manifold with compact boundary ∂M and N is a complete Riemannian manifold. The analysis is also applicable to halfspaces and strips. The extension is based on a tiling of the domain of the considered applications by suitably chosen dyadic cubes to construct the desired extension. Along the way, we obtain asymptotic results concerning the L^1-mappings into manifolds.

5.1 Introduction and main results

We consider two Riemannian manifolds M and N, such that M has a compact boundary ∂M. We assume that N is isometrically embedded in $\mathbb{R}^\nu$, $\nu \in \mathbb{N}$. The trace operator

$$\operatorname{tr}_{\partial M} : \dot{\mathrm{W}}^{1,p}(M, N) \to \mathrm{L}^1_{\mathrm{loc}}(\partial M, N)$$

extends the restriction of continuous maps $\mathrm{C}(M, N) \to \mathrm{C}(\partial M, N)$ to mappings belonging to the homogeneous Sobolev space

$$\dot{\mathrm{W}}^{1,p}(M, N) = \{U : M \to N \text{ is weakly differentiable and } |\mathrm{D}U| \in \mathrm{L}^p(M)\}.$$

Here and after, $|\mathrm{D}U|$ is the pointwise Frobenius or Hilbert-Schmidt norm of the weak differential $\mathrm{D}U$ of U.

For $p > 1$, the range of the trace of $\dot{\mathrm{W}}^{1,p}(M, \mathbb{R}^\nu)$ is known to be the fractional space $\dot{\mathrm{W}}^{1-1/p,p}(\partial M, \mathbb{R}^\nu)$ since Gagliardo's seminal work [8]; when N is a compact manifold, the cases where the range is the fractional space $\dot{\mathrm{W}}^{1-1/p,p}(\partial M, N)$ has been characterized in a series of

This chapter originally appeared as J. Van Schaftingen and B. Van Vaerenbergh. *Extensions of traces for Sobolev mappings into manifolds at the endpoint* $p = 1$. 2025

works over the last decades [10, Section 6][4][3][16][22]. For our purpose it is interesting to note that the surjectivity always hold for $1 < p < 2$.

Going back to $p = 1$, Gagliardo has also proved [8] (see also [15] and [2, Teorema 10]) that the range of the trace of $\dot{\mathrm{W}}^{1,1}(M, \mathbb{R}^\nu)$ is $\mathrm{L}^1(\partial M, \mathbb{R}^\nu)$. However, even in the case $N = \mathbb{R}^\nu$, $\nu \in \mathbb{N}\setminus\{0\}$, the existence of a continuous and linear extension, *i.e.* the existence of a continuous and linear right inverse to $\mathrm{tr}_{\mathbb{R}^d \times \{0\}} : \mathrm{W}^{1,1}(\mathbb{R}^d \times (0,1)) \to \mathrm{L}^1(\mathbb{R}^d)$, is prevented by a result of Peetre [19]; see also [20, Section 5].

Hardt and Lin's proof [10] works straightforwardly for $p = 1$ [4, Proof of Theorem 7] and [16, Footnote 2] and shows the surjectivity of the trace when the target N is a compact Riemannian manifold.

The starting point of this work is to provide a direct proof of the surjectivity of the trace through a fairly explicit construction of the extension. As a byproduct, we get a slight weakening of the hypotheses from which we remove the compactness assumption on N.

Theorem 5.1.1. *Let M be a compact Riemannian manifold with compact boundary, let N be complete. Every $U \in \dot{\mathrm{W}}^{1,1}(M, N)$ has a trace $u = \mathrm{tr}_{\partial M} U$ such that*

$$\iint_{\partial M \times \partial M} \mathrm{dist}_N(u(x), u(y))\,\mathrm{d}x\,\mathrm{d}y \le C_M \iint_{\partial M \times M} \mathrm{dist}_N(u(x), U(y))\,\mathrm{d}x\,\mathrm{d}y$$
$$\le C'_M \int_M |\mathrm{D}U|,$$

where $C_M, C'_M > 0$ only depend on M. Conversely, if $u : \partial M \to N$ is measurable, then there exists a map $U \in \dot{\mathrm{W}}^{1,1}(M, N)$ satisfying $\mathrm{tr}_{\partial M} U = u$ and

$$\int_M |\mathrm{D}U| \le C''_M \iint_{\partial M \times \partial M} \mathrm{dist}_N(u(x), u(y))\,\mathrm{d}x\,\mathrm{d}y. \tag{5.1.1}$$

where $C''_M > 0$ only depends on M, provided the right-hand side is finite.

On the half-space $\mathbb{R}^d \times (0, \infty)$, the statement is a little more delicate. Indeed, it turns out that the condition

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \mathrm{dist}_N(u(x), u(y))\,\mathrm{d}x\,\mathrm{d}y \text{ is finite}$$

implies that u is *constant* (see Proposition 5.2.2).

Theorem 5.1.2. *Let N be a complete separable Riemannian manifold. Every $U \in \dot{\mathrm{W}}^{1,1}(\mathbb{R}^d \times (0,\infty), N)$ has a locally integrable trace*

$u = \operatorname{tr}_{\mathbb{R}^d \times \{0\}} U : \mathbb{R}^d \times \{0\} \to N$ such that for every $R > 0$

$$\iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{R^d} \,\mathrm{d}x \,\mathrm{d}y \le C_d \iint\limits_{\substack{\mathbb{R}^d \times (\mathbb{R}^d \times (0,R)) \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), U(y))}{R^{d+1}} \,\mathrm{d}x \,\mathrm{d}y$$

$$\le C_d' \iint_{\mathbb{R}^d \times (0,R)} |\mathrm{D}U|, \tag{5.1.2}$$

where $C_d, C_d' > 0$ only depend d.
Conversely, if $u : \mathbb{R}^d \to N$ is measurable and if

$$\liminf_{R \to +\infty} \iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{R^d} \,\mathrm{d}x \,\mathrm{d}y < \infty \tag{5.1.3}$$

then, there exists $U \in \dot{\mathrm{W}}^{1,1}(\mathbb{R}^d \times (0, \infty), N)$ such that $\operatorname{tr}_{\mathbb{R}^d \times \{0\}} U = u$ and

$$\int_{\mathbb{R}^d \times (0,\infty)} |\mathrm{D}U| \le C_d'' \liminf_{R \to \infty} \iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{R^d} \,\mathrm{d}x \,\mathrm{d}y.$$

where $C_d'' > 0$ only depends on d.

Let us explain how the double integral in (5.1.6) and its limit in (5.1.3) relate to more usual notions of integrability. First one always has for $R > 0$ by the triangle inequality

$$\iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0, R))} \,\mathrm{d}x \,\mathrm{d}y \le 2 \inf_{b \in N} \int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b) \,\mathrm{d}x. \tag{5.1.4}$$

The reverse inequality is also true asymptotically.

Theorem 5.1.3. *If $u : \mathbb{R}^d \to N$, there exists $b_* \in N$ such that*

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b_*) \,\mathrm{d}x = \frac{1}{2} \lim_{R \to +\infty} \iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0, R))} \,\mathrm{d}x \,\mathrm{d}y. \tag{5.1.5}$$

The limit (5.1.5) is reminiscent to Bourgain-Brezis-Mironescu-Maz′ya-Shaposhnikova-type formulae [5][14]. The volume of the d-dimensional ball of radius R is denoted $\mathcal{L}^d(\mathbb{B}^d(0, R))$ and sometimes $|\mathbb{B}^d_R|$.

Our methods of proof also yield an extension result on strips which is the manifold constrained counterpart of Leoni and Tice's [13, Theorem 1.8].

Theorem 5.1.4. *Every $U \in \dot{\mathrm{W}}^{1,1}(\mathbb{R}^d \times (0,1), N)$ has traces then $u_0 = \mathrm{tr}_{\mathbb{R}^d \times \{0\}} U$ and $u_1 = \mathrm{tr}_{\mathbb{R}^d \times \{1\}} U$ satisfying*

$$\begin{aligned}\int_{\mathbb{R}^d} \mathrm{dist}_N(u_0(x), u_1(x)) \, \mathrm{d}x + \sum_{i \in \{0,1\}} \iint_{\substack{\mathbb{R}^d \times (\mathbb{R}^d \times (0,1)) \\ |x-y| \le 1}} \mathrm{dist}_N(u_i(x), u_i(y)) \, \mathrm{d}x \, \mathrm{d}y \\ \le C_d \sum_{i \in \{0,1\}} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le 1}} \mathrm{dist}_N(u_i(x), U(y)) \, \mathrm{d}x \, \mathrm{d}y \\ \le C_d' \iint_{\mathbb{R}^d \times (0,1)} |\mathrm{D}U|, \end{aligned} \tag{5.1.6}$$

where $C_d, C_d' > 0$ only depend d.
Conversely, given measurable $u_0, u_1 : \mathbb{R}^d \to N$, there exists $U \in \dot{\mathrm{W}}^{1,1}(\mathbb{R}^d \times (0,1), N)$ such that $\mathrm{tr}_{\mathbb{R}^d \times \{0\}} U = u_0$, $\mathrm{tr}_{\mathbb{R}^d \times \{1\}} U = u_1$ and

$$\begin{aligned}\int_{\mathbb{R}^d \times (0,1)} |\mathrm{D}U| \le C_d'' \int_{\mathbb{R}^d} \mathrm{dist}_N(u_0(x), u_1(x)) \, \mathrm{d}x \\ + C_d'' \sum_{i \in \{0,1\}} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le 1}} \mathrm{dist}_N(u_i(x), u_i(y)) \, \mathrm{d}x \, \mathrm{d}y\end{aligned} \tag{5.1.7}$$

where $C_d'' > 0$ only depends on d, provided the right-hand side is finite.

Plan of the paper. In Section 5.2, we explain the precise assumptions we make on N throughout the paper. We also explain the slightly weaker assumptions we will work with in the extension results, see (5.4.1), and prove Theorem 5.1.3. In Section 5.3 we write the proof of Theorem 5.1.2 as well as a variant with manifold domain. The proof of Theorem 5.1.4 consists in two main steps: in Section 5.4 we extend by a map which is constant on a countable union of rectangles that is only of bounded variation, *i.e.* $\mathrm{BV}(\mathbb{R}^d \times (-1,1), N)$ and in Section 5.5 we smoothen the BV-extension to get a $\dot{\mathrm{W}}^{1,1}$ map and therefore prove Theorem 5.1.4. Theorem 5.1.1 is then proven in Section 5.6 using a variant of Theorem 5.1.4 on cubes.

5.2 Manifold-valued integrable mappings

In this section, we write the precise assumptions on N, we define integrability, and prove Theorem 5.1.3 in the form of Proposition 5.2.3.

The manifold N. We assume that N is a (path-)connected complete manifold endowed with a distance function $\mathrm{dist}_N : N \times N \to [0, \infty)$ on it. The manifold is assumed to be a submanifold $N \subset \mathbb{R}^\nu, \nu \in \mathbb{N} \setminus \{0\}$ isometrically embedded by Nash isometric embedding theorem [18]; by

[17], even if the manifold N is not assumed to be compact, we may assume that N is a closed set of $\mathbb{R}^\nu$. The completeness assumption ensures that any two points can be connected by a geodesic of minimal length. This result is known as the Hopf-Rinow theorem [11, Theorem 6.19]. We also assume separability of N : there exists a countable dense set in N for the metric dist_N; this assumption is of use in Proposition 5.2.6 to ensure that the essential range of a measurable map is separable. As said in Section 5.1, M is a Riemannian manifold that carries a compact boundary ∂M.

A measurable map $u : \mathbb{R}^d \to N$ is said *integrable* whenever

$$\inf_{b \in N} \int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b)\, \mathrm{d}x < \infty. \tag{5.2.1}$$

When $N = \mathbb{R}^\nu$ and $\operatorname{dist}_N(p, q) = |p - q|$ for all $p, q \in N$, we recover classical integrable functions augmented by their translation in $\mathbb{R}^\nu$.

It follows from the next proposition that the point b appearing in the condition (5.2.1) is unique.

Proposition 5.2.1. *Let $u : \mathbb{R}^d \to N$ and $v : \mathbb{R}^d \to N$ be measurable, and let $b, c \in N$. If*

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b)\, \mathrm{d}x, \int_{\mathbb{R}^d} \operatorname{dist}_N(v(x), c)\, \mathrm{d}x \text{ and } \int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), v(x))\, \mathrm{d}x$$

are all finite then $b = c$.

Proof of Proposition 5.2.1. By the triangle inequality,

$$\begin{aligned} \int_{\mathbb{R}^d} \operatorname{dist}_N(b, c)\, \mathrm{d}x \le \int_{\mathbb{R}^d} &\operatorname{dist}_N(b, u(x))\, \mathrm{d}x \\ &+ \int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), v(x))\, \mathrm{d}x + \int_{\mathbb{R}^d} \operatorname{dist}_N(v(x), c)\, \mathrm{d}x \end{aligned}$$

is finite and the conclusion follows. □

Proposition 5.2.2. *If $u : \mathbb{R}^d \to N$ is measurable and if*

$$\iint_{\mathbb{R}^d \times \mathbb{R}^d} \operatorname{dist}_N(u(x), u(y))\, \mathrm{d}x\, \mathrm{d}y < \infty,$$

then u is constant.

Proof of Proposition 5.2.2. For every $n \in \mathbb{N}_*$, by Fubini's theorem and comparison, there exists some point $y_n \in \mathbb{R}^d$ such that

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), u(y_n))\, \mathrm{d}x \le \frac{1}{n}. \tag{5.2.2}$$

By Proposition 5.2.1, for every $m \in \mathbb{N}_*$, $u(y_n) = u(y_m)$. Setting $b = u(y_n)$, we have

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(b, u(x)) \,\mathrm{d}x = 0, \tag{5.2.3}$$

and thus $u = b$ almost everywhere on $\mathbb{R}^d$. □

The integrability can in fact be recovered from a variant of the Bourgain-Brezis-Minorescu-Maz'ya-Shaposhnikova formula [5, 6, 14] (see also [9, 7]).

Proposition 5.2.3. *If $u : \mathbb{R}^d \to N$ is measurable, then for every $R > 0$,*

$$\iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0, R))} \,\mathrm{d}x \,\mathrm{d}y \le 2 \inf_{b \in N} \int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b) \,\mathrm{d}x \tag{5.2.4}$$

and there exists $b_* \in N$ such that

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b_*) \,\mathrm{d}x = \frac{1}{2} \lim_{R \to \infty} \iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0, R))} \,\mathrm{d}x \,\mathrm{d}y. \tag{5.2.5}$$

This proves Theorem 5.1.3.

Proof of Proposition 5.2.3. We write $\mathbb{B}^d_R = \mathbb{B}^d(0, R)$. We have, by the triangle inequality,

$$\begin{aligned} &\iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}x \,\mathrm{d}y \\ &\qquad \le \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \operatorname{dist}_N(u(x), b) \,\mathrm{d}x \,\mathrm{d}y + \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \operatorname{dist}_N(u(y), b) \,\mathrm{d}x \,\mathrm{d}y \\ &\qquad = 2\mathcal{L}(\mathbb{B}^d_R) \int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b) \,\mathrm{d}x \end{aligned} \tag{5.2.6}$$

which proves (5.2.4).

Set for $R > 0$,

$$\Theta(R) \doteq \iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0, R))} \,\mathrm{d}x \,\mathrm{d}y.$$

We assume without loss of generality that we have an increasing positive sequence $(R_n)_{n\in\mathbb{N}}$ such that

$$\lim_{n\to\infty} \Theta(R_n) = \liminf_{R\to\infty} \Theta(R) < \infty$$

and we set

$$\eta_n = \frac{1}{\max(2, R_n^{1/2})}.$$

We have, since $\eta_n \le 1/2$,

$$(\mathbb{B}^d_{(1-\eta_n)R_n} \times \mathbb{B}^d_{\eta_n R_n}) \cup (\mathbb{B}^d_{\eta_n R_n} \times \mathbb{B}^d_{(1-\eta_n)R_n}) \subseteq \{(x, y) \in \mathbb{R}^d \times \mathbb{R}^d : |x - y| \le R_n\}$$

and

$$(\mathbb{B}^d_{(1-\eta_n)R_n} \times \mathbb{B}^d_{\eta_n R_n}) \cap (\mathbb{B}^d_{\eta_n R_n} \times \mathbb{B}^d_{(1-\eta_n)R_n}) = \mathbb{B}^d_{\eta_n R_n} \times \mathbb{B}^d_{\eta_n R_n},$$

so that, by symmetry,

$$\begin{aligned}(5.2.7)\quad 2\int_{\mathbb{B}^d_{(1-\eta_n)R_n}} \int_{\mathbb{B}^d_{\eta_n R_n}} & \operatorname{dist}_N(u(x), u(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\le \int_{\mathbb{B}^d_{\eta_n R_n}} \int_{\mathbb{B}^d_{\eta_n R_n}} \operatorname{dist}_N(u(x), u(y))\,\mathrm{d}x\,\mathrm{d}y \\ &\qquad + \iint\limits_{\substack{\mathbb{R}^d\times\mathbb{R}^d\\ |x-y|\le R_n}} \operatorname{dist}_N(u(x), u(y))\,\mathrm{d}x\,\mathrm{d}y\end{aligned}$$

and thus

$$\frac{2}{\mathcal{L}^d(\mathbb{B}^d_{(1-\eta_n)R_n})} \int_{\mathbb{B}^d_{(1-\eta_n)R_n}} \int_{\mathbb{B}^d_{\eta_n R_n}} \operatorname{dist}_N(u(x), u(y))\,\mathrm{d}x\,\mathrm{d}y \le \theta_n \doteq \frac{\Theta(R_n) + (2\eta_n)^d \Theta(2\eta_n R_n)}{(1-\eta_n)^d}.$$

Hence there exists $y_n \in \mathbb{B}^d_{(1-\eta_n)R_n}$ such that

$$(5.2.8)\qquad \int_{\mathbb{B}^d_{\eta_n R_n}} \operatorname{dist}_N(u(x), u(y_n))\,\mathrm{d}x \le \frac{\theta_n}{2}.$$

If $n \le m$, we have $\eta_n R_n \le \eta_m R_m$. Hence by the triangle inequality and by (5.2.8),

$$\begin{aligned}
\operatorname{dist}_N(u(y_n), u(y_m)) &\le \int_{\mathbb{B}^d_{\eta_n R_n}} \frac{\operatorname{dist}_N(u(y_n), u(y_m))}{\mathcal{L}^d(\mathbb{B}^d_{\eta_n R_n})} \,\mathrm{d}x \\
&\le \int_{\mathbb{B}^d_{\eta_n R_n}} \frac{\operatorname{dist}_N(u(x), u(y_n))}{\mathcal{L}^d(\mathbb{B}^d_{\eta_n R_n})} \,\mathrm{d}x \\
&\quad + \int_{\mathbb{B}^d_{\eta_m R_m}} \frac{\operatorname{dist}_N(u(x), u(y_m))}{\mathcal{L}^d(\mathbb{B}^d_{\eta_n R_n})} \,\mathrm{d}x \\
&\le \frac{\theta_n + \theta_m}{2(\eta_n R_n)^d \mathcal{L}^d(\mathbb{B}^d_1)}.
\end{aligned}$$

Since $\eta_n R_n = \min(R_n/2, R_n^{1/2}) \to \infty$, the sequence $(u(y_n))_n$ is a Cauchy sequence that converges to some b by completeness of N. By Fatou's lemma and (5.2.8)

$$\begin{aligned}
(5.2.9)\quad & \int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b) \,\mathrm{d}x \\
&\le \lim_{n\to\infty} \int_{\mathbb{B}^d_{R_n}} \operatorname{dist}_N(u(x), u(y_n)) \,\mathrm{d}x \le \lim_{n\to\infty} \frac{\theta_n}{2} = \lim_{n\to\infty} \frac{\Theta(R_n)}{2},
\end{aligned}$$

which, combined with (5.2.4), gives (5.2.5). □

Remark 5.2.4. It follows from Proposition 5.2.3 that if $A \subset \mathbb{R}^d$ is measurable, then

$$\mathcal{L}^d(A) = \frac{1}{2} \lim_{R\to+\infty} \frac{\mathcal{L}^{2d}(\{(x,y) \in A \times (\mathbb{R}^d \setminus A) : |x-y| \le R\})}{\mathcal{L}^d(\mathbb{B}^d(0,R))}.$$

We will conclude this section by proving Proposition 5.2.5.

Proposition 5.2.5. *If $u : \mathbb{R}^d \to N$ is measurable and if for some $R > 0$,*

$$(5.2.10)\qquad \iint_{\substack{\mathbb{R}^d\times\mathbb{R}^d \\ |x-y|\le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0,R))} \,\mathrm{d}x \,\mathrm{d}y < \infty$$

then

$$(5.2.11)\qquad \lim_{r\searrow 0} \iint_{\substack{\mathbb{R}^d\times\mathbb{R}^d \\ |x-y|\le r}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0,r))} \,\mathrm{d}x \,\mathrm{d}y = 0.$$

Proposition 5.2.6. *If $u : \mathbb{R}^d \to N$ is locally integrable, then for a.e. $x \in \mathbb{R}^d$ one has*

$$(5.2.12)\qquad \lim_{r\searrow 0} \fint_{\mathbb{B}^d(x,r)} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}y = 0.$$

The dashed-integral refers to the average integral.

Proof of Proposition 5.2.6. Given $b \in N$, we consider the function $f_b = \operatorname{dist}(u, b) \colon \mathbb{R}^d \to \mathbb{R}$. Since

$$|f_b(x) - f_b(y)| = |\operatorname{dist}_N(u(x), b) - \operatorname{dist}(u(y), b)| \le \operatorname{dist}_N(u(x), u(y)),$$

the function f_b is locally integrable. By the classical Lebesgue's differentiation theorem applied to f_b, there exists a set $E_b \subset \mathbb{R}^d$ such that $\mathcal{L}^d(E_b) = 0$ and such that for every $x \in \mathbb{R}^d \setminus E_b$,

$$\lim_{r \searrow 0} \fint_{\mathbb{B}^d(x,r)} \operatorname{dist}_N(u(y), b) \,\mathrm{d}y = \operatorname{dist}_N(u(x), b),$$

and thus

$$\begin{aligned}
&\limsup_{r \searrow 0} \fint_{\mathbb{B}^d(x,r)} \operatorname{dist}_N(u(y), u(x)) \,\mathrm{d}y \\
&\qquad \le \lim_{r \searrow 0} \fint_{\mathbb{B}^d(x,r)} \operatorname{dist}_N(u(y), b) + \operatorname{dist}_N(u(x), b) \,\mathrm{d}y \\
&\qquad = 2 \operatorname{dist}_N(u(x), b).
\end{aligned} \tag{5.2.13}$$

By our separability assumption on N, considering a countable dense set $B \subset N$, we write $E = \bigcup_{b \in B} E_b$ and observe that $\mathcal{L}^d(E) = 0$. Moreover, for every $b \in B$ and $x \in \mathbb{R}^d \setminus E$, (5.2.13) holds and its right right-hand side can be made arbitrary small. We have shown that for almost every $x \in \mathbb{R}^d$, (5.2.12) holds. □

Proposition 5.2.7. *If $u : \mathbb{R}^d \to N$ is measurable, then for every $u : \mathbb{R}^d \to N$, $R > 0$ and $h \in \mathbb{B}^d(0, R)$, one has*

$$\begin{aligned}
&\int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), u(x + h)) \,\mathrm{d}x \\
&\qquad\qquad \le C_d \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0, R))} \,\mathrm{d}x \,\mathrm{d}y
\end{aligned} \tag{5.2.14}$$

where $C_d > 0$ only depends on d.

Lemma 5.2.8. *If $x, y \in \mathbb{R}^d$ and $|x - y| \le R$, then*

$$\begin{aligned}
&\operatorname{dist}_N(u(x), u(y)) \\
&\le A_d \Bigl(\fint_{\mathbb{B}^d(x,R)} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}z + \fint_{\mathbb{B}^d(y,R)} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}z \Bigr),
\end{aligned}$$

where $A_d > 0$ only depends on d.

Proof of Lemma 5.2.8. By the triangle inequality, we have successively

$$
\begin{aligned}
&\operatorname{dist}_N(u(x), u(y))\\
&\qquad = \fint_{\mathbb{B}^d(x,R)\cap\mathbb{B}^d(y,R)} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}z\\
&\qquad \le \fint_{\mathbb{B}^d(x,R)\cap\mathbb{B}^d(y,R)} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}z\\
&\qquad\quad + \fint_{\mathbb{B}^d(x,R)\cap\mathbb{B}^d(y,R)} \operatorname{dist}_N(u(y), u(z)) \,\mathrm{d}z\\
&\qquad \le A_d\Big(\fint_{\mathbb{B}^d(x,R)} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}z + \fint_{\mathbb{B}^d(y,R)} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}z\Big),
\end{aligned}
$$

where

$$
A_d \doteq \sup\left\{ \frac{|\mathbb{B}^d(x;R)|}{|\mathbb{B}^d(x;R)\cap\mathbb{B}^d(y;R)|} : x, y \in \mathbb{R}^d, R > 0, |x-y| \le R \right\}
$$

is finite. □

Proof of Proposition 5.2.7. Applying Lemma 5.2.8 and integrating over $x \in \mathbb{R}^d$, we obtain

$$
\begin{aligned}
&\int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), u(x+h)) \,\mathrm{d}x\\
&\le \frac{A_d}{\mathcal{L}(\mathbb{B}^d(0;R))} \int_{\substack{\mathbb{R}^d\times\mathbb{B}^d\\ |x-z|\le R}} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}x \,\mathrm{d}z\\
&\quad + \frac{A_d}{\mathcal{L}(\mathbb{B}^d(0;R))} \int_{\substack{\mathbb{R}^d\times\mathbb{R}^d\\ |x+h-z|\le R}} \operatorname{dist}_N(u(x+h), u(z)) \,\mathrm{d}x \,\mathrm{d}z\\
&= \frac{A_d}{\mathcal{L}(\mathbb{B}^d(0;R))} \int_{\substack{\mathbb{R}^d\times\mathbb{B}^d\\ |x-z|\le R}} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}x \,\mathrm{d}z.
\end{aligned}
$$

which is (5.2.14). □

Proof of Proposition 5.2.5. Given $R > 0$, we define the function

$$
f(x) = \fint_{\mathbb{B}^d(x,R)} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}z.
$$

In view of (5.2.10), we have

$$
\int_{\mathbb{R}^d} f < \infty
$$

whereas by Lemma 5.2.8, for every $x, y \in \mathbb{R}^d$ such that $|x-y| \le R$,

$$\operatorname{dist}_N(u(x), u(y)) \le A_d(f(x) + f(y)).$$

Averaging the latter inequality over $x \in \mathbb{B}^d(x, r)$, we get

$$\fint_{\mathbb{B}^d(x,r)} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}y \le A_d f(x) + A_d \fint_{\mathbb{B}^d(x,r)} f. \tag{5.2.15}$$

Since the left-hand side of (5.2.15) converges almost-everywhere and its right-hand side converges in $\mathrm{L}^1(\mathbb{R}^d)$ as $r \to 0$ by approximation by averages in L^1, (5.2.11) follows from Lebesgue's dominated convergence. □

Remark 5.2.9. From (5.2.14), we see that the finiteness (5.2.10) implies that

$$\begin{aligned}\sup_{r \in (0,R)} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le r}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0,r))} \,\mathrm{d}x \,\mathrm{d}y \\ \le C_d \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0,R))} \,\mathrm{d}x \,\mathrm{d}y.\end{aligned}$$

We finally record the following easy-to-state consequence of Propositions 5.2.5 and 5.2.7.

Corollary 5.2.10. *If $u : \mathbb{R}^d \to N$ is measurable, then*

$$\begin{aligned}\lim_{h \to 0} \int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), u(x+h)) \,\mathrm{d}x \\ = \lim_{r \searrow 0} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le r}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0,r))} \,\mathrm{d}x \,\mathrm{d}y \in \{0, +\infty\}.\end{aligned} \tag{5.2.16}$$

5.3 Trace inequalities

In this section, we write a proof of Theorem 5.1.2 as well as a variant on balls (Proposition 5.3.1) and when the domain is a manifold, see Proposition 5.3.2.

Proposition 5.3.1. *Fix $R > 0$. If $U \in \dot{\mathrm{W}}^{1,1}(\mathbb{B}^d(0,R) \times (0,1), N)$, the trace $u = \operatorname{tr}_{\mathbb{B}^d(0,R)\{0\}} U : \mathbb{B}^d(0,R) \times \{0\} \to N$ is locally integrable and satisfies for each $r \in (0,R)$*

$$\iint_{\substack{\mathbb{B}^d(0,R) \times \mathbb{B}^d(0,R) \\ |x-y| \le r}} \frac{\operatorname{dist}_N(u(x), u(y))}{r^d} \,\mathrm{d}x \,\mathrm{d}y \le C_d \iint_{\mathbb{B}^d(0,R) \times (0,r)} |\mathrm{D}U| \tag{5.3.1}$$

and

$$\iint\limits_{\substack{\mathbb{B}^d(0,R)\times(\mathbb{B}^d(0,R)\times(0,R))\\ |x-y|\le r}} \frac{\operatorname{dist}_N(u(x),U(y))}{r^{d+1}}\,\mathrm{d}x\,\mathrm{d}y \le C_d \iint_{\mathbb{B}^d(0,R)\times(0,r)} |\mathrm{D}U| \tag{5.3.2}$$

where $C_d > 0$ *only depends on* d.

In the proof of Proposition 5.3.1, we use the fact that if $C \subset \mathbb{R}^d$ is a convex and $U \in \dot{\mathrm{W}}^{1,1}_{\mathrm{loc}}(C, N)$ then for almost every $x, y \in C$,

$$\operatorname{dist}_N(U(x),U(y)) \le |x-y| \int_0^1 |\mathrm{D}U((1-t)x+ty))|\,\mathrm{d}t \tag{5.3.3}$$

and the right-hand side is finite.

Proof of Proposition 5.3.1. By scaling, we assume $R = 1$ and set $\mathbb{B}^d = \mathbb{B}^d(0,1)$ the unit ball centered at the origin. For almost every $x \in \mathbb{B}^d \simeq \mathbb{B}^d \times \{0\}$ and $y \in \mathbb{B}^d \times (0,r)$, we have, by (5.3.3),

$$\operatorname{dist}_N(u(x),U(y)) \le |x-y| \int_0^1 |\mathrm{D}U((1-t)x+ty)|\,\mathrm{d}t.$$

and thus, with $y = (y', y_{d+1})$,

$$\begin{aligned}
&\iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(r/2,r))} \operatorname{dist}_N(u(x),U(y))\mathbb{1}_{\{|x-y'|<r\}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \int_0^1 \iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(0,r))} |x-y|\mathbb{1}_{\{|x-y'|<r\}}|\mathrm{D}U((1-t)y+tx)|\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}t.
\end{aligned} \tag{5.3.4}$$

Changing the variable y to $z = (1-t)x + ty$ so that the Jacobian determinant of the deformation is equal to t^{d+1}, $x - z = t(x-y)$ and $z_{d+1} = ty_{d+1}$, the right-hand side of (5.3.4) is controlled by

$$\begin{aligned}
&\int_0^1 \iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(0,r))} |x-z|\mathbb{1}_{\{\max(|x-z'|,z_{d+1})<tr<2z_{d+1}\}}|\mathrm{D}U(z)|\frac{\mathrm{d}x\,\mathrm{d}z}{t^{d+2}}\,\mathrm{d}t \\
&\quad\le \int_{\mathbb{B}^d\times(r/2,r)} |\mathrm{D}U(z)| \int_{\mathbb{B}^d} \int_{\max(|x-z|,z_{d+1})/r}^{\infty} \frac{|x-z|\mathbb{1}_{\{|x-z'|<r\}}}{t^{d+2}}\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}z \\
&\quad\le \frac{r^{d+1}}{d+1}\int_{\mathbb{B}^d\times(r/2,r)} |\mathrm{D}U(z)| \int_{\mathbb{B}^d} \frac{|x-z|\mathbb{1}_{\{|x-z'|\le 2z_{d+1}\}}}{\max(|x-z'|,z_{d+1})^{d+1}}\,\mathrm{d}x\,\mathrm{d}z \\
&\quad\le (\tfrac{1}{2}+\tfrac{1}{d})r^{d+1}|\mathbb{B}^d| \int_{\mathbb{B}^d\times(0,r)} |\mathrm{D}U(z)|\,\mathrm{d}z,
\end{aligned} \tag{5.3.5}$$

since

$$\begin{aligned}
&\int_{\mathbb{B}^d} \frac{\mathbb{1}_{\{|x-z'|\le 2z_{d+1}\}}}{\max(|x-z'|, z_{d+1})^{d+1}}\,\mathrm{d}x \\
&\quad\le \int_{\mathbb{R}^d} \frac{\mathbb{1}_{\{|x-z'|\le 2z_{d+1}\}}}{\max(|x-z'|, z_{d+1})^{d+1}}\,\mathrm{d}x \\
&\quad\le (d+1)|\mathbb{B}^d| \int_0^{2z_{d+1}} \frac{r^{d-1}}{\max(r, z_{d+1})^{d+1}}\,\mathrm{d}r \\
&\quad= (d+1)|\mathbb{B}^d| \biggl(\int_0^{z_{d+1}} \frac{r^{d-1}}{z_{d+1}^{d+1}}\,\mathrm{d}r + \int_{z_{d+1}}^{2z_{d+1}} \frac{1}{r^2}\,\mathrm{d}r\biggr) \\
&\quad= (d+1)|\mathbb{B}^d| \biggl(\frac{1}{dz_{d+1}} + \frac{1}{2z_{d+1}}\biggr).
\end{aligned}$$

Next, we have

(5.3.6)

$$\begin{aligned}
&\iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(0,r/2))} \operatorname{dist}_N(U(y), U(y', y_{d+1}+r/2))\mathbb{1}_{\{|x-y'|<r\}}\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(0,r/2))} \int_0^{r/2} |\mathrm{D}U(y', y_{d+1}+t)\mathbb{1}_{\{|x-y'|<r\}}|\,\mathrm{d}t\,\mathrm{d}x\,\mathrm{d}y \\
&\quad\le \frac{r^{d+1}|\mathbb{B}^d|}{2} \int_{\mathbb{B}^d\times(0,r)} |\mathrm{D}U|.
\end{aligned}$$

Combining (5.3.4), (5.3.5) and (5.3.6), we get

$$\begin{aligned}
&\iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(0,r/2))} \operatorname{dist}_N(u(x), U(y))\mathbb{1}_{\{|x-y'|<r\}}\,\mathrm{d}x\,\mathrm{d}z \\
&\qquad\qquad\le \bigl(\tfrac{1}{d}+1\bigr)|\mathbb{B}^d| r^{d+1} \int_{\mathbb{B}^d\times(0,r)} |\mathrm{D}U|,
\end{aligned}$$

and

$$\begin{aligned}
&\iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(0,r))} \operatorname{dist}_N(u(x), U(y))\mathbb{1}_{\{|x-y'|<r\}}\,\mathrm{d}x\,\mathrm{d}z \\
&\qquad\qquad\le \bigl(\tfrac{2}{d}+\tfrac{3}{2}\bigr)|\mathbb{B}^d| r^{d+1} \int_{\mathbb{B}^d\times(0,r)} |\mathrm{D}U|,
\end{aligned}$$

from which (5.3.2) follows since

$$\begin{aligned}
&\{(x,y) \in \mathbb{B}^d(0,R) \times (\mathbb{B}^d(0,R) \times (0,R) \colon |x-y| \le r\} \\
&\qquad\subseteq \{(x,y) \in \mathbb{B}^d(0,R) \times \mathbb{B}^d(0,R) \times (0,r) \colon |x-y'| \le r\}.
\end{aligned}$$

Finally, we have by the triangle inequality

(5.3.7)
$$\begin{aligned}\iiint_{\mathbb{B}^d\times\mathbb{B}^d\times(\mathbb{B}^d\times(0,r))} &\operatorname{dist}_N(u(x),u(y))\mathbb{1}_{\{|x-z|<r,|y-z|<r\}}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\\
&\le |\mathbb{B}^d|r^d\iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(0,r))}\operatorname{dist}_N(u(x),U(z))\mathbb{1}_{\{|x-z|<r\}}\,\mathrm{d}x\,\mathrm{d}z\\
&\qquad+|\mathbb{B}^d|r^d\iint_{\mathbb{B}^d\times(\mathbb{B}^d\times(0,r))}\operatorname{dist}_N(u(y),U(z))\mathbb{1}_{\{|y-z|<r\}}\,\mathrm{d}y\,\mathrm{d}z\\
&\le \big(\tfrac{2}{d}+\tfrac{3}{2}\big)|\mathbb{B}^d|^2r^{2d+1}\int_{\mathbb{B}^d\times(0,r)}|\mathrm{D}U|\end{aligned}$$

where the last is justified by (5.3.2). We note that for every $x,y\in\mathbb{B}^d$,

$$|z-x|\le|z-\tfrac{x+y}{2}|+\tfrac{1}{2}|x-y|$$

there exists $B_d>0$ depending only on d such that

(5.3.8)
$$\begin{aligned}\int_{\mathbb{B}^d\times(0,r)}\mathbb{1}_{\{|x-z|<r,|y-z|<r\}}\,\mathrm{d}z &\ge \int_{\mathbb{B}^d\times(0,r)}\mathbb{1}_{\{|z-\frac{x+y}{2}|<r-\frac{1}{2}|x-y|\}}\,\mathrm{d}z\\
&\ge B_d r^{d+1}\mathbb{1}_{\{|x-y|<r\}}.\end{aligned}$$

Finally, (5.3.1) follows from (5.3.7) and (5.3.8). □

We record that taking $R\to+\infty$ one obtains a similar result on $\mathbb{R}^d$ (instead of $\mathbb{B}^d(0,R)$) holding for all $r>0$; this result is written as Theorem 5.1.2.

Proposition 5.3.2. *Let M a Riemannian manifold with compact boundary ∂M. There exists $R_0>0$ and such that if $U\in\dot{\mathrm{W}}^{1,1}(M,N)$, the trace $u=\operatorname{tr}_{\mathbb{R}^d\times\{0\}}U:\partial M\to N$ is integrable and satisfies for each $r\in(0,R_0)$,*

$$\iint\limits_{\substack{\partial M\times\partial M\\ \operatorname{dist}_{\partial M}(x,y)\le r}}\frac{\operatorname{dist}_N(u(x),u(y))}{r^{\dim(\partial M)}}\,\mathrm{d}x\,\mathrm{d}y\le C_M\iint\limits_{M\cap\{\operatorname{dist}_M(x,\partial M)\le r\}}|\mathrm{D}U|$$

and

$$\iint\limits_{\substack{\partial M\times M\\ \operatorname{dist}_{\partial M}(x,y)\le r}}\frac{\operatorname{dist}_N(u(x),U(y))}{r^{\dim(\partial M)+1}}\,\mathrm{d}x\,\mathrm{d}y\le C_M\iint\limits_{M\cap\{\operatorname{dist}_M(x,\partial M)\le r\}}|\mathrm{D}U|$$

where $C_M>0$ only depends on M.

Proof. This is a consequence of Proposition 5.3.1 on a finite subcovering of $\partial M\times\partial M\subset\bigcup_{x\in\partial M}\mathbb{B}^M(x,r)\times\mathbb{B}^M(x,r)$ by geodesic balls $\mathbb{B}^M(x,r)$, $x\in\partial M$ and the fact that $\mathbb{B}^M(x,r)$ and $\mathbb{B}^{\dim(\partial M)}(0,1)\times(0,1)$ are Lipschitz deformations of each other. □

5.4 Joining two maps by a map of bounded variation

In this section we provide the first step of the proof of the extensions results: Proposition 5.4.1 joins two integrable mappings by a map of bounded variation.

Fix $L > 0$ and $k \in \mathbb{Z}$ and let us denote by $\mathcal{Q}_k = \{Q(a, 2^{-k}L) : a \in 2^{1-k}L\mathbb{Z}^d\}$ the *dyadic cubes* of radius $2^{-k}L$. We will crucially use that $\mathcal{L}^d(Q) = (2^{1-k}L)^d$ which we will sometimes write $|Q|$.

Proposition 5.4.1. *If $u_0, u_1 : \mathbb{R}^d \to N$ satisfy for $i \in \{0, 1\}$*

$$\inf_{r \in \mathbb{R}} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le r}} \frac{\operatorname{dist}_N(u_i(x), u_i(y))}{r^d} \,\mathrm{d}x \,\mathrm{d}y < \infty \tag{5.4.1}$$

and if

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(u_0(x), u_1(x)) \,\mathrm{d}x < \infty, \tag{5.4.2}$$

then, for each $L > 0$, there exists $U \in \dot{\mathrm{BV}}(\mathbb{R}^d \times (-L, L), N)$ such that

$$\operatorname{tr}_{\mathbb{R}^d \times \{-L\}} U = u_0 \quad \text{and} \quad \operatorname{tr}_{\mathbb{R}^d \times \{L\}} U = u_1. \tag{5.4.3}$$

Moreover, there exists an increasing sequence $(k_n)_{n \in \mathbb{N}} \subset \mathbb{N}$ depending on $u_i, i \in \{0, 1\}$ such that $k_0 = 0$ and such that setting

$$\begin{aligned} I_{n,0} &= (-(1 - 2^{-k_{n+1}})L, -(1 - 2^{-k_n})L) \\ I_{n,1} &= ((1 - 2^{-k_{n+1}})L, (1 - 2^{-k_n})L), \end{aligned} \tag{5.4.4}$$

and we have for each $t \in I_{n,i}$ and each $x \in Q \in \mathcal{Q}_{k_n}$, $U(x, t) = u_i(x_Q)$ for some Lebesgue point $x_Q \in Q$ of u_i. Moreover, if $t \in I_{n,i}$,

$$\begin{aligned} \int_{\mathbb{R}^d} &\operatorname{dist}_N(u_i(x), \operatorname{tr}_{\mathbb{R}^d \times \{t\}} U(x)) \,\mathrm{d}x \\ &\le \frac{2^{-n}}{L^d} \sum_{i \in \{0,1\}} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le L}} \operatorname{dist}_N(u_i(x), u_i(y)) \,\mathrm{d}x \,\mathrm{d}y. \end{aligned} \tag{5.4.5}$$

If we write J_U for the union of the faces F of the rectangular cuboïds $Q \times I_n$, $Q \in \mathcal{Q}_{k_{n,i}}$ and denote by $\operatorname{tr}^+_{\mathrm{J}_U} U$ and $\operatorname{tr}^-_{\mathrm{J}_U} U$ the value of U on the *two cuboïds sharing a same face F, we have*

$$\begin{aligned}(5.4.6)\quad &\int_{\mathrm{J}_U\cap\mathbb{R}^d\times(-L,L)} \operatorname{dist}_N(\operatorname{tr}^+_{\mathrm{J}_U}U(x),\operatorname{tr}^-_{\mathrm{J}_U}U(x))\,\mathrm{d}x\\ &\qquad\le \int_{\mathbb{R}^d}\operatorname{dist}_N(u_0(x),u_1(x))\,\mathrm{d}x\\ &\qquad\quad+\frac{C_d}{L^d}\sum_{i\in\{0,1\}}\iint_{\substack{\mathbb{R}^d\times\mathbb{R}^d\\|x-y|\le L}}\operatorname{dist}_N(u_i(x),u_i(y))\,\mathrm{d}x\,\mathrm{d}y,\end{aligned}$$

where $C_d > 0$ only depends on d.

The homogeneous space of manifold constrained mappings of bounded variation $\mathrm{BV}(\mathbb{R}^d\times(-L,L),N)$ is defined through $N\subset\mathbb{R}^\nu$:

$$\begin{aligned}&\dot{\mathrm{B}\mathrm{V}}(\mathbb{R}^d\times(-L,L),N)\\ &=\{u\in\dot{\mathrm{B}\mathrm{V}}(\mathbb{R}^d\times(-L,L),\mathbb{R}^\nu): u(x)\in N \text{ for almost everywhere } x\in N\}.\end{aligned}$$

We refer to [1] for the definition of $\mathrm{BV}(\mathbb{R}^d\times(-L,L),\mathbb{R}^\nu)$; the corresponding homogeneous space $\dot{\mathrm{B}\mathrm{V}}(\mathbb{R}^d\times(-L,L),N)$ neglects the L^1 part of the definition. One can show that J_U and $\operatorname{tr}^\pm_{\mathrm{J}_U}$ in Proposition 5.4.1 are respectively the jump set and the one sided traces of BV maps as defined in [1, Theorem 3.77]. The trace of mappings of bounded variations appearing in (5.4.3) is defined in [2]. For our purposes, one has by (5.4.5), for $i\in\{0,1\}$,

$$\lim_{n\to\infty}\sup_{t\in I_{n,i}}\int_{\mathbb{R}^d}\operatorname{dist}_N(u_i(x),\operatorname{tr}_{\mathbb{R}^d\times\{t\}}U(x))\,\mathrm{d}x=0.$$

The trace operator $\mathrm{BV}\to\mathrm{L}^1_{\mathrm{loc}}$ being continuous under translations, we deduce (5.4.3).

Proof of Proposition 5.4.1. Let $i\in\{0,1\}$. For each $k\in\mathbb{Z}$, for each $Q\in\mathcal{Q}_k$, there exists a Lebesgue point $x_Q\in Q$ of u_i such that

$$(5.4.7)\quad \fint_Q \operatorname{dist}_N(u_i(x_Q),u_i(y))\,\mathrm{d}y\le\fint_Q\fint_Q\operatorname{dist}_N(u_i(x),u_i(y))\,\mathrm{d}y\,\mathrm{d}x,$$

where here and after for sets Ω of finite measure, we write

$$\fint_\Omega f\,\mathrm{d}\mu=\frac{1}{\mu(\Omega)}\int_\Omega f\,\mathrm{d}\mu,$$

the mean of $f:\Omega\to\mathbb{R}$ on Ω.

For each $k \in \mathbb{Z}$, we define for $x \in \mathbb{R}^d$,

$$\mathcal{E}_k(u_i)(x) = \mathcal{E}_k(u_i)(Q) = u_i(x_Q) \qquad \text{if } x \in Q \in \mathcal{Q}_k. \tag{5.4.8}$$

We will use the second expression $\mathcal{E}_k(u_i)(Q) \in N$ to emphasize that $\mathcal{E}_k(u_i)(x)$ is constant on the cube Q.

By our assumption (5.4.1) and Proposition 5.2.5

$$\begin{aligned}
&\varlimsup_{k\to+\infty} \int_{\mathbb{R}^d} \operatorname{dist}_N(\mathcal{E}_k(u_i)(x), u_i(x))\,\mathrm{d}x \\
&\qquad \le \varlimsup_{k\to+\infty} \sum_{Q\in\mathcal{Q}_k} \int_Q \fint_Q \operatorname{dist}_N(u_i(x), u_i(y))\,\mathrm{d}y\,\mathrm{d}x \\
&\qquad \le \lim_{k\to\infty} \frac{1}{2^{(1-k)d}L^d} \iint\limits_{\substack{\mathbb{R}^d\times\mathbb{R}^d \\ |x-y|_\infty \le 2^{1-k}L}} \operatorname{dist}_N(u_i(x), u_i(y))\,\mathrm{d}x\,\mathrm{d}y \\
&\qquad = 0.
\end{aligned} \tag{5.4.9}$$

For later we set $k_0 = 0$ and define

$$\Gamma \doteq \sum_{i\in\{0,1\}} \frac{1}{L^d} \iint\limits_{\substack{\mathbb{R}^d\times\mathbb{R}^d \\ |x-y|\le L}} \operatorname{dist}_N(u_i(x), u_i(y))\,\mathrm{d}x\,\mathrm{d}y. \tag{5.4.10}$$

In view of (5.4.9), there exists an increasing sequence of natural numbers $(k_n)_n$ so that for each $n \in \mathbb{N}_*$, $k \ge k_n$ and $i \in \{0,1\}$,

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(\mathcal{E}_k(u_i)(x), u_i(x))\,\mathrm{d}x \le \frac{\Gamma}{2^n}. \tag{5.4.11}$$

By (5.4.1) and a change of variable,

$$\lim_{k\to+\infty} \int_{\mathbb{R}^d} \fint_{|h|_\infty \le 2^{1-k}L} \operatorname{dist}_N(u_i(x), u_i(x+h))\,\mathrm{d}h\,\mathrm{d}x = 0. \tag{5.4.12}$$

By (5.4.12), we may further assume that there exists a subsequence $(k_{n,i})_n$ satisfying for each $n \in \mathbb{N}_*$

$$\int_{\mathbb{R}^d} \fint_{|h|_\infty \le 2^{1-k_n}L} \operatorname{dist}_N(u_i(x), u_i(x+h))\,\mathrm{d}h\,\mathrm{d}x \le \frac{\Gamma}{2^{k_{n-1}}} \tag{5.4.13}$$

by mathematical induction.

We set

$$U(x,t) \doteq \mathcal{E}_{k_n}(u_i)(x) \ \text{ if } \ i \in \{0,1\},\ t \in I_{n,i},\ x \in \mathcal{Q}_{k_n},$$

where $I_{n,i}$ is defined in (5.4.4). We obtain a map $U : \mathbb{R}^d \times (-L, L) \to N$ which is constant on rectangular cuboïds; the manifold constraint is satisfied as $\mathcal{E}_{k_n}(u_i)(x) \in N$ by (5.4.7). The map is in fact only defined almost everywhere as no value was given on the interfaces of the cuboïds. By the choice of $\Gamma > 0$ it proves (5.4.5). We claim that $\mathrm{tr}_{\mathbb{R}^d\times\{-L\}}U = u_0$ and $\mathrm{tr}_{\mathbb{R}^d\times\{L\}}U = u_1$. The situation is symmetric so we only consider the case $i = 0$ in $t = R$. The trace operator $\mathrm{tr}_{\mathbb{R}^d\times\{0\}} : \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d \times (-L, L)) \to \mathrm{L}^1(\mathbb{R}^d)$ is continuous with respect with translations. We have $\mathrm{tr}_{\mathbb{R}^d\times\{\tau\}}U = \mathcal{E}_{k_n}(u_0)$ if $\tau \in I_{n,0}$. By (5.4.9), we deduce that $\mathrm{tr}_{\mathbb{R}^d\times\{L\}}U = u_1$ and (5.4.5). Therefore it only remains to prove (5.4.6).

Since U is constant on rectangular cuboïds by construction and jumps on the interfaces of the rectangles, we deduce that $U \in \mathrm{BV}_{\mathrm{loc}}(\mathbb{R}^d \times (-L, L))$. The measure $|\mathrm{D}U|$ is absolutely continuous with respect to $\mathcal{H}^d$ and we will denote its support J_U. We therefore deduce that the left-hand side in the Proposition 5.4.1(5.4.6), is controlled by the three following contributions: we will estimate (i), (ii) and (iii) separately.

(i) Contributions at the interface $\mathbb{R}^d \times \{0\}$ in $t = 0$

$$\sum_{Q\in\mathcal{Q}_{k_0}} \mathcal{H}^d(Q)\,\mathrm{dist}_N(\mathcal{E}_{k_0}(u_0)(Q), \mathcal{E}_{k_0}(u_1)(Q)). \tag{5.4.14}$$

(ii) *Parallel* contributions to $\mathbb{R}^d \times \{0\}$ of each u_i, $i \in \{0, 1\}$, (except the jump on $\mathbb{R}^d \times \{0\}$, see (i))

$$\sum_{i\in\{0,1\}}\sum_{n\in\mathbb{N}}\sum_{Q\in\mathcal{Q}_{k_n}}\sum_{Q'\in\mathcal{Q}_{k_{n+1}} Q'\subset Q} \mathcal{H}^d(Q')\times \mathrm{dist}_N(\mathcal{E}_{k_n}(u_i)(Q), \mathcal{E}_{k_{n+1}}(u_i)(Q')). \tag{5.4.15}$$

(iii) *Perpendicular* contributions to $\mathbb{R}^d \times \{0\}$

$$\sum_{i\in\{0,1\}}\sum_{n\in\mathbb{N}}\sum_{Q\in\mathcal{Q}_{k_n}}\sum_{Q'\in\mathcal{Q}_{k_n}\cap\mathrm{neigh}(Q)} \mathcal{H}^d(\bar{Q}\cap\bar{Q}'\times I_{k_n})\times \mathrm{dist}_N(\mathcal{E}_{k_n}(u_i)(Q'), \mathcal{E}_{k_n}(u_i)(Q)) \tag{5.4.16}$$

where the fourth sum runs over the $2d$ neighbors of the dyadic cube $Q \in \mathcal{Q}_{k_n}$. We record that the measure of a intersecting lateral face of two cuboïds, denoted $\bar{Q}\cap\bar{Q}'\times I_{k_n}$, is $\mathcal{H}^d(\bar{Q}\cap\bar{Q}'\times I_{k_n}) = 2^{(1-k_n)(d-1)}(2^{-k_{n+1}} - 2^{-k_n})L^d$.

The analysis of (i) relies on the triangle inequality and (5.4.7). Indeed, the sum over cubes $\mathcal{Q}_{k_0}$ of (i) equals

$$\begin{aligned}
&\int_{\mathbb{R}^d} \operatorname{dist}_N(\mathcal{E}_{k_0}(u_0)(x), \mathcal{E}_{k_0}(u_1)(x)) \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \operatorname{dist}_N(\mathcal{E}_{k_0}(u_1)(x), u_0(x)) \,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^d} \operatorname{dist}_N(u_0(x), u_1(x)) \,\mathrm{d}x + \int_{\mathbb{R}^d} \operatorname{dist}_N(u_1(x), \mathcal{E}_{k_{1,0}}(u_1)(x)) \,\mathrm{d}x \\
&\le \int_{\mathbb{R}^d} \operatorname{dist}_N(u_0(x), u_1(x)) \,\mathrm{d}x \\
&\quad + \sum_{i\in\{0,1\}} \sum_{Q\in\mathcal{Q}_{k_0}} |Q|^{-1} \iint_{Q\times Q} \operatorname{dist}_N(u_i(x), u_i(y)) \,\mathrm{d}x \,\mathrm{d}y \\
&\le \int_{\mathbb{R}^d} \operatorname{dist}_N(u_0(x), u_1(x)) \,\mathrm{d}x \\
&\quad + L^{-d} \sum_{i\in\{0,1\}} 2^{d(k_0-1)} \iint_{|x-y|_\infty\le 2L} \operatorname{dist}_N(u_i(x), u_i(y)) \,\mathrm{d}x \,\mathrm{d}y. \qquad (5.4.17)
\end{aligned}$$

For (ii), we have that by the triangle inequality, for each $i \in \{0,1\}$, $n \in \mathbb{N}$,

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}_{k_n}} \sum_{\substack{Q'\in\mathcal{Q}_{k_{n+1}}\\ Q'\subset Q}} \int_{Q'} \operatorname{dist}_N(\mathcal{E}_{k_n}(u_i)(x), \mathcal{E}_{k_{n+1}}(u_i)(x)) \,\mathrm{d}x \\
&\le \sum_{Q\in\mathcal{Q}_{k_n}} \sum_{\substack{Q'\in\mathcal{Q}_{k_{n+1}}\\ Q'\subset Q}} \Bigg[\int_{Q'} \operatorname{dist}_N(\mathcal{E}_{k_n}(u_i)(x), u_i(x)) \,\mathrm{d}x \\
&\qquad + \int_{Q'} \operatorname{dist}_N(u_i(x), \mathcal{E}_{k_{n+1,i}}(u_i)(x)) \,\mathrm{d}x \Bigg].
\end{aligned}$$

We will sum over all the subcubes $Q' \subset Q$, noting that $Q = \bigcup\{Q' : Q' \in \mathcal{Q}_{k_{n+1}}, Q' \subset Q\}$. One can control further by

$$\begin{aligned}
&\sum_{Q\in\mathcal{Q}_{k_n}} \int_Q \operatorname{dist}_N(\mathcal{E}_{k_{n,1}}(u_i)(x), u_i(x)) \,\mathrm{d}x \\
&\qquad + \sum_{Q\in\mathcal{Q}_{k_n}} \int_Q \operatorname{dist}_N(\mathcal{E}_{k_{n+1}}(u_i)(x), u_i(x)) \,\mathrm{d}x \\
&\qquad\le \int_{\mathbb{R}^d} \operatorname{dist}_N(\mathcal{E}_{k_n}(u_i)(x), u_i(x)) \,\mathrm{d}x \\
&\qquad\qquad + \int_{\mathbb{R}^d} \operatorname{dist}_N(\mathcal{E}_{k_{n+1}}(u_i)(x), u_i(x)) \,\mathrm{d}x
\end{aligned} \tag{5.4.18}$$

which is controlled using (5.4.11) and (5.4.7). We deduce that (ii) is controlled by

$$
\begin{aligned}
\text{(5.4.19)}\quad \Gamma &\sum_{i\in\{0,1\}}\sum_{n\in\mathbb{N}\setminus\{0\}}(2^{-n}+2^{-(n+1)})\\
&+\sum_{i\in\{0,1\}}\sum_{Q\in\mathcal{Q}_{k_0}}\fint_Q\int_Q \operatorname{dist}_N(u_i(x),u_i(y))\,\mathrm{d}x\,\mathrm{d}y\\
&\le 6\Gamma+\sum_{i\in\{0,1\}}\frac{1}{L^d}\iint_{\substack{\mathbb{R}^d\times\mathbb{R}^d\\ |x-y|_\infty\le 2L}}\operatorname{dist}_N(u_i(x),u_i(y))\,\mathrm{d}x\,\mathrm{d}y.
\end{aligned}
$$

Next, for (iii), we have for any two cubes $Q,Q'\in\mathcal{Q}_{k_n}$ adjacent, using in order constantness on cubes and the triangle inequality,

$$2^{(1-k_n)(d-1)}(2^{-k_n}-2^{-k_{n+1}})L^d\fint_Q\fint_{Q'}\operatorname{dist}_N(\mathcal{E}_{k_n}(u_i)(x),\mathcal{E}_{k_n}(u_i)(y))\,\mathrm{d}x\,\mathrm{d}y$$

$$\le 2^{(1-k_n)(d-1)}(2^{-k_n}-2^{-k_{n+1}})L^d\times \tag{5.4.20}$$

$$\Big[\fint_Q\fint_{Q'}\operatorname{dist}_N(u_i(x),u_i(y))\,\mathrm{d}x\,\mathrm{d}y \tag{5.4.21}$$

$$+\fint_Q\fint_{Q'}\operatorname{dist}_N(u_i(x),\mathcal{E}_{k_n}(u_i)(y))\,\mathrm{d}x\,\mathrm{d}y \tag{5.4.22}$$

$$+\fint_Q\fint_{Q'}\operatorname{dist}_N(\mathcal{E}_{k_n}(u_i)(x),u_i(y))\,\mathrm{d}x\,\mathrm{d}y\Big]. \tag{5.4.23}$$

The two terms (5.4.22)–(5.4.23) are treated noting that the prefactor can be estimated using

$$2^{(1-k_n)(d-1)}(2^{-k_n}-2^{-k_{n+1}})L^d\le 2^{(1-k_n)(d-1)}2^{-k_n}L^d=2^{d-1}2^{-dk_n}=|Q|/2$$

so that

$$
\begin{aligned}
&\frac{|Q|}{2}\fint_Q\fint_{Q'}\operatorname{dist}_N(u_i(x),\mathcal{E}_{k_n}(u)(Q'))\,\mathrm{d}x\,\mathrm{d}y\\
\text{(5.4.24)}\qquad &\qquad\le\frac12\fint_Q\int_{Q'}\operatorname{dist}_N(u_i(x),\mathcal{E}_{k_n}(u_i)(Q'))\,\mathrm{d}x\,\mathrm{d}y\\
&\qquad=\frac12\int_{Q'}\operatorname{dist}_N(u_i(x),\mathcal{E}_{k_n}(u_i)(Q'))\,\mathrm{d}x
\end{aligned}
$$

and this will be estimated below similarly to (5.4.18), see (5.4.26).

So, we focus on the term (5.4.20). Since Q' is a neighbour of Q, we first observe $Q \times Q' \subset Q \times 3Q$ so that (5.4.20) is controlled by

$$
\begin{aligned}
&2^{(1-k_n)(d-1)}(2^{-k_n} - 2^{-k_{n+1}})L^d \times \\
&\quad \fint_Q \fint_{Q'} \operatorname{dist}_N(u_i(x), u_i(y))\,\mathrm{d}x\,\mathrm{d}y \\
&\le 2^{(1-k_n)(d-1)}(2^{-k_n} - 2^{-k_{n+1,i}})L^d|Q|^{-2} \int_Q \int_{3Q} \operatorname{dist}_N(u_i(x), u_i(y))\,\mathrm{d}x\,\mathrm{d}y \\
&\le 2^{(1-k_n)(d-1)}2^d(2^{-k_n} - 2^{-k_{n+1}})(2^{1-k_n})^{-d}L^{-d}\times \\
&\quad \int_Q \int_{3Q} \operatorname{dist}_N(u_i(x), u_i(y))\,\mathrm{d}x\,\mathrm{d}y \\
&\le 3^d \cdot 2^{k_n}(2^{-k_n} - 2^{-k_{n+1}})L^{-d}\times \\
&\quad \int_Q \fint_{|h|_\infty \le 3\cdot 2^{1-k_n}L} \operatorname{dist}_N(u_i(x), u_i(x+h))\,\mathrm{d}h\,\mathrm{d}x
\end{aligned}
\tag{5.4.25}
$$

by the change of variable $y = x + h$, $|h|_\infty \le |x - y|_\infty \le \operatorname{diam}(3Q) \le 2 \cdot 3 \cdot 2^{-k_n}L$. So, the term (iii) is estimated in view of (5.4.24) and (5.4.25) by

$$
\begin{aligned}
&2d \sum_{i\in\{0,1\}} \sum_{n\in\mathbb{N}} \int_{\mathbb{R}^d} \operatorname{dist}_N(u_i(x), \mathcal{E}_{k_n}(u_i)(x))\,\mathrm{d}x \\
&\qquad + 2d3^d \sum_{i\in\{0,1\}} \sum_{n\in\mathbb{N}} (2^{-k_n} - 2^{-k_{n+1}})2^{k_n}\times \\
&\qquad\qquad \int_{\mathbb{R}^d} \fint_{|h|_\infty \le 3\cdot 2^{1-k_n}L} \operatorname{dist}_N(u_i(x), u_i(x+h))\,\mathrm{d}h\,\mathrm{d}x
\end{aligned}
\tag{5.4.26}
$$

where the $2d$ arises as the sum over the $2d$ neighbors of the cube $Q \in \mathcal{Q}_{k_n}$.

Using (5.4.11) for the first term and (5.4.13) for the second one when $n \ge 1$, we control (5.4.26) by

$$
\begin{aligned}
&2d \sum_{i\in\{0,1\}} \sum_{n\in\mathbb{N}} \frac{\Gamma}{2^n} + 2d \sum_{i\in\{0,1\}} \sum_{n\in\mathbb{N}_*} (2^{-k_n} - 2^{-k_{n+1}})2^{k_n} \frac{\Gamma}{2^{k_{n-1}}} \\
&\qquad + 2d \sum_{i\in\{0,1\}} \int_{\mathbb{R}^d} \fint_{|h|_\infty \le 6L} \operatorname{dist}_N(u_i(x), u_i(x+h))\,\mathrm{d}h\,\mathrm{d}x.
\end{aligned}
\tag{5.4.27}
$$

The first term of (5.4.27) is equal to $8d\Gamma$. The second one can be controlled by

$$
\begin{aligned}
&2d\Gamma \sum_{i\in\{0,1\}} \sum_{n\in\mathbb{N}_*} \left(\frac{1}{2^{k_{n-1}}} - \frac{2^{k_n}}{2^{k_{n-1}}2^{k_{n+1}}}\right)\\
&\qquad\le 2d\Gamma \sum_{i\in\{0,1\}} \sum_{n\in\mathbb{N}_*} \left(\frac{1}{2^{k_{n-1}}} - \frac{1}{2^{k_{n+1}}}\right)\\
&\qquad\le 4d\Gamma \sum_{i\in\{0,1\}} \frac{1}{2^{k_0}} + \frac{1}{2^{k_1}} \le 3d\Gamma.
\end{aligned}
\tag{5.4.28}
$$

The last term of (5.4.27) is estimated by

$$
\frac{d}{L^d} \sum_{i\in\{0,1\}} \iint_{\substack{\mathbb{R}^d\times\mathbb{R}^d\\ |x-y|_\infty\le 6L}} \operatorname{dist}_N(u_i(x),u_i(y))\,\mathrm{d}x\,\mathrm{d}y.
$$

We deduce that (5.4.27) is estimated by

$$
(8d+3d)\Gamma + \frac{d}{L^d} \sum_{i\in\{0,1\}} \iint_{\substack{\mathbb{R}^d\times\mathbb{R}^d\\ |x-y|_\infty\le 6L}} \operatorname{dist}_N(u_i(x),u_i(y))\,\mathrm{d}x\,\mathrm{d}y
\tag{5.4.29}
$$

so that in turn (iii) is controlled by it.

We conclude that (5.4.6) is controlled by (i), (ii) and (iii) which are respectively controlled by (5.4.17), (5.4.19) and (5.4.29)

$$
\begin{aligned}
&C_d\Gamma + \int_{\mathbb{R}^d} \operatorname{dist}_N(u_0(x),u_1(x))\,\mathrm{d}x\\
&\qquad\qquad + \frac{C_d}{L^d} \sum_{i\in\{0,1\}} \iint_{|x-y|_\infty\le 6L} \operatorname{dist}_N(u_i(x),u_i(y))\,\mathrm{d}x\,\mathrm{d}y
\end{aligned}
$$

by the choice of Γ in (5.4.10), this concludes the proof of the estimate (5.4.6) of the Proposition 5.4.1 but with "$|x-y|_\infty \le 6L$"; Lemma 5.4.2 below allows us to write "$|x-y| \le L$" at the price of increasing the constant C_d. □

Inspection of the proof of Proposition 5.4.1 shows that one could have chosen if N was convex

$$
\mathcal{E}_k(u_i)(Q) = \fint_Q u_i.
$$

However our present choice of $\mathcal{E}_k(u_i)(Q)$ implies that the essential images $\operatorname{Im}(U) \subset \operatorname{Im}(u_0) \cup \operatorname{Im}(u_1) \subset N$.

Lemma 5.4.2. *Let $u : \mathbb{R}^d \to N$. If $0 < \ell \leq L$,*

$$\frac{1}{L^{d+1}} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \leq L}} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}x \,\mathrm{d}y \leq \frac{2^{d+1}}{\ell^{d+1}} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \leq \ell}} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}x \,\mathrm{d}y. \tag{5.4.30}$$

Proof of Lemma 5.4.2. We have, by the triangle inequality and a change of variable

$$\iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \leq L}} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}x \,\mathrm{d}y \tag{5.4.31}$$

$$\leq \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \leq L}} \mathrm{d}x \,\mathrm{d}y \times \tag{5.4.32}$$

$$\operatorname{dist}_N(u(x), u(\tfrac{x+y}{2})) + \operatorname{dist}_N(u(\tfrac{x+y}{2}), u(y)) \tag{5.4.33}$$

$$\leq 2^{1+d} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-z| \leq L/2}} \operatorname{dist}_N(u(x), u(z)) \,\mathrm{d}x \,\mathrm{d}z. \tag{5.4.34}$$

By mathematical induction, for each $n \in \mathbb{N}$,

$$\iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \leq L}} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}x \,\mathrm{d}y \leq 2^{n(d+1)} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \leq L/2^n}} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}x \,\mathrm{d}y.$$

Given $\ell \leq L$, there exists $n \in \mathbb{N}$ such that $L \leq 2^n \ell \leq 2L$. We deduce (5.4.30). $\square$

5.5 Connecting two maps by a Sobolev map

Proposition 5.5.1 smooths the map obtained by proposition 5.4.1. So does Proposition 5.5.3 in a localized way.

Proposition 5.5.1. *If $u_0, u_1 : \mathbb{R}^d \to N$ satisfy* (5.4.1) *and* (5.4.2), *then, for each $L > 0$, there exists $U \in \dot{\mathrm{W}}^{1,1}(\mathbb{R}^d \times (-L, L), N)$ such that $\operatorname{tr}_{\mathbb{R}^d \times \{-L\}} U = u_0$, $\operatorname{tr}_{\mathbb{R}^d \times \{L\}} U = u_1$ and*

$$\int_{\mathbb{R}^d \times (-L,L)} |\mathrm{D}U| \leq C_d \int_{\mathbb{R}^d} \operatorname{dist}_N(u_0(x), u_1(x)) \,\mathrm{d}x + \frac{C_d}{L^d} \sum_{i \in \{0,1\}} \iint_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \leq L}} \operatorname{dist}_N(u_i(x), u_i(y)) \,\mathrm{d}x \,\mathrm{d}y \tag{5.5.1}$$

where $C_d > 0$ only depends on d.

Remark 5.5.2. In [13, Thm 1.8], in the framework of

$$N = \mathbb{R}^\nu, \operatorname{dist}_N(p,q) = |p-q|$$

the same result is obtained by other means. The authors assume the integrability condition (5.4.2); the equivalent (see Proposition 5.2.7) condition than (5.4.1), for $i \in \{0,1\}$,

$$\lim_{\epsilon \to 0} \sup_{|h| \le \epsilon} \int_{\mathbb{R}^d} |u_i(x) - u_i(x+h)| \,\mathrm{d}x = 0,$$

is required there; corresponding estimates are obtained.

Proposition 5.5.3. *Let $Q = Q(0,1) \subset \mathbb{R}^{d+1}$ be the unit cube of radius one centered at the origin. If $u : \partial Q \to N$ is integrable and equals to $b \in N$ on each face except two opposed faces, then there exists a map $U \in \dot{\mathrm{W}}^{1,1}(Q,N)$ of trace $\operatorname{tr}_{\partial Q} U = u$ and*

$$\int_Q |\mathrm{D}U| \le C_d \int_{\partial Q} \operatorname{dist}_N(u(x), b) \,\mathrm{d}x$$

where $C_d > 0$ is a constant depending only on d.

We write an interpolating tool for the $\dot{\mathrm{W}}^{1,1}$-extension, Proposition 5.5.1.

Lemma 5.5.4. *If $P = \prod_{i=1}^d [0, |\ell_i|]$ and $a, b \in N$, there exists $u \in \dot{\mathrm{W}}^{1,1}(P \times \mathbb{R}, N)$ satisfying*

(i) If $t < -\operatorname{dist}(x, \partial P)/2$, $u(x,t) = a$ and if $t > \operatorname{dist}(x, \partial P)/2$, $u(x,t) = b$,

(ii) $\operatorname{dist}_N(a, u(x,t)) \le \operatorname{dist}_N(a,b)$ and $\operatorname{dist}_N(b, u(x,t)) \le \operatorname{dist}_N(a,b)$.

Moreover,

$$\int_{P \times \mathbb{R}} |\mathrm{D}u| \le 4 \operatorname{dist}_N(a,b) \mathcal{L}^d(P) \tag{5.5.2}$$

and

$$\sup_{t \in \mathbb{R}} \int_P \operatorname{dist}_N(a \mathbb{1}_{(-\infty,0)}(t) + b \mathbb{1}_{[0,+\infty)}(t), u(x,t)) \,\mathrm{d}x \le \operatorname{dist}_N(a,b) \mathcal{L}^d(P). \tag{5.5.3}$$

The proof follows the argument for $\mathrm{W}^{1,p}$ with $1 < p < 2$ [21].

Proof of Lemma 5.5.4. Since d is a geodesic distance, we can choose $\gamma \in C^\infty(\mathbb{R}, N)$ to be a mapping such that $\gamma(t) = a$ if $t \le -1$, $\gamma(t) = b$ if $t \ge 1$ and $|\gamma'(t)| \le \operatorname{dist}_N(a, b)$ if $-1 \le t \le 1$, and $\gamma(t) = b$ if $t \ge 1$. We define $u : P \times \mathbb{R} \to N$ for every $(x, t) \in P \times \mathbb{R}$ by

$$u(t, x) = \gamma\left(\frac{2t}{\operatorname{dist}(x, \partial P)}\right).$$

By definition, we have (i) and (ii). Then, we record that $u \in \mathrm{W}^{1,1}_{\mathrm{loc}}(P \times \mathbb{R}, N)$, with for $(t, x) \in \mathbb{R} \times P$:

$$|\mathrm{D}u(x, t)| \le \left|\gamma'\left(\frac{2t}{\operatorname{dist}(x, \partial P)}\right)\right| \left(\frac{|t|}{\operatorname{dist}(x, \partial P)^2} + \frac{1}{\operatorname{dist}(x, \partial P)}\right).$$

Defining the set $\Sigma = \{(x, t) \in P \times \mathbb{R} : 2|t| \le \operatorname{dist}(x, \partial P)\}$, we integrate and estimate

$$\begin{aligned}
\int_{P \times \mathbb{R}} |\mathrm{D}u| &\le \operatorname{dist}_N(a, b) \iint_\Sigma \frac{|t|}{\operatorname{dist}(x, \partial P)^2} + \frac{1}{\operatorname{dist}(x, \partial P)} \,\mathrm{d}x \,\mathrm{d}t \\
&\le 2 \operatorname{dist}_N(a, b) \iint_\Sigma \frac{1}{\operatorname{dist}(x, \partial P)} \,\mathrm{d}x \,\mathrm{d}t = 4 \operatorname{dist}_N(a, b) \int_P \mathrm{d}x \\
&= 4 \prod_{i=1}^d |\ell_i| \operatorname{dist}_N(a, b).
\end{aligned}$$

which establish (5.5.2). For (5.5.3), we assume $t > 0$

$$\begin{aligned}
&\int_P \operatorname{dist}_N(a \mathbb{1}_{[-\infty, 0)}(t) + b \mathbb{1}_{[0, +\infty)}(t), u(x, t)) \,\mathrm{d}x \\
&\qquad \le \int_{\Sigma(t)} \operatorname{dist}\left(b, \gamma\left(\frac{2t}{\operatorname{dist}(x, \partial P)}\right)\right) \mathrm{d}x \\
&\le \operatorname{dist}_N(a, b) \int_{\Sigma(t)} 1 - \frac{2t}{\operatorname{dist}(x, \partial P)} \,\mathrm{d}x \\
&\qquad \le \operatorname{dist}_N(a, b) \mathcal{L}^d(\Sigma(t) \cap P) \le \operatorname{dist}_N(a, b) \prod_{i=1}^d |\ell_i|
\end{aligned}$$

and similarly for $t \le 0$. ☐

Lemma 5.5.5. *If $P = \prod_{i=1}^d [0, |\ell_i|]$ and $a, b \in N$, there exists $u \in \dot{\mathrm{W}}^{1,1}(P \times (0, \infty), N)$ satisfying*

(i) If $t > \operatorname{dist}(x, \partial P)/2$, $u(x, t) = b$.

(ii) $\operatorname{dist}_N(a, u(x, t)) \le \operatorname{dist}_N(a, b)$ and $\operatorname{dist}_N(b, u(x, t)) \le \operatorname{dist}_N(a, b)$.

(iii) $\operatorname{tr}_{P\times\{0\}} u = a$.

Moreover,

$$\int_{P\times\mathbb{R}} |\mathrm{D}u| \leq 8 \operatorname{dist}_N(a,b)\mathcal{L}^d(P) \tag{5.5.4}$$

and

$$\sup_{t>0} \int_P \operatorname{dist}_N(b, u(x,t))\,\mathrm{d}x \leq 2 \operatorname{dist}_N(a,b)\mathcal{L}^d(P). \tag{5.5.5}$$

Proof of Lemma 5.5.5. Take $\gamma \in C^\infty(\mathbb{R}, \mathcal{N})$ to be a mapping such that $\gamma(0) = a$, $|\gamma'(t)| \leq 2 \operatorname{dist}_N(b,a)$ and $\gamma(t) = b, t \geq 1$. We proceed as in the proof of Lemma 5.5.4. $\square$

Proof of Proposition 5.5.1. By Proposition 5.4.1, there exists a map $U_* : \mathbb{R}^d \times (-L, L) \to N$ constant on a countable family of rectangular cuboïds.

By Lemma 5.5.4 applied to each faces *i.e.* element of

$$\big\{\partial Q \times I_{n,i} \cap \partial Q \times I_{i',n'} : \\ i, i' \in \{0,1\}, n, n' \in \mathbb{N}, Q \in \mathcal{Q}_{k_n}, Q' \in \mathcal{Q}_{k_{n'}} \text{ s.t } \partial Q \times I_{n,i} \neq \partial Q \times I_{n',i'}\big\}$$

we obtain a map $U : \mathbb{R}^d \to N$. First, by Lemma 5.5.4 (i) this is well defined: the support of the modifications is disjoint. By Lemma 5.5.4(5.5.2) we get

$$\int_{\mathbb{R}^d\times(-L,L)} |\mathrm{D}U|\,\mathrm{d}\mathcal{L}^{d+1} \leq 4 \int_{\mathrm{J}_{U_*}\cap\mathbb{R}^d\times(-L,L)} \operatorname{dist}_N(\operatorname{tr}^+_{\mathrm{J}_{U_*}} U_*, \operatorname{tr}^-_{\mathrm{J}_{U_*}} U_*)\,\mathrm{d}\mathcal{H}^d.$$

By the choice of U_* and in particular Proposition 5.5.1(5.4.6) we get the estimate (5.5.1).

The estimate (5.5.3) will only be used to the faces perpendicular $\mathcal{F}^\perp$ to $\mathbb{R}^d \times \{0\}$. If for $i \in \{0,1\}, n \in \mathbb{N}$, $I_{n,i} = (a,b)$, we set $t_{i,n} = (a+b)/2$. The choice of $t \in \mathbb{R}$ ensures that the plane $\mathbb{R}^d \times \{t\}$ does not intersect the support of the modification of the faces parallel to $\mathbb{R}^d \times \{0\}$. The choice of t allows us to ensure that $\mathbb{R}^d \times \{t\}$ only crosses points in $\{x \in \mathbb{R}^d \times (-L,L) : U(x) \neq \tilde{U}(x)\}$ arising from modification of the perpendicular faces by Lemma 5.5.4. The estimate (5.5.3) implies

$$\begin{aligned} \int_{\{x\in\mathbb{R}^d\,:\,U(x)\neq\tilde{U}(x)\}} & \operatorname{dist}_N(\operatorname{tr}_{\mathbb{R}^d\times\{t_{i,n}\}} U(x), \operatorname{tr}_{\mathbb{R}^d\times\{t_{i,n}\}} U_*(x))\,\mathrm{d}x \\ & \leq \int_{\mathrm{J}_{U_*}} \operatorname{dist}_N(\operatorname{tr}^+_{\mathrm{J}} U_*, \operatorname{tr}^-_{\mathrm{J}} U_*)\,\mathrm{d}\mathcal{H}^d. \end{aligned} \tag{5.5.6}$$

By the triangle inequality, (5.5.6) and Proposition 5.4.1(5.4.5), for all $n \in \mathbb{N}$, $i \in \{0, 1\}$,

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(u_i(x), \operatorname{tr}_{\mathbb{R}^d \times \{t_{i,n}\}} U(x)) \, \mathrm{d}x \le 2^{-n} C(d, u_0, u_1, L) \tag{5.5.7}$$

where $C(d, u_0, u_1, L) > 0$ only depends on d, u_0, u_1 and L. By continuity of the trace under translations, since $t_{i,n} \to (-1)^{i+1} L$ when $n \to +\infty$, we conclude that $\operatorname{tr}_{\mathbb{R}^d \times \{-L\}} U = u_0$ and $\operatorname{tr}_{\mathbb{R}^d \times \{L\}} U = u_1$. □

The proof of Proposition 5.5.3 is similar to Proposition 5.5.1 and uses the Lemma 5.5.5 to handle the values on the map on the boundary of the cube.

Proof of Proposition 5.5.3. We assume that u is not constant on the faces $\partial Q \setminus (\mathbb{R}^d \times \{-1, 1\})$. We define $v_0 \in \mathrm{L}^1(\mathbb{R}^d, N)$ by $v_0(x) = u(x)$, $x \in \partial Q \setminus (\mathbb{R}^d \times \{-1\})$ and $v_0(x) = p$ elsewhere. We define $v_1 \in \mathrm{L}^1(\mathbb{R}^d, N)$ by $v_1(x) = u(x)$, $x \in \partial Q \setminus (\mathbb{R}^d \times \{1\})$ and $v_1(x) = p$ elsewhere. By Proposition 5.4.1 there exists $V \in \dot{\mathrm{B}}\mathrm{V}(\mathbb{R}^d \times (-1, 1), N)$ such that $\operatorname{tr}_{\mathbb{R}^d \times \{-1\}} V = v_0$ and $\operatorname{tr}_{\mathbb{R}^d \times \{1\}} = v_1$ satisfying $V|_{\mathbb{R}^d \times (-1,1) \setminus Q} = p$ and by (5.4.6) and the triangle inequality

$$\int_{\mathbb{R}^d \times (-1,1)} \operatorname{dist}_N(\operatorname{tr}^+_{\mathrm{J}_V} V, \operatorname{tr}^-_{\mathrm{J}_V} V) \, \mathrm{d}\mathcal{H}^d \le C_d \int_{\partial Q} \operatorname{dist}_N(u(x), b) \, \mathrm{d}x.$$

Considering $V|_Q$, we then argue as in the proof of Proposition 5.5.1 using Lemma 5.5.5 instead of Lemma 5.5.4 for faces of cuboïds adjacent to $\partial Q \setminus (\mathbb{R}^d \times \{-1, +1\})$. □

5.6 Extension to a halfspace and to a manifold

Based on Proposition 5.5.3, we prove the extension part of Theorem 5.1.2.

Proposition 5.6.1. *If $u : \mathbb{R}^d \to N$ is measurable and the right-hand side of (5.6.1) is finite, then there exists $U \in \dot{\mathrm{W}}^{1,1}(\mathbb{R}^d \times (0, \infty), N)$ such that $\operatorname{tr}_{\mathbb{R}^d \times \{0\}} U = u$ and*

$$\int_{\mathbb{R}^d \times (0,\infty)} |\mathrm{D}U| \le C_d \liminf_{R \to +\infty} \iint\limits_{\substack{\mathbb{R}^d \times \mathbb{R}^d \\ |x-y| \le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0, R))} \, \mathrm{d}x \, \mathrm{d}y, \tag{5.6.1}$$

where $C_d > 0$ only depends on d.

Proof. Since u is integrable by Proposition 5.2.3, there exists b such that

$$\int_{\mathbb{R}^d} \operatorname{dist}_N(u(x), b) \,\mathrm{d}x \le C_d \liminf_{R\to\infty} \iint\limits_{\substack{\mathbb{R}^d\times\mathbb{R}^d \\ |x-y|\le R}} \frac{\operatorname{dist}_N(u(x), u(y))}{\mathcal{L}^d(\mathbb{B}^d(0,R))} \,\mathrm{d}x \,\mathrm{d}y < \infty.$$

We set $u_0 = u$ and $u_1 = b$ and we apply Proposition 5.5.3. □

We finally construct the extension in Theorem 5.1.1.

Proposition 5.6.2. *Let M be a Riemannian manifold with compact boundary ∂M. If $u : \partial M \to N$ is integrable, there exists $U \in \dot{\mathrm{W}}^{1,1}(M, N)$ satisfying $\operatorname{tr}_{\partial M} U = u$ and*

$$\int_M |\mathrm{D}U| \le C_M \iint_{\partial M\times\partial M} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}x$$

where $C_M > 0$ only depends on M.

The integration on M and on ∂M are performed with respect to the measure induced by their Riemannian metric or, equivalently, by the Hausdorff measure, and can be computed in local charts thanks to the introduction of the appropriate Jacobian.

Proof of Proposition 5.6.2. Let m denote the dimension of the manifold M. By smoothness and compactness of the boundary ∂M, we realize ∂M as a simplicial complex [24, Theorem 7] $S_{\partial M} = \bigcup_{n=0}^{m-1} S_n$ of dimension $m-1$ with S_n containing the cells of dimension n; a cell $C \in S_n$ is bi-Lipschitz homeomorphic to a cube $Q(0,1) \subset \mathbb{R}^n$. Let $V = \{x \in M \cup \partial M : \operatorname{dist}_M(x, \partial M) \le \rho\}$ for $\rho > 0$ sufficiently small so that there exists a diffeomorphism $\Psi : V \to \partial M \times [0,1]$. We realize V as a finite simplicial complex S_M of dimension m, such that $S_{\partial M}$ is the boundary of S_M meaning that for a k-dimension cell C in S_M, $C \cap \partial M$ is a $(k-1)$-dimensional cell in S_{k-1} ; this can be done by taking $C \in S_{\partial M}$ and defining $C' = C \times [0,1] \subset \partial M \times [0,1]$ and defining the cell in V by $\Psi^{-1}(C')$. Fix $b \in N$ so that

$$\int_{\partial M} \operatorname{dist}_N(u(x), b) \,\mathrm{d}x \le \frac{1}{\mathcal{H}^d(\partial M)} \iint_{\partial M\times\partial M} \operatorname{dist}_N(u(x), u(y)) \,\mathrm{d}x \,\mathrm{d}y$$

If $C \in S_{m-1}$, on $\partial C' \setminus C \times \{0\}$ we extend the domain of definition of u by setting $u \circ \Psi^{-1} = p$. Therefore C' is a cell such that $\operatorname{tr}_{\partial C'} u \in \mathrm{L}^1(\partial C', N)$ and if two cells C' are adjacent they share the same constant on their boundary. Using the bi-Lipschitz equivalence of cells with cubes, the situation reduces to extend a map $u \in \mathrm{L}^1(\partial Q, N)$, $Q = Q(0,1) \subset \mathbb{R}^m$

equals to p on each face exept one to a map $U \in \dot{\mathrm{W}}^{1,1}(Q, N)$ of trace $\mathrm{tr}_{\partial Q} U = u$, which is done by Proposition 5.5.3 applied to each cube. We set $U = p$ on $M \setminus V$. Since the same trace is shared on the faces of the m-dimensional cells in V, we obtain, by integration by part (*cfr* [12, Theorem 18.1(ii)][21]), a map $U \in \dot{\mathrm{W}}^{1,1}(M, N)$ such that $\mathrm{tr}_{\partial M} U = u$ such that

$$\begin{aligned}
\int_M |\mathrm{D}U| = \int_V |\mathrm{D}U| &= \sum_{\substack{C' = C \times [0,1] \\ C \in S_{m-1}}} \int_{C'} |\mathrm{D}U| \\
&\le C_M \sum_{\substack{C' = C \times [0,1] \\ C \in S_{m-1}}} \int_{\partial C'} \mathrm{dist}_N(u(x), b) \\
&\le C_M \sum_{C \in S_{m-1}} \int_C \mathrm{dist}_N(u(x), b) \\
&= C_M \int_{\partial M} \mathrm{dist}_N(u(x), b)
\end{aligned}$$

where $C_M > 0$ depends on the Lipschitz constant of Ψ and absorbs the constant arising from Proposition 5.5.3, which concludes the proof. $\square$

Bibliography of Chapter 5

[1] L. Ambrosio, N. Fusco, and D. Pallara. *Functions of bounded variation and free discontinuity problems.* Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000, pp. xviii+434.

[2] G. Anzellotti and M. Giaquinta. "Funzioni BV e tracce". In: *Rend. Semin. Mat. Univ. Padova* 60 (1978), pp. 1–21.

[3] F. Bethuel. "A new obstruction to the extension problem for Sobolev maps between manifolds". In: *J. Fixed Point Theory Appl.* 15.1 (2014), pp. 155–183.

[4] F. Bethuel and F. Demengel. "Extensions for Sobolev mappings between manifolds". In: *Calc. Var. Partial Differential Equations* 3.4 (1995), pp. 475–491.

[5] J. Bourgain, H. Brezis, and P. Mironescu. "Another look at Sobolev spaces". In: *Optimal control and partial differential equations. In honour of Professor Alain Bensoussan's 60th birthday. Proceedings of the conference, Paris, France, December 4, 2000.* Amsterdam: IOS Press; Tokyo: Ohmsha, 2001, pp. 439–455.

[6] H. Brezis. "How to recognize constant functions. Connections with Sobolev spaces". In: *Russ. Math. Surv.* 57.4 (2002), pp. 693–708.

[7] O. Domínguez and M. Milman. "Bourgain-Brezis-Mironescu-Maz′ya-Shaposhnikova limit formulae for fractional Sobolev spaces via interpolation and extrapolation". In: *Calc. Var. Partial Differential Equations* 62.2 (2023), Paper No. 43, 37.

[8] E. Gagliardo. "Caratterizzazioni delle tracce sulla frontiera relative ad alcune classi di funzioni in n variabili". In: *Rend. Sem. Mat. Univ. Padova* 27 (1957), pp. 284–305.

[9] Q. Gu and P.-L. Yung. "A new formula for the L^p norm". In: *J. Funct. Anal.* 281.4 (2021), Paper No. 109075, 19.

[10] R. Hardt and F. Lin. “Mappings minimizing the L^p norm of the gradient”. In: *Commun. Pure Appl. Math.* 40.5 (1987), pp. 555–588.

[11] J. M. Lee. *Introduction to Riemannian manifolds.* 2nd edition. Vol. 176. Grad. Texts Math. Cham: Springer, 2018.

[12] G. Leoni. *A first course in Sobolev spaces.* 2nd edition. Vol. 181. Grad. Stud. Math. Providence, RI: American Mathematical Society (AMS), 2017.

[13] G. Leoni and I. Tice. “Traces for homogeneous Sobolev spaces in infinite strip-like domains”. In: *J. Funct. Anal.* 277.7 (2019), pp. 2288–2380.

[14] V. Maz′ ya and T. Shaposhnikova. “On the Bourgain, Brezis, and Mironescu theorem concerning limiting embeddings of fractional Sobolev spaces”. In: *J. Funct. Anal.* 195.2 (2002), pp. 230–238.

[15] P. Mironescu. “Note on Gagliardo’s theorem “$\mathrm{tr}\mathrm{W}^{1,1} = \mathrm{L}^1$””. In: *Ann. Univ. Buchar. Math. Ser.* 6(LXIV).1 (2015), pp. 99–103.

[16] P. Mironescu and J. Van Schaftingen. “Trace theory for Sobolev mappings into a manifold”. In: *Ann. Fac. Sci. Toulouse, Math. (6)* 30.2 (2021), pp. 281–299.

[17] O. Müller. “A note on closed isometric embeddings”. In: *J. Math. Anal. Appl.* 349.1 (2009), pp. 297–298.

[18] J. Nash. “The imbedding problem for Riemannian manifolds”. In: *Ann. of Math. (2)* 63 (1956), pp. 20–63.

[19] J. Peetre. “A counter-example connected with Gagliardo’s trace theorem”. In: *Commentat. Math.* (1979), pp. 277–282.

[20] A. Pełczyński and M. Wojciechowski. “Sobolev spaces in several variables in L^1-type norms are not isomorphic to Banach lattices”. In: *Ark. Mat.* 40.2 (2002), pp. 363–382.

[21] J. Van Schaftingen. *Lectures on Sobolev mappings.* to appear.

[22] J. Van Schaftingen. *The extension of traces for Sobolev mappings between manifolds.* 2024.

[23] J. Van Schaftingen and B. Van Vaerenbergh. *Extensions of traces for Sobolev mappings into manifolds at the endpoint $p = 1$.* 2025.

[24] J. H. C. Whitehead. “On C^1-Complexes”. In: *Annals of Mathematics* 41.4 (1940), pp. 809–824.

Contents

Gratitude

Gratitude envers mes pairs et maîtres en mathématiques !

Privilège immense d'avoir été ton étudiant, Jean. Je te suis reconnaissant pour ton dévouement, tes conseils et ton soutien. Ordinaire est un épithète qui sied peu à celui qui a une si grande culture mathématique et des vues si profondes sur les problèmes de notre temps.

Reconnaissance ! aux artisans des mathématiques croisés sur mon parcours universitaire. En particulier, à ces professeurs et assistants qui m'ont formé à la pensée mathématique pendant ces 5+4 ans à l'UCLouvain. Je suis également l'obligé de chaque membre du jury pour leur temps et leur expertise. Je salue ici en particulier le professeur Augusto Ponce pour, notamment, avoir, il y a 6 ans, attiré pour la première fois mon attention sur les applications de Sobolev à valeur variété.

Mes camarades des mathématiques de Belgique ou d'ailleurs, je vous dis le plus chaleureux merci. Chers collègues de cours, je mesure la chance que j'ai eue de vous connaître et la bienveillante entraide qui nous a animé. Chers collègues doctorants, la thèse est parfois une épreuve et la camaraderie ambiante de la tour b m'a porté pendant ces quatre ans. Leon, colocataire du bureau b.325, je te remercie du fond du cœur d'avoir pu célébrer avec toi chaque instant de cette épreuve-thèse.

Chers amis, merci de m'avoir soutenu dans l'adversité et élevé par nos discussions.

Ma famille et en particulier Joyce, merci d'avoir été le roc sur lequel j'ai pu bâtir cette thèse, merci pour votre soutien et votre amour.

Que restera-t-il des espaces de Sobolev à valeur variété dans 1000 ans, que restera-t-il des applications p-harmoniques quand p tend vers 2 dans 3000 ans ? Souhaitons ici que l'humanité puisse toujours s'occuper de mathématiques.