

VALUE-AT-RISK FOR LONG AND SHORT TRADING POSITIONS

Pierre Giot^{1,3} and Sébastien Laurent²

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Abstract

In this paper we model Value-at-Risk (VaR) for daily stock index returns using a collection of parametric models of the ARCH family based on the skewed Student distribution. We show that models that rely on a symmetric density distribution for the error term underperform with respect to skewed density models when the left and right tails of the distribution of returns must be modelled. Thus, VaR for traders having both long *and short* positions is not adequately modelled using usual Normal or Student distributions. We suggest using an APARCH model based on the skewed Student distribution to fully take into account the fat left and right tails of the returns distribution. This allows for an adequate modelling of large returns defined on long and short trading positions. The performances of all models are assessed on daily data for the CAC40, DAX, NASDAQ, NIKKEI and SMI stock indexes. We also compute the expected short-fall and the average multiple of tail event to risk measure for the new model.

Keywords: Value-at-Risk, Expected short-fall, Skewed Student distribution, APARCH, short trading

JEL classification: C52, C53, G15

¹Department of Quantitative Economics, Maastricht University and Center for Operations Research and Econometrics, UCL; email: giot@core.ucl.ac.be or p.giot@ke.unimaas.nl

²Département des Sciences Economiques, Université de Liège. This research was done when S. Laurent was visiting the Department of Quantitative Economics at Maastricht University. email: S.Laurent@ulg.ac.be

³Corresponding author

1 Introduction

In recent years, the tremendous growth of trading activity and the well-publicized trading loss of well known financial institutions (see Jorion, 2000, for a brief history of these events) has led financial regulators and supervisory committee of banks to favor quantitative techniques which appraise the possible loss that these institutions can incur. Value-at-Risk has become one of the most sought-after techniques as it provides a simple answer to the following question: with a given probability (say α), what is my predicted financial loss over a given time horizon? The answer is the VaR at level α , which gives an amount in the currency of the traded assets (in dollar terms for example) and is thus easily understandable.

It turns out that the VaR has a simple statistical definition: the VaR at level α for a sample of returns is defined as the corresponding empirical quantile at $\alpha\%$. Because of the definition of the quantile, we have that, with probability $1 - \alpha$, the returns will be larger than the VaR. In other words, with probability $1 - \alpha$, the losses will be smaller than the dollar amount given by the VaR.¹ From an empirical point of view, the computation of the VaR for a collection of returns thus requires the computation of the empirical quantile at level α of the distribution of the returns of the portfolio.

Most models in the literature focus on the computation of the VaR for negative returns (see van den Goorbergh and Vlaar, 1999 or Jorion, 2000). Indeed, it is assumed that traders or portfolio managers have long trading positions, i.e. they bought the traded asset and are concerned when the price of the asset falls. In this paper we focus on modelling VaR for portfolios defined on long *and short* trading positions. Thus we model VaR for traders having either bought the asset (long position) or short-sold it (short position).² In the first case, the risk comes from a drop in the price of the asset, while the trader loses money when the price increases in the second case (because he would have to buy back the asset at a higher price than the one he got when he sold it). Correspondingly, one focuses in the first case on the left side of the distribution of returns, and on the right side of the distribution in the second case.

Because the distribution of returns is often not symmetric (see Section 3), we show that ‘usual’ parametric VaR models of the RiskMetrics and ARCH class have a tough job in modelling correctly the left and right tails of the distribution of returns. This is also true for the so-called asymmetric GARCH models where the asymmetry refers to the relationship between the conditional variance and the lagged squared error term. Indeed, as pointed out by El Babsiri and Zakoian (1999), although such asymmetric GARCH models allow positive and negative changes to have different

¹Contrary to some wide-spread beliefs, the VaR does not specify the maximum amount that can be lost.

²An asset is short-sold by a trader when it is first borrowed and subsequently sold on the market. By doing this, the trader hopes that the price will fall, so that he can then buy the asset at a lower price and give it back to the lender. See Sharpe, Alexander, and Bailey (1999) for general information on trading procedures.

impacts on future volatilities, the two components of the innovation have - up to a constant - the same volatilities, while it is desirable to allow an asymmetric confidence interval around the predicted volatility in the VaR application.

To alleviate these problems, we introduce in this paper a skewed Student Asymmetric Power ARCH (APARCH) model (Ding, Granger, and Engle, 1993) to model the VaR for portfolios defined on long (long VaR) and short (short VaR) trading positions. We compare the performance of this new model with the ones of the RiskMetrics, normal and Student APARCH models and show that the new model brings about considerable improvements in correctly forecasting one-day-ahead VaR for long and short trading positions on daily stock indexes (French CAC40, German DAX, US NASDAQ, Japanese NIKKEI and Swiss SMI data). For the skewed Student APARCH model, we also compute the expected short-fall and the average multiple of tail event to risk measure as these two measures supplement the information given by the empirical failure rates.

While we focus exclusively on parametric models, other approaches are possible, such as Danielsson and de Vries (2000) who combine an historical simulation method (i.e. non parametric technique) for the interior of the distribution of returns with a fitted distribution based on extreme value theory for the most extreme returns. In this setting, normal and extreme events are thus modelled using two different methods. With the skewed Student APARCH model we aim to model left and right tail VaRs with a single parametric method for a wide range of values for α .

Recently, Mittnik and Paoletta (2000) have introduced an APARCH model coupled with an asymmetric generalized Student distribution to model VaR for negative returns. While the analysis in their paper is sometimes similar to ours, there are some significant differences. First, we focus on the joint behavior of VaR models for long *and short* trading positions, i.e. we look at both how large negative and positive returns are taken into account by the model (Mittnik and Paoletta, 2000, focus on long VaR only). Secondly, our empirical analysis deals with daily data for stock indexes, in contrast to exchange rate data for the other paper. That usual datasets such as the daily returns for European and US indexes indicate the need for these types of models is an important issue, as most studies usually focus on ‘exotic’ series for justifying the use of these models. Thirdly, we assess the performances of the models by computing Kupiec (1995)’s LR tests on the empirical failure rates. For the new model, we also compute the expected short-fall and the average multiple of tail event to risk measure. Last, from a methodological point of view, following Lambert and Laurent (2000) we re-express the estimated parameters in terms of the mean and variance of the skewed Student distribution (instead of the mode and the dispersion).

As indicated in Christoffersen and Diebold (2000), volatility forecastability (such as featured by ARCH class models) decays quickly with the time horizon of the forecasts. An immediate consequence is that volatility forecastability is relevant for short time horizons (such as daily trading), but not for long time horizons on which portfolio managers usually focus. In this paper,

we are consistent with these characteristics of volatility forecastability as we focus on daily returns and analyze VaR performance for daily trading portfolios made up of long and short positions.

The rest of the paper is organized in the following way. In Section 2, we describe the symmetric and asymmetric VaR models. These models are applied to daily stock indexes data in Section 3 where we assess their performances and characterize the long and short VaR.

2 VaR models

In this section we present parametric VaR models of the ARCH class. ARCH class models were first introduced by Engle (1982) with the ARCH model. Since then, numerous extensions have been put forward, see Engle (1995), Bera and Higgins (1993) or Palm (1996), but they all share the same goal, i.e. modelling the conditional variance as a function of past (squared) returns and associated characteristics. Because quantiles are direct functions of the variance in parametric models, ARCH class models immediately translate into conditional VaR models. As mentioned in the introduction, these conditional VaR models are important for characterizing short term risk for intradaily or daily trading positions.

In the first sub-section we characterize the symmetric (RiskMetrics, normal and Student APARCH) and asymmetric (skewed Student APARCH) volatility models, while we detail corresponding VaR results for negative and positive returns in the second sub-section. We stress that, by symmetric and asymmetric models, we mean a possible asymmetry in the distribution of the error term (i.e. whether it is skewed or not), and not the asymmetry in the relationship between the conditional variance and the lagged squared innovations (the APARCH model features this kind of ‘conditional’ asymmetry whatever the chosen error term).

2.1 Symmetric and asymmetric volatility models

To characterize the models, we consider a collection of daily returns, r_t , with $t = 1 \dots T$. Because daily returns are known to exhibit some serial autocorrelation³, we fit an AR(p) structure on the r_t series for all specifications:

$$r_t = \rho_0 + \rho_1 r_{t-1} + \dots + \rho_p r_{t-p} + e_t \quad (1)$$

We now consider several specifications for the conditional variance of e_t .

³The serial autocorrelation found in daily returns is not necessarily at odds with the efficient market hypothesis. See Campbell, Lo, and MacKinlay (1997) for a detailed discussion.

2.1.1 RiskMetrics

In its most simple form, it can be shown that the basic RiskMetrics model is equivalent to a normal IGARCH model where the autoregressive parameter is set at a prespecified value λ and the coefficient of e_{t-1}^2 is equal to $1 - \lambda$. In the RiskMetrics specification, $\lambda = 0.94$ and we then have:

$$e_t = \epsilon_t h_t \quad (2)$$

where ϵ_t is IID $N(0, 1)$ and h_t^2 is defined as:

$$h_t^2 = (1 - \lambda)e_{t-1}^2 + \lambda h_{t-1}^2 \quad (3)$$

2.1.2 Normal APARCH

The normal APARCH (Ding, Granger, and Engle, 1993) is an extension of the GARCH model of Bollerslev (1986). It is probably one of the most promising ARCH-type model. Indeed, it nests at least seven GARCH specifications. The APARCH(1,1) is:

$$h_t^\delta = \omega + \alpha_1 (|e_{t-1}| - \alpha_n e_{t-1})^\delta + \beta_1 h_{t-1}^\delta \quad (4)$$

where $\omega, \alpha_1, \alpha_n, \beta_1$ and δ are parameters to be estimated. δ ($\delta > 0$) plays the role of a Box-Cox transformation of h_t , while α_n ($-1 < \alpha_n < 1$), reflects the so-called leverage effect. A positive (resp. negative) value of α_n means that past negative (resp. positive) shocks have a deeper impact on current conditional volatility than past positive shocks (see Black, 1976; French, Schwert, and Stambaugh, 1987; Pagan and Schwert, 1990). The properties of the APARCH model have been studied recently by He and Terasvirta (1999a, 1999b).

2.1.3 Student APARCH

Previous empirical studies on VaR (van den Goorbergh and Vlaar, 1999; Giot, 2000) have shown that models based on the normal distribution usually cannot fully take into account the ‘fat tails’ of the returns distribution. To alleviate this problem, the Student APARCH (or t APARCH) is introduced:

$$e_t = \epsilon_t h_t \quad (5)$$

where ϵ_t is IID $t(0, 1, \nu)$ and h_t is defined as in (4).

2.1.4 Skewed Student APARCH

To accommodate the excess of skewness and kurtosis, Fernández and Steel (1998) propose to extend the Student distribution by adding a skewness parameter.⁴ Their procedure allows the introduction of skewness in any continuous unimodal and symmetric (about 0) distribution $g(\cdot)$ by changing the scale at each side of the mode. The main drawback of this density is that it is expressed in terms of the mode and the dispersion. In order to keep in the ARCH ‘tradition’, Lambert and Laurent (2001) re-expressed the skewed Student density in terms of the mean and the variance, i.e. reparametrize this density in such a way that the innovation process has zero mean and unit variance. Otherwise, it will be difficult to separate the fluctuations in the mean and variance from the fluctuations in the shape of the conditional density (Hansen, 1994).

The innovation process ε is said to be (standardized) skewed Student distributed if:

$$f(\varepsilon|\xi, v) = \begin{cases} \frac{2}{\xi + \frac{1}{\xi}} sg[\xi(s\varepsilon + m)|v] & \text{if } \varepsilon < -\frac{m}{s} \\ \frac{2}{\xi + \frac{1}{\xi}} sg[(s\varepsilon + m)/\xi|v] & \text{if } \varepsilon \geq -\frac{m}{s} \end{cases} \quad (6)$$

where $g(\cdot|v)$ is the symmetric (unit variance) Student density and ξ is the asymmetry coefficient.⁵ m and s^2 are respectively the mean and the variance of the non-standardized skewed Student:

$$m = \frac{\Gamma(\frac{v-1}{2})\sqrt{v-2}}{\sqrt{\pi}\Gamma(\frac{v}{2})} \left(\xi - \frac{1}{\xi} \right) \quad (7)$$

and

$$s^2 = \left(\xi^2 + \frac{1}{\xi^2} - 1 \right) - m^2 \quad (8)$$

Note that the density $f(\varepsilon|1/\xi, v)$ is the symmetric of $f(\varepsilon|\xi, v)$ with respect to the mean. Therefore, working with $\log(\xi)$ might be preferable to indicate the sign of the skewness.

Lambert and Laurent (2000) show that the quantile function $st_{\alpha, v, \xi}^*$ of a non standardized skewed student density is:

$$st_{\alpha, v, \xi}^* = \begin{cases} \frac{1}{\xi} t_{\alpha, v} \left[\frac{\alpha}{2} (1 + \xi^2) \right] & \text{if } \alpha < \frac{1}{1 + \xi^2} \\ -\xi t_{\alpha, v} \left[\frac{1 - \alpha}{2} (1 + \xi^{-2}) \right] & \text{if } \alpha \geq \frac{1}{1 + \xi^2} \end{cases} \quad (9)$$

where $t_{\alpha, v}$ is the quantile function of the (unit variance) Student-t density. It is straightforward to obtain the quantile function of the standardized skewed Student: $st_{\alpha, v, \xi} = \frac{st_{\alpha, v, \xi}^* - m}{s}$.

Following Ding, Granger, and Engle (1993) and Paoletta (1997), if it exists, a stationary solution of (4) is given by:

⁴Other (but very similar) asymmetric Student densities have been proposed by Hansen (1994) and Paoletta (1997).

⁵The asymmetry coefficient $\xi > 0$ is defined such that the ratio of probability masses above and below the mean is $\frac{\Pr(\varepsilon \geq 0|\xi)}{\Pr(\varepsilon < 0|\xi)} = \xi^2$

$$E(h_t^\delta) = \frac{\omega}{1 - \alpha_1 E(|\epsilon| - \alpha_n \epsilon)^\delta - \beta_1} \quad (10)$$

which depends on the density of ϵ . Such a solution exists if $\alpha_1 E(|\epsilon| - \alpha_n \epsilon)^\delta + \beta_1 < 1$. Ding, Granger, and Engle (1993) derived the expression for $E(|\epsilon| - \alpha_n \epsilon)^\delta$ in the Gaussian case. Paoletta (1997) did the same thing for various non standardized densities. It is straightforward to show that for the standardized skewed Student:⁶

$$E(|\epsilon| - \gamma \epsilon)^\delta = \left\{ \xi^{-(1+\delta)} (1 + \gamma)^\delta + \xi^{1+\delta} (1 - \gamma)^\delta \right\} \frac{\Gamma\left(\frac{\delta+1}{2}\right) \Gamma\left(\frac{v-\delta}{2}\right) (v-2)^{\frac{1+\delta}{2}}}{\left(\xi + \frac{1}{\xi}\right) \sqrt{(v-2)} \pi \Gamma\left(\frac{v}{2}\right)} \quad (11)$$

2.2 VaR for long and short positions

Because the goal of our paper is to check the performance of the models on both the long and short sides of daily trading, we are particularly interested in comparing the Student APARCH model with the skewed Student APARCH model regarding their performance in forecasting one step ahead long and short VaR. As indicated in the introduction, the long side of the daily VaR is defined as the VaR level for traders having long positions in the relevant equity index: this is the ‘usual’ VaR where traders incur losses when negative returns are observed. Correspondingly, the short side of the daily VaR is the VaR level for traders having short positions, i.e. traders who incur losses when stock prices increase. How good a model is at predicting long VaR is thus related to its ability to model large negative returns, while its performance regarding the short side of the VaR is based on its ability to take into account large positive returns.

For the RiskMetrics and normal APARCH models, the one-step-ahead VaR as computed in $t - 1$ for long trading positions is given by $z_\alpha h_t$, for short trading positions it is equal to $z_{1-\alpha} h_t$, with z_α being the left quantile at $\alpha\%$ for the normal distribution and $z_{1-\alpha}$ is the right quantile at $\alpha\%$.⁷ For the Student APARCH model, the VaR for long and short positions is given by $t_{\alpha,v} h_t$ and $t_{1-\alpha,v} h_t$, with $t_{\alpha,v}$ being the left quantile at $\alpha\%$ for the Student distribution with v degrees of freedom and $t_{1-\alpha,v}$ is the right quantile at $\alpha\%$ for this same distribution. Because $z_\alpha = -z_{1-\alpha}$ for the normal distribution and $t_{\alpha,v} = -t_{1-\alpha,v}$ for the Student distribution, the forecasted long and short VaR will be equal in both cases.

For the skewed Student APARCH model, the VaR for long and short positions is given by $st_{\alpha,v,\xi} h_t$ and $st_{1-\alpha,v,\xi} h_t$, with $st_{\alpha,v,\xi}$ being the left quantile at $\alpha\%$ for the skewed Student distribution with v degrees of freedom and asymmetry coefficient ξ ; $st_{1-\alpha,v,\xi}$ is the corresponding

⁶Notice that setting $\xi = 1$ leads to the stationarity condition of the symmetric Student density (with unit variance).

⁷All VaR expressions are reported for the residuals e_t , which is equivalent to reporting the VaR centered around the expected return based on (1).

right quantile. If $\log(\xi)$ is smaller than zero (or $\xi < 1$), $|st_{\alpha,v,\xi}| > |st_{1-\alpha,v,\xi}|$ and the VaR for long trading positions will be larger (for the same conditional variance) than the VaR for short trading positions. When $\log(\xi)$ is positive, we have the opposite result.

3 Empirical application

3.1 Data

In this empirical application we consider daily data for a collection of 5 stock market indexes: the French CAC 40 stock index (CAC, 1/1/1990 - 21/12/2000), the German DAX stock index (DAX, 26/11/1990 - 21/12/2000), the U.S. NASDAQ stock index (NASDAQ, 11/10/1984 - 21/12/2000), the Japanese NIKKEI stock index (NIKKEI, 4/1/1984 - 21/12/2000) and the Swiss SMI stock index (SMI, 9/11/1990 - 21/12/2000), where the numbers in parentheses are the start and end dates for the sample at hand and the first symbol inside the parentheses designates the short notation for the index that will be used in the tables and comments below. The VaR models introduced in Section 2 are tested on these five datasets.

For all price series p_t , daily returns are defined as $r_t = \ln(p_t) - \ln(p_{t-1})$. Descriptive characteristics for the returns series are given in Table 1. While the time spans for the five stock indexes are different, the five returns series share similar statistical properties as far as third and fourth moments are concerned. More specifically, the returns series are negatively skewed and the large returns (either positive or negative) lead to a large degree of kurtosis. The Ljung-Box Q-statistics of order 10 on the squared series indicate a high serial correlation in the second moment.

Descriptive graphs (level of index, daily returns, density of the daily returns and QQ-plot against the normal distribution) for each index are given in Figures 1-5. Volatility clustering is immediately apparent from the graphs of daily returns. The density graphs and the QQ-plot against the normal distribution show that all returns distributions exhibit fat tails. Moreover, the QQ-plots indicate that fat tails are not symmetric.

3.2 Estimating the models

In order to perform the VaR analysis in Section 3.3, the normal APARCH, RiskMetrics, Student APARCH and skewed Student APARCH are estimated in this section. We do not report full estimation results of the normal and Student APARCH models as they are quite similar to what has been documented in the literature. Furthermore, these specifications are encompassed by the skewed Student APARCH model which we fully detail below. The RiskMetrics model does not require any estimation for the conditional volatility specification as it is tantamount to an IGARCH model with some predefined values.

Table 2 presents the results for the (approximate quasi-maximum likelihood) estimation of the skewed Student APARCH model on the CAC, DAX, NASDAQ, NIKKEI and SMI data. An AR(3) was found to be sufficient to correct the serial correlation in the conditional mean. Note that to save some space, the estimated mean parameters are not reported. All the computations have been done using GAUSS.

The model is particularly successful in taking into account the heteroskedasticity exhibited by the data as the Ljung-Box Q-statistic computed on the squared standardized residuals is never significant.⁸ The five stock market indexes feature relatively similar volatility specifications:

- the autoregressive effect in the volatility specification is strong as β_1 is around 0.9, suggesting a strong memory effects. Indeed, $\alpha_1 E(|\epsilon| - \alpha_n \epsilon)^\delta + \beta_1$ is just below 1 for four indexes and equals 1 for the NASDAQ (indicating that h_t^δ may be integrated).
- α_n is positive and significant for all datasets, indicating a leverage effect for negative returns in the *conditional* variance specification;
- $\log(\xi)$ is negative and significant for all datasets, which implies that the *asymmetry in the Student distribution* is needed to fully model the distribution of returns. Likelihood ratio tests (not reported) also clearly favor the skewed Student density;
- δ is between 1.002 and 1.378 and always significantly different from 2. The results suggest that instead of modelling the conditional variance (GARCH) it is more relevant to model the conditional standard deviation (indeed, δ is not significantly different from 1). This result is in line with those of Taylor (1986), Schwert (1990) and Ding, Granger, and Engle (1993) who indicate that there is substantially more correlation between absolute returns than squared returns, a stylized fact of high frequency financial returns (often called ‘long memory’).

These results indicate the need for a model featuring a negative leverage effect (conditional asymmetry) for the conditional variance combined with an asymmetric distribution for the underlying error term (unconditional asymmetry). The skewed Student APARCH model delivers such specifications and we study in Section 3.3 if this model improves on symmetric GARCH models when the VaR for long and short returns is needed.

3.3 In-sample VaR computation

In this section, we use the estimation results of Section 3.2 and the expressions of Section 2.2 to compute the one-step-ahead VaR for all models. As financial returns are known to exhibit fat tails

⁸For NASDAQ data, the decrease in the $Q^2(10)$ is impressive as it goes down from more than 3,000 to about 12.

(this was confirmed in the descriptive properties of the data given in Table 1), we expect poor performance by the models based on the normal distribution.

All models are tested with a VaR level α which ranges from 5% to 0.25% and their performance is then assessed by computing the failure rate for the returns r_t . By definition, the failure rate is the number of times returns exceed (in absolute value) the forecasted VaR. If the VaR model is correctly specified, the failure rate should be equal to the prespecified VaR level. In our empirical application, we define a failure rate f_l for the long trading positions, which is equal to the percentage of negative returns smaller than one-step-ahead VaR for long positions. Correspondingly, we define f_s as the failure rate for short trading positions as the percentage of positive returns larger than the one-step-ahead VaR for short positions.

Because the computation of the empirical failure rate defines a sequence of yes/no observations, it is possible to test $H_0 : f = \alpha$ against $H_1 : f \neq \alpha$, where f is the failure rate (estimated by $\hat{f}$, the empirical failure rate).⁹ At the 5% level and if T yes/no observations are available, a confidence interval for $\hat{f}$ is given by $\left[\hat{f} - 1.96\sqrt{\hat{f}(1-\hat{f})/T}, \hat{f} + 1.96\sqrt{\hat{f}(1-\hat{f})/T} \right]$. In this paper these tests are successively applied to the failure rate f_l for long trading positions and then to f_s , the failure rate for short trading positions.

In Table 3 we present complete VaR results (i.e. P-values for the Kupiec LR test) for the NASDAQ and NIKKEI stock indexes.¹⁰ In Table 4 we give summary results for the five stock indexes. These results indicate that:

- VaR models based on the normal distribution (RiskMetrics and normal APARCH model) have a difficult job in modelling large returns, with large positive returns being somewhat better handled than large negative returns.
- the symmetric Student APARCH model improves considerably on the performance of normal based models but its performance is still not satisfactory for large positive returns. For the NASDAQ index, its performance in general is even worse than normal based models. The reason is that the critical values of the Student distribution $t_{\alpha,v}$ and $t_{1-\alpha,v}$ are very large in this case, which leads to a high level of long and short VaR: the model is often rejected because it is too conservative.¹¹
- the skewed Student APARCH model improves on all other specifications for both negative and positive returns. For the NASDAQ the improvement is substantial as the switch to a

⁹In the literature on VaR models, this test is also called the Kupiec LR test, if the hypothesis is tested using a likelihood ratio test. See Kupiec (1995).

¹⁰Complete VaR results are available for all indexes on request.

¹¹For example, the empirical failure rates for the short VaR are equal to 3.59%, 1.39%, 0.37%, 0.10% and 0.05% when α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%: in all cases the model is rejected because it is too conservative.

skewed Student distribution alleviates almost all problems due to the ‘conservativeness’ of the symmetric Student distribution. The model performs correctly in 100% of all cases for the negative returns (long VaR) and for the positive returns (short VaR). As indicated in Table 4, the skewed Student APARCH model correctly models nearly all VaR levels for long and short positions (the success rate is 100% for four stock indexes and 80% for one index). In all cases, this is a significant improvement on the VaR performances of symmetric models.

3.4 Out-of-sample VaR computation

The testing methodology in the previous subsection is equivalent to back-testing the model on the estimation sample. Therefore it can be argued that this should be favorable to the tested model. In a ‘real life situation’, VaR models are used to deliver out-of-sample forecasts, where the model is estimated on the known returns (up to time t for example) and the VaR forecast is made for period $[t + 1, t + h]$, where h is the time horizon of the forecasts. In this subsection we implement this testing procedure for the long and short VaR with $h = 1$ day.

We use an iterative procedure where the skewed Student APARCH model is estimated to predict the one-day-ahead VaR. The first estimation sample is the complete sample for which the data is available less the last five years. The predicted one-day-ahead VaR (both for long and short positions) is then compared with the observed return and both results are recorded for later assessment using the statistical tests. At the second iteration, the estimation sample is augmented to include one more day, the model is *re-estimated* and the VaRs are forecasted and recorded. We iterate the procedure until all days (less the last one) have been included in the estimation sample. Corresponding failure rates are then computed by comparing the long and short forecasted VaR_{t+1} with the observed return e_{t+1} for all days in the five years period. We use the same statistical tests as in the subsection dealing with the in-sample VaR.

Empirical results for the five stock indexes are given in Table 5. Broadly speaking, these results are quite similar (although not as good) to those obtained for the in-sample testing procedure as the skewed Student APARCH model performs rather well for out-of-sample VaR prediction. Its combined (i.e. for long and short VaR) success rate (at the 5% level) is equal to 70% (CAC), 90% (DAX), 70% (NASDAQ), 90% (NIKKEI) and 80% (SMI, almost 90% as one P-value is very close to the 0.05 level).

3.5 Expected short-fall and related measures

Our analysis in sub-sections 3.3 and 3.4 focused on the computation of empirical failure rates. In the last part of the empirical application, we now characterize the skewed Student APARCH model with respect to two other VaR related measures: the expected short-fall and the average

multiple of tail event to risk measure.

The expected short-fall (see Scaillet, 2000) is defined as the expected value of the losses conditioned on the loss being larger than the VaR. The average multiple of tail event to risk measure “measures the degree to which events in the tail of the distribution typically exceed the VaR measure by calculating the average multiple of these outcomes to their corresponding VaR measures” (Hendricks, 1996). Both measures are computed for the in-sample estimation of the long and short VaR performed in sub-section 3.3.¹²

For the expected short-fall, we report full estimation results for the NASDAQ and NIKKEI stock indexes in Table 6.¹³ These results indicate that the expected short-fall is in most cases larger (in absolute value) for the models based on the Student distribution than for the models based on the normal distribution. This is easily understood if one remembers that these models ‘fail’ less than the ones based on the normal distribution, but, when they fail, it happens for large (in absolute value) returns: the average of these returns is correspondingly large. It should be stressed that the expected short-fall is not a tool to rank VaR models or assess models’ performances. Nevertheless it is useful for risk managers as it answers the following question: “when my model fails, how much do I lose on average?”.

A related measure is the average multiple of tail event to risk measure, which is reported in Table 7 for the NASDAQ and NIKKEI stock indexes. The figures in this table indicate what is the average loss/predicted loss when the VaR model fails. For example, the 1.38 for the long VaR with NASDAQ data and the skewed Student APARCH models indicates that, at the 1% level, one expects to lose 1.38 the amount given by the VaR when returns are smaller than the VaR. As for the expected short-fall, this measure does not allow a ranking of the VaR models.

4 Conclusion

Over short-term time horizons, conditional VaR models are usually found to be good candidates for quantifying possible trading losses. In this paper, we extended this analysis by introducing a VaR model that could take into account losses arising from long and short trading positions. Because of the nature of long and short trading, this translates into bringing forward a statistical model that correctly models the left and right tails of the distribution of returns. The proposed model is the skewed Student APARCH model. Because density distribution of returns are usually not symmetric, it is shown that models¹⁴ that rely on symmetric normal or Student distributions

¹²The expected short-fall for the long VaR is computed as the average of the observed returns smaller than the long VaR. The expected short-fall for the short VaR is computed as the average of the observed returns larger than the short VaR. Computations are similar for the average multiple of tail event to risk measure.

¹³Estimation results for the other 3 indexes are very similar to those given in Table 6 and are not reported.

¹⁴We considered three symmetric volatility models: the RiskMetrics, normal and Student APARCH models.

underperform with respect to the new model when the one-day-ahead VaR is to be forecasted. All models were applied to daily data for five stock indexes (CAC40, DAX, NASDAQ, NIKKEI and SMI), with an out-of-sample testing procedure confirming the results of the in-sample backtesting method: in all cases the skewed Student APARCH model performed rather well. In the last part of the paper, we also computed the expected short-fall and the average multiple of tail event to risk measure for the models.

At this stage, several extensions can be considered. First, the performance of the new VaR model could also be assessed on multi-days period forecasts. While VaR models based on ARCH class specifications perform rather well for one-day time horizons, it is known that their performance is not as good for long time periods. Some recent work in this field is Christoffersen and Diebold (2000). Secondly, the VaR for long and short trading positions could be computed using non-parametric VaR models. Computation times and quality of VaR forecasts could be compared with the results given by the skewed Student APARCH. Finally, as argued recently by Engle and Patton (1999), time varying higher conditional moments are clearly of interest. In this respect, Hansen (1994), Harvey and Siddique (1999) and Lambert and Laurent (2000) have had some success in introducing dynamics in the third and fourth moments.

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Table 1: Descriptive statistics

	CAC	DAX	NASDAQ	NIKKEI	SMI
Annual mean	10.66	14.53	13.90	1.79	17.44
Annual s.d.	19.87	19.67	20.03	21.38	16.87
Skewness	-0.16	-0.39	-0.74	-0.14	-0.41
Excess Kurtosis	2.09	4.15	11.25	10.15	5.13
Minimum	-7.57	-9.87	-12.04	-16.14	-8.38
Maximum	6.83	7.29	9.96	12.43	7.46
$Q^2(10)$	444.8	428.7	3269.8	635.1	849.7

Descriptive statistics for the daily returns of the corresponding stock index expressed in %. All values are computed using PcGive. $Q^2(10)$ is the Ljung-Box Q-statistic of order 10 on the squared series.

Table 2: Skewed Student APARCH

	CAC	DAX	NASDAQ	NIKKEI	SMI
ω	0.032 (0.013)	0.013 (0.006)	0.016 (0.006)	0.024 (0.005)	0.029 (0.010)
α_1	0.056 (0.011)	0.081 (0.012)	0.128 (0.022)	0.105 (0.012)	0.089 (0.016)
α_n	0.501 (0.126)	0.248 (0.083)	0.291 (0.063)	0.493 (0.079)	0.408 (0.080)
β_1	0.915 (0.017)	0.926 (0.012)	0.887 (0.023)	0.897 (0.012)	0.902 (0.022)
ν	11.325 (2.793)	7.808 (1.441)	6.737 (0.708)	6.519 (0.703)	8.184 (1.562)
$\log(\xi)$	-0.067 (0.029)	-0.073 (0.029)	-0.185 (0.023)	-0.054 (0.023)	-0.124 (0.031)
δ	1.378 (0.207)	1.133 (0.132)	1.122 (0.178)	1.185 (0.133)	1.002 (0.153)
$Q^2(10)$	9.5	0.7	12.7	13.3	2.1
$\alpha_1 E(\epsilon - \alpha_n \epsilon)^\delta + \beta_1$	0.981	0.992	1.003	0.986	0.979

Estimation results for the volatility specification of the skewed Student APARCH model. Robust standard errors are reported in parenthesis. $Q^2(10)$ is the Ljung-Box Q-statistic of order 10 computed on the squared standardized residuals.

Table 3: VaR results for NASDAQ and NIKKEI (in-sample)

α	5%	2.5%	1%	0.5%	0.25%
VaR for long positions (NASDAQ)					
N APARCH	0.084	0	0	0	0
RiskMetrics	0.002	0	0	0	0
t APARCH	0	0	0.069	0.075	0.028
st APARCH	0.498	0.743	0.533	0.920	0.409
VaR for long positions (NIKKEI)					
N APARCH	0.563	0.336	0	0.046	0
RiskMetrics	0	0.006	0	0	0
t APARCH	0.063	0.563	0.592	0.789	0.477
st APARCH	0.563	0.838	0.043	0.472	0.675
VaR for short positions (NASDAQ)					
N APARCH	0	0.	0.032	0.305	0.811
RiskMetrics	0.001	0.011	0.863	0.166	0.162
t APARCH	0	0	0	0	0.002
st APARCH	0.593	0.743	0.533	0.205	0.151
VaR for short positions (NIKKEI)					
N APARCH	0	0.018	0.399	0.006	0
RiskMetrics	0.004	0.094	0.023	0	0
t APARCH	0.004	0.024	0.486	0.423	0.904
st APARCH	0.071	0.189	0.809	0.214	0.204

P-values for the null hypothesis $f_t = \alpha$ (i.e. failure rate for the long trading positions is equal to α , top of the table) and $f_s = \alpha$ (i.e. failure rate for the short trading positions is equal to α , bottom of the table). α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%. The models are successively the normal APARCH, RiskMetrics, Student APARCH and skewed Student APARCH models.

Table 4: VaR results for all indexes (in-sample)

VaR for long positions					
	CAC	DAX	NASDAQ	NIKKEI	SMI
N APARCH	40	0	20	40	20
RiskMetrics	0	0	0	0	0
t APARCH	100	100	40	100	80
st APARCH	100	100	100	80	100
VaR for short positions					
	CAC	DAX	NASDAQ	NIKKEI	SMI
N APARCH	20	0	40	20	80
RiskMetrics	20	40	60	20	80
t APARCH	60	60	0	60	20
st APARCH	100	80	100	100	100

Number of times (out of 100) that the null hypothesis $f_t = \alpha$ (i.e. failure rate for the long trading positions is equal to α , top of the table) is not rejected and $f_s = \alpha$ (i.e. failure rate for the short trading positions is equal to α , bottom of the table) is not rejected for the combined five possible values of α (the level of significance is 5%). The models are successively the normal APARCH, RiskMetrics, Student APARCH and skewed Student APARCH models.

Table 5: VaR results
(Skewed Student APARCH, out-of-sample)

α	5%	2.5%	1%	0.5%	0.25%
VaR for long positions					
CAC	0.103	0.014	0.016	0.590	0.486
DAX	0.207	0.427	0.864	0.783	0.645
NASDAQ	0.005	0.101	0.355	0.311	0.646
NIKKEI	0.035	0.427	0.643	0.590	0.156
SMI	0.132	0.022	0.910	0.325	0.645
VaR for short positions					
CAC	0.007	0.334	0.697	0.311	0.645
DAX	0.026	0.520	0.163	0.784	0.932
NASDAQ	0	0.001	0.151	0.515	0.337
NIKKEI	0.797	0.786	0.355	0.515	0.932
SMI	0.045	0.101	0.697	0.784	0.337

P-values for the null hypothesis $f_t = \alpha$ (i.e. failure rate for the long trading positions is equal to α , top of the table) and $f_s = \alpha$ (i.e. failure rate for the short trading positions is equal to α , bottom of the table). α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%. The failure rates are computed for the skewed Student APARCH model (out-of-sample estimation).

Table 6: Expected short-fall for NASDAQ and NIKKEI
(in-sample)

α	5%	2.5%	1%	0.5%	0.25%
Expected short-fall for long positions (NASDAQ)					
N APARCH	-2.42	-2.60	-3.00	-3.40	-3.58
RiskMetrics	-2.28	-2.40	-2.75	-2.84	-3.03
t APARCH	-2.32	-2.54	-3.25	-3.76	-4.31
st APARCH	-2.42	-2.75	-3.62	-4.07	-4.42
Expected short-fall for long positions (NIKKEI)					
N APARCH	-2.66	-3.09	-3.58	-4.03	-4.28
RiskMetrics	-2.56	-2.86	-3.23	-3.52	-3.59
t APARCH	-2.61	-3.13	-3.88	-4.48	-5.27
st APARCH	-2.68	-3.23	-4.06	-4.62	-5.49
Expected short-fall for short positions (NASDAQ)					
N APARCH	2.41	2.58	3.02	3.26	3.24
RiskMetrics	2.22	2.76	2.99	2.98	3.55
t APARCH	2.33	2.66	3.38	2.95	3.66
st APARCH	2.08	2.52	2.82	3.42	2.48
Expected short-fall for short positions (NIKKEI)					
N APARCH	2.80	3.37	3.85	4.08	4.26
RiskMetrics	2.67	3.04	3.92	4.17	4.24
t APARCH	2.73	3.43	4.34	4.37	5.03
st APARCH	2.65	3.28	4.20	4.44	4.63

Expected short-fall (in-sample evaluation) for the long and short VaR (at level α) given by the normal APARCH, Student APARCH, RiskMetrics and skewed Student APARCH models. α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%.

Table 7: Average multiple of tail event to risk measure
for NASDAQ and NIKKEI (in-sample)

α	5%	2.5%	1%	0.5%	0.25%
AMTERM for long positions (NASDAQ)					
N APARCH	1.42	1.36	1.37	1.35	1.35
RiskMetrics	1.51	1.46	1.41	1.39	1.35
t APARCH	1.45	1.36	1.39	1.40	1.38
st APARCH	1.42	1.36	1.38	1.39	1.35
AMTERM for long positions (NIKKEI)					
N APARCH	1.38	1.36	1.32	1.46	1.49
RiskMetrics	1.44	1.40	1.36	1.41	1.40
t APARCH	1.33	1.36	1.41	1.49	1.49
st APARCH	1.39	1.35	1.47	1.49	1.48
AMTERM for short positions (NASDAQ)					
N APARCH	1.27	1.23	1.20	1.20	1.16
RiskMetrics	1.29	1.28	1.22	1.16	1.20
t APARCH	1.27	1.22	1.15	1.27	1.26
st APARCH	1.29	1.24	1.20	1.20	1.22
AMTERM for short positions (NIKKEI)					
N APARCH	1.37	1.37	1.32	1.28	1.24
RiskMetrics	1.40	1.35	1.32	1.30	1.31
t APARCH	1.39	1.35	1.28	1.19	1.17
st APARCH	1.39	1.36	1.29	1.23	1.17

Average multiple of tail event to risk measure (AMTERM, in-sample evaluation) for the long and short VaR (at level α) given by the normal APARCH, Student APARCH, RiskMetrics and skewed Student APARCH models. α is equal successively to 5%, 2.5%, 1%, 0.5% and 0.25%.

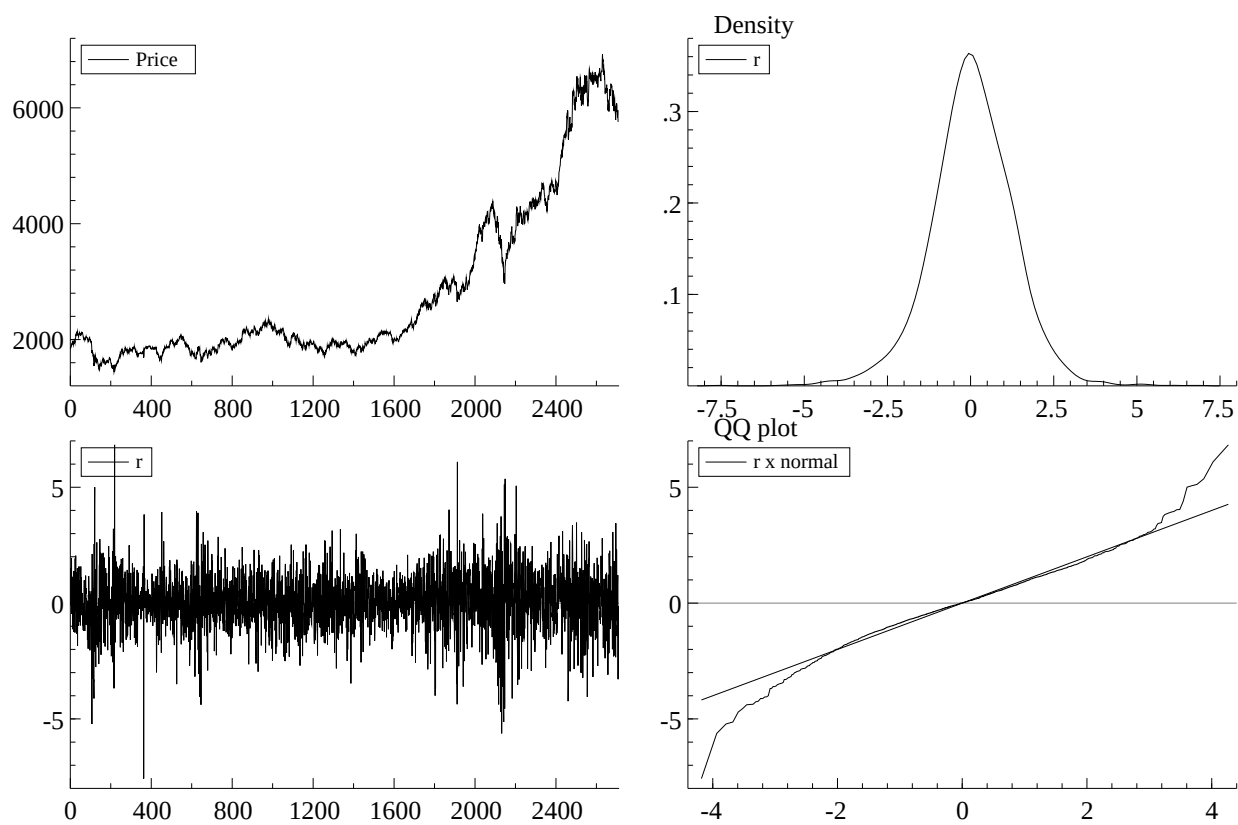


Figure 1: CAC 40 stock index in level, daily returns, daily returns density and QQ-plot against the normal distribution. The time period is 1/1/1990 - 21/12/2000.

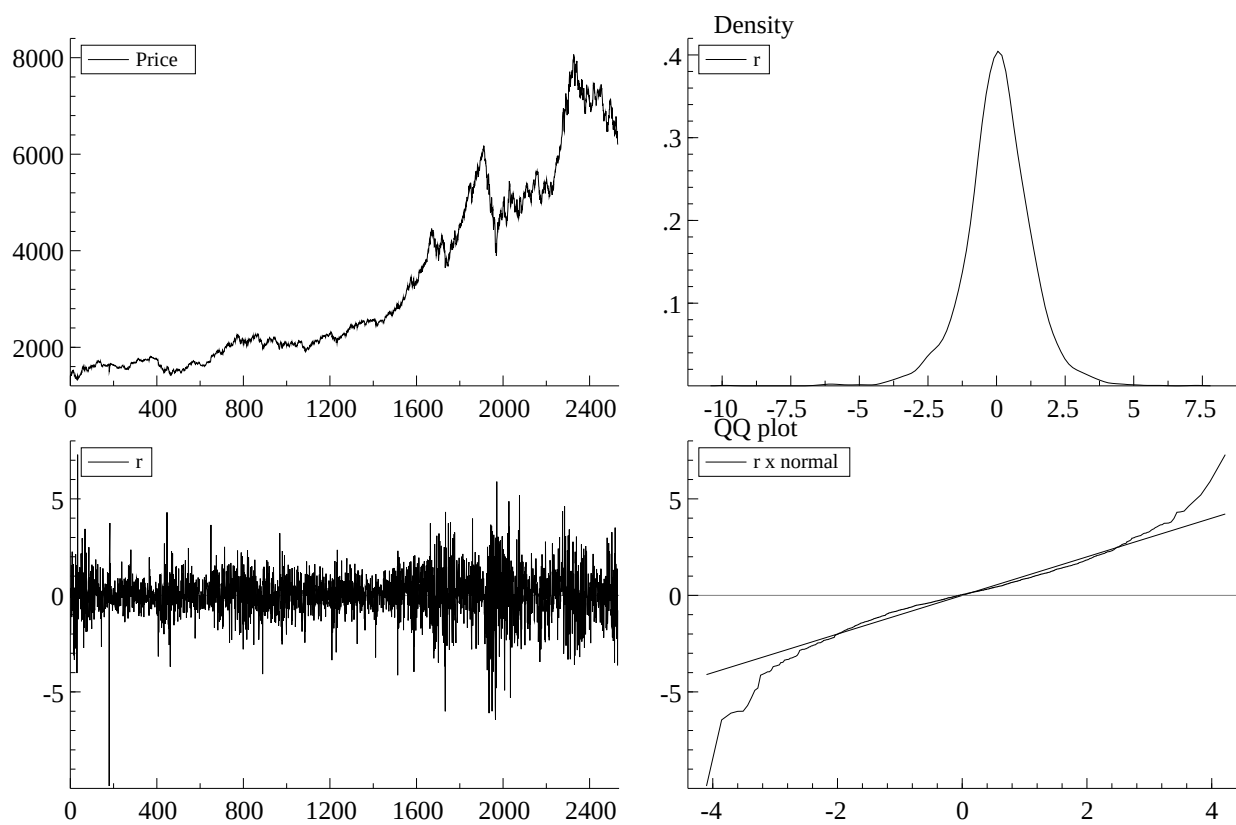


Figure 2: DAX stock index in level, daily returns, daily returns density and QQ-plot against the normal distribution. The time period is 26/11/1990 - 21/12/2000.

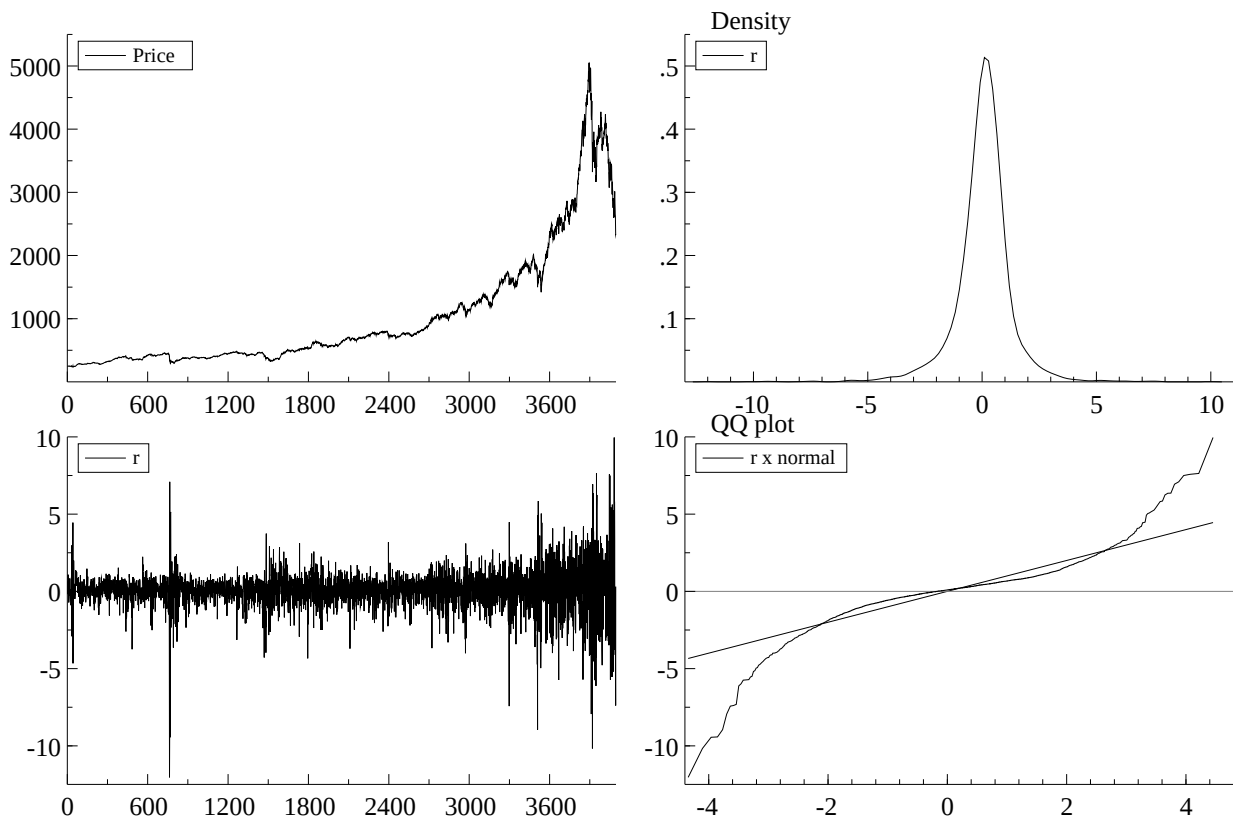


Figure 3: NASDAQ stock index in level, daily returns, daily returns density and QQ-plot against the normal distribution. The time period is 11/10/1984 - 21/12/2000.

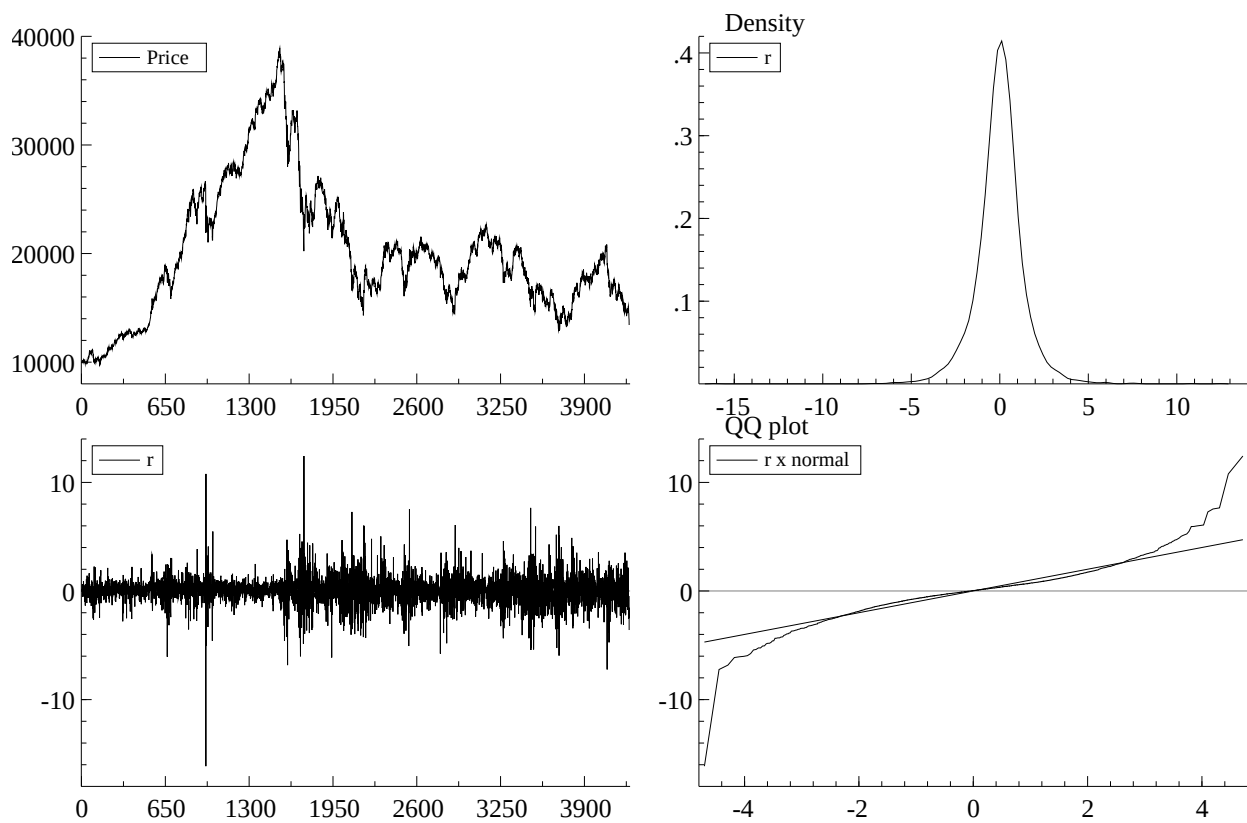


Figure 4: NIKKEI stock index in level, daily returns, daily returns density and QQ-plot against the normal distribution. The time period is 4/1/1984 - 21/12/2000.

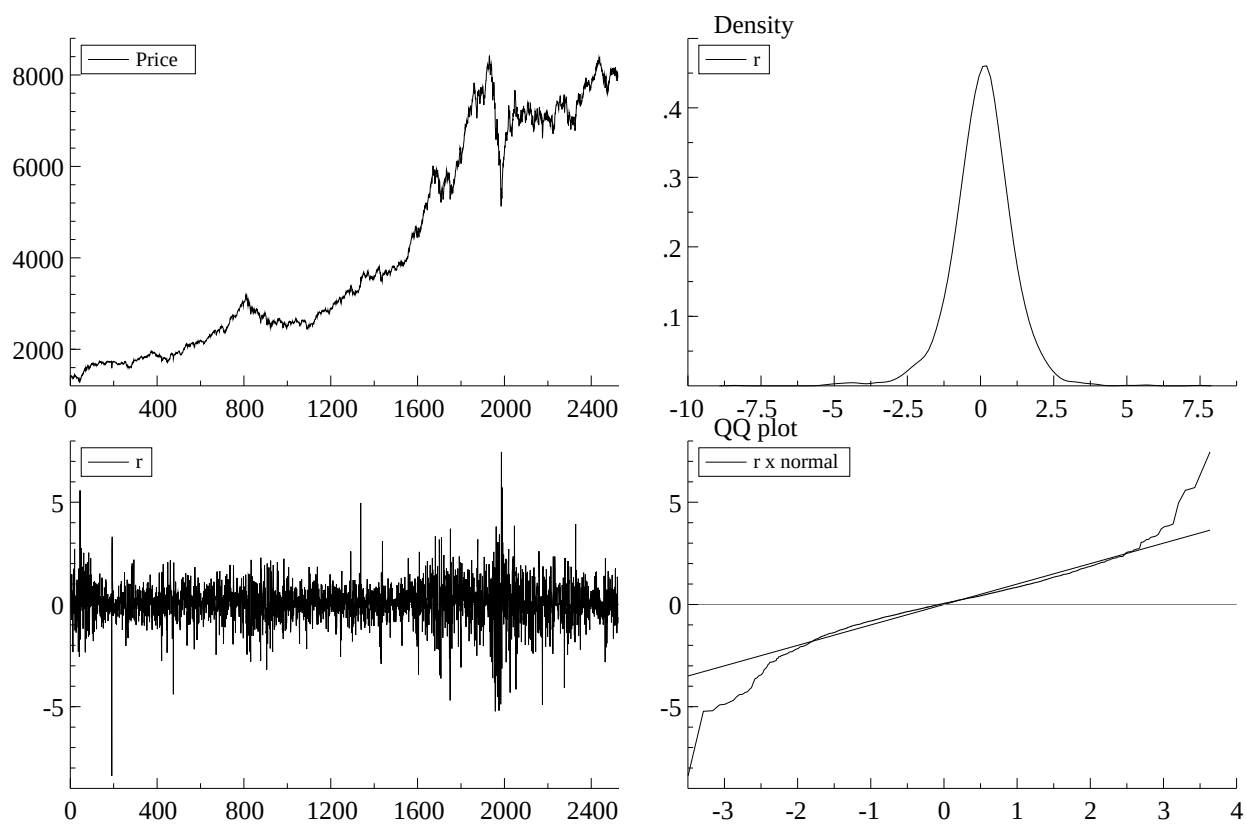


Figure 5: SMI stock index in level, daily returns, daily returns density and QQ-plot against the normal distribution. The time period is 9/11/1990 - 21/12/2000.