

THE DISCRIMINANT PFISTER FORM OF AN ALGEBRA WITH INVOLUTION OF CAPACITY FOUR

KARIM JOHANNES BECHER, NICOLAS GRENIER-BOLEY,
AND JEAN-PIERRE TIGNOL

ABSTRACT. To an orthogonal or unitary involution on a central simple algebra of degree 4, or to a symplectic involution on a central simple algebra of degree 8, we associate a Pfister form that characterises the decomposability of the algebra with involution. In this way we obtain a unified approach to known decomposability criteria for several cases, and a new result for symplectic involutions on degree 8 algebras in characteristic 2.

Classification (MSC 2010): 11E04, 11E81, 12G05, 16H05, 16R50, 16W10

Keywords: Central simple algebra, involution, decomposability, étale algebra, quadratic form, isotropy, hyperbolicity, discriminant, cohomological invariant, composition formula, characteristic two

1. INTRODUCTION

The decomposability of central simple algebras with involution into tensor products of quaternion algebras with involution has been in the focus of much study, motivated notably by the analogy between decomposable involutions and quadratic Pfister forms; see [3], [4], and the references in [23]. When the characteristic is different from 2, vanishing of the first cohomological invariant yields a necessary condition for decomposability, which is also sufficient for algebras of low degree; see [23]. The aim of this article is to establish in arbitrary characteristic a decomposability criterion for algebras with involution of low degree in terms of a canonically associated quadratic Pfister form. The condition on the degree depends on the type of the involution, and can be better expressed in terms of the capacity of the Jordan algebra of symmetric elements, which is the maximal dimension of an étale algebra consisting of symmetric elements (see Section 3 and [6, p. 389]). In a nutshell, our main result functorially associates to every algebra with involution of capacity four a Pfister quadratic form that detects

Date: 28 March, 2022.

This work was supported by the FWO Odysseus Programme (project *Explicit Methods in Quadratic Form Theory*), funded by the Fonds Wetenschappelijk Onderzoek – Vlaanderen. The third author acknowledges support from the Fonds de la Recherche Scientifique–FNRS under CDR grants 1.5054.12F, J.0014.15, J.0149.17, J.0159.19. Work on this paper was initiated in 2010 while the first and the third author were, respectively, Fellow and Senior Fellow of the Zukunftskolleg, Universität Konstanz, whose hospitality is gratefully acknowledged.

when the algebra decomposes into a tensor product of quaternion algebras with involution.

The following statement spells out explicitly the various cases that we consider:

Main Theorem. *Let F be a field. Let $n \in \{1, 2, 3\}$. Let A be an F -algebra with $\dim_F A = 2^{n+3}$ and σ an F -linear involution on A such that the following holds, depending on n :*

$n = 1$: $\text{char } F \neq 2$, (A, σ) is a central simple F -algebra with orthogonal involution of degree 4.

$n = 2$: (A, σ) is a central simple F -algebra with unitary involution of degree 4 (in the sense of [17, §2.B]: the centre of A is thus a quadratic étale F -algebra).

$n = 3$: (A, σ) is a central simple F -algebra with symplectic involution of degree 8.

To (A, σ) is associated a quadratic n -fold Pfister form \mathfrak{P}_σ over F which has the following characteristic property:

For any field extension F'/F , the form \mathfrak{P}_σ is hyperbolic over F' if and only if the F' -algebra with involution $(A, \sigma)_{F'}$ obtained from (A, σ) by scalar extension to F' is totally decomposable, which means that

$$(A, \sigma) \simeq (Q_1, \sigma_1) \otimes \cdots \otimes (Q_r, \sigma_r)$$

for some quaternion algebras Q_1, \dots, Q_r over the centre of A , with involutions $\sigma_1, \dots, \sigma_r$. (Here, $r = 2$ if $n = 1$ or 2 , and $r = 3$ if $n = 3$.)

In most of the cases the Main Theorem gives a reinterpretation of some previously known decomposability criteria in terms of quadratic Pfister forms. Here our principal aim is to handle these different cases by a new uniform approach. In the case where $n = 3$ and $\text{char } F = 2$ the result is itself novel.

Note that when $n = 1$ we assume that $\text{char } F \neq 2$. In this case a criterion for decomposability was established by Knus–Parimala–Sridharan [18]. In [19] the same authors proved an analogous statement in arbitrary characteristic. This criterion could also be formulated in terms of a bilinear 1-fold Pfister form (given by the determinant of the involution, see [17, §7.A]). On the other hand, a corresponding criterion for degree 4 algebras with quadratic pair can be obtained using the exceptional isomorphism $D_2 = A_1 \times A_1$; see [17, (15.12)]. This criterion could be formulated in terms of a quadratic 1-fold Pfister form.

The case $n = 1$ is the least difficult and interesting one, but at the same time it would be the most cumbersome to cover if characteristic 2 were included, in view of the necessary distinction between orthogonal involutions and quadratic pairs. For this reason we decided to include the case $n = 1$ only when $\text{char } F \neq 2$, mainly in order to highlight the analogy with the other two cases. Nevertheless, we will have cause to consider orthogonal involutions in characteristic 2 in some auxiliary results.

In the case $n = 2$, a criterion for decomposability was first obtained by Karpenko–Quéguiner [15]. Their result is stated in terms of the discriminant

algebra, and it is obtained using the exceptional isomorphism $A_3 = D_3$. The 2-fold Pfister form \mathfrak{P}_σ turns out to be the norm form of the F -quaternion algebra that is Brauer equivalent to the discriminant algebra of (A, σ) , see Proposition 8.2. For the case $n = 3$ a criterion for decomposability was established in characteristic different from 2 by Garibaldi–Parimala–Tignol [13] in terms of a cohomology class of degree 3, which gives the first nontrivial cohomological invariant of (A, σ) . This class is given by the Arason invariant of the 3-fold Pfister form \mathfrak{P}_σ , see Proposition 8.4.

What unites the three cases in the Main Theorem, in spite of the different dimensions of the algebra, is the fact that biquadratic étale subalgebras on which σ restricts to the identity are maximal for this property, by [6, Theorem 4.1].

The core of the proof of the Main Theorem is carried out in Section 7. The construction of the quadratic form \mathfrak{P}_σ is inspired by the treatment in [13] of the symplectic case in characteristic different from two. It further relies on a peculiar property of certain biquadratic étale subalgebras, which was first used by Rost–Serre–Tignol [21] to define a cohomological invariant of degree 4 for central simple algebras of degree 4.

A priori the construction depends on the choice of a biquadratic étale F -subalgebra L of A contained in the space $\text{Symd}(\sigma) = \{x + \sigma(x) \mid x \in A\}$. In Lemma 7.4 and Theorem 7.5 we show that such an L induces a decomposition

$$\text{Symd}(\sigma) = L \oplus W_1 \oplus W_2 \oplus W_3$$

where each of the F -subspaces W_1, W_2, W_3 is naturally endowed with a quadratic form (determined up to a similarity factor). Moreover, we show that these three quadratic forms are related by a composition formula. Hence they are similar to a quadratic Pfister form, which is determined by σ and L and which we denote by $\mathfrak{P}_{\sigma, L}$. In Proposition 7.8 we show that this form $\mathfrak{P}_{\sigma, L}$ is hyperbolic if and only if the algebra with involution decomposes along L (this notion is introduced in Section 4). We then prove in Proposition 7.12 that this Pfister form does not depend on the choice of the biquadratic subalgebra L . Hence it is an invariant of σ , which we denote by \mathfrak{P}_σ . The fact that \mathfrak{P}_σ has the properties stipulated in the Main Theorem is then achieved by Theorem 7.13.

Showing the independence of the Pfister form \mathfrak{P}_σ from the choices made in its construction is the most delicate part of the proof of the Main Theorem. This is based on a reduction to the case where σ is hyperbolic, and hence it relies on a comprehensive study of the decomposability of algebras with hyperbolic involution. This is carried out in some more generality in Section 5, and then specialised in Section 6 to the situation of capacity 4.

Our treatment also leads to a new result on the possible decompositions of a totally decomposable algebra with involution (A, σ) such as in the Main Theorem. In Theorem 7.10 we obtain that any biquadratic étale F -subalgebra of A to which σ restricts to the identity can be distributed nicely over the quaternion factors of a certain decomposition of (A, σ) . In Corollary 7.11 we further show that, when

A is simple but contains zero-divisors (or when A is unitary of inner type and the simple components contain zero-divisors), then a total decomposition of (A, σ) can be found which contains a split F -quaternion algebra.

In the final Section 8 we relate the quadratic Pfister forms arising in the three cases of the Main Theorem between each other and to other, previously known invariants. We further give some examples where the Pfister form invariant of an algebra with involution is computed explicitly.

We are grateful to the referee for the large number of suggestions that allowed us to improve the wording significantly.

2. ALGEBRAS WITH INVOLUTION

In this section we recall some basic facts and objects associated with involutions on central simple algebras. We recall the distinction of involutions into two kinds and further into three different types. We also include some notation from [6]. Our main reference for involutions is [17].

Let F always be a field. Let A be an F -algebra. We denote by $Z(A)$ the centre of A . For an F -subalgebra B of A we denote by $C_A(B)$ the centraliser of B in A . Assume now that $\dim_F(A) < \infty$. If A is simple, then $Z(A)$ is a field and A is a central simple $Z(A)$ -algebra. In this case we denote respectively by $\deg A$, $\text{ind } A$ and $\exp A$ the degree, the index and the exponent of A , and we further set $\text{coind } A = \frac{\deg A}{\text{ind } A}$, and we call A *split* if $\text{ind } A = 1$. A central simple F -algebra of degree 2 is called an *F -quaternion algebra*.

By an *F -involution on A* we mean an F -linear anti-automorphism $\sigma : A \rightarrow A$ such that $\sigma \circ \sigma = \text{id}_A$. Consider an F -involution σ on A . We set

$$\begin{aligned} \text{Sym}(\sigma) &= \{x \in A \mid \sigma(x) = x\}, \\ \text{Skew}(\sigma) &= \{x \in A \mid \sigma(x) = -x\}, \\ \text{Symd}(\sigma) &= \{x + \sigma(x) \mid x \in A\}. \end{aligned}$$

Using the F -linear map $A \rightarrow A, x \mapsto x + \sigma(x)$ one obtains that

$$\dim_F A = \dim_F \text{Skew}(\sigma) + \dim_F \text{Symd}(\sigma).$$

An *F -algebra with involution* is a pair (A, σ) of a finite-dimensional F -algebra A and an F -involution σ on A such that $F = Z(A) \cap \text{Sym}(\sigma)$ and such that A has no nontrivial two-sided ideals I with $\sigma(I) = I$.

For an étale extension L/F we denote by $[L : F]$ the dimension of L as an F -vector space.

In the sequel let (A, σ) be an F -algebra with involution. Then either $Z(A) = F$ or $Z(A)$ is a quadratic étale extension of F with nontrivial automorphism $\sigma|_{Z(A)}$. We say that the involution σ —or the algebra with involution (A, σ) —is

$$\begin{aligned} &\text{of the } \textit{first kind} && \text{if } [Z(A) : F] = 1, \\ &\text{of the } \textit{second kind} && \text{if } [Z(A) : F] = 2. \end{aligned}$$

If $Z(A)$ field, then A is a central simple $Z(A)$ -algebra. If (A, σ) is of the second kind, then either $Z(A)$ is a quadratic field extension of F , or $Z(A) \simeq F \times F$.

Involutions on central simple algebras are further distinguished into three types. Involutions of the second kind are said to be of *unitary type*. Involutions of the first kind are of *orthogonal type* (resp. of *symplectic type*) if after scalar extension to a splitting field they are adjoint to a symmetric bilinear form (resp. to an alternating bilinear form), see [17, §2]. If $Z(A) \simeq F \times F$ then we call (A, σ) *unitary of inner type*. (The term is motivated by a corresponding notion for algebraic groups.) In this case we obtain that $(A, \sigma) \simeq (A_0 \times A_0^{\text{op}}, \text{sw})$ for a central simple F -algebra A_0 , its opposite algebra A_0^{op} and the *switch involution* $\text{sw}: A_0 \times A_0^{\text{op}} \rightarrow A_0 \times A_0^{\text{op}}$, given by $\text{sw}(a_1, a_2) = (a_2, a_1)$ (see [17, (2.14)]). In this situation we set $\deg A = \deg A_0$, $\text{ind } A = \text{ind } A_0$ and $\text{coind } A = \text{coind } A_0$.

2.1. Remarks. (1) We have $\text{Symd}(\sigma) \subseteq \text{Sym}(\sigma)$ and this is an equality unless $\text{char } F = 2$ and (A, σ) is of the first kind. (See [17, (2.17)] for the case where $\text{char } F = 2$ and σ unitary.)

(2) We have $1 \notin \text{Symd}(\sigma)$ if and only if σ is orthogonal and $\text{char}(F) = 2$. Orthogonal involutions in characteristic two are peculiar. When we need to exclude this case, we will simply assume that $1 \in \text{Symd}(\sigma)$.

2.2. Proposition. *Set $d = \deg A$, hence $\dim_F A = [Z(A) : F] \cdot d^2$. If $1 \in \text{Symd}(\sigma)$, then*

$$\dim_F \text{Symd}(\sigma) = \begin{cases} \frac{d(d+1)}{2} & \text{if } \sigma \text{ is orthogonal,} \\ d^2 & \text{if } \sigma \text{ is unitary,} \\ \frac{d(d-1)}{2} & \text{if } \sigma \text{ is symplectic.} \end{cases}$$

If $1 \notin \text{Symd}(\sigma)$, then $\dim_F \text{Symd}(\sigma) = \frac{d(d-1)}{2}$.

Proof: This follows from the definitions together with [17, (2.6)] or [17, (2.17)], depending on whether σ is of the first or second kind. \square

We recall the most basic examples of involutions.

2.3. Examples.

- (1) The identity map id_F is the unique orthogonal involution on F , viewed as a central simple F -algebra.
- (2) Consider a quadratic étale extension K/F and let $\text{can}_{K/F}$ denote the nontrivial F -automorphism of K . Then $(K, \text{can}_{K/F})$ is an F -algebra with unitary involution.
- (3) Let Q be an F -quaternion algebra. The unique symplectic involution on Q is given by $x \mapsto \text{Trd}_Q(x) - x$, where $\text{Trd}_Q: Q \rightarrow F$ denotes the reduced trace form of Q . We denote this involution by can_Q and call it the *canonical involution on Q* .

By an F -algebra with *canonical involution* we mean an F -algebra with involution of one of the three types in Examples 2.3.

3. CAPACITY

Let (A, σ) be an F -algebra with involution. Following [6, Section 5], we call an F -subalgebra L of A *neat in (A, σ)* or a *neat subalgebra of (A, σ)* if L is an étale F -algebra contained in $\mathbf{Sym}(\sigma)$ and such that A is free as a left L -module and, for all primitive idempotents e of L , the F -algebras with involution $(eAe, \sigma|_{eAe})$ have the same degree and the same type; this type coincides with the type of σ .

3.1. Remark. In this article we mostly (but not always) avoid the case of orthogonal involutions in characteristic 2, which is the most troublesome in the study of stable subalgebras, as demonstrated in [6]. In particular, in the cases which we consider the definition of neat subalgebra simplifies as follows: A commutative F -subalgebra L of A is *neat in (A, σ)* if L is étale, consists of σ -symmetric elements, and A is free as a left (or right) L -module. In particular it then follows that, if L is neat in (A, σ) and L is a free module over some subalgebra L' , then L' is also neat in (A, σ) .

Following [6], we define

$$\mathrm{cap}(A, \sigma) = \begin{cases} \mathrm{deg}A & \text{if } \sigma \text{ is orthogonal or unitary,} \\ \frac{1}{2}\mathrm{deg}A & \text{if } \sigma \text{ is symplectic,} \end{cases}$$

and we call this integer the *capacity of (A, σ)* .

Algebras with involution of capacity 1 are easily determined:

3.2. Proposition. *The following are equivalent:*

- (i) (A, σ) is an F -algebra with canonical involution (in the sense of Examples 2.3).
- (ii) $\mathbf{Symd}(\sigma) = F$ or $\{0\}$.
- (iii) $\mathrm{cap}(A, \sigma) = 1$.

Proof: This follows from Proposition 2.2. □

By [6, Theorem 4.1] we have

$$\mathrm{cap}(A, \sigma) = \max\{[L : F] \mid L \text{ étale } F\text{-algebra with } L \subseteq \mathbf{Sym}(\sigma)\}.$$

Furthermore, by [6, Proposition 5.6], every étale F -subalgebra L of A contained in $\mathbf{Sym}(\sigma)$ and with $[L : F] = \mathrm{cap}(A, \sigma)$ is neat in (A, σ) . These commutative algebras are maximal in a strong sense, as demonstrated by the following proposition.

3.3. Proposition. *Let L be an étale F -subalgebra of A contained in $\mathbf{Sym}(\sigma)$ with $[L : F] = \mathrm{cap}(A, \sigma)$. Then $\mathbf{C}_A(L) \cap \mathbf{Symd}(\sigma) \subset L$.*

Proof: As we just observed, L is neat in (A, σ) . If σ is orthogonal, then [6, Proposition 2.3] yields $\dim_F A = [\mathbf{C}_A(L) : F] \cdot [L : F]$, hence $[\mathbf{C}_A(L) : F] = [L : F]$ since $[L : F] = \mathrm{cap}(A, \sigma) = \mathrm{deg}A$. Therefore, $\mathbf{C}_A(L) = L$.

If σ is unitary, then upon extending scalars we may assume σ is unitary of inner type and identify (A, σ) with $(A_0 \times A_0^{\text{op}}, \text{sw})$ for some central simple F -algebra A_0 . Then there is an étale F -subalgebra $L_0 \subset A_0$ such that

$$L = \{(\ell, \ell) \mid \ell \in L_0\},$$

and the condition $[L : F] = \text{cap}(A, \sigma)$ implies $[L_0 : F] = \text{deg} A_0$. Then $\mathbf{C}_{A_0}(L_0) = L_0$ by [6, Proposition 2.3], hence $L = \mathbf{C}_A(L) \cap \text{Symd}(\sigma)$.

If σ is symplectic, then $[L : F] = \frac{1}{2} \text{deg} A$, hence $[\mathbf{C}_A(L) : F] = 4[L : F]$. Moreover, all the simple components of $\mathbf{C}_A(L)$ have the same degree by [6, Proposition 5.4], and the restriction of σ to eAe is symplectic for every nonzero symmetric idempotent $e \in A$ by [6, Proposition 3.1]. Extending scalars, we may assume L is split. Let e_1, \dots, e_r be the primitive idempotents of L , so $L = e_1 F \oplus \dots \oplus e_r F$ and $r = [L : F] = \frac{1}{2} \text{deg} A$. Then

$$\mathbf{C}_A(L) = (e_1 A e_1) \oplus \dots \oplus (e_r A e_r)$$

where each $e_i A e_i$ is a quaternion F -algebra on which the restriction of σ is the canonical involution. Let $x \in \mathbf{C}_A(L) \cap \text{Symd}(\sigma)$, and write $x = y + \sigma(y)$ for some $y \in A$. Since $e_1 + \dots + e_r = 1$ and x commutes with each e_i , we have

$$x = e_1 x + \dots + e_r x = e_1 x e_1 + \dots + e_r x e_r.$$

Now, for $i = 1, \dots, r$,

$$e_i x e_i = e_i y e_i + \sigma(e_i y e_i) \in \text{Symd}(\sigma|_{e_i A e_i}),$$

and $\text{Symd}(\sigma|_{e_i A e_i}) = e_i F$ because $\sigma|_{e_i A e_i}$ is the canonical involution. Therefore, $x \in L$. \square

The following proposition will be needed in Section 4 to prove that étale algebras along which an algebra with involution decomposes are neat.

3.4. Proposition. *Let L be an étale F -subalgebra of A contained in $\text{Sym}(\sigma)$. Assume that there exists a σ -stable central simple F -subalgebra B of A such that $L \subseteq B$ and $\mathbf{C}_B(L) = L$. Then L is neat in (A, σ) .*

Proof: Let $C = \mathbf{C}_A(B)$, which is a simple σ -stable F -subalgebra of A . We set $\sigma_B = \sigma|_B$ and $\sigma_C = \sigma|_C$. Then (B, σ_B) and (C, σ_C) are F -algebras with involution such that $(A, \sigma) = (B, \sigma_B) \otimes (C, \sigma_C)$. Since $\mathbf{C}_B(L) = L$, we have $[L : F] = \text{deg}(B)$ by [6, Proposition 2.3], hence $\text{cap}(B, \sigma_B) = \text{deg}(B)$, which shows that σ_B is orthogonal. It follows by [17, (2.23)] that σ_C is of the same type as σ , and consequently $\text{cap}(A, \sigma) = \text{cap}(B, \sigma_B) \cdot \text{cap}(C, \sigma_C)$.

We choose an étale F -subalgebra M of (C, σ_C) contained in $\text{Sym}(\sigma_C)$ with $[M : F] = \text{cap}(C, \sigma_C)$. Then LM is an étale extension of F contained in $\text{Sym}(\sigma)$ with

$$[LM : F] = [L : F] \cdot [M : F] = \text{cap}(B, \sigma_B) \cdot \text{cap}(C, \sigma_C) = \text{cap}(A, \sigma).$$

Hence LM is neat in (A, σ) , by [6, Proposition 5.6]. Since LM is free as an L -module, it follows by [6, Lemma 5.8] that L is neat in (A, σ) . \square

4. DECOMPOSABILITY

In this section we discuss decompositions of algebras with involution that are compatible with a given multiquadratic étale subalgebra. We further recall the notion of an algebra with involution being totally decomposable.

Unless explicitly mentioned, all tensor products of algebras and vector spaces are taken over F and simply denoted by \otimes .

Let (A, σ) be an F -algebra with involution. We call (A, σ) *totally decomposable* if, for some $n \in \mathbb{N}$, there exist F -quaternion algebras Q_i with involutions σ_i of the first kind for $1 \leq i \leq n$ such that

$$(A, \sigma) \simeq (Z(A), \sigma|_{Z(A)}) \otimes \bigotimes_{i=1}^n (Q_i, \sigma_i);$$

if $Z(A) = F$, then this simply means that $(A, \sigma) \simeq \bigotimes_{i=1}^n (Q_i, \sigma_i)$. Recall from [17, (2.22)] that every quaternion algebra with unitary involution is totally decomposable. Therefore, an F -algebra with unitary involution (A, σ) is totally decomposable if and only if it has a decomposition

$$(A, \sigma) \simeq (H_1, \sigma_1) \otimes_{Z(A)} \cdots \otimes_{Z(A)} (H_n, \sigma_n)$$

for some quaternion $Z(A)$ -algebras H_1, \dots, H_n with unitary involutions $\sigma_1, \dots, \sigma_n$ fixing F . The degree of any totally decomposable F -algebra with involution is a power of 2.

Let $r \in \mathbb{N}$ and let B_1, \dots, B_r be central simple F -subalgebras of A . We call B_1, \dots, B_r *independent* if the F -subalgebra of A generated by $B_1 \cup \dots \cup B_r$, is F -isomorphic to the tensor product $B_1 \otimes \cdots \otimes B_r$; this is equivalent to having that B_1, \dots, B_r are centralizing one another, that is, for any $i, j \in \{1, \dots, r\}$ with $i \neq j$, we have $xy = yx$ for all $x \in B_i$ and $y \in B_j$.

4.1. Proposition. *Let $n \in \mathbb{N}$ be such that $\text{cap}(A, \sigma) = 2^n$. Then the following are equivalent:*

- (i) (A, σ) is totally decomposable.
- (ii) There exist n independent σ -stable F -quaternion subalgebras of A .
- (iii) There exist independent σ -stable F -quaternion subalgebras Q_1, \dots, Q_{n-1} of A such that $\sigma|_{Q_i}$ is orthogonal for $1 \leq i \leq n-1$.
- (iv) There exist independent σ -stable F -quaternion algebras Q_1, \dots, Q_n of A such that $\sigma|_{Q_i}$ is orthogonal for $1 \leq i \leq n$.

Proof: The implication (i) \Rightarrow (ii) is trivial.

(ii) \Rightarrow (iii) This implication follows by induction on n , starting with the trivial cases where $n \leq 1$. For the induction step, it suffices to observe that a tensor product of two F -quaternion algebras with symplectic involutions can up to isomorphism be rearranged to have an orthogonal involution on at least one of the two factors. See e.g. [5, Proposition 5.5] for a proof in arbitrary characteristic for this fact.

(iii \Rightarrow iv) Assume that Q_1, \dots, Q_{n-1} are independent σ -stable F -subalgebras of A such that $\sigma|_{Q_i}$ is orthogonal for $1 \leq i \leq n-1$. Let $B = Q_1 \cdots Q_{n-1}$ and $C = \mathbf{C}_A(B)$. We set $\sigma_B = \sigma|_B$, $\sigma_C = \sigma|_C$ and $\sigma_i = \sigma|_{Q_i}$ for $1 \leq i \leq n-1$. Then (C, σ_C) is an F -algebra with involution, and we have

$$(B, \sigma_B) \simeq \bigotimes_{i=1}^{n-1} (Q_i, \sigma_i) \quad \text{and} \quad (A, \sigma) \simeq (B, \sigma_B) \otimes (C, \sigma_C).$$

Since $\sigma_1, \dots, \sigma_{n-1}$ are orthogonal, so is σ_B , by [17, (2.23)]. Hence the types of the involutions σ and σ_C coincide, and we have that $\text{cap}(B, \sigma_B) = 2^{n-1}$ and $\text{cap}(A, \sigma) = \text{cap}(B, \sigma_B) \cdot \text{cap}(C, \sigma_C)$. Since $\text{cap}(A, \sigma) = 2^n = 2 \cdot \text{cap}(B, \sigma_B)$, we obtain that $\text{cap}(C, \sigma_C) = 2$. Hence there exists an étale quadratic extension K of F contained in $\text{Sym}(\sigma_C)$, and by [6, Corollary 6.6], K is contained in a σ -stable F -quaternion subalgebra Q_n of C . Then Q_1, \dots, Q_n are independent and $\sigma|_{Q_n}$ is orthogonal.

(iv \Rightarrow i) Assume that Q_1, \dots, Q_n are independent σ -stable F -subalgebras of A such that $\sigma|_{Q_i}$ is orthogonal for $1 \leq i \leq n$. We set $B = Q_1 \cdots Q_n$, $C = \mathbf{C}_A(B)$, $\sigma_B = \sigma|_B$, $\sigma_C = \sigma|_C$ and $\sigma_i = \sigma|_{Q_i}$ for $1 \leq i \leq n$. Then

$$(B, \sigma_B) \simeq \bigotimes_{i=1}^n (Q_i, \sigma_i) \quad \text{and} \quad (A, \sigma) \simeq (B, \sigma_B) \otimes (C, \sigma_C).$$

Since $\sigma_1, \dots, \sigma_n$ are orthogonal, σ_B is orthogonal as well, by [17, (2.23)]. We conclude that $\text{cap}(B, \sigma_B) = 2^n = \text{cap}(A, \sigma)$ and that (C, σ_C) is an F -algebra with involution with $\text{cap}(C, \sigma_C) = 1$, thus an algebra with canonical involution, by Proposition 3.2. If σ is symplectic, then C is an F -quaternion algebra, and otherwise $C = \mathbf{Z}(A)$. Therefore (A, σ) is totally decomposable. \square

We retrieve the following well-known fact, which is trivial in the orthogonal case, and which is contained in [17, (2.22)] in the unitary case, and in [17, (16.16)] in the symplectic case.

4.2. Corollary. *If $\text{cap}(A, \sigma) = 2$, then (A, σ) is totally decomposable.*

Proof: This is the implication (iii \Rightarrow i) in Proposition 4.1 for $n = 1$. (Alternatively the statement is obtained directly from [6, Corollary 6.6], which was also used in the proof of (iii \Rightarrow iv) in Proposition 4.1.) \square

In order to investigate decomposability of (A, σ) , we first try to establish the existence of an appropriate neat subextension and then to obtain criteria for the decomposability of (A, σ) along this subextension—in a sense to be defined.

Let L be an étale F -subalgebra of A contained in $\text{Sym}(\sigma)$. We say that (A, σ) *decomposes along L* or is *decomposable along L* if we have $[L : F] = 2^r$ for some $r \in \mathbb{N}$ and there exist independent σ -stable F -quaternion subalgebras Q_1, \dots, Q_r of A such that $Q_i \cap L$ is a quadratic F -algebra for $1 \leq i \leq r$. Note that in this case L is necessarily neat in (A, σ) , by Proposition 3.4.

4.3. Corollary. *For $n \in \mathbb{N}$ such that $\text{cap}(A, \sigma) = 2^n$, the following are equivalent:*

- (i) (A, σ) is totally decomposable.
- (ii) (A, σ) decomposes along some neat F -subalgebra L with $[L : F] = 2^{n-1}$.
- (iii) (A, σ) decomposes along some neat F -subalgebra M with $[M : F] = 2^n$.

Proof: This is immediate from Proposition 4.1. □

5. HYPERBOLIC INVOLUTIONS AND QUATERNION FACTORS

Let (A, σ) be an F -algebra with involution. We call the involution σ *isotropic* if $\sigma(x)x = 0$ for some $x \in A \setminus \{0\}$, and *anisotropic* otherwise. We call σ *metabolic* if there exists $e \in A$ such that $e^2 = e$, $\sigma(e)e = 0$ and $\dim_F eA = \frac{1}{2}\dim_F A$. We further call σ *hyperbolic* if there exists $e \in A$ such that $e^2 = e$, $\sigma(e) = 1 - e$. Note that every hyperbolic involution is metabolic and every metabolic involution is isotropic. Note further that every symplectic involution on a split algebra and every unitary involution of inner type is hyperbolic. We recollect from [9, Lemma A.3] and [10, Proposition 4.10] the following fact.

5.1. Proposition. *The involution σ is hyperbolic if and only if σ is metabolic and $1 \in \text{Symd}(\sigma)$.*

Proof: If σ is hyperbolic, then for $e \in A$ with $e^2 = e$ and $\sigma(e) = 1 - e$, we obtain that $1 = e + \sigma(e) \in \text{Symd}(\sigma)$ and $\sigma(e)e = 0$, so in particular σ is metabolic.

Assume now that σ is metabolic and $1 \in \text{Symd}(\sigma)$. If $\text{char}(F) \neq 2$, then it follows by [10, Proposition 4.10] that σ is hyperbolic. If $\text{char}(F) = 2$, then the condition that $1 \in \text{Symd}(\sigma)$ says that σ is not orthogonal, and it follows by [9, Lemma A.3] that σ is hyperbolic. □

We are going to characterise the hyperbolicity of the involution σ by the existence of certain σ -stable F -quaternion subalgebras. The following statement provides the basis to this approach.

5.2. Proposition. *If $\text{cap}(A, \sigma) = 2$, then σ is either anisotropic or metabolic.*

Proof: If (A, σ) is unitary of inner type or symplectic with A split, then σ is hyperbolic. Assume now that (A, σ) is not unitary of inner type. If σ is symplectic, assume further that A is not split. Thus, the F -dimension of every nontrivial right ideal of A is $\frac{1}{2}\dim_F A$. If σ is isotropic, pick $x \in A \setminus \{0\}$ such that $\sigma(x)x = 0$, and choose $e \in A$ with $e^2 = e$ and $xA = eA$. Then $\sigma(e)e = 0$, and furthermore $\dim_F eA = \frac{1}{2}\dim_F A$, because xA is a nontrivial right ideal of A , hence σ is metabolic. □

The following statement is a variation of [8, Theorem 2.2], without restriction on the characteristic.

5.3. Proposition. *Assume that $\text{cap}(A, \sigma)$ is even and $1 \in \text{Symd}(\sigma)$. Then the following are equivalent:*

- (i) σ is hyperbolic and $\text{coind } A$ is even.
- (ii) There exists a split σ -stable F -quaternion subalgebra $Q \subseteq A$ such that $\sigma|_Q$ is orthogonal and metabolic.
- (iii) There exist elements $u, v \in A$ such that $u^2 = u$, $v^2 = 1$, $uv + vu = v$, $\sigma(u) = 1 - u + uv$ and $\sigma(v) = -1 + 2u + v - uv$.

Note that $\text{coind } A$ is necessarily even if σ is hyperbolic, except in the unitary case of inner type, so the second condition in (i) is relevant only in this case. Also, each of the properties (i), (ii), (iii) readily implies that $\text{ind } A$ is even. Therefore, the condition that $\text{cap}(A, \sigma)$ is even is needed only when σ is symplectic.

Proof of Proposition 5.3: (iii \Rightarrow ii) Assume that $u, v \in A$ satisfy the relations in (iii). Then $Q = F \oplus uF \oplus vF \oplus uvF$ is a σ -stable F -quaternion subalgebra of A . Note that $\sigma|_Q$ is not the canonical involution on Q , because $u^2 = u$ and $\sigma(u) \neq 1 - u$. Hence $\sigma|_Q$ is orthogonal. Since

$$\sigma(u)u = (1 - u + uv)u = uvu = u(v - uv) = (u - u^2)v = 0,$$

we have that $\sigma|_Q$ is isotropic. As Q is an F -quaternion algebra, it follows by Proposition 5.2 that $\sigma|_Q$ is metabolic.

(ii \Rightarrow i) Let Q be a σ -stable F -quaternion subalgebra of A such that $\sigma|_Q$ is orthogonal and metabolic. Let C be the centralizer of Q in A . Then C is σ -stable and

$$(A, \sigma) \simeq (Q, \sigma|_Q) \otimes (C, \sigma|_C).$$

Since $\sigma|_Q$ is metabolic, it follows that Q is split and σ is metabolic. Hence $\text{coind } A$ is even, and as $1 \in \text{Symd}(\sigma)$, we obtain by Proposition 5.1 that σ is hyperbolic.

(i \Rightarrow iii) Suppose that (A, σ) is hyperbolic and $\text{coind } A$ is even. Since $\text{cap}(A, \sigma)$ is even as well, there exists an F -algebra with involution (B, τ) of the same type as (A, σ) and such that $A \simeq \mathbb{M}_2(F) \otimes B$. The matrices $u = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ and $v = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ in $\mathbb{M}_2(F)$ satisfy the relations in (iii) with respect to the involution $\sigma' = \text{Int}(u+v) \circ t$ on $\mathbb{M}_2(F)$, where t denotes the transposition involution on $\mathbb{M}_2(F)$. As in the proof of (iii \Rightarrow ii) we obtain that $(\mathbb{M}_2(F), \sigma')$ is metabolic. It follows that $(\mathbb{M}_2(F), \sigma') \otimes (B, \tau)$ is metabolic. Since $\sigma' \otimes \tau$ is of the same type as σ , we have that $1 \in \text{Symd}(\sigma' \otimes \tau)$ and conclude by Proposition 5.1 that $(\mathbb{M}_2(F), \sigma') \otimes (B, \tau)$ is hyperbolic. It follows from [17, (12.35)] that all algebras with hyperbolic involution of the same type and with the same underlying algebra are isomorphic. Since $A \simeq \mathbb{M}_2(F) \otimes B$, we conclude that $(A, \sigma) \simeq (\mathbb{M}_2(F), \sigma') \otimes (B, \tau)$. Hence A contains elements u and v satisfying the equations in (iii) with respect to σ . \square

5.4. Corollary. *Let K be a separable field extension of F contained in $\text{Sym}(\sigma)$. Let $C = \mathbf{C}_A(K)$ and assume that $\text{coind } C$ and $\text{cap}(C, \sigma|_C)$ are even and $\sigma|_C$ is hyperbolic. Then there exists a σ -stable split F -quaternion subalgebra Q of A such that $\sigma|_Q$ is orthogonal and metabolic and such that $K \subseteq \mathbf{C}_A(Q)$.*

Proof: By Proposition 5.3, there exist elements $u, v \in C$ such that $u^2 = u$, $v^2 = 1$, $uv + vu = v$, $\sigma(u) = 1 - u + uv$ and $\sigma(v) = -1 + 2u + v - uv$. Then

$Q = F \oplus Fu \oplus Fv \oplus Fuv$ is a σ -stable F -quaternion subalgebra of A such that $\sigma|_Q$ is orthogonal and metabolic and such that $K \subseteq C_A(Q)$. \square

Our last two results in this section provide conditions under which a quadratic étale F -algebra K contained in $\mathbf{Sym}(\sigma)$ can be embedded into a split σ -stable F -quaternion subalgebra Q such that $\sigma|_Q$ is orthogonal and metabolic. We first consider in Proposition 5.5 the case where K is split, and then in Theorem 5.6 the case where K is a field. In the following proof we interpret an involution as adjoint to a hermitian form and use the correspondence between the concepts of hyperbolicity for the two sorts of objects.

5.5. Proposition. *Let K be a neat subalgebra of (A, σ) such that $K \simeq F \times F$. Assume that σ is hyperbolic. Then K is contained in a split σ -stable F -quaternion subalgebra Q of A such that $\sigma|_Q$ is orthogonal and metabolic.*

Proof: We first assume that A is simple. In this case we may identify A with $\mathbf{End}_D(V)$ for a finite-dimensional F -division algebra D and a finite-dimensional D -right vector space V . We fix an involution τ on D of the same kind as σ . By Proposition 5.1, since σ is hyperbolic, σ is not orthogonal if $\mathbf{char} F = 2$. Note further that, if $\mathbf{char} F \neq 2$ or if τ is unitary, then every hermitian or skew-hermitian form over (D, τ) is even, by [16, Chap. I, Lemma 6.6.1]. Using the identification of A with $\mathbf{End}_D(V)$, we therefore obtain by [17, (4.2)] that the involution σ is adjoint to a nondegenerate even hermitian or skew-hermitian form $h: V \times V \rightarrow D$ with respect to τ .

Let e and e' be the primitive idempotents in K . Hence $e' = 1 - e$ and we have that e and e' are the two projections given by a direct decomposition $V = W \oplus W'$ for two D -subspaces W and W' of V . Since $e, e' \in K \subseteq \mathbf{Sym}(\sigma)$ we obtain that W and W' are orthogonal to one another with respect to h . Hence, we have an orthogonal decomposition

$$(V, h) \simeq (W, h|_W) \perp (W', h|_{W'}).$$

Since K is neat in (A, σ) , we have that $(eAe, \sigma|_{eAe})$ and $(e'Ae', \sigma|_{e'Ae'})$ have the same degree and the same type. Since these F -algebras with involution correspond to the D -endomorphism algebras of W and of W' with their involutions adjoint to the restrictions of h , we conclude that $\dim_D W = \dim_D W' = \frac{1}{2} \dim_D V$ and that the restrictions of h to W and W' have the same type as h , hermitian or skew-hermitian.

Since σ is hyperbolic, so is (V, h) . Since also $(W', -h|_{W'}) \perp (W', h|_{W'})$ is hyperbolic and of the same dimension as (V, h) , it follows by [22, Proposition 7.7.3] that $(V, h) \simeq (W', -h|_{W'}) \perp (W', h|_{W'})$. Since h is even, so is $h|_{W'}$, and hence we may apply Witt Cancellation [16, Chap. I, Proposition 6.4.5] and conclude that

$$(W, h|_W) \simeq -(W', h|_{W'}).$$

Let $g: W \rightarrow W'$ be an isometry between $h|_W$ and $-h|_{W'}$. Let $f: V \rightarrow V$ be the D -automorphism of V determined by $f(w + w') = g^{-1}(w') + g(w)$ for $w \in W$ and

$w' \in W'$. It follows that $\sigma(f) = -f$, $ef + fe = f$ and $f^2 = \text{id}_V$. We conclude that e and f generate a split F -quaternion subalgebra Q which is σ -stable and such that $\sigma|_Q$ is orthogonal. Furthermore, since $f \neq 1$ and

$$\sigma(1 - f) \cdot (1 - f) = (1 + f)(1 - f) = 1 - f^2 = 0,$$

we conclude that $\sigma|_Q$ is isotropic. Since Q is an F -quaternion algebra, it follows by Proposition 5.2 that $\sigma|_Q$ is metabolic. This concludes the proof for the case where A is simple.

Suppose finally that (A, σ) is unitary of inner type. Hence, we may identify (A, σ) with $(B \times B^{\text{op}}, \text{sw})$ for some central simple F -algebra B . We obtain that $K = \{(x, x) \mid x \in K_0\}$ for an F -algebra K_0 contained in B , isomorphic to $F \times F$ and such that B is free as a K_0 -left module. Hence $\text{coind } B$ is even. We use a variation of the above argument, without involutions and hermitian forms. We identify B with $\text{End}_D(V)$ for a finite-dimensional F -division algebra D and a finite-dimensional D -right vector space V . As in the previous case the two primitive idempotents of K_0 give rise to a decomposition $V = W \oplus W'$ where the D -subspaces W and W' of V are of the same dimension and therefore isomorphic. We fix a D -isomorphism $g: W \rightarrow W'$ and then define a D -automorphism f of V determined by letting $f(w + w') = g^{-1}(w') + g(w)$ for $w \in W$ and $w' \in W'$. Let $e: V \rightarrow W$ be the projection on the first component for the decomposition $V = W \oplus W'$. Then f and e generate a split F -quaternion subalgebra Q_0 of B . We fix an orthogonal metabolic involution τ on Q_0 . Then $Q = \{(x, \tau(x)) \mid x \in Q_0\}$ is a split F -quaternion subalgebra of $A = B \times B^{\text{op}}$ containing K and stable under $\sigma = \text{sw}$. Furthermore, $(Q, \sigma|_Q) \simeq (Q_0, \tau)$, whereby $\sigma|_Q$ is orthogonal and metabolic. \square

5.6. Theorem. *Let K be a separable quadratic field extension of F contained in $\text{Sym}(\sigma)$ and let $C = \mathbf{C}_A(K)$. Assume that σ is hyperbolic and that $\sigma|_C$ is anisotropic. Then K is contained in a σ -stable split F -quaternion subalgebra Q such that $\sigma|_Q$ is orthogonal and metabolic.*

Proof: Let γ be the nontrivial F -automorphism of K and

$$C' = \{x \in A \mid xk = \gamma(k)x \text{ for every } k \in K\}.$$

By [6, Proposition 6.1] we have that $\sigma(C) = C$, $\sigma(C') = C'$ and $A = C \oplus C'$.

Let $e \in A$ be such that $e^2 = e$ and $\sigma(e) = 1 - e$. We write $e = v + w$ with $v \in C$ and $w \in C'$. We have

$$(v^2 + w^2) + (vw + wv) = e^2 = e = v + w.$$

Note that $v^2, w^2 \in C$ and $vw, wv \in C'$. As $A = C \oplus C'$, we obtain that

$$v^2 + w^2 = v \quad \text{and} \quad vw + wv = w.$$

Furthermore $\sigma(v) + \sigma(w) = \sigma(e) = 1 - e = (1 - v) - w$. As $\sigma(C) = C$ and $\sigma(C') = C'$, it follows that

$$\sigma(v) = 1 - v \quad \text{and} \quad \sigma(w) = -w.$$

We conclude that

$$wv = w - vw = \sigma(v)w.$$

We claim that there exists $v' \in C$ with $vv' = 1$. Suppose $x \in C$ is such that $vx = 0$. Then $\sigma(x) \cdot (1 - v) = \sigma(vx) = 0$, and therefore

$$\sigma(x) = \sigma(x)v.$$

It follows that

$$\sigma(x)x = \sigma(x)vx = 0.$$

Since $\sigma|_C$ is anisotropic we conclude that $x = 0$. Therefore the F -linear map $C \rightarrow C$ given by multiplication with v from the left is injective. Since C is finite-dimensional, this map is also surjective. Hence there exists an element $v' \in C$ with $vv' = 1$. Using that $wv = \sigma(v)w$ we obtain that

$$\sigma(v')w = \sigma(v')wvv' = \sigma(v')\sigma(v)wv' = \sigma(vv')wv' = wv'.$$

As $\sigma(w) = -w$ we conclude that $\sigma(wv') = -wv'$ and

$$(wv')^2 = \sigma((wv')^2) = \sigma(v')w\sigma(v')w = \sigma(v')w^2v' = \sigma(v')(v - v^2)v' = \sigma(v')(1 - v).$$

As $\sigma(v) = 1 - v$ we obtain that $(wv')^2 = 1$. Since $wv' \in C'$ it follows that $Q = K \oplus Kwv'$ is a split σ -stable F -quaternion subalgebra. Since K is a quadratic étale F -algebra and $\sigma|_K = \text{id}_K$, the involution $\sigma|_Q$ is orthogonal. Since

$$\sigma(1 + wv')(1 + wv') = (1 - wv')(1 + wv') = 1 - (wv')^2 = 0,$$

we have that $\sigma|_Q$ is metabolic. □

6. HYPERBOLICITY IN CAPACITY FOUR

Our study of algebras with involution of capacity 4 in Section 7 will crucially rely on the special case where the involution is hyperbolic, which we study in this section. We start by showing that an algebra with hyperbolic involution in capacity four is decomposable along any neat quadratic subalgebra (Proposition 6.1), except in one special case. We will then show that any decomposable unitary or symplectic involution of capacity four can be made hyperbolic by passing to the function field of some quadratic form of dimension at least five. This will allow us in Section 7 to show a certain Pfister form which we will attach to an algebra with involution of capacity 4 is independent from certain choices which we make in its construction.

Let (A, σ) be an F -algebra with involution. If σ is unitary, $\text{exp}A$ is not necessarily 1 or 2. Example 6.2 below shows that we need to exclude the case where $\text{exp}A > 2$ in the following statement.

6.1. Proposition. *Assume that $\text{cap}(A, \sigma) = 4$, $\text{exp}A \leq 2$ and σ is hyperbolic. Then (A, σ) is decomposable along every neat quadratic F -subalgebra of (A, σ) .*

Proof: We have $1 \in \text{Symd}(\sigma)$, because σ is hyperbolic. Let K be an arbitrary neat quadratic F -subalgebra of (A, σ) . To prove the statement we need to show that K is contained in a σ -stable F -quaternion subalgebra of A . When $K \simeq F \times F$ this already follows by Proposition 5.5, so we may assume that K is a field.

Suppose first that A is not simple. In this case we may identify (A, σ) with $(B \times B^{\text{op}}, \text{sw})$ for some central simple F -algebra B . Then $K = \{(x, x) \mid x \in K_0\}$ for a separable quadratic field extension K_0 of F contained in B . Since we have $\text{exp}B = \text{exp}A \leq 2$, it follows by a theorem of Albert, [17, (16.2)], that K_0 is contained in an F -quaternion subalgebra Q_0 of B . We fix an orthogonal involution τ on Q_0 with $\tau|_{K_0} = \text{id}_{K_0}$. Then $Q = \{(x, \tau(x)) \mid x \in Q_0\}$ is an F -quaternion subalgebra of $A = B \times B^{\text{op}}$ containing K and Q is stable under the involution $\sigma = \text{sw}$. Furthermore, $(Q, \sigma|_Q) \simeq (Q_0, \tau)$, whereby $\sigma|_Q$ is orthogonal.

Assume now that A is simple. Let $C = \mathbf{C}_A(K)$ and $\sigma_C = \sigma|_C$. Then C is simple and (C, σ_C) is a K -algebra with involution with $\text{deg}C = \frac{1}{2}\text{deg}A$. Since σ_C has the same type as σ by [6, Proposition 3.3], it follows that $\text{cap}(C, \sigma_C) = 2$. If σ_C is anisotropic, then we obtain the desired conclusion by Theorem 5.6. Assume now that σ_C is isotropic. As C is simple and σ_C is isotropic, $\text{coind}C$ is even. As further $\text{cap}(C, \sigma_C) = 2$, it follows by Proposition 5.2 and Proposition 5.1 that (C, σ_C) is hyperbolic. Hence, by Corollary 5.4 it follows that C contains a σ -stable split F -quaternion subalgebra Q' such that $(Q', \sigma|_{Q'})$ is orthogonal and metabolic. Then $D = \mathbf{C}_A(Q')$ is a σ -stable F -subalgebra of A containing K . Let $\sigma_D = \sigma|_D$. Then (D, σ_D) is an F -algebra with involution with $\text{cap}(D, \sigma_D) = 2$ and σ_D is of the same type as σ . Since $K \subseteq \text{Sym}(\sigma_D)$, it follows by [6, Corollary 6.6] that K is contained in a σ -stable F -quaternion subalgebra Q of D , and hence of A . \square

The following example explains why the condition on the exponent cannot be omitted in the statement of Proposition 6.1.

6.2. Example. Let B be a central F -division algebra with $\text{exp}B = \text{deg}B = 4$. Then $(B \times B^{\text{op}}, \text{sw})$ is an F -algebra with unitary involution of capacity 4, which is hyperbolic and indecomposable.

We will use standard notation from [12] for diagonal quadratic forms in characteristic different from 2 and for nonsingular binary quadratic forms in arbitrary characteristic. We recall some quadratic form terminology from [12, (7.17)], in particular concerning the radicals of a quadratic form and of its polar form.

Let $q : V \rightarrow F$ be a quadratic form over F , defined on a finite-dimensional F -vector space V . We denote by b_q the *polar form of q* given by

$$V \times V \rightarrow F, \quad (x, y) \mapsto q(x + y) - q(x) - q(y).$$

We further set

$$\begin{aligned}\mathrm{rad}(b_q) &= \{x \in V \mid b_q(x, y) = 0 \text{ for all } y \in V\} \\ \mathrm{rad}(q) &= \{x \in \mathrm{rad}(b_q) \mid q(x) = 0\}\end{aligned}$$

and observe that these are F -subspaces of V with $\mathrm{rad}(q) \subseteq \mathrm{rad}(b_q)$. Moreover, if $\mathrm{char} F \neq 2$ then $q(x) = \frac{1}{2}b_q(x, x)$ for all $x \in V$ and thus $\mathrm{rad}(q) = \mathrm{rad}(b_q)$. We call the quadratic form q *regular* if $\mathrm{rad}(q) = \{0\}$ and *nondegenerate* if q is regular and $\dim_F \mathrm{rad}(b_q) \leq 1$.

6.3. Lemma. *Assume that $\mathrm{cap}(A, \sigma) = 4$, σ is unitary or symplectic and (A, σ) is totally decomposable. Then there exists an F -subspace V of $\mathrm{Sym}(\sigma)$ with*

$$\dim_F V = \begin{cases} 6 & \text{if } \sigma \text{ is symplectic,} \\ 5 & \text{if } \sigma \text{ is unitary,} \end{cases}$$

such that $x^2 \in F$ for all $x \in V$ and such that $\psi: V \rightarrow F$, $x \mapsto x^2$ is a nondegenerate quadratic form. Furthermore, if for such an F -space V the corresponding form ψ is isotropic, then σ is hyperbolic.

The inclusion $V \hookrightarrow A$ yields an F -algebra homomorphism from the Clifford algebra $\mathbb{C}(\psi)$ to A , which will be used in Lemma 6.4.

Proof: Assume first that σ is symplectic. Since (A, σ) is totally decomposable, we obtain that

$$(A, \sigma) \simeq (Q_1, \mathrm{can}_{Q_1}) \otimes (Q_2, \mathrm{can}_{Q_2}) \otimes (Q_3, \mathrm{can}_{Q_3})$$

for three F -quaternion algebras Q_1, Q_2, Q_3 . For $i = 1, 2, 3$ we fix $u_i, v_i \in Q_i$ with

$$(6.3.1) \quad u_i^2, v_i^2 \in F^\times \text{ and } u_i v_i + v_i u_i = \begin{cases} 0 & \text{if } \mathrm{char}(F) \neq 2, \\ 1 & \text{if } \mathrm{char}(F) = 2. \end{cases}$$

(The existence of u_i and v_i in $\mathrm{char}(F) = 2$ follows from the presentation of quaternion algebras in [17, p. 25].) Set

$$V = \begin{cases} (Fu_1 \oplus Fv_1 \oplus Fu_1 v_1)u_3 \oplus (Fu_2 \oplus Fv_2 \oplus Fu_2 v_2)v_3 & \text{if } \mathrm{char}(F) \neq 2, \\ Fu_1 \oplus Fv_1 \oplus Fu_2 \oplus Fv_2 \oplus Fu_3 \oplus Fv_3 & \text{if } \mathrm{char}(F) = 2. \end{cases}$$

Then $\dim_F V = 6$ and $V \subseteq \mathrm{Sym}(\sigma)$, and one can easily check that $x \mapsto x^2$ defines a nondegenerate quadratic form $V \rightarrow F$: In fact, letting $a_i = u_i^2, b_i = v_i^2 \in F^\times$ for $i = 1, 2, 3$, we obtain that

$$\psi \simeq \begin{cases} a_3 \langle a_1, b_1, -a_1 b_1 \rangle \perp b_3 \langle a_2, b_2, -a_2 b_2 \rangle & \text{if } \mathrm{char}(F) \neq 2, \\ [a_1, b_1] \perp [a_2, b_2] \perp [a_3, b_3] & \text{if } \mathrm{char}(F) = 2. \end{cases}$$

Assume now that σ is unitary. Then

$$(A, \sigma) \simeq (K, \sigma_K) \otimes (Q_1, \mathrm{can}_{Q_1}) \otimes (Q_2, \mathrm{can}_{Q_2})$$

for $K = Z(A)$, $\sigma_K = \sigma|_K$ and two F -quaternion algebras Q_1, Q_2 . If $\text{char}(F) \neq 2$ then let $d \in F^\times$ be such that $K \simeq F(\sqrt{d})$. Hence there exists $w \in K$ such that $K = F \oplus Fw$ and $w^2 = d$. For $i = 1, 2$ we fix $u_i, v_i \in Q_i$ satisfying (6.3.1). Set

$$V = \begin{cases} (Fu_1 \oplus Fv_1 \oplus Fu_1v_1)u_2 \oplus Fwv_2 \oplus Fwu_2v_2 & \text{if } \text{char}(F) \neq 2, \\ Fu_1 \oplus Fv_1 \oplus Fu_2 \oplus Fv_2 \oplus F & \text{if } \text{char}(F) = 2. \end{cases}$$

Then $\dim_F V = 5$ and $V \subseteq \text{Sym}(\sigma)$, and one can easily check that $x \mapsto x^2$ defines a nondegenerate quadratic form $\psi: V \rightarrow F$: In fact, letting $a_i = u_i^2, b_i = v_i^2 \in F^\times$ for $i = 1, 2$, we obtain that

$$\psi \simeq \begin{cases} a_2 \langle a_1, b_1, -a_1b_1 \rangle \perp db_2 \langle 1, -a_2 \rangle & \text{if } \text{char}(F) \neq 2, \\ [a_1, b_1] \perp [a_2, b_2] \perp \langle 1 \rangle & \text{if } \text{char}(F) = 2. \end{cases}$$

After having chosen V and ψ case by case, the rest of the argument can be given uniformly for all four cases. Note that

$$\psi(x + y) - \psi(x) - \psi(y) = xy + yx \quad \text{for any } x, y \in V.$$

Suppose now that ψ is isotropic. Fix $x \in V \setminus \{0\}$ such that $\psi(x) = 0$. Since ψ is nondegenerate, there exists $y \in V \setminus \{0\}$ such that $xy + yx = 1$. It easily follows that $Q = F \oplus Fx \oplus Fy \oplus Fxy$ is a σ -stable F -quaternion subalgebra of A . Since $x\sigma(x) = x^2 = \psi(x) = 0$, it follows that $\sigma|_Q$ is isotropic, and since Q is an F -quaternion algebra, we conclude that $\sigma|_Q$ is metabolic. This implies that σ is metabolic. As σ is symplectic or unitary, we obtain by Proposition 5.1 that σ is hyperbolic. \square

6.4. Lemma. *Assume that (A, σ) is totally decomposable, $\text{cap}(A, \sigma) = 4$ and $1 \in \text{Symd}(\sigma)$. Let $n = 1$ if σ is orthogonal, $n = 2$ if σ is unitary, and $n = 3$ if σ is symplectic. There exists a field extension F'/F such that $(A, \sigma)_{F'}$ is hyperbolic and every anisotropic quadratic n -fold Pfister form over F remains anisotropic over F' .*

Proof: Assume first that σ is orthogonal. Then the hypothesis implies that $\text{char}(F) \neq 2$. It is well-known (see e.g. [17, Corollary 15.12] or [7, Proposition 3.9]) that every tensor product of two F -quaternion algebras with orthogonal involution is isomorphic to a tensor product of two F -quaternion algebras with canonical involution. In particular, since (A, σ) is totally decomposable, there exists a σ -stable F -quaternion subalgebra Q of A on which σ restricts to the canonical involution. Let F'/F be the function field of the Severi–Brauer variety associated with Q , or equivalently, the function field of the projective conic given by the pure part of the norm form of Q . Then $Q_{F'}$ is split, whereby $\sigma|_{Q_{F'}}$ is hyperbolic. Hence $(A, \sigma)_{F'}$ is hyperbolic. Since F is relatively algebraically closed in F' , every anisotropic 1-fold Pfister form over F stays anisotropic over F' .

Suppose now that σ is unitary or symplectic. Since (A, σ) is totally decomposable, by Lemma 6.3 there exists an F -subspace V of $\mathbf{Sym}(\sigma)$ with

$$\dim_F V = \begin{cases} 6 & \text{if } \sigma \text{ is symplectic,} \\ 5 & \text{if } \sigma \text{ is unitary,} \end{cases}$$

such that $x^2 \in F$ for all $x \in V$ and such that $V \rightarrow F, x \mapsto x^2$ is a nondegenerate quadratic form. We denote this quadratic form by ψ . Let $F(\psi)$ be the function field of the projective quadric defined by ψ over F . Since $\psi_{F(\psi)}$ is isotropic, it follows by Lemma 6.3 that the $F(\psi)$ -algebra with involution $(A, \sigma)_{F(\psi)}$ is hyperbolic.

Our first attempt for choosing F' is to take $F(\psi)$. Suppose that this choice is not satisfying the claim. Then there must exist an anisotropic quadratic n -fold Pfister form π over F such that $\pi_{F(\psi)}$ is isotropic. Since quadratic Pfister forms are either anisotropic or hyperbolic, it follows that $\pi_{F(\psi)}$ is hyperbolic. By the Subform Theorem [12, (22.5)], this implies that ψ is similar to a subform of π . In particular, $5 \leq \dim(\psi) \leq \dim(\pi)$, whereby $n = 3$. Hence the involution σ is symplectic and $\dim(\psi) = 6$. It follows that π is similar to $\psi \perp \beta$ for a regular 2-dimensional quadratic form β over F . Hence β is similar to the norm form of a separable quadratic extension K/F . Since π is anisotropic over F , so is β , whereby K is a field. Moreover β_K is hyperbolic, and since π_K is a Pfister form containing a form similar to β_K , it follows that π_K is hyperbolic. By Witt Cancellation [12, (8.4)] we obtain that ψ_K is hyperbolic.

Let $\mathbf{C}(\psi)$ denote the Clifford algebra of ψ . Since ψ is nondegenerate, $\mathbf{C}(\psi)$ is a central simple F -algebra with $\dim_F \mathbf{C}(\psi) = 2^{\dim(\psi)} = 2^6$. Moreover, the definition of ψ as the squaring map on an F -subspace of A gives rise to an F -algebra homomorphism $\mathbf{C}(\psi) \rightarrow A$, which is injective, because $\mathbf{C}(\psi)$ is simple. Note that $\deg A = 8$ as (A, σ) is symplectic of capacity 4. Hence $\dim_F \mathbf{C}(\psi) = \dim_F A < \infty$. We conclude that $A \simeq \mathbf{C}(\psi)$.

In particular, as ψ_K is hyperbolic, A_K is split. Therefore $\text{ind } A \leq [K : F] = 2$. Hence A is Brauer equivalent to an F -quaternion algebra Q . Since (A, σ) is totally decomposable, it follows by [11, Theorem 6.1] that $(A, \sigma) \simeq \text{Ad}(\varphi) \otimes (Q, \text{can}_Q)$ for some bilinear 2-fold Pfister form φ over F .

Let ρ be the quadratic 4-fold Pfister form over F given by the tensor product of φ with the norm form of Q . If ρ is isotropic, then since it is a quadratic Pfister form, it is hyperbolic, and it follows by [11, Theorem 5.2] that (A, σ) is hyperbolic. Denoting by $F(\rho)$ the function field of the projective quadric defined by ρ over F , we obtain in any case that $(A, \sigma)_{F(\rho)}$ is hyperbolic. On the other hand, since quadratic Pfister forms are either anisotropic or hyperbolic, it follows by the Subform Theorem [12, (22.5)] that every anisotropic 3-fold Pfister form over F remains anisotropic over $F(\rho)$. Hence we may take $F' = F(\rho)$. \square

7. CONSTRUCTION OF THE DISCRIMINANT FROM A BIQUADRATIC ALGEBRA

In this section we study an F -algebra with involution (A, σ) with

$$1 \in \text{Symd}(\sigma) \quad \text{and} \quad \text{cap}(A, \sigma) = 4.$$

The first hypothesis excludes the case where σ is orthogonal and $\text{char}(F) = 2$. The second hypothesis means that

$$\dim_F A = 2^{n+3} \quad \text{for} \quad n = \begin{cases} 1 & \text{if } \sigma \text{ is orthogonal,} \\ 2 & \text{if } \sigma \text{ is unitary,} \\ 3 & \text{if } \sigma \text{ is symplectic.} \end{cases}$$

We will attach to (A, σ) a quadratic form over F of dimension 2^n and study its properties. This form will yield a criterion for the decomposability of (A, σ) . The construction of a candidate for this form crucially relies on the existence of a neat biquadratic F -subalgebra L of (A, σ) , which was proven in [6, Theorem 7.4]. Note that the condition $1 \in \text{Symd}(\sigma)$ implies that every neat subalgebra of (A, σ) is contained in $\text{Symd}(\sigma)$, by [6, Theorem 4.1]. It also implies that $x^2 \in \text{Symd}(\sigma)$ for all $x \in \text{Sym}(\sigma)$, for if $\ell \in A$ is such that $\ell + \sigma(\ell) = 1$, then $x^2 = x\ell x + \sigma(x\ell x)$.

By a *biquadratic F -algebra* we mean a commutative F -algebra of dimension 4 which is isomorphic to a tensor product of two quadratic F -algebras. Let L be an étale biquadratic F -algebra, and for an F -automorphism γ , let

$$L^\gamma = \{x \in L \mid \gamma(x) = x\}.$$

There are three F -automorphisms γ for which L^γ is a quadratic étale extension of F . Together with id_L , they form a subgroup G of the automorphism group of L such that the fixed field L^G is equal to F . (In other terms, L is a *G -Galois algebra* or a *Galois F -algebra* if G is clear from the context, see [17, (18.15)].) We refer to G as the *Galois group of L over F* .

7.1. Lemma. *Assume that $\text{cap}(A, \sigma) = 4$. Let L be a neat biquadratic F -subalgebra of (A, σ) and let γ be a nontrivial element of the Galois group of L/F . Let $s : L^\gamma \rightarrow F$ be an F -linear form with $\text{Ker}(s) = F$ and let*

$$W = \{x \in \text{Symd}(\sigma) \mid yx = x\gamma(y) \text{ for all } y \in L\}.$$

Then $\dim_F W = 2^n$ for $n = \log_2 \dim_F A - 3$. Moreover, $x^2 \in L^\gamma$ for all $x \in W$, and

$$q : W \rightarrow F, \quad x \mapsto s(x^2)$$

is a nondegenerate quadratic form over F . This form is isotropic if and only if there exists an element $x \in W$ with $x^2 \in F^\times$.

Proof: We set $K = L^\gamma$, $C = \mathbf{C}_A(K)$, $\sigma_C = \sigma|_C$ and $V = C \cap \text{Symd}(\sigma)$. Note that W is an F -subspace of V . If $x \in W$, then x^2 centralises L , hence $x^2 \in L$ by Proposition 3.3, and actually $x^2 \in K$ because x^2 commutes with x . In order to

prove that q is nondegenerate, we relate it to a nondegenerate quadratic form \tilde{q} on V . We will see that \tilde{q} restricts to q on W and is Witt equivalent to q . For the definition of \tilde{q} , we consider separately the cases where K is a field and where K is split.

Assume first that K is a field. Then (C, σ_C) is a K -algebra with involution of the same type as (A, σ) and with $\text{cap}(C, \sigma_C) = 2$, and L is a maximal neat subalgebra of (C, σ_C) . We obtain by [6, Proposition 4.6, Proposition 6.1 and Proposition 6.5] that

$$V = \text{Symd}(\sigma_C) = L \oplus W, \quad \dim_F W = \frac{1}{8} \dim_F A,$$

and that there exists a canonical nondegenerate quadratic form $c_2: V \rightarrow K$ with the following properties: W is the orthogonal complement of L , the restriction $c_2|_L$ is the norm form $\mathbf{N}_{L/K}$ of the quadratic étale extension L/K , and $c_2(x) = -x^2$ holds for all $x \in W$. It follows that the transfer $\tilde{q} = -(s \circ c_2): V \rightarrow F$ is a nondegenerate quadratic form over F such that $\tilde{q}(x) = s(x^2)$ for all $x \in W$, whereby q is the restriction of \tilde{q} to W . Moreover, W and L are orthogonal to one another with respect to \tilde{q} , and since $\dim_F V = \dim_F W + \dim_F L$, it follows that W is the orthogonal complement of L in V with respect to \tilde{q} . This implies that q is nondegenerate and

$$\tilde{q} = -(s \circ \mathbf{N}_{L/K}) \perp q.$$

Since L is a biquadratic F -algebra, the form $\mathbf{N}_{L/K}$ is extended from a quadratic form defined over F . Hence $s \circ \mathbf{N}_{L/K}$ is hyperbolic and \tilde{q} is Witt equivalent to q .

Since $s(1) = 0$, it is clear that q is isotropic whenever there exists $x \in W$ such that $x^2 \in F^\times$. Conversely, assume that q is isotropic. Hence there exists $y \in W \setminus \{0\}$ such that $q(y) = 0$, whereby $y^2 \in \text{Ker}(s) = F$. If $y^2 \neq 0$, then let $x = y$, and otherwise, the regular quadratic form $c_2|_W: W \rightarrow K$, $x \mapsto x^2$ is isotropic and hence universal, so we can find an element $x \in W$ with $x^2 = 1$. This shows that q is isotropic if and only if there exists $x \in W$ with $x^2 \in F^\times$.

Consider now the case where $K \simeq F \times F$. We denote by e_1 and e_2 the two primitive idempotents of K . Since L is a biquadratic étale F -algebra, there exists an F -automorphism γ' of L which is of order 2 and different from γ . Then $\gamma'|_K \neq \text{id}_K$, and it follows that γ' interchanges e_1 and e_2 . Since $L = e_1 L \oplus e_2 L$, we conclude by [6, Lemma 5.8] that K is neat in (A, σ) . For $i = 1, 2$ we set $E_i = e_i A e_i$ and denote by τ_i the restriction of σ to E_i . Then $C = E_1 \oplus E_2$ and the F -algebras E_1 and E_2 are isomorphic. Moreover, (E_1, τ_1) and (E_2, τ_2) are F -algebras with involution, and by [6, Proposition 3.1], the involutions τ_1 and τ_2 have the same type as σ . It follows that $\text{cap}(E_1, \tau_1) = \text{cap}(E_2, \tau_2) = 2$. For $i = 1, 2$, since $e_i L$ is a quadratic étale F -subalgebra of E_i contained in $\text{Sym}(\tau_i)$ and with nontrivial automorphism $\gamma|_{e_i L}$, we obtain by [6, Proposition 4.6, Proposition 6.1 and Proposition 6.5] that

$$e_i V e_i = e_i V = e_i L \oplus e_i W, \quad \dim_F e_i W = \frac{1}{16} \dim_F A,$$

and that there exists a canonical nondegenerate quadratic form $c_{2,i}: e_iV \rightarrow e_iF$ with the following properties: e_iW is the orthogonal complement of e_iL , the restriction $c_{2,i}|_{e_iL}$ is the norm form \mathbf{N}_{e_iL/e_iF} of the quadratic étale extension e_iL/e_iF , and $c_{2,i}(x) = -x^2$ holds for all $x \in e_iW$. We consider the quadratic maps $c_2: V \rightarrow K$, $x \mapsto c_{2,1}(e_1x) + c_{2,2}(e_2x)$ and $\tilde{q} = -(s \circ c_2): V \rightarrow F$. The restriction of c_2 to $L = e_1L \oplus e_2L$ is the norm form $\mathbf{N}_{L/K}$, and its restriction to $W = e_1W \oplus e_2W$ is the map $x \mapsto -x^2 \in K$. The definition of W implies that W and L are orthogonal to one another with respect to \tilde{q} . Since $\dim_F V = \dim_F W + \dim_F L$, we obtain that

$$\tilde{q} = -(s \circ \mathbf{N}_{L/K}) \perp q.$$

Since $\text{Ker}(s) = F$, we have that $s(e_1) = -s(e_2) = \alpha$ for some $\alpha \in F^\times$, whereby $\tilde{q} \simeq -\alpha c_{2,1} \perp \alpha c_{2,2}$. Therefore, the form \tilde{q} is nondegenerate, and the same holds for q . Note further that $\dim_F W = \dim_F e_1W + \dim_F e_2W = \frac{1}{8} \dim_F A$ and that the form $\mathbf{N}_{L/K}$ is extended from F . We conclude that $s \circ \mathbf{N}_{L/K}$ is hyperbolic and q is Witt equivalent to \tilde{q} .

Since $\text{Ker}(s) = F$, it is clear that q is isotropic whenever there exists $x \in W$ such that $x^2 \in F^\times$. Conversely, if q is isotropic, we may find $w_1, w_2 \in W$ with $(w_1, w_2) \neq (0, 0)$ and $\lambda \in F$ such that $(e_i w_i)^2 = e_i \lambda$ for $i = 1, 2$. If $\lambda \neq 0$, then for $x = e_1 w_1 + e_2 w_2 \in W$ we have $x^2 = \lambda \in F^\times$. If $\lambda = 0$, then for $i = 1, 2$, the form $c_{2,i}|_{e_iW}$ is isotropic and hence universal, so we may find w'_i such that $(e_i w'_i)^2 = e_i$, and then $x = e_1 w'_1 + e_2 w'_2 \in W$ satisfies $x^2 = 1$. \square

7.2. Remark. The construction of the form \tilde{q} in the proof of Lemma 7.1 goes back to [13, Section 7], for the case where $\text{char } F \neq 2$ and σ is symplectic. There it also turns out, by a very different argument, that \tilde{q} contains a hyperbolic plane and therefore is Witt equivalent to a smaller subform q . We will use in the sequel quite a different method than in [13] to determine the properties of q .

New: Henceforth, we use the following notation when we consider a neat bi-quadratic F -subalgebra L of (A, σ) : we write $G = \{\text{id}_L, \gamma_1, \gamma_2, \gamma_3\}$ for the Galois group of L/F and for $i = 1, 2, 3$, we set

$$(7.2.1) \quad L_i = L^{\gamma_i} \quad \text{and} \quad \Delta_i = \text{Ker}(\text{T}_i) \subset L_i$$

where $\text{T}_i: L_i \rightarrow F$ is the trace map. Thus, Δ_i is a 1-dimensional F -subspace of L_i , and $\Delta_i = F$ if $\text{char } F = 2$.

7.3. Lemma. *Every nonzero element in Δ_i is invertible, for $i = 1, 2, 3$, and $\Delta_1 \Delta_2 = \Delta_3$.*

Proof: The first part is clear if L_i is a field. If $L_i \simeq F \times F$, then

$$\Delta_i \simeq \{(a, -a) \mid a \in F\},$$

hence every nonzero element in Δ_i is invertible.

To prove $\Delta_1\Delta_2 = \Delta_3$, take $\delta_1 \in \Delta_1$ and $\delta_2 \in \Delta_2$. Since γ_1 restricts to the nontrivial automorphism of L_3/F and of L_2/F , we have

$$\mathsf{T}_3(\delta_1\delta_2) = \delta_1\delta_2 + \gamma_1(\delta_1\delta_2) = \delta_1 \mathsf{T}_2(\delta_2) = 0.$$

□

Now, for $i = 1, 2, 3$, let

$$(7.3.1) \quad W_i = \{x \in \text{Symd}(\sigma) \mid yx = x\gamma_i(y) \text{ for all } y \in L\}.$$

Lemma 7.1 shows that $x_i^2 \in L_i$ for all $x_i \in W_i$. We may therefore define

$$(7.3.2) \quad \varphi_i(x_i) = \mathsf{T}_i(x_i^2) - 2x_i^2 \in L_i \quad \text{for } x_i \in W_i,$$

and observe that $\varphi_i(x_i) \in \Delta_i$.

Define an operation $*$: $A \times A \rightarrow A$ by

$$x * y = xy + yx \quad \text{for } x, y \in A.$$

(If $\text{char}(F) \neq 2$, then $*$ is up to a factor 2 the Jordan operation on A .) Note that for $x, y \in \text{Sym}(\sigma)$ we have $\sigma(xy) = yx$ and thus $x * y \in \text{Symd}(\sigma)$.

7.4. Lemma. *We have*

$$\text{Symd}(\sigma) = L \oplus W_1 \oplus W_2 \oplus W_3$$

and, for $x_1 \in W_1$ and $x_2 \in W_2$,

$$x_1 * x_2 \in W_3, \quad x_1x_2x_1x_2 + x_2x_1x_2x_1 \in F \quad \text{and} \quad \varphi_3(x_1 * x_2) = \varphi_1(x_1)\varphi_2(x_2).$$

Proof: Let $n = \log_2 \dim_F A - 3$. Lemma 7.1 shows that $\dim_F W_i = 2^n$ for $i = 1, 2, 3$, hence

$$\dim_F L + \dim_F W_1 + \dim_F W_2 + \dim_F W_3 = 4 + 3 \cdot 2^n = \dim_F \text{Symd}(\sigma).$$

In order to show that $\text{Symd}(\sigma) = L \oplus W_1 \oplus W_2 \oplus W_3$, it therefore suffices to prove that the sum $L + W_1 + W_2 + W_3$ is direct.

We fix an element $a \in L_1 \setminus F$ such that $\mathsf{T}_1(a) = 1$. Then $2a - 1 \in \Delta_1$ is nonzero, hence it is invertible by Lemma 7.3, and $\gamma_2(a) = \gamma_3(a) = 1 - a$. For $x \in L + W_1$ we have $ax + xa = 2ax$, whereas for $x \in W_2 + W_3$ we obtain that $ax + xa = x$. Since $2a - 1 \in L^\times$ it follows that $(L + W_1) \cap (W_2 + W_3) = 0$. In the same way we obtain that $(L + W_2) \cap (W_1 + W_3) = 0$. Together this implies that the sum $L + W_1 + W_2 + W_3$ is direct.

Now, consider $x_1 \in W_1$ and $x_2 \in W_2$. Using that $\gamma_1 \circ \gamma_2 = \gamma_3 = \gamma_2 \circ \gamma_1$, we obtain for all $z \in L$ that

$$z(x_1x_2) = zx_1x_2 = x_1\gamma_1(z)x_2 = x_1x_2\gamma_2(\gamma_1(z)) = (x_1x_2)\gamma_3(z).$$

In the same way we obtain that $z(x_2x_1) = (x_2x_1)\gamma_3(z)$ for all $z \in L$, hence $z(x_1 * x_2) = (x_1 * x_2)\gamma_3(z)$ for all $z \in L$. Since $x_1 * x_2 \in \text{Symd}(\sigma)$, it follows that $x_1 * x_2 \in W_3$.

Now, for $x_1 \in W_1$ and $x_2 \in W_2$, we have

$$x_1x_2x_1x_2 + x_2x_1x_2x_1 = x_1x_2x_1x_2 + \sigma(x_1x_2x_1x_2) \in \text{Symd}(\sigma).$$

Moreover, $x_1x_2x_1x_2$ and $x_2x_1x_2x_1$ centralise L , because for $z \in L$

$$z(x_1x_2x_1x_2) = (x_1x_2)\gamma_3(z)(x_1x_2) = (x_1x_2x_1x_2)z,$$

and similarly for $x_2x_1x_2x_1$. Therefore, Proposition 3.3 shows that $x_1x_2x_1x_2 + x_2x_1x_2x_1 \in L$. Furthermore, $x_1x_2x_1x_2 + x_2x_1x_2x_1$ commutes with x_1 and x_2 , because x_1^2 commutes with $x_2x_1x_2$ and x_2^2 with $x_1x_2x_1$. Therefore, $x_1x_2x_1x_2 + x_2x_1x_2x_1$ is fixed under γ_1 and γ_2 , and lies therefore in F .

Finally, we compute

$$\begin{aligned} (x_1 * x_2)^2 &= (x_1x_2x_1x_2 + x_2x_1x_2x_1) + x_1x_2^2x_1 + x_2x_1^2x_2 \\ &= (x_1x_2x_1x_2 + x_2x_1x_2x_1) + x_1^2\gamma_1(x_2^2) + \gamma_2(x_1^2)x_2^2. \end{aligned}$$

We know $x_1x_2x_1x_2 + x_2x_1x_2x_1 \in F$, whereby

$$\mathbb{T}_3(x_1x_2x_1x_2 + x_2x_1x_2x_1) = 2(x_1x_2x_1x_2 + x_2x_1x_2x_1),$$

hence

$$\begin{aligned} \varphi_3(x_1 * x_2) &= \mathbb{T}_3(x_1^2\gamma_1(x_2^2) + \gamma_2(x_1^2)x_2^2) - 2(x_1^2\gamma_1(x_2^2) + \gamma_2(x_1^2)x_2^2) \\ &= x_1^2x_2^2 + \gamma_2(x_1^2)\gamma_1(x_2^2) - x_1^2\gamma_1(x_2^2) - \gamma_2(x_1^2)x_2^2 \\ &= (\gamma_2(x_1^2) - x_1^2)(\gamma_1(x_2^2) - x_2^2). \end{aligned}$$

Since, for $i = 1, 2$,

$$\varphi_i(x_i) = \mathbb{T}_i(x_i^2) - 2x_i^2 = \gamma_{3-i}(x_i^2) - x_i^2,$$

we see that $\varphi_3(x_1 * x_2) = \varphi_1(x_1)\varphi_2(x_2)$, and the proof is complete. \square

7.5. Theorem. *Assume that $\text{cap}(A, \sigma) = 4$, let $n = \log_2 \dim_F A - 3$, let L be a neat biquadratic F -subalgebra of (A, σ) and use the notation set up in (7.2.1), (7.3.1) and (7.3.2). For $i = 1, 2, 3$, let $s_i: L_i \rightarrow F$ be an F -linear form with $\text{Ker}(s_i) = F$, and define the quadratic form*

$$q_i: W_i \rightarrow F, \quad x \mapsto s_i(x^2).$$

There exists an n -fold Pfister form over F which is similar to q_1, q_2 and q_3 . This Pfister form is uniquely determined by L .

Proof: We first observe that each of the linear forms s_i is uniquely determined by the condition $\text{Ker}(s_i) = F$ up to a scalar factor, hence the quadratic forms q_i also are determined up to scalars. Since the statement is invariant under scaling the forms q_i , it suffices to prove it for a specific choice of s_1, s_2, s_3 .

For $i = 1, 2$, pick a nonzero element $\delta_i \in \Delta_i$, and let $\delta_3 = \delta_1\delta_2$. Lemma 7.3 shows that δ_1, δ_2 and δ_3 are invertible, and since $\dim_F \Delta_i = 1$ we have $\lambda\delta_i^{-1} \in F$ for all $\lambda \in \Delta_i$. Therefore, we may set

$$s_i(\ell) = (\mathbb{T}_i(\ell) - 2\ell)\delta_i^{-1} \in F \quad \text{for } \ell \in L_i,$$

and obtain $\text{Ker}(s_i) = F$ because for $\ell \in L_i$ the equation $\mathbb{T}_i(\ell) = 2\ell$ holds if and only if $\ell \in F$. With this choice of s_i ,

$$q_i(x) = (\mathbb{T}_i(x^2) - 2x^2)\delta_i^{-1} = \varphi_i(x)\delta_i^{-1} \quad \text{for } x \in W_i.$$

Lemma 7.4 yields $\varphi_3(x_1 * x_2) = \varphi_1(x_1)\varphi_2(x_2)$ for $x_1 \in W_1$ and $x_2 \in W_2$, hence

$$q_3(x_1 * x_2) = \varphi_3(x_1 * x_2)\delta_3^{-1} = \varphi_1(x_1)\delta_1^{-1} \cdot \varphi_2(x_2)\delta_2^{-1} = q_1(x_1)q_2(x_2).$$

As the forms q_i are nondegenerate by Lemma 7.1, we have thus shown that

$$*: (W_1, q_1) \times (W_2, q_2) \rightarrow (W_3, q_3)$$

is a composition of nondegenerate quadratic forms in the sense of [17, p. 488]. As these forms are of dimension 2^n , it follows by [17, (33.18) and (33.27)] that they are similar to one and the same quadratic n -fold Pfister form π over F .

Since π is a Pfister form, it is uniquely determined up to isometry by its similarity class, which is also given by each of the forms q_1 , q_2 and q_3 . Since the linear forms $s_i: L^{\gamma_i} \rightarrow F$ for $i = 1, 2, 3$ are determined up to a scalar by the property that $\text{Ker}(s_i) = F$, it follows that π does also not depend on the choices of s_1, s_2, s_3 . Therefore π is determined up to isometry by L . \square

7.6. Remark. If $\text{char}(F) = 2$, then $\Delta_i = F$ for $i = 1, 2, 3$, so we may take $\delta_i = 1$ in the proof of Theorem 7.5. With this choice, the map s_i is the trace map \mathbb{T}_i for $i = 1, 2, 3$.

Back to the old stuff

For a neat biquadratic F -subalgebra L of (A, σ) , we denote by $\mathfrak{P}_{\sigma, L}$ the quadratic Pfister form over F which is characterised in Theorem 7.5. We will see later that this Pfister form does actually not depend on the choice of L . It is clearly functorial:

7.7. Proposition. *Assume that $\text{cap}(A, \sigma) = 4$. Let L be a neat biquadratic F -subalgebra of (A, σ) and let F'/F be a field extension. Then $L \otimes F'$ is a neat biquadratic F' -subalgebra of $(A, \sigma)_{F'}$ and the associated Pfister form $\mathfrak{P}_{\sigma_{F'}, L \otimes F'}$ is obtained by scalar extension to F' from $\mathfrak{P}_{\sigma, L}$.*

Recall in this context that all tensor products are taken over F .

Proof: See [6, Proposition 5.5] for the fact that $L \otimes F'$ is neat in $(A, \sigma)_{F'}$. The remaining part follows directly from the definitions and from Theorem 7.5. \square

We next give a criterion for the hyperbolicity of $\mathfrak{P}_{\sigma, L}$.

7.8. Proposition. *Assume that $\text{cap}(A, \sigma) = 4$. Let L be a neat biquadratic F -subalgebra of (A, σ) . Let K be a neat quadratic F -subalgebra of (A, σ) contained in L . The following conditions are equivalent:*

- (i) $\mathfrak{P}_{\sigma, L}$ is hyperbolic.
- (ii) (A, σ) is decomposable along L .
- (iii) (A, σ) is decomposable along K .

Proof: Let G be the Galois group of L/F . Since K and L are neat in (A, σ) , it follows that L is free as a K -module and hence K is the fixed subalgebra of some element of G . We write $G = \{\text{id}_L, \gamma_1, \gamma_2, \gamma_3\}$ with $K = L^{\gamma_1}$. For $i = 1, 2, 3$ we set $W_i = \{x \in \text{Symd}(\sigma) \mid yx = x\gamma_i(y) \text{ for all } y \in L\}$ and fix an F -linear form $s_i: L^{\gamma_i} \rightarrow F$ with $\text{Ker}(s_i) = F$, and we consider the quadratic form $q_i: W_i \rightarrow F$, $x \mapsto s_i(x^2)$. Then q_1, q_2 and q_3 are similar to the Pfister form $\mathfrak{P}_{\sigma, L}$. Hence $\mathfrak{P}_{\sigma, L}$ is hyperbolic if and only if any of the forms q_1, q_2 or q_3 is isotropic, and in this case they are all isotropic.

($i \Rightarrow ii$): Suppose that $\mathfrak{P}_{\sigma, L}$ is hyperbolic. Then q_1 is isotropic, and by Lemma 7.1, there exists an element $x \in W_1$ with $x^2 \in F^\times$. Then $Q_2 = L^{\gamma_2} \oplus L^{\gamma_2}x$ is a σ -stable F -quaternion algebra contained in (A, σ) . Since σ is the identity on L^{γ_2} , the involution $\sigma|_{Q_2}$ is orthogonal. Let $A' = \mathbb{C}_A(Q_2)$ and $\sigma' = \sigma|_{A'}$. It follows that (A', σ') is an F -algebra with involution with $\text{cap}(A', \sigma') = 2$. Since $K = L^{\gamma_1} \subseteq A'$, it follows by [17, Corollary 6.6] that K is contained in a σ -stable F -quaternion subalgebra Q_1 of A' , and hence of A . Hence Q_1 and Q_2 are independent σ -stable F -quaternion subalgebras of A such that $L = (L \cap Q_1) \cdot (L \cap Q_2)$. Therefore (A, σ) is decomposable along L .

($ii \Rightarrow iii$): This implication is obvious.

($iii \Rightarrow i$): Recall the quadratic form $\tilde{q}: \mathbb{C}_A(K) \cap \text{Symd}(\sigma) \rightarrow F$ defined in the proof of Lemma 7.1 as a transfer with respect to an F -linear form $s: K \rightarrow F$. This form \tilde{q} is Witt equivalent to q_1 and hence similar to $\mathfrak{P}_{\sigma, L}$. Hence to prove (i) it suffices to show that \tilde{q} is hyperbolic. Condition (iii) yields a decomposition $A = Q \otimes B$ with a σ -stable F -subalgebra B of A and a σ -stable F -quaternion subalgebra of A containing K . For $C = \mathbb{C}_A(K)$ we obtain that

$$C = K \otimes B.$$

Let $\sigma_B = \sigma|_B$. Since σ is the identity on K , the involution $\sigma|_Q$ is orthogonal. Hence σ_B has the same type as σ and $\text{cap}(B, \sigma_B) = 2$. The quadratic form c_2 on $\text{Symd}(\sigma_C)$ over K given by [6, Proposition 4.6] is extended from the corresponding quadratic form c_2 on $\text{Symd}(\sigma_B)$ over F , hence its transfer with respect to $s: K \rightarrow F$ is hyperbolic. Since $\tilde{q} = -s \circ c_2$, it follows that (i) holds. \square

7.9. Corollary. *Assume that $\text{cap}(A, \sigma) = 4$. If σ is hyperbolic and $\text{exp}A \leq 2$, then $\mathfrak{P}_{\sigma, L}$ is hyperbolic for every neat biquadratic F -subalgebra L of (A, σ) .*

Proof: Let L be a neat biquadratic F -subalgebra of (A, σ) . We fix a neat quadratic F -subalgebra K of (A, σ) contained in L . If σ is hyperbolic, then Proposition 6.1 shows that (A, σ) is decomposable along K , and hence $\mathfrak{P}_{\sigma, L}$ is hyperbolic by Proposition 7.8. \square

7.10. Theorem. *Assume that (A, σ) is totally decomposable and $\text{cap}(A, \sigma) = 4$. Then (A, σ) is decomposable along every neat biquadratic F -subalgebra of (A, σ) .*

Proof: Let L be a neat biquadratic F -subalgebra of (A, σ) . By Lemma 6.4 there exists a field extension F'/F such that $(A, \sigma)_{F'}$ is hyperbolic and such

that every anisotropic n -fold Pfister form over F remains anisotropic over F' . Note that $\exp A_{F'} \leq \exp A \leq 2$, because (A, σ) is totally decomposable. Since $\sigma_{F'}$ is hyperbolic, it follows by Proposition 7.7 and Corollary 7.9 that $(\mathfrak{P}_{\sigma, L})_{F'}$ is hyperbolic. By the choice of F'/F , we conclude that $\mathfrak{P}_{\sigma, L}$ is hyperbolic. Therefore (A, σ) is decomposable along L , by Proposition 7.8. \square

7.11. Corollary. *Assume (A, σ) is totally decomposable, $\text{cap}(A, \sigma) = 4$ and $\text{coind } A$ is even. Then (A, σ) has a total decomposition involving a split F -quaternion algebra with orthogonal involution.*

Proof: If σ is hyperbolic then the statement follows directly from Proposition 5.3. Hence we may assume that σ is not hyperbolic. Since $\text{coind}(A, \sigma)$ is even, it follows from [6, Corollary 5.12] that (A, σ) contains a split quadratic neat F -subalgebra K . By [6, Theorem 6.10], there is a neat quadratic subalgebra K' of (A, σ) such that $K \otimes K'$ is a neat biquadratic subalgebra of (A, σ) , which we denote by L . It follows from Theorem 7.10 that (A, σ) is decomposable along L , hence there are independent σ -stable F -quaternion algebras Q_1, Q_2 in A such that $Q_1 \cap L$ and $Q_2 \cap L$ are quadratic subalgebras of L . Thus, writing $L_i = Q_i \cap L$ for $i = 1, 2$, we have

$$K \otimes K' = L = L_1 \otimes L_2.$$

If $K \neq L_1, L_2$, we claim that there exists a σ -stable F -quaternion algebra $Q_3 \subset Q_1 \otimes Q_2$ containing K . To see this, observe that for $i = 1, 2$ we may find $j_i \in Q_i^\times$ such that $\sigma(j_i) = -j_i$, $j_i^2 \in F^\times$ and conjugation by j_i restricts to the nontrivial F -automorphism of L_i . Then K is the subalgebra of L centralised by $j_1 \otimes j_2$, and the subalgebra $Q_3 \subset Q_1 \otimes Q_2$ generated by K and j_1 is an F -quaternion algebra stable under σ . Thus, in all cases we may find an F -quaternion algebra $Q \subset A$ containing K and stable under σ . Since K is split, the quaternion algebra Q is split. Since σ is not hyperbolic, we conclude that $\sigma|_Q$ is not hyperbolic. Hence $\sigma|_Q$ is orthogonal, for the unique symplectic involution on a split quaternion algebra is hyperbolic. \square

We are now in the position to show that the form $\mathfrak{P}_{\sigma, L}$ given by a neat biquadratic subalgebra L of (A, σ) is independent of the choice of this subalgebra.

7.12. Proposition. *Assume that $\text{cap}(A, \sigma) = 4$ and let $n = \log_2 \dim_F A - 3$. There exists a unique n -fold Pfister form π over F such that $\pi \simeq \mathfrak{P}_{\sigma, L}$ for every neat biquadratic F -subalgebra L of (A, σ) .*

Proof: By [6, Theorem 7.4] there exists a neat biquadratic F -subalgebra L of (A, σ) . Consider another neat biquadratic F -subalgebra L' of (A, σ) . We need to show that $\mathfrak{P}_{\sigma, L'} = \mathfrak{P}_{\sigma, L}$.

If (A, σ) is totally decomposable, then we obtain by Theorem 7.10 and Proposition 7.8 that $\mathfrak{P}_{\sigma, L}$ and $\mathfrak{P}_{\sigma, L'}$ are hyperbolic, whereby $\mathfrak{P}_{\sigma, L'} = \mathfrak{P}_{\sigma, L}$. Assume now that (A, σ) is not totally decomposable. Then it follows by Proposition 7.8 that $\mathfrak{P}_{\sigma, L}$ and $\mathfrak{P}_{\sigma, L'}$ are both anisotropic. Let F' denote the function field of the

projective quadric over F given by $\mathfrak{P}_{\sigma,L}$. Then $\mathfrak{P}_{\sigma,L}$ becomes hyperbolic over F' . By Proposition 7.7 and Proposition 7.8, it follows that $(A, \sigma)_{F'}$ is decomposable along $L \otimes F'$. By Theorem 7.10, then $(A, \sigma)_{F'}$ is also decomposable along $L' \otimes F'$. By Proposition 7.7 and Proposition 7.8, it follows that $\mathfrak{P}_{\sigma,L'}$ becomes hyperbolic over F' . Since $\mathfrak{P}_{\sigma,L}$ and $\mathfrak{P}_{\sigma,L'}$ are anisotropic n -fold Pfister forms and since F' is the function field of $\mathfrak{P}_{\sigma,L}$ over F , we conclude by the Subform Theorem [12, (22.5)] that $\mathfrak{P}_{\sigma,L'} = \mathfrak{P}_{\sigma,L}$. \square

When $\text{cap}(A, \sigma) = 4$, we denote by \mathfrak{P}_σ the quadratic Pfister form over F which is characterised in Proposition 7.12 and we call it the *discriminant Pfister form* of (A, σ) .

7.13. Theorem. *Assume that $\text{cap}(A, \sigma) = 4$. The form \mathfrak{P}_σ is hyperbolic if and only if (A, σ) is totally decomposable. Moreover, for any field extension F'/F , we have that $(\mathfrak{P}_\sigma)_{F'} = \mathfrak{P}_{\sigma_{F'}}$.*

Proof: By [6, Theorem 7.4], there exists a neat biquadratic F -subalgebra of (A, σ) . Hence the first part follows from Proposition 7.8 and Theorem 7.10. The second part follows from Proposition 7.7. \square

This finishes the proof of our main theorem stated in the introduction.

8. DETERMINATION OF THE DISCRIMINANT PFISTER FORM

In this final section we relate the discriminant Pfister form of an algebra with involution of capacity 4 to other known invariants and we compute it in some special situations: we show in Proposition 8.2 that in the unitary case the discriminant Pfister form is the norm form of a quaternion algebra that is Brauer equivalent to the discriminant algebra, and in Corollary 8.3 we use this result to give an alternative proof of the Karpenko–Quéguiner criterion for total decomposability of algebras with unitary involution of degree 4. In the symplectic case, we show in Proposition 8.4 that the cohomological invariant defined by Garibaldi–Parimala–Tignol when the characteristic is different from 2 is the Arason invariant of the discriminant Pfister form, and we give an alternative proof of the result characterising totally decomposable involutions by the vanishing of the invariant.

We further aim to relate the notions of discriminant Pfister form for the different types of algebras with involution of capacity 4. We will see in Lemma 8.5 that an embedding between two such algebras with involution leads naturally to a factorisation relation between the corresponding Pfister forms.

In the sequel, we shall use the notion of quaternion K -algebra also in cases where K is an étale F -algebra, but not necessarily a field. More generally, quaternion K -algebras can be defined over an arbitrary commutative ring K : see [16, p. 4] for the general case or [2, p. 27] in the case where K is a semilocal ring (which therefore covers the case where K is an étale F -algebra). The Skolem-Noether theorem, familiar in the case where K is a field, extends to the case where K is

an étale F -algebra: this follows directly by applying it from the situation where the center is a field to the simple components of K (whose centers are the simple factors of K , hence fields).

8.1. Lemma. *Let (A, σ) be an F -algebra with unitary involution of capacity 4. Let L be a neat biquadratic F -subalgebra of (A, σ) with Galois group $\{\text{id}_L, \gamma_1, \gamma_2, \gamma_3\}$ and let $Z = Z(A)$. The following hold:*

- (1) *There exists $w \in \text{Sym}(\sigma) \cap A^\times$ such that $w\ell = \gamma_1(\ell)w$ for all $\ell \in L$.*
- (2) *For every w as in (1), we have $\mathfrak{A}_\sigma \simeq \langle 1, -\text{Nrd}_A(w) \rangle \otimes \mathbf{N}_{N/F}$ for the étale quadratic F -algebra $N = (L^{\gamma_2} \otimes Z)^{\gamma_1 \otimes \sigma|_Z}$.*

Proof: First observe that, after fixing an element $z \in Z \setminus F$ such that $z + \sigma(z) = 1$, every element $a \in A$ can be decomposed as $a = s_1 + s_2z$ with $s_1, s_2 \in \text{Sym}(\sigma)$, given explicitly by $s_2 = (z - \sigma(z))^{-1}(a - \sigma(a))$ and $s_1 = a - s_2z$. Therefore multiplication in A induces an isomorphism of F -vector spaces

$$\text{Sym}(\sigma) \otimes Z \rightarrow A.$$

This allows us to identify $L \otimes Z$ with the F -subalgebra LZ of A . We obtain that LZ is a Galois F -algebra: the nontrivial elements of its Galois group are $\sigma|_{LZ}$ and the maps $\gamma_i \otimes \text{id}_Z$ and $\gamma_i \otimes \sigma|_Z$ for $i = 1, 2, 3$.

To simplify notation we set $K = L^{\gamma_1}$. Let $C = \mathbf{C}_A(K) = \mathbf{C}_A(KZ)$, which is a KZ -quaternion algebra and to which σ restricts as a unitary involution. Let further

$$(8.1.1) \quad D = \{x \in C \mid \sigma(x) = \text{can}_C(x)\}.$$

Then D is a K -quaternion algebra: if KZ is a field, then this follows by [17, (2.22)], using that K is the F -subalgebra of KZ fixed under $\sigma|_{KZ}$. Clearly, $LZ \subseteq C = DZ$, and can_C restricts to $\gamma_1 \otimes \text{id}_Z$ on LZ because KZ is the center of C . Therefore the F -algebra $M = (LZ)^{\gamma_1 \otimes \sigma|_Z}$ lies in D , and it is a quadratic Galois extension of K with Galois group generated by $\gamma_1 \otimes \text{id}_Z$. By the Skolem–Noether Theorem, $(\gamma_1 \otimes \text{id}_Z)|_M$ extends to an inner automorphism $\text{Int}_D(y)$ of D for some $y \in D^\times$. Since $(\gamma_1 \otimes \text{id}_Z)^2|_M = \text{id}_M$, and $D = M[y]$, it follows that $y^2 \in \mathbf{C}(D) = K$. Since $(\gamma_1 \otimes \text{id}_Z)|_M \neq \text{id}_M$, we have $y \notin \mathbf{C}(D) = K$, and as $\sigma|_D = \text{can}_D$ and $y^2 \in K$, we conclude that $\sigma(y) = -y$. As $C = DZ$ and $MZ = LZ$, it follows that $\text{Int}_C(y)|_L = \gamma_1$.

We pick an element $z' \in Z^\times$ such that $\sigma(z') = -z'$ and set $w_0 = yz'$. (If $\text{char}(F) = 2$ then we may take $z' = 1$.) Then $\sigma(w_0) = w_0$ and $w_0 \in A^\times$, and for any $\ell \in L$ we have

$$w_0\ell w_0^{-1} = \text{Int}_C(y)(\ell) = \gamma_1(\ell).$$

Hence w_0 is a possible choice for an element w satisfying the conditions in (1).

To prove part (2), we now consider an arbitrary element w as in (1) while keeping our choice for w_0 . We set

$$W = \{x \in \text{Sym}(\sigma) \mid x\ell = \gamma_1(\ell)x \text{ for all } \ell \in L\},$$

and fix an F -linear form $s: K \rightarrow F$ with $\text{Ker}(s) = F$. Since σ is unitary, we have $\text{Sym}(\sigma) = \text{Symd}(\sigma)$, hence

$$q: W \rightarrow F, \quad x \mapsto s(x^2)$$

is the quadratic form from Lemma 7.1, and by definition the Pfister form \mathfrak{P}_σ is similar to q . Since $M = (LZ)^{\gamma_1 \otimes \sigma|_Z}$, we have $\sigma|_M = (\gamma_1 \otimes \text{id}_Z)|_M = \text{Int}_A(w)|_M$, hence

$$\sigma(wm) = \sigma(m)w = wm \quad \text{for all } m \in M,$$

which proves that $wm \in \text{Sym}(\sigma)$. Since moreover M centralizes L it follows that $wM \subseteq W$. As $w \in A^\times$ we have $\dim_F wM = \dim_F M = 4 = \dim_F W$, so we conclude that

$$W = wM.$$

As $w_0 \in W \cap A^\times$, we may write $w = w_0 m_0$ for some $m_0 \in M$, hence

$$w^2 = w_0 m_0 w_0 m_0 = \sigma(m_0) w_0^2 m_0 = \sigma(m_0) y^2 z'^2 m_0.$$

Since $y^2 \in K$, $z'^2 \in F$ and $\sigma(m_0) m_0 = \mathbf{N}_{M/K}(m_0) \in K$, it follows that $w^2 \in K$. For $m \in M$ we have

$$(wm)^2 = w^2 \sigma(m) m = w^2 \mathbf{N}_{M/K}(m) \in K.$$

We conclude that q is isometric to the quadratic form

$$q': M \rightarrow F, \quad m \mapsto s(w^2 \mathbf{N}_{M/K}(m)).$$

Note that $M = (LZ)^{\gamma_1 \otimes \sigma|_Z}$ is a Galois F -algebra whose Galois group is generated by $\sigma|_M = \gamma_1 \otimes \text{id}_Z$ and $\gamma_2 \otimes \text{id}_Z$. Since $\sigma|_K = \text{id}_K$, it follows that $M = KN$ for

$$N = M^{\gamma_2 \otimes \text{id}_Z} = (L^{\gamma_2} \otimes Z)^{\gamma_1 \otimes \sigma|_Z}$$

and M is naturally isomorphic to $K \otimes N$. Therefore the quadratic form $\mathbf{N}_{M/K}$ extends $\mathbf{N}_{N/F}$. By Frobenius Reciprocity [12, (20.2), (20.3c)] it follows that q' is isometric to $s_*(\langle w^2 \rangle) \otimes \mathbf{N}_{N/F}$. Set $d = \text{Nrd}_A(w)$. Since $w \notin K$ and $w^2 \in K$, we obtain that $d = \mathbf{N}_{K/F}(w^2)$. Since $s_*(\langle 1 \rangle)$ is hyperbolic, it follows from [12, (34.19)] that $s_*(\langle w^2 \rangle)$ is similar to $\langle 1, -d \rangle$. Therefore q' is similar to $\langle 1, -d \rangle \otimes \mathbf{N}_{N/F}$. This shows that the forms \mathfrak{P}_σ and $\langle 1, -d \rangle \otimes \mathbf{N}_{N/F}$ are similar, and since they are both Pfister forms, we conclude that they are isometric. \square

8.2. Proposition. *Let (A, σ) be an F -algebra with unitary involution of capacity 4. Then $\mathfrak{P}_\sigma \simeq \text{Nrd}_Q$ for an F -quaternion algebra Q , which is Brauer equivalent to the discriminant algebra of (A, σ) .*

Proof: We fix a neat biquadratic F -subalgebra L of (A, σ) , whose existence is guaranteed by [6, Theorem 7.4]. We use the same notation as in the proof of Lemma 8.1. By [17, p. 129], the K -quaternion algebra D defined in (8.1.1) is the discriminant algebra of $(C, \sigma|_C)$. By [9, Lemma 3.1(2)], the discriminant algebra of (A, σ) is Brauer equivalent to the corestriction of D to F .

The proof of Lemma 8.1 yields that

$$D = M \oplus My = KN \oplus KNy$$

where M is a biquadratic Galois F -algebra and K and N are the subalgebras fixed by two different nontrivial elements of the Galois group of M/F and where $y \in D^\times$ is such that $\text{Int}_D(y)$ extends the nontrivial K -automorphism of M , $\sigma(y) = -y$ and $y^2 \in K$. Hence D is isomorphic to the crossed product algebra $(KN/K, y^2)$ over K . Since $M = KN$, which is a free compositum over F , it follows by the projection formula (see [14, Prop. 3.4.10]) that the corestriction of D is Brauer equivalent to the crossed product algebra $Q = (N/F, \mathbf{N}_{K/F}(y^2))$ over F , which is an F -quaternion algebra with norm form $\langle 1, -\mathbf{N}_{K/F}(y^2) \rangle \otimes \mathbf{N}_{N/F}$.

By the proof of Lemma 8.1, after choosing $z' \in \mathbf{C}(A)^\times$ with $\sigma(z') = -z'$ and letting $w_0 = yz'$, we obtain that $\mathfrak{P}_\sigma \simeq \langle 1, -\text{Nrd}_A(w_0) \rangle \otimes \mathbf{N}_{N/F}$. Since $z'^2 \in F^\times$ and $\text{Nrd}_A(w_0) = \mathbf{N}_{K/F}(y^2)z'^4$, it follows that \mathfrak{P}_σ is isometric to Nrd_Q . \square

We can now retrieve from Theorem 7.13 the criterion from [15, Section 3] for total decomposability of an algebra with unitary involution of capacity 4.

8.3. Corollary (Karpenko–Quéguiner). *An F -algebra with unitary involution of capacity 4 is totally decomposable if and only if its discriminant algebra is split.*

Proof: Let (A, σ) be an F -algebra with unitary involution of capacity 4 and let D be its discriminant algebra. By Proposition 8.2, D is Brauer equivalent to an F -quaternion algebra Q such that $\mathfrak{P}_\sigma \simeq \text{Nrd}_Q$. It follows that D is split if and only if \mathfrak{P}_σ is hyperbolic, and by Theorem 7.13 this is equivalent to (A, σ) being totally decomposable. \square

When $\text{char } F \neq 2$, for an F -algebra with symplectic involution of degree a multiple of 8, a cohomological invariant $\Delta(A, \sigma) \in H^3(F, \mu_2)$ was defined in [13]. In the case where $\text{deg } A = 8$ this invariant is related to the discriminant Pfister form.

For a 3-fold Pfister form π over a field F of characteristic different from 2, and for $a, b, c \in F^\times$ such that $\pi \simeq \langle 1, -a \rangle \otimes \langle 1, -b \rangle \otimes \langle 1, -c \rangle$, the cup product $(a) \cup (b) \cup (c)$ in $H^3(F, \mu_2)$ is an invariant of π , by [1, Satz 1.6], also called the *Arason invariant* of π .

8.4. Proposition (Garibaldi–Parimala–Tignol). *Suppose that $\text{char } F \neq 2$. Let (A, σ) be an F -algebra with symplectic involution with $\text{cap}(A, \sigma) = 4$. Then $\Delta(A, \sigma)$ is the Arason invariant of \mathfrak{P}_σ . Furthermore $\Delta(A, \sigma) = 0$ if and only if (A, σ) is totally decomposable.*

Proof: It is proven in [13, Proposition 8.1] that $\Delta(A, \sigma)$ is the Arason invariant of a 10-dimensional quadratic form of trivial discriminant and Clifford invariant: this is the form \tilde{q} appearing the proof of Lemma 7.1. This form \tilde{q} is Witt equivalent to the 8-dimensional quadratic form q in Lemma 7.1, and the Pfister form

\mathfrak{P}_σ is by definition similar to q , whereby its Arason invariant is the same as for q and \tilde{q} . This relates the two invariants in the way as it is claimed here.

The equivalence of the vanishing of $\Delta(A, \sigma)$ with the decomposability of (A, σ) is shown in [13, Section 9]; it can now alternatively be obtained from Theorem 7.13. In either way one relies on the fact that a quadratic 3-fold Pfister form is hyperbolic if and only if its Arason invariant is trivial, which follows from [1, Satz 5.6]. \square

We return to the situation where the field F is of arbitrary characteristic. For an involution σ on an F -algebra A one defines

$$\text{Alt}(\sigma) = \{x - \sigma(x) \mid x \in A\}.$$

Recall from [17, (7.2)] that the discriminant of an orthogonal involution σ on a central simple F -algebra A of even degree $2m$ is the square class $(-1)^m \text{Nrd}_A(y) F^{\times 2}$ in $F^\times / F^{\times 2}$ given by an arbitrary element $y \in A^\times \cap \text{Alt}(\sigma)$, and that there always exists such an element.

8.5. Lemma. *Let (B, τ) be an F -algebra with orthogonal involution of degree 4. Let $d \in F^\times$ be such that the discriminant of τ is $dF^{\times 2}$. Then d is represented by $\mathbf{N}_{E/F}$ for every neat quadratic F -subalgebra E of (B, τ) .*

Proof: Fix a neat quadratic F -subalgebra E of (B, τ) . Then $C = \mathbf{C}_B(E)$ is a quaternion E -algebra. We fix an element $y \in C^\times \cap \text{Alt}(\tau|_C)$. Since $\tau|_C$ is orthogonal, we have $y \notin E$. On the other hand $y^2 \in E$. Hence $\deg B = 4 = [E[y] : F]$, and it follows that $\text{Nrd}_B(y) = \mathbf{N}_{E/F}(y^2)$. By the choice of d , we obtain that $dF^{\times 2} = \text{Nrd}_B(y)F^{\times 2} = \mathbf{N}_{E/F}(y^2)F^{\times 2}$, which shows the claim. \square

8.6. Proposition. *Let (B, τ) be an F -algebra with orthogonal involution of capacity 4. Let $d \in F^\times$ be such that $dF^{\times 2}$ is the discriminant of (B, τ) . The following hold:*

- (1) *If $\text{char } F \neq 2$, then $\mathfrak{P}_\tau = \langle 1, -d \rangle$.*
- (2) *For any quadratic étale F -algebra Z , the discriminant Pfister form of the F -algebra with unitary involution $(B, \tau) \otimes (Z, \text{can}_{Z/F})$ of capacity 4 is given by $\langle 1, -d \rangle \otimes \mathbf{N}_{Z/F}$.*
- (3) *For any F -quaternion algebra Q , the discriminant Pfister form of the F -algebra with symplectic involution $(B, \tau) \otimes (Q, \text{can}_Q)$ of capacity 4 is given by $\langle 1, -d \rangle \otimes \text{Nrd}_Q$ where Nrd_Q is the reduced norm form of Q .*

Proof: By [6, Theorem 7.4], (B, τ) contains a neat biquadratic F -subalgebra L . Let $\{\text{id}_L, \gamma_1, \gamma_2, \gamma_3\}$ be the Galois group of L viewed as a Galois F -algebra. We set $K = L^{\gamma_1}$ and fix an F -linear functional $s : K \rightarrow F$ with $\text{Ker}(s) = F$. We further set $C_0 = \mathbf{C}_B(K)$ and observe that C_0 is a K -quaternion algebra containing L and that τ restricts to an orthogonal involution on C_0 . We fix $y \in C_0^\times \cap \text{Alt}(\tau|_{C_0})$. Then $y^2 \in K$ and $y \in B^\times \cap \text{Alt}(\tau)$, hence

$$(8.6.1) \quad \text{can}_{C_0}(y) = -y, \quad \text{Nrd}_B(y) = \mathbf{N}_{K/F}(y^2) \quad \text{and} \quad dF^{\times 2} = \text{Nrd}_B(y)F^{\times 2}.$$

Moreover, since $y \in \text{Alt}(\tau|_{C_0})$ and $L \subseteq \text{Sym}(\tau|_{C_0})$, it follows that $\text{Trd}_{C_0}(yL) = 0$, by [17, (2.3)]. As $\text{can}_{C_0}|_L = \gamma_1|_L$, this implies that

$$(8.6.2) \quad y\ell = \gamma_1(\ell)y \quad \text{for every } \ell \in L.$$

(1) Recall that (only) for part (1) we assume that $\text{char } F \neq 2$. Thus, $\text{Symd}(\tau) = \text{Sym}(\tau)$ and $L = K \oplus vK$ for some element $v \in L^\times$ with $v^2 \in F^\times$ and $\gamma_1(v) = -v$. The latter implies that $yv = -vy$. Let

$$W_0 = \{x \in \text{Sym}(\tau) \mid x\ell = \gamma_1(\ell)x \text{ for all } \ell \in L\}.$$

By definition, the Pfister form \mathfrak{P}_τ is similar to the quadratic form

$$q_0: W_0 \rightarrow F, \quad x \mapsto s(x^2).$$

As $\tau(y) = -y$, $\tau(v) = v$ and $yv = -vy$, it follows that $yv \in \text{Sym}(\tau)$, and as $v \in L$, we conclude by (8.6.2) that $yv \in W_0$. Hence $yvK \subseteq W_0$, and since $yv \in B^\times$ and therefore $\dim_F yvK = \dim_F K = 2 = \dim_F W_0$, we obtain that

$$W_0 = yvK.$$

For $x \in K$, we have

$$q_0(yvx) = s((yv)^2x^2) = -v^2s(y^2x^2).$$

Therefore, q_0 , hence also \mathfrak{P}_τ , is similar to $s_*(\langle y^2 \rangle)$. Now, the discriminant of $s_*(\langle y^2 \rangle)$ is $\mathbf{N}_{K/F}(y^2)F^{\times 2}$ by [12, (34.19)], so $s_*(\langle y^2 \rangle)$ is similar to $\langle 1, -\mathbf{N}_{K/F}(y^2) \rangle$, hence also to $\langle 1, -d \rangle$ by (8.6.1). We conclude that \mathfrak{P}_τ is similar to $\langle 1, -d \rangle$, and since both binary forms represent 1, it follows that they are isometric.

(2) For proving part (2), we set $(A, \sigma) = (B, \tau) \otimes (Z, \text{can}_{Z/F})$ and fix an element $z \in Z^\times$ such that $\text{can}_{Z/F}(z) = -z$. (If $\text{char}(F) = 2$ we may choose $z = 1$.) Then $yz \in \text{Sym}(\sigma) \cap A^\times$ and (8.6.2) shows that $yz\ell = \gamma_1(\ell)yz$ for every $\ell \in L$. Hence it follows from Lemma 8.1 that $\mathfrak{P}_\sigma \simeq \langle 1, -\text{Nrd}_A(yz) \rangle \otimes \mathbf{N}_{N/F}$ for the étale quadratic F -algebra $N = (L^{\gamma_2} \otimes Z)^{\gamma_1 \otimes \text{can}_{Z/F}}$. Since $z^2 \in F^\times$, we have by (8.6.1) $\text{Nrd}_A(yz) = \text{Nrd}_A(y)z^4 \in dF^{\times 2}$, whereby $\langle 1, -\text{Nrd}_A(yz) \rangle \simeq \langle 1, -d \rangle$. To complete the proof, it suffices to show that $\langle 1, -d \rangle \otimes \mathbf{N}_{N/F}$ is isometric to $\langle 1, -d \rangle \otimes \mathbf{N}_{Z/F}$.

For this we note that $(LZ)^{\gamma_2 \otimes \text{id}_Z}$ is a Galois F -algebra with Galois group isomorphic to $(\mathbb{Z}/2\mathbb{Z})^2$, and the quadratic F -subalgebras fixed under the nontrivial elements of the Galois group are N , Z and L^{γ_2} . Hence $N \otimes L^{\gamma_2} \simeq Z \otimes L^{\gamma_2}$. If L^{γ_2} splits, then $N \simeq Z$, hence $\mathbf{N}_{N/F} \simeq \mathbf{N}_{Z/F}$ and the proof is complete. If L^{γ_2} is a field, the isomorphism $N \otimes L^{\gamma_2} \simeq Z \otimes L^{\gamma_2}$ implies that $\mathbf{N}_{N/F}$ and $\mathbf{N}_{Z/F}$ become isometric after scalar extension to L^{γ_2} , hence by [12, (34.9)] the form $\mathbf{N}_{N/F} \perp -\mathbf{N}_{Z/F}$ is Witt equivalent to a multiple of $\mathbf{N}_{L^{\gamma_2}/F}$. But since L^{γ_2} is a neat quadratic subfield of (B, τ) , Lemma 8.5 implies that $\langle 1, -d \rangle \otimes \mathbf{N}_{L^{\gamma_2}/F}$ is hyperbolic. Hence $\langle 1, -d \rangle \otimes (\mathbf{N}_{N/F} \perp -\mathbf{N}_{Z/F})$ is hyperbolic. Therefore we have $\langle 1, -d \rangle \otimes \mathbf{N}_{N/F} \simeq \langle 1, -d \rangle \otimes \mathbf{N}_{Z/F}$.

(3) Refreshing the notation, we set $(A, \sigma) = (B, \tau) \otimes (Q, \text{can}_Q)$, which is an F -algebra with symplectic involution. We fix a quadratic étale F -subalgebra Z

of Q . The non-trivial F -automorphism of Z extends to $\text{Int}_Q(j)$ for some $j \in Q^\times$, and we obtain that

$$Q = Z \oplus Zj$$

and $j^2 \in F^\times$. Let $A' = \mathbf{C}_A(Z) = B \otimes Z$ and note that A' is σ -stable. We set $\sigma' = \sigma|_{A'}$. Then (A', σ') is an F -algebra with unitary involution. Note that $\text{cap}(A, \sigma) = \text{cap}(A', \sigma') = 4$ and L is a neat biquadratic F -subalgebra of (A', σ') and of (A, σ) . By (2) we have

$$\mathfrak{P}_{\sigma'} \simeq \langle 1, -d \rangle \otimes \mathbf{N}_{Z/F}.$$

We set

$$\begin{aligned} W &= \{x \in \text{Symd}(\sigma) \mid x\ell = \gamma_1(\ell)x \text{ for all } \ell \in L\} \quad \text{and} \\ W' &= \{x \in \text{Symd}(\sigma') \mid x\ell = \gamma_1(\ell)x \text{ for all } \ell \in L\}. \end{aligned}$$

Since σ' is unitary we have $\text{Symd}(\sigma') = \text{Sym}(\sigma')$ and thus $W' = W \cap A'$. By the definition, \mathfrak{P}_σ is similar to the quadratic form $q: W \rightarrow F, x \mapsto s(x^2)$, and $\mathfrak{P}_{\sigma'}$ is similar to $q': W' \rightarrow F, x \mapsto s(x^2)$, which is the restriction of q to W' .

We first look at the case where q' is isotropic. In this case $\mathfrak{P}_{\sigma'}$ and \mathfrak{P}_σ are isotropic, and hence hyperbolic, because they are Pfister forms. Since $\mathfrak{P}_{\sigma'} \simeq \langle 1, -d \rangle \otimes \mathbf{N}_{Z/F}$, and since $\mathbf{N}_{Z/F}$ is a subform of \mathbf{Nrd}_Q , it follows that the 3-fold Pfister form $\langle 1, -d \rangle \otimes \mathbf{Nrd}_Q$ is hyperbolic, and hence isometric to \mathfrak{P}_σ .

We may now assume for the rest of the proof that q' is anisotropic. Note that $\text{Int}_A(j)$ commutes with σ' , because $\sigma(j) = -j$ and $j^2 \in Z = \mathbf{C}(A')$. Moreover $j \in \mathbf{C}_A(L)$. It follows that W' is preserved under $\text{Int}_A(j)$.

Note that $L \cap W' = 0$ and $L \oplus W' \subseteq \mathbf{C}_{A'}(K)$. However $\text{Sym}(\tau) \not\subseteq \mathbf{C}_A(K)$, so in particular $\text{Sym}(\tau) \neq L \oplus W'$. Since $\dim_F L \oplus W' = 8 = \dim_F \text{Sym}(\tau)$ and $L \subseteq \text{Sym}(\tau)$, we obtain that $W' \not\subseteq \text{Sym}(\tau) = \text{Sym}(\sigma') \cap B$. As $W' \subseteq \text{Sym}(\sigma')$, we conclude that $W' \not\subseteq B$.

Since $Q = Z \oplus jZ$ and $W' \subseteq A' = \mathbf{C}_A(Z)$, it follows that there exists $w_0 \in W'$ such that $kw_0 \neq w_0j$. Let $w = kw_0j^{-1} - w_0 \in W'$. Then $w \neq 0$ and

$$wj = kw_0 - w_0j = kw_0 + \sigma(jw_0) \in \text{Symd}(\sigma).$$

Moreover, $wj \in W$ because $w \in W' \subseteq W$, $j \in \mathbf{C}_A(L)$ and $kw = -wj$.

We fix an element $z \in Z \setminus F$ with $z^2 - z \in F$. Then $kwzj^{-1} = 1 - z$.

For every $w' \in W'$ we have

$$(8.6.3) \quad (w'wj + wjw')z = (1 - z)(w'wj + wjw').$$

On the other hand, since $x^2 \in K$ for every $x \in W$, it follows that

$$w'wj + wjw' = (w' + wj)^2 - w'^2 - (wj)^2 \in K,$$

whereby

$$(8.6.4) \quad (w'wj + wjw')z = z(w'wj + wjw')$$

because $K \subseteq A' = C_A(Z)$. By comparing (8.6.3) and (8.6.4), we obtain for every $w' \in W'$ that $w'wj + wjw' = 0$. This proves that wj lies in the orthogonal complement of W' with respect to the quadratic form q .

We set $a = q(w)$. As $w \in W' \setminus \{0\}$ and $q' = q|_{W'}$ is anisotropic, we have that $a \in F^\times$. Since $q|_{W'}$ is similar to $\mathfrak{P}_{\sigma'}$, we obtain that $\mathfrak{P}_{\sigma'} \simeq aq'$. Similarly, since q is similar to \mathfrak{P}_σ and represents a , we obtain that $\mathfrak{P}_\sigma \simeq aq$.

Set $b = j^2$. Then $b \in F^\times$ and $\text{Nrd}_Q \simeq \langle 1, -b \rangle \otimes \text{N}_{Z/F}$. We further have

$$q(wj) = s((wj)^2) = s(-bw^2) = -bq(w) = -ab.$$

Since wj lies in the orthogonal complement of W' with respect to q , it follows that $q' \perp \langle -ab \rangle$ is a subform of q . Therefore $\mathfrak{P}_{\sigma'} \perp \langle -b \rangle$ is a subform of \mathfrak{P}_σ . On the other hand, having $\mathfrak{P}_{\sigma'} \simeq \langle 1, -d \rangle \otimes \text{N}_{Z/F}$ and $\langle 1, -b \rangle \otimes \text{N}_{Z/F} \simeq \text{Nrd}_Q$, we also have that $\mathfrak{P}_{\sigma'} \perp \langle -b \rangle$ is a subform of $\langle 1, -d \rangle \otimes \text{Nrd}_Q$. Hence the quadratic 3-fold Pfister forms \mathfrak{P}_σ and $\langle 1, -d \rangle \otimes \text{Nrd}_Q$ share a common 5-dimensional subform. In view of [12, Lemma 23.1] this readily yields that they are isometric. \square

8.7. Remark. In view of Proposition 8.2, one can derive part (2) of Proposition 8.6 alternatively from the description of the Brauer class of the discriminant algebra of $(B, \tau) \otimes (Z, \text{can}_{Z/F})$ in [17, (10.33)].

We round up by computing the discriminant Pfister form in some special cases of Proposition 8.6 where the algebra B is split.

8.8. Examples. Let $B = \text{End}_F(V)$ for some 4-dimensional F -vector space V . Let $\beta: V \times V \rightarrow F$ be a nondegenerate symmetric bilinear form over F , and let ad_β denote the adjoint involution on $\text{End}_F(V)$, which is determined by

$$\beta(u, f(v)) = \beta(\text{ad}_\beta(f)(u), v) \quad \text{for all } f \in \text{End}_F(V), u, v \in V.$$

Let $d \in F^\times$ be the determinant of β (determined up to a square factor). Applying Proposition 8.6, we obtain the following results:

- (1) If $\text{char}(F) \neq 2$, then $\mathfrak{P}_{\text{ad}_\beta} \simeq \langle 1, -d \rangle$.
- (2) For $(A, \sigma) = (\text{End}_F(V), \text{ad}_\beta) \otimes (Z, \text{can}_{Z/F})$, where Z is a quadratic étale F -algebra, we obtain that $\mathfrak{P}_\sigma \simeq \langle 1, -d \rangle \otimes \text{N}_{Z/F}$.
- (3) Let Q be an F -quaternion algebra. For $(A, \sigma) = (\text{End}_F(V), \text{ad}_\beta) \otimes (Q, \text{can}_Q)$, we obtain that that $\mathfrak{P}_\sigma \simeq \langle 1, -d \rangle \otimes \text{Nrd}_Q$.

REFERENCES

- [1] J.Kr. Arason. Cohomologische Invarianten quadratischer Formen. *J. Algebra* **36** (1975): 448–491.
- [2] R. Baeza. *Quadratic forms over semilocal rings*. Springer-Verlag, Berlin, 1978, Lecture Notes in Mathematics, Vol. 655.
- [3] E. Bayer-Fluckiger, R. Parimala and A. Quéguiner-Mathieu. Pfister involutions. *Proc. Indian Acad. Sci. Math. Sci.* **113** (2003): 365–377.
- [4] K.J. Becher, A proof of the Pfister Factor Conjecture. *Invent. Math.* **173** (2008): 1–6.

- [5] K.J. Becher, A. Dolphin. Totally decomposable quadratic pairs. *Math. Z.* **284** (2016): 117–129.
- [6] K.J. Becher, N. Grenier-Boley, J.-P. Tignol. Involutions and stable subalgebras. *J. Algebra*, **493** (2018): 381–409.
- [7] K.J. Becher, T. Unger. Weakly hyperbolic involutions. *Expo. Math.* **36** (2018): 78–97.
- [8] E. Bayer-Fluckiger, D.B. Shapiro, J.-P. Tignol. Hyperbolic involutions. *Math. Z.* **214** (1993): 461–476.
- [9] G. Berhuy, C. Frings, J.-P. Tignol. Galois cohomology of the classical groups over imperfect fields. *J. Pure App. Algebra* **211** (2007): 307–341.
- [10] A. Dolphin. Metabolic involutions. *J. Algebra* **336** (2011): 286–300.
- [11] A. Dolphin. Totally decomposable symplectic and unitary involutions. *Manuscripta Math.* **153** (2017): 523–534.
- [12] R. Elman, N. Karpenko, A. Merkurjev. *The algebraic and geometric theory of quadratic forms*. American Mathematical Society Colloquium Publications, **56**, Amer. Math. Soc., Providence, RI, 2008.
- [13] S. Garibaldi, R. Parimala, J.-P. Tignol. Discriminant of symplectic involutions. *Pure Appl. Math. Q.* **5** (2009): 349–374.
- [14] P. Gille and T. Szamuely. *Central simple algebras and Galois cohomology*. Cambridge University Press, 2006.
- [15] N. Karpenko and A. Quéguiner, A criterion of decomposability for degree 4 algebras with unitary involution, *J. Pure Appl. Algebra* **147** (2000), no. 3, 303–309.
- [16] M.-A. Knus. *Quadratic and Hermitian forms over rings*. Grundlehren der Mathematischen Wissenschaften **294**. Springer-Verlag, Berlin, 1991.
- [17] M.-A. Knus, A. S. Merkurjev, M. Rost, and J.-P. Tignol. *The book of involutions*. American Mathematical Society Colloquium Publications **44**. American Mathematical Society, Providence, RI, 1998.
- [18] M.-A. Knus, R. Parimala and R. Sridharan, Pfaffians, central simple algebras and similitudes, *Math. Z.* **206** (1991), no. 4, 589–604.
- [19] M.-A. Knus, R. Parimala and R. Sridharan, Involutions on rank 16 central simple algebras, *J. Indian Math. Soc. (N.S.)* **57** (1991), no. 1-4, 143–151.
- [20] T.Y. Lam. *Introduction to quadratic forms over fields*. Graduate Studies in Mathematics, **67**, Amer. Math. Soc., Providence, RI, 2005.
- [21] M. Rost, J.-P. Serre and J.-P. Tignol, La forme trace d’une algèbre simple centrale de degré 4, *C.R. Math. Acad. Sci. Paris* **342** (2006): 83–87.
- [22] W. Scharlau. *Quadratic and Hermitian Forms*. Grundlehren, vol. **270**, Springer, Berlin, 1985.
- [23] J.-P. Tignol. Cohomological invariants of central simple algebras with involution. In: *Quadratic forms, linear algebraic groups, and cohomology*, 137–171, Dev. Math., **18**, Springer, New York.

UNIVERSITEIT ANTWERPEN, DEPARTEMENT WISKUNDE, MIDDELHEIMLAAN 1, B-2020 ANTWERPEN, BELGIUM

Email address: karimjohannes.becher@uantwerpen.be

NORMANDIE UNIV, UNIROUEN, LDAR (EA 4434), 76000 ROUEN, FRANCE, UNIVERSITÉS DE PARIS, ARTOIS, CERGY PONTOISE, PARIS-EST CRÉTEIL.

Email address: nicolas.grenier-boleyn@univ-rouen.fr

UCLouvain, ICTeam Institute, Avenue G. Lemaître 4, Box L4.05.01, B-1348
Louvain-la-Neuve, Belgium.

Email address: `jean-pierre.tignol@uclouvain.be`