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UNIVERSITÉ CATHOLIQUE DE LOUVAIN



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P A P E R

2016/03

On the Maximum Likelihood Estimator for the
Generalized Extreme-Value Distribution

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ON THE MAXIMUM LIKELIHOOD ESTIMATOR FOR THE GENERALIZED EXTREME-VALUE DISTRIBUTION

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ABSTRACT. The vanilla method in univariate extreme-value theory consists of fitting the three-parameter Generalized Extreme-Value (GEV) distribution to a sample of block maxima. Despite claims to the contrary, the asymptotic normality of the maximum likelihood estimator has never been established. In this paper, a formal proof is given using a general result on the maximum likelihood estimator for parametric families that are differentiable in quadratic mean but whose support depends on the parameter. An interesting side result concerns the (lack of) differentiability in quadratic mean of the GEV family.

Key words. Differentiability in quadratic mean; M-estimator; maximum likelihood; empirical process; Fisher information; Generalized Extreme-Value distribution; Lipschitz condition; support.

1. INTRODUCTION

Asymptotic normality of maximum likelihood estimators in regular parametric models is a classic subject, but when the support of the distribution depends on the parameter, the mathematics are not routine. The family of univariate, three-parameter Generalized Extreme-Value distributions is a case in point. Its cumulative distribution function G_θ at a parameter vector $\theta = (\gamma, \mu, \sigma) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$ is given by

$$G_{\gamma, \mu, \sigma}(x) = G_{\gamma, 0, 1}((x - \mu)/\sigma), \quad x \in \mathbb{R}, \quad (1.1)$$

where

$$G_{\gamma, 0, 1}(z) = \begin{cases} \exp(-e^{-z}) & \text{if } \gamma = 0, \\ \exp\{-(1 + \gamma z)^{-1/\gamma}\} & \text{if } \gamma \neq 0 \text{ and } 1 + \gamma z > 0, \\ 0 & \text{if } \gamma > 0 \text{ and } z \leq -1/\gamma, \\ 1 & \text{if } \gamma < 0 \text{ and } z \geq -1/\gamma. \end{cases}$$

The support of G_θ is an interval, $S_\theta = \{x \in \mathbb{R} : \sigma + \gamma(x - \mu) > 0\}$, whose endpoints depend on θ . The above parametrization, due to [von Mises \(1936\)](#) and [Jenkinson \(1955\)](#), generalizes and unifies the Fréchet/Gumbel/Weibull trichotomy in [Fisher and Tippet \(1928\)](#) across all signs of the shape parameter γ .

Fitting a generalized extreme-value distribution to a sample of annual maxima is the earliest statistical method in extreme-value theory. Various inference methods have been explored, including quantile or probability matching ([Gumbel, 1958](#)), probability weighted moments ([Hosking et al., 1985](#)), and maximum likelihood ([Prescott and](#)

Walden, 1980; Hosking, 1985). The present paper revisits the maximum likelihood estimator and its asymptotic distribution.

In general, deriving large-sample asymptotics of the maximum likelihood estimator for distribution families with varying supports is a difficult problem. The classical regularity conditions of Cramér (1946) are not fulfilled, and the same is true for the weaker Lipschitz conditions stemming from empirical process theory (van der Vaart, 1998, Theorem 5.39). Up to our knowledge, a general theory does not exist.

Smith (1985) studies maximum likelihood estimation for parametric families of densities on the real line of the form

$$f(x; \mu, \phi) = (x - \mu)^{\alpha-1} g(x - \mu; \phi), \quad x \in (\mu, \infty).$$

Here, μ is a location parameter, ϕ is a parameter vector, $\alpha = \alpha(\phi)$ is a smooth function of ϕ , and g is a known function. The formulation is general enough to include location versions of the Weibull, Gamma, Beta and log-Gamma distributions. It does not, however, include the three-parameter Generalized-Extreme Value distribution. Still, after a reparametrization, the formulation does include the case $-1/2 < \gamma < 0$. For $\gamma > 0$, Smith (1985, p. 88) claims that similar arguments will work too, but details are omitted. The case $\gamma = 0$ is not mentioned at all.

It could be argued that the case $\gamma < 0$ treated in Smith (1985) is indeed the most difficult one. However, we believe that some of the proofs in Smith (1985) are incorrect in a way which cannot be easily remedied. Uniformity in certain convergence statements is claimed but not proven, and we doubt that it can be done with the techniques used in the article. We justify our point of view at the end of Section 2.

Our contribution consists of a general result on the asymptotic normality of the maximum likelihood estimator for parametric models whose support may depend on the parameter. We apply the result to the three-parameter GEV family. An interesting side result concerns the differentiability in quadratic mean of that family; for the Gumbel case, see Marohn (1994, 2000).

As in Smith (1985), we need to show that certain limit relations hold uniformly over specific subsets of the parameter space. To do so, we use empirical process machinery borrowed from van der Vaart (1998). The approach requires exercising control on the entropy of certain function classes. We obtain such control via carefully formulated Lipschitz conditions. Checking these conditions for the three-parameter GEV family turns out to be surprisingly tedious.

The set-up in our paper is that of independent random samples drawn from a distribution within the parametric family itself. For the GEV-family, Dombry (2015) considers the more realistic setting of triangular arrays of block maxima extracted from independent random variables sampled from a distribution in the domain of attraction of a GEV distribution. He shows consistency of the maximum likelihood estimator, and our Proposition 3.1 below is not far from being a special case of his Theorem 2. In Bücher and Segers (2015), we go one step further and also establish asymptotic normality, even for block maxima extracted from time series. In that paper, however, we do not consider the full three-parameter GEV family but focus on the two-parameter Fréchet sub-family. The support of the latter is equal to the positive half-line for all values of the parameter vector. The complications that motivate the present paper do therefore not arise.

Asymptotic normality of the maximum likelihood estimator in general parametric families with parameter-dependent support is asserted in Section 2. The proof is given in Appendix A. We specialize the theory to the GEV family in Section 3. The reasonings and calculations needed to work out this case are sufficiently complicated to fill Appendices B to E.

2. MAXIMUM LIKELIHOOD ESTIMATOR OF A SUPPORT-DETERMINING PARAMETER

Let $(P_\theta : \theta \in \Theta)$ be a family of distributions on some measurable space $(\mathcal{X}, \mathcal{A})$, where $\Theta \subset \mathbb{R}^k$. Suppose that each P_θ has a density p_θ with respect to some common dominating measure μ . The model $(P_\theta : \theta \in \Theta)$ is said to be *differentiable in quadratic mean* at an inner point $\theta_0 \in \Theta$ if there exists a measurable function $\dot{\ell}_{\theta_0} : \mathcal{X} \rightarrow \mathbb{R}^k$ such that

$$\int_{\mathcal{X}} \left\{ \sqrt{p_{\theta_0+h}} - \sqrt{p_{\theta_0}} - \frac{1}{2} h^T \dot{\ell}_{\theta_0} \sqrt{p_{\theta_0}} \right\}^2 d\mu = o(\|h\|^2), \quad h \rightarrow 0. \quad (2.1)$$

The function $\dot{\ell}_{\theta_0}$ is referred to as the *score vector*. Its components are square-integrable with respect to P_{θ_0} and have mean zero. The covariance matrix $I_{\theta_0} = P_{\theta_0} \dot{\ell}_{\theta_0} \dot{\ell}_{\theta_0}^T$ is called *Fisher information matrix*. Differentiability in quadratic mean with invertible Fisher information matrix implies an asymptotic expansion of the log-likelihood ratio statistic implying *local asymptotic normality* of the associated sequence of statistical experiments. For more on the statistical implications of this property, see for instance [Le Cam \(1986\)](#) and [van der Vaart \(1998, Chapters 7–8\)](#). In [Marohn \(2000\)](#), local asymptotic normality is exploited to construct asymptotically efficient tests of the Gumbel hypothesis.

Let $X_1, \dots, X_n$ be an i.i.d. sample from P_{θ_0} . By definition, any global maximizer of the $\bar{\mathbb{R}}$ -valued function $\theta \mapsto \mathbb{P}_n \log p_\theta = \sum_{i=1}^n \log p_\theta(X_i)$ is called a maximum likelihood estimator. Here, it is implicitly assumed that the set of global maximizers is non-empty. This is easily satisfied in most situations where Θ is compact. Otherwise, a compactification argument often works ([van der Vaart, 1998, Chapter 5](#)) or one can restrict attention to local maximizers instead.

Under “regularity conditions”, the asymptotic distribution of $\sqrt{n}(\hat{\theta}_n - \theta_0)$ is $N_k(0, I_{\theta_0}^{-1})$. Various sets of sufficient conditions have been proposed in the literature: Cramér’s classical conditions require the existence of and bounds on the third-order derivatives of $\theta \mapsto \ell_\theta(x) = \log p_\theta(x)$. Theorem 5.39 in [van der Vaart \(1998\)](#) only demands differentiability in quadratic mean plus a Lipschitz condition on $\theta \mapsto \ell_\theta(x)$, for all x in the support of P_{θ_0} and all θ in a neighbourhood of θ_0 .

However, if the support, $\{x : p_\theta(x) > 0\}$, of P_θ depends on θ , an approach based on smoothness of $\theta \mapsto \ell_\theta(x)$ is not possible. Indeed, for any neighbourhood of θ_0 , we may find θ in that neighbourhood and $x \in \mathcal{X}$ such that $p_{\theta_0}(x) > 0$ but $p_\theta(x) = 0$ and thus $\ell_\theta(x) = -\infty$. For the same reason, empirical processes indexed by θ in a neighbourhood of θ_0 will have unbounded trajectories with probability tending to one. But the weak convergence of such empirical processes in the space of bounded functions is a crucial ingredient in the proofs of many M-estimator theorems.

A possible way to circumvent these problems consists of replacing the criterion function ℓ_θ by

$$m_\theta = 2 \log \left(\frac{p_\theta + p_{\theta_0}}{2p_{\theta_0}} \right), \quad (2.2)$$

a real-valued function on $\{x : p_{\theta_0}(x) > 0\}$ satisfying $m_\theta(x) \geq -2 \log(2)$ for all such x . Using Corollary 5.53 in [van der Vaart \(1998\)](#), the function m_θ can be used to obtain the $O_{\mathbb{P}}(1/\sqrt{n})$ rate of convergence of the maximum likelihood estimator for the GEV parameter θ , see Proposition [D.1](#) below. The method of proof does not allow to obtain the asymptotic distribution of the estimator, however.

Yet, if the rate of convergence $O_{\mathbb{P}}(1/\sqrt{n})$ has been established, then our next proposition provides alternative conditions (on ℓ_θ) which can be used to prove asymptotic normality of the maximum likelihood estimator. We thereby follow the well-known three-step strategy for deriving the asymptotic distribution of M-estimators (see, e.g., [van de Geer, 2009](#)): (1) prove consistency, then (2) derive the rate of convergence and finally (3) derive the exact limiting distribution. In Section [3](#), the three steps are worked out for the GEV family by an application of the following proposition for step (3).

The support of P_θ is $S_\theta = \{x : p_\theta(x) > 0\}$, a subset of $\mathcal{X}$ which may vary with $\theta \in \Theta$. Let $U_\varepsilon(\theta) = \{\theta' \in \Theta : \|\theta - \theta'\| < \varepsilon\}$ and put

$$\bar{S}(\varepsilon) = \bar{S}(\theta_0, \varepsilon) = \bigcap_{\theta \in U_\varepsilon(\theta_0)} S_\theta = \{x : p_\theta(x) > 0 \text{ for all } \theta \text{ such that } \|\theta - \theta_0\| < \varepsilon\}.$$

The set $\bar{S}(\varepsilon)$ is the intersection of the supports of the distributions P_θ for all θ in an ε -ball centered at θ_0 . The following proposition claims the asymptotic normality of the maximum likelihood estimator under a Lipschitz condition on the function $\theta \mapsto \ell_\theta(x)$ for x and θ in a certain range.

Proposition 2.1. *Let $(P_\theta : \theta \in \Theta)$ be a parametric model on $(\mathcal{X}, \mathcal{A})$, with $\Theta \subset \mathbb{R}^k$, and let θ_0 be an inner point of Θ . Let $X_1, X_2, \dots$ be a sequence of independent and identically distributed random elements of $\mathcal{X}$ with common distribution P_{θ_0} and let $\hat{\theta}_n$ be a maximum likelihood estimator based on $X_1, \dots, X_n$. Assume the following conditions:*

- (a) *The model is differentiable in quadratic mean at θ_0 with score vector $\dot{\ell}_{\theta_0}$ and non-singular Fisher information matrix I_{θ_0} .*
- (b) *We have*

$$P_{\theta_0}(\mathcal{X} \setminus \bar{S}(\varepsilon)) = o(\varepsilon^2), \quad \varepsilon \downarrow 0. \quad (2.3)$$

- (c) *There exist $\dot{\ell} \in L_2(P_{\theta_0})$ and $\varepsilon_0 > 0$ such that, for every $0 < \varepsilon < \varepsilon_0$,*

$$|\ell_{\theta_1}(x) - \ell_{\theta_2}(x)| \leq \dot{\ell}(x) \|\theta_1 - \theta_2\| \quad \text{for all } x \in \bar{S}(2\varepsilon) \text{ and } \theta_1, \theta_2 \in U_\varepsilon(\theta_0). \quad (2.4)$$

- (d) *We have $\sqrt{n}(\hat{\theta}_n - \theta_0) = O_{\mathbb{P}}(1)$ as $n \rightarrow \infty$.*

Then

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = I_{\theta_0}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) + o_{\mathbb{P}}(1) \rightsquigarrow N_k(0, I_{\theta_0}^{-1}), \quad n \rightarrow \infty. \quad (2.5)$$

Condition (b) controls the size of that part of the support of P_{θ_0} that is not contained in the support of P_θ for some θ in a neighbourhood of θ_0 . Condition (c) is a Lipschitz condition on $\theta \mapsto \ell_\theta(x)$ in which the range of x and θ is specified in such a way that $p_{\theta'}(x) > 0$ for all θ' in a neighbourhood of θ . In view of (b) and (d), the part of the range of x that has been left out is asymptotically negligible.

The proof of Proposition 2.1 adapts arguments from the proofs of Theorems 5.23, 5.39 and 7.2 and Lemma 19.31 in [van der Vaart \(1998\)](#). It is given in detail in Appendix A.

The role of uniformity. Theorem 3(i) in [Smith \(1985\)](#) states the asymptotic normality of the maximum likelihood estimator for a certain class of parametric families whose support depends in a smooth way on the parameter. As our paper may have the appearance of needlessly redoing known results, we feel obliged to expose what we believe to be a difficultly repairable flaw in the proof of that theorem. The matter is not easy to explain, and we invite the reader to consult Smith's article while reading this remark. The pages in [Smith \(1985\)](#) we will be needing are 71, 75–76, 80, and 82, and it will be convenient to discuss them in more or less reverse order.

The proof of Theorem 3(i), stated on page 80, is surprisingly short: just three lines on the bottom of page 82. The heart of the argument is the claim that the second-order derivatives of the loglikelihood are asymptotically constant in a sequence of shrinking neighbourhoods of the true parameter. Although the argument is not further developed, we follow the author that if the claim on the second-order derivatives is true, then the asymptotic normality can be proven too. The link is made via a second-order Taylor expansion of the loglikelihood around the true parameter. A good example of this technique is the proof of Theorem 5.41 in [van der Vaart \(1998\)](#).

As a justification of the assertion on the asymptotic constancy of the second-order derivatives of the loglikelihood, the author refers to the proof of his Theorem 1, without giving a specific pointer. Studying the proof of that theorem, we find the assertion in Lemma 4, items I(i), II(i), and III, on page 75. Only for item I(i), a proof is given (page 76). The proof of the other items is said to be similar, a point which we do not wish to argue.

The proof of item I(i) in Lemma 4 relies on Assumption 7 on page 71. This assumption concerns the L_1 -continuity of the second-order derivatives of the loglikelihood of a single observation as a function of the parameter vector. It is here that we wish to formulate our objection.

After careful reflection, we are convinced that Assumption 7 is not sufficient to conclude the convergence to zero of the *supremum of the sample averages* on line 4 of page 76. Indeed, the *expectation of the supremum* of a collection of random variables is in general larger than the *supremum of the expectations* of those random variables. In fact, the difference is fundamental and lies at the heart of what makes proving uniform laws of large numbers and uniform central limit theorems so challenging. Controlling expectations of suprema is the topic of so-called maximal inequalities. In [van der Vaart \(1998, Section 19.6\)](#), such control is exercised by bracketing integrals of the function classes over which suprema are to be taken.

Bracketing integrals appear in the proof of our Proposition 3.3. We find bounds on such integrals by relying on the Donsker theory in Chapter 19 in [van der Vaart \(1998\)](#). The complexity of the function classes under consideration is tempered by

means of uniform Lipschitz conditions. This justifies the appearance of our Lipschitz condition (2.4).

3. MAXIMUM LIKELIHOOD ESTIMATOR FOR THE GEV FAMILY

Let $X_1, \dots, X_n$ be an independent random sample from the Generalized Extreme-Value distribution G_{θ_0} in (1.1) with unknown parameter $\theta_0 = (\gamma_0, \mu_0, \sigma_0)$, to be estimated. Expressions for the log-density $\ell_\theta(x) = \log p_\theta(x)$ and the score vector $\dot{\ell}_\theta(x) = (\partial_\gamma \ell_\theta(x), \partial_\mu \ell_\theta(x), \partial_\sigma \ell_\theta(x))^T$ are given in Appendix B.

Consider the maximum likelihood estimator, $\hat{\theta}_n$, defined by maximizing the log-likelihood $\theta \mapsto n^{-1} \sum_{i=1}^n \ell_\theta(X_i) = \mathbb{P}_n \ell_\theta$. As argued by Dombry (2015), it is not guaranteed that the log-likelihood attains a unique, global maximum. For that reason, he rather defines any local maximizer of the log-likelihood over $\Theta_0 = \mathbb{R} \times \mathbb{R} \times (0, \infty)$ as a maximum likelihood estimator. His main result then states that, for block maxima constructed from an underlying i.i.d. series, one can always find a strongly consistent local maximizer, as long as γ_0 is strictly larger than -1 . Going into his proofs, we see how that local maximizer is constructed: by restricting the parameter space to some arbitrary compact set $\Theta \subset (-1, \infty) \times \mathbb{R} \times (0, \infty)$ which contains θ_0 as an interior point, it is guaranteed that the $[-\infty, \infty)$ -valued, continuous function $\theta \mapsto \mathbb{P}_n \ell_\theta$ attains its global maximum over Θ . That global maximum is then shown to be a local maximum over $\Theta_0 = \mathbb{R} \times \mathbb{R} \times (0, \infty)$, eventually, and to be strongly consistent.

In the subsequent parts of this paper, we could also work along Dombry's lines, redefining maximum likelihood estimators as local rather than global maxima. However, we prefer to keep track of the above-mentioned construction within his and our proofs. For the following proposition concerning consistency, we therefore consider the restriction of the parameter space to an arbitrarily large compact set $\Theta \subset (-1, \infty) \times \mathbb{R} \times (0, \infty)$ that contains the true value $\theta_0 \in (-1, \infty) \times \mathbb{R} \times (0, \infty)$ in its interior. Note that, in practice, this is hardly a restriction, at least if γ_0 is known to be larger than -1 ; see Dombry (2015, Remark 4) for the case $\gamma_0 \leq -1$.

Proposition 3.1 (Consistency). *Let $X_1, X_2, \dots$ be an independent random sample from the G_{θ_0} distribution, with $\theta_0 = (\gamma_0, \mu_0, \sigma_0) \in (-1, \infty) \times \mathbb{R} \times (0, \infty)$. For any compact set $\Theta \subset (-1, \infty) \times \mathbb{R} \times (0, \infty)$ such that θ_0 is in the interior of Θ and for any estimator sequence $\hat{\theta}_n$ such that $\mathbb{P}_n \ell_{\hat{\theta}_n} = \max_{\theta \in \Theta} \mathbb{P}_n \ell_\theta$, such maximizers always existing, we have $\hat{\theta}_n \rightarrow \theta_0$ almost surely as $n \rightarrow \infty$.*

Proof of Proposition 3.1. Since an extreme-value distribution is in its own domain of attraction, Proposition 3.1 is in fact a combination of Theorem 2 and Proposition 2 in Dombry (2015) with block size sequence $m(n) \equiv 1$ and with $a_m = \sigma_0$ and $b_m = \mu_0$. However, such a choice for $m(n)$ is prohibited in the cited theorem because of its condition (5), requiring that $m(n)/\log n \rightarrow \infty$ as $n \rightarrow \infty$. Therefore, we need to pass through the proof of the theorem and adapt where necessary.

First, the hypothesis $m(n) \rightarrow \infty$ is used in Lemmas 2 and 3 to apply the domain-of-attraction condition that $F^m(a_m x + b_m) \rightarrow \bar{G}_{\gamma_0}(x)$ as $m \rightarrow \infty$, where $\bar{G}_\gamma = G_{\gamma, 0, 1}$. But in our case, $F = G_{\gamma_0, \mu_0, \sigma_0}$, and thus $F(\sigma_0 x + \mu_0) = \bar{G}_{\gamma_0}(x)$ already, without having to pass through the limit.

Further, as explained in Remark 3 in [Dombry \(2015\)](#), the condition $m(n)/\log n \rightarrow \infty$ appears only in the proof of Lemma 4 in the same paper. Writing $\bar{\ell}_\gamma = \ell_{\gamma,0,1}$ and $Z_i = (X_i - \mu_0)/\sigma_0$, the claim of that lemma is that

$$\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{i=1}^n \bar{\ell}_{\gamma_0}(Z_i) = \int_{\mathbb{R}} \bar{\ell}_{\gamma_0}(z) d\bar{G}_{\gamma_0}(z), \quad \text{almost surely.}$$

But the random variables $Z_1, Z_2, \dots$ are independent and identically distributed with common distribution $\bar{G}_{\gamma_0}$, so that the claim follows from the strong law of large numbers. Note that the random variables $E_i = (1 + \gamma Z_i)^{-1/\gamma}$ have a unit Exponential distribution and that $\bar{\ell}_{\gamma_0}(Z_i) = (1 + \gamma) \log(E_i) - E_i$, the absolute value of which has a finite expectation. $\square$

To prove the asymptotic normality of the maximum likelihood estimator, we apply Proposition 2.1. To that end, we need to check a number of conditions, one of which, differentiability in quadratic mean, is interesting in its own right.

Proposition 3.2 (Differentiability in quadratic mean). *The three-parameter GEV family $\{G_{(\gamma, \mu, \sigma)} : (\gamma, \mu, \sigma) \in \Theta_0 = \mathbb{R} \times \mathbb{R} \times (0, \infty)\}$ is differentiable in quadratic mean at $\theta_0 = (\gamma_0, \mu_0, \sigma_0)$ if and only if $\gamma_0 > -1/2$. In that case, the score vector $\dot{\ell}_{\theta_0}(x)$ is equal to the gradient of the map $\theta \mapsto \ell_\theta(x)$ at $\theta = \theta_0$ for x such that $\sigma_0 + \gamma_0(x - \mu_0) > 0$ and equal to 0 otherwise.*

The proof of Proposition 3.2 requires showing (2.1). For every real x such that $\sigma_0 + \gamma_0(x - \mu_0) \neq 0$, the order of the integrand in (2.1) is $o(\|h\|^2)$ as $h \rightarrow \infty$ by the differentiability of the map $\theta \mapsto \sqrt{p_\theta(x)}$ at $\theta = \theta_0$ for such x . This asymptotic relation is true pointwise in x , and it remains to be shown that it can be integrated over x . The most delicate case occurs when $-1/2 < \gamma_0 \leq -1/3$, because in that case, the first-order partial derivatives of the map $\theta \mapsto \sqrt{p_\theta(x)}$ are not continuous at the boundary situation $\sigma + \gamma(x - \mu) = 0$. A detailed proof is given in Appendix C.

Proposition 3.3 (Asymptotic normality). *Let $X_1, X_2, \dots$ be independent and identically distributed random variables with common GEV law G_{θ_0} , with $\theta_0 = (\gamma_0, \mu_0, \sigma_0) \in (-1/2, \infty) \times \mathbb{R} \times (0, \infty)$. Then, for any compact parameter set $\Theta \subset (-1/2, \infty) \times \mathbb{R} \times (0, \infty)$, any sequence of maximum likelihood estimators over Θ is strongly consistent and asymptotically normal:*

$$\sqrt{n}(\hat{\theta}_n - \theta_0) = I_{\theta_0}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) + o_{\mathbb{P}}(1) \rightsquigarrow N_3(0, I_{\theta_0}^{-1}), \quad n \rightarrow \infty.$$

Proof. Strong consistency follows from Proposition 3.1. The proof of Proposition 3.3 therefore consists of checking the four conditions (a)–(d) of Proposition 2.1. Property (a) is just Proposition 3.2. The rate of convergence (d) is the topic of Proposition D.1, the proof of which consists of an application of Corollary 5.53 in [van der Vaart \(1998\)](#) to the modified criterion function m_θ in (2.2); a Lipschitz property of the latter function is established in Lemma D.3. The remaining points (b) and (c) are treated in Appendix E: the condition (b) on the support is given in Lemma E.1, while the Lipschitz condition (c) is given in Lemma E.2. $\square$

Remark 3.4. If $\gamma_0 = -1/2$, then $(d/dx)p_\theta(x)$ converges to a negative constant as x increases to the upper endpoint of the support, and the results in [Woodroffe \(1972\)](#) suggest that the maximum likelihood estimator is still asymptotically Normal, but at a rate faster than $O_{\mathbb{P}}(1/\sqrt{n})$; see also [Smith \(1985, Theorem 3\(ii\)\)](#). For $-1 < \gamma_0 < -1/2$, the results in [Woodroffe \(1974\)](#) suggest that the maximum likelihood estimator converges weakly to a certain non-Gaussian limit and at a rate depending on γ_0 ; see also [Smith \(1985, Theorem 3\(iii\)\)](#). Since at $\gamma_0 \leq -1/2$, the GEV family is not differentiable in quadratic mean, our Proposition 2.1 does not apply, and we do not pursue the matter further.

APPENDIX A. PROOF OF PROPOSITION 2.1

For $0 < \varepsilon < \varepsilon_0$ and $\theta \in U_\varepsilon(\theta_0)$, define a real-valued function $\ell_{\theta,\varepsilon}$ on S_{θ_0} by

$$\ell_{\theta,\varepsilon}(x) = \ell_\theta(x) \mathbb{1}\{x \in \bar{S}(2\varepsilon)\}.$$

The Lipschitz condition (2.4) on ℓ implies that

$$|\ell_{\theta_1,\varepsilon}(x) - \ell_{\theta_2,\varepsilon}(x)| \leq \dot{\ell}(x) \|\theta_1 - \theta_2\| \quad (\text{A.1})$$

for all θ_1 and θ_2 in $U_\varepsilon(\theta_0)$ and for all $x \in S_{\theta_0}$.

We claim that, for any $C > 0$, we have

$$\mathbb{P}_n \ell_{\hat{\theta}_n} = \sup_{\theta \in \Theta} \mathbb{P}_n \ell_\theta \geq \sup_{\|\theta - \theta_0\| < C/\sqrt{n}} \mathbb{P}_n \ell_\theta \geq \sup_{\|\theta - \theta_0\| < C/\sqrt{n}} \mathbb{P}_n \ell_{\theta,C/\sqrt{n}} - o_{\mathbb{P}}(1/n). \quad (\text{A.2})$$

Only the last inequality is non-trivial. Write, for an arbitrary θ with $\|\theta - \theta_0\| < C/\sqrt{n}$,

$$\begin{aligned} \mathbb{P}_n \ell_\theta &= \mathbb{P}_n \ell_{\theta,C/\sqrt{n}} + \mathbb{P}_n \ell_\theta \mathbb{1}\{\cdot \in S_{\theta_0} \setminus \bar{S}(2C/\sqrt{n})\} \\ &\geq \mathbb{P}_n \ell_{\theta,C/\sqrt{n}} - \sup_{\|\theta' - \theta_0\| < C/\sqrt{n}} |\mathbb{P}_n \ell_{\theta'} \mathbb{1}\{\cdot \in S_{\theta_0} \setminus \bar{S}(2C/\sqrt{n})\}|. \end{aligned}$$

The supremum on the second line is $o_{\mathbb{P}}(r_n)$ for any sequence $r_n \downarrow 0$: for all $\eta > 0$,

$$\begin{aligned} \mathbb{P} \left[r_n^{-1} \sup_{\|\theta' - \theta_0\| < C/\sqrt{n}} |\mathbb{P}_n \ell_{\theta'} \mathbb{1}\{\cdot \in S_{\theta_0} \setminus \bar{S}(2C/\sqrt{n})\}| > \eta \right] \\ \leq \mathbb{P}\{\exists i = 1, \dots, n : X_i \in \mathcal{X} \setminus \bar{S}(2C/\sqrt{n})\} \\ \leq nP_{\theta_0}(\mathcal{X} \setminus \bar{S}(2C/\sqrt{n})) = o(1), \quad n \rightarrow \infty, \quad (\text{A.3}) \end{aligned}$$

by the assumption (2.3). Equation (A.2) has thus been proved.

Now, let us show that, for fixed $C > 0$ and all converging sequences $h_n \rightarrow h \in \mathbb{R}^k$ with $\|h_n\| < C$ for all n , we have

$$\sqrt{n}(\ell_{\theta_0+h_n/\sqrt{n},C/\sqrt{n}} - \ell_{\theta_0,C/\sqrt{n}}) \rightarrow h^T \dot{\ell}_{\theta_0} \quad \text{in } L_2(P_{\theta_0}). \quad (\text{A.4})$$

For that purpose, write the left-hand side as

$$A_{n1} - A_{n2} = \sqrt{n}(\ell_{\theta_0+h_n/\sqrt{n}} - \ell_{\theta_0}) - \sqrt{n}(\ell_{\theta_0+h_n/\sqrt{n},C/\sqrt{n}} - \ell_{\theta_0}) \mathbb{1}\{\cdot \in S_{\theta_0} \setminus \bar{S}(2C/\sqrt{n})\}.$$

Because of differentiability in quadratic mean, condition (a), the term A_{n1} converges to $h^T \dot{\ell}_{\theta_0}$ in P_{θ_0} -probability; see the proof of Theorem 5.39 in [van der Vaart \(1998\)](#). Moreover, A_{n2} converges to zero in P_{θ_0} -probability: for all $\eta > 0$, we have

$$P_{\theta_0}(|A_{n2}| > \eta) \leq P_{\theta_0}\{S_{\theta_0} \setminus \bar{S}(2C/\sqrt{n})\} = o(1), \quad n \rightarrow \infty.$$

Hence, the convergence in (A.4) holds in P_{θ_0} -probability. In view of the Lipschitz property of $\ell_{\theta, C/\sqrt{n}}$ in (A.1) and the fact that $P_{\theta_0} \dot{\ell}^2 < \infty$ by assumption (c), convergence in P_{θ_0} -probability can be strengthened to $L_2(P_{\theta_0})$ convergence by applying the dominated convergence theorem.

Next, we shall show that, for any converging sequence $h_n \rightarrow h$ with $\|h_n\| < C$ for all n ,

$$n P_{\theta_0}(\ell_{\theta_0+h_n/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}) \rightarrow -\frac{1}{2} h^T I_{\theta_0} h, \quad n \rightarrow \infty. \quad (\text{A.5})$$

Recall the empirical process $\mathbb{G}_n = \sqrt{n}(\mathbb{P}_n - P_{\theta_0})$. In view of (A.4), computing means and variances, we have

$$B_n(h_n) = \mathbb{G}_n\{\sqrt{n}(\ell_{\theta_0+h_n/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}) - h^T \dot{\ell}_{\theta_0}\} = o_{\mathbb{P}}(1), \quad n \rightarrow \infty. \quad (\text{A.6})$$

We can rewrite B_n as

$$\begin{aligned} B_n(h_n) &= n \mathbb{P}_n(\ell_{\theta_0+h_n/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}) - \mathbb{G}_n h^T \dot{\ell}_{\theta_0} \\ &\quad - n P_{\theta_0}(\ell_{\theta_0+h_n/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}) \\ &= n \mathbb{P}_n(\ell_{\theta_0+h_n/\sqrt{n}} - \ell_{\theta_0}) - \mathbb{G}_n h^T \dot{\ell}_{\theta_0} \\ &\quad - n \mathbb{P}_n(\ell_{\theta_0+h_n/\sqrt{n}} - \ell_{\theta_0}) \mathbb{1}\{\cdot \in \mathcal{X} \setminus \bar{S}(2C/\sqrt{n})\} \\ &\quad - n P_{\theta_0}(\ell_{\theta_0+h_n/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}). \end{aligned}$$

Note that $n \mathbb{P}_n(\ell_{\theta_0+h_n/\sqrt{n}} - \ell_{\theta_0}) = \sum_{i=1}^n \log(p_{\theta_0+h_n/\sqrt{n}}/p_{\theta_0})(X_i)$ is a likelihood ratio statistic. Differentiability in quadratic mean [assumption (a)] implies

$$n \mathbb{P}_n(\ell_{\theta_0+h_n/\sqrt{n}} - \ell_{\theta_0}) - \mathbb{G}_n h^T \dot{\ell}_{\theta_0} = -\frac{1}{2} h^T I_{\theta_0} h + o_{\mathbb{P}}(1), \quad n \rightarrow \infty;$$

see [van der Vaart \(1998, Theorem 7.2\)](#). Moreover, by the assumption (b),

$$\begin{aligned} \mathbb{P} \left[\left| n \mathbb{P}_n(\ell_{\theta_0+h_n/\sqrt{n}} - \ell_{\theta_0}) \mathbb{1}\{\cdot \in \mathcal{X} \setminus \bar{S}(2C/\sqrt{n})\} \right| \geq \eta \right] \\ \leq n P_{\theta_0}(\mathcal{X} \setminus \bar{S}(2C/\sqrt{n})) = o(1), \quad n \rightarrow \infty. \end{aligned}$$

The last four displays imply (A.5).

We can now follow the lines of the proof of Theorem 5.23 in [van der Vaart \(1998\)](#) to prove the following reinforcement of (A.6): for any random sequence $\tilde{h}_n$ such that $\|\tilde{h}_n\| < C$ almost surely for all n , we have

$$B_n(\tilde{h}_n) = o_{\mathbb{P}}(1), \quad n \rightarrow \infty. \quad (\text{A.7})$$

To see this, note that it follows from (A.6) that $B_n(h) = o_{\mathbb{P}}(1)$ for any fixed $h \in \mathbb{R}^k$ with $\|h\| < C$. As a consequence, the finite-dimensional distributions of the stochastic process B_n converge indeed to 0 in probability. It remains to show asymptotic tightness of the sequence B_n in the space $\ell^\infty(\{h : \|h\| < C\})$ equipped with the supremum distance; note that by the Lipschitz property (A.1), the trajectories of B_n are indeed bounded almost surely. Obviously, the sequence of linear processes $h \mapsto \mathbb{G}_n h^T \dot{\ell}_{\theta_0} = h^T \mathbb{G}_n \dot{\ell}_{\theta_0}$ is asymptotically tight, so that it suffices to show the same property for the processes

$$h \mapsto M_n(h) = \sqrt{n} \mathbb{G}_n(\ell_{\theta_0+h/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}).$$

This can be done along the lines of the proof of Lemma 19.31 in [van der Vaart \(1998\)](#), relying on a result for empirical processes indexed by sequences of function classes. More precisely, define a sequence of function classes on S_{θ_0} through

$$\mathcal{M}_n = \{\sqrt{n}(\ell_{\theta_0+h/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}) : \|h\| < C\}.$$

By the Lipschitz property (A.1), the functions in $\mathcal{M}_n$ are bounded by the envelope function $C\dot{\ell} \in L_2(P_{\theta_0})$. From Example 19.7 in [van der Vaart \(1998\)](#), we obtain the following bound on the bracketing number $N_{[]}(\varepsilon, \mathcal{M}_n, L_2(P_{\theta_0}))$: for some constant $K > 0$ not depending on n , we have, for all sufficiently small $\varepsilon > 0$,

$$N_{[]}(\varepsilon, \mathcal{M}_n, L_2(P_{\theta_0})) \leq K \left(\varepsilon^{-1} (2C^2) \|\dot{\ell}\|_{L_2(P_{\theta_0})} \right)^k,$$

where k denotes the dimension of the Euclidean space of which Θ is a subset; here we used the property that the diameter of the ball $\{h \in \mathbb{R}^k : \|h\| < C\}$ is equal to $2C$. As a consequence, if $\delta_n \downarrow 0$, the bracketing integrals converge to zero:

$$J_{[]}(\delta_n, \mathcal{M}_n, L_2(P_{\theta_0})) = \int_0^{\delta_n} \sqrt{\log N_{[]}(\varepsilon, \mathcal{M}_n, L_2(P_{\theta_0}))} d\varepsilon \rightarrow 0, \quad n \rightarrow \infty.$$

The Lindeberg condition $P_{\theta_0}(C\dot{\ell})^2 \mathbb{1}\{C\dot{\ell} > \varepsilon\sqrt{n}\} \rightarrow 0$ ($n \rightarrow \infty$) is satisfied because the envelope function $C\dot{\ell}$ belongs to $L_2(P_{\theta_0})$. Asymptotic tightness of B_n then follows from Theorem 19.28 in [van der Vaart \(1998\)](#). Equation (A.7) is thus proven.

To complete the proof of Proposition 2.1, we can now proceed similarly as in the proof of Theorem 5.23 in [van der Vaart \(1998\)](#), with some additional effort needed to get rid of the constant C . In view of (A.5), the convergence property (A.7) can be rewritten as

$$n \mathbb{P}_n(\ell_{\theta_0+\tilde{h}_n/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}) = -\frac{1}{2} \tilde{h}_n^T I_{\theta_0} \tilde{h}_n + \tilde{h}_n^T \mathbb{G}_n \dot{\ell}_{\theta_0} + o_{\mathbb{P}}(1), \quad n \rightarrow \infty, \quad (\text{A.8})$$

where $\tilde{h}_n$ denotes an arbitrary random sequence in $\mathbb{R}^k$ with $\|\tilde{h}_n\| < C$.

Define $\hat{h}_{n1} = \sqrt{n}(\hat{\theta}_n - \theta_0)$ and $\hat{h}_{n2} = I_{\theta_0}^{-1} \mathbb{G}_n \dot{\ell}_{\theta_0}$. For $C > 0$, let $A_{n,C}$ denote the event $\{\max(\|\hat{h}_{n1}\|, \|\hat{h}_{n2}\|) < C\}$, and set $\tilde{h}_{nj} = \hat{h}_{nj} \mathbb{1}(A_{n,C})$, for $j \in \{1, 2\}$. Inserting both tilde-expressions into (A.8), we get, as $n \rightarrow \infty$,

$$\begin{aligned} n \mathbb{P}_n(\ell_{\theta_0+\tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}) &= -\frac{1}{2} \tilde{h}_{n1}^T I_{\theta_0} \tilde{h}_{n1} + \tilde{h}_{n1}^T \mathbb{G}_n \dot{\ell}_{\theta_0} + o_{\mathbb{P}}(1), \\ n \mathbb{P}_n(\ell_{\theta_0+\tilde{h}_{n2}/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0, C/\sqrt{n}}) &= \frac{1}{2} \mathbb{G}_n \dot{\ell}_{\theta_0}^T I_{\theta_0}^{-1} \mathbb{G}_n \dot{\ell}_{\theta_0} \mathbb{1}(A_{n,C}) + o_{\mathbb{P}}(1). \end{aligned}$$

Subtracting the second equation from the first one yields

$$n \mathbb{P}_n(\ell_{\theta_0+\tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}} - \ell_{\theta_0+\tilde{h}_{n2}/\sqrt{n}, C/\sqrt{n}}) = -Q_n \mathbb{1}(A_{n,C}) + o_{\mathbb{P}}(1), \quad n \rightarrow \infty,$$

$$\text{where } Q_n = \frac{1}{2} (\hat{h}_{n1} - I_{\theta_0}^{-1} \mathbb{G}_n \dot{\ell}_{\theta_0})^T I_{\theta_0} (\hat{h}_{n1} - I_{\theta_0}^{-1} \mathbb{G}_n \dot{\ell}_{\theta_0}).$$

We will show that

$$Q_n = o_{\mathbb{P}}(1), \quad n \rightarrow \infty. \quad (\text{A.9})$$

Since the Fisher information matrix I_{θ_0} is positive definite, this will imply that

$$\sqrt{n}(\hat{\theta}_n - \theta_0) - I_{\theta_0}^{-1} \frac{1}{\sqrt{n}} \sum_{i=1}^n \dot{\ell}_{\theta_0}(X_i) = \hat{h}_{n1} - I_{\theta_0}^{-1} \mathbb{G}_n \dot{\ell}_{\theta_0} = o_{\mathbb{P}}(1), \quad n \rightarrow \infty,$$

which is the first part of (2.5). The asymptotic normality then follows from the multivariate central limit theorem.

It remains to show (A.9). Note that $Q_n \geq 0$, since I_{θ_0} is positive definite. By the maximization property (A.2), we have, as $n \rightarrow \infty$,

$$\begin{aligned} R_n &= n \mathbb{P}_n \ell_{\hat{\theta}_n} \geq n \mathbb{P}_n \ell_{\theta_0 + \tilde{h}_{n2}/\sqrt{n}, C/\sqrt{n}} - o_{\mathbb{P}}(1) \\ &= n \mathbb{P}_n \ell_{\theta_0 + \tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}} + Q_n \mathbb{1}(A_{n,C}) - o_{\mathbb{P}}(1). \end{aligned}$$

Isolating $Q_n \mathbb{1}(A_{n,C})$, we find

$$\begin{aligned} 0 \leq Q_n &= Q_n \mathbb{1}(A_{n,C}) + Q_n \mathbb{1}(A_{n,C}^c) \\ &\leq R_n - n \mathbb{P}_n \ell_{\theta_0 + \tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}} + Q_n \mathbb{1}(A_{n,C}^c) + o_{\mathbb{P}}(1), \quad n \rightarrow \infty. \end{aligned}$$

Note that $\ell_{\hat{\theta}_n}(x) \mathbb{1}\{x \in \bar{S}(2C/\sqrt{n})\} \mathbb{1}(A_{n,C}) = \ell_{\theta_0 + \tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}}(x) \mathbb{1}(A_{n,C})$. Therefore,

$$\begin{aligned} R_n &= R_n \mathbb{1}(A_{n,C}) + R_n \mathbb{1}(A_{n,C}^c) \\ &= [n \mathbb{P}_n \ell_{\hat{\theta}_n} \mathbb{1}\{\cdot \in \bar{S}(2C/\sqrt{n})\} + n \mathbb{P}_n \ell_{\hat{\theta}_n} \mathbb{1}\{\cdot \notin \bar{S}(2C/\sqrt{n})\}] \mathbb{1}(A_{n,C}) + R_n \mathbb{1}(A_{n,C}^c) \\ &= n \mathbb{P}_n \ell_{\theta_0 + \tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}} \mathbb{1}(A_{n,C}) + o_{\mathbb{P}}(1) + R_n \mathbb{1}(A_{n,C}^c), \quad n \rightarrow \infty; \end{aligned}$$

the $o_{\mathbb{P}}(1)$ term appears because of the argument in (A.3). Substituting the expansion for R_n into the upper bound for Q_n yields, as $n \rightarrow \infty$,

$$\begin{aligned} 0 \leq Q_n &\leq n \mathbb{P}_n \ell_{\theta_0 + \tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}} \mathbb{1}(A_{n,C}) + R_n \mathbb{1}(A_{n,C}^c) \\ &\quad - n \mathbb{P}_n \ell_{\theta_0 + \tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}} + Q_n \mathbb{1}(A_{n,C}^c) + o_{\mathbb{P}}(1) \\ &= \{R_n + Q_n - n \mathbb{P}_n \ell_{\theta_0 + \tilde{h}_{n1}/\sqrt{n}, C/\sqrt{n}}\} \mathbb{1}(A_{n,C}^c) + o_{\mathbb{P}}(1). \end{aligned}$$

Hence, for any $\varepsilon > 0$,

$$\limsup_{n \rightarrow \infty} \mathbb{P}(|Q_n| > \varepsilon) \leq \limsup_{n \rightarrow \infty} \mathbb{P}(A_{n,C}^c),$$

which can be made arbitrary small by increasing C . This finishes the proof of (A.9) and thus of Proposition 2.1. $\square$

APPENDIX B. DENSITY AND SCORE FUNCTIONS OF THE GEV FAMILY

The density, log-density, score functions and Fisher information matrix of the GEV family can of course be found in many articles and textbooks. Here we present some of these objects in a form which is convenient for later analysis. In the notation of Section 2, the state space $(\mathcal{X}, \mathcal{A}, \mu)$ is the real line equipped with its Borel sets and the Lebesgue measure. The probability density function of the GEV distribution P_{θ} with parameter $\theta = (\gamma, \mu, \sigma) \in \mathbb{R} \times \mathbb{R} \times (0, \infty)$ is given by

$$p_{\theta}(x) = \sigma^{-1} e^{-u} u^{\gamma+1} \mathbb{1}(1 + \gamma z > 0), \quad x \in \mathbb{R},$$

where

$$\begin{aligned} z &= z_{\mu, \sigma}(x) = \frac{x - \mu}{\sigma}, \\ u &= u_{\gamma}(z) = \exp\left(-\int_0^z \frac{1}{1 + \gamma t} dt\right) = \begin{cases} (1 + \gamma z)^{-1/\gamma} & \text{if } \gamma \neq 0, \\ e^{-z} & \text{if } \gamma = 0. \end{cases} \end{aligned} \quad (\text{B.1})$$

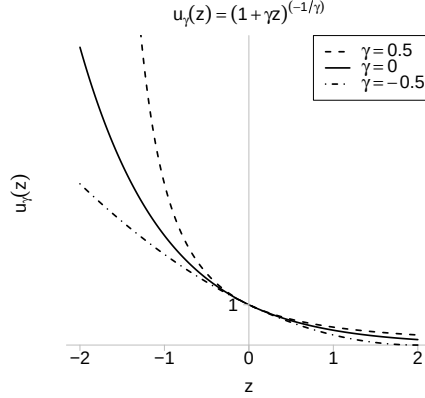


FIGURE 1. Graph of $z \mapsto u_\gamma(z)$ in (B.1) for $\gamma \in \{-0.5, 0, 0.5\}$ and $z \in [-2, 2]$.

The expression $u_\gamma(z)$ is convex and decreasing in z and is increasing in γ . See Figure 1 for graphs of the functions $z \mapsto u_\gamma(z)$ for $\gamma \in \{-0.5, 0, 0.5\}$.

The support, S_θ , of the distribution is (defined as) the open interval

$$S_\theta = \{x : p_\theta(x) > 0\} = \{x \in \mathbb{R} : \sigma + \gamma(x - \mu) > 0\} = \begin{cases} (-\infty, \mu - \sigma/\gamma) & \text{if } \gamma < 0, \\ \mathbb{R} & \text{if } \gamma = 0, \\ (\mu - \sigma/\gamma, \infty) & \text{if } \gamma > 0. \end{cases}$$

The log-density is given by

$$\ell_\theta(x) = -\log(\sigma) - u + (\gamma + 1) \log(u), \quad x \in S_\theta,$$

with $\log(u) = -\int_0^z (1 + \gamma t)^{-1} dt$. The partial derivatives of the map $\theta \mapsto \ell_\theta(x)$ at θ such that $p_\theta(x) > 0$ are given by

$$\partial_\gamma \ell_\theta(x) = (1 - u) \partial_\gamma \log(u) - \frac{z}{1 + \gamma z}, \quad (\text{B.2})$$

$$\partial_\mu \ell_\theta(x) = \frac{\gamma + 1 - u}{\sigma(1 + \gamma z)}, \quad (\text{B.3})$$

$$\partial_\sigma \ell_\theta(x) = \frac{(1 - u)z - 1}{\sigma(1 + \gamma z)}, \quad (\text{B.4})$$

with $\partial_\gamma \log(u)$ given by

$$0 \leq \partial_\gamma \log(u) = \int_0^z \frac{t}{(1 + \gamma t)^2} dt = \begin{cases} \frac{1}{\gamma} \left(\frac{1}{\gamma} \log(1 + \gamma z) - \frac{z}{1 + \gamma z} \right) & \text{if } \gamma \neq 0, \\ \frac{z^2}{2} & \text{if } \gamma = 0. \end{cases} \quad (\text{B.5})$$

For $\gamma > -1/2$, these partial derivatives have zero expectation and finite second moments with respect to P_θ . For such θ , the Fisher information matrix $I_\theta \in \mathbb{R}^{3 \times 3}$ is equal to the covariance matrix of the score vector

$$\dot{\ell}_\theta = (\partial_\gamma \ell_\theta, \partial_\mu \ell_\theta, \partial_\sigma \ell_\theta)^T,$$

the three entries of which are viewed as elements of $L_2(P_\theta)$. Explicit expressions for I_θ are given in [Prescott and Walden \(1980\)](#), but we will not be needing those here. The only properties of I_θ that are relevant for our current study are that I_θ is symmetric, positive definite and non-singular and that the map $\theta \mapsto I_\theta$ is continuous.

We continue with a number of properties of the GEV densities and score functions.

Lemma B.1. *Let γ and z be such that $1 + \gamma z > 0$. Let $u = u_\gamma(z)$ be as in (B.1). If $\gamma z > 0$, then*

$$\partial_\gamma \log(u) \leq \begin{cases} \frac{z^2}{2}, \\ \frac{1}{\gamma} \log(1 + \gamma z) \frac{z}{1 + \gamma z} \leq \begin{cases} \frac{z^2}{1 + \gamma z}, \\ \frac{1}{\gamma^2} \log(1 + \gamma z). \end{cases} \end{cases} \quad (\text{B.6})$$

If $-1 < \gamma z < 0$, then also

$$\partial_\gamma \log(u) \leq \frac{z^2}{1 + \gamma z}. \quad (\text{B.7})$$

As a consequence, for any γ and z such that $1 + \gamma z > 0$, we have

$$0 \leq \partial_\gamma \log(u) \leq \frac{z^2}{1 + \gamma z}. \quad (\text{B.8})$$

Proof. If $\gamma = 0$ or $z = 0$, the stated inequalities are clearly satisfied, so suppose that $\gamma \neq 0$ and $z \neq 0$. Substituting $\gamma t = v$ in (B.5), we find

$$\partial_\gamma \log(u) = \frac{1}{\gamma^2} \int_0^{\gamma z} \frac{v}{(1 + v)^2} dv.$$

If $\gamma z > 0$, since $v \mapsto v/(1 + v)$ is increasing in $v \geq 0$ and $v \mapsto 1/(1 + v)$ is decreasing in $v \geq 0$, we obtain the bounds in the first display of the lemma: not only

$$\partial_\gamma \log(u) \leq \frac{1}{\gamma^2} \int_0^{\gamma z} v dv = \frac{z^2}{2}$$

but also

$$\partial_\gamma \log(u) \leq \frac{z}{1 + \gamma z} \frac{1}{\gamma} \int_0^{\gamma z} \frac{1}{1 + v} dv \leq \frac{z^2}{1 + \gamma z}.$$

If $\gamma z < 0$, then we have $\gamma z = -|\gamma z|$ and

$$\partial_\gamma \log(u) = \frac{1}{\gamma^2} \int_0^{|\gamma z|} \frac{v}{(1 - v)^2} dv \leq \frac{|z|}{|\gamma|} \int_0^{|\gamma z|} \frac{1}{(1 - v)^2} dv = \frac{|z|^2}{1 - |\gamma z|}. \quad \square$$

Lemma B.2. *Let θ and x be such that $1 + \gamma z > 0$, where $z = (x - \mu)/\sigma$. We have*

$$|\partial_\mu \ell_\theta(x)| \leq \begin{cases} \sigma^{-1}(1 + |\gamma|) u^{1+\gamma} & \text{if } z \leq 0, \\ \sigma^{-1}(1 + |\gamma|) u^\gamma & \text{if } z \geq 0. \end{cases}$$

Proof. We have $(1 + \gamma z)^{-1} = u^\gamma$ and thus $\partial_\mu \ell_\theta(x) = \sigma^{-1}(\gamma + 1 - u)u^\gamma$. If $z \leq 0$, then $u \geq 1$, so that $|\gamma + 1 - u| \leq u + |\gamma| \leq (1 + |\gamma|)u$. If $z \geq 0$, then $0 < u \leq 1$, so that $|\gamma + 1 - u| \leq 1 + |\gamma|$. $\square$

Lemma B.3. *Let θ and x be such that $1 + \gamma z > 0$, where $z = (x - \mu)/\sigma$. We have*

$$|\partial_\sigma \ell_\theta(x)| \leq \begin{cases} \frac{z+1}{\sigma(1+\gamma z)} & \text{if } z \geq 0, \\ \sigma^{-1}(1+u \log(u)) u^{\max(\gamma, 0)} & \text{if } z \leq 0. \end{cases}$$

Proof. If $z \geq 0$, then $0 < u \leq 1$, so that $|(1-u)z - 1| \leq z+1$, yielding the stated bound.

If $z \leq 0$, then $u \geq 1$, so that $|(1-u)z - 1| \leq u|z| + 1$. Further, $(1+\gamma z)^{-1} = u^\gamma$ as well as

$$\frac{|z|}{1+\gamma z} = \frac{u^\gamma - 1}{\gamma} = \int_1^u t^{\gamma-1} dt \leq u^{\max(\gamma, 0)} \int_1^u t^{-1} dt = u^{\max(\gamma, 0)} \log(u). \quad (\text{B.9})$$

The stated bound now follows from $|\partial_\sigma \ell_\theta(x)| \leq \sigma^{-1} \frac{u|z|+1}{(1+\gamma z)}$. $\square$

Lemma B.4. *Let θ and x be such that $1 + \gamma z > 0$, where $z = (x - \mu)/\sigma$.*

If $\gamma \geq 0$ and $z \geq 0$, then, with $\gamma^{-2} \log(1 + \gamma z)$ and γ^{-1} being defined as $+\infty$ for $\gamma = 0$,

$$|\partial_\gamma \ell_\theta(x)| \leq \max \left\{ \min \left(\frac{z^2}{2}, \frac{1}{\gamma^2} \log(1 + \gamma z) \right), \min \left(z, \frac{1}{\gamma} \right) \right\}. \quad (\text{B.10})$$

If $\gamma \leq 0$ and $z \geq 0$, then

$$|\partial_\gamma \ell_\theta(x)| \leq \frac{\max(z^2, z)}{1 + \gamma z}. \quad (\text{B.11})$$

If $\gamma \geq 0$ and $z \leq 0$, then

$$|\partial_\gamma \ell_\theta(x)| \leq u^{1+\gamma} \max\{(\log u)^2, \log u\}. \quad (\text{B.12})$$

If $\gamma \leq 0$ and $z \leq 0$,

$$|\partial_\gamma \ell_\theta(x)| \leq u \max\{(\log u)^2, \log u\}. \quad (\text{B.13})$$

Proof. If $\gamma = 0$, then $u = e^{-z}$ and $|\partial_\gamma \ell_\theta(x)| = (1 - e^{-z})z^2/2 - z$, and all stated inequalities are satisfied. In the remainder of the proof, we assume therefore that $\gamma \neq 0$.

Suppose first that $z \geq 0$, so that $0 < u \leq 1$. Since $|a - b| \leq \max(a, b)$ for nonnegative numbers a, b , we have

$$|\partial_\gamma \ell_\theta(x)| \leq \max \left\{ \partial_\gamma \log(u), \frac{z}{1 + \gamma z} \right\}.$$

Use (B.6) to find (B.10) and use (B.7) to find (B.11).

Next suppose $z \leq 0$, so that $u \geq 1$. If $\gamma < 0$, then, by (B.6) and (B.9),

$$\partial_\gamma \log(u) \leq \left(-\frac{1}{\gamma}\right) \log(1 + \gamma z) \frac{|z|}{1 + \gamma z} \leq (\log u)^2.$$

We find, again using (B.9), as stated in (B.13), that

$$|\partial_\gamma \ell_\theta(x)| \leq \max \left\{ u \partial_\gamma \log(u), \frac{|z|}{1 + \gamma z} \right\} \leq u \max\{(\log u)^2, \log u\}.$$

Finally, suppose $z \leq 0$ and $\gamma > 0$. By (B.7) and (B.9)

$$\partial_\gamma \log(u) \leq \frac{z^2}{1 + \gamma z} = (1 + \gamma z) \left(\frac{|z|}{1 + \gamma z} \right)^2 \leq u^{-\gamma} (u^\gamma \log(u))^2 = u^\gamma (\log u)^2.$$

We obtain, again using (B.9),

$$|\partial_\gamma \ell_\theta(x)| \leq \max \left\{ u \partial_\gamma \log(u), \frac{|z|}{1 + \gamma z} \right\} \leq u^{1+\gamma} \max\{(\log u)^2, \log u\},$$

which is (B.12). $\square$

Lemma B.5. *Fix $a \in [1/2, 1)$. For any $x \in \mathbb{R}$, the function $\theta \mapsto p_\theta^a(x)$ is continuously differentiable on $(-a/(1+a), \infty) \times \mathbb{R} \times (0, \infty)$ and the partial derivatives are continuous in $(x, \theta) \in \mathbb{R} \times (-a/(1+a), \infty) \times \mathbb{R} \times (0, \infty)$.*

Proof. Fix $\theta_0 = (\gamma_0, \mu_0, \sigma_0)$ with $\gamma_0 > -a/(1+a)$ and choose $x_0 \in \mathbb{R}$.

If $\sigma_0 + \gamma_0(x_0 - \mu_0) > 0$, then also $\sigma + \gamma(x - \mu) > 0$ for all (x, θ) in a neighbourhood of (x_0, θ_0) and thus $p_\theta^a(x) = \sigma^{-a} e^{-au} u^{a(\gamma+1)}$ for such (x, θ) . All functions arising in the expression of $p_\theta^a(x)$ are continuously differentiable in the three parameters and are continuous in (x, θ) . Since

$$\partial_{\theta_k} p_\theta^a(x) = a p_\theta^a(x) \partial_{\theta_k} \ell_\theta(x),$$

the score function formulas in (B.2), (B.3), and (B.4) imply that

$$\begin{aligned} \partial_\gamma p_\theta^a(x) &= \left\{ (1-u) \partial_\gamma \log(u) - \frac{z}{1+\gamma z} \right\} \frac{a}{\sigma^a} e^{-au} u^{a(\gamma+1)}, \\ \partial_\mu p_\theta^a(x) &= \frac{\gamma+1-u}{\sigma(1+\gamma z)} \frac{a}{\sigma^a} e^{-au} u^{a(\gamma+1)}, \\ \partial_\sigma p_\theta^a(x) &= \frac{(1-u)z-1}{\sigma(1+\gamma z)} \frac{a}{\sigma^a} e^{-au} u^{a(\gamma+1)}. \end{aligned}$$

If $\sigma_0 + \gamma_0(x_0 - \mu_0) < 0$, then also $\sigma + \gamma(x - \mu) < 0$ for all (x, θ) in a neighbourhood of (x_0, θ_0) and thus $p_\theta^a(x) = 0$ for all such (x, θ) , whence the partial derivatives vanish too.

The difficult case is $\sigma_0 + \gamma_0(x_0 - \mu_0) = 0$, that is, if $\gamma_0 \neq 0$ and $x_0 = \mu_0 - \sigma_0/\gamma_0$. We need to show that, for every $k \in \{1, 2, 3\}$,

$$\lim_{\substack{(x, \theta) \rightarrow (x_0, \theta_0) \\ \sigma + \gamma(x - \mu) > 0}} \partial_{\theta_k} p_\theta^a(x) = 0$$

where $(\theta_1, \theta_2, \theta_3) = (\gamma, \mu, \sigma)$. Recall $u = (1 + \gamma z)^{-1/\gamma}$ with $z = (x - \mu)/\sigma$.

First, suppose that $\gamma_0 > 0$. Then $u \rightarrow \infty$ as $(x, \theta) \rightarrow (x_0, \theta_0)$ and convergence to zero is assured by the exponential factor e^{-au} in each of the three partial derivatives.

Second, suppose that $\gamma_0 < 0$. Then $u \rightarrow 0$ as $(x, \theta) \rightarrow (x_0, \theta_0)$. Using the bound in (B.8), we see that the limit behaviour of the three partial derivatives is dominated by the factor

$$(1 + \gamma z)^{-\frac{a}{\gamma} - a - 1}$$

as $1 + \gamma z \rightarrow 0$. The exponent must be positive eventually: $-(a/\gamma_0) - a - 1 > 0$. But this is equivalent to $-a/(1+a) < \gamma_0 < 0$. $\square$

APPENDIX C. DIFFERENTIABILITY IN QUADRATIC MEAN OF THE GEV FAMILY

Proof of Proposition 3.2. Let $s_\theta = \sqrt{p_\theta}$. We distinguish between three cases: $\gamma_0 > -1/3$, $-1/2 < \gamma_0 \leq -1/3$, and $\gamma_0 \leq -1/2$.

Case $\gamma_0 > -1/3$. Lemma B.5 implies that for all $x \in \mathbb{R}$, the function $\theta \mapsto s_\theta(x)$ is continuously differentiable in a neighbourhood of θ_0 . Differentiability in quadratic mean then follows from an application of Lemma 7.6 in van der Vaart (1998). Note that the existence and the continuity of the Fisher information matrix in a neighbourhood of θ_0 has been derived Prescott and Walden (1980).

Case $-1/2 < \gamma_0 \leq -1/3$. Recall $s_\theta = \sqrt{p_\theta}$. The conditions of Lemma 7.6 in van der Vaart (1998) are no longer fulfilled, but an adaptation of that proof still yields differentiability in quadratic mean.

Since the unit ball in $\mathbb{R}^3$ is compact, it suffices to show that, if $h_n \rightarrow h$ in $\mathbb{R}^3$ and if $t_n \downarrow 0$ as $n \rightarrow \infty$, then

$$\lim_{n \rightarrow \infty} \int_{\mathbb{R}} \left(\frac{s_{\theta_0 + t_n h_n}(x) - s_{\theta_0}(x)}{t_n} - \frac{1}{2} h_n^T \dot{\ell}_{\theta_0}(x) s_{\theta_0}(x) \right)^2 dx = 0.$$

It is sufficient to show the same equality with $h_n^T \dot{\ell}_{\theta_0}(x) s_{\theta_0}(x)$ replaced by $h^T \dot{\ell}_{\theta_0}(x) s_{\theta_0}(x)$; to see why, use the elementary inequality $(a+b)^2 \leq 2a^2 + 2b^2$, the fact that the score vector has P_{θ_0} -square integrable components, and the assumption that $h_n \rightarrow h$ as $n \rightarrow \infty$. For every $x \in \mathbb{R}$ except $x = \mu_0 - \sigma_0/\gamma_0$, the integrand of the resulting integral converges to zero by differentiability of the map $\theta \mapsto s_\theta(x)$ at $\theta = \theta_0$. What remains is to show convergence of the integral itself. To this end, we apply Proposition 2.29 in van der Vaart (1998) with $p = 2$; the condition to check is that

$$\limsup_{n \rightarrow \infty} \int_{\mathbb{R}} \left(\frac{s_{\theta_0 + t_n h_n}(x) - s_{\theta_0}(x)}{t_n} \right)^2 dx \leq \int_{\mathbb{R}} \left(\frac{1}{2} h^T \dot{\ell}_{\theta_0}(x) s_{\theta_0}(x) \right)^2 dx. \quad (\text{C.1})$$

The right-hand side in (C.1) is equal to $(1/4)h^T (\int \dot{\ell}_{\theta_0} \dot{\ell}_{\theta_0}^T p_{\theta_0}) h = (1/4)h^T I_{\theta_0} h$. We need to control the left-hand side.

Write $h_n = (h_{n1}, h_{n2}, h_{n3})^T$ and $\theta_n = \theta_0 + t_n h_n = (\gamma_n, \mu_n, \sigma_n)$. Without loss of generality, assume that n is large enough such that $-1/2 < \gamma_n < 0$. The upper endpoints of the supports of the GEV distributions with parameter vectors θ_0 , θ_n , $(\gamma_0, \mu_n, \sigma_n)$ and $(\gamma_n, \mu_0, \sigma_0)$ are equal to $\mu_0 - \sigma_0/\gamma_0$, $\mu_n - \sigma_n/\gamma_n$, $\mu_n - \sigma_n/\gamma_0$ and $\mu_0 - \sigma_0/\gamma_n$, respectively. Let α_n and β_n be the minimum and the maximum of these four endpoints, respectively. Write

$$f_n = \frac{s_{\theta_0 + t_n h_n} - s_{\theta_0}}{t_n}.$$

Since $f_n(x) = 0$ for $x > \beta_n$, we have

$$\int_{\mathbb{R}} f_n(x)^2 dx = \int_{-\infty}^{\alpha_n} f_n(x)^2 dx + \int_{\alpha_n}^{\beta_n} f_n(x)^2 dx.$$

We will show that the limit superior of the first term on the right-hand side is bounded by the right-hand side of (C.1), while the second term on the right-hand side converges to zero.

To bound the integral of $f_n(x)^2$ from $-\infty$ to α_n , we can proceed as in the proof of Lemma 7.6 in [van der Vaart \(1998\)](#). See in particular the display on top of page 96. Each point $x \in (-\infty, \alpha_n)$ is an element of the support of s_θ for each θ on the line segment connecting θ_0 and θ_n ; this was the reason for introducing the additional parameter vectors $(\gamma_0, \mu_n, \sigma_n)$ and $(\gamma_n, \mu_0, \sigma_0)$ in the previous paragraph. Let $\dot{s}_\theta(x)$ denote the gradient of the map $\theta \mapsto s_\theta(x)$, for x in the support of P_θ . Note that $\dot{s}_\theta(x) = (1/2)s_\theta(x) \dot{\ell}_\theta(x)$. Since the map $u \mapsto s_{\theta_0+ut_n h_n}(x)$, for $u \in [0, 1]$, is continuously differentiable, we have $f_n(x) = \int_{u=0}^1 \dot{s}_{\theta_0+ut_n h_n}(x)^T h_n du$. Applying the Cauchy-Schwarz inequality and the Fubini theorem, we find

$$\begin{aligned} \int_{-\infty}^{\alpha_n} f_n(x)^2 dx &\leq \int_{x=-\infty}^{\alpha_n} \int_{u=0}^1 (\dot{s}_{\theta_0+ut_n h_n}(x)^T h_n)^2 du dx \\ &\leq \frac{1}{4} \int_{u=0}^1 h_n^T I_{\theta_0+ut_n h_n} h_n du. \end{aligned}$$

By continuity of the map $\theta \mapsto I_\theta$, the right-hand side converges to $(1/4)h^T I_{\theta_0} h$.

To show that $\int_{\alpha_n}^{\beta_n} f_n(x)^2 dx$ converges to zero, observe that

$$f_n(x)^2 \leq 2t_n^{-2} (p_{\theta_n}(x) + p_{\theta_0}(x))$$

Moreover, there exists $c_0 > 0$ such that $\alpha_n \geq \beta_n - c_0 t_n$ for all sufficiently large n . As a consequence, it is sufficient to show that, whenever $\tilde{\theta}_n \rightarrow \theta_0$ and $u_n \downarrow 0$, we have

$$\lim_{n \rightarrow \infty} u_n^{-2} P_{\tilde{\theta}_n} [\tilde{\omega}_n - u_n, \tilde{\omega}_n] = 0, \quad (\text{C.2})$$

with $\tilde{\omega}_n$ the upper endpoint of the support of $P_{\tilde{\theta}_n}$. Writing $\tilde{\theta}_n = (\tilde{\gamma}_n, \tilde{\mu}_n, \tilde{\sigma}_n)$, we have

$$\begin{aligned} P_{\tilde{\theta}_n} [\tilde{\omega}_n - u_n, \tilde{\omega}_n] &= 1 - \exp \left[- \left\{ 1 + \tilde{\gamma}_n \frac{\tilde{\mu}_n - \tilde{\sigma}_n / \tilde{\gamma}_n - u_n - \tilde{\mu}_n}{\tilde{\sigma}_n} \right\}^{-1/\tilde{\gamma}_n} \right] \\ &= 1 - \exp \left\{ - (|\tilde{\gamma}_n| u_n / \tilde{\sigma}_n)^{1/|\tilde{\gamma}_n|} \right\} \\ &\leq (|\tilde{\gamma}_n| u_n / \tilde{\sigma}_n)^{1/|\tilde{\gamma}_n|}. \end{aligned}$$

On the last line, we used the elementary inequality $1 - \exp(-z) \leq z$ for all $z \in \mathbb{R}$. Since $1/|\tilde{\gamma}_n| \rightarrow 1/|\gamma_0| > 2$, equation (C.2) follows.

Case $\gamma_0 \leq -1/2$. We will show that for $h = (0, t, 0)^T$ with $t \downarrow 0$, we have

$$\liminf_{t \downarrow 0} t^{-2} P_{\theta_0+h} \{p_{\theta_0} = 0\} > 0. \quad (\text{C.3})$$

Here, $\{p_{\theta_0} = 0\}$ is short-hand for $\{x \in \mathbb{R} : p_{\theta_0}(x) = 0\} = [\mu_0 - \sigma_0/\gamma_0, \infty)$. Restricting the integral on the left-hand side in (2.1) to the set $\{p_{\theta_0} = 0\}$ then shows that the asymptotic relation in (2.1) cannot hold.

We show (C.3). The upper endpoint of P_{θ_0+h} is equal to $\mu_0 + t - \sigma_0/\gamma_0$, which is larger than the one of P_{θ_0} . The mass assigned by P_{θ_0+h} to the difference of the two supports

is given by

$$\begin{aligned} P_{\theta_0+h}[\mu_0 - \sigma_0/\gamma_0, \infty) &= 1 - \exp \left[- \left\{ 1 + \gamma_0 \frac{(\mu_0 - \sigma_0/\gamma_0) - \mu_0 - t}{\sigma_0} \right\}^{-1/\gamma_0} \right] \\ &= 1 - \exp \left\{ - (|\gamma_0| t / \sigma_0)^{1/|\gamma_0|} \right\}. \end{aligned}$$

Since $0 < 1/|\gamma_0| \leq 2$ and since $1 - \exp(-u) = (1 + o(1))u$ as $u \rightarrow 0$, the inequality (C.3) follows. $\square$

APPENDIX D. RATE OF CONVERGENCE

To apply Proposition 2.1, the $O_{\mathbb{P}}(1/\sqrt{n})$ rate of convergence of the maximum likelihood estimator needs to be established first.

Proposition D.1 (Rate of convergence). *Let $X_1, X_2, \dots$ be independent and identically distributed random variables with common GEV law G_{θ_0} , with $\theta_0 = (\gamma_0, \mu_0, \sigma_0) \in (-1/2, \infty) \times \mathbb{R} \times (0, \infty)$. Then, for any compact parameter set $\Theta \subset (-1/2, \infty) \times \mathbb{R} \times (0, \infty)$, any sequence of maximum likelihood estimators over Θ satisfies $\hat{\theta}_n - \theta_0 = O_{\mathbb{P}}(1/\sqrt{n})$ as $n \rightarrow \infty$.*

Proof of Proposition D.1. We apply Corollary 5.53 in van der Vaart (1998) to the function m_{θ} in (2.2). To do so, we need to check a number of things:

- $\hat{\theta}_n = \theta_0 + o_{\mathbb{P}}(1)$ as $n \rightarrow \infty$: this is okay by Proposition 3.1.
- $\mathbb{P}_n m_{\hat{\theta}_n} \geq \mathbb{P}_n m_{\theta_0} - O_{\mathbb{P}}(n^{-1})$: By concavity of the logarithm, we have

$$\mathbb{P}_n m_{\hat{\theta}_n} \geq \mathbb{P}_n \left(\log \frac{p_{\hat{\theta}_n}}{p_{\theta_0}} \right) + \mathbb{P}_n \log 1 \geq 0 = \mathbb{P}_n m_{\theta_0}.$$

- There exists $\dot{m} \in L_2(P_{\theta_0})$ such that $|m_{\theta_1}(x) - m_{\theta_2}(x)| \leq \dot{m}(x) \|\theta_1 - \theta_2\|$ for P_{θ_0} -almost all x and all θ_1 and θ_2 in a neighbourhood of θ : this Lipschitz condition is the topic of Lemma D.3.
- The map $\theta \mapsto P_{\theta_0} m_{\theta}$ admits a second-order Taylor expansion at the point of maximum θ_0 with non-singular second derivative: this is established in Lemma D.4.

The cited corollary now yields $\sqrt{n}(\hat{\theta}_n - \theta_0) = O_{\mathbb{P}}(1)$ as $n \rightarrow \infty$, as required. $\square$

Lemma D.2 (Relative errors). *Let $(\mu_0, \sigma_0) \in \mathbb{R} \times (0, \infty)$ and $\varepsilon \in (0, 1]$. Let $(\mu, \sigma) \in \mathbb{R} \times (0, \infty)$ be such that $|\mu - \mu_0|/\sigma_0 \leq \varepsilon$ and $|\sigma/\sigma_0 - 1| \leq \varepsilon$. Let $x \in \mathbb{R}$ and write $z_0 = (x - \mu_0)/\sigma_0$ and $z = (x - \mu)/\sigma$. If $|z_0| \geq 2$, then $|z/z_0 - 1| \leq 2\varepsilon$.*

Proof. Since $(1+b)(1-c) - 1 = b - c + bc$ for $b, c \in \mathbb{R}$, we have

$$\begin{aligned} \left| \frac{z}{z_0} - 1 \right| &= \left| \left(1 + \frac{\sigma_0}{\sigma} - 1 \right) \left(1 - \frac{(\mu - \mu_0)/\sigma_0}{(x - \mu_0)/\sigma_0} \right) - 1 \right| \\ &\leq \left| \frac{\sigma_0}{\sigma} - 1 \right| + \left| \frac{\mu - \mu_0}{2\sigma_0} \right| + \left| \frac{\sigma_0}{\sigma} - 1 \right| \left| \frac{\mu - \mu_0}{2\sigma_0} \right| \leq \varepsilon + \frac{\varepsilon}{2} + \frac{\varepsilon^2}{2} \leq 2\varepsilon. \quad \square \end{aligned}$$

Lemma D.3 (Lipschitz condition). *Let m_θ be as in (2.2) with p_θ the GEV density function. For fixed $\theta_0 = (\gamma_0, \mu_0, \sigma_0) \in (-1/2, \infty) \times \mathbb{R} \times (0, \infty)$, there exists $\dot{m}$ such that $P_{\theta_0} \dot{m}^2 < \infty$ and such that*

$$|m_{\theta_1} - m_{\theta_2}| \mathbb{1}_{\{p_{\theta_0} > 0\}} \leq \dot{m} \|\theta_1 - \theta_2\| \quad (\text{D.1})$$

for all θ_1 and θ_2 in a neighborhood of θ_0 .

Proof of Lemma D.3. The function $\dot{m}$ can be constructed along the following lines. First, fix $a \in [1/2, 1)$, to be determined later in terms of γ_0 . Since $|\log x - \log y| \leq |x - y| / \min(x, y)$ and since the map $z \mapsto |(x + z)^a - (y + z)^a|$ is decreasing, we have, on $\{x : p_{\theta_0}(x) > 0\}$,

$$\begin{aligned} \frac{1}{2} |m_{\theta_1} - m_{\theta_2}| &= |\log(p_{\theta_1} + p_{\theta_0}) - \log(p_{\theta_2} + p_{\theta_0})| \\ &= \frac{1}{a} |\log\{(p_{\theta_1} + p_{\theta_0})^a\} - \log\{(p_{\theta_2} + p_{\theta_0})^a\}| \\ &\leq \frac{1}{a} \times \frac{|(p_{\theta_1} + p_{\theta_0})^a - (p_{\theta_2} + p_{\theta_0})^a|}{\min\{(p_{\theta_1} + p_{\theta_0})^a, (p_{\theta_2} + p_{\theta_0})^a\}} \\ &\leq \frac{1}{a} \times \frac{|p_{\theta_1}^a - p_{\theta_2}^a|}{p_{\theta_0}^a}. \end{aligned}$$

Suppose we can find a nonnegative function $\dot{p}_a$ such that, for some neighbourhood V of θ_0 , we have, for each $k \in \{1, 2, 3\}$,

$$\sup_{\theta \in V} |\partial_{\theta_k} p_\theta^a(x)| \leq \dot{p}_a(x), \quad x \in \{p_{\theta_0} > 0\}. \quad (\text{D.2})$$

Then we find, on $\{p_{\theta_0} > 0\}$,

$$\frac{1}{2} |m_{\theta_1} - m_{\theta_2}| \leq \frac{3}{a} \frac{\dot{p}_a}{p_{\theta_0}^a} \|\theta_1 - \theta_2\|.$$

Hence, the Lipschitz condition (D.1) is satisfied, with $\dot{m} = (6/a) \dot{p}_a p_{\theta_0}^{-a}$, provided that

$$P_{\theta_0}[(\dot{p}_a p_{\theta_0}^{-a})^2] = \int_{\{p_{\theta_0} > 0\}} \dot{p}_a^2(x) p_{\theta_0}^{1-2a}(x) dx < \infty. \quad (\text{D.3})$$

We will split the domain $\{p_{\theta_0} > 0\}$ into certain intervals and we will use the previous construction on each of these intervals separately, with possibly different values of a . For bounded intervals, a simplification may occur. Recall Lemma B.5 and let $a \in [1/2, 1)$ be large enough such that $\gamma_0 > -a/(1+a)$. Let V be a compact neighbourhood of θ_0 within $(-a/(1+a), \infty) \times \mathbb{R} \times (0, \infty)$ and let I be a bounded interval. Since continuous functions are uniformly bounded on compacta, we have

$$\sup_{(x, \theta) \in I \times V} \max_{k \in \{1, 2, 3\}} |\partial_{\theta_k} p_\theta^a(x)| < \infty. \quad (\text{D.4})$$

Hence, on any bounded interval of the support of P_{θ_0} on which $p_{\theta_0}^{1-2a}$ is integrable, we may choose $\dot{m}$ equal to a constant times $p_{\theta_0}^{-a}$. Such intervals need therefore not be looked into further. For $a = 1/2$, a choice which will occur often, the condition that $p_{\theta_0}^{1-2a}$ is integrable on bounded intervals is trivially satisfied.

To further control the partial derivatives of p_θ^a , we will use the identity

$$\partial_{\theta_k} p_\theta^a(x) = a p_\theta^a(x) \partial_{\theta_k} \ell_\theta(x), \quad k \in \{1, 2, 3\}.$$

We will seek bounds for the functions p_θ^a and $|\partial_{\theta_k} \ell_\theta|$ separately.

To facilitate writing, let us say that positive functions A and B of (x, θ) satisfy

$$A \lesssim B$$

if and only if there exists $c(\theta_0) > 0$ such that $A(x, \theta) \leq c(\theta_0) B(x, \theta)$ for all (x, θ) in the proper range. Here $c(\theta_0)$ is a positive constant whose value may depend on θ_0 . For instance, for all $\theta = (\gamma, \mu, \sigma)$ in a compact neighbourhood of θ_0 within $\mathbb{R} \times \mathbb{R} \times (0, \infty)$, we have $1/\sigma \lesssim 1$. The relation $\lesssim$ is transitive (and reflexive, but not anti-symmetric) and behaves well under multiplication of positive quantities.

I. Case $\gamma_0 > 0$. Fix $\theta_0 = (\gamma_0, \mu_0, \sigma_0) \in (0, \infty) \times \mathbb{R} \times (0, \infty)$ and set $a = 1/2$ in (D.2) and (D.3). Fix $\varepsilon \in (0, 1/4]$ and consider the following neighbourhood V_ε of θ_0 :

$$V_\varepsilon = [(1 - \varepsilon)\gamma_0, (1 + \varepsilon)\gamma_0] \times [\mu_0 - \varepsilon\sigma_0, \mu_0 + \varepsilon\sigma_0] \times \left[\frac{\sigma_0}{1 + \varepsilon}, \frac{\sigma_0}{1 - \varepsilon} \right].$$

The support of P_{θ_0} is $(\mu_0 - \sigma_0/\gamma_0, \infty)$. In view of (D.4), it is sufficient to construct $\dot{p}_{1/2}(x)$ for $x \geq \mu_0 + 2\sigma_0$, i.e., $z_0 = (x - \mu_0)/\sigma_0 \geq 2$. For such x and for $\theta \in V_\varepsilon$, we have $p_\theta(x) > 0$ as well as $|z/z_0 - 1| \leq 2\varepsilon$ by Lemma D.2. In particular, $1 \leq z_0/2 \leq z \leq 2z_0$.

Put $u = (1 + \gamma z)^{-1/\gamma}$; note that $0 < u < 1$. By Lemmas B.2, B.3 and B.4, all three score functions $\partial_{\theta_k} \ell_\theta(x)$ can be bounded in absolute value by a multiple of $1 + \log(1/u)$, uniformly in $\theta \in V_\varepsilon$ and for all $x \geq \mu_0 + 2\sigma_0$. Further, the density can be bounded by

$$p_\theta(x) = \frac{1}{\sigma} e^{-u} u^{\gamma+1} \lesssim u^{\gamma+1}.$$

Hence

$$p_\theta(x) |\partial_{\theta_k} \ell_\theta(x)| \lesssim (1 + \log(1/u)) u^{\gamma+1} \lesssim u^{\gamma+1/2} = (1 + \gamma z)^{-1-1/2\gamma}.$$

Since $z > z_0/2$ and $0 < \gamma_0/2 < \gamma < 2\gamma_0$, the supremum over $\theta \in V_\varepsilon$ of the previous upper bound is easily seen to be integrable over $z_0 \geq 2$.

II. Case $\gamma_0 \in (-1/2, 0)$. Fix $\varepsilon \in (0, 1/12)$ sufficiently small such that $-1/2 < \gamma_0 - 2\varepsilon < \gamma_0 + 2\varepsilon < 0$. Consider the following neighbourhood V_ε of θ_0 :

$$V_\varepsilon = [\gamma_0 - \varepsilon, \gamma_0 + \varepsilon] \times [\mu_0 - \varepsilon\sigma_0, \mu_0 + \varepsilon\sigma_0] \times \left[\frac{\sigma_0}{1 + \varepsilon}, \frac{\sigma_0}{1 - \varepsilon} \right].$$

The support of P_{θ_0} is $(-\infty, \mu_0 + \sigma_0/|\gamma_0|)$. We split this set into two intervals:

$$(-\infty, \mu_0 - 2\sigma_0], \quad (\mu_0 - 2\sigma_0, \mu_0 + \sigma_0/|\gamma_0|).$$

II.1. The interval $(\mu_0 - 2\sigma_0, \mu_0 + \sigma_0/|\gamma_0|)$. We follow the construction leading to (D.2) and (D.3). To this end, we choose $a = a(\gamma_0) \in (1/2, 1)$ in such a way that

$$\frac{|\gamma_0|}{1 - |\gamma_0|} < a < \frac{1}{2(1 - |\gamma_0|)}.$$

Define

$$\dot{p}_a(x) = \sup_{\theta \in V} \max_{k \in \{1, 2, 3\}} |\partial_{\theta_k} p_\theta^a(x)|.$$

Our choice of a entails that $\gamma_0 > -a/(1+a)$. Hence, in view of (D.4), the function $\dot{p}_a$ is bounded on the interval $(\mu_0 - 2\sigma_0, \mu_0 + \sigma_0/|\gamma_0|)$. We need to show that the integral in (D.3) is finite when we restrict the domain to $(\mu_0 - 2\sigma_0, \mu_0 + \sigma_0/|\gamma_0|)$. It is then sufficient to show that the function $p_{\theta_0}^{1-2a}$ is integrable on that set. Write down the integral and make a change of variable $z_0 = (x - \mu_0)/\sigma_0$ to obtain that

$$\begin{aligned} \int_{\mu_0 - 2\sigma_0}^{\mu_0 + \sigma_0/|\gamma_0|} p_{\theta_0}^{1-2a}(x) dx &= \int_{-2}^{1/|\gamma_0|} \sigma_0^{2a} \exp \left\{ -(1-2a)(1 + \gamma_0 z_0)^{-1/\gamma_0} \right\} \\ &\quad \times (1 + \gamma_0 z_0)^{(1-2a)(1/|\gamma_0|-1)} dz_0 \\ &\lesssim \int_{-2}^{1/|\gamma_0|} (1 + \gamma_0 z_0)^{(1-2a)(1/|\gamma_0|-1)} dz_0. \end{aligned}$$

The proportionality constant in the last inequality only depends on θ_0 . The right-hand side is finite since the exponent is larger than -1 by our choice of a .

II.2. The interval $(-\infty, \mu_0 - 2\sigma_0]$. We construct $\dot{m}(x)$ for $x \leq \mu_0 - 2\sigma_0$, i.e., for $z_0 = (x - \mu_0)/\sigma_0 \leq -2$. We will again use the construction leading to (D.2) and (D.3), this time choosing $a = 1/2$. In that case, it is sufficient to construct $\dot{p}_{1/2}$ such that it is square-integrable on $(-\infty, \mu_0 - 2\sigma_0)$ with respect to the Lebesgue measure.

For $x \leq \mu_0 - 2\sigma_0$ and for $\theta \in V_\varepsilon$ we have $p_\theta(x) > 0$ and, by Lemma D.2, $|z/z_0 - 1| \leq 2\varepsilon$ and therefore $2z_0 \leq z \leq z_0/2 \leq -1$.

Since z is negative, we have $u \geq 1$. The density satisfies

$$p_\theta(x) = \frac{1}{\sigma} e^{-u} u^{\gamma+1} \lesssim e^{-u} u.$$

By Lemmas B.2, B.3 and B.4, the three score functions $\partial_{\theta_k} \ell_\theta(x)$ can all be bounded by a multiple of u^2 , the proportionality constant not depending on $x \leq \mu_0 - 2\sigma_0$ nor on $\theta \in V_\varepsilon$. Hence, for all $k \in \{1, 2, 3\}$, all $\theta \in V_\varepsilon$ and all $x \leq \mu_0 - 2\sigma_0$,

$$p_\theta(x) |\partial_{\theta_k} \ell_\theta(x)|^2 \lesssim e^{-u} u^5 \lesssim e^{-u/2}.$$

(Note that $\sup_{u \geq 1} u^m e^{-u/2} < \infty$ for any scalar m .) Since $\gamma \geq \gamma_0 - \varepsilon$ and $z \leq z_0/2 < 0$, we have $u \geq \{1 + (\gamma_0 - \varepsilon)z_0/2\}^{-1/(\gamma_0 - \varepsilon)}$. Inserting this bound into the last display yields a function which is integrable over $z_0 \in (-\infty, -2)$.

III. Case $\gamma_0 = 0$. For fixed $\varepsilon \in (0, 1/6]$, consider the following compact neighbourhood of $\theta_0 = (0, \mu_0, \sigma_0)$:

$$V_\varepsilon = [-\varepsilon, \varepsilon] \times [\mu_0 - \sigma_0\varepsilon, \mu_0 + \sigma_0\varepsilon] \times \left[\frac{\sigma_0}{1+\varepsilon}, \frac{\sigma_0}{1-\varepsilon} \right].$$

Partition this set in two pieces, according to the sign of γ :

$$V_{\varepsilon,+} = \{\theta \in V_\varepsilon : \gamma \geq 0\}, \quad V_{\varepsilon,-} = \{\theta \in V_\varepsilon : \gamma \leq 0\}.$$

The support of the Gumbel distribution is $\mathbb{R}$, which we will decompose into three intervals:

$$(-\infty, \mu_0 - \sigma_0/\varepsilon], \quad [\mu_0 - \sigma_0/\varepsilon, \mu_0 + \sigma_0/\varepsilon], \quad [\mu_0 + \sigma_0/\varepsilon, \infty).$$

The middle interval is bounded. By (D.4) with $a = 1/2$, we only need to consider the cases $z_0 \geq 1/\varepsilon$ and $z_0 \leq -1/\varepsilon$, where $z_0 = (x - \mu_0)/\sigma_0$. Put $z = (x - \mu)/\sigma$ and note that $|z/z_0 - 1| \leq 2\varepsilon$ by Lemma D.2.

III.1. Case $z_0 \geq 1/\varepsilon$. We have $z_0 \geq 1/\varepsilon \geq 6$ and $4 \leq (1 - 2\varepsilon)z_0 \leq z \leq (1 + 2\varepsilon)z_0$. Moreover, $0 < u < 1$. We will write the supremum over $\theta \in V_\varepsilon$ as the maximum of the suprema over $\theta \in V_{\varepsilon,+}$ and $\theta \in V_{\varepsilon,-}$.

III.1.1. Case $\theta \in V_{\varepsilon,+}$. In this case, always $1 + \gamma z > 1$. The bounds on the scores in Lemmas B.2, B.3, and B.4 imply that

$$|\partial_\gamma \ell_\theta(x)| \lesssim z^2 \lesssim z_0^2, \quad |\partial_\mu \ell_\theta(x)| \lesssim 1, \quad |\partial_\sigma \ell_\theta(x)| \lesssim z \lesssim z_0.$$

Further, we have

$$p_\theta(x) \lesssim (1 + \gamma z)^{-1/\gamma-1} \leq (1 + \gamma z)^{-1/\gamma} \leq (1 + \varepsilon z)^{-1/\varepsilon} \leq \{1 + \varepsilon(1 - 2\varepsilon)z_0\}^{-6}.$$

It follows that, for each $k \in \{1, 2, 3\}$,

$$\sup_{\theta \in V_{\varepsilon,+}} p_\theta(x) |\partial_{\theta_k} \ell_\theta(x)|^2 \lesssim \{1 + \varepsilon(1 - 2\varepsilon)z_0\}^{-6} z_0^4.$$

The right-hand side is integrable over $z_0 \in [1/\varepsilon, \infty)$.

III.1.2. Case $\theta \in V_{\varepsilon,-}$. If $-\varepsilon \leq \gamma \leq -1/z$, then $1 + \gamma z \leq 0$ and thus $p_\theta(x) = 0$. So suppose $\gamma > -1/z$. By Lemmas B.2, B.3, and B.4, the scores can be bounded as follows:

$$\max_{k \in \{1, 2, 3\}} |\partial_{\theta_k} \ell_\theta(x)| \lesssim \frac{z^2}{1 + \gamma z}. \quad (\text{D.5})$$

Moreover, the density is bounded by

$$p_\theta(x) \lesssim (1 + \gamma z)^{-1/\gamma-1}.$$

Hence, for $k \in \{1, 2, 3\}$, since $\gamma \mapsto (1 + \gamma z)^{-1/\gamma}$ is increasing in γ , on $\{\gamma : 1 + \gamma z\}$,

$$\begin{aligned} \sup_{\theta \in V_{\varepsilon,+}} p_\theta(x) |\partial_{\theta_k} \ell_\theta(x)|^2 &\lesssim z^4 (1 + \gamma z)^{-1/\gamma-3} \\ &= z^4 \{(1 + \gamma z)^{-1/\gamma}\}^{1+3\gamma} \leq z^4 e^{-(1+3\gamma)z} \leq z^4 e^{-z/2} \lesssim e^{-z/4}, \end{aligned}$$

as $\sup_{z \geq 0} z^4 e^{-z/4} < \infty$. Bounding z in $e^{-z/4}$ from below by $z_0/2$ yields a function which is integrable over $z_0 \in [1/\varepsilon, \infty)$.

III.2. Case $z_0 \leq -1/\varepsilon$. We have $z_0 \leq -1/\varepsilon \leq -6$ and, by Lemma D.2, $(1 + 2\varepsilon)z_0 \leq z \leq (1 - 2\varepsilon)z_0 \leq -4$. Moreover, $u > 1$.

III.2.1. Case $\theta \in V_{\varepsilon,+}$. If $\gamma \geq 1/|z|$, then $p_\theta(x) = 0$. Assume $0 \leq \gamma < 1/|z|$, so that $1 + \gamma z > 0$. By (B.12),

$$|\partial_\gamma \ell_\theta(x)| \leq u^{1+\gamma} \{(\log u)^2 + \log u\} \lesssim u^{1+2\varepsilon}. \quad (\text{D.6})$$

Further, by Lemmas B.2 and B.3,

$$|\partial_\mu \ell_\theta(x)| \lesssim u^{1+\gamma} \leq u^{1+\varepsilon}, \quad |\partial_\sigma \ell_\theta(x)| \lesssim u^{1+\gamma} \{\log(u) + 1\} \lesssim u^{1+2\varepsilon}. \quad (\text{D.7})$$

The density is bounded by

$$p_\theta(x) \lesssim e^{-u} u^{1+\gamma}.$$

In total, since $\sup_{u \geq 1} u^m e^{-u/2} < \infty$ for any scalar m ,

$$p_\theta(x) |\partial_{\theta_k} \ell_\theta(x)|^2 \lesssim e^{-u} u^{(1+\gamma)+2(1+2\varepsilon)} \lesssim e^{-u} u^{3+5\varepsilon} \lesssim e^{-u/2}.$$

Since $\gamma \mapsto (1 + \gamma z)^{-1/\gamma}$ is increasing in γ , a lower bound for u is given by $u \geq e^{-z} = e^{|z|} \geq e^{|z_0|/2}$. Plugging this bound into $e^{-u/2}$ yields a function which is integrable over $z_0 \in (-\infty, -1/\varepsilon]$.

III.2.2. Case $\theta \in V_{\varepsilon,-}$. By Lemmas B.2, B.3 and B.4,

$$|\partial_\gamma \ell_\theta(x)| \lesssim u^{1+\varepsilon}, \quad |\partial_\mu \ell_\theta(x)| \lesssim u^{1+\varepsilon}, \quad |\partial_\sigma \ell_\theta(x)| \lesssim u \log(u).$$

The density is bounded by

$$p_\theta(x) \lesssim e^{-u} u^{1+\gamma}.$$

We get

$$\sup_{\theta \in V_{\varepsilon,-}} p_\theta(x) |\partial_{\theta_k} \ell_\theta(x)|^2 \lesssim e^{-u} u^{3+3\varepsilon} \lesssim e^{-u/2}.$$

Further, u can be bounded from below by $(1 + (-\varepsilon)(1 - 2\varepsilon)z_0)^{-1/(-\varepsilon)}$. Inserting this into $e^{-u/2}$ yields an integrable function in $z_0 \in (-\infty, -1/\varepsilon]$. $\square$

Lemma D.4. *Let P_θ denote the three-parameter GEV distribution with parameter θ and let m_θ be as in (2.2). For fixed θ_0 , the map $\theta \mapsto P_{\theta_0} m_\theta$ attains a unique maximum over $(-1/2, \infty) \times \mathbb{R} \times (0, \infty)$ at $\theta = \theta_0$. If moreover $\gamma_0 > -1/2$, then whenever $h_t \rightarrow h$ in $\mathbb{R}^k$ as $t \rightarrow 0$, we have*

$$t^{-2} P_{\theta_0}(m_{\theta_0+th_t} - m_{\theta_0}) \rightarrow -\frac{1}{4} h^T I_{\theta_0} h, \quad t \rightarrow 0.$$

Proof. The fact that θ_0 is the unique maximizer of the map $\theta \mapsto P_{\theta_0}$ follows from Lemma 5.35 in van der Vaart (1998) applied to the mixture densities $(p_\theta + p_{\theta_0})/2$.

The second-order Taylor expansion can now be shown along similar lines as in the proof of Theorem 5.39 in van der Vaart (1998). In fact, the same technique was used to show (A.5) in the proof of Proposition 2.1. $\square$

APPENDIX E. REMAINING STEPS FOR THE PROOF OF PROPOSITION 3.3

In view of the outline of the proof given right after the statement of the Proposition 3.3, it remains to check the condition on the support in (2.3) and the Lipschitz property (2.4). This is the content of the present section.

Lemma E.1. *For any $\theta_0 \in (-1/2, \infty) \times \mathbb{R} \times (0, \infty)$, the three-parameter GEV family satisfies Condition (2.3).*

Proof. First, consider $\theta_0 \in \Theta_{0,-} = (-\infty, 0) \times \mathbb{R} \times (0, \infty)$ such that $\gamma_0 > -1/2$. Let $\omega(\theta) = \mu - \sigma/\gamma = \mu + \sigma/|\gamma|$ and $\omega_0 = \omega(\theta_0)$. Since $\theta \mapsto \omega(\theta)$ is increasing in each component of $\theta \in \Theta_{0,-}$, we have

$$h_-(\varepsilon) := \inf\{\omega(\theta) : \theta \in U_\varepsilon(\theta_0)\} = \omega(\gamma_0 - \varepsilon, \mu_0 - \varepsilon, \sigma_0 - \varepsilon)$$

whence $\bar{S}(\varepsilon) = (-\infty, h_-(\varepsilon))$. The function $t \mapsto h_-(t)$ is continuously differentiable on a neighbourhood of $t = 0$ and with negative derivative at $t = 0$. Hence, we can find constants $0 < b_1 < c_1$ and $t_1 > 0$ such that

$$\omega_0 - t \cdot c_1 \sigma_0 / |\gamma_0| \leq h_-(t) \leq \omega_0 - t \cdot b_1 \sigma_0 / |\gamma_0|, \quad t \in [0, t_1]. \quad (\text{E.1})$$

As a consequence of (E.1), for sufficiently small $\varepsilon > 0$,

$$\mathcal{X} \setminus \bar{S}(\varepsilon) \subset [\omega_0 - \varepsilon c_1 \sigma_0 / |\gamma_0|, \infty).$$

Therefore, using the substitution $u = \{1 + \gamma_0(x - \mu_0)/\sigma_0\}^{1/|\gamma_0|}$,

$$P_{\theta_0}(\mathcal{X} \setminus \bar{S}(\varepsilon)) \leq \int_{\omega_0 - c_1 \varepsilon \sigma_0 / |\gamma_0|}^{\omega_0} p_{\theta_0}(x) dx = \int_0^{(\varepsilon c_1)^{1/|\gamma_0|}} e^{-u} du = O(\varepsilon^{1/|\gamma_0|}) = o(\varepsilon^2)$$

as $\varepsilon \downarrow 0$, since $-1/2 < \gamma_0 < 1$.

Now, consider $\theta_0 \in \Theta_{0,+} = (0, \infty) \times \mathbb{R} \times (0, \infty)$, i.e., $\gamma_0 > 0$. Then, for any $\varepsilon < \min(\sigma_0, \gamma_0)$ and any $\theta \in U_\varepsilon(\theta_0)$, we have

$$\omega(\theta) = \mu - \frac{\sigma}{\gamma} \leq \mu_0 + \varepsilon - \frac{\sigma_0 - \varepsilon}{\gamma_0 + \varepsilon} =: h_+(\varepsilon).$$

whence $\bar{S}(\varepsilon) = (h_+(\varepsilon), \infty)$. The function $t \mapsto h_+(\varepsilon)$ is continuously differentiable on a neighbourhood of $t = 0$, with positive derivative at $t = 0$. As a consequence, we can find constants $0 < b_2 < c_2$ and $t_2 > 0$ such that

$$\omega_0 + t \cdot b_2 \sigma_0 / \gamma_0 \leq h_+(t) \leq \omega_0 + t \cdot c_2 \sigma_0 / \gamma_0, \quad t \in [0, t_2]. \quad (\text{E.2})$$

Hence, for sufficiently small $\varepsilon > 0$, we obtain that

$$\mathcal{X} \setminus \bar{S}(\varepsilon) \subset (-\infty, \omega_0 + \varepsilon \cdot c_2 \sigma_0 / \gamma_0].$$

Therefore, using the substitution $u = \{1 + \gamma_0(x - \mu_0)/\sigma_0\}^{-1/\gamma_0}$,

$$P_{\theta_0}(\mathcal{X} \setminus \bar{S}(\varepsilon)) \leq \int_{\omega_0}^{\omega_0 + \varepsilon c_2 \sigma_0 / \gamma_0} p_{\theta_0}(x) dx = \int_{(c_2 \varepsilon)^{-1/\gamma_0}}^{\infty} e^{-u} du = \exp\{-(c_2 \varepsilon)^{-1/\gamma_0}\},$$

which clearly is of the order $o(\varepsilon^2)$ as $\varepsilon \downarrow 0$, since $\gamma_0 > 0$.

Finally, consider $\theta_0 \in \Theta_0$ with $\gamma_0 = 0$. Then, for any $\varepsilon < \sigma_0/2$, we have

$$\omega(\theta) = \mu - \frac{\sigma}{\gamma} \leq \mu_0 + \varepsilon - \frac{\sigma_0 - \varepsilon}{\varepsilon} =: h_{0,+}(\varepsilon), \quad \theta \in U_\varepsilon(\theta_0) \cap \Theta_{0,+},$$

as well as

$$\omega(\theta) = \mu + \frac{\sigma}{|\gamma|} \geq \mu_0 - \varepsilon + \frac{\sigma_0 - \varepsilon}{\varepsilon} =: h_{0,-}(\varepsilon), \quad \theta \in U_\varepsilon(\theta_0) \cap \Theta_{0,-}.$$

As a consequence, for sufficiently small ε , $\bar{S}(\varepsilon) = (h_{0,+}(\varepsilon), h_{0,-}(\varepsilon))$. Clearly, we can find constants $d_1, d_2 > 0$ such that $h_{0,+}(\varepsilon) \leq -d_1/\varepsilon$ and $h_{0,-}(\varepsilon) \geq d_2/\varepsilon$ for all sufficiently small ε . Hence,

$$\begin{aligned} P_{\theta_0}(\mathcal{X} \setminus \bar{S}(\varepsilon)) &\leq \int_{-\infty}^{-d_1/\varepsilon} p_{\theta_0}(x) dx + \int_{d_2/\varepsilon}^{\infty} p_{\theta_0}(x) dx \\ &= \exp\left\{-\exp\left(\frac{d_1}{\sigma_0 \varepsilon} + \frac{\mu_0}{\sigma_0}\right)\right\} + 1 - \exp\left\{-\exp\left(-\frac{d_2}{\sigma_0 \varepsilon} + \frac{\mu_0}{\sigma_0}\right)\right\}, \end{aligned}$$

which can be easily seen to be of the order $o(\varepsilon^2)$ as $\varepsilon \downarrow 0$, since $1 - \exp(-y) = (1 + o(1))y$ as $y \rightarrow 0$. $\square$

Lemma E.2. *For any $\theta_0 \in (-1/2, \infty) \times \mathbb{R} \times (0, \infty)$, the three-parameter GEV family satisfies Condition (2.4) with $\Theta = (-1/2, \infty) \times \mathbb{R} \times (0, \infty)$ (and hence also with any compact subset containing θ_0 in its interior).*

Proof. Throughout, let $\|\cdot\|$ denote the maximum norm on $\mathbb{R}^3$. The proof will be split up in three cases, according to the sign of γ_0 . For some suitable $\varepsilon_0 > 0$ to be specified below, and for any $x \in S_{\theta_0}$, define a set of *admissible parameter vectors* θ as

$$\Theta_0(x) = \{\theta \in U_{\varepsilon_0}(\theta_0) : \text{if } \theta' \in \Theta \text{ satisfies } \|\theta' - \theta_0\| \leq 2\|\theta - \theta_0\|, \text{ then } x \in S_{\theta'}\}.$$

Note that, if $\theta \in \Theta_0(x)$, then the entire ball with center θ_0 and radius $\|\theta - \theta_0\|$ is a subset of $\Theta_0(x)$. In other words, $\Theta_0(x)$ is the union of all neighbourhoods $U_\varepsilon(\theta_0)$ for those $\varepsilon \in (0, \varepsilon_0]$ such that $x \in S_{\theta'}$ for every $\theta' \in U_{2\varepsilon}(\theta_0)$. [The multiplicative constant 2 is not essential and could have been replaced by an arbitrary constant $c_0 > 1$.]

It follows from the definition of $\Theta_0(x)$ that $\theta \mapsto \ell_\theta(x)$ is continuously differentiable on $\Theta_0(x)$. Hence, we may define

$$\dot{\ell}(x) = 3 \sup\{\|\dot{\ell}_\theta(x)\| : \theta \in \Theta_0(x)\}, \quad x \in S_{\theta_0}, \quad (\text{E.3})$$

and, by the mean-value theorem, immediately obtain that

$$\forall x \in S_{\theta_0} : \forall \theta_1, \theta_2 \in \Theta_0(x) : |\ell_{\theta_1}(x) - \ell_{\theta_2}(x)| \leq \dot{\ell}(x) \|\theta_1 - \theta_2\|. \quad (\text{E.4})$$

[The constant 3 appears because of the use of the max-norm.] It can be seen easily that (E.4) implies (2.4). The proof will be finished once we will have constructed ε_0 and will have showed that the function in (E.3) is square-integrable with respect to P_{θ_0} .

I. Case $\gamma_0 \in (0, \infty)$. Write $\omega_0 = \mu_0 - \sigma_0/\gamma_0$ and recall that $S_{\theta_0} = (\omega_0, \infty)$. For $x \in S_{\theta_0}$, write $z_0 = z_0(x) = (x - \mu_0)/\sigma_0$. Note that $z_0 > -1/\gamma_0$.

For a parameter vector θ with $\gamma > 0$, we have $S_\theta = (\omega(\theta), \infty)$ where $\omega(\theta) = \mu - \sigma/\gamma$. A monotonicity and differentiability argument similar to that yielding (E.2) shows that there exist positive constants b_0 and t_0 with $b_0 t_0 \leq 1/2$ and $t_0 < \min(\sigma_0, \gamma_0)/2$ such that for all $t \in [0, t_0]$,

$$h_+(t) = \sup_{\theta \in U_t(\theta_0)} \omega(\theta) = \mu_0 + t - \frac{\sigma_0 - t}{\gamma_0 + t} \quad \begin{cases} \leq & \omega_0 + (\sigma_0/\gamma_0) t b_0 (4/3), \\ \geq & \omega_0 + (\sigma_0/\gamma_0) t b_0. \end{cases}$$

Partition the interval S_{θ_0} into three sub-intervals:

$$S_{\theta_0} = (\omega_0, \omega_0 + (\sigma_0/\gamma_0) t_0 b_0) \cup [\omega_0 + (\sigma_0/\gamma_0) t_0 b_0, \mu_0 + 2\sigma_0] \cup (\mu_0 + 2\sigma_0, \infty).$$

Choose $0 < \varepsilon_0 < \min\{1/4, t_0/2, \sigma_0/(2\gamma_0)\}$ small enough such that

$$\sigma/\sigma_0, \gamma/\gamma_0 \in [5/6, 7/6], \quad (\sigma/\gamma)/(\sigma_0/\gamma_0) \in [3/4, 4/3], \quad \text{for all } \theta \in U_{\varepsilon_0}(\theta_0).$$

We will provide an upper bound for $\dot{\ell}(x)$ in (E.3) for each x in each piece of the partition separately.

First, suppose that $x \in S_{\theta_0}$ is such that $x = \omega_0 + (\sigma_0/\gamma_0)y$ with $0 < y < t_0 b_0$. If θ is such that $t := \|\theta - \theta_0\|$ satisfies $t < \varepsilon_0 \leq t_0/2$ and $2tb_0 \geq y$, then $h_+(2t) \geq \omega_0 + (\sigma_0/\gamma_0)(2tb_0) \geq x$ and thus $\theta \notin \Theta_0(x)$. As a consequence, $\Theta_0(x) \subset U_{y/(2b_0)}(\theta_0)$. But for $\theta \in U_{y/(2b_0)}(\theta_0)$, we have $\omega(\theta) \leq \omega_0 + (\sigma_0/\gamma_0)(y/2)(4/3) = \omega_0 + (\sigma_0/\gamma_0)(2/3)y$ and thus $x - \omega(\theta) \geq (\sigma_0/\gamma_0)(1/3)y$. For such θ , writing $z = (x - \mu)/\sigma$, we find, using $y = 1 + \gamma_0 z_0$, that $1 + \gamma z = (\gamma/\sigma)(x - \omega(\theta)) \geq (\gamma/\sigma)(\sigma_0/\gamma_0)(1/3)y \geq (1/4)(1 + \gamma_0 z_0)$. In addition, $x < \omega_0 + (\sigma_0/\gamma_0)(1/2) = \mu_0 - (\sigma_0/\gamma_0)(1/2) < \mu_0 - \varepsilon_0 \leq \mu$ and thus $z < 0$. Apply (B.3), (B.4) and (B.12) to arrive at a P_{θ_0} square-integrable bound.

Second, suppose that $x \in S_{\theta_0}$ is such that $\omega_0 + (\sigma_0/\gamma_0) t_0 b_0 \leq x \leq \mu_0 + 2\sigma_0$. The partial derivatives of $\ell_\theta(x)$ with respect to the three components of θ being continuous functions of (θ, x) in the compact domain $\{\theta : \|\theta - \theta_0\| \leq \varepsilon_0\} \times [\omega_0 + (\sigma_0/\gamma_0) t_0 b_0, \mu_0 + 2\sigma_0]$, they are also uniformly bounded and thus square-integrable with respect to P_{θ_0} .

Third, suppose that $x \in S_{\theta_0}$ is such that $z_0 \geq 2$. Let $\theta \in U_{\varepsilon_0}(\theta_0)$ and write $z = (x - \mu)/\sigma$. By Lemma D.2 we have $|(z/z_0) - 1| \leq 1/2$ and thus $z_0/2 \leq z \leq 3z_0/2$. The inequalities in Lemmas B.2, B.3 and B.4 then combine into a P_{θ_0} square-integrable upper bound for $\dot{\ell}$.

II. Case $\gamma_0 \in (-1/2, 0)$. Write $\omega_0 = \mu_0 + \sigma_0/|\gamma_0|$ and recall that $S_{\theta_0} = (-\infty, \omega_0)$. For $x \in S_{\theta_0}$, write $z_0 = z_0(x) = (x - \mu_0)/\sigma_0$. Note that $z_0 < 1/|\gamma_0|$.

For a parameter vector θ with $\gamma < 0$, we have $S_\theta = (-\infty, \omega(\theta))$ where $\omega(\theta) = \mu + \sigma/|\gamma|$. A monotonicity and differentiability argument similar to that yielding (E.1) shows that there exist positive constants b_0 and t_0 with $b_0 t_0 \leq 1/2$ and $t_0 < \min(\sigma_0, |\gamma_0|)/2$ such that for all $t \in [0, t_0]$,

$$h_-(t) = \inf_{\theta \in U_t(\theta_0)} \omega(\theta) = \mu_0 - t - \frac{\sigma_0 - t}{\gamma_0 - t} \begin{cases} \leq \omega_0 - (\sigma_0/|\gamma_0|) t b_0, \\ \geq \omega_0 - (\sigma_0/|\gamma_0|) t b_0 (4/3). \end{cases}$$

Partition the interval S_{θ_0} into three sub-intervals:

$$S_{\theta_0} = (-\infty, \mu_0 - 2\sigma_0) \cup [\mu_0 - 2\sigma_0, \omega_0 - (\sigma_0/|\gamma_0|) t_0 b_0] \cup (\omega_0 - (\sigma_0/|\gamma_0|) t_0 b_0, \omega_0).$$

Choose $0 < \varepsilon_0 < \min\{1/4, t_0/2, \sigma_0/(2|\gamma_0|), \sigma_0/7\}$ small enough such that

$$\sigma/\sigma_0, \gamma/\gamma_0 \in [5/6, 7/6], \quad (\sigma/\gamma)/(\sigma_0/\gamma_0) \in [3/4, 4/3], \quad \text{for all } \theta \in U_{\varepsilon_0}(\theta_0).$$

We will provide an upper bound for $\dot{\ell}(x)$ in (E.3) for each x in each piece of the partition separately.

First, consider $x \in (\omega_0 - (\sigma_0/|\gamma_0|) t_0 b_0, \omega_0)$, which can be rewritten as $x = \omega_0 - (\sigma_0/|\gamma_0|) y$ for some $0 < y < t_0 b_0$. Then $\Theta_0(x) \subset U_{y/(2b_0)}(\theta_0)$, for, if $\theta \in U_{\varepsilon_0}(\theta_0) \subset U_{t_0/2}(\theta_0)$ satisfies $t = \|\theta - \theta_0\| \geq y/(2b_0)$, then there exists θ' with $\|\theta' - \theta_0\| \leq 2t$ such that $\omega(\theta') = h_-(2t) \leq \omega_0 - (\sigma_0/|\gamma_0|) 2t b_0 \leq x$. Now, for $\theta \in U_{y/(2b_0)}(\theta_0)$, we have $\omega(\theta) - x \geq \omega_0 - \sigma_0/|\gamma_0| (y/2)(4/3) - x = \omega_0 - (2/3)(\omega_0 - x) - x = (1/3)(\omega_0 - x)$. In addition, $x > \omega_0 - \sigma_0/|\gamma_0|/2 = \mu_0 + \sigma_0/|\gamma_0| > \mu_0 + \varepsilon_0 > \mu$, whence $z = (x - \mu)/\sigma > 0$. Therefore, as a consequence of Lemmas B.2, B.3 and B.4, we obtain the upper bound

$$\|\dot{\ell}_\theta(x)\| \lesssim \frac{1}{1 + \gamma z} = \frac{\sigma}{|\gamma|} \frac{1}{\omega(\theta) - x} \leq \frac{(4/3)\sigma_0}{(1/3)|\gamma_0|} \frac{1}{\omega_0 - x}.$$

Using that $\gamma_0 \in (-1/2, 0)$ implies $1/|\gamma_0| > 2$, the bound can be seen to be square-integrable with respect to P_{θ_0} .

Second, consider $x \in [\mu_0 - 2\sigma_0, \omega_0 - (\sigma_0/|\gamma_0|) t_0 b_0]$. The partial derivatives of $\ell_\theta(x)$ with respect to the three components of θ being continuous functions of (θ, x) in the compact domain $\{\theta : \|\theta - \theta_0\| \leq \varepsilon_0\} \times [\mu_0 - 2\sigma_0, \omega_0 - (\sigma_0/|\gamma_0|) t_0 b_0]$, they are also uniformly bounded and thus square-integrable with respect to P_{θ_0} .

Third, consider $x \in (-\infty, \mu_0 - 2\sigma_0)$. Let $\theta \in U_{\varepsilon_0}(\theta_0)$ and write $z = (x - \mu)/\sigma$. By Lemma D.2, we have $|(z/z_0) - 1| \leq 1/2$ and thus $3z_0/2 \leq z \leq z_0/2 \leq -1$. The inequalities in Lemmas B.2, B.3 and B.4 then combine into a P_{θ_0} square-integrable upper bound for $\dot{\ell}$.

III. Case $\gamma_0 = 0$. Recall that $S_{\theta_0} = \mathbb{R}$. Find $t_0 > 0$ small enough such that $(1 + t_0)t_0 \leq \sigma_0/4$. Then we have

$$(3/4)\sigma_0/t \leq (\sigma_0/t) - t - 1 \leq \sigma_0/t, \quad \text{for } t \in (0, t_0].$$

It follows that, for all $t \in (0, t_0]$,

$$h_{0,+}(t) = \sup_{\theta \in U_t(\theta_0) \cap \Theta_{0,+}} \omega(\theta) = \mu_0 + t - \frac{\sigma_0 - t}{t} \quad \begin{cases} \leq & \mu_0 - (3/4)\sigma_0/t, \\ \geq & \mu_0 - \sigma_0/t. \end{cases} \quad (\text{E.5})$$

$$h_{0,-}(t) = \inf_{\theta \in U_t(\theta_0) \cap \Theta_{0,-}} \omega(\theta) = \mu_0 - t + \frac{\sigma_0 - t}{t} \quad \begin{cases} \leq & \mu_0 + \sigma_0/t, \\ \geq & \mu_0 + (3/4)\sigma_0/t. \end{cases} \quad (\text{E.6})$$

Choose $0 < \varepsilon_0 < \min\{1/4, t_0/2\}$ small enough such that $\sigma/\sigma_0 \in [5/6, 7/6]$ whenever $\theta \in U_{\varepsilon_0}(\theta_0)$. Partition the real line into three sub-intervals:

$$\mathbb{R} = (-\infty, \mu_0 - \sigma_0/\varepsilon_0) \cup [\mu_0 - \sigma_0/\varepsilon_0, \mu_0 + \sigma_0/\varepsilon_0] \cup (\mu_0 + \sigma_0/\varepsilon_0, \infty).$$

We will provide an upper bound for $\dot{\ell}(x)$ in (E.3) for each x in each piece of the partition separately. Write $x = \mu_0 + \sigma_0 z_0$.

III.1 – Case $x < \mu_0 - \sigma_0/\varepsilon_0$. We have $z_0 < -1/\varepsilon_0$. For any $\theta \in \Theta_0(x) \subset U_{\varepsilon_0}(\theta_0)$, we have $x < \mu_0 - \sigma_0/\varepsilon_0 < \mu_0 - \varepsilon_0 \leq \mu$; note that $\varepsilon_0^2 < t_0^2 < \sigma_0$. Therefore, $z = z_\theta(x) = (x - \mu)/\sigma < 0$ and thus $u = u_\gamma(z) > 1$, see Figure 1.

We claim that $\Theta_0(x) \subset U_{1/(2|z_0|)}(\theta_0)$. To prove this, it suffices to show that we can find a point θ' such that $\|\theta' - \theta_0\| = 2/(2|z_0|) = 1/|z_0|$ and such that $x \notin S_{\theta'}$. But this is easy: just set $\theta' = (1/|z_0|, \mu_0, \sigma_0)$ and note that $1 + (1/|z_0|)z_0 = 1 - 1 = 0$.

Let $\theta \in \Theta_0(x) \subset U_{1/(2|z_0|)}(\theta_0)$. Write $z = (x - \mu)/\sigma$. Our choice of ε_0 and the fact that $|z_0| > 1/\varepsilon_0 > 2$ imply that $|z/z_0 - 1| \leq 1/2$; see Lemma D.2. Consider three subcases: $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$.

- Suppose $\gamma > 0$. By (E.5), we have $\omega(\theta) \leq \mu_0 - (3/4)\sigma_0(2|z_0|) = \mu_0 + (3/2)\sigma_0 z_0 < \mu_0 + \sigma_0 z_0 = x$. Hence, $x \in S_\theta$. As $\gamma > 0$ and $z < 0$, the bounds in (D.6) and (D.7) then imply that $\|\dot{\ell}_\theta(x)\| \lesssim u^{1+2\varepsilon_0}$. We need to bound u from above. Recall that $u_\gamma(z)$ is decreasing in z and increasing in γ . Moreover, $(3/2)z_0 \leq z < 0$ and $0 < \gamma < 1/(2|z_0|)$. We find

$$1 \leq u \leq \left(1 + \frac{1}{2|z_0|} \frac{3}{2} z_0\right)^{-2|z_0|} = (1 - (3/4))^{-2|z_0|} = 16^{-z_0}$$

For arbitrary $\rho > 1$, the function $z_0 \mapsto \rho^{-z_0}$ is integrable with respect to the standard Gumbel density $z_0 \mapsto \exp(-e^{-z_0}) e^{-z_0}$.

- Suppose $\gamma = 0$. The expressions for the score functions are $\partial_\gamma \ell_\theta(x) = (1 - e^{-z})z^2/2 - z$, $\partial_\mu \ell_\theta(x) = (1 - e^{-z})/\sigma$, and $\partial_\sigma \ell_\theta(x) = ((1 - e^{-z})z - 1)/\sigma$. Since $(3/2)z_0 \leq z \leq (1/2)z_0 < 0$ for all θ considered, these expressions can be easily bounded by P_{θ_0} -square integrable functions.
- Suppose $\gamma < 0$. Since $z < 0$, the components of the score vector can be bounded by a multiple of $u \max\{z, (\log u)^2\}$. But since $u_\gamma(z)$ is increasing in γ and since $\gamma < 0$, we can bound u by its value at $\gamma = 0$, which is e^{-z} . Now continue as for the case $\gamma = 0$.

III.2 – Case $\mu_0 - \sigma_0/\varepsilon_0 \leq x \leq \mu_0 + \sigma_0/\varepsilon_0$. This case is trivial, since the three components of the score vector $\dot{\ell}_\theta(x)$ are continuous and thus uniformly bounded on the closure of the bounded domain $\{(x, \theta) : |x - \mu_0| \leq \sigma_0/\varepsilon_0, \theta \in \Theta_0(x)\}$.

III.3 – Case $x > \mu_0 + \sigma_0/\varepsilon_0$. This case is partially similar to the case $x < \mu_0 - \sigma_0/\varepsilon_0$. For $\theta \in \Theta_0(x) \subset U_{\varepsilon_0}(\theta_0)$, we have $x > \mu_0 + \varepsilon_0 > \mu$ and thus $z = (x - \mu)/\sigma > 0$ and $0 < u = u_\gamma(z) < 1$. Moreover, $z_0 > 1/\varepsilon_0$.

We claim that $\Theta_0(x) \subset U_{1/(2z_0)}(\theta_0)$. Indeed, the point $\theta' = (-1/z_0, \mu_0, \sigma_0)$ is such that $\|\theta - \theta_0\| = 1/z_0 = 2/(2z_0)$ and still $x \notin S_{\theta'}$.

Let $\theta \in \Theta_0(x) \subset U_{1/(2z_0)}(\theta_0)$ and write $z = (x - \mu)/\sigma$. Again, we have $|z/z_0 - 1| \leq 1/2$. Consider three subcases: $\gamma > 0$, $\gamma = 0$, and $\gamma < 0$.

- Suppose $\gamma > 0$. Since $0 < u < 1$ and $1 + \gamma z > 1$, all three components of the score vector can be bounded by a constant multiple of z^2 and thus by a constant multiple of z_0^2 . Now it suffices to observe that all moments of the Gumbel distribution are finite.
- Suppose $\gamma = 0$. Then apply the same reasoning as for the subcase $\gamma = 0$ in the case III.1 above.
- Suppose $\gamma < 0$. By (E.6), we have $\omega(\theta) \geq \mu_0 + (3/4)\sigma_0(2z_0) = \mu_0 + (3/2)\sigma_0 z_0 \leq \mu_0 + \sigma_0 z_0 = x$, so that $x \in S_\theta$. Since $0 > \gamma > -1/(2z_0) \geq -3/(4z) > -1/z$, the bound in (D.5) applies. We need to find a square-integrable bound on $z^2/(1 + \gamma z)$. But this is immediate, since the denominator is larger than $1 - 3/4 = 1/4$ and the numerator is smaller than a multiple of z_0^2 . $\square$

ACKNOWLEDGMENTS

The research by A. Bücher has been supported by the Collaborative Research Center “Statistical modeling of nonlinear dynamic processes” (SFB 823, Project A7) of the German Research Foundation, which is gratefully acknowledged. Parts of this paper were written when A. Bücher was a postdoctoral researcher at Ruhr-Universität Bochum.

J. Segers gratefully acknowledges funding by contract “Projet d’Actions de Recherche Concertées” No. 12/17-045 of the “Communauté française de Belgique” and by IAP research network Grant P7/06 of the Belgian government (Belgian Science Policy).

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